Symmetries in complex cobordism theory, related to stable equivalence classes of SU-bundles

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Abstract

Let $\mathcal{Z}$ (resp. $\widehat{\Xi}$) be the universal line bundle over $\mathbb{C}P^\infty$ (resp. the universal U-bundle over BU). Since BU is homotopy equivalent to BSU $\times \mathbb{C}P^\infty$, we have $\sigma_1(\widehat{\Xi}) = \mathfrak{h}(\sigma_1(\mathcal{Z}))$ for some formal power series $\mathfrak{h}(u) \in \mathrm{MU}^*(\mathrm{BSU})[[u]]$, where σ_1 denotes the first Conner-Floyd class. Let $\mathfrak{H}(x,y)$ be the formal group law $\mathfrak{h}(F(\mathfrak{h}^{-1}(x), \mathfrak{h}^{-1}(y))) \in \mathrm{MU}^*(\mathrm{BSU})[[x,y]]$, and $\mathrm{K} \subset \mathrm{MU}^*(\mathrm{BSU})$ the subring, generated by its coefficients, where $F(u,v) \in \mathrm{MU}^*(\mathrm{pt})[[u,v]]$ is the formal product in complex cobordism theory. Clearly, $\mathrm{K} \cong \mathrm{MU}^*(\mathrm{pt})$. Thus for any stable equivalence class $\{\xi\} \in \widetilde{\mathrm{K\,SU}}(X)$ of SU-bundles over a finite CW-complex X we get a formal group law $(\varphi_\xi^*\mathfrak{H})(x,y) \in \mathrm{MU}^*(X)[[x,y]]$, where $\varphi_\xi \colon X \to \mathrm{BSU}$ is a classifying map for $\xi \in \{\xi\}$. Suppose ξ has virtual dimension 1. It was shown in [6] that $\sigma_1(\xi \otimes \zeta_1 \otimes \zeta_2) = (\varphi_\xi^*\mathfrak{H})(\sigma_1(\xi \otimes \zeta_1), \sigma_1(\xi \otimes \zeta_2))$ for any geometric line bundles ζ_1, ζ_2 over X. The coefficients of $(\varphi_\xi^*\mathfrak{H})(x,y)$ generate the direct summand subring $\varphi_\xi^*(\mathrm{K}) \cong \mathrm{MU}^*(\mathrm{pt})$ in $\mathrm{MU}^*(X)$.

In the present paper for a given pair (X, ξ) we construct a ring automorphism $\mathfrak{T}_{\{\xi\}} \colon \mathrm{MU}[\frac{1}{2}]^*(X) \to \mathrm{MU}[\frac{1}{2}]^*(X)$ such that

- it is natural on (X, ξ);
- $\mathfrak{T}_{\{\xi\}}(\mathrm{R}) = \varphi_\xi^*(\mathrm{K})$, where $\mathrm{R} := \mathrm{MU}^*(\mathrm{pt}) \subset \mathrm{MU}^*(X)$;
- $\mathfrak{T}_{\{\xi\}}(\sigma_1(\zeta)) = \sigma_1(\xi \otimes \zeta)$ for any line bundle ζ over X.

Furthermore, we prove that the assignment $\{\xi\} \mapsto \mathfrak{T}_{\{\xi\}}$ determines a group homomorphism

$$\widetilde{\mathrm{K\,SU}}(X)_\otimes \to \mathrm{Aut}_{ring}(\mathrm{MU}[\tfrac{1}{2}]^*(X)),$$

i.e. $\mathfrak{T}_{\{\eta\}} \circ \mathfrak{T}_{\{\xi\}} = \mathfrak{T}_{\{\eta\otimes\xi\}}$ for all virtual SU-bundles ξ, η over X of virtual dimension 1.

Contents

*Partially supported by RFFI Grant 02-01-00572.

Introduction

This paper is closely connected to Preprint [6], therefore first let us give a review of its results.

Let $\mathrm{BU}_\otimes$ (resp. $\mathrm{BSU}_\otimes$) be the space BU (resp. BSU) with the structure of H-group, related to the tensor product of virtual U-bundles (resp. SU-bundles) of virtual dimension 1. By $\widehat{\mathrm{H}}$ (resp. H) denote the Hopf algebra $\mathrm{MU}^*(\mathrm{BU}_\otimes)$ (resp. $\mathrm{MU}^*(\mathrm{BSU}_\otimes)$) over $\mathrm{R} := \mathrm{MU}^*(\mathrm{pt})$. Since the H-groups $\mathrm{BU}_\otimes$ and $\mathrm{BSU}_\otimes \times \mathbb{C}P^\infty$ are isomorphic (see the Appendix), we have an isomorphism of Hopf algebras $\widehat{\mathrm{H}} \cong \mathrm{H} \mathbin{\widehat{\otimes}_{\mathrm{R}}} \mathrm{R}[[u]] = \mathrm{H}[[u]]$. Clearly, the comultiplication $\widehat{\Delta}$ and the antipode $\widehat{S}$ in $\widehat{\mathrm{H}}$ are determined as follows:

$$\widehat{\Delta}(u) = F(u \otimes 1, 1 \otimes u),\ \widehat{\Delta}|_{\mathrm{H}} = \Delta, \quad \widehat{S}(u) = \theta(u),\ \widehat{S}|_{\mathrm{H}} = S, \tag{1}$$

where Δ, S are the comultiplication and the antipode in H and $\theta(u) \in \mathrm{R}[[u]]$ is the formal inverse in the formal product in complex cobordism theory $F(u, v)$.

Let $\widehat{\Xi}$ (resp. Ξ) be the universal U-bundle over BU of virtual dimension 1 (resp. the universal SU-bundle over BSU of virtual dimension 1), and $\mathcal{Z}$ the canonical line bundle over $\mathbb{C}P^\infty$. It can be proved that $\widehat{\Xi}$ is identified with $\Xi \boxtimes \mathcal{Z}$ under the H-group isomorphism (see the Appendix). This implies that the element $x := \sigma_1(\widehat{\Xi}) \in \mathrm{MU}^2(\mathrm{BU})$ can be expressed as a formal power series $\mathfrak{h}(u) \in \mathrm{H}[[u]]$ of the variable $u := \sigma_1(\mathcal{Z})$, where σ_1 denotes the first Conner-Floyd class. Using the splitting principle, one can verify the following explicit form of this power series [6]:

$$\mathfrak{h}(u) = 1 \otimes u + \sum_{i,j \geq 1} \alpha_{i,j} \widetilde{s}_i \otimes u^j \ \in \mathrm{H} \mathbin{\widehat{\otimes}_{\mathrm{R}}} \mathrm{R}[[u]], \tag{2}$$

where $\alpha_{i,j} \in \mathrm{MU}^{2(1-i-j)}(\mathrm{pt})$ are the coefficients of the formal product in complex cobordism theory $F(u, v)$ and $\widetilde{s}_i := s_i(\Xi) \in \mathrm{MU}^{2i}(\mathrm{BSU})$ is the i-th Newton characteristic class in complex cobordism theory of the universal SU-bundle Ξ (recall that $s_1(\Xi) = \sigma_1(\Xi) = 0$).

Remark 1. In particular, we see that $\mathfrak{h}(u)$ is functionally invertible, $(\varepsilon\mathfrak{h})(u) = u$, and $\mathfrak{h}(0) = 0$, where $\varepsilon\colon \mathrm{H} \to \mathrm{R}$ is the counit in the Hopf algebra H. Thus, $\mathrm{MU}^*(\mathrm{BU}) = \mathrm{H}[[u]] = \mathrm{H}[[x]]$.

Now consider the formal power series $\mathfrak{F}(x \otimes 1, 1 \otimes x) := \widehat{\Delta}(x) \in (\mathrm{H} \mathbin{\widehat{\otimes}_{\mathrm{R}}} \mathrm{H})[[x \otimes 1, 1 \otimes x]] = \mathrm{MU}^*(\mathrm{BU} \times \mathrm{BU})$ and $\Theta(x) := \widehat{\iota}^*(x) \in \mathrm{H}[[x]]$, where $\widehat{\iota}\colon \mathrm{BU} \to \mathrm{BU}$ is the inversion in $\mathrm{BU}_\otimes$, i.e. $\widehat{\iota}^*(\widehat{\Xi}) = \widehat{\Xi}^{-1}$. The series $\mathfrak{F}(x \otimes 1, 1 \otimes x)$ is a *formal group law over the Hopf algebra* H with the formal inverse $\Theta(x)$, i.e. it satisfies the following conditions [6]:

1) (*associative*)

$$\begin{aligned}&((\mathrm{id}_{\mathrm{H}} \otimes \Delta)\mathfrak{F})(x \otimes 1 \otimes 1, 1 \otimes \mathfrak{F}(x \otimes 1, 1 \otimes x)) \\ &= ((\Delta \otimes \mathrm{id}_{\mathrm{H}})\mathfrak{F})(\mathfrak{F}(x \otimes 1, 1 \otimes x) \otimes 1, 1 \otimes 1 \otimes x).\end{aligned}$$

2) (*identity*)

$$((\mathrm{id}_{\mathrm{H}} \otimes \varepsilon)\mathfrak{F})(x \otimes 1, 0) = x \otimes 1,$$

$$((\varepsilon \otimes \mathrm{id}_{\mathrm{H}})\mathfrak{F})(0, 1 \otimes x) = 1 \otimes x.$$

3) (*formal inverse*)

$$(\mu \circ (\mathrm{id}_{\mathrm{H}} \otimes S)\mathfrak{F})(x, \Theta(x)) = 0 = (\mu \circ (S \otimes \mathrm{id}_{\mathrm{H}})\mathfrak{F})(\Theta(x), x),$$

where $\mu\colon\ \mathrm{H} \mathbin{\widehat{\otimes}_{\mathrm{R}}} \mathrm{H} \to \mathrm{H}$ is the multiplication in H.

Moreover, the formal group law $\mathfrak{F}(x \otimes 1, 1 \otimes x)$ is *commutative*, i.e. $(\tau\mathfrak{F})(1 \otimes x, x \otimes 1) = \mathfrak{F}(x \otimes 1, 1 \otimes x)$, where $\tau\colon\ \mathrm{H} \mathbin{\widehat{\otimes}_{\mathrm{R}}} \mathrm{H} \to \mathrm{H} \mathbin{\widehat{\otimes}_{\mathrm{R}}} \mathrm{H}$ is the *switching map* $\tau(a \otimes b) = b \otimes a$.

Using (1), we can obtain the explicit form of $\mathfrak{F}(x \otimes 1, 1 \otimes x)$ and $\Theta(x)$:

$$\mathfrak{F}(x \otimes 1, 1 \otimes x) = \widehat{\Delta}(x) = \widehat{\Delta}(\mathfrak{h}(u)) = (\Delta\mathfrak{h})(F(u \otimes 1, 1 \otimes u))$$

$$= (\Delta\mathfrak{h})(F(\mathfrak{h}^{-1}(x) \otimes 1, 1 \otimes \mathfrak{h}^{-1}(x)));$$

$$\Theta(x) = \widehat{S}(x) = \widehat{S}(\mathfrak{h}(u)) = (S\mathfrak{h})(\theta(u)) = (S\mathfrak{h})(\theta(\mathfrak{h}^{-1}(x))).$$

Let $p\colon\ \mathrm{BU}_{\otimes} \to \mathrm{BSU}_{\otimes}$ be the composition of the isomorphism $\varpi\colon\ \mathrm{BU} \to \mathrm{BSU} \times \mathbb{C}P^{\infty}$, $\varpi^*(\Xi \boxtimes \mathcal{Z}) = \widehat{\Xi}$ with the projection onto the first factor (note that $\varpi = (\phi \circ (\lambda \times \kappa))^{-1}$, see the Appendix). Then p is a morphism of H-groups. Let ξ be a virtual vector U-bundle of virtual dimension 1 over a finite CW-complex X, and $\varphi_{\xi}\colon X \to \mathrm{BU}$ its classifying map. Hence the composite map $\varphi_{\overline{\xi}} := p \circ \varphi_{\xi}\colon X \to \mathrm{BSU}$ is the classifying map for the corresponding virtual SU-bundle $\overline{\xi}$ of virtual dimension 1 (let us remark that for any ξ there exists a unique line bundle ζ such that $c_1(\xi \otimes \zeta) = 0$, where c_1 is the first Chern class; clearly $\overline{\xi} = \xi \otimes \zeta$). By μ_X denote the multiplication $\mathrm{MU}^*(X) \otimes_{\mathrm{R}} \mathrm{MU}^*(X) \to \mathrm{MU}^*(X)$.

Theorem 2. *For any virtual complex vector bundles ξ, η of virtual dimension 1 over X we have*

$$\sigma_1(\xi \otimes \eta) = (\mu_X \circ (\varphi^*_{\overline{\xi}} \otimes \varphi^*_{\overline{\eta}})\mathfrak{F})(\sigma_1(\xi),\ \sigma_1(\eta)). \tag{3}$$

Suppose ξ^{-1} is the virtual bundle of virtual dimension 1 over X such that $\xi \otimes \xi^{-1} = [1]$, where $[1]$ is the trivial bundle. Then

$$\sigma_1(\xi^{-1}) = (\varphi^*_{\overline{\xi}}\Theta)(\sigma_1(\xi))$$

and we have

$$(\mu_X \circ (\varphi^*_{\overline{\xi}} \otimes (\varphi^*_{\overline{\xi}} \circ S))\mathfrak{F})(\sigma_1(\xi), \sigma_1(\xi^{-1})) = 0$$

(note that $\varphi^*_{\overline{\xi}} \circ S = \varphi^*_{\overline{\xi}^{-1}}$, where $\overline{\xi}^{-1}$ is the inverse bundle for $\overline{\xi}$, i.e. $\overline{\xi} \otimes \overline{\xi}^{-1} = [1]$).

Remark 3. If ξ, η are geometric line bundles, then (3) reduces to the well-known formula $\sigma_1(\xi \otimes \eta) = F(\sigma_1(\xi), \sigma_1(\eta))$ (note that in the last case $\varphi_{\overline{\xi}}$, $\varphi_{\overline{\eta}}$ are maps to a point). Moreover, in this case $\sigma_1(\xi^{-1}) = (\varphi^*_{\overline{\xi}}\Theta)(\sigma_1(\xi)) = \theta(\sigma_1(\xi))$.

Remark 4. It follows from Remark 1 that the constant term of $\mathfrak{F}(x \otimes 1, 1 \otimes x)$ equals 0. Therefore, using (3) we see that $\sigma_1(\xi \otimes \eta)$ belongs to the ideal $\mathrm{I} = \{\sigma_1(\xi),\ \sigma_1(\eta)\} \subset \mathrm{MU}^*(X)$, generated by $\sigma_1(\xi)$ and $\sigma_1(\eta)$.

Set $\mathfrak{H}(x,y) := \mathfrak{h}(F(\mathfrak{h}^{-1}(x),\mathfrak{h}^{-1}(y))) \in \mathrm{H}[[x,y]]$. Obviously, $\mathfrak{H}(x,y)$ is a formal group law over the ring H, with the formal inverse $\mathfrak{I}(x) := \mathfrak{h}(\theta(\mathfrak{h}^{-1}(x))) \in \mathrm{H}[[x]]$. Moreover, the formal group law $\mathfrak{H}(x,y)$ is isomorphic to $F(u,v)$ (although $\mathfrak{h}(u)$ is not a strict isomorphism of formal group laws, it is not difficult to find a strict one).

Write $\mathfrak{H}(x,y) = x+y+\sum_{i,j\geq 1}\widetilde{\alpha}_{i,j}x^iy^j$. Recall that the formal group law $F(u,v)\in \mathrm{R}[[u,v]]$ is the universal one and the ring R is the Lasard ring [4]. Hence there is a unique ring homomorphism $\lambda\colon\ \mathrm{R}\to\mathrm{H}$ such that $\lambda(\alpha_{i,j}) = \widetilde{\alpha}_{i,j},\ i,j\geq 1$. Clearly, $\varepsilon(\widetilde{\alpha}_{i,j}) = \alpha_{i,j},\ i,j\geq 1$. By P denote the composite homomorphism $\lambda\circ\varepsilon\colon\ \mathrm{H}\to\mathrm{H}$.

Let $\mathrm{K}\subset\mathrm{H}$ be the subring, generated by $\{\widetilde{\alpha}_{i,j},\ i,j\geq 1\}$.

Theorem 5. *λ determines the canonical ring isomorphism* $\lambda\colon\ \mathrm{R}\overset{\cong}{\to}\mathrm{K}\subset\mathrm{H},\ \alpha_{i,j}\overset{\lambda}{\mapsto}\widetilde{\alpha}_{i,j}$. *Furthermore, P is a projector (i.e. $P^2=P$ and $\mathrm{H}=\operatorname{im}P\oplus\ker P$) and $\operatorname{im}P=\mathrm{K}\subset\mathrm{H}$.*

Moreover, it can be shown that $\mathrm{K}\neq\mathrm{R}$ as subrings in H.

Thus we get the formal group law $\mathfrak{H}(x,y)\in\mathrm{K}[[x,y]]$ with the formal inverse $\mathfrak{I}(x)\in\mathrm{K}[[x]]$ and the ring isomorphism $\lambda\colon\ \mathrm{R}\to\mathrm{K}$, realizing the isomorphism of formal group laws $F\to\mathfrak{H}$.

Now let us fix a virtual SU-bundle ξ of virtual dimension 1 over a finite CW-complex X. We say that a virtual bundle $\widehat{\xi}$ of the form $\xi\otimes\zeta$, where ζ is a geometric line bundle over X, is a lift of ξ. By $\mathcal{L}(\xi)\subset\mathrm{K}_{\mathbb{C}}(X)$ denote the set of stable equivalence classes of lifts of ξ.

Proposition 6. *The set $\mathcal{L}(\xi)$ is an Abelian group with respect to the operation, induced by the following operation on lifts:*

$$\widehat{\xi_1}\circ\widehat{\xi_2} := \widehat{\xi_1}\otimes\widehat{\xi_2}\otimes\xi^{-1}.$$

Furthermore, for all ξ the corresponding groups $\mathcal{L}(\xi)$ are isomorphic to each other (in particular, all of them are isomorphic to the group $\mathrm{H}^2(X;\mathbb{Z})$ of line bundles over X with respect to the tensor product).

Theorem 7. *Let $\widehat{\xi_1}$, $\widehat{\xi_2}$ be two lifts of ξ, and $\varphi_\xi\colon X\to\mathrm{BSU}$ a classifying map for ξ. Then*

$$\sigma_1(\widehat{\xi_1}\circ\widehat{\xi_2}) = (\varphi_\xi^*\mathfrak{H})(\sigma_1(\widehat{\xi_1}),\sigma_1(\widehat{\xi_2})). \tag{4}$$

Moreover, $\sigma_1(\widehat{\xi}^{-1}) = (\varphi_\xi^*\mathfrak{I})(\sigma_1(\widehat{\xi}))$, where $\widehat{\xi}^{-1}$ is a lift of ξ, whose stable equivalence class is the inverse element for the class $\{\widehat{\xi}\}$ of $\widehat{\xi}$ with respect to the operation on $\mathcal{L}(\xi)$ (note that the class of ξ is the identity element in $\mathcal{L}(\xi)$). The assignment $\widehat{\xi}\mapsto\sigma_1(\widehat{\xi})$ determines a group isomorphism from $\mathcal{L}(\xi)$ to the group $\{\sigma_1(\widehat{\xi})\,|\,\widehat{\xi}$ is a lift of $\xi\}\subset\mathrm{MU}^2(X)$ with respect to the operation $\sigma_1(\widehat{\xi_1})\circ\sigma_1(\widehat{\xi_2}) := (\varphi_\xi^*\mathfrak{H})(\sigma_1(\widehat{\xi_1}),\sigma_1(\widehat{\xi_2}))$.

Remark 8. Note that (4) can be written as follows:

$$\sigma_1(\xi\otimes\zeta_1\otimes\zeta_2) = (\varphi_\xi^*\mathfrak{H})(\sigma_1(\xi\otimes\zeta_1),\sigma_1(\xi\otimes\zeta_2)),$$

where $\widehat{\xi_1}=\xi\otimes\zeta_1$, $\widehat{\xi_2}=\xi\otimes\zeta_2$ for line bundles ζ_1, ζ_2. In the case $\xi=[1]$ it shows that we recur to the relation $\sigma_1(\zeta_1\otimes\zeta_2) = F(\sigma_1(\zeta_1),\sigma_1(\zeta_2))$.

By $\mathrm{MU}^{ev}(X)$ denote the subring $\underset{n\in\mathbb{Z}}{\oplus}\mathrm{MU}^{2n}(X)\subset\mathrm{MU}^*(X)$. Thus the formal group law $\mathfrak{H}_\xi(x,y):=(\varphi_\xi^*\mathfrak{H})(x,y)\in\mathrm{MU}^{ev}(X)[[x,y]]$ describes the relation between first Conner-Floyd classes of lifts of the fixed SU-bundle ξ. Recall that by K we denote the subring, generated by the coefficients $\widetilde{\alpha}_{i,j}$ of $\mathfrak{H}(x,y)$. Hence the coefficients $\varphi_\xi^*(\widetilde{\alpha}_{i,j})\in\mathrm{MU}^{2(1-i-j)}(X)$ of $\mathfrak{H}_\xi(x,y)$ generate the subring $\mathrm{K}_\xi:=\varphi_\xi^*(\mathrm{K})\subset\mathrm{MU}^{ev}(X)$, which is a direct summand in $\mathrm{MU}^{ev}(X)$, isomorphic to R. Note that K_ξ is equipped with the ring isomorphism $\lambda_\xi:=\varphi_\xi^*\circ\lambda\colon\ \mathrm{R}\to\mathrm{K}_\xi$.

Thus we have the following situation: for any $\{\xi\}\in\widetilde{\mathrm{KSU}}(X)$ we get the formal group law $\mathfrak{H}_\xi(x,y)$ over $\mathrm{K}_\xi\subset\mathrm{MU}^{ev}(X)$, which determines the group structure on the set $\{\sigma_1(\widehat{\xi})\,|\,\widehat{\xi}\text{ is a lift of }\xi\}\subset\mathrm{MU}^2(X)$. Note that for all $\{\xi\}$ the corresponding formal group laws $\mathfrak{H}_\xi(x,y)$, the corresponding subrings K_ξ, and the corresponding sets $\{\sigma_1(\widehat{\xi})\,|\,\widehat{\xi}\text{ is a lift of }\xi\}$ are isomorphic to each other. Note also that having a pair $(\mathrm{K}_\xi,\,\lambda_\xi)$ one can restore the corresponding formal group law $\mathfrak{H}_\xi(x,y)$. Indeed, $\mathfrak{H}_\xi(x,y)=(\lambda_{\xi*}F)(x,y)$.

We say that the triple, consisting of the subring $\mathrm{K}_\xi\subset\mathrm{MU}^{ev}(X)$, the ring isomorphism $\lambda_\xi\colon\ \mathrm{R}\to\mathrm{K}_\xi$, and the set $\{\sigma_1(\widehat{\xi})\,|\,\widehat{\xi}\text{ is a lift of }\xi\}\subset\mathrm{MU}^2(X)$ is a *data*, corresponding to the stable equivalence class $\{\xi\}\in\widetilde{\mathrm{K\,SU}}(X)$.

The aim of the present paper is to give some explanation of the described results. We shall show that for any pair $(X,\,\{\eta\})$ there exists a natural ring automorphism $\mathfrak{T}^{ev}_{\{\eta\}}\colon\ \mathrm{MU}^{ev}(X)\to\mathrm{MU}^{ev}(X)$ such that for any $\{\xi\}\in\widetilde{\mathrm{K\,SU}}(X)$ with a representative ξ of virtual dimension 1 it transform the *data* corresponding to $\{\xi\}$ isomorphically to the *data*, corresponding to $\{\eta\otimes\xi\}$. More precisely, $\mathrm{K}_{\eta\otimes\xi}=\mathfrak{T}^{ev}_{\{\eta\}}(\mathrm{K}_\xi)$, $\lambda_{\eta\otimes\xi}=\mathfrak{T}^{ev}_{\{\eta\}}\circ\lambda_\xi$, $\sigma_1(\eta\otimes\widehat{\xi})=\mathfrak{T}^{ev}_{\{\eta\}}(\sigma_1(\widehat{\xi}))$ for any lift $\widehat{\xi}$ of ξ.

Let $\widetilde{\mathrm{K\,SU}}(X)_\otimes$ be the set $\widetilde{\mathrm{K\,SU}}(X)$ with respect to the group structure, induced by the tensor product of virtual bundles of virtual dimension 1, i.e. $\widetilde{\mathrm{K\,SU}}(X)_\otimes=[X,\,\mathrm{BSU}_\otimes]$. It will be proved that the assignment $\{\xi\}\mapsto\mathfrak{T}^{ev}_{\{\xi\}}$ determines a group homomorphism $\widetilde{\mathrm{K\,SU}}(X)_\otimes\to\mathrm{Aut}_{ring}(\mathrm{MU}^{ev}(X))$, i.e. $\mathfrak{T}^{ev}_{\{\eta\}}\circ\mathfrak{T}^{ev}_{\{\xi\}}=\mathfrak{T}^{ev}_{\{\eta\otimes\xi\}}$ for all virtual SU-bundles ξ, η over X of virtual dimension 1.

Now let us introduce an algebraic view on some results of Sections 2, 3. Let $\mathfrak{Alg}_\mathrm{R}$ be a subcategory of the category of associative commutative unitary algebras over R, equipped with the tensor product, i.e. for any two objects $\mathrm{C},\,\mathrm{C}'\in\mathfrak{Alg}_\mathrm{R}$ we have $\mathrm{C}\underset{\mathrm{R}}{\otimes}\mathrm{C}'\in\mathfrak{Alg}_\mathrm{R}$. In particular, $\mathfrak{Alg}_\mathrm{R}$ can be the category of $\mathbf{S}$-modules, where $\mathbf{S}$ is a Hopf algebra over $\mathbb{Z}$. Let $\mathrm{H}\in\mathfrak{Alg}_\mathrm{R}$ be a Hopf algebra over R. Assume that for any $\mathrm{C}\in\mathfrak{Alg}_\mathrm{R}$ (in particular, for $\mathrm{C}=\mathrm{R}$) there is a natural homomorphism $\Delta_\mathrm{C}\colon\ \mathrm{C}\to\mathrm{H}\underset{\mathrm{R}}{\otimes}\mathrm{C}$ (i.e. $\Delta_{\mathrm{C}'}\circ f=(\mathrm{id}_\mathrm{H}\otimes f)\circ\Delta_\mathrm{C}$ for $f\in\mathrm{Mor}_{\mathfrak{Alg}_\mathrm{R}}(\mathrm{C},\,\mathrm{C}')$) such that the diagrams

$$\begin{tikzcd}
\mathrm{R}\underset{\mathrm{R}}{\otimes}\mathrm{C} & \mathrm{H}\underset{\mathrm{R}}{\otimes}\mathrm{C} \arrow[l, "\varepsilon\otimes\mathrm{id}_\mathrm{C}"'] \\
 & \mathrm{C}, \arrow[ul, "\cong"] \arrow[u, "\Delta_\mathrm{C}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{C} \arrow[r, "\Delta_\mathrm{C}"] \arrow[d, "\Delta_\mathrm{C}"'] & \mathrm{H}\underset{\mathrm{R}}{\otimes}\mathrm{C} \arrow[d, "\Delta_{\mathrm{H}\underset{\mathrm{R}}{\otimes}\mathrm{C}}"] \\
\mathrm{H}\underset{\mathrm{R}}{\otimes}\mathrm{C} \arrow[r, "\Delta\otimes\mathrm{id}_\mathrm{C}"'] & \mathrm{H}\underset{\mathrm{R}}{\otimes}\mathrm{H}\underset{\mathrm{R}}{\otimes}\mathrm{C}
\end{tikzcd}$$

commute, where $\Delta\colon\ \mathrm{H}\to\mathrm{H}\underset{\mathrm{R}}{\otimes}\mathrm{H}$, $\varepsilon\colon\ \mathrm{H}\to\mathrm{R}$ are the comultiplication and the counit map in H respectively (let us remark that (i) $\Delta_{\mathrm{H}\underset{\mathrm{R}}{\otimes}\mathrm{C}}\colon\ \mathrm{H}\underset{\mathrm{R}}{\otimes}\mathrm{C}\to\mathrm{H}\underset{\mathrm{R}}{\otimes}\mathrm{H}\underset{\mathrm{R}}{\otimes}\mathrm{C}$ makes sense because of $\mathrm{H}\underset{\mathrm{R}}{\otimes}\mathrm{C}\in\mathfrak{Alg}_\mathrm{R}$, and (ii) Δ_C induces an isomorphism from C onto a direct summand in

$\mathrm{H}\underset{\mathrm{R}}{\otimes}\mathrm{C}$. Indeed, it follows from the first diagram above that $\Delta_{\mathrm{C}}\circ(\varepsilon\otimes\mathrm{id}_{\mathrm{C}})$ is a projector onto $\mathrm{im}(\Delta_{\mathrm{C}})$). Then the assignment

$$\mathrm{Hom}_{\mathrm{R}}(\mathrm{H},\,\mathrm{C})\ni\varphi\mapsto\mu_{\mathrm{C}}\circ(\varphi\otimes\mathrm{id}_{\mathrm{C}})\circ\Delta_{\mathrm{C}},$$

where $\mu_{\mathrm{C}}\colon\ \mathrm{C}\underset{\mathrm{R}}{\otimes}\mathrm{C}\to\mathrm{C}$ is the multiplication in C, determines a group homomorphism

$$\mathfrak{G}\colon\ \mathrm{Hom}_{\mathrm{R}}(\mathrm{H},\,\mathrm{C})\to\mathrm{Aut}_{ring}(\mathrm{C}),$$

which is natural on C (here $\mathrm{Hom}_{\mathrm{R}}(\mathrm{H},\,\mathrm{C})$ is a group with respect to the convolution $\star$: $\varphi\star\psi:=\mu_{\mathrm{C}}\circ(\varphi\otimes\psi)\circ\Delta$). The assertion that $\mathfrak{G}$ is a group homomorphism follows from the diagram of Corollary 21. The commutative diagram

$$\begin{tikzcd}
\mathrm{C} \arrow[r, "\Delta_{\mathrm{C}}"] \arrow[d, "f"'] & \mathrm{H}\underset{\mathrm{R}}{\otimes}\mathrm{C} \arrow[r, "\varphi\otimes\mathrm{id}_{\mathrm{C}}"] \arrow[d, "\mathrm{id}_{\mathrm{H}}\otimes f"] & \mathrm{C}\underset{\mathrm{R}}{\otimes}\mathrm{C} \arrow[r, "\mu_{\mathrm{C}}"] \arrow[d, "f\otimes f"] & \mathrm{C} \arrow[d, "f"] \\
\mathrm{C}' \arrow[r, "\Delta_{\mathrm{C}'}"'] & \mathrm{H}\underset{\mathrm{R}}{\otimes}\mathrm{C}' \arrow[r, "f_*(\varphi)\otimes\mathrm{id}_{\mathrm{C}'}"'] & \mathrm{C}'\underset{\mathrm{R}}{\otimes}\mathrm{C}' \arrow[r, "\mu_{\mathrm{C}'}"'] & \mathrm{C}',
\end{tikzcd}$$

where $f\in\mathrm{Mor}_{\mathfrak{Alg}_{\mathrm{R}}}(\mathrm{C},\,\mathrm{C}')$, $\varphi\in\mathrm{Hom}_{\mathrm{R}}(\mathrm{H},\,\mathrm{C})$ and $f_*(\varphi):=f\circ\varphi$, shows that $\mathfrak{G}(f_*(\varphi))\circ f=f\circ\mathfrak{G}(\varphi)$, i.e. that $\mathfrak{G}$ is natural. We do not assert that $\mathfrak{G}(\varphi)$ is an automorphism of the R-algebra C (it depends on the homomorphism $\Delta_{\mathrm{R}}\colon\ \mathrm{R}\to\mathrm{H}\underset{\mathrm{R}}{\otimes}\mathrm{R}=\mathrm{H}$).

Now let us describe roughly a construction of the natural homomorphism $\Delta_{\mathrm{C}}\colon\ \mathrm{C}\to\mathrm{H}\underset{\mathrm{R}}{\otimes}\mathrm{C}$ under the following additional assumption: $\mathfrak{Alg}_{\mathrm{R}}$ is a category of $\mathbf{S}$-modules, where $\mathbf{S}$ is a Hopf algebra over $\mathbb{Z}$. That is, the Hopf algebra $\mathbf{S}$ acts on any $\mathrm{C}\in\mathfrak{Alg}_{\mathrm{R}}$ (in particular, on R) and on the tensor product $\mathrm{C}\underset{\mathrm{R}}{\otimes}\mathrm{C}'$, $\mathrm{C},\,\mathrm{C}'\in\mathfrak{Alg}_{\mathrm{R}}$ the action is defined as follows: $S(\alpha\otimes\beta)=\sum_i S_i'(\alpha)\otimes S_i''(\beta)$, for any $\alpha\in\mathrm{C}$, $\beta\in\mathrm{C}'$, $S\in\mathbf{S}$, where $\Delta_{\mathbf{S}}S=\sum_i S_i'\otimes S_i''$ and $\Delta_{\mathbf{S}}\colon\ \mathbf{S}\to\mathbf{S}\underset{\mathbb{Z}}{\otimes}\mathbf{S}$ is the comultiplication in $\mathbf{S}$.

Assume that there is an element $\mathfrak{T}=\sum_{i\geq 0}\gamma_i\otimes S_i\in\mathrm{H}\underset{\mathbb{Z}}{\otimes}\mathbf{S}$ such that

- $\gamma_0\otimes S_0=1_{\mathrm{H}}\otimes 1_{\mathbf{S}}$, $\varepsilon(\gamma_i)=0$, $i\geq 1$ ($\Rightarrow(\varepsilon\otimes\mathrm{id}_{\mathbf{S}})\mathfrak{T}=1\otimes S_0=1\otimes 1_{\mathbf{S}}\in\mathrm{R}\underset{\mathbb{Z}}{\otimes}\mathbf{S}$);
- for any $\mathrm{C}\in\mathfrak{Alg}_{\mathrm{R}}$ $\mathfrak{T}$ defines a ring homomorphism $\mathfrak{T}_{\mathrm{C}}\colon\ \mathrm{C}\to\mathrm{H}\underset{\mathrm{R}}{\otimes}\mathrm{C}$ by setting $\mathfrak{T}_{\mathrm{C}}(\alpha):=\mathfrak{T}(\alpha)=\sum_i\gamma_i\otimes S_i(\alpha)\in\mathrm{H}\underset{\mathrm{R}}{\otimes}\mathrm{C}$ for any $\alpha\in\mathrm{C}$;
- the identity $(\mathrm{id}_{\mathrm{H}}\otimes\Delta_{\mathbf{S}})\mathfrak{T}\circ\mathfrak{T}=(\Delta\otimes\mathrm{id}_{\mathbf{S}})\mathfrak{T}\ \in\mathrm{H}\underset{\mathrm{R}}{\otimes}\mathrm{H}\underset{\mathbb{Z}}{\otimes}\mathbf{S}$ holds. More explicitly,

$$\sum_i\gamma_i\otimes(\Delta_{\mathbf{S}}S_i(\sum_j\gamma_j\otimes S_j))=\sum_k\Delta(\gamma_k)\otimes S_k.$$

Then, setting $\Delta_{\mathrm{C}}=\mathfrak{T}_{\mathrm{C}}$ for any $\mathrm{C}\in\mathfrak{Alg}_{\mathrm{R}}$, we obtain a natural transformation with the required properties (i.e. $(\varepsilon\otimes\mathrm{id}_{\mathrm{C}})\circ\Delta_{\mathrm{C}}=\mathrm{id}_{\mathrm{C}}$ and $\Delta_{\mathrm{H}\underset{\mathrm{R}}{\otimes}\mathrm{C}}\circ\Delta_{\mathrm{C}}=(\Delta\otimes\mathrm{id}_{\mathrm{C}})\circ\Delta_{\mathrm{C}}$, see the diagrams above).

Roughly, below we shall deal with the following situation: $\mathfrak{Alg}_{\mathrm{R}}$ is the category whose objects are cobordism rings $\mathrm{MU}^*(X)$ with morphisms $\mathrm{MU}^*(X)\to\mathrm{MU}^*(Y)$, induced by continuous maps $Y \to X$; $\mathrm{H} = \mathrm{MU}^*(\mathrm{BSU}_\otimes)$, $\mathbf{S}$ is Landweber-Novikov's algebra, and the tensor product $\otimes$ must be replaced by the completed graded tensor product $\widehat{\otimes}$.

Acknowledgments The author is grateful to Professor E.V. Troitsky for constant attention to this work. I would like to thank the Max Planck Institute for Mathematics in Bonn for its kind support and hospitality while this work has been completed. A number of related questions were discussed with Professors V.G. Gorbounov and S.V. Konyagin and I would like to express my gratitude to them.

1 A natural transformation $\mathfrak{T}^{ev}\colon\ \mathrm{MU}^{ev}(\_) \to \mathrm{H} \mathbin{\widehat{\otimes}_{\mathrm{R}}} \mathrm{MU}^{ev}(\_)$

The main tool of this paper is a multiplicative natural transformation $\mathfrak{T}^{ev}\colon\ \mathrm{MU}^{ev}(X) \to \mathrm{H} \mathbin{\widehat{\otimes}_{\mathrm{R}}} \mathrm{MU}^{ev}(X)$, which is uniquely determined by the following condition: for $X = \mathbb{C}P^\infty$ and $u = \sigma_1(\mathcal{Z}) \in \mathrm{MU}^2(\mathbb{C}P^\infty)$ $\mathfrak{T}^{ev}(u) = \mathfrak{h}(u) = \sigma_1(\widehat{\Xi}) \in \mathrm{H} \mathbin{\widehat{\otimes}_{\mathrm{R}}} \mathrm{MU}^*(\mathbb{C}P^\infty) = \mathrm{MU}^*(\mathrm{BU})$. Let us remember that Landweber-Novikov's operations act on the "geometric cobordism" u in the following way: $S_\omega u = 0$ if $\mathrm{l}(\omega) \geq 2$ and $S_{(k)}u = u^{k+1}$, where $\mathrm{l}(\_)$ is the length of a partition (if $\omega = (k_1, \dots, k_l)$, then $\mathrm{l}(\omega) = l$) [3]. The idea of the construction of the natural transformation $\mathfrak{T}^{ev}$ is based on the following observation. It follows from the just described action of S_ω on u that the element $\sigma_1(\widehat{\Xi}) = \sigma_1(\Xi \otimes \mathcal{Z}) = 1 \otimes u + \sum_{i,j\geq 1} \alpha_{i,j}\widetilde{s}_i \otimes u^j$ can be obtained by application of $1 \otimes S_0 + \sum_{i,j\geq 1} \alpha_{i,j}\widetilde{s}_i \otimes S_{j-1} \in \mathrm{H} \mathbin{\widehat{\otimes}_{\mathbb{Z}}} \mathbf{S}$ (here $\mathbf{S}$ is Landweber-Novikov's algebra, see below) to the element $u = \sigma_1(\mathcal{Z}) \in \mathrm{MU}^2(\mathbb{C}P^\infty)$. Thus, the goal is to extend this transformation to a multiplicative transformation of whole cobordism theory.

Now we recall some results on Landweber-Novikov's operations and Landweber-Novikov's algebra (see [2], [3], [5]).

Let $s_\omega \in \mathrm{MU}^{2|\omega|}(\mathrm{BU})$ be the characteristic class in complex cobordism theory, corresponding to the partition $\omega = (k_1, \dots, k_l)$, where $|\omega| = \sum_{1\leq q\leq l} k_q$. Let $\mathfrak{t}\colon\ \mathrm{MU}^*(\mathrm{BU}) \to \mathrm{MU}^*(\mathbf{MU})$ be the stable Thom isomorphism; then $\mathfrak{t}(s_\omega) = S_\omega \in \mathrm{MU}^{2|\omega|}(\mathbf{MU})$ is Landweber-Novikov's operation, corresponding to the partition ω. Landweber-Novikov's algebra $\mathbf{S}$ is a $2\mathbb{Z}^+$-graded Hopf algebra over $\mathbb{Z}$, $\mathbf{S} = \bigoplus_{j\geq 0} \mathbf{S}^{2j}$, with the basis $\{S_\omega\}$, $S_\omega \in \mathbf{S}^{2|\omega|}$, where ω is either a partition $(k_1, \dots, k_l)$, $k_q \in \mathbb{Z}^+$ or the symbol of an empty set (note that $S_\emptyset = S_{(0)} = 1 \in \mathbf{S}^0$). The comultiplication $\Delta_{\mathbf{S}}$ in $\mathbf{S}$ is defined as follows:

$$\Delta_{\mathbf{S}}(S_\omega) = \sum_{\omega' \cup \omega'' = \omega} S_{\omega'} \otimes S_{\omega''}.$$

Furthermore, $\mathbf{S}$ acts on any cobordism ring $\mathrm{MU}^*(X)$, turning it into Milnor's module:

$$S_\omega(\alpha\beta) = \sum_{\omega' \cup \omega'' = \omega} S_{\omega'}(\alpha) S_{\omega''}(\beta).$$

For a CW-complex X by $\mathrm{MU}^{ev}(X)$ we denote the subalgebra $\bigoplus_{n\in\mathbb{Z}} \mathrm{MU}^{2n}(X) \subset \mathrm{MU}^*(X)$. Suppose $\alpha \in \mathrm{MU}^{2n}(X)$; then $S_\omega(\alpha) \in \mathrm{MU}^{2(n+|\omega|)}(X)$. Therefore $\mathbf{S}$ preserves the subset $\mathrm{MU}^{ev}(X) \subset \mathrm{MU}^*(X)$.

Let A be a graded topological algebra over R with an augmentation $\varepsilon_{\mathrm{A}}\colon\ \mathrm{A}\to\mathrm{R}$. Assume that A is complete with respect to the augmentation ideal $\ker(\varepsilon_{\mathrm{A}})$. For example, we can take $\mathrm{A}=\mathrm{MU}^*(\mathrm{BSU})$ or $A=\mathrm{MU}^*(X)$, where X is a finite CW-complex. Note that in the last case the topology on A is discrete. By u we denote the element $\sigma_1(\mathcal{Z})\in\mathrm{MU}^2(\mathbb{C}P^\infty)$, where $\mathcal{Z}$ is the canonical line bundle over $\mathbb{C}P^\infty$. Recall that $\mathrm{MU}^*(\mathbb{C}P^\infty)=\mathrm{R}[[u]]$.

Theorem 9. *Given a formal power series $\mathfrak{f}(u)=\sum_{i\geq 1}a_i\otimes u^i\in\mathrm{A}\mathop{\widehat{\otimes}}\limits_{\mathrm{R}}\mathrm{MU}^*(\mathbb{C}P^\infty)$ such that*

(i) $\deg a_i=2-2i$;

(ii) $\varepsilon_{\mathrm{A}}(a_1)=1,\ \varepsilon_{\mathrm{A}}(a_i)=0,\ i\geq 2$

there is a unique multiplicative natural transformation $\mathfrak{T}^{ev}_{\mathfrak{f}}\colon\ \mathrm{MU}^{ev}(X)\to\mathrm{A}\mathop{\widehat{\otimes}}\limits_{\mathrm{R}}\mathrm{MU}^{ev}(X)$ such that $\mathfrak{T}^{ev}_{\mathfrak{f}}(u)=\mathfrak{f}(u)$ for $X=\mathbb{C}P^\infty$ and $u=\sigma_1(\mathcal{Z})$. Furthermore, $\mathfrak{T}^{ev}_{\mathfrak{f}}$ induces a ring isomorphism from $\mathrm{MU}^{ev}(X)$ onto the direct summand $\mathfrak{T}^{ev}_{\mathfrak{f}}(\mathrm{MU}^{ev}(X))\subset\mathrm{A}\mathop{\widehat{\otimes}}\limits_{\mathrm{R}}\mathrm{MU}^{ev}(X)$.

Proof. Consider the symmetric formal power series of N variables $u_1,\dots,u_N$:

$$\prod_{1\leq j\leq N}\left(\sum_{i\geq 0}\alpha_i\otimes(u_j)^i\right)\ \in\mathrm{A}\mathop{\widehat{\otimes}}\limits_{\mathrm{R}}\mathrm{MU}^*((\mathbb{C}P^\infty)^N),$$

where $\alpha_i=\frac{a_{i+1}}{a_1},\ i\geq 0,\ \deg(\alpha_i)=-2i$ (since $\varepsilon_{\mathrm{A}}(a_1)=1$ and A is complete with respect to $\ker(\varepsilon_{\mathrm{A}})$, we see that there is $a_1^{-1}\in\mathrm{A}$). Let us remark that this power series has degree 0 as an element of the tensor product of graded rings.

Let

$$\sum_\omega\alpha^{(N)}_\omega\otimes s_\omega\in\mathrm{A}\mathop{\widehat{\otimes}}\limits_{\mathrm{R}}\mathrm{MU}^*(\mathrm{BU}(N))$$

be the element such that

$$(\mathrm{id}_{\mathrm{A}}\otimes\widehat{\mu}^*_N)(\sum_\omega\alpha^{(N)}_\omega\otimes s_\omega)=\prod_{1\leq j\leq N}\left(\sum_{i\geq 0}\alpha_i\otimes(u_j)^i\right),$$

where $\mathrm{id}_{\mathrm{A}}\otimes\widehat{\mu}^*_N\colon\ \mathrm{A}\mathop{\widehat{\otimes}}\limits_{\mathrm{R}}\mathrm{MU}^*(\mathrm{BU}(N))\ \to\ \mathrm{A}\mathop{\widehat{\otimes}}\limits_{\mathrm{R}}\mathrm{MU}^*((\mathbb{C}P^\infty)^N)$ is the homomorphism, induced by the "splitting map" $\widehat{\mu}_N\colon(\mathbb{C}P^\infty)^N\to\mathrm{BU}(N)$ and $\alpha^{(N)}_\omega\in\mathrm{A},\ \deg(\alpha^{(N)}_\omega)=-2|\omega|$. Note that $\alpha^{(N)}_\omega=0$ if $\mathrm{l}(\omega)\geq N+1$ and $\alpha^{(N)}_\omega=\alpha_{k_1}\dots\alpha_{k_l}$ if $\mathrm{l}(\omega)\leq N$ (in particular, $\alpha^{(N)}_\omega$ is independed of N under the last condition), where $\mathrm{l}(\_)$ denotes the length of a partition, $\mathrm{l}(k_1,\dots,k_l)=l$. Therefore, passing to the limit $N\to\infty$, we get the element $\sum_\omega\alpha_\omega\otimes s_\omega\in\mathrm{A}\mathop{\widehat{\otimes}}\limits_{\mathrm{R}}\mathrm{MU}^*(\mathrm{BU})$ of degree 0.

Let $\mathfrak{t}\colon\ \mathrm{MU}^*(\mathrm{BU})\to\mathrm{MU}^*(\mathbf{MU})$ be the stable Thom isomorphism. Applying the operator $\mathrm{id}_{\mathrm{A}}\otimes\mathfrak{t}$ to $\sum_\omega\alpha_\omega\otimes s_\omega$, we obtain the element $\mathfrak{T}_{\mathfrak{f},0}:=\sum_\omega\alpha_\omega\otimes S_\omega\in\mathrm{A}\mathop{\widehat{\otimes}}\limits_{\mathrm{R}}\mathrm{MU}^*(\mathbf{MU})$ of degree 0, where $S_\omega=\mathfrak{t}(s_\omega)$ are Landweber-Novikov's operations. Note that $\mathfrak{T}_{\mathfrak{f},0}$ can be considered as an element of $\mathrm{A}\mathop{\widehat{\otimes}}\limits_{\mathbb{Z}}\mathbf{S}$, where $\mathbf{S}$ is Landweber-Novikov's algebra.

For any $n\in\mathbb{Z}$ set $\mathfrak{T}_{\mathfrak{f},2n}:=(a_1^n\otimes 1)\mathfrak{T}_{\mathfrak{f},0}=\sum_\omega(a_1^n\alpha_\omega)\otimes S_\omega\in\mathrm{A}\mathop{\widehat{\otimes}}\limits_{\mathbb{Z}}\mathbf{S}$. We define the natural transformation $\mathfrak{T}^{ev}_{\mathfrak{f}}$ in the following way. Given a homogeneous element $\alpha\in\mathrm{MU}^{2n}(X)$ of degree $2n$ we set $\mathfrak{T}^{ev}_{\mathfrak{f}}(\alpha):=\mathfrak{T}_{\mathfrak{f},2n}(\alpha)=\sum_\omega(a_1^n\alpha_\omega)\otimes S_\omega(\alpha)\in(\mathrm{A}\mathop{\widehat{\otimes}}\limits_{\mathrm{R}}\mathrm{MU}^{ev}(X))^{(2n)}$.

Let $\alpha \in \mathrm{MU}^{2m}(X)$, $\beta \in \mathrm{MU}^{2n}(X)$ be two homogeneous elements. Let us show that $\mathfrak{T}_{\mathfrak{f}}^{ev}(\alpha\beta) = \mathfrak{T}_{\mathfrak{f}}^{ev}(\alpha)\mathfrak{T}_{\mathfrak{f}}^{ev}(\beta)$. We have:

$$
\begin{aligned}
\mathfrak{T}_{\mathfrak{f}}^{ev}(\alpha\beta) = \mathfrak{T}_{\mathfrak{f},2(m+n)}(\alpha\beta) &= \sum_{\omega} (a_1^{m+n}\alpha_\omega) \otimes S_\omega(\alpha\beta) \\
&= \sum_{\omega} (a_1^{m+n}\alpha_\omega \otimes \sum_{\omega' \cup \omega'' = \omega} S_{\omega'}(\alpha) S_{\omega''}(\beta)) \\
&= \sum_{\omega} \sum_{\omega' \cup \omega'' = \omega} (a_1^m \alpha_{\omega'})(a_1^n \alpha_{\omega''}) \otimes S_{\omega'}(\alpha) S_{\omega''}(\beta) \\
&= (\sum_{\omega'} a_1^m \alpha_{\omega'} \otimes S_{\omega'}(\alpha))(\sum_{\omega''} a_1^n \alpha_{\omega''} \otimes S_{\omega''}(\beta)) \\
&= \mathfrak{T}_{\mathfrak{f},2m}(\alpha)\mathfrak{T}_{\mathfrak{f},2n}(\beta) = \mathfrak{T}_{\mathfrak{f}}^{ev}(\alpha)\mathfrak{T}_{\mathfrak{f}}^{ev}(\beta),
\end{aligned}
$$

where we used the following well-known identity for Landweber-Novikov's operations

$$S_\omega(\alpha\beta) = \sum_{\omega' \cup \omega'' = \omega} S_{\omega'}(\alpha) S_{\omega''}(\beta)$$

and the following obvious identity: $\alpha_\omega = \alpha_{\omega'}\alpha_{\omega''}$ for any partitions ω', ω'' such that $\omega = \omega' \cup \omega''$. This implies the multiplicativity of $\mathfrak{T}_{\mathfrak{f}}^{ev}$ for any α, β, not necessarily homogeneous.

Now by using $S_\omega u = 0$ if $\mathrm{l}(\omega) > 1$ and $S_{(k)}u = u^{k+1}$, the required property of $\mathfrak{T}_{\mathfrak{f}}^{ev}$: $\mathfrak{T}_{\mathfrak{f}}^{ev}(u) = \mathfrak{f}(u) = \sum_{i \geq 1} a_i \otimes u^i$ for $u = \sigma_1(\mathcal{Z}) \in \mathrm{MU}^2(\mathbb{C}P^\infty)$ can easily be verified.

The naturality of $\mathfrak{T}_{\mathfrak{f}}^{ev}$ is clear. Since the representation of Landweber-Novikov's algebra $\mathbf{S}$ is asymptotically precise on the product of copies of "geometric cobordism" $u_1 \dots u_N$, $N \to \infty$ it follows that $\mathfrak{T}_{\mathfrak{f}}^{ev}$ is unique. More explicitly, take $X = (\mathbb{C}P^\infty)^N$ and set $\mathbf{u}_N := u_1 \dots u_N \in \mathrm{MU}^{2N}(X) = \mathrm{R}[[u_1, \dots, u_N]]$. For $\omega = (k_1, \dots, k_l)$, $\mathrm{l}(\omega) = l \leq N$ let s_ω be the smallest symmetric polynomial of $u_1, \dots, u_N$ which contains $u_1^{k_1} \dots u_l^{k_l}$. We have $S_\omega(\mathbf{u}_N) = s_\omega \mathbf{u}_N$. Hence

$$
\begin{aligned}
\mathfrak{T}_{\mathfrak{f}}^{ev}(\mathbf{u}_N) &= \sum_{\omega} a_1^N \alpha_\omega^{(N)} \otimes S_\omega(\mathbf{u}_N) \\
&= \sum_{\omega} a_1^N \alpha_\omega^{(N)} \otimes s_\omega \mathbf{u}_N \ \in \mathrm{A} \mathbin{\widehat{\otimes}_{\mathrm{R}}} \mathrm{R}[[u_1, \dots, u_N]] \cong \mathrm{A}[[u_1, \dots, u_N]].
\end{aligned}
$$

Thus the element $a_1^N \alpha_\omega \in \mathrm{A}$, $\mathrm{l}(\omega) \leq N$ ($\Rightarrow \alpha_\omega$, $\mathrm{l}(\omega) \leq N$) is uniquely determined as the coefficient of $s_\omega \mathbf{u}_N$ in the formal power series $\mathfrak{T}_{\mathfrak{f}}^{ev}(\mathbf{u}_N) \in \mathrm{A}[[u_1, \dots, u_N]]$.

We now prove that $\mathfrak{T}_{\mathfrak{f}}^{ev}$ induces a ring isomorphism from $\mathrm{MU}^{ev}(X)$ onto the direct summand $\mathfrak{T}_{\mathfrak{f}}^{ev}(\mathrm{MU}^{ev}(X)) \subset \mathrm{A} \mathbin{\widehat{\otimes}_{\mathrm{R}}} \mathrm{MU}^{ev}(X)$. It can easily be checked that for any $n \in \mathbb{Z}$ $(\varepsilon_{\mathrm{A}} \otimes \mathrm{id}_{\mathbf{S}})\mathfrak{T}_{\mathfrak{f},2n} = S_\emptyset = S_{(0)} = 1$. This implies that the following composition:

$$(\varepsilon_{\mathrm{A}} \otimes \mathrm{id}_{\mathrm{MU}^{ev}(X)}) \circ \mathfrak{T}_{\mathfrak{f}}^{ev} \colon \mathrm{MU}^{ev}(X) \to \mathrm{MU}^{ev}(X)$$

is the identity map $\mathrm{id}_{\mathrm{MU}^{ev}(X)}$. Therefore $\mathfrak{T}_{\mathfrak{f}}^{ev}$ is a monomorphism. By P we denote the composite homomorphism $\mathfrak{T}_{\mathfrak{f}}^{ev} \circ (\varepsilon_{\mathrm{A}} \otimes \mathrm{id}_{\mathrm{MU}^{ev}(X)}) \colon \mathrm{A} \mathbin{\widehat{\otimes}_{\mathrm{R}}} \mathrm{MU}^{ev}(X) \to \mathrm{A} \mathbin{\widehat{\otimes}_{\mathrm{R}}} \mathrm{MU}^{ev}(X)$. It is easy to see that P is a projector onto the subring $\mathfrak{T}_{\mathfrak{f}}^{ev}(\mathrm{MU}^{ev}(X)) \subset \mathrm{A} \mathbin{\widehat{\otimes}_{\mathrm{R}}} \mathrm{MU}^{ev}(X)$. In particular, P is an idempotent. $\square$

Corollary 10. *Suppose X is a CW-complex such that $\mathrm{MU}^{2n+1}(X) = 0$ for any $n \in \mathbb{Z}$. Then $\mathfrak{T}_{\mathfrak{f}}^{ev}$ is a transformation $\mathrm{MU}^*(X) \to \mathrm{A} \mathbin{\widehat{\otimes}_{\mathrm{R}}} \mathrm{MU}^*(X)$ which determines a ring isomorphism from $\mathrm{MU}^*(X)$ onto the direct summand $\mathfrak{T}_{\mathfrak{f}}^{ev}(\mathrm{MU}^*(X)) \subset \mathrm{A} \mathbin{\widehat{\otimes}_{\mathrm{R}}} \mathrm{MU}^*(X)$.*

Since $\mathrm{MU}^{2n+1}(\mathrm{pt}) = 0$, $\mathrm{MU}^{2n+1}(\mathbb{C}P^\infty) = 0$ $\forall n \in \mathbb{Z}$ we see that $\mathfrak{T}_{\mathfrak{f}}^{ev}(\mathrm{MU}^*(\mathrm{pt})) \subset \mathrm{A} \mathbin{\otimes_{\mathrm{R}}} \mathrm{MU}^*(\mathrm{pt}) = \mathrm{A}$ and $\mathfrak{T}_{\mathfrak{f}}^{ev}(\mathrm{MU}^*(\mathbb{C}P^\infty)) \subset \mathrm{A} \mathbin{\widehat{\otimes}_{\mathrm{R}}} \mathrm{MU}^*(\mathbb{C}P^\infty) = \mathrm{A}[[u]]$.

Let $F(u,v) \in \mathrm{MU}^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) = \mathrm{R}[[u,v]]$ be the formal product in complex cobordism theory. Clearly, $\mathfrak{F}(x,y) := \mathfrak{T}_{\mathfrak{f}}^{ev}(F(u,v)) \in \mathrm{A}[[x,y]] = \mathrm{A}[[u,v]]$ is a formal group law, where $x := \mathfrak{f}(u)$, $y := \mathfrak{f}(v)$. By the universality of $F(u,v)$ we have the ring homomorphism $\lambda\colon \mathrm{R} \to \mathrm{A}$ such that $(\lambda_* F)(x,y) = \mathfrak{F}(x,y)$.

Proposition 11. *For $X = \mathrm{pt}$ the homomorphism $\mathfrak{T}_{\mathfrak{f}}^{ev}\colon \mathrm{MU}^*(\mathrm{pt}) \to \mathrm{A}$ coincides with the homomorphism $\lambda\colon \mathrm{R} \to \mathrm{A}$, determined by the formal group law $\mathfrak{F}(x,y) \in \mathrm{A}[[x,y]]$.*

Proof. Since the transformation $\mathfrak{T}_{\mathfrak{f}}^{ev}$ is multiplicative, we have:

$$\mathfrak{T}_{\mathfrak{f}}^{ev}(F(u,v)) = \mathfrak{T}_{\mathfrak{f}}^{ev}(u) + \mathfrak{T}_{\mathfrak{f}}^{ev}(v) + \sum_{i,j\geq 1} \mathfrak{T}_{\mathfrak{f}}^{ev}(\alpha_{i,j}) \mathfrak{T}_{\mathfrak{f}}^{ev}(u)^i \mathfrak{T}_{\mathfrak{f}}^{ev}(v)^j$$

$$= x + y + \sum_{i,j\geq 1} \mathfrak{T}_{\mathfrak{f}}^{ev}(\alpha_{i,j}) x^i y^j = x + y + \sum_{i,j\geq 1} \lambda(\alpha_{i,j}) x^i y^j.$$

Now we must only note that $\alpha_{i,j}$, $i,j \geq 1$ generate R. $\square$

From the other hand, for the geometric cobordism $w := F(u,v) \in \mathrm{MU}^2(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$ we have:

$$\mathfrak{T}_{\mathfrak{f}}^{ev}(w) = \mathfrak{f}(w) = \mathfrak{f}(F(u,v)) = \mathfrak{f}(F(\mathfrak{f}^{-1}(x), \mathfrak{f}^{-1}(y))) = x + y + \sum_{i,j\geq 1} \widetilde{\alpha}_{i,j} x^i y^j.$$

It gives us another description of $\mathfrak{F}(x,y)$.

Since $(\varepsilon_{\mathrm{A}} \mathfrak{f})(u) = u$, we see that $\varepsilon_{\mathrm{A}}(\widetilde{\alpha}_{i,j}) = \varepsilon_{\mathrm{A}}(\lambda(\alpha_{i,j})) = \alpha_{i,j}$. Hence $\varepsilon_{\mathrm{A}} \circ \lambda = \mathrm{id}_{\mathrm{R}}$. This implies that $P := \lambda \circ \varepsilon_{\mathrm{A}}\colon \mathrm{A} \to \mathrm{A}$ is a projector onto $\operatorname{im}\lambda$ and λ induces a ring isomorphism $\mathrm{R} \cong \operatorname{im}\lambda \subset \mathrm{A}$. Set $\mathrm{K} := \operatorname{im}\lambda$. In particular, K is a direct summand in A, generated by the coefficients of $\mathfrak{F}(x,y)$.

By $\mathrm{H} = (\mathrm{H}, \mu, \eta, \Delta, \varepsilon, S)$ we denote the Hopf algebra $\mathrm{MU}^*(\mathrm{BSU}_\otimes)$, where $\mu\colon \mathrm{H} \mathbin{\widehat{\otimes}_{\mathrm{R}}} \mathrm{H} \to \mathrm{H}$ is the multiplication, $\eta\colon \mathrm{R} \to \mathrm{H}$ is the unit (map), $\Delta\colon \mathrm{H} \to \mathrm{H} \mathbin{\widehat{\otimes}_{\mathrm{R}}} \mathrm{H}$ is the comultiplication, $\varepsilon\colon \mathrm{H} \to \mathrm{R}$ is the counit (map), and $S\colon \mathrm{H} \to \mathrm{H}$ is the antipode. It is a graded commutative topological Hopf algebra over R. It is well known that $\mathrm{MU}^*(\mathrm{BSU}) = \mathrm{R}[[\sigma_2(\Xi), \sigma_3(\Xi), \ldots]]$ as an R-algebra. Clearly, we can take H as A.

Corollary 12. *There is a unique multiplicative natural transformation $\mathfrak{T}^{ev} := \mathfrak{T}_{\mathfrak{h}}^{ev}\colon \mathrm{MU}^{ev}(X) \to \mathrm{H} \mathbin{\widehat{\otimes}_{\mathrm{R}}} \mathrm{MU}^{ev}(X)$ such that $\mathfrak{T}^{ev}(u) = \mathfrak{h}(u) = 1 \otimes u + \sum_{k,i\geq 1} \alpha_{k,i} \widetilde{s}_k \otimes u^i \in \mathrm{H} \mathbin{\widehat{\otimes}_{\mathrm{R}}} \mathrm{MU}^*(\mathbb{C}P^\infty)$ for $X = \mathbb{C}P^\infty$ and $u = \sigma_1(\mathcal{Z}) \in \mathrm{MU}^2(\mathbb{C}P^\infty)$.*

Proof. We have: $a_1 = 1 + \sum_{k\geq 1} \alpha_{k,1} \widetilde{s}_k$, $a_i = \sum_{k\geq 1} \alpha_{k,i} \widetilde{s}_k$, $i \geq 2$. We see that the formal power series $\mathfrak{f}(u) = \mathfrak{h}(u)$ satisfies the conditions of Theorem 9. $\square$

Remark 13. Let us remember that $\mathrm{H}\,\widehat{\underset{\mathrm{R}}{\otimes}}\,\mathrm{MU}^*(\mathbb{C}P^\infty) = \mathrm{MU}^*(\mathrm{BSU})\,\widehat{\underset{\mathrm{R}}{\otimes}}\,\mathrm{MU}^*(\mathbb{C}P^\infty) = \mathrm{MU}^*(\mathrm{BU})$ and $\mathfrak{h}(u) = \sigma_1(\widehat{\Xi}) \in \mathrm{MU}^2(\mathrm{BU})$, where $u = \sigma_1(\mathcal{Z})$ as above. Therefore $\mathfrak{T}^{ev}(\sigma_1(\mathcal{Z})) = \sigma_1(\widehat{\Xi}) \in \mathrm{MU}^2(\mathrm{BU})$. Clearly, in the case $\mathfrak{f}(u) = \mathfrak{h}(u)$ the formal group law $\mathfrak{H}(x,y) := \mathfrak{T}^{ev}(F(u,v)) \in \mathrm{K}[[x,y]] \subset \mathrm{H}[[x,y]]$ coincides with the one, defined in Subsection 2.3 of [6].

Corollary 14. *For $X = \mathrm{pt}$ the homomorphism $\mathfrak{T}^{ev}\colon \mathrm{MU}^*(\mathrm{pt}) \to \mathrm{H}\underset{\mathrm{R}}{\otimes}\mathrm{MU}^*(\mathrm{pt}) = \mathrm{H}$ coincides with the homomorphism $\lambda\colon \mathrm{R} \to \mathrm{H}$, determined by $\mathfrak{H}(x,y) \in \mathrm{K}[[x,y]]$.*

Proof follows from Proposition 11. Another version of the proof. By $\mathrm{R}_{\mathbb{Q}}$ (resp. $\mathrm{H}_{\mathbb{Q}}$) denote $\mathrm{R}\underset{\mathbb{Z}}{\otimes}\mathbb{Q}$ (resp. $\mathrm{H}\,\widehat{\underset{\mathbb{Z}}{\otimes}}\,\mathbb{Q}$). It is known [1] that $S_\omega(g(u)) = 0$ for any $\omega \neq \emptyset$, where $g(u) \in \mathrm{R}_{\mathbb{Q}}[[u]]$ is the logarithm of the formal product in complex cobordism theory. Hence $\mathfrak{T}^{ev}(g(u)) = a_1 \otimes g(u)$. From the other hand, the elements $g(u) \in \mathrm{R}_{\mathbb{Q}}[[u]] \subset \mathrm{H}_{\mathbb{Q}}[[x]]$ and $\mathfrak{g}(x) \in \mathrm{H}_{\mathbb{Q}}[[x]]$ coincide (as elements of $\mathrm{H}_{\mathbb{Q}}[[x]]$), where $\mathfrak{g}(x) \in \mathrm{H}_{\mathbb{Q}}[[x]]$ is the formal power series, which has been defined in [6]. It is easily shown that the formal power series $a_1\mathfrak{g}(x) \in \mathrm{H}_{\mathbb{Q}}[[x]]$ is the logarithm of the formal group law $\mathfrak{H}(x,y)$. This (together with the multiplicativity of $\mathfrak{T}^{ev}$) implies the required assertion. $\square$

Remark 15. Actually we obtain the following explicit form of the homomorphism $\lambda\colon \mathrm{R} \to \mathrm{H}$. Suppose $\alpha \in \mathrm{MU}^{-2n}(\mathrm{pt})$; then $S_\omega(\alpha) \in \mathrm{MU}^{2(|\omega|-n)}(\mathrm{pt})$. Recall that $\mathrm{MU}^m(\mathrm{pt}) = 0$ if $m > 0$. Therefore we have:

$$\lambda(\alpha) = \sum_{\omega,\,|\omega|\le n} a_1^{-n}\alpha_\omega \otimes S_\omega(\alpha) \in \mathrm{H}\underset{\mathrm{R}}{\otimes}\mathrm{R} = \mathrm{H}\,.$$

2 A natural transformation $\widetilde{\mathrm{K}\,\mathrm{SU}}(\_)_\otimes \to \mathrm{Aut}_{ring}(\mathrm{MU}^{ev}(\_))$

Let X be a finite CW-complex. Let ξ be a virtual SU-bundle over X of virtual dimension 1 and let $\{\xi\} \in \widetilde{\mathrm{KSU}}(X)$ be its stable equivalence class. By $\mathfrak{T}^{ev}_{\{\xi\}}$ denote the composite homomorphism

$$\mathrm{MU}^{ev}(X) \overset{\mathfrak{T}^{ev}}{\to} \mathrm{H}\underset{\mathrm{R}}{\otimes}\mathrm{MU}^{ev}(X) \overset{\varphi_\xi^*\otimes\mathrm{id}_{\mathrm{MU}^{ev}(X)}}{\longrightarrow} \mathrm{MU}^{ev}(X)\underset{\mathrm{R}}{\otimes}\mathrm{MU}^{ev}(X) \overset{\mu_X^{ev}}{\to} \mathrm{MU}^{ev}(X),$$

where the homomorphism $\varphi_\xi^*\colon \mathrm{H} \to \mathrm{MU}^{ev}(X)$ is induced by a classifying map $\varphi_\xi\colon X \to \mathrm{BSU}$ for the bundle ξ and $\mu_X^{ev}\colon \mathrm{MU}^{ev}(X)\underset{\mathrm{R}}{\otimes}\mathrm{MU}^{ev}(X) \to \mathrm{MU}^{ev}(X)$ is the multiplication, restricted to the subalgebra $\mathrm{MU}^{ev}(X) \subset \mathrm{MU}^*(X)$. Note that for X such that $\mathrm{MU}^{2n+1}(X) = 0\ \forall n \in \mathbb{Z}$ $\mathfrak{T}^{ev}_{\{\xi\}}$ is a homomorphism $\mathrm{MU}^*(X) \to \mathrm{MU}^*(X)$.

Theorem 16. *For any stable equivalence class of* SU*-bundles $\{\xi\}$ over X the homomorphism $\mathfrak{T}^{ev}_{\{\xi\}}\colon \mathrm{MU}^{ev}(X) \to \mathrm{MU}^{ev}(X)$ is a ring isomorphism such that*

(i) *it is natural on $(X,\,\xi)$;*

(ii) $\mathfrak{T}^{ev}_{\{\xi\}}(\mathrm{R}) = \varphi_\xi^*(\mathrm{K})$*;*

(iii) *for any line bundle* ζ *over* X *we have* $\mathfrak{T}^{ev}_{\{\xi\}}(\sigma_1(\zeta)) = \sigma_1(\xi\otimes\zeta)$, *where* ξ *is a representative of* $\{\xi\}$ *of virtual dimension 1 as above.*

(Let us remember that $\mathrm{K} \subset \mathrm{H}$ *is the subring, generated by the coefficients of the formal group law* $\mathfrak{H}(x,y) \in \mathrm{H}[[x,y]]$*).*

Proof. (i) Suppose (Y,η) is another pair with a morphism $f\colon (Y,\eta) \to (X,\xi)$ (i.e. f is a continuous map $Y \to X$ such that $f^*(\xi) = \eta$). We claim that $f^* \circ \mathfrak{T}^{ev}_{\{\xi\}} = \mathfrak{T}^{ev}_{\{\eta\}} \circ f^*$. Indeed, since $\varphi_\eta = \varphi_\xi \circ f\colon Y \to \mathrm{BSU}$ is a classifying map for the bundle η, we see that the following digram

$$\begin{tikzcd}
\mathrm{MU}^{ev}(X) \arrow[r, "\mathfrak{T}^{ev}"] \arrow[d, "f^*"'] & \mathrm{H}\underset{\mathrm{R}}{\otimes}\mathrm{MU}^{ev}(X) \arrow[r, "\varphi_\xi^*\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}_{\mathrm{H}}\otimes f^*"] & \mathrm{MU}^{ev}(X)\underset{\mathrm{R}}{\otimes}\mathrm{MU}^{ev}(X) \arrow[r, "\mu_X^{ev}"] \arrow[d, "f^*\otimes f^*"] & \mathrm{MU}^{ev}(X) \arrow[d, "f^*"] \\
\mathrm{MU}^{ev}(Y) \arrow[r, "\mathfrak{T}^{ev}"] & \mathrm{H}\underset{\mathrm{R}}{\otimes}\mathrm{MU}^{ev}(Y) \arrow[r, "\varphi_\eta^*\otimes\mathrm{id}"] & \mathrm{MU}^{ev}(Y)\underset{\mathrm{R}}{\otimes}\mathrm{MU}^{ev}(Y) \arrow[r, "\mu_Y^{ev}"] & \mathrm{MU}^{ev}(Y)
\end{tikzcd}$$

commutes.

Assertion (ii) follows easily from the naturality of $\mathfrak{T}^{ev}$ and Proposition 11.

(iii) Because of the naturality of $\mathfrak{T}^{ev}$, we have the following commutative diagram:

$$\begin{tikzcd}
\mathrm{MU}^2(\mathbb{C}P^\infty) \arrow[r, "\mathfrak{T}_2"] \arrow[d, "\varphi_\zeta^*"'] & \mathrm{H}\underset{\mathrm{R}}{\widehat{\otimes}}\mathrm{MU}^{ev}(\mathbb{C}P^\infty) \arrow[d, "\mathrm{id}_{\mathrm{H}}\otimes\varphi_\zeta^*"] \\
\mathrm{MU}^2(X) \arrow[r, "\mathfrak{T}_2"] & \mathrm{H}\underset{\mathrm{R}}{\otimes}\mathrm{MU}^{ev}(X),
\end{tikzcd}$$

where $\varphi_\zeta\colon X \to \mathbb{C}P^\infty$ is a classifying map for the line bundle ζ. Since $\mathfrak{T}^{ev}(u) = \mathfrak{h}(u) = \sigma_1(\widehat{\Xi})$, we have: $\mathfrak{T}^{ev}(\sigma_1(\zeta)) = \mathfrak{h}(\sigma_1(\zeta)) = 1\otimes\sigma_1(\zeta) + \sum_{i,j\geq 1}\alpha_{i,j}\widetilde{s}_i\otimes\sigma_1(\zeta)^j \in \mathrm{H}\underset{\mathrm{R}}{\otimes}\mathrm{MU}^{ev}(X)$. Therefore $\mathfrak{T}^{ev}_{\{\xi\}}(\sigma_1(\zeta)) = \sigma_1(\zeta) + \sum_{i,j\geq 1}\alpha_{i,j}\varphi_\xi^*(\widetilde{s}_i)\sigma_1(\zeta)^j = (\varphi_\xi^*\mathfrak{h})(\sigma_1(\zeta)) = \varphi_{\xi\otimes\zeta}^*(\sigma_1(\widehat{\Xi})) = \sigma_1(\xi\otimes\zeta)$, where $\varphi_{\xi\otimes\zeta}\colon X \to \mathrm{BU}$ is a classifying map for the U-bundle $\xi\otimes\zeta$ over X. $\square$

Remark 17. We do not assert that $\mathfrak{T}^{ev}_{\{\xi\}}(\sigma_i(\zeta)) = \sigma_i(\xi\otimes\zeta)$ for $i \geq 2$. Otherwise, using Theorem 9 and $\sigma_i(\zeta) = 0$ for $i \geq 2$ for any line bundle ζ, we obtain $\sigma_i(\xi\otimes\zeta) = 0$, $i \geq 2$, but clearly this is not true in general.

For example, using formulas, obtained in [6], one can easily verify that

$$s_r(\Xi\otimes\mathcal{Z}) = s_r(\Xi)\otimes 1 + 1\otimes s_r(\mathcal{Z}) + \sum_{m,n\geq 1}\gamma^{(r)}_{m,n}s_m(\Xi)\otimes s_n(\mathcal{Z})$$

$$= s_r(\Xi)\otimes 1 + 1\otimes u^r + \sum_{m,n\geq 1}\gamma^{(r)}_{m,n}s_m(\Xi)\otimes u^n, \tag{5}$$

where the coefficients $\gamma^{(r)}_{m,n} \in \mathrm{MU}^{2(r-m-n)}(\mathrm{pt})$ are determined by the equation $F(u,v)^r = u^r + v^r + \sum_{m,n\geq 1,\, m+n\geq r}\gamma^{(r)}_{m,n}u^m v^n$. On the other hand, $\mathfrak{T}^{ev}_{\{\xi\}}(s_r(\mathcal{Z})) = \mathfrak{T}^{ev}_{\{\xi\}}(u^r) = s_1(\Xi\otimes\mathcal{Z})^r = \mathfrak{h}(u)^r$, but it is clear that $\mathfrak{h}(u)^r$ does not coincide with expression (5).

Actually, we do not use the Hopf algebra structure on $\mathrm{H} = \mathrm{MU}^*(\mathrm{BSU}_\otimes)$ up to now. But it gives us, for example, the following identity:

$$(\varphi^*_{\eta\otimes\xi}\mathfrak{H})(x, y) = ((\varphi^*_\eta \star \varphi^*_\xi)\mathfrak{H})(x, y)$$

for two virtual SU-bundles ξ, η over X of virtual dimension 1, where $\star$ denotes the convolution in $\mathrm{Hom}_{\mathrm{R}}(\mathrm{H},\, \mathrm{MU}^*(X))$, $\varphi^*_\eta \star \varphi^*_\xi = \mu_{\mathrm{MU}^*(X)} \circ (\varphi_\eta \otimes \varphi_\xi) \circ \Delta\colon\ \mathrm{H} \to \mathrm{MU}^*(X)$.

Let us define the composite multiplicative natural transformation

$$(\Delta \otimes \mathrm{id}_{\mathbf{S}})\mathfrak{T}^{ev} = (\Delta \otimes \mathrm{id}_{\mathrm{MU}^{ev}(\_)}) \circ \mathfrak{T}^{ev}\colon\ \mathrm{MU}^{ev}(\_) \to \mathrm{H} \mathbin{\widehat{\otimes}_{\mathrm{R}}} \mathrm{H} \mathbin{\widehat{\otimes}_{\mathrm{R}}} \mathrm{MU}^{ev}(\_),$$

where $\Delta\colon\ \mathrm{H} \to \mathrm{H} \mathbin{\widehat{\otimes}_{\mathrm{R}}} \mathrm{H}$ is the comultiplication in H. Let us remember that the transformation $\mathfrak{T}^{ev}$ was defined by means of the elements $\mathfrak{T}^{ev}_{2n} = \sum_\omega a_1^n \alpha_\omega \otimes S_\omega \in \mathrm{H} \mathbin{\widehat{\otimes}_{\mathbb{Z}}} \mathbf{S}$, $n \in \mathbb{Z}$. Similarly, $(\Delta \otimes \mathrm{id}_{\mathbf{S}})\mathfrak{T}^{ev}$ can be defined by means of the elements $(\Delta \otimes \mathrm{id}_{\mathbf{S}})\mathfrak{T}^{ev}_{2n} = \sum_\omega \Delta(a_1^n \alpha_\omega) \otimes S_\omega \in \mathrm{H} \mathbin{\widehat{\otimes}_{\mathrm{R}}} \mathrm{H} \mathbin{\widehat{\otimes}_{\mathbb{Z}}} \mathbf{S}$, $n \in \mathbb{Z}$.

Since Landweber-Novikov's algebra $\mathbf{S}$ acts on $\mathrm{H} = \mathrm{MU}^*(\mathrm{BSU}_\otimes)$, we can define the composition $((\mathrm{id}_{\mathrm{H}} \otimes \Delta_{\mathbf{S}})\mathfrak{T}^{ev}) \circ \mathfrak{T}^{ev}$, where $\Delta_{\mathbf{S}}\colon\ \mathbf{S} \to \mathbf{S} \otimes_{\mathbb{Z}} \mathbf{S}$ is the comultiplication in $\mathbf{S}$ [2], [3]. More precisely, the multiplicative natural transformation

$$((\mathrm{id}_{\mathrm{H}} \otimes \Delta_{\mathbf{S}})\mathfrak{T}^{ev}) \circ \mathfrak{T}^{ev}\colon\ \mathrm{MU}^{ev}(\_) \to \mathrm{H} \mathbin{\widehat{\otimes}_{\mathrm{R}}} \mathrm{H} \mathbin{\widehat{\otimes}_{\mathrm{R}}} \mathrm{MU}^{ev}(\_)$$

is given by the elements

$$((\mathrm{id}_{\mathrm{H}} \otimes \Delta_{\mathbf{S}})\mathfrak{T}^{ev}_{2n}) \circ \mathfrak{T}^{ev}_{2n} = (\sum_\omega a_1^n \alpha_\omega \otimes \Delta_{\mathbf{S}}(S_\omega)) \circ (\sum_\tau a_1^n \alpha_\tau \otimes S_\tau)$$

$$= \sum_\omega (a_1^n \alpha_\omega \otimes \sum_{\omega' \cup \omega'' = \omega,\, \tau} S_{\omega'}(a_1^n \alpha_\tau) \otimes S_{\omega''} S_\tau) \in \mathrm{H} \mathbin{\widehat{\otimes}_{\mathrm{R}}} \mathrm{H} \mathbin{\widehat{\otimes}_{\mathbb{Z}}} \mathbf{S}, \quad n \in \mathbb{Z},$$

where we used the following well-known fact [5]: for every pair of partitions ω', ω'', the composition $S_{\omega''} S_{\omega'}$ is an integral linear combination of elements S_ω.

Theorem 18. $((\mathrm{id}_{\mathrm{H}} \otimes \Delta_{\mathbf{S}})\mathfrak{T}^{ev}) \circ \mathfrak{T}^{ev}$ *and* $(\Delta \otimes \mathrm{id}_{\mathbf{S}})\mathfrak{T}^{ev}$ *coincide as a natural transformations*

$$\mathrm{MU}^{ev}(\_) \to \mathrm{H} \mathbin{\widehat{\otimes}_{\mathrm{R}}} \mathrm{H} \mathbin{\widehat{\otimes}_{\mathrm{R}}} \mathrm{MU}^{ev}(\_).$$

Remark 19. Now let us introduce some additional notation. For a CW-complex X by C we denote the R-algebra $\mathrm{MU}^{ev}(X)$ and by Δ_{C} we denote the homomorphism $\mathfrak{T}^{ev}\colon\ \mathrm{C} \to \mathrm{H} \mathbin{\widehat{\otimes}_{\mathrm{R}}} \mathrm{C}$. Using the comultiplication in the Hopf algebra $\mathbf{S}$ we can define the homomorphism

$$\Delta_{\mathrm{H} \widehat{\otimes}_{\mathrm{R}} \mathrm{C}}\colon\ \mathrm{H} \mathbin{\widehat{\otimes}_{\mathrm{R}}} \mathrm{C} \to \mathrm{H} \mathbin{\widehat{\otimes}_{\mathrm{R}}} \mathrm{H} \mathbin{\widehat{\otimes}_{\mathrm{R}}} \mathrm{C},$$

corresponding to $(\mathrm{id}_{\mathrm{H}} \otimes \Delta_{\mathbf{S}})\mathfrak{T}^{ev}$ and such that the following diagram

$$\begin{tikzcd}
\mathrm{H} \widehat{\otimes}_{\mathrm{R}} \mathrm{C} \arrow[r, "\Delta_{\mathrm{H} \widehat{\otimes}_{\mathrm{R}} \mathrm{C}}"] \arrow[d, "m^{ev}_{\mathrm{BSU},\, X}"'] & \mathrm{H} \widehat{\otimes}_{\mathrm{R}} \mathrm{H} \widehat{\otimes}_{\mathrm{R}} \mathrm{C} \arrow[d, "\mathrm{id}_{\mathrm{H}} \otimes m^{ev}_{\mathrm{BSU},\, X}"] \\
\mathrm{D} \arrow[r, "\Delta_{\mathrm{D}}"] & \mathrm{H} \widehat{\otimes}_{\mathrm{R}} \mathrm{D}
\end{tikzcd}$$

commutes, where D is the R-algebra $\mathrm{MU}^{ev}(\mathrm{BSU}\times X)$ and $m^{ev}_{\mathrm{BSU},X}\colon \mathrm{MU}^*(\mathrm{BSU})\mathop{\widehat{\otimes}}_{\mathrm{R}}\mathrm{MU}^{ev}(X)\to\mathrm{MU}^{ev}(\mathrm{BSU}\times X)$ is the canonical homomorphism. We see that theorem's statement is equivalent to the commutativity of the following diagram (for any X):

$$\begin{tikzcd}
\mathrm{C} \arrow[r, "\Delta_{\mathrm{C}}"] \arrow[d, "\Delta_{\mathrm{C}}"'] & \mathrm{H}\mathop{\widehat{\otimes}}_{\mathrm{R}}\mathrm{C} \arrow[d, "\Delta_{\mathrm{H}\widehat{\otimes}_{\mathrm{R}}\mathrm{C}}"] \\
\mathrm{H}\mathop{\widehat{\otimes}}_{\mathrm{R}}\mathrm{C} \arrow[r, "\Delta\otimes\mathrm{id}_{\mathrm{C}}"'] & \mathrm{H}\mathop{\widehat{\otimes}}_{\mathrm{R}}\mathrm{H}\mathop{\widehat{\otimes}}_{\mathrm{R}}\mathrm{C}\,.
\end{tikzcd}$$

Together with the commutative diagram

$$\begin{tikzcd}
\mathrm{R}\mathop{\otimes}_{\mathrm{R}}\mathrm{C} & \mathrm{H}\mathop{\widehat{\otimes}}_{\mathrm{R}}\mathrm{C} \arrow[l, "\varepsilon\otimes\mathrm{id}_{\mathrm{C}}"'] \\
 & \mathrm{C}, \arrow[ul, "\cong"] \arrow[u, "\Delta_{\mathrm{C}}"']
\end{tikzcd}$$

where $\varepsilon\colon \mathrm{H}\to\mathrm{R}$ is the counit in H, this shows that Δ_{C} defines a structure on C which is similar to the structure of a comodule over the Hopf algebra $\mathrm{H}=\mathrm{MU}^*(\mathrm{BSU}_{\otimes})$.

Recall that by $\widehat{\Xi}$ (resp. Ξ) we denote the universal *virtual* U-bundle over BU of *virtual dimension* 1 (the universal *virtual* SU-bundle over BSU of *virtual dimension* 1 respectively). Theorem's proof is based on the following lemma.

Lemma 20. For $u=\sigma_1(\mathcal{Z})\in\mathrm{MU}^2(\mathbb{C}P^\infty)$ as above we have:

$$(((\mathrm{id}_{\mathrm{H}}\otimes\Delta_{\mathbf{S}})\mathfrak{T}^{ev})\circ\mathfrak{T}^{ev})(u)=((\Delta\otimes\mathrm{id}_{\mathbf{S}})\mathfrak{T}^{ev})(u)=\sigma_1(\Xi\boxtimes\widehat{\Xi})\in\mathrm{MU}^2(\mathrm{BSU}\times\mathrm{BU}).$$

Proof of the theorem. Theorem's assertion follows from the lemma. Indeed, applying Theorem 9 for $\mathrm{A}=\mathrm{H}\mathop{\widehat{\otimes}}_{\mathrm{R}}\mathrm{H}$, $\mathfrak{f}(u)=(((\mathrm{id}_{\mathrm{H}}\otimes\Delta_{\mathbf{S}})\mathfrak{T}^{ev})\circ\mathfrak{T}^{ev})(u)=((\Delta\otimes\mathrm{id}_{\mathbf{S}})\mathfrak{T}^{ev})(u)=(\Delta\mathfrak{h})(u)\in(\mathrm{H}\mathop{\widehat{\otimes}}_{\mathrm{R}}\mathrm{H})[[u]]=\mathrm{A}[[u]]$ we see that the multiplicative natural transformations $((\mathrm{id}_{\mathrm{H}}\otimes\Delta_{\mathbf{S}})\mathfrak{T}^{ev})\circ\mathfrak{T}^{ev}$ and $(\Delta\otimes\mathrm{id}_{\mathbf{S}})\mathfrak{T}^{ev}\colon \mathrm{MU}^{ev}(\_)\to\mathrm{A}\mathop{\widehat{\otimes}}_{\mathrm{R}}\mathrm{MU}^{ev}(\_)$ coincide. $\square$

Proof of the lemma. Let us remember that the elements $s_1(\widehat{\Xi})=\sigma_1(\widehat{\Xi})\in\mathrm{MU}^2(\mathrm{BU})$ and $\mathfrak{T}^{ev}(u)=\mathfrak{h}(u)\in\mathrm{H}[[u]]$ coincide, if we identify $\mathrm{MU}^*(\mathrm{BU})$ with $\mathrm{H}[[u]]$, $u=\sigma_1(\mathcal{Z})$. From Cartan's formula it follows that the following diagram

$$\begin{tikzcd}
\mathrm{MU}^*(\mathrm{BSU}\times\mathbb{C}P^\infty) \arrow[r, "(\mathrm{id}_{\mathrm{H}}\otimes\Delta_{\mathbf{S}})\mathfrak{T}^{ev}"] \arrow[d, "\cong"'] & \mathrm{H}\mathop{\widehat{\otimes}}_{\mathrm{R}}\mathrm{MU}^*(\mathrm{BSU}\times\mathbb{C}P^\infty) \arrow[d, "\cong"] \\
\mathrm{MU}^*(\mathrm{BU}) \arrow[r, "\mathfrak{T}^{ev}"] & \mathrm{H}\mathop{\widehat{\otimes}}_{\mathrm{R}}\mathrm{MU}^*(\mathrm{BU})
\end{tikzcd}$$

commutes. Hence $(((\mathrm{id}_{\mathrm{H}}\otimes\Delta_{\mathbf{S}})\mathfrak{T}^{ev})\circ\mathfrak{T}^{ev})(u)=\mathfrak{T}^{ev}(\sigma_1(\widehat{\Xi}))\in\mathrm{H}\mathop{\widehat{\otimes}}_{\mathrm{R}}\mathrm{MU}^*(\mathrm{BU})=\mathrm{H}\mathop{\widehat{\otimes}}_{\mathrm{R}}\mathrm{H}\mathop{\widehat{\otimes}}_{\mathrm{R}}\mathrm{MU}^*(\mathbb{C}P^\infty)$. Therefore at first we must compute the element $\mathfrak{T}^{ev}(\sigma_1(\widehat{\Xi}))$.

Let $\widehat{\Xi}_m$ be the universal m-dimensional $\mathrm{U}(m)$-bundle over $\mathrm{BU}(m)$. Consider the diagram

$$\begin{tikzcd}
\widehat{\Xi}_m \arrow[d] & \mathcal{Z}_1 \oplus \ldots \oplus \mathcal{Z}_m \arrow[l] \arrow[d] \\
\mathrm{BU}(m) & \mathbb{C}P^\infty \times \ldots \times \mathbb{C}P^\infty, \arrow[l, "\widehat{\mu}_m"']
\end{tikzcd}$$

where $\widehat{\mu}_m\colon (\mathbb{C}P^\infty)^m \to \mathrm{BU}(m)$ is the "splitting map", $\widehat{\mu}_m^*(\widehat{\Xi}_m) = \mathcal{Z}_1 \oplus \ldots \oplus \mathcal{Z}_m$. Using the splitting principle, we have:

$$\widehat{\mu}_m^*(s_1(\widehat{\Xi}_m)) = \sigma_1(\mathcal{Z}_1) + \ldots + \sigma_1(\mathcal{Z}_m) = u_1 + \ldots + u_m.$$

Because of the naturality of $\mathfrak{T}^{ev}$, we have:

$$(\mathrm{id}_{\mathrm{H}} \otimes \widehat{\mu}_m^*)(\mathfrak{T}^{ev}(s_1(\widehat{\Xi}_m))) = \mathfrak{h}(u_1) + \ldots + \mathfrak{h}(u_m) = 1 \otimes \left(\sum_{1 \le i \le m} u_i\right) + \sum_{k,l \ge 1} \alpha_{k,l} \widetilde{s}_k \otimes \left(\sum_{1 \le i \le m} u_i^l\right).$$

Therefore we get:

$$\mathfrak{T}^{ev}(s_1(\widehat{\Xi}_m)) = 1 \otimes s_1(\widehat{\Xi}_m) + \sum_{k,l \ge 1} \alpha_{k,l} \widetilde{s}_k \otimes s_l(\widehat{\Xi}_m).$$

Now, passing to the limit as $m \to \infty$, we obtain:

$$\mathfrak{T}^{ev}(s_1(\widehat{\Xi})) = 1 \otimes s_1(\widehat{\Xi}) + \sum_{k,l \ge 1} \alpha_{k,l} \widetilde{s}_k \otimes s_l(\widehat{\Xi}). \tag{6}$$

It was proved in [6] that (6) coincides with $\sigma_1(\Xi \boxtimes \widehat{\Xi}) \in \mathrm{MU}^2(\mathrm{BSU} \times \mathrm{BU})$, i.e. with $\sigma_1(\Xi \boxtimes \Xi \boxtimes \mathcal{Z}) \in \mathrm{MU}^2(\mathrm{BSU} \times \mathrm{BSU} \times \mathbb{C}P^\infty)$.

From the other hand, we have $\phi^*(\Xi) = \Xi \boxtimes \Xi$, where $\phi\colon \mathrm{BSU}_\otimes \times \mathrm{BSU}_\otimes \to \mathrm{BSU}_\otimes$ is the multiplication in the H-group $\mathrm{BSU}_\otimes$ and $\Delta = \phi^*\colon \mathrm{H} \to \mathrm{H} \mathbin{\widehat{\otimes}_{\mathrm{R}}} \mathrm{H}$. Therefore we obtain $((\Delta \otimes \mathrm{id}_{\mathbf{S}})\mathfrak{T}^{ev})(u) = \sigma_1(\Xi \boxtimes \Xi \boxtimes \mathcal{Z})$ (let us remark that the element $((\Delta \otimes \mathrm{id}_{\mathbf{S}})\mathfrak{T}^{ev})(u)$ can also be expressed as $(\Delta\mathfrak{h})(u) \in \mathrm{H} \mathbin{\widehat{\otimes}_{\mathrm{R}}} \mathrm{H} \mathbin{\widehat{\otimes}_{\mathrm{R}}} \mathrm{MU}^*(\mathbb{C}P^\infty) = \mathrm{MU}^*(\mathrm{BSU} \times \mathrm{BU})$). $\square$

Recall that by $\widetilde{\mathrm{K}\,\mathrm{SU}}(X)_\otimes$ we denote the group $[X,\, \mathrm{BSU}_\otimes]$ of stable equivalence classes of SU-bundles over X with respect to the operation, induced by the tensor product of virtual bundles of virtual dimension 1.

Corollary 21. *For any virtual* SU*-bundles ξ, η over X of virtual dimension 1 we have $\mathfrak{T}^{ev}_{\{\eta\}} \circ \mathfrak{T}^{ev}_{\{\xi\}} = \mathfrak{T}^{ev}_{\{\eta\otimes\xi\}}$. In particular, we obtain the group homomorphism $\widetilde{\mathrm{K}\,\mathrm{SU}}(X)_\otimes \to \mathrm{Aut}_{ring}(\mathrm{MU}^{ev}(X))$, $\{\xi\} \mapsto \mathfrak{T}^{ev}_{\{\xi\}}$, which is natural on X.*

Proof. We shall use the following notation: $\mathrm{C} := \mathrm{MU}^{ev}(X)$, $\Delta_{\mathrm{C}} := \mathfrak{T}^{ev}\colon \mathrm{C} \to \mathrm{H} \mathbin{\widehat{\otimes}_{\mathrm{R}}} \mathrm{C}$, $\varphi := \varphi_\xi^*$, $\psi := \varphi_\eta^*\colon \mathrm{H} \to \mathrm{C}$, $\tau_\varphi := \mu_{\mathrm{C}} \circ (\varphi \otimes \mathrm{id}_{\mathrm{C}})$, $\tau_\psi := \mu_{\mathrm{C}} \circ (\psi \otimes \mathrm{id}_{\mathrm{C}})\colon \mathrm{H} \mathbin{\widehat{\otimes}_{\mathrm{R}}} \mathrm{C} \to \mathrm{C}$, $\mathfrak{T}^{ev}_\varphi := \mathfrak{T}^{ev}_{\{\xi\}} = \tau_\varphi \circ \Delta_{\mathrm{C}}$, $\mathfrak{T}^{ev}_\psi := \mathfrak{T}^{ev}_{\{\eta\}} = \tau_\psi \circ \Delta_{\mathrm{C}}$. Because of $\varphi^*_{\eta\otimes\xi} = \varphi^*_\eta \star \varphi^*_\xi$, we see that $\mathfrak{T}^{ev}_{\{\eta\otimes\xi\}} = \mathfrak{T}^{ev}_{\psi\star\varphi}$ in the just introduced notation, where $\star$ is the convolution in $\mathrm{Hom}_{\mathrm{R}}(\mathrm{H},\, \mathrm{C})$.

It can easily be checked that the diagram

$$
\begin{tikzcd}
\mathrm{C} \arrow[r, "\Delta_{\mathrm{C}}"] \arrow[dr, "\mathfrak{T}^{ev}_{\varphi}"'] & \mathrm{H} \mathbin{\widehat{\otimes}_{\mathrm{R}}} \mathrm{C} \arrow[r, "\Delta_{\mathrm{H} \widehat{\otimes}_{\mathrm{R}} \mathrm{C}}"] \arrow[d, "\tau_{\varphi}"] & \mathrm{H} \mathbin{\widehat{\otimes}_{\mathrm{R}}} \mathrm{H} \mathbin{\widehat{\otimes}_{\mathrm{R}}} \mathrm{C} \arrow[r, "\psi\otimes\varphi\otimes\mathrm{id}_{\mathrm{C}}"] \arrow[d, "\mathrm{id}_{\mathrm{H}} \otimes \tau_{\varphi}"] & \mathrm{C} \mathbin{\widehat{\otimes}_{\mathrm{R}}} \mathrm{C} \mathbin{\widehat{\otimes}_{\mathrm{R}}} \mathrm{C} \arrow[d, "\mu_{\mathrm{C}}\otimes\mathrm{id}_{\mathrm{C}}"] \\
 & \mathrm{C} \arrow[r, "\Delta_{\mathrm{C}}"] \arrow[dr, "\mathfrak{T}^{ev}_{\psi}"'] & \mathrm{H} \mathbin{\widehat{\otimes}_{\mathrm{R}}} \mathrm{C} \arrow[d, "\tau_{\psi}"] & \mathrm{C} \mathbin{\widehat{\otimes}_{\mathrm{R}}} \mathrm{C} \arrow[dl, "\mu_{\mathrm{C}}"] \\
 & & \mathrm{C} &
\end{tikzcd}
$$

commutes. For example, the commutativity of the square easily follows from the commutative diagram

$$
\begin{tikzcd}
\mathrm{H} \mathbin{\widehat{\otimes}_{\mathrm{R}}} \mathrm{C} \arrow[r, "\Delta_{\mathrm{H} \widehat{\otimes}_{\mathrm{R}} \mathrm{C}}"] \arrow[d, "\varphi\otimes\mathrm{id}_{\mathrm{C}}"'] & \mathrm{H} \mathbin{\widehat{\otimes}_{\mathrm{R}}} \mathrm{H} \mathbin{\widehat{\otimes}_{\mathrm{R}}} \mathrm{C} \arrow[d, "\mathrm{id}_{\mathrm{H}} \otimes\varphi\otimes\mathrm{id}_{\mathrm{C}}"] \\
\mathrm{C} \mathbin{\widehat{\otimes}_{\mathrm{R}}} \mathrm{C} \arrow[r, "\Delta_{\mathrm{C} \widehat{\otimes}_{\mathrm{R}} \mathrm{C}}"] \arrow[d, "\mu_{\mathrm{C}}"'] & \mathrm{H} \mathbin{\widehat{\otimes}_{\mathrm{R}}} \mathrm{C} \mathbin{\widehat{\otimes}_{\mathrm{R}}} \mathrm{C} \arrow[d, "\mathrm{id}_{\mathrm{H}} \otimes\mu_{\mathrm{C}}"] \\
\mathrm{C} \arrow[r, "\Delta_{\mathrm{C}}"'] & \mathrm{H} \mathbin{\widehat{\otimes}_{\mathrm{R}}} \mathrm{C}\,.
\end{tikzcd}
$$

Then, using Remark 19, we have:

$$\mathfrak{T}^{ev}_{\psi} \circ \mathfrak{T}^{ev}_{\varphi} = \mu_{\mathrm{C}} \circ (\mu_{\mathrm{C}} \otimes \mathrm{id}_{\mathrm{C}}) \circ (\psi \otimes \varphi \otimes \mathrm{id}_{\mathrm{C}}) \circ \Delta_{\mathrm{H} \widehat{\otimes}_{\mathrm{R}} \mathrm{C}} \circ \Delta_{\mathrm{C}}$$

$$= \mu_{\mathrm{C}} \circ (\mu_{\mathrm{C}} \otimes \mathrm{id}_{\mathrm{C}}) \circ (\psi \otimes \varphi \otimes \mathrm{id}_{\mathrm{C}}) \circ (\Delta \otimes \mathrm{id}_{\mathrm{C}}) \circ \Delta_{\mathrm{C}} = \mathfrak{T}^{ev}_{\psi\star\varphi}. \quad \square$$

Recall that by K_{ξ} we denote the subring $\varphi^*_{\xi}(\mathrm{K}) \subset \mathrm{MU}^{ev}(X)$.

Corollary 22. *Suppose ξ, η are virtual* SU*-bundles over X of virtual dimension 1. Then $\mathfrak{T}^{ev}_{\eta}(\sigma_1(\xi\otimes\zeta)) = \sigma_1(\eta\otimes\xi\otimes\zeta)$ for any line bundle ζ over X. Furthermore, $\mathfrak{T}^{ev}_{\eta}(\mathrm{K}_{\xi}) = \mathrm{K}_{\eta\otimes\xi}$.*

For a pair (X, ξ) consisting of a CW-complex X and of a virtual SU-bundle ξ over X of virtual dimension 1 we can also define the natural transformation

$$\mathfrak{T}^{ev}_{X,\{\xi\}} := (\varphi^*_{\xi} \otimes \mathrm{id}_{\mathbf{S}})\mathfrak{T}^{ev} \colon\ \mathrm{MU}^{ev}(Z) \to \mathrm{MU}^{ev}(X) \mathbin{\widehat{\otimes}_{\mathrm{R}}} \mathrm{MU}^{ev}(Z),$$

where Z is an arbitrary CW-complex. In a different way $\mathfrak{T}^{ev}_{X,\{\xi\}}$ can be defined by application of Theorem 9 for $\mathrm{A} = \mathrm{MU}^{ev}(X)$ and $\mathfrak{f}(u) = (\varphi^*_{\xi}\mathfrak{h})(u) \in \mathrm{A}[[u]]$. If we take $Z = \mathbb{C}P^{\infty} \times \mathbb{C}P^{\infty}$, then the formal power series $\mathfrak{T}^{ev}_{X,\{\xi\}}(F(u,v)) = (\varphi^*_{\xi}\mathfrak{H})(x,y) \in \mathrm{MU}^{ev}(X)[[x,y]] \subset \mathrm{MU}^*(X)[[x,y]]$ describes the relation between first Conner-Floyd classes of lifts $\widehat{\xi}$ of the bundle ξ (see [6]), where $F(u,v)$ is the formal product in complex cobordism theory and $x = \mathfrak{T}^{ev}_{X,\{\xi\}}(u) = (\varphi^*_{\xi}\mathfrak{h})(u)$, $y = \mathfrak{T}^{ev}_{X,\{\xi\}}(v) = (\varphi^*_{\xi}\mathfrak{h})(v)$ (recall that a virtual U-bundle $\widehat{\xi}$ of virtual dimension 1 over X is a lift of ξ if $\widehat{\xi} = \xi\otimes\zeta$ for some geometric line bundle ζ over X).

Let $\mathfrak{R}^{ev}_{X,\{\xi\}}$ be the natural transformation

$$\mathrm{MU}^{ev}(Z) \overset{\mathfrak{T}^{ev}_{X,\{\xi\}}}{\to} \mathrm{MU}^{ev}(X) \mathbin{\widehat{\otimes}_{\mathrm{R}}} \mathrm{MU}^{ev}(Z) \overset{m^{ev}_{X,Z}}{\to} \mathrm{MU}^{ev}(X \times Z),$$

where $m^{ev}_{X,\,Z}$ is the canonical homomorphism.

Let $(Y,\,\eta)$ be another pair. Using the commutativity of the diagram

$$\begin{tikzcd}
 & \mathrm{H}\mathbin{\widehat{\otimes}_{\mathrm{R}}}\mathrm{MU}(Z) \arrow[r, "(\mathrm{id}_{\mathrm{H}}\otimes\Delta_{\mathbf{S}})\mathfrak{T}^{ev}"] \arrow[d, "\varphi^*_\xi\otimes\mathrm{id}"] & \mathrm{H}\mathbin{\widehat{\otimes}_{\mathrm{R}}}\mathrm{H}\mathbin{\widehat{\otimes}_{\mathrm{R}}}\mathrm{MU}(Z) \arrow[dr, "\varphi^*_\eta\otimes\varphi^*_\xi\otimes\mathrm{id}"] & & \\
\mathrm{MU}(Z) \arrow[ur, "\mathfrak{T}^{ev}"] \arrow[r, "\mathfrak{T}^{ev}_{X,\{\xi\}}"] \arrow[dr, "\mathfrak{R}^{ev}_{X,\{\xi\}}"'] & \mathrm{MU}(X)\mathbin{\widehat{\otimes}_{\mathrm{R}}}\mathrm{MU}(Z) \arrow[rr, "(\varphi^*_\eta\otimes\Delta_{\mathbf{S}})\mathfrak{T}^{ev}"] \arrow[d, "m^{ev}_{X,Z}"] & & \mathrm{MU}(Y)\mathbin{\widehat{\otimes}_{\mathrm{R}}}\mathrm{MU}(X)\mathbin{\widehat{\otimes}_{\mathrm{R}}}\mathrm{MU}(Z) \arrow[dl, "\mathrm{id}\otimes m^{ev}_{X,Z}"'] \arrow[dr, "m^{ev}_{Y,X}\otimes\mathrm{id}"] & \\
 & \mathrm{MU}(X\times Z) \arrow[r, "(\varphi^*_\eta\otimes\mathrm{id}_{\mathbf{S}})\mathfrak{T}^{ev}"] \arrow[drr, "\mathfrak{R}^{ev}_{Y,\{\eta\}}"'] & \mathrm{MU}(Y)\mathbin{\widehat{\otimes}_{\mathrm{R}}}\mathrm{MU}(X\times Z) \arrow[dr, "m^{ev}_{Y,X\times Z}"] & & \mathrm{MU}(Y\times X)\mathbin{\widehat{\otimes}_{\mathrm{R}}}\mathrm{MU}(Z) \arrow[dl, "m^{ev}_{Y\times X,Z}"] \\
 & & & \mathrm{MU}(Y\times X\times Z) &
\end{tikzcd}$$

(here we write $\mathrm{MU}(\_)$ instead of $\mathrm{MU}^{ev}(\_)$) together with Theorem 18 and the commutative diagram

$$\begin{tikzcd}
\mathrm{H} \arrow[r, "\Delta"] \arrow[d, "\varphi^*_{\eta\boxtimes\xi}"'] & \mathrm{H}\mathbin{\widehat{\otimes}_{\mathrm{R}}}\mathrm{H} \arrow[d, "\varphi^*_\eta\otimes\varphi^*_\xi"] \\
\mathrm{MU}^{ev}(Y\times X) & \mathrm{MU}^{ev}(Y)\mathbin{\widehat{\otimes}_{\mathrm{R}}}\mathrm{MU}^{ev}(X) \arrow[l, "m^{ev}_{Y,X}"]
\end{tikzcd}$$

one can easily prove the following proposition.

Proposition 23. *For any pairs $(X,\,\xi)$ and $(Y,\,\eta)$ as above we have*

$$\mathfrak{R}^{ev}_{Y,\{\eta\}}\circ\mathfrak{R}^{ev}_{X,\{\xi\}}=\mathfrak{R}^{ev}_{Y\times X,\{\eta\boxtimes\xi\}},$$

where $\eta\boxtimes\xi$ is the exterior tensor product of bundles (which is a bundle over $Y\times X$).

3 An extended natural transformation $\widetilde{\mathrm{KSU}}(\_)_\otimes\to\mathrm{Aut}_{ring}(\mathrm{MU}[\frac{1}{2}]^*(\_))$

Now we want to extend the natural transformation $\mathfrak{T}^{ev}\colon\ \mathrm{MU}^{ev}(\_)\to\mathrm{H}\mathbin{\widehat{\otimes}_{\mathrm{R}}}\mathrm{MU}^{ev}(\_)$ to whole cobordism ring $\mathrm{MU}^*(\_)$. Let A be a graded topological algebra over R as in the statement of Theorem 9. By $\mathrm{MU}[\frac{1}{2}]^*(\_)$ denote the localization in 2 (i.e. 2 is invertible) of complex cobordism theory. Set $\mathrm{A}[\frac{1}{2}]:=\mathrm{A}\mathbin{\widehat{\otimes}_{\mathbb{Z}}}\mathbb{Z}[\frac{1}{2}]$.

The following theorem generalizes Theorem 9.

Theorem 24. *Given a formal power series $\mathfrak{f}(u)=\sum_{i\geq1}a_i\otimes u^i\in\mathrm{A}\mathbin{\widehat{\otimes}_{\mathrm{R}}}\mathrm{MU}^*(\mathbb{C}P^\infty)$ such that*

(i) $\deg a_i=2-2i$;

(ii) $\varepsilon_{\mathrm{A}}(a_1)=1,\ \varepsilon_{\mathrm{A}}(a_i)=0,\ i\geq2$

there is a multiplicative natural transformation $\mathfrak{T}_{\mathfrak{f}}\colon\ \mathrm{MU}^*(X)\to\mathrm{A}[\frac{1}{2}]\mathbin{\widehat{\otimes}_{\mathrm{R}}}\mathrm{MU}^*(X)$ such that

a) $\mathfrak{T}_{\mathfrak{f}}(u) = \mathfrak{f}(u)$ *for* $X = \mathbb{C}P^\infty$ *and* $u \in \mathrm{MU}^2(\mathbb{C}P^\infty)$ *as above;*

b) *its restriction* $\mathfrak{T}_{\mathfrak{f}}|_{\mathrm{MU}^{ev}(X)}\colon \mathrm{MU}^{ev}(X) \to \mathrm{A}[\frac{1}{2}] \mathop{\widehat{\otimes}}\limits_{\mathrm{R}} \mathrm{MU}^*(X)$ *can be decomposed into the composition of the natural transformation* $\mathfrak{T}_{\mathfrak{f}}^{ev}\colon \mathrm{MU}^{ev}(X) \to \mathrm{A} \mathop{\widehat{\otimes}}\limits_{\mathrm{R}} \mathrm{MU}^{ev}(X)$ *and of the homomorphism, induced by the canonical homomorphism* $\mathrm{A} \to \mathrm{A}[\frac{1}{2}]$.

Furthermore, using the universal property of the localization, we obtain the natural ring homomorphism $\mathfrak{T}_{\mathfrak{f}}[\frac{1}{2}]\colon \mathrm{MU}[\frac{1}{2}]^*(X) \to \mathrm{A}[\frac{1}{2}] \mathop{\widehat{\otimes}}\limits_{\mathrm{R}} \mathrm{MU}^*(X)$ *which is a ring isomorphism onto a direct summand in* $\mathrm{A}[\frac{1}{2}] \mathop{\widehat{\otimes}}\limits_{\mathrm{R}} \mathrm{MU}^*(X)$.

In order to prove the theorem we need the following lemma.

Lemma 25. For any element $a \in \mathrm{A}$ such that $\varepsilon_{\mathrm{A}}(a) = 1$ there is $\kappa \in \mathrm{A}[\frac{1}{2}]$ such that $\kappa^2 = a$.

Proof of the lemma. We have:

$$r(t) := \sqrt{1+t} = 1 + \frac{t}{2} + \sum_{n\geq 2} (-1)^{n-1} \frac{(2n-3)!!}{2^n n!} t^n \in \mathbb{Z}[\tfrac{1}{2}][[t]],$$

where $(2n-3)!! = \prod_{k=1}^{n-1}(2k-1)$. In other words, the minimal subring in $\mathbb{Q}$, containing the coefficients of $r(t)$, is $\mathbb{Z}[\frac{1}{2}]$. Now note that $a = 1 + b$, where $b \in \ker(\varepsilon_{\mathrm{A}})$. Finally, using the completeness of A with respect to the ideal $\ker(\varepsilon_{\mathrm{A}})$, we see that $r(b) \in \mathrm{A}[\frac{1}{2}]$ is the element such that $r(b)^2 = a$. □

Proof of the theorem. The proof is very similar to the one of Theorem 9 and we give only a sketch.

Note that a_1 satisfies the condition of the previous lemma. Therefore we have the element $\kappa \in \mathrm{A}[\frac{1}{2}]$ such that $\kappa^2 = a_1$. For any $n \in \mathbb{Z}$ we set $\mathfrak{T}_{\mathfrak{f},n} := (\kappa^n \otimes 1)\mathfrak{T}_{\mathfrak{f},0} = \sum_\omega (\kappa^n \alpha_\omega) \otimes S_\omega \in \mathrm{A}[\frac{1}{2}] \mathop{\widehat{\otimes}}\limits_{\mathbb{Z}} \mathbf{S}$. Given a homogeneous element $\alpha \in \mathrm{MU}^n(X)$ we define the value $\mathfrak{T}_{\mathfrak{f}}(\alpha)$ as $\mathfrak{T}_{\mathfrak{f},n}(\alpha) = \sum_\omega (\kappa^n \alpha_\omega) \otimes S_\omega(\alpha) \in \mathrm{A}[\frac{1}{2}] \mathop{\widehat{\otimes}}\limits_{\mathrm{R}} \mathrm{MU}^*(X)$. Clearly, if $n = 2m$, $m \in \mathbb{Z}$, then $\mathfrak{T}_{\mathfrak{f}}(\alpha) = \mathfrak{T}_{\mathfrak{f}}^{ev}(\alpha)$.

The multiplicativity and the naturality of the transformation $\mathfrak{T}_{\mathfrak{f}}$ can be verified in just the same way as in Theorem 9. □

Corollary 26. *There is a unique multiplicative natural transformation* $\mathfrak{T} := \mathfrak{T}_{\mathfrak{h}}\colon \mathrm{MU}^*(X) \to \mathrm{H}[\frac{1}{2}] \mathop{\widehat{\otimes}}\limits_{\mathrm{R}} \mathrm{MU}^*(X)$ *such that 1)* $\mathfrak{T}(u) = \mathfrak{h}(u) = 1 \otimes u + \sum_{k,i\geq 1} \alpha_{k,i} \widetilde{s}_k \otimes u^i \in \mathrm{H}[\frac{1}{2}] \mathop{\widehat{\otimes}}\limits_{\mathrm{R}} \mathrm{MU}^*(\mathbb{C}P^\infty)$ *for* $X = \mathbb{C}P^\infty$, $u = \sigma_1(\mathcal{Z}) \in \mathrm{MU}^2(\mathbb{C}P^\infty)$ *and 2) the composition* $(\varepsilon[\frac{1}{2}] \otimes \mathrm{id}_{\mathrm{MU}^*(X)}) \circ \mathfrak{T}$ *coincides with the localization homomorphism* $\mathrm{MU}^*(X) \to \mathrm{MU}[\frac{1}{2}]^*(X)$ *(an equivalent condition: the composition* $(\varepsilon[\frac{1}{2}] \otimes \mathrm{id}_{\mathrm{MU}^*(X)}) \circ \mathfrak{T}[\frac{1}{2}]$ *is the identity map* $\mathrm{id}_{\mathrm{MU}[\frac{1}{2}](X)}$*).*

Let X be a finite CW-complex. Let ξ be a virtual SU-bundle over X of virtual dimension 1 and let $\{\xi\} \in \widetilde{\mathrm{KSU}}(X)$ be its stable equivalence class. By $\mathfrak{T}_{\{\xi\}}$ denote the composite homomorphism

$$\mathrm{MU}^*(X) \xrightarrow{\mathfrak{T}} \mathrm{H}[\tfrac{1}{2}] \mathop{\otimes}_{\mathrm{R}} \mathrm{MU}^*(X) \xrightarrow{\varphi_\xi^* \otimes \mathrm{id}_{\mathrm{MU}^*(X)}} \mathrm{MU}[\tfrac{1}{2}]^*(X) \mathop{\otimes}_{\mathrm{R}} \mathrm{MU}^*(X) \xrightarrow{\mu_X} \mathrm{MU}[\tfrac{1}{2}]^*(X),$$

where $\varphi_\xi^*\colon \mathrm{H}[\frac{1}{2}] \to \mathrm{MU}[\frac{1}{2}]^*(X)$ is induced by a classifying map $\varphi_\xi\colon X \to \mathrm{BSU}$ for the bundle ξ and $\mu_X\colon \mathrm{MU}[\frac{1}{2}]^*(X) \otimes_{\mathrm{R}} \mathrm{MU}^*(X) \to \mathrm{MU}[\frac{1}{2}]^*(X)$ is the multiplication. By $\mathfrak{T}_{\{\xi\}}[\frac{1}{2}]$ denote the corresponding homomorphism $\mathrm{MU}[\frac{1}{2}]^*(X) \to \mathrm{MU}[\frac{1}{2}]^*(X)$.

Theorem 27. *For any stable equivalence class of* SU*-bundles* $\{\xi\}$ *over* X *we have:*

(i) $\mathfrak{T}_{\{\xi\}}(\mathrm{R}) = \varphi_\xi^*(\mathrm{K})$*;*

(ii) *for any line bundle* ζ *over* X $\mathfrak{T}_{\{\xi\}}(\sigma_1(\zeta)) = \sigma_1(\xi \otimes \zeta)$, *where* ξ *is a representative of* $\{\xi\}$ *as above (note that here we abuse notation by letting* $\sigma_1(\xi \otimes \zeta)$ *denote the element in* $\mathrm{MU}[\frac{1}{2}]^*(X)$*).*

Moreover, the homomorphism $\mathfrak{T}_{\{\xi\}}[\frac{1}{2}]\colon \mathrm{MU}[\frac{1}{2}]^*(X) \to \mathrm{MU}[\frac{1}{2}]^*(X)$ *is a ring automorphism.*

Note that the pair $(\mathrm{MU}[\frac{1}{2}]^*(X),\, \mathfrak{T}_{\{\xi\}}[\frac{1}{2}])$ is natural on $(X,\, \xi)$. Indeed, suppose $(Y,\, \eta)$ is another pair with a morphism $f\colon (Y,\, \eta) \to (X,\, \xi)$ (i.e. f is a continuous map $Y \to X$ such that $f^*(\xi) = \eta$). We claim that $f^* \circ \mathfrak{T}^X_{\{\xi\}} = \mathfrak{T}^Y_{\{\eta\}} \circ f^*$, where to avoid a confusion we denoted $\mathfrak{T}_{\{\xi\}}$, $\mathfrak{T}_{\{\eta\}}$ by $\mathfrak{T}^X_{\{\xi\}}$, $\mathfrak{T}^Y_{\{\eta\}}$ respectively. Indeed, since $\varphi_\eta = \varphi_\xi \circ f\colon Y \to \mathrm{BSU}$ is a classifying map for the bundle η, we see that the diagram

$$\begin{tikzcd}
\mathrm{MU}^*(X) \arrow[r, "\mathfrak{T}"] \arrow[d, "f^*"'] & \mathrm{H}[\frac{1}{2}] \otimes_{\mathrm{R}} \mathrm{MU}^*(X) \arrow[r, "\varphi_\xi^*\otimes \mathrm{id}"] \arrow[d, "\mathrm{id}_{\mathrm{H}[\frac{1}{2}]}\otimes f^*"'] & \mathrm{MU}[\frac{1}{2}]^*(X) \otimes_{\mathrm{R}} \mathrm{MU}^*(X) \arrow[r, "\mu_X"] \arrow[d, "f^*\otimes f^*"'] & \mathrm{MU}[\frac{1}{2}]^*(X) \arrow[d, "f^*"] \\
\mathrm{MU}^*(Y) \arrow[r, "\mathfrak{T}"] & \mathrm{H}[\frac{1}{2}] \otimes_{\mathrm{R}} \mathrm{MU}^*(Y) \arrow[r, "\varphi_\eta^*\otimes \mathrm{id}"] & \mathrm{MU}[\frac{1}{2}]^*(Y) \otimes_{\mathrm{R}} \mathrm{MU}^*(Y) \arrow[r, "\mu_Y"] & \mathrm{MU}[\frac{1}{2}]^*(Y)
\end{tikzcd}$$

commutes. This implies that the pair $(\mathrm{MU}[\frac{1}{2}]^*(X),\, \mathfrak{T}_{\{\xi\}}[\frac{1}{2}])$ is natural on $(X,\, \xi)$.

Let us remember that the transformation $\mathfrak{T}$ was defined by means of the elements $\mathfrak{T}_n \in \mathrm{H}[\frac{1}{2}] \mathbin{\widehat{\otimes}_{\mathbb{Z}}} \mathbf{S}$, $n \in \mathbb{Z}$. Therefore the elements $(\Delta[\frac{1}{2}] \otimes \mathrm{id}_{\mathbf{S}})\mathfrak{T}_n \in \mathrm{H}[\frac{1}{2}] \mathbin{\widehat{\otimes}_{\mathrm{R}[\frac{1}{2}]}} \mathrm{H}[\frac{1}{2}] \mathbin{\widehat{\otimes}_{\mathbb{Z}}} \mathbf{S}$, $n \in \mathbb{Z}$ (where $\Delta[\frac{1}{2}]\colon \mathrm{H}[\frac{1}{2}] \to \mathrm{H}[\frac{1}{2}] \mathbin{\widehat{\otimes}_{\mathrm{R}[\frac{1}{2}]}} \mathrm{H}[\frac{1}{2}]$ is the extension of Δ to $\mathrm{H}[\frac{1}{2}]$) determine the multiplicative natural transformation

$$(\Delta[\tfrac{1}{2}] \otimes \mathrm{id}_{\mathbf{S}})\mathfrak{T}\colon \mathrm{MU}^*(\_) \to \mathrm{H}[\tfrac{1}{2}] \mathbin{\widehat{\otimes}_{\mathrm{R}[\frac{1}{2}]}} \mathrm{H}[\tfrac{1}{2}] \mathbin{\widehat{\otimes}_{\mathrm{R}}} \mathrm{MU}^*(\_).$$

Since Landweber-Novikov's algebra acts on H and therefore on $\mathrm{H}[\frac{1}{2}]$, we can define the composition $((\mathrm{id}_{\mathrm{H}[\frac{1}{2}]} \otimes \Delta_{\mathbf{S}})\mathfrak{T}) \circ \mathfrak{T}$. More explicitly, we have

$$((\mathrm{id}_{\mathrm{H}[\frac{1}{2}]} \otimes \Delta_{\mathbf{S}})\mathfrak{T}_n) \circ \mathfrak{T}_n = (\sum_{\omega} \kappa^n \alpha_\omega \otimes \Delta_{\mathbf{S}}(S_\omega)) \circ (\sum_{\tau} \kappa^n \alpha_\tau \otimes S_\tau)$$

$$= \sum_{\omega} (\kappa^n \alpha_\omega \otimes \sum_{\omega' \cup \omega'' = \omega,\, \tau} S_{\omega'}(\kappa^n \alpha_\tau) \otimes S_{\omega''} S_\tau) \in \mathrm{H}[\tfrac{1}{2}] \mathbin{\widehat{\otimes}_{\mathrm{R}[\frac{1}{2}]}} \mathrm{H}[\tfrac{1}{2}] \mathbin{\widehat{\otimes}_{\mathbb{Z}}} \mathbf{S},$$

where we used the following well-known fact [5]: for every pair of partitions ω', ω'', the composition $S_{\omega''} S_{\omega'}$ is an integral linear combination of elements S_ω.

Theorem 28. *$((\mathrm{id}_{\mathrm{H}}\otimes\Delta_{\mathbf{S}})\mathfrak{T})\circ\mathfrak{T}$ and $(\Delta[\frac{1}{2}]\otimes\mathrm{id}_{\mathbf{S}})\mathfrak{T}$ coincide as a natural transformations*

$$\mathrm{MU}^*(\_)\to \mathrm{H}[\tfrac{1}{2}]\underset{\mathrm{R}[\frac{1}{2}]}{\widehat{\otimes}}\mathrm{H}[\tfrac{1}{2}]\underset{\mathrm{R}}{\widehat{\otimes}}\mathrm{MU}^*(\_).$$

Corollary 29. *For any virtual* SU*-bundles ξ, η over X of virtual dimension 1 $\mathfrak{T}_{\{\eta\}}[\frac{1}{2}]\circ\mathfrak{T}_{\{\xi\}}[\frac{1}{2}]=\mathfrak{T}_{\{\eta\otimes\xi\}}[\frac{1}{2}]\colon\ \mathrm{MU}[\frac{1}{2}]^*(X)\to\mathrm{MU}[\frac{1}{2}]^*(X)$. In particular, we obtain a group homomorphism $\widetilde{\mathrm{K}\,\mathrm{SU}}(X)_{\otimes}\to\mathrm{Aut}_{ring}(\mathrm{MU}[\frac{1}{2}]^*(X))$, $\{\xi\}\mapsto\mathfrak{T}_{\{\xi\}}[\frac{1}{2}]$, which is natural on X.*

Remark 30. It can be reasonable to treat the functor

$$(X,\,\xi)\mapsto\{\mathrm{K}_{\xi}\text{-algebra }\ \mathrm{MU}[\tfrac{1}{2}]^*(X)\}$$

as an extension of the functor of (localized) complex cobordism theory $X\mapsto\mathrm{MU}[\frac{1}{2}]^*(X)$ to the homotopy category of pairs $(X,\,\xi)$. Using the isomorphism $\mathfrak{T}_{\{\xi\}}[\frac{1}{2}]$, we can define relevant cohomology operations $S_\omega^{\{\xi\}}$ acting on K_ξ-algebra $\mathrm{MU}[\frac{1}{2}]^*(X)$. More precisely, set $S_\omega^{\{\xi\}}:=\mathfrak{T}_{\{\xi\}}[\frac{1}{2}]\circ S_\omega\circ(\mathfrak{T}_{\{\xi\}}[\frac{1}{2}])^{-1}$, where S_ω is usual Landweber-Novikov's operation. Clearly, $S_\omega^{\{\xi\}}$ preserves $\mathrm{K}_\xi\subset\mathrm{MU}[\frac{1}{2}]^*(X)$. Furthermore, such operations $S_\omega^{\{\xi\}}$ form an algebra $\mathbf{S}^{\{\xi\}}$, which is isomorphic to Landweber-Novikov's algebra $\mathbf{S}$. Moreover, K_ξ-algebra $\mathrm{MU}[\frac{1}{2}]^*(X)$ is Milnor's module over $\mathbf{S}^{\{\xi\}}$. Indeed, we have:

$$\begin{aligned}
S_\omega^{\{\xi\}}(\alpha\beta)&=(\mathfrak{T}_{\{\xi\}}[\tfrac{1}{2}]\circ S_\omega)((\mathfrak{T}_{\{\xi\}}[\tfrac{1}{2}])^{-1}(\alpha\beta))\\
&=(\mathfrak{T}_{\{\xi\}}[\tfrac{1}{2}]\circ S_\omega)((\mathfrak{T}_{\{\xi\}}[\tfrac{1}{2}])^{-1}(\alpha)(\mathfrak{T}_{\{\xi\}}[\tfrac{1}{2}])^{-1}(\beta))\\
&=\mathfrak{T}_{\{\xi\}}[\tfrac{1}{2}](\sum_{\omega'\cup\omega''=\omega}S_{\omega'}((\mathfrak{T}_{\{\xi\}}[\tfrac{1}{2}])^{-1}(\alpha))S_{\omega''}((\mathfrak{T}_{\{\xi\}}[\tfrac{1}{2}])^{-1}(\beta))\\
&=\sum_{\omega'\cup\omega''=\omega}\mathfrak{T}_{\{\xi\}}[\tfrac{1}{2}](S_{\omega'}((\mathfrak{T}_{\{\xi\}}[\tfrac{1}{2}])^{-1}(\alpha)))\mathfrak{T}_{\{\xi\}}[\tfrac{1}{2}](S_{\omega''}((\mathfrak{T}_{\{\xi\}}[\tfrac{1}{2}])^{-1}(\beta)))\\
&=\sum_{\omega'\cup\omega''=\omega}S_{\omega'}^{\{\xi\}}(\alpha)S_{\omega''}^{\{\xi\}}(\beta).
\end{aligned}$$

For a pair $(X,\,\xi)$ consisting of a CW-complex X and of a virtual SU-bundle ξ over X of virtual dimension 1 we can also define the natural transformation

$$\mathfrak{T}_{X,\{\xi\}}:=(\varphi_\xi^*\otimes\mathrm{id}_{\mathbf{S}})\mathfrak{T}\colon\ \mathrm{MU}^*(Z)\to\mathrm{MU}[\tfrac{1}{2}]^*(X)\underset{\mathrm{R}}{\widehat{\otimes}}\mathrm{MU}^*(Z),$$

where Z is an arbitrary CW-complex. In a different way $\mathfrak{T}_{X,\{\xi\}}$ can be defined by application of Theorem 24 for $\mathrm{A}=\mathrm{MU}^*(X)$ and $\mathfrak{f}(u)=(\varphi_\xi^*\mathfrak{h})(u)\in\mathrm{A}[[u]]$.

Let $\mathfrak{R}_{X,\{\xi\}}$ be the composite natural transformation

$$\mathrm{MU}[\tfrac{1}{2}]^*(Z)\overset{\mathfrak{T}_{X,\{\xi\}}[\frac{1}{2}]}{\to}\mathrm{MU}[\tfrac{1}{2}]^*(X)\underset{\mathrm{R}[\frac{1}{2}]}{\widehat{\otimes}}\mathrm{MU}[\tfrac{1}{2}]^*(Z)\overset{m_{X,Z}}{\to}\mathrm{MU}[\tfrac{1}{2}]^*(X\times Z),$$

where $m_{X,Z}$ is the canonical homomorphism.

Corollary 31. *For any pairs $(X,\,\xi)$ and $(Y,\,\eta)$ as above we have*

$$\mathfrak{R}_{Y,\{\eta\}}\circ\mathfrak{R}_{X,\{\xi\}}=\mathfrak{R}_{Y\times X,\{\eta\boxtimes\xi\}},$$

where $\eta\boxtimes\xi$ is the exterior tensor product of bundles.

Remark 32. Let us remember that the spectrum $(\mathbf{MU},\, u)$ (where $u \in \mathrm{MU}^2(\mathbb{C}P^\infty)$ as above) is the universal $\mathbb{C}$-oriented spectrum [5]. Therefore all previous constructions can be extended to any $\mathbb{C}$-oriented spectrum $(\mathrm{E},\, u^{\mathrm{E}})$. Clearly, in the case $(\mathrm{H}(\mathbb{Z}),\, u^{\mathrm{H}})$ (where u^{H} is the generator of $\mathrm{H}^2(\mathbb{C}P^\infty;\, \mathbb{Z}) = \mathrm{H}^2(\mathrm{K}(\mathbb{Z}\,, 2);\, \mathbb{Z}) \cong \mathbb{Z}$, corresponding to $1 \in \mathbb{Z}$) our theory is trivial. But this is also true in the case of complex K-theory $(\mathrm{K}_{\mathbb{C}},\, u^{\mathrm{K}_{\mathbb{C}}})$. Indeed, it is well known [5] that the formal product in $(\mathrm{K}_{\mathbb{C}},\, u^{\mathrm{K}_{\mathbb{C}}})$ has the form $F^{\mathrm{K}_{\mathbb{C}}}(x,y) = x + y + \sigma^{-1}xy$, where $\sigma \in \widetilde{\mathrm{K}}_{\mathbb{C}}^{2}(S^0)$ is the Bott element. Since Ξ is an SU-bundle, we have $\widetilde{s}_1^{\mathrm{K}_{\mathbb{C}}} := s_1^{\mathrm{K}_{\mathbb{C}}}(\Xi) = 0$, where $s_1^{\mathrm{K}_{\mathbb{C}}}$ is the first Newton characteristic class in the theory $(\mathrm{K}_{\mathbb{C}},\, u^{\mathrm{K}_{\mathbb{C}}})$. Now using the universality of $(\mathbf{MU},\, u)$ and the formula (2) we see that $\sigma_1^{\mathrm{K}_{\mathbb{C}}}(\widehat{\Xi}) = \sigma_1^{\mathrm{K}_{\mathbb{C}}}(\mathcal{Z}) = u^{\mathrm{K}_{\mathbb{C}}}$.

4 Appendix: A short proof of an isomorphism of some H-groups

Lemma 33. There exists an isomorphism of H-groups: $\mathrm{BSU}_{\otimes} \times \mathbb{C}P^\infty \cong \mathrm{BU}_{\otimes}$.

Proof. Let $\lambda\colon\, \mathrm{BSU} \to \mathrm{BU}$ be a classifying map for the universal SU-bundle Ξ, $\kappa\colon \mathbb{C}P^\infty \to \mathrm{BU}$ a classifying map for the universal line bundle $\mathcal{Z}$. Consider the composite map

$$\mathrm{BSU} \times \mathbb{C}P^\infty \xrightarrow{\lambda\times\kappa} \mathrm{BU} \times \mathrm{BU} \xrightarrow{\phi} \mathrm{BU},$$

where ϕ is the multiplication in the H-group BU. Clearly, $\phi \circ (\lambda \times \kappa)$ is a classifying map for the virtual bundle $\Xi \boxtimes \mathcal{Z}$ (recall that Ξ assumed to be of virtual dimension 1). Now it follows easily that $\phi \circ (\lambda \times \kappa)$ is a homotopy equivalence $\mathrm{BSU} \times \mathbb{C}P^\infty \simeq \mathrm{BU}$ and moreover it is an isomorphism of H-groups. $\square$

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