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A K-THEORETIC APPROACH TO MULTIPLICITIES

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INTRODUCTION

In Algebre Locale: Multiplicites , Serre proposed a purely algebraic definition of the intersection product $\chi(M,N)$ of two finitely generated modules M and N over a regular local ring R; namely, supposing the length $l(M\otimes_R N)$ to be finite, then

$$\chi(M,N) = \sum_{i=0}^{\dim(R)} (-1)^i l(\mathrm{Tor}_i^R(M,N)) \quad .$$

He then considered two properties which would justify calling $\chi(M,N)$ an intersection multiplicity:

1) if $\dim(M) + \dim(N) = \dim(R)$, then $\chi(M,N) > 0$ (non-vanishing

2) if $\dim(M) + \dim(N) < \dim(R)$, then $\chi(M,N) = 0$ (vanishing) ,

and showed that (1) and (2) held in case R contains a field, or if R is regular over a DVR of mixed characteristic. The remaining so called ramified case is still open. Most of the progress in proving (1) and (2) in this case has been in the direction of increasing the dimension of R. M. Hochester [H] has verified both the vanishing and non-vanishing conjectures for regular local rings of dimension at most four. S. Dutta([D1], [D2], and [D3]) has also proven this result by different methods, and has pushed the vanishing result to dimension five.

Another important conjecture about regular local rings is Gersten's conjecture. Denoting the abelian category of finitely generated R modules supported in codimension i by $M_R^{(i)}$, and the p^{th} Quillen K-group of this category by $K_p(M_R^{(i)})$, Gersten's conjecture states that the inclusion functor $M_R^{(i)} \rightarrow M_R^{(i-1)}$ induces the zero map on K-groups

$$K_p(M_R^{(i)}) \to K_p(M_R^{(i-1)}) \quad ,$$

if R is a regular local ring. This was proven in case R contains a field by Quillen in [Q]. If R is smooth over a DVR Λ (possibly of mixed characteristic), a similar statement about the relative K-groups $K_p(M_{R/\Lambda}^{(i)})$ was proved by the author and H. Gillet [G-L], where $M_{R/\Lambda}^{(i)}$ is the subcategory of $M_R^{(i)}$ consisting of modules flat over Λ . This result reduces Gersten's conjecture for such rings to the case of a DVR. In the ramified case, essentially nothing is known.

The relation between Gersten's conjecture and the vanishing part of the multiplicity conjecture was first pointed out by Gillet [G]. It was noted in [G-L] that only a small portion of Gersten's conjecture is needed to prove the vanishing theorem; in fact, the vanishing of the maps $K_0(M_R^{(i)}) \to K_0(M_R^{(i-1)})$ for $i=1,\ldots,\dim(R)$ is sufficient (see corollary 1.2). The vanishing of the maps on K_0 also gives a nice structure theorem for the modules supported in codimension i, namely that $K_0(M_R^{(i)})$ is generated by cyclic modules of the form $R/(f_1,\ldots,f_i)$, where $f_1,\ldots,f_i$ forms a regular sequence (see proposition 1.1).

In this paper, we consider the K-theory of a special type of ramified local ring, the ring $R_{n,m} = \Lambda[[x_1,\ldots,x_n]]/\sum_{i=1}^{m} x_i^2-\pi$, where Λ is a (mixed characteristic) DVR with maximal ideal (π), and the residue field k is algebraically closed of characterist different from 2. This type of ring is in a sense the simplest regular ramified local ring. Using a modification of Quillen argument in [Q], we prove the vanishing of the maps $K_0(M_{R_{n,m}}^{(i)}) \to K_0(M_{R_{n,m}}^{(i-1)})$, and prove the vanishing theorem (2) for all local rings with completion isomorphic to some $R_{n,m}$.

Section 1.

We start out by reformulating some of the results of [G] and [G-L] regarding the relationship between Gersten's conjecture and the vanishing part of the multiplicity conjecture.

If R is a ring, we denote the category of finitely generated R-modules supported in codimension i by $M_R^{(i)}$, and the quotient category $M_R^{(i)}/M_R^{(j)}$, $i<j$, by $M_R^{(i/j)}$. We will drop the subscript R if the ring is clear from the context, and we will use a similar notation for the corresponding categories of coherent sheaves on a scheme X. If C is a closed subset of a scheme X, we denote the category of coherent $\mathcal{O}_X$ modules supported on C by $M_X(C)$, or $M_R(C)$ if $X = \mathrm{Spec}(R)$. For a scheme X, the group $K_0(M_X^{(i/i+1)})$ can be identified with the group of cycles of codimension i on X, $Z^i(X)$, by sending a reduced irreducible codimension i subscheme Z of X to the class $[\mathcal{O}_Z]$ in $K_0(M_X^{(i/i+1)})$, and extending by linearity. If Z is a pure codimension i subscheme of X, we let $|Z|$ denote the associated cycle.

<u>Proposition 1.1</u>. Let R be a regular domain, and suppose the maps $K_0(M^{(i)}) \to K_0(M^{(i-1)})$ are zero for $i=1,\ldots,\dim(R)$. Then $K_0(M^{(i)})$ is generated by the cyclic modules $R/(f_1,\ldots,f_i)$, where $f_1,\ldots,f_i$ forms a regular sequence.

<u>Proof</u>. We proceed by induction on i, the case i=0 being trivial. We have the exact sequence

$$\to K_1(M^{(i-1)}) \to K_1(M^{(i-1/i)}) \xrightarrow{\delta} K_0(M^{(i)}) \xrightarrow{\alpha} K_0(M^{(i-1)}) \to \quad .$$

By assumption, α is the zero map, hence δ is surjective. In addition, we have the isomorphism ($X = \mathrm{Spec}(R)$)

$$K_1(M^{(i-1/i)}) = \bigoplus_{x \text{ in } X^{i-1}} k(x)^* .$$

Furthermore, if Z is a reduced irreducible subscheme of X of codimension i, defined by a prime ideal $\mathfrak{p}$ of R, and if f is in $R/\mathfrak{p} - \{0\}$, then

(1.1) $\delta(f) = [\bar{R}/f\bar{R}]$ in $K_0(M^{(i)})$; $\bar{R} = R/\mathfrak{p}$.

Fix such a quotient ring $\bar{R} = R/\mathfrak{p}$, and an element f of $\bar{R} - \{0\}$. By induction, there are complete intersection ideals $I_1,\ldots,I_k$ and and integers $n_1,\ldots,n_k$ such that

$$[\bar{R}] = \sum_{j=1}^{k} n_j [R/I_j] \qquad \text{in } K_0(M^{(i-1)}) .$$

Lift f to an element F of R so that F acts as a non-zerodivisor on R/I_j for each j. We can do this as R is Cohen-Macauley, and hence the R/I_j have no embedded components. Then

$$[\bar{R}/f] = \sum_{j=1}^{k} n_j [R/I_j,F] \qquad \text{in } K_0(M^{(i)}) .$$

Thus $[\bar{R}/f]$ is in the subgroup of $K_0(M^{(i)})$ generated by complete intersections. The proposition then follows from the computation (1.1), and the surjectivity of δ.

q.e.d.

Corollary 1.2. Let R be a regular local ring, and let M and N be finitely generated R-modules with $l(M \otimes_R N) < \infty$. Suppose the maps $K_0(M^{(i)}) \to K_0(M^{(i-1)})$ are all zero, for $i=1,\ldots,\dim(R)$. Suppose further that $\dim(M)+\dim(N) < \dim(R)$. Then

$$\sum_{i=0}^{\dim(R)} (-1)^i l(\mathrm{Tor}_i^R(M,N)) = 0$$

Proof. Suppose codim(supp(M)) = i, codim(supp(N)) = j. We first note that, if N' is a finitely generated R-module, with codim(supp(N'))=j, and

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

is an exact sequence of finitely generated R-modules with codim(supp(M_k))=i,and $l(M_k \otimes_R N') < \infty$, then vanishing holds for one pair (M_k,N') if and only if it holds for the other two. A similar remark holds for a module M' with support in codimension i, and an exact sequence of modules $0 \to N_1 \to N_2 \to N_3 \to 0$ of modules with support in codimension j, and with $l(M' \otimes N_k) < \infty$, for k=1,2,3. In addition, the vanishing statement is easily proven if either M or N is a complete intersection (see [S] pp.137-8).

From the proposition, [M] is a sum of complete intersections in $K_0(M^{(i)})$, hence a sum of complete intersections in $K_0(M(C))$ for some suitable C of codimension i on Spec(R). In addition, by induction, we may assume that vanishing holds for all pairs (M,N') with $l(M \otimes N') < \infty$, and N' supported in codimension j+1. Thus we may assume that N = R/J for some ideal J of R. We can embed N in a complete intersection R/I,

$$0 \to N = R/J \to R/I \to N'' \to 0 \qquad ,$$

such that $l(R/I \otimes M) < \infty$, codim(R/I)=j, and supp(N") $\cap$ C is just the closed point of Spec(R).

Since I is a complete intersection, vanishing holds for (M,R/I), and since [M] is a sum of complete intersection in $K_0(M(C))$, vanishing holds for (M,N"). Thus vanishing holds for (M,N), which proves the corollary.

q.e.d.

Section 2.

We now begin our study of the particular rings $R_{n,m}$,

$$R_{n,m} = \Lambda[[x_1,\ldots,x_n]] \, / \sum_{i=1}^{m} x_i^2 - \pi \qquad ,$$

where Λ is a complete DVR with maximal ideal (π), and residue field k. We assume that k is algebraically closed, and that char(k)$\neq$2. The method of Quillen consists in taking a generic fibering of Spec($R_{n,m}$) by curves; one can easily see that a generic such fibering consists of a family of deformations of a curve with an ordinary double point (o.d.p.). We therefore begin with a study of the versal deformation space of such a singularity. We include the proof of the following result, lemma 2.1, for the sake of completeness; one can also recover this result from the general theory using the less explicit infinitesmal methods.

Notation: Let B = Spec(A), T = Spec(R), and U = Spec(S), where A is a complete local ring with residue field k, and R and S are complete local A-algebras. Since k is algebraically closed, the tensor product $R\otimes_A S$ is again local, and we denote the completion of this ring at the maximal ideal by $R\hat{\otimes}_A S$. We denote Spec($R\hat{\otimes}_A S$) by $T\hat{\times}_B U$.

Lemma 2.1. Let $p:U \to B$ be the morphism of Λ-schemes $\mathrm{Spec}(\Lambda[[x,y,t]]/xy-t) \to \mathrm{Spec}(\Lambda[[t]])$. Then $p:U \to B$ is the versal deformation space (amoung Λ-schemes) of an ordinary double point, i.e., if $q:X \to T$ is a flat map of complete local Λ-schemes, with T and X the completion of Λ-schemes of finite type, and if $q^{-1}(0)$ is isomorphic to $\mathrm{Spec}(k[[x,y]]/xy=0)$, then there is a map $f:T \to B$ such that $X = U\hat{\times}_B T$.

Proof. Since $q^{-1}(0)$ has embedding dimension two, we may assume that X is a closed subscheme of $T\hat{\times}\,\mathrm{Spec}(\Lambda[[u,v]])$. If $T = \mathrm{Spec}(R)$, this latter is just $\mathrm{Spec}(R[[u,v]])$. X is of relative dimension one over T, and is flat, hence X is defined by a single equation F=0,

$$F = \sum_{i,j} a_{ij}u^i v^j \quad ; \quad a_{ij} \text{ in } R \quad .$$

Since the characteristic is different from two, we may assume, after a linear change of coordinates, that $a_{10}=a_{01}=a_{20}=a_{02}=0$, and $a_{11}=1$. Then $F-a_{00} = uv$ + higher order terms. Since R is complete, we can choose new coordinates x and y for $R[[u,v]]$ so that $F - a_{00} = xy$. Defining f by $f^*(t) = a_{00}$ completes the proof.

q.e.d.

Lemma 2.2. Let $D \subset B$ be the discriminant locus t=0, $D_U = p^{-1}(D)$ the inverse image of D in U, $L \subset D_U$ the subscheme defined by t=x=0, and $\Delta \subset U\hat{\times}_B U$ the diagonal. Then the divisor $Q = L\hat{\times}_B L + \Delta$ is a principal divisor on $U\hat{\times}_B U$.

Proof. One sees easily that Q is defined by $x \otimes 1 = 1 \otimes x$ in $\Lambda[[x,y,t]]\hat{\otimes}_{\Lambda[[t]]}\Lambda[[x,y,t]]$.

If $f:Y \to B$ is a morphism, Y complete and local, and $p:X \to Y$ the pullback family $U\hat{\times}_B Y$, we denote the discriminant subscheme $f^{-1}(D)$ of Y by D_Y, and the subscheme $p_1^{-1}(L)$ of X by L_Y. We note that p restricts to a regular, surjective morphism $p:L_Y \to D_Y$.

As $U \to B$ is the versal deformation space for an o.d.p. over Λ, it follows from general nonsense that the complete fiber product $p_2:U\hat{\times}_B U \to U$, together with the diagonal section $s_\Delta:U \to U\hat{\times}_B U$ is the versal deformation space for deformations of an o.d.p., together with a section. If $q:W \to Z$ is such a family, with section $s:Z \to W$, we therefore have a map $g:Z \to U$ such that W is the pullback $(U\hat{\times}_B U)\hat{\times}_U Z$, and s is the pullback of s_Δ. In such a case, we have the divisors on W, $\Delta_Z=s(Z)$, and $Q_Z=p_1^{-1}(Q)$. Q_Z is principal, and $Q_Z = \Delta_Z + L_Z$.

<u>Lemma 2.3</u>. Let $Y = Y_{n,m} = \mathrm{Spec}(R_{n,m})$, and let $g:Y \to \mathrm{Spec}\,\Lambda[[z_1,\ldots z_{n-2}]]$ be a generic linear projection, mapping the closed point 0* of Y to the closed point 0 of $\hat{\mathbb{A}}^{n-2}_\Lambda \underset{\mathrm{def}}{=} \mathrm{Spec}(\Lambda[[z_1,\ldots,z_{n-2}]])$. Then

1) $g^{-1}(0)$ has an o.d.p. at 0*
2) the discriminant subscheme D_A contained in $\hat{\mathbb{A}}^{n-2}_\Lambda$ is isomorphic to $\mathrm{Spec}(R_{n-2,m-2})$ for $n\geq 3$, $m\geq 3$. If $m=2$, $n \geq 2$, then D_A is defined by $\pi=0$.

<u>Proof</u>. Define the linear projection $g:\hat{\mathbb{A}}^n_\Lambda \to \hat{\mathbb{A}}^{n-2}_\Lambda$ by $z_i=L_i(X)$, $i=1,\ldots,,n-2$, and let z_{n-1}, z_n be new linear functions of $x_1,\ldots,x_n$ so that $z_1,\ldots,z_n$ are coordinates for $\hat{\mathbb{A}}^n_\Lambda$. Write L_i as

$$L_i(x) = \sum_j a_{ij}x_j \; ; \; i=1,\ldots,n \; ; \; a_{ij} \text{ in } \Lambda \; .$$

Substituting this into $P(x) = \sum_1^m x_i^2$, we may write $P(z)$ as

$$P(z) = \sum_{i,j} b_{ij}z_iz_j \quad ; \quad b_{ij} \text{ in } \Lambda \quad .$$

Denoting by $\bar{b}_{ij}$ the reduction of b_{ij} to an element of k, the matrices (b_{ij}) and $(\bar{b}_{ij})$ both have rank m. The ramification locus of g is the simultaneous solution of

$$P(z) = \pi; \quad \frac{\partial P}{\partial z_{n-1}} = \frac{\partial P}{\partial z_n} = 0 \quad .$$

As $\frac{\partial P}{\partial z_k} = 2\sum_i b_{ik}z_i$, we can (for a general cohice of (a_{ij})) solve for z_{n-1} and z_n in terms of $z_1,\ldots,z_{n-2}$ using $\frac{\partial P}{\partial z_{n-1}} = \frac{\partial P}{\partial z_n} = 0$, and substitute into $P(z)=\pi$ to yield the equation

$$\sum_{i,j=1}^{n-2} c_{ij}z_iz_j = \pi$$

for the discriminant locus. In addition, we have

$$(*) \qquad \begin{aligned} &\operatorname{rank}(c_{ij}) \leq m-2 \\ &\operatorname{rank}(\bar{c}_{ij}) \leq m-2 \end{aligned} \quad .$$

Specializing to the projection $z_i = x_i$, $i=1,\ldots,n$, we have $\frac{\partial P}{\partial z_k}=2z_k$, and the discriminant locus is given by $\sum_{i=1}^{m-2} z_i^2 = \pi$. Thus, in this case, we have equality in (*), hence we get equality in (*) for a general choice of (a_{ij}). This proves the lemma.

q.e.d

We now proceed to the study of the K-theory of the rings $R_{n,m}$. We first consider a rather trivial case.

<u>Lemma 2.4</u>. Let $R = k[[x_1,\ldots,x_n]]$. Then, for each $i=1,\ldots,n$, the map $K_0(M_R^{(i)}) \to K_0(M_R^{(i-1)})$ is the zero map.

<u>Proof</u>. As this is essentially the same as the original argument of Quillen, we only give a sketch. Let Z be a codim i subvariety of $\hat{\mathbb{A}}^n_k = \operatorname{Spec}(R)$, and let $g: \hat{\mathbb{A}}^n_k \to \hat{\mathbb{A}}^{n-1}_k$ be a general linear projection. Then Z is quasi-finite over its image in $\hat{\mathbb{A}}^{n-1}_k$, and since $\hat{\mathbb{A}}^{n-1}_k$ is Hensel, Z is in fact finite over its image. Form the pullback diagram

$$\begin{array}{ccc} \hat{\mathbb{A}}^n_k & \overset{p}{\leftarrow} & W = \hat{\mathbb{A}}^n_k \times_{\hat{\mathbb{A}}^{n-1}_k} Z \\ g\downarrow & & s\uparrow \ \downarrow p_2 \\ \hat{\mathbb{A}}^{n-1}_k & \underset{g}{\leftarrow} & Z \end{array} ,$$

where s is the section induced by the inclusion of Z in $\hat{\mathbb{A}}^n_k$. Since p_2 is smooth, and W is local, s(Z) is defined by a single equation u=0 for some u in k[W]. This gives the exact sequence of coherent sheaves on $\hat{\mathbb{A}}^n_k$

$$0 \to p_*(\mathcal{O}_W) \overset{\times u}{\to} p_*(\mathcal{O}_W) \to \mathcal{O}_Z \to 0 \qquad .$$

As $p_*(\mathcal{O}_W)$ is in $M_R^{(i-1)}$, this shows $[\mathcal{O}_Z] = 0$ in $K_0(M_R^{(i-1)})$. Finally, if M is in $M_R^{(i)}$, then M has a composition series with relative quotients of the form k[Z] , where Z is a subvariety of $\hat{\mathbb{A}}^n_k$ of codimension at least i. This proves the lemma.

q.e.d.

We are now ready to prove our main result. We fix integers $n \geq m \geq 2$, and let R denote the ring $R_{n,m}$.

<u>Theorem 2.5</u>. The map $K_0(M_R^{(i)}) \to K_0(M_R^{(i-1)})$ is zero for $i=1,\ldots,\dim(R)$.

<u>Proof</u>. We proceed by descending induction on i. For $i=\dim(R)=n+1$, $K_0(M_R^{(i)})$ is just $\mathbb{Z}$, generated by the class of the residue field k. Let $\bar{R}$ be the finite Λ-algebra

$$\bar{R} = R/(x_2,\ldots,x_n) = \Lambda[[x]]/x^2 - \pi \quad .$$

$\bar{R}$ defines an element of $K_0(M^{(n)})$, and the exact sequence

$$0 \to \bar{R} \underset{\times x}{\to} \bar{R} \to k \to 0$$

shows that k goes to zero in $K_0(M^{(n)})$.

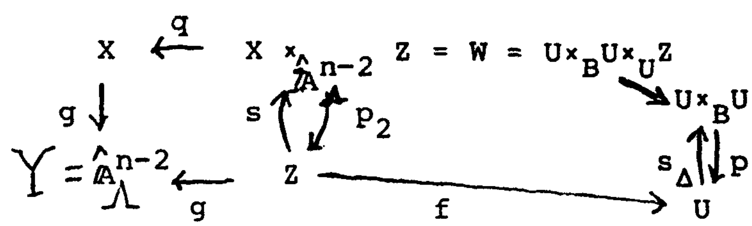
We now consider the case $i>1$, assuming the result for $i+1,\ldots,\dim(R)$. Using induction on m, we assume the result for the rings $R_{n',m'}$, with $n'<n$, the case $n' = 0$ being trivial. From the exact sequence

$$\to K_0(M^{(i+1)}) \to K_0(M^{(i)}) \to K_0(M^{(i/i+1)}) \to 0 \quad ,$$

and induction, we see that $K_0(M^{(i)})$ is isomorphic to $K_0(M^{(i/i+1)})$, which in turn is isomorphic to the group of codimension i cycles on $X=\mathrm{Spec}(R)$. Let Z be a reduced, irreducible subscheme of X of codimension i. We need only show that the class of $\mathcal{O}_Z$ in $K_0(M^{(i-1)}$

is zero.

We first assume that Z is not contained in the singular locus of $X\otimes_{\Lambda}k$. Let $g:X \to \hat{\mathbb{A}}^{n-2}_{\Lambda}$ be a general linear projection, chosen so that Z is quasi-finite over $\hat{\mathbb{A}}^{n-2}_{\Lambda}$, and which is smooth at a generic point of Z. As in lemma 2.4, Z is then finite over $\hat{\mathbb{A}}^{n-2}_{\Lambda}$. We have the diagram



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s being the section induced by the inclusion of Z in X, and $f:Z \to U$ the map induced by viewing $p_2:W \to Z$ as a family of deformations of an o.d.p

Let u be a defining equation for the divisor Q_Z of W. We have the exact sequence of coherent $\mathcal{O}_X$ modules

$$0 \to q_*(\mathcal{O}_W) \overset{\times u}{\to} q_*(\mathcal{O}_W) \to q_*(\mathcal{O}_{Q_Z}) \to 0 \ ,$$

hence $[q_*(\mathcal{O}_{Q_Z})] = 0$ in $K_0(M_R^{(i-1)})$. On the other hand, as a (Weil) divisor on W, we have

$$|Q_Z| = |L_Z| + |\Delta_Z| \qquad ,$$

hence in $K_0(M_R^{(i/i+1)})$ we have

$$|Z| = q_*(|Q_Z|) - q_*(|L_Z|) \qquad .$$

Viewing $g:X \to Y=\hat{A}^{n-2}_{\Lambda}$ as a family of deformations of an o.d.p., there is a map $h:Y \to B$, compatible with the map $f:Z \to U$ such that $X \cong U\times_B Y$. The cycle $Z' = q_*(|L_Z|)$ is supported on L_Y and is mapped via g to $q(D_Z) \subset D_Y$. Let $\bar{g}:L_Y \to D_Y$ be the restriction of g to L_Y. Then the support of Z' is $\bar{g}^{-1}(q(D_Z))$, and the fibers of $\bar{g}$ are irreducible, hence Z' is of the form $\bar{g}^{-1}(Z_0)$ for some cycle Z_0 of codimension i-1 on D_Y. Since D_Y is isomorphic to $\mathrm{Spec}(R_{n-2,m-2})$ if $m\geq 3$, or to $\mathrm{Spec}(k[[x_1,\dots,x_{n-2}]])$ if m=2, we apply induction in the first case, or lemma 2.4 in the second case to see that

$$Z_0 = 0 \text{ in } K_0(M_{D_Y}^{(i-2/i)}) .$$

Thus $Z' = 0$ in $K_0(M_{L_Y}^{(i-2/i)})$, and hence $Z' = 0$ in $K_0(M_X^{(i-1/i+1)})$. Therefore,

$$[\mathcal{O}_Z] = 0 \text{ in } K_0(M_X^{(i-1/i+1)}) .$$

As $K_0(M_X^{(i+1)})$ already goes to zero in $K_0(M_X^{(i)})$ by induction, we have $K_0(M_X^{(i-1)}) = K_0(M_X^{(i-1/i+1)})$, which proves the theorem in this case.

If Z is contained in the singular locus of $X\otimes_{\Lambda} k$, then for a general linear projection $g:X \to Y=\hat{A}^{n-2}_{\Lambda}$, Z is contained in $L_Y \cap \hat{A}^{n-2}_k$. As this is isomorphic to an $\hat{A}^r_k$ for suitable r, lemma 2.4 takes care of this case. Finally if i=1, then Z is a Cartier divisor on X. X being local, Z is a principal divisor, hence $[\mathcal{O}_Z] = 0$ in $K_0(M_X)$. This completes the proof of the theorem.

q.e.d.

Corollary 2.6. For $n\geq m\geq 2$, and for $i=1,\ldots,n+1$, $K_0(M^{(i)}_{R_{n,m}})$ is generated by the cyclic modules $R_{n,m}/(f_1,\ldots,f_i)$, where $f_1,\ldots,f_i$ form a regular sequence.

Proof. This follows immediately from proposition 1.1 and the theorem.

Corollary 2.7. Let R be a noetherian, local ring, and suppose the completion of R is isomorphic to $R_{n,m}$ for some n and m. Then the vanishing theorem holds for R.

Proof. Let M and N be two finitely generated R-modules. Denote by $\hat{M}$ and $\hat{N}$ the completed modules $M\otimes_R R_{n,m}$, and $N\otimes_R R_{n,m}$. Then $\dim(R)=\dim(R_{n,m})$, $\dim(M)=\dim(M)$, and $\dim(N)=\dim(N)$. Suppose that $l(M\otimes_R N)$ is finite. Then $l(\hat{M}\otimes_{R_{n,m}}\hat{N})$ is also finite, and

$$l(\mathrm{Tor}_i^R(M,N)) = l(\mathrm{Tor}_i^{R_{n,m}}(\hat{M},\hat{N}))$$

for each i. The corollary then follows from the theorem, and corollary 1.2.

q.e.d.

Essentially the same technique can be used to prove the analogue of Theorem 2.5 for the Henselization at $(x_1,\ldots,x_n)$ of

$$\Lambda[x_1,\ldots,x_n]/\sum_{i=1}^{m} x_i^2 - \pi ,$$

and for the local rings $\Lambda[x_1,\ldots,x_n]_{(x_1,\ldots,x_n)}/\sum_{i=1}^{m} x_i^2 - \pi$

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