Topics in Noncommutative Algebraic Geometry, Homological Algebra and K-theory.

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Introduction

This text is based on my lectures delivered at the School on Algebraic K-Theory and Applications which took place at the International Center for Theoretical Physics (ICTP) in Trieste during the last two weeks of May of 2007. It might be regarded as an introduction to some basic facts of noncommutative algebraic geometry and the related chapters of homological algebra and (as a part of it) a non-conventional version of higher K-theory of noncommutative 'spaces'. Arguments are mostly replaced by sketches of the main steps, or references to complete proofs, which makes the text an easier reading than the ample accounts on different topics discussed here indicated in the bibliography.

Lecture 1 is dedicated to the first notions of noncommutative algebraic geometry – preliminaries on 'spaces' represented by categories and morphisms of 'spaces' represented by (isomorphism classes of) functors. We introduce *continuous*, *affine*, and *locally affine* morphisms which lead to definitions of noncommutative schemes and more general locally affine 'spaces'. The notion of a noncommutative scheme is illustrated with important examples related to quantized enveloping algebras: the quantum base affine spaces and flag varieties and the associated *quantum D-schemes* represented by the categories of (twisted) quantum D-modules introduced in [LR] (see also [T]). Noncommutative projective 'spaces' introduced in [KR1] and more general Grassmannians and flag varieties studied in [KR3] are examples of *smooth* locally affine noncommutative 'spaces' which are not schemes.

In Lecture 2, we recover some fragments of geometry behind the pseudo-geometric picture outlined in the first lecture. We start with introducing underlying topological spaces (spectra) of 'spaces' represented by abelian categories and describing their main properties. One of the consequences of these properties is the reconstruction theorem for commutative schemes [R4] which can be regarded as one of the major tests for the noncommutative theory. It says, in particular, that any quasi-separated commutative scheme can be *canonically* reconstructed uniquely up to isomorphism from its category of quasi-coherent sheaves. The noncommutative fact behind the reconstruction theorem is the *geometric realization* of a noncommutative scheme as a *locally affine stack of local categories* on its underlying topological space. The latter is a noncommutative analog of a locally affine locally ringed topological space, that is a geometric scheme.

Lecture 3 complements this short introduction to the geometry of noncommutative 'spaces' and schemes with a sketch of the first notions and facts of pseudo-geometry (in particular, descent) and spectral theory of 'spaces' represented by triangulated categories. This is a simple, but quite revelative piece of derived noncommutative algebraic geometry.

Lectures 4, 5, 6 are based on some parts of the manuscript [R8] created out of attempts to find natural frameworks for homological theories which appear in noncommutative algebraic geometry. We start, in Lecture 4, with a version of non-abelian (and often non-additive) homological algebra which is based on presites (– categories endowed with a Grothendieck pretopology) whose covers consist of one morphism. Although it does not matter for most of constructions and facts, we assume that the pretopologies are subcanonical (i.e. representable presheaves are sheaves), or, equivalently, covers are *strict* epimorphisms (– cokernels of pairs of arrows). We call such presites *right exact* categories. The dual structures, *left* exact categories, appear naturally and play a crucial role in the version of higher K-theory sketched in Lecture 5. We develop standard tools of higher K-theory starting with the long 'exact' sequence (in Lecture 5) followed by reductions by resolutions and devissage which are discussed in Lecture 6.

The first version of these lectures was sketched at the Institute des Hautes Études Scientifiques, Bures-sur-Yvette, in the late Spring of 2007 (in the process of preparation to actual lectures in Trieste). The present text, which is an extended version of [R9], was written at the Max Planck Institut für Mathematik in Bonn. I am grateful to both Institutes for hospitality and excellent working conditions.

References:

Bibliographical references and cross references inside of one lecture are as usual. References to results in another lecture are preceded by the lecture number, e.g. II.3.1.

Lecture 1. Noncommutative locally affine 'spaces' and schemes.

In Section 1, we review the first notions of noncommutative algebraic geometry – preliminaries on 'spaces' represented by categories, morphisms represented by their inverse image functors. We recall the notions of continuous, flat and affine morphisms and illustrate them with a couple of examples. In Section 2, we remind Beck's theorem characterizing monadic morphisms and apply it to study of affine relative schemes. In Section 3, we introduce the notions of a *locally affine* morphism and a *scheme* over a 'space'. Section 4 is dedicated to *flat descent* which is one of the basic tools of noncommutative algebraic geometry. In Section 5, we sketch several examples of noncommutative schemes and more general locally affine spaces which are among illustrations and/or motivations of constructions of this work.

1. Noncommutative 'spaces' represented by categories and morphisms between them. Continuous, affine and locally affine morphisms.

1.1. Categories and 'spaces'. As usual, Cat, or $Cat_{\mathfrak{U}}$, denotes the bicategory of categories which belong to a fixed universum $\mathfrak{U}$. We call objects of Cat^{op} 'spaces'. For any 'space' X, the corresponding category C_X is regarded as the category of quasi-coherent sheaves on X. For any $\mathfrak{U}$-category $\mathcal{A}$, we denote by $|\mathcal{A}|$ the corresponding object of Cat^{op} (the underlying 'space') defined by $C_{|\mathcal{A}|} = \mathcal{A}$.

We denote by $|Cat|^o$ the category having same objects as Cat^{op}. Morphisms from X to Y are isomorphism classes of functors $C_Y \longrightarrow C_X$. For a morphism $X \xrightarrow{f} Y$, we denote by f^* any functor $C_Y \longrightarrow C_X$ representing f and call it an *inverse image functor of the morphism* f. We shall write $f = [F]$ to indicate that f is a morphism having an inverse image functor F. The composition of morphisms $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$ is defined by $g \circ f = [f^* \circ g^*]$.

1.2. Localizations and conservative morphisms. Let Y be an object of $|Cat|^o$ and Σ a class of arrows of the category C_Y. We denote by $\Sigma^{-1}Y$ the object of $|Cat|^o$ such that the corresponding category coincides with (the standard realization of) the quotient of the category C_Y by Σ (cf. [GZ, 1.1]): $C_{\Sigma^{-1}Y} = \Sigma^{-1}C_Y$. The canonical *localization functor* $C_Y \xrightarrow{p^*_\Sigma} \Sigma^{-1}C_Y$ is regarded as an inverse image functor of a morphism, $\Sigma^{-1}Y \xrightarrow{p_\Sigma} Y$.

For any morphism $X \xrightarrow{f} Y$ in $|Cat|^o$, we denote by Σ_f the family of all arrows s of the category C_Y such that $f^*(s)$ is invertible (notice that Σ_f does not depend on the choice of an inverse image functor f^*). Thanks to the universal property of localizations, f^* is represented as the composition of the localization functor $p^*_f = p^*_{\Sigma_f} : C_Y \longrightarrow \Sigma_f^{-1}C_Y$ and a uniquely determined functor $\Sigma^{-1}C_Y \xrightarrow{f^*_{\mathfrak{c}}} C_X$. In other words, $f = p_f \circ f_{\mathfrak{c}}$ for a uniquely determined morphism $X \xrightarrow{f_{\mathfrak{c}}} \Sigma_f^{-1}Y$.

A morphism $X \xrightarrow{f} Y$ is called *conservative* if Σ_f consists of isomorphisms, or, equivalently, p_f is an isomorphism.

A morphism $X \xrightarrow{f} Y$ is called a *localization* if $f_{\mathfrak{c}}$ is an isomorphism, i.e. the functor $f^*_{\mathfrak{c}}$ is an equivalence of categories.

Thus, $f = p_f \circ f_{\mathfrak{c}}$ is a unique decomposition of a morphism f into a localization and a conservative morphism.

1.3. Continuous, flat, and affine morphisms. A morphism is called *continuous* if its inverse image functor has a right adjoint (called a direct image functor), and *flat* if, in addition, the inverse image functor is left exact (i.e. preserves finite limits). A continuous morphism is called *affine* if its direct image functor is conservative (i.e. it reflects isomorphisms) and has a right adjoint.

1.4. Categoric spectrum of a unital ring. For an associative unital ring R, we define the *categoric spectrum* of R as the object $\mathbf{Sp}(R)$ of $|Cat|^o$ represented by the category $R-mod$ of left R-modules; i.e. $C_{\mathbf{Sp}(R)} = R - mod$.

Let $R \xrightarrow{\phi} S$ be a unital ring morphism and $R-mod \xrightarrow{\bar{\phi}^*} S-mod$ the functor $S \otimes_R -$. The canonical right adjoint to $\bar{\phi}^*$ is the pull-back functor $\bar{\phi}_*$ along the ring morphism ϕ. A right adjoint to $\bar{\phi}_*$ is given by

$$\phi^! : S - mod \xrightarrow{\bar{\phi}^!} R - mod, \quad L \longmapsto Hom_R(\phi_*(S), L).$$

The map

$$\Big(R \xrightarrow{\phi} S\Big) \longmapsto \Big(\mathbf{Sp}(S) \xrightarrow{\bar{\phi}} \mathbf{Sp}(R)\Big)$$

is a functor

$$Rings^{op} \xrightarrow{\mathbf{Sp}} |Cat|^o$$

which takes values in the subcategory of $|Cat|^o$ formed by affine morphisms.

The image $\mathbf{Sp}(R) \xrightarrow{\bar{\phi}} \mathbf{Sp}(T)$ of a ring morphism $T \xrightarrow{\phi} R$ is flat (resp. faithful) iff ϕ turns R into a flat (resp. faithful) right T-module.

1.4.1. Continuous, flat, and affine morphisms from $\mathbf{Sp}(S)$ to $\mathbf{Sp}(R)$. Let R and S be associative unital rings. A morphism $\mathbf{Sp}(S) \xrightarrow{f} \mathbf{Sp}(R)$ with an inverse image functor f^* is continuous iff

$$f^* \simeq \mathcal{M} \otimes_R \ : L \longmapsto \mathcal{M} \otimes_R L \tag{1}$$

for an (S, R)-bimodule $\mathcal{M}$ defined uniquely up to isomorphism. The functor

$$f_* = Hom_S(\mathcal{M}, -) : N \longmapsto Hom_S(\mathcal{M}, N) \tag{2}$$

is a direct image of f.

By definition, the morphism f is conservative iff $\mathcal{M}$ is *faithful* as a right R-module, i.e. the functor $\mathcal{M} \otimes_R -$ is faithful.

The direct image functor (2) is conservative iff $\mathcal{M}$ is a generator in the category of left S-modules, i.e. for any nonzero S-module N, there exists a nonzero S-module morphism $\mathcal{M} \longrightarrow N$.

The morphism f is flat iff $\mathcal{M}$ is flat as a right R-module.

The functor (2) has a right adjoint, $f^!$, iff f_* is isomorphic to the tensoring (over S) by a bimodule. This happens iff $\mathcal{M}$ is a projective S-module of finite type. The latter is equivalent to the condition: the natural functor morphism $\mathcal{M}^*_S \otimes_S - \longrightarrow Hom_S(\mathcal{M}, -)$ is an isomorphism. Here $\mathcal{M}^*_S = Hom_S(M, S)$. In this case, $f^! \simeq Hom_R(\mathcal{M}^*_S, -)$.

1.5. Example. Let $\mathcal{G}$ be a monoid and R a $\mathcal{G}$-graded unital ring. We define the 'space' $\mathbf{Sp}_{\mathcal{G}}(R)$ by taking as $C_{\mathbf{Sp}_{\mathcal{G}}(R)}$ the category $gr_{\mathcal{G}}R-mod$ of left $\mathcal{G}$-graded R-modules. There is a natural functor $gr_{\mathcal{G}}R - mod \xrightarrow{\phi_*} R_0 - mod$ which assigns to each graded R-module its zero component ('zero' is the unit element of the monoid $\mathcal{G}$). The functor ϕ_* has a left adjoint, ϕ^*, which maps every R_0-module M to the graded R-module $R \otimes_{R_0} M$. The adjunction arrow $Id_{R_0-mod} \longrightarrow \phi_*\phi^*$ is an isomorphism. This means that the functor ϕ^* is fully faithful, or, equivalently, the functor ϕ_* is a localization.

The functors ϕ_* and ϕ^* are regarded as respectively a direct and an inverse image functor of a morphism $\mathbf{Sp}_{\mathcal{G}}(R) \xrightarrow{\phi} \mathbf{Sp}(R_0)$. It follows from the above that the morphism ϕ is affine iff ϕ is an isomorphism (i.e. ϕ^* is an equivalence of categories).

In fact, if ϕ is affine, the functor ϕ_* should be conservative. Since ϕ_* is a localization, this means, precisely, that ϕ_* is an equivalence of categories.

1.6. The cone of a non-unital ring. Let R_0 be a unital associative ring, and let R_+ be an associative ring, non-unital in general, in the category of R_0-bimodules; i.e. R_+ is endowed with an R_0-bimodule morphism $R_+ \otimes_{R_0} R_+ \xrightarrow{m} R_+$ satisfying the associativity condition. Let $R = R_0 \oplus R_+$ denote the augmented ring described by this data. Let $\mathcal{T}_{R_+}$ denote the full subcategory of the category $R - mod$ whose objects are all R-modules annihilated by R_+. Let $\mathcal{T}^-_{R_+}$ be the Serre subcategory (that is a full subcategory closed by taking subquotients, extensions, and arbitrary direct sums) of the category $R - mod$ spanned by $\mathcal{T}_{R_+}$.

We define the 'space' *cone of* R_+ by taking as $C_{\mathbf{Cone}(R_+)}$ the quotient category $R - mod/\mathcal{T}^-_{R_+}$. The localization functor $R-mod \xrightarrow{u^*} R-mod/\mathcal{T}^-_{R_+}$ is an inverse image functor of a morphism of 'spaces' $\mathbf{Cone}(R_+) \xrightarrow{u} \mathbf{Sp}(R)$. The functor u^* has a (necessarily fully faithful) right adjoint, i.e. the morphism u is continuous. If R_+ is a unital ring, then u is an isomorphism (see C3.2.1). The composition of the morphism u with the canonical affine morphism $\mathbf{Sp}(R) \longrightarrow \mathbf{Sp}(R_0)$ is a continuous morphism $\mathbf{Cone}(R_+) \longrightarrow \mathbf{Sp}(R_0)$. Its direct image functor is (regarded as) the *global sections functor*.

1.7. The graded version: $\mathbf{Proj}_{\mathcal{G}}$. Let $\mathcal{G}$ be a monoid and $R = R_0 \oplus R_+$ a $\mathcal{G}$-graded ring with zero component R_0. Then we have the category $gr_{\mathcal{G}}R - mod$ of $\mathcal{G}$-graded R-modules and its full subcategory $gr_{\mathcal{G}}\mathcal{T}_{R_+} = \mathcal{T}_{R_+} \cap gr_{\mathcal{G}}R - mod$ whose objects are graded modules annihilated by the ideal R_+. We define the 'space' $\mathbf{Proj}_{\mathcal{G}}(R)$ by setting

$$C_{\mathbf{Proj}_{\mathcal{G}}(R)} = gr_{\mathcal{G}}R - mod/gr_{\mathcal{G}}\mathcal{T}^-_{R_+}.$$

Here $gr_{\mathcal{G}}\mathcal{T}^-_{R_+}$ is the Serre subcategory of the category $gr_{\mathcal{G}}R-mod$spanned by $gr_{\mathcal{G}}\mathcal{T}_{R_+}$. One can show that $gr_{\mathcal{G}}\mathcal{T}^-_{R_+} = gr_{\mathcal{G}}R - mod \cap \mathcal{T}^-_{R_+}$. Therefore, we have a canonical projection

$$\mathbf{Cone}(R_+) \xrightarrow{\mathfrak{p}} \mathbf{Proj}_{\mathcal{G}}(R).$$

The localization functor $gr_{\mathcal{G}}R - mod \longrightarrow C_{\mathbf{Proj}_{\mathcal{G}}(R_+)}$ is an inverse image functor of a continuous morphism $\mathbf{Proj}_{\mathcal{G}}(R) \xrightarrow{\mathfrak{v}} \mathbf{Sp}_{\mathcal{G}}(R)$. The composition $\mathbf{Proj}_{\mathcal{G}}(R) \xrightarrow{\mathfrak{v}} \mathbf{Sp}(R_0)$ of the morphism $\mathfrak{v}$ with the canonical morphism $\mathbf{Sp}_{\mathcal{G}}(R) \xrightarrow{\phi} \mathbf{Sp}(R_0)$ defines $\mathbf{Proj}_{\mathcal{G}}(R)$ as a 'space' over $\mathbf{Sp}(R_0)$. Its direct image functor is called the *global sections functor*.

1.7.1. Example: cone and Proj of a $\mathbb{Z}_+$-graded ring. Let $R = \oplus_{n\ge 0}R_n$ be a $\mathbb{Z}_+$-graded ring, $R_+ = \oplus_{n\ge 1}R_n$ its 'irrelevant' ideal. Thus, we have the *cone* of R_+, $\mathbf{Cone}(R_+)$, and $\mathbf{Proj}(R) = \mathbf{Proj}_{\mathbb{Z}}(R)$, and a canonical morphism $\mathbf{Cone}(R_+) \longrightarrow \mathbf{Proj}(R)$.

2. Beck's Theorem and affine morphisms.

2.1. The Beck's Theorem. Let $X \xrightarrow{f} Y$ be a continuous morphism with inverse image functor f^*, direct image functor f_*, and adjunction morphisms

$$Id_{C_Y} \xrightarrow{\eta_f} f_*f^* \quad \text{and} \quad f^*f_* \xrightarrow{\epsilon_f} Id_{C_X}.$$

Let $\mathcal{F}_f$ denote the monad (F_f, μ_f) on Y, where $F_f = f_*f^*$ and $\mu_f = f_*\epsilon_f f^*$.

We denote by $\mathcal{F}_f - mod$, or by $(\mathcal{F}_f/Y) - mod$ the category of $\mathcal{F}_f$-modules. Its objects are pairs (M, ξ), where $M \in ObC_Y$ and ξ is a morphism $F_f(M) \longrightarrow M$ such that the diagram

$$\begin{array}{ccc} F_f^2(M) & \xrightarrow{\mu_f(M)} & F_f(M) \\ {\scriptstyle F_f(\xi)}\downarrow & & \downarrow{\scriptstyle \xi} \\ F_f(M) & \xrightarrow{\xi} & M \end{array}$$

commutes and $\xi \circ \eta_f(M) = id_M$. Morphisms from (M, ξ) to $\widetilde{M}, \widetilde{\xi})$ are given by morphisms $M \xrightarrow{g} \widetilde{M}$ of the category C_Y such that the diagram

$$\begin{array}{ccc} F_f(M) & \xrightarrow{F_f(g)} & F_f(\widetilde{M}) \\ {\scriptstyle \xi}\downarrow & & \downarrow{\scriptstyle \widetilde{\xi}} \\ M & \xrightarrow{g} & \widetilde{M} \end{array}$$

commutes. The composition is defined in a standard way.

We denote by $\mathbf{Sp}(\mathcal{F}_f/Y)$ the 'space' represented by the category of $\mathcal{F}_f$-modules and call it the *categoric spectrum* of the monad $\mathcal{F}_f$.

There is a commutative diagram

$$\begin{array}{ccc} C_X & \xrightarrow{\widetilde{f_*}} & (\mathcal{F}_f/Y) - mod \\ & {\scriptstyle f_*}\searrow \quad \swarrow{\scriptstyle \mathfrak{f}_*} & \\ & C_X & \end{array} \qquad (3)$$

Here $\widetilde{f_*}$ is the canonical functor

$$C_X \longrightarrow (\mathcal{F}_f/Y) - mod, \quad M \longmapsto (f_*(M), f_*\epsilon_f(M)),$$

and $\mathfrak{f}^*$ is the forgetful functor $(\mathcal{F}_f/Y) - mod \longrightarrow C_Y$.

The following assertion is one of the versions of Beck's theorem.

2.1.1. Theorem. *Let $X \xrightarrow{f} Y$ be a continuous morphism.*

(a) If the category C_Y has cokernels of reflexive pairs of arrows, then the functor $\bar{f}_*$ has a left adjoint, $\bar{f}^*$; hence $\bar{f}_*$ is a direct image functor of a continuous morphism $\bar{X} \xrightarrow{f} \mathbf{Sp}(\mathcal{F}_f/Y)$.

(b) If, in addition, the functor f_* preserves cokernels of reflexive pairs, then the adjunction arrow $\bar{f}^*\bar{f}_* \longrightarrow Id_{C_X}$ is an isomorphism, i.e. $\bar{f}_*$ is a localization.

(c) If, in addition to (a) and (b), the functor f_* is conservative, then $\bar{f}_*$ is a category equivalence.

Proof. See [MLM], IV.4.2, or [ML], VI.7. ∎

2.1.2. Corollary. *Let $X \xrightarrow{f} Y$ be an affine morphism (cf. 1.3). If the category C_Y has cokernels of reflexive pairs of arrows (e.g. C_Y is an abelian category), then the canonical morphism $X \xrightarrow{\mathfrak{f}} \mathbf{Sp}(\mathcal{F}_f/Y)$ is an isomorphism.*

2.1.3. Monadic morphisms. A continuous morphism $X \xrightarrow{f} Y$ is called *monadic* if the functor

$$C_X \xrightarrow{\widetilde{f_*}} \mathcal{F}_f - mod, \quad M \longmapsto (f_*(M), f_*\epsilon_f(M)),$$

is an equivalence of categories.

2.2. Continuous monads and affine morphisms. A functor F is called *continuous* if it has a right adjoint. A monad $\mathcal{F} = (F, \mu)$ on a 'space' Y (i.e. on the category C_Y) is called *continuous* if the functor F is continuous.

2.2.1. Proposition. *A monad $\mathcal{F} = (F, \mu)$ on Y is continuous iff the canonical morphism $\mathbf{Sp}(\mathcal{F}/Y) \xrightarrow{\hat{f}} Y$ is affine.*

Proof. A proof in the case of a continuous monad can be found in [KR2, 6.2], or in [R3, 4.4.1] (see also [R4, 2.2]). ∎

2.2.2. Corollary. *Suppose that the category C_Y has cokernels of reflexive pairs of arrows. A continuous morphism $X \xrightarrow{f} Y$ is affine iff its direct image functor $C_X \xrightarrow{f_*} C_Y$ is the composition of a category equivalence*

$$C_X \longrightarrow (\mathcal{F}_f/Y) - mod$$

for a continuous monad $\mathcal{F}_f$ on Y and the forgetful functor $(\mathcal{F}_f/Y) - mod \longrightarrow C_Y$. The monad $\mathcal{F}_f$ is determined by f uniquely up to isomorphism.

Proof. The conditions of the Beck's theorem are fullfiled if f is affine, hence f_* is the composition of an equivalence $C_X \longrightarrow (\mathcal{F}_f/Y) - mod$ for a monad $\mathcal{F}_f = (f_*f^*, \mu_f)$ in C_Y and the forgetful functor $(\mathcal{F}_f/Y) - mod \longrightarrow C_Y$ (see (1)). The functor $F_f = f_*f^*$ has a right adjoint $f_*f^!$, where $f^!$ is a right adjoint to f_*. The rest follows from 2.2.1. ∎

2.3. The category of affine schemes over a 'space' and the category of monads on this 'space'.

2.3.1. Proposition. *Let*

$$\begin{array}{ccc} X & \xrightarrow{h} & Y \\ & f \searrow \quad \swarrow g & \\ & S & \end{array}$$

be a commutative diagram in $|Cat|^o$. Suppose C_Z has cokernels of reflexive pairs of arrows. If f and g are affine, then h is affine.

Let Aff_S denote the full subcategory of the category $|Cat|^o/S$ of 'spaces' over S whose objects are pairs $(X, X \xrightarrow{f} S)$, where f is an affine morphism. On the other hand, we have the category $\mathfrak{Mon}_\mathfrak{c}(S)$ of continuous monads on the 'space' S (i.e. on the category C_S) and the functor

$$\mathfrak{Mon}_\mathfrak{c}(S)^{op} \longrightarrow Aff_S \tag{1}$$

which assigns to every continuous monad $\mathcal{F}$ the object $(\mathbf{Sp}(\mathcal{F}/S, \mathfrak{f})$, where $\mathbf{Sp}(\mathcal{F}/S)$ is the 'space' represented by the category $\mathcal{F} - mod$ and the morphism $\mathfrak{f}$ has the forgetful functor $\mathcal{F} - mod \longrightarrow C_S$ as a direct image functor. It follows from 2.3.1 and 2.2.2 that this functor is essentially full (that is its image is equivalent to the category Aff_S).

For every endofunctor $C_S \xrightarrow{G} C_S$, let $|G|$ denote the set $Hom(Id_{C_S}, G)$ of *elements* of G. If $\mathcal{F} = (F, \mu)$ is a monad, then the set of elements of F has a natural monoid structure; we denote this monoid by $|\mathcal{F}|$. And we denote by $|\mathcal{F}|^*$ the group of the invertible elements of the monoid $|\mathcal{F}|$. We say that two monad morphisms $\mathcal{F} \underset{\psi}{\overset{\phi}{\rightrightarrows}} \mathcal{G}$ are *conjugate* to each other of $\phi = t \cdot \psi \cdot t^{-1}$ for some $t \in |\mathcal{G}|^*$.

Let $\mathfrak{Mon}_\mathfrak{c}^\mathfrak{r}(S)$ denote the category whose objects are continuous monads on C_S and morphisms are *conjugacy classes* of morphisms of monads.

2.3.2. Proposition *The functor (1) induces an equivalence between the category $\mathfrak{Mon}_\mathfrak{c}^\mathfrak{r}(S)$ and the category Aff_S of affine schemes over S.*

2.3.3. Example. Let $S = \mathbf{Sp}(R)$ for an associative ring R. Then the category $\mathfrak{Mon}_\mathfrak{c}(S)$ of monads on $C_S = R - mod$ is naturally equivalent to the category $R\backslash Rings$ of associative rings over R. The conjugacy classes of monad morphisms correspond to conjugacy classes of ring morphisms. Let $\mathfrak{Ass}$ denote the category whose objects are associative rings and morphisms the conjugacy classes of ring morphisms.

One deduces from 2.3.2 the following assertion:

2.3.3.1. Proposition. *The category Aff_S of affine schemes over $S = \mathbf{Sp}(R)$ is naturally equivalent to the category $(R\backslash\mathfrak{Ass})^{op}$.*

3. Noncommutative schemes and locally affine 'spaces'.

3.1. Covers. We call a family $\{U_i \xrightarrow{u_i} X \mid i \in J\}$ of morphisms of 'spaces' a *cover* if

– all inverse image functors u_i^* are *exact* (i.e. the functors u_i^* preserve finite limits and colimits),

– the family $\{u_i^* \mid i \in J\}$ is *conservative* (i.e. if $u_i^*(s)$ is an isomorphism for all $i \in J$, then s is an isomorphism).

3.2. Locally affine morphisms of 'spaces'. We call a morphism $X \xrightarrow{f} S$ of 'spaces' *locally affine* if there exists a cover $\{U_i \xrightarrow{u_i} X \mid i \in J\}$ of the 'space' X such that all the compositions $f \circ u_i$ are affine.

3.2.1. Semiseparated covers and semiseparated locally affine 'spaces'. A cover $\{U_i \xrightarrow{u_i} X \mid i \in J\}$ is called *semiseparated* if each of the morphisms u_i is affine.

A locally affine 'space' with a semiseparated affine cover is called *semiseparated.*

3.3. Weak schemes over S. *Weak schemes* over a 'space' S are locally affine morphisms $X \longrightarrow S$ which have an affine cover $\{U_i \xrightarrow{u_i} X \mid i \in J\}$ formed by *localizatios.* The latter means that each inverse image functor u_i^* is the composition of the localization functor $C_X \longrightarrow \Sigma_{u_i^*}^{-1} C_X$, where $\Sigma_{u_i^*} = \{s \in HomC_X \mid u_i^*(s) \text{ is invertible}\}$, and an equivalence of categories $\Sigma_{u_i^*}^{-1} C_X \longrightarrow C_{U_i}$.

3.4. Schemes. For a multiplicative system Σ of arrows of a svelte category C_X, we shall denote by $\Sigma^{-1}X$ the 'space' represented by the category $\Sigma^{-1}C_X$ and call it the *localization of X at Σ.*

For any pair of localizations $U_1 \xrightarrow{\mathfrak{u}_1} X \xleftarrow{\mathfrak{u}_2} U_2$, we have a commutative square

$$\begin{array}{ccc} U_1 \cap U_2 & \xrightarrow{u_{12}} & U_1 \\ {\scriptstyle u_{21}}\big\downarrow & & \big\downarrow {\scriptstyle u_1} \\ U_2 & \xrightarrow{u_2} & X \end{array} \qquad (1)$$

where $U_1 \cap U_2$ is the localization of X at the multiplicative system $\Sigma_{u_1^*} \vee \Sigma_{u_2^*}$ generated by $\Sigma_{u_1^*}$ and $\Sigma_{u_2^*}$ and u_{ij} are corresponding localizations. The square (1) is cartesian in the subcategory of 'spaces' formed by morphisms with exact inverse image functors.

3.4.1. Definition. A weak scheme $X \xrightarrow{f} S$ with an affine cover $\{U_i \xrightarrow{u_i} X \mid i \in J\}$ is a *scheme*, if for every $i \in J$ and any exact localization $\mathcal{V} \xrightarrow{\mathfrak{v}} X$ which is affine over S (that is $f \circ \mathfrak{v}$ is affine), the inverse image functor of the morphism

$$U_i \cap \mathcal{V} = (\Sigma_{u_i^*} \vee \Sigma_{\mathfrak{v}^*})^{-1} X \xrightarrow{\widetilde{\mathfrak{u}}_i} \mathcal{V} = \Sigma_{\mathfrak{v}^*}^{-1} X$$

is a localization at a finitely generated multiplicative system.

3.5. Open immersions. Let $X \xrightarrow{f} S$ be a morphism and $U \xrightarrow{\mathfrak{u}} X$ an exact localization. We call it a *quasi-compact open immersion* if for every localization $\mathcal{V} \xrightarrow{\mathfrak{v}} X$ which is affine over S, the inverse image functor of the projection $U \cap \mathcal{V} \longrightarrow \mathcal{V}$ is a localization at a finitely generated multiplicative system.

We call a localization $U \xrightarrow{\mathfrak{u}} X$ an *open immersion* to $X \xrightarrow{f} S$, if there is a cover $\{U_i \xrightarrow{\mathfrak{u}_i} U \mid i \in J\}$ such that the composition of arrows $U_i \xrightarrow{\mathfrak{u}\mathfrak{u}_i} X$ is a quasi-compact open immersion for each $i \in J$.

Thus, $X \xrightarrow{f} S$ is a scheme if it has an affine cover consisting of open immersions.

4. Descent: "covers", comonads, and glueing.

4.1. Comonads associated with "covers". Let $\{U_i \xrightarrow{u_i} X \mid i \in J\}$ be a family of continuous morphisms and $\mathfrak{u}$ the corresponding morphism $\mathcal{U} = \coprod_{i\in J} U_i \xrightarrow{\mathfrak{u}} X$ with the inverse image functor

$$C_X \xrightarrow{\mathfrak{u}^*} \prod_{i\in J} C_{U_i} = C_{\mathcal{U}}, \quad M \longmapsto (u_i^*(M)|i \in J).$$

It follows that the family of inverse image functors $\{C_X \xrightarrow{u_i^*} C_{U_i} \mid i \in J\}$ is conservative iff the functor $\mathfrak{u}^*$ is conservative.

Suppose that the category C_X has products of $|J|$ objects. Then the morphism $\mathcal{U} = \coprod_{i\in J} U_i \xrightarrow{\mathfrak{u}} X$ is continuous: its direct image functor assigns to every object $(L_i|i \in J)$ of the category $C_{\mathcal{U}} = \prod_{i\in J} C_{U_i}$ the product $\prod_{i\in J} u_{i*}(L_i)$.

The adjunction morphism $Id_{C_X} \xrightarrow{\eta_{\mathfrak{u}}} \mathfrak{u}_*\mathfrak{u}^*$ assigns to each object M of C_X the morphism $M \longrightarrow \prod_{i\in J} u_{i*}u_i^*(M)$ determined by adjunction arrows $Id_{C_X} \xrightarrow{\eta_{u_i}} u_{i*}u_i^*$.

The adjunction morphism $\mathfrak{u}^*\mathfrak{u}_* \xrightarrow{\epsilon_{\mathfrak{u}}} Id_{C_{\mathcal{U}}}$ assigns to each object $\mathcal{L} = (L_i|i \in J)$ of $C_{\mathcal{U}}$ the morphism $(\epsilon_{\mathfrak{u},i}(\mathcal{L})|i \in J)$, where

$$u_i^*(\prod_{j\in J} u_{j*}(L_j)) \xrightarrow{\epsilon_{\mathfrak{u},i}(\mathcal{L})} L_i$$

is the composition of the image

$$u_i^*(\prod_{j\in J} u_{j*}(L_j)) \xrightarrow{u_i^*(p_i)} u_i^*u_{i*}(L_i)$$

of the image of the projection p_i and the adjunction arrow $u_i^*u_{i*}(L_i) \xrightarrow{\epsilon_{u_i}(L_i)} L_i$.

4.2. Beck's theorem and glueing. Suppose that for each $i \in J$, the category C_{U_i} has kernels of coreflexive pairs of arrows and the functor u_i^* preserves them. Then the inverse and direct image functors of the morphism $\mathfrak{u}$ satisfy the conditions of Beck's theorem, hence the category C_X is equivalent to the category of comodules over the comonad $\mathcal{G}_{\mathfrak{u}} = (G_{\mathfrak{u}}, \delta_{\mathfrak{u}}) = (\mathfrak{u}^*\mathfrak{u}_*, \mathfrak{u}^*\eta_{\mathfrak{u}}\mathfrak{u}_*)$ associated with the choice of inverse and direct image functors of $\mathfrak{u}$ together with an adjunction morphism $Id_{C_X} \xrightarrow{\eta_{\mathfrak{u}}} \mathfrak{u}_*\mathfrak{u}^*$.

Recall that $\mathcal{G}_{\mathfrak{u}}$-comodule is a pair $(\mathcal{L}, \zeta)$, where $\mathcal{L}$ is an object of $C_{\mathcal{U}}$ and ζ a morphism $\mathcal{L} \longrightarrow G_{\mathfrak{u}}(\mathcal{L})$ such that $\epsilon_{\mathfrak{u}}(\mathcal{L}) \circ \zeta = id_{\mathcal{L}}$ and $G_{\mathfrak{u}}(\zeta) \circ \zeta = \delta_{\mathfrak{u}}(\mathcal{L}) \circ \zeta$. Beck's theorem says that if the category $C_{\mathcal{U}}$ has kernels of coreflexive pairs of arrows and the functor $\mathfrak{u}^*$ preserves and reflects them, then the functor $C_X \xrightarrow{\widetilde{\mathfrak{u}^*}} (\mathcal{U}\backslash\mathcal{G}_{\mathfrak{u}}) - \mathit{comod}$ which assigns to each object M of C_X the $\mathcal{G}_{\mathfrak{u}}$-comodule $(\mathfrak{u}^*(M), \delta_{\mathfrak{u}}(M))$ is an equivalence of categories.

In terms of our local data – the "cover" $\{U_i \xrightarrow{u_i} X \mid i \in J\}$, a $\mathcal{G}_{\mathfrak{u}}$-comodule $(\mathcal{L}, \zeta)$ is the data $(L_i, \zeta_i | i \in J)$, where $(L_i | i \in J) = \mathcal{L}$ and ζ_i is a morphism

$$L_i \longrightarrow u_i^* \mathfrak{u}_*(\mathcal{L}) = u_i^*(\prod_{j\in J} u_{j*}(L_j))$$

which equalizes the pair of arrows

$$u_i^*\mathfrak{u}_*(\mathcal{L}) = u_i^*(\prod_j u_{j*}(L_j)) \underset{u_i^*(u_{j*}\zeta_j)}{\overset{u_i^*\eta_{\mathfrak{u}}\mathfrak{u}_*(\mathcal{L})}{\rightrightarrows}} u_i^*(\prod_m u_{m*}u_m^*(\prod_j u_{j*}(L_j))) = u_i^*\mathfrak{u}_*\mathfrak{u}^*\mathfrak{u}_*(\mathcal{L})$$

and such that $\epsilon_{\mathfrak{u},i}(\mathcal{L}) \circ \zeta_i = id_{L_i}, \ i \in J$.

The exactness of the diagram

$$\mathcal{L} \xrightarrow{\ \zeta\ } G_{\mathfrak{u}}(\mathcal{L}) \underset{G_{\mathfrak{u}}(\zeta)}{\overset{\delta_{\mathfrak{u}}(\mathcal{L})}{\rightrightarrows}} G_{\mathfrak{u}}^2(\mathcal{L})$$

is equivalent to the exactness of the diagram

$$L_i \xrightarrow{\ \zeta_i\ } u_i^*(\prod_{j\in J} u_{j*}(L_j)) \underset{u_i^*(u_{j*}\zeta_j)}{\overset{u_i^*\eta_{\mathfrak{u}}\mathfrak{u}_*(\mathcal{L})}{\rightrightarrows}} u_i^*(\prod_{m\in J} u_{m*}u_m^*(\prod_{j\in J} u_{j*}(L_j))) \qquad (1)$$

for every $i \in J$. If the functors u_k^* preserve products of J objects (or just the products involved into (1)), then the diagram (1) is isomorphic to the diagram

$$L_i \xrightarrow{\ \zeta_i\ } \prod_{j\in J} u_i^* u_{j*}(L_j) \underset{u_i^*(u_{j*}\zeta_j)}{\overset{u_i^*\eta_{\mathfrak{u}}\mathfrak{u}_*(\mathcal{L})}{\rightrightarrows}} \prod_{j,m\in J} u_i^* u_{m*} u_m^* u_{j*}(L_j) \qquad (2)$$

4.3. Remark. The exactness of the diagram (1) might be viewed as a sort of sheaf property. This interpretation looks more plausible (or less streched) when the diagram (1) is isomorphic to the diagram (2), because $u_i^*u_{j*}(L_j)$ can be regarded as the section of L_j over the 'intersection' of U_i and U_j and $u_i^*u_{m*}u_m^*u_{j*}(L_j)$ as the section of L_j over the intersection of the elements U_j, U_m, and U_i of the "cover".

4.4. The condition of the continuity of the comonad associated with a "cover". Suppose that each direct image functor $C_{U_i} \xrightarrow{u_{i*}} C_X, \ i \in J$, has a right adjoint, $u_i^!$; and let $\mathfrak{u}^!$ denote the functor $C_X \longrightarrow C_{\mathcal{U}} = \prod_{i\in J} C_{U_i}$ which maps every object M to

$(u_i^!(M)|i \in J)$. If the category C_X has coproducts of $|J|$ objects, then the functor $\mathfrak{u}^!$ has a left adjoint which maps every object $(L_i|i \in J)$ of $C_{\mathcal{U}}$ to the coproduct $\coprod_{i \in J} u_{i*}(L_i)$.

Therefore, if the canonical morphism $\coprod_{i\in J} u_{i*}(L_i) \longrightarrow \prod_{i\in J} u_{i*}(L_i)$ is an isomorphism for every object $(L_i|i \in J)$ of the category $C_{\mathcal{U}}$, then (and only then) the functor $\mathfrak{u}^!$ is a right adjoint to the functor $\mathfrak{u}_*$.

In particular, $\mathfrak{u}^!$ is a right adjoint to $\mathfrak{u}_*$, if the category C_X is additive and J is finite.

4.5. Note. If, in addition, the functors u_{i*} are conservative for all $i \in J$, then the functor $\mathfrak{u}_*$ is conservative, and the category $C_{\mathcal{U}}$ is equivalent to the category of modules over the continuous monad $\mathcal{F}_{\mathfrak{u}} = (F_{\mathfrak{u}}, \mu_{\mathfrak{h}})$, where $F_{\mathfrak{u}} = \mathfrak{u}_*\mathfrak{u}^*$ and $\mu_{\mathfrak{u}} = \mathfrak{u}_*\epsilon_{\mathfrak{u}}\mathfrak{u}^*$ for an adjunction morphism $\mathfrak{u}^*\mathfrak{u}_* \xrightarrow{\epsilon_{\mathfrak{u}}} Id_{C_{\mathcal{U}}}$.

5. Some motivating examples.

5.1. The base affine 'space' and the flag variety of a reductive Lie algebra from the point of view of noncommutative algebraic geometry. Let $\mathfrak{g}$ be a reductive Lie algebra over $\mathbb{C}$ and $U(\mathfrak{g})$ the enveloping algebra of $\mathfrak{g}$. Let $\mathcal{G}$ be the group of integral weights of $\mathfrak{g}$ and $\mathcal{G}_+$ the semigroup of nonnegative integral weights. Let $R = \bigoplus_{\lambda \in \mathcal{G}_+} R_\lambda$, where R_λ is the vector space of the (canonical) irreducible finite dimensional representation with the highest weight λ. The module R is a $\mathcal{G}$-graded algebra with the multiplication determined by the projections $R_\lambda \otimes R_\nu \longrightarrow R_{\lambda+\nu}$, for all $\lambda, \nu \in \mathcal{G}_+$. It is well known that the algebra R is isomorphic to the algebra of regular functions on the *base affine space* of $\mathfrak{g}$. Recall that G/U, where G is a connected simply connected algebraic group with the Lie algebra $\mathfrak{g}$, and U is its maximal unipotent subgroup.

The category $C_{\mathbf{Cone}(R)}$ is equivalent to the category of quasi-coherent sheaves on the base affine space Y of the Lie algebra $\mathfrak{g}$. The category $C_{\mathbf{Proj}_{\mathcal{G}}(R)}$ is equivalent to the category of quasi-coherent sheaves on the flag variety of $\mathfrak{g}$.

5.2. The quantized base affine 'space' and quantized flag variety of a semisimple Lie algebra. Let now $\mathfrak{g}$ be a semisimple Lie algebra over a field k of zero characteristic, and let $U_q(\mathfrak{g})$ be the quantized enveloping algebra of $\mathfrak{g}$. Define the $\mathcal{G}$-graded algebra $R = \bigoplus_{\lambda \in \mathcal{G}_+} R_\lambda$ the same way as above. This time, however, the algebra R is not commutative. Following the classical example (and identifying spaces with categories of quasi-coherent sheaves on them), we call $\mathbf{Cone}(R)$ *the quantum base affine 'space'* and $\mathbf{Proj}_{\mathcal{G}}(R)$ the *quantum flag variety* of $\mathfrak{g}$.

5.2.1. Canonical affine covers of the base affine 'space' and the flag variety. Let W be the Weyl group of the Lie algebra $\mathfrak{g}$. Fix a $w \in W$. For any $\lambda \in \mathcal{G}_+$, choose a nonzero w-extremal vector $e^\lambda_{w\lambda}$ generating the one dimensional vector subspace of R_λ formed by the vectors of the weight $w\lambda$. Set $S_w = \{k^* e^\lambda_{w\lambda} | \lambda \in \mathcal{G}_+\}$. It follows from the Weyl character formula that $e^\lambda_{w\lambda} e^\mu_{w\mu} \in k^* e^{\lambda+\mu}_{w(\lambda+\mu)}$. Hence S_w is a multiplicative set. It was proved by Joseph [Jo] that S_w is a left and right Ore subset in R. The Ore sets $\{S_w | w \in W\}$ determine a conservative family of affine localizations

$$\mathbf{Sp}(S_w^{-1}R) \longrightarrow \mathbf{Cone}(R), \quad w \in W, \tag{4}$$

of the quantum base affine 'space' and a conservative family of affine localizations

$$\mathbf{Sp}_{\mathcal{G}}(S_w^{-1}R) \longrightarrow \mathbf{Proj}_{\mathcal{G}}(R), \quad w \in W, \tag{5}$$

of the quantum flag variety. We claim that the category $gr_{\mathcal{G}} S_w^{-1}R - mod$ of $\mathcal{G}$-graded $S_w^{-1}R$-modules is naturally equivalent to the category $(S_w^{-1}R)_0 - mod$.

In fact, by 1.5, it suffices to verify that the canonical functor

$$gr_{\mathcal{G}} S_w^{-1}R - mod \longrightarrow S_w^{-1}R)_0 - mod$$

which assigns to every graded $S_w^{-1}R$-module its zero component is faithful; i.e. the zero component of every nonzero $\mathcal{G}$-graded $S_w^{-1}R$-module is nonzero. This is, really, the case, because if z is a nonzero element of the λ-component of a $\mathcal{G}$-graded $S_w^{-1}R$-module, then $(e^{\lambda}_{w\lambda})^{-1}z$ is a nonzero element of the zero component of this module.

Thus, we obtain an affine cover

$$\mathbf{Sp}((S_w^{-1}R)_0) \longrightarrow \mathbf{Proj}_{\mathcal{G}}(R), \quad w \in W, \tag{6}$$

of the quantum flag variety $\mathbf{Proj}_{\mathcal{G}}(R)$ of the Lie algebra $\mathfrak{g}$.

The covers (4) and (6) are scheme structures on respectively quantum base affine 'space' and quantum flat variety. One can check that all morphisms (4) and (6) are affine, i.e. the covers (4) and (5) are semiseparated.

5.3. Noncommutative Grassmannians. Fix an associative unital k-algebra R. Let $R\backslash Alg_k$ be the category of associative *k-algebras over R* (i.e. pairs $(S, R \to S)$, where S is a k-algebra and $R \to S$ a k-algebra morphism). We call them for convenience R-rings. We denote by R^e the k-algebra $R \otimes_k R^o$. Here R^o is the algebra opposite to R.

5.3.1. The functor $Gr_{M,V}$. Let M, V be left R-modules. Consider the functor, $Gr_{M,V} : R\backslash Alg_k \longrightarrow \mathbf{Sets}$, which assigns to any R-ring $(S, R \xrightarrow{s} S)$ the set of isomorphism classes of epimorphisms $s^*(M) \longrightarrow s^*(V)$ (here $s^*(M) = S \otimes_R M$) and to any R-ring morphism $(S, R \xrightarrow{s} S) \xrightarrow{\phi} (T, R \xrightarrow{t} T)$ the map $Gr_{M,V}(S,s) \longrightarrow Gr_{M,V}(T,t)$ induced by the inverse image functor $S - mod \xrightarrow{\phi^*} T - mod$, $\mathcal{N} \longmapsto T \otimes_S \mathcal{N}$.

5.3.2. The functor $G_{M,V}$. Denote by $G_{M,V}$ the functor $R\backslash Alg_k \longrightarrow \mathbf{Sets}$ which assigns to any R-ring $(S, R \xrightarrow{s} S)$ the set of pairs of morphisms $s^*(V) \xrightarrow{v} s^*(M) \xrightarrow{u} s^*(V)$ such that $u \circ v = id_{s^*(V)}$ and acts naturally on morphisms. Since V is a projective module, the map

$$\pi = \pi_{M,V} : G_{M,V} \longrightarrow Gr_{M,V}, \ (v,u) \longmapsto [u], \tag{1}$$

is a (strict) functor epimorphism.

5.3.3. Relations. Denote by $\mathfrak{R}_{M,V}$ the "functor of relations" $G_{M,V} \times_{Gr_{M,V}} G_{M,V}$. By definition, $\mathfrak{R}_{M,V}$ is a subfunctor of $G_{M,V} \times G_{M,V}$ which assigns to each R-ring, $(S, R \xrightarrow{s} S)$, the set of all 4-tuples $(u_1, v_1; u_2, v_2) \in G_{M,V} \times G_{M,V}$ such that the epimorphisms u_1, u_2 are equivalent. The latter means that there exists an isomorphism $s^*(V) \xrightarrow{\varphi} s^*(V)$ such that $u_2 = \varphi \circ u_1$, or, equivalently, $\varphi^{-1} \circ u_2 = u_1$. Since $u_i \circ v_i = id$, $i = 1, 2$, these equalities imply that $\varphi = u_2 \circ v_1$ and $\varphi^{-1} = u_1 \circ v_2$. Thus, $\mathfrak{R}_{M,V}(S, s)$ is a subset of all $(u_1, v_1; u_2, v_2) \in G_{M,V}(S, s) \times G_{M,V}(S, s)$ satisfying the following relations:

$$u_2 = (u_2 \circ v_1) \circ u_1, \quad u_1 = (u_1 \circ v_2) \circ u_2 \tag{2}$$

in addition to the relations describing $G_{M,V}(S, s) \times G_{M,V}(S, s)$:

$$u_1 \circ v_1 = id_{S \otimes_R V} = u_2 \circ v_2 \tag{3}$$

Denote by $p_1,\ p_2$ the canonical projections $\mathfrak{R}_{M,V} \rightrightarrows G_{M,V}$. It follows from the surjectivity of $G_{M,V} \longrightarrow Gr_{M,V}$ that the diagram

$$\mathfrak{R}_{M,V} \ \underset{p_2}{\overset{p_1}{\rightrightarrows}}\ G_{M,V} \ \xrightarrow{\ \pi\ }\ Gr_{M,V} \tag{4}$$

is exact.

5.3.4. Proposition. *If both M and V are projective modules of a finite type, then the functors $G_{M,V}$ and $\mathfrak{R}_{M,V}$ are corepresentable.*

Proof. See [KR2, 10.4.3]. ∎

5.3.5. Quasi-coherent presheaves on presheaves of sets. Consider the category $\mathbf{Aff}_k$ of *affine k-schemes* which we identify with the category of representable functors on the category Alg_k of k-algebras, and the fibered category with the base $\mathbf{Aff}_k$ whose fibers are categories of left modules over corresponding algebras. Let X be a presheaf of sets on $\mathbf{Aff}_k$. Then we have a fibered category $\widetilde{\mathbf{Aff}_k/X}$ with the base $\mathbf{Aff}_k/X$ induced by the forgetful functor $\mathbf{Aff}_k/X \longrightarrow \mathbf{Aff}_k$. The category $Qcoh(X)$ of *quasi-coherent presheaves* on X is the opposite to the category of *cartesian sections* of $\widetilde{\mathbf{Aff}_k/X}$.

5.3.6. Quasi-coherent presheaves on $Gr_{M,V}$. Suppose that M and V are projective modules of a finite type, hence the functors $G_{M,V}$ and $\mathfrak{R}_{M,V}$ are corepresentable by R-rings resp. $(\mathfrak{G}_{M,V}, R \to \mathfrak{G}_{M,V})$ and $(\mathcal{R}_{M,V}, R \to \mathcal{R}_{M,V})$. Then the category $Qcoh(G_{M,V})$ (resp. $Qcoh(\mathfrak{R}_{M,V})$) is equivalent to $\mathfrak{G}_{M,V} - mod$ (resp. $\mathcal{R}_{M,V} - mod$), and the category $Qcoh(Gr_{M,V})$ of quasi-coherent presheaves on $Gr_{M,V}$ is equivalent to the kernel of the diagram

$$Qcoh(G_{M,V}) \ \underset{p_2^*}{\overset{p_1^*}{\rightrightarrows}}\ Qcoh(\mathfrak{R}_{M,V}) \tag{5}$$

This means that, after identifying categories of quasi-coherent presheaves in (5) with corresponding categories of modules, quasi-coherent presheaves on $Gr_{M,V}$ can be realized as pairs (L,ϕ), where L is a $\mathfrak{G}_{M,V}$-module and ϕ is an isomorphism $p_1^*(L) \xrightarrow{\sim} p_2^*(L)$. Morphisms $(L,\phi) \longrightarrow (N,\psi)$ are given by morphisms $L \xrightarrow{g} N$ such that the diagram

$$\begin{array}{ccc} p_1^*(L) & \xrightarrow{p_1^*(g)} & p_1^*(N) \\ \phi \big\downarrow \wr & & \wr \big\downarrow \psi \\ p_2^*(L) & \xrightarrow{p_2^*(g)} & p_2^*(N) \end{array}$$

commutes. The functor

$$Qcoh(Gr_{M,V}) \xrightarrow{\pi^*} Qcoh(G_{M,V}), \quad (L,\phi) \longmapsto L,$$

is an inverse image functor of the projection $G_{M,V} \xrightarrow{\pi} Gr_{M,V}$ (see 5.3.3(4)).

One can show that the functor π^* is an inverse image functor of a faithfully flat affine morphism $\bar{\pi}$ from an affine 'space' $\mathbf{Sp}(\mathcal{G}_{\mathcal{M},\mathcal{V}})$ (where $\mathcal{G}_{\mathcal{M},\mathcal{V}}$ is a ring representing the functor $G_{\mathcal{M},\mathcal{V}}$) to the 'space' $\mathfrak{Grass}_{\mathcal{M},\mathcal{V}}$ represented by the category $Qcoh(Gr_{M,V})$ of quasi-coherent sheaves on $Gr_{M,V}$. In our terminology, this means that $\bar{\pi}$ is an affine semiseparated cover of $\mathfrak{Grass}_{\mathcal{M},\mathcal{V}}$.

5.3.7. Quasi-coherent sheaves of sets. Let X be a presheaf of sets on $\mathbf{Aff}_k$ Given a *(pre)topology* τ on $\mathbf{Aff}_k/X$, we define the subcategory $Qcoh(X,\tau)$ of *quasi-coherent sheaves* on (X,τ) [KR4].

5.3.7.1. Theorem ([KR4]). *(a) A topology τ on $\mathbf{Aff}_k$ is subcanonical (i.e. all representable presheaves are sheaves) iff $Qcoh(X) = Qcoh(X,\tau)$ for every presheaf of sets X on $\mathbf{Aff}_k$ (in other words, 'descent' topologies on $\mathbf{Aff}_k$ are precisely subcanonical topologies). In this case, $Qcoh(X) = Qcoh(X,\tau) \hookrightarrow Qcoh(X^\tau) = Qcoh(X^\tau,\tau)$, where X^τ is the sheaf associated to X and $\hookrightarrow$ is a natural full embedding.*

(b) If τ is a topology of effective descent [KR4] (e.g. the **fpqc** *or smooth topology [KR2]), then the categories $Qcoh(X,\tau)$ and $Qcoh(X^\tau)$ are naturally equivalent.*

This theorem says, roughly speaking, that the category $Qcoh(X)$ of quasi-coherent presheaves knows which topologies to choose. A topology that seems to be the most plausible for Grassmannians, in particular, for $N\mathbb{P}_k^n$, is the *smooth* topology introduced in [KR2]. It is of effective descent, and the category of quasi-coherent sheaves on $N\mathbb{P}_k^n$ defined in [KR1] is naturally equivalent to the category of quasi-coherent sheaves of the projective space defined via smooth topology on $\mathbf{Aff}_k$.

Lecture 2. Underlying topological spaces of noncommutative 'spaces' and schemes.

Section 1 contains necessary preliminaries on topologizing, thick and Serre subcategories of an abelian category. In Section 2, we introduce the main notion for the geometric study of ('spaces' represented by) abelian categories – the spectrum $\mathbf{Spec}(-)$ and define Zariski topology on the spectrum. In Section 3, we introduce *local* 'spaces' and related to them spectrum $\mathbf{Spec}^-(X)$. This spectrum is naturally isomorphic to the *Gabriel spectrum* in the case when C_X is a Grothendieck locally noetherian category. There is a natural embedding $\mathbf{Spec}(X) \hookrightarrow \mathbf{Spec}^-(X)$ whose image, $\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X)$, consists of all Serre subcategories $\mathcal{P}$ such that the intersection $\mathcal{P}^{\mathfrak{t}}$ of all topologizing subcategories properly containing $\mathcal{P}$ is not equal to $\mathcal{P}$. In Section 4, we study local properties of the spectra with respect to finite covers. In Section 5, we discuss noncommutative k-'spaces' and schemes, where k is an associative unital ring. In Section 6, we introduce the *geometric* center of a 'space' X, which is a locally ringed topological space. The geometric center of a 'space' X turns to be a (conventional) scheme if X has a structure of a noncommutative (that is not necessarily commutative) k-scheme for an associative ring k.

The Reconstruction Theorem for quasi-compact schemes sounds as follows:

If C_X is the category of quasi-coherent sheaves on a quasi-compact quasi-separated commutative scheme, then the geometric center of X is isomorphic to the scheme.

If a scheme (commutative or not) is not quasi-compact, the spectrum $\mathbf{Spec}(X)$ should be replaced by the spectrum $\mathbf{Spec}^0_{\mathfrak{c}}(X)$. Its definition and general properties, as well as the Reconstruction Theorem for non-quasi-compact schemes, are given in Section 7.

1. Topologizing, thick, and Serre subcategories.

1.1. Topologizing subcategories. A full subcategory $\mathbb{T}$ of an abelian category C_X is called *topologizing* if it is closed under finite coproducts and subquotients.

A subcategory $\mathbb{S}$ of C_X is called *coreflective* if the inclusion functor $\mathbb{S} \hookrightarrow C_X$ has a right adjoint; that is every object of C_X has a biggest subobject which belongs to $\mathbb{S}$. Dually, a subcategory $\mathbb{T}$ of C_X is called *reflective* if the inclusion functor $\mathbb{T} \hookrightarrow C_X$ has a left adjoint. We denote by $\mathfrak{T}(X)$ the preorder with respect to $\subseteq$ of topologizing subcategories and by $\mathfrak{T}_{\mathfrak{c}}(X)$ (resp. $\mathfrak{T}^{\mathfrak{c}}(X)$) the preorder of coreflective (resp. reflective) topologizing subcategories of C_X.

1.1.1. The Gabriel product and infinitesimal neighborhoods of topologizing categories. The *Gabriel product*, $\mathbb{S} \bullet \mathbb{T}$, of the pair of subcategories $\mathbb{S}$, $\mathbb{T}$ of C_X is the full subcategory of C_X spanned by all objects M such that there exists an exact sequence

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

with $L \in Ob\mathbb{T}$ and $N \in Ob\mathbb{S}$. It follows that $0 \bullet \mathbb{T} = \mathbb{T} = \mathbb{T} \bullet 0$ for any strictly full subcategory $\mathbb{T}$. The Gabriel product of two topologizing subcategories is a topologizing subcategory, and its restriction to topologizing categories is associative; i.e. $(\mathfrak{T}(X), \bullet)$ is a monoid. Similarly, the Gabriel product of coreflective topologizing subcategories is a coreflective topologizing subcategory, hence $\mathfrak{T}_{\mathfrak{c}}(X)$ is a submonoid of $(\mathfrak{T}(X), \bullet)$. Dually, the preorder $\mathfrak{T}^{\mathfrak{c}}(X)$ of reflective topologizing subcategories of C_X is a submonoid of $(\mathfrak{T}(X), \bullet)$.

The n^{th} *infinitesimal neighborhood*, $\mathbb{T}^{(n+1)}$, of a subcategory $\mathbb{T}$ is defined by $\mathbb{T}^{(0)} = 0$ and $\mathbb{T}^{(n+1)} = \mathbb{T}^{(n)} \bullet \mathbb{T}$ for $n \geq 0$.

1.2. The preorder $\succ$ and topologizing subcategories. For any two objects, M and N, of an abelian category C_X, we write $M \succ N$ if N is a subquotient of a finite coproduct of copies of M. For any object M of the category C_X, we denote by $[M]$ the full subcategory of C_X whose objects are all $L \in ObC_X$ such that $M \succ L$. It follows that $M \succ N \Leftrightarrow [N] \subseteq [M]$. In particular, M and N are equivalent with respect to $\succ$ (i.e. $M \succ N \succ M$) iff $[M] = [N]$. Thus, the preorder $\big(\{[M] \mid M \in ObC_X\}, \supseteq\big)$ is a canonical realization of the quotient of $(ObC_X, \succ)$ by the equivalence relation associated with $\succ$.

1.2.1. Lemma. *(a) For any object M of C_X, the subcategory $[M]$ is the smallest topologizing subcategory containing M.*

(b) The smallest topologizing subcategory spanned by a family of objects $\mathcal{S}$ coincides with $\bigcup_{N \in \mathcal{S}_\Sigma} [N]$, where $\mathcal{S}_\Sigma$ denotes the family of all finite coproducts of objects of $\mathcal{S}$.

Proof. (a) Since $\succ$ is a transitive relation, the subcategory $[M]$ is closed with respect to taking subquotients. If $M \succ M_i$, $i = 1, 2$, then $M \succ M \oplus M \succ M_1 \oplus M_2$, which shows that $[M]$ is closed under finite coproducts, hence it is topologizing. Clearly, any topologizing subcategory containing M contains the subcategory $[M]$.

(b) The union $\bigcup_{N \in \mathcal{S}_\Sigma} [N]$ is contained in every topologizing subcategory containing the family $\mathcal{S}$. It is closed under taking subquotients, because each $[N]$ has this property. It is closed under finite coproducts, because if $N_1, N_2 \in \mathcal{S}_\Sigma$ and $N_i \succ M_i$, $i = 1, 2$, then $N_1 \oplus N_2 \succ M_1 \oplus M_2$. ■

For any subcategory (or a class of objects) $\mathcal{S}$, we denote by $[\mathcal{S}]$ (resp. by $[\mathcal{S}]_{\mathfrak{c}}$) the smallest topologizing resp. coreflective topologizing) subcategory containing $\mathcal{S}$.

1.2.2. Proposition. *Suppose that C_X is an abelian category with small coproducts. Then a topologizing subcategory of C_X is coreflective iff it is closed under small coproducts. The smallest coreflective topologizing subcategory spanned by a set of objects $\mathcal{S}$ coincides with $\bigcup_{N \in \widetilde{\mathcal{S}}} [N] = \bigcup_{N \in \widetilde{\mathcal{S}}} [N]$, where $\widetilde{\mathcal{S}}$ is the family of all small coproducts of objects of $\mathcal{S}$.*

Suppose that C_X satisfies (AB4), i.e. it has infinite coproducts and the coproduct of a set of monomorphisms is a monomorphism. Then, for any object M of C_X, the smallest coreflective topologizing subcategory $[M]_{\mathfrak{c}}$ spanned by M is generated by subquotients of coproducts of sets of copies of M.

Proof. The argument is similar to that of 1.2.1 and left to the reader as an exercise. ■

1.3. Thick subcategories. A topologizing subcategory $\mathbb{T}$ of the category C_X is called *thick* if $\mathbb{T} \bullet \mathbb{T} = \mathbb{T}$; in other words, $\mathbb{T}$ is thick iff it is closed under extensions.

We denote by $\mathfrak{Th}(X)$ the preorder of thick subcategories of C_X. For a thick subcategory $\mathcal{T}$ of C_X, we denote by $X/\mathcal{T}$ the *quotient 'space'* defined by $C_{X/\mathcal{T}} = C_X/\mathcal{T}$.

1.4. Serre subcategories. We recall the notion of a Serre subcategory of an abelian category as it is defined in [R, III.2.3.2]. For a subcategory $\mathbb{T}$ of C_X, let $\mathbb{T}^-$ denote the full subcategory of C_X generated by all objects L of C_X such that any nonzero subquotient of L has a nonzero subobject which belongs to $\mathbb{T}$.

1.4.1. Proposition. *Let* $\mathbb{T}$ *be a subcategory of* C_X*. Then*
(a) The subcategory $\mathbb{T}^-$ *is thick.*
(b) $(\mathbb{T}^-)^- = \mathbb{T}^-$.
(c) $\mathbb{T} \subseteq \mathbb{T}^-$ *iff any subquotient of an object of* $\mathbb{T}$ *is isomorphic to an object of* $\mathbb{T}$.

Proof. See [R, III.2.3.2.1]. ■

1.4.2. Remark. It follows from 1.4.1 and the definition of $\mathbb{T}^-$ that, for any subcategory $\mathbb{T}$ of an abelian category C_X, the associated Serre subcategory $\mathbb{T}^-$ is the largest topologizing (or the largest thick) subcategory of C_X such that every its nonzero object has a nonzero subobject from $\mathbb{T}$.

1.4.3. Definition. A subcategory $\mathbb{T}$ of an abelian category C_X is called a *Serre subcategory* if $\mathbb{T}^- = \mathbb{T}$. We denote by $\mathfrak{Se}(X)$ the preorder (with respect to $\supseteq$) of all Serre subcategories of C_X.

The following characterization of Serre subcategories turns to be quite useful.

1.4.4. Proposition. *Let* $\mathbb{T}$ *be a subcategory of an abelian category* C_X *closed under taking subquotients. The following conditions are equivalent:*
(a) $\mathbb{T}$ *is a Serre subcategory.*
(b) If $\mathbb{S}$ *is a subcategory of the category* C_X *which is closed under subquotients and is not contained in* $\mathbb{T}$*, then* $\mathbb{S} \bigcap \mathbb{T}^\perp \neq 0$.

Proof. $(a) \Rightarrow (b)$. Let $\mathbb{T}$ be a subcategory of C_X closed under taking quotients. By the definition of $\mathbb{T}^-$, an object M does not belong to $\mathbb{T}^-$ iff it has a nonzero subquotient, L, which does not have a nonzero subobject from $\mathbb{T}$. Since $\mathbb{T}$ is closed under taking quotients, the latter means precisely that $Hom(N, L) = 0$ for every $N \in Ob\mathbb{T}$, i.e. $L \in Ob\mathbb{T}^\perp$. Thus, M does not belong to $\mathbb{T}^-$ iff it has a nonzero subquotient which belongs to $\mathbb{T}^\perp$.

$(b) \Rightarrow (a)$. By the condition (b), if an object M does not belong to $\mathbb{T}$, then it has a nonzero subquotient which belongs to $\mathbb{T}^\perp$. But, by the observation above, this means that the object M does not belong to $\mathbb{T}^-$. So that $\mathbb{T}^- \subseteq \mathbb{T}$. The inverse inclusion holds, because $\mathbb{T}$ is closed under taking subquotients (see 1.4.1(c)). ■

1.4.5. The property (sup). Recall that X (or the corresponding category C_X) has the property (sup) if for any object M of C_X and for any ascending chain, Ω, of subobjects of M, the supremum of Ω exists, and for any subobject L of M, the natural morphism

$$sup(N \cap L \mid N \in \Omega) \longrightarrow (sup\Omega) \cap L$$

is an isomorphism.

1.4.6. Coreflective thick subcategories and Serre subcategories. Recall that a full subcategory $\mathcal{T}$ of a category C_X is called *coreflective* if the inclusion functor $\mathcal{T} \hookrightarrow C_X$

has a right adjoint. In other words, each object of C_X has the largest subobject which belongs to $\mathcal{T}$.

1.4.6.1. Lemma. *Any coreflective thick subcategory is a Serre subcategory. If C_X has the property (sup), then any Serre subcategory of C_X is coreflective.*

Proof. See [R, III.2.4.4]. ∎

1.4.7. Proposition. *Let C_X have the property (sup). Then for any thick subcategory $\mathbb{T}$ of C_X, all objects of $\mathbb{T}^-$ are supremums of their subobjects contained in $\mathbb{T}$.*

Proof. Since C_X has the property (sup), the full subcategory $\mathbb{T}_{\mathfrak{s}}$ of C_X whose objects are supremums of objects from $\mathbb{T}$ is thick and coreflective, hence Serre, subcategory containing $\mathbb{T}$ and contained in $\mathbb{T}^-$. Therefore it coincides with $\mathbb{T}^-$. ∎

2. The spectrum Spec(X). We denote by $Spec(X)$ the family of all nonzero objects M of the category C_X such that $L \succ M$ for any nonzero subobject L of M.

The spectrum $\mathbf{Spec}(X)$ of the 'space' X is the family of topologizing subcategories $\{[M] \mid M \in Spec(X)\}$ endowed with the *specialization* preorder $\supseteq$.

Let $\tau^{\succ}$ denote the topology on $\mathbf{Spec}(X)$ associated with the specialization preorder: the closure of $W \subseteq \mathbf{Spec}(X)$ consists of all $[M]$ such that $[M] \subseteq [M']$ for some $[M'] \in W$.

2.1. Proposition. *(a) Every simple object of the category C_X belongs to $Spec(X)$. The inclusion $Simple(X) \hookrightarrow Spec(X)$ induces an embedding of the set of the isomorphism classes of simple objects of C_X into the set of closed points of $(\mathbf{Spec}(X), \tau^{\succ})$.*

(b) If every nonzero object of C_X has a simple subquotient, then each closed point of $(Spec(X), \tau^{\succ})$ is of the form $[M]$ for some simple object M of the category C_X.

Proof. (a) If M is a simple object, then $Ob[M]$ consists of all objects isomorphic to coproducts of finite number of copies of M. In particular, if M and N are simple objects, then $[M] \subseteq [N]$ iff $M \simeq N$.

(b) If L is a subquotient of M, then $[L] \subseteq [M]$. If $[M]$ is a closed point of $\mathbf{Spec}(X)$, this implies the equality $[M] = [L]$. ∎

Notice that the notion of a simple object of an abelian category is selfdual, i.e. $Simple(X) = Simple(X^o)$, where X^o is the dual 'space' defined by $C_{X^o} = C_X^{op}$. In particular, the map $M \longmapsto [M]$ induces an embedding of isomorphism classes of simple objects of C_X into the intersection $\mathbf{Spec}(X) \bigcap \mathbf{Spec}(X^o)$.

2.1.1. Proposition. *If the category C_X has enough objects of finite type, then the set of closed points of $(\mathbf{Spec}(X), \tau^{\succ})$ coincides with $\mathbf{Spec}(X) \bigcap \mathbf{Spec}(X^o)$ and is in a natural bijective correspondence with the set $\mathbf{Simple}(X)$ of isomorphism classes of simple objects of the category C_X.*

Proof. Since every nonzero object of C_X has a nonzero subobject of finite type, $\mathbf{Spec}(X)$ consists of $[M]$ such that M is of finite type and belongs to $Spec(X)$. On the other hand, if M is of finite type and $[M]$ belongs to $\mathbf{Spec}(X^o)$, then $[M] = [M_1]$, where M_1 is a simple quotient of M. This proves that the set of closed points of $\mathbf{Spec}(X)$ coincides with $\mathbf{Spec}(X) \bigcap \mathbf{Spec}(X^o)$. It follows from 2.2(b) that the set closed points of $\mathbf{Spec}(X)$ coincides with the set of isomorphism classes of simple objects of C_X. ∎

2.2. Supports of objects. For any object M of the category C_X, the support of M is defined by $Supp(M) = \{\mathcal{Q} \in \mathbf{Spec}(X) \mid \mathcal{Q} \subseteq [M]\}$. This notion enjoys the usual properties:

2.2.1. Proposition. *(a) If* $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$, *is a short exact sequence, then*

$$Supp(M) = Supp(M') \bigcup Supp(M'').$$

(b) Suppose the category C_X has the property (sup). Then

(b1) If M is the supremum of a filtered system $\{M_i \mid i \in J\}$ of its subobjects, then

$$Supp(M) = \bigcup_{i \in J} Supp(M_i).$$

(b2) As a consequence of (a) and (b1), we have

$$Supp(\bigoplus_{i \in J} M_i) = \bigcup_{i \in J} Supp(M_i).$$

Proof. (a) Since $[M'] \subseteq [M] \supseteq [M'']$, we have the inclusion

$$Supp(M') \bigcup Supp(M'') \subseteq Supp(M).$$

In order to show the inverse inclusion, notice that for any object L of the subcategory $[M]$, there exists an exact sequence $0 \longrightarrow L' \longrightarrow L \longrightarrow L'' \longrightarrow 0$ such that L' is an object of $[M']$ and L'' belongs to $[M'']$. This follows from the fact that L is a subquotient of a coproduct $M^{\oplus n}$ of n copies of M the related commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M'^{\oplus n} & \longrightarrow & M^{\oplus n} & \longrightarrow & M''^{\oplus n} & \longrightarrow & 0 \\
 & & \uparrow & cart & \uparrow & & \uparrow & & \\
0 & \longrightarrow & K' & \longrightarrow & K & \longrightarrow & K'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L' & \longrightarrow & L & \longrightarrow & L'' & \longrightarrow & 0
\end{array} \qquad (1)$$

whose rows are exact sequences, the upper vertical arrows are monomorphisms, the lower ones epimorphisms, and the left upper square is cartesian.

Now if $[L] \in \mathbf{Spec}(X)$ and the object L' in the diagram (1) is nonzero, then, by the definition of the spectrum, $[L'] = [L]$, hence $[L] \in Supp(M')$. If $L' = 0$, then the arrow $L \longrightarrow L''$ is an isomorphism, in particular, $[L] = [L''] \in Supp(M'')$.

(b1) The inclusion $Supp(M) \supseteq \bigcup_{i \in J} Supp(M_i)$ is obvious. It follows from the property (sup) that if an object L is a nonzero subquotient of $M^{\oplus n}$ for some n, then it contains a nonzero subobject, L', which is a subquotient of M_i for some $i \in J$. If $[L] \in \mathbf{Spec}(X)$, this implies that $[L] = [L'] \in Supp(M_i)$.

(b2) If J is finite, the assertion follows from (a). If J is infinite, it is a consequence of (a) and (b1). ■

2.3. Topologies on $\mathbf{Spec}(X)$. We are interested in topologies compatible with the specializations: every closed set should contain specializations of all its points.

2.3.1. The topologies $\tau^{\succ}$ and $\tau_{\succ}$. The finest such topology is the topology $\tau^{\succ}$ introduced at the beginning of the section. The coarsest reasonable topology is the topology $\tau_{\succ}$ having the set of specializations of points of the spectrum as a base of closed sets.

2.3.2. A general construction: supports and topologies. Let Ξ be a class of objects of C_X closed under finite coproducts. For any set E of objects of Xi, let $\mathcal{V}(E)$ denote the intersection $\bigcap_{M\in E} Supp(M)$. Then, for any family $\{E_i \mid i \in \mathfrak{I}\}$ of such sets, we have, evidently,

$$\mathcal{V}(\bigcup_{i\in J} E_i) = \bigcap_{i\in J} \mathcal{V}(E_i).$$

It follows from the equality $Supp(M \oplus N) = Supp(M) \bigcup Supp(N)$ (see 2.2.1(a)) that $\mathcal{V}(E \oplus \widetilde{E}) = \mathcal{V}(E) \bigcup \mathcal{V}(\widetilde{E})$. Here $E \oplus \widetilde{E} \stackrel{\text{def}}{=} \{M \oplus N \mid M \in E,\ N \in \widetilde{E}\}$.

This shows that $\mathbf{Spec}(X)$, $\emptyset$, and the subsets $\mathcal{V}(E)$ of $\mathbf{Spec}(X)$, where E runs through subsets of Ξ, form the family of all closed sets of a topology, τ_{Ξ}, on $\mathbf{Spec}(X)$.

2.3.3. Examples. Taking as Ξ the class of finite coproducts of the representatives of the elements of $\mathbf{Spec}(X)$, we recover the topology $\tau_{\succ}$. If C_X is a category with small coproducts, then, taking $\Xi = ObC_X$, we recover the topology $\tau^{\succ}$.

2.4. Zariski topology on the spectrum. Notice that the class $\Xi_{\mathfrak{f}}(X)$ of objects of finite type is closed under finite coproducts, hence it defines a topology on $\mathbf{Spec}(X)$ which we denote by $\tau_{\mathfrak{f}}$. If the category C_X has enough objects of finite type, then we shall call the topology $\tau_{\mathfrak{f}}$ the *Zariski* topology.

2.4.1. Example. Let R be a commutative unital ring and C_X the category $R-mod$ of R-modules. Then $\mathbf{Spec}(X)$ is isomorphic to the prime spectrum $Spec(R)$ of the ring R and the topology $\tau_{\mathfrak{f}}$ corresponds to the Zariski topology on $Spec(R)$.

2.4.2. Note. If the category C_X does not have enough objects of finite type, the topology $\tau_{\mathfrak{f}}$ is not a "right" candidate for the Zariski topology, because it should be finer than the topology $\tau_{\succ}$ (introduced in 2.3.1) and it might be not. Zariski topology can be naturally extended to 'spaces' X for which the category C_X has "locally" enough objects of finite type. In general, "locally" is an additional data. This data is naturally available in the case of (not necessarily commutative) schemes.

2.5. Complement: topologizing subcategories and topologies on the spectrum. There is another, more universal, way to define topologies on $\mathbf{Spec}(X)$.

2.5.1. Lemma. *Let $\mathfrak{W}$ be a family of topologizing subcategories of an abelian category C_X which contains C_X, the zero subcategory, is closed under arbitrary intersections and such that for any pair $\mathcal{T}_1$, $\mathcal{T}_2$ of elements of $\mathfrak{W}$, there exists $\mathcal{T}_3 \in \mathfrak{W}$ such that $\mathcal{T}_1 \cup \mathcal{T}_2 \subseteq$*

$\mathcal{T}_3 \subseteq (\mathcal{T}_1 \cup \mathcal{T}_2)^-$. *Then* $\tau^o_{\mathfrak{W}} = \{\mathcal{V}(\mathcal{T}) = \mathbf{Spec}(|\mathcal{T}|) \mid \mathcal{T} \in \mathfrak{W}\}$ *is a set of closed subsets of a topology,* $\tau_{\mathfrak{W}}$.

Proof. In fact, for any set $\{\mathcal{T}_i \mid i \in J\}$ of topologizing subcategories of the category C_X, we have

$$\mathcal{V}(\bigcap_{i\in J} \mathcal{T}_i) = \bigcap_{i\in J} \mathcal{V}(\mathcal{T}_i). \tag{1}$$

If $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ are topologizing subcategories of C_X such that

$$\mathcal{T}_1 \cup \mathcal{T}_2 \subseteq \mathcal{T}_3 \subseteq (\mathcal{T}_1 \cup \mathcal{T}_2)^-, \tag{2}$$

then $\mathcal{V}(T_3) = \mathcal{V}(T_1) \cup \mathcal{V}(\mathcal{T}_2)$. This is a consequence of the inclusions $\mathcal{V}(\mathcal{T}_1) \cup \mathcal{V}(T_2) \subseteq \mathcal{V}(T_3) \subseteq \mathcal{V}((\mathcal{T}_1 \cup \mathcal{T}_2)^-)$ following from (2) and the equality $\mathcal{V}(\mathcal{T}) = \mathcal{V}(\mathcal{T}^-)$ which holds for any topologizing subcategory $\mathcal{T}$. ∎

2.5.2. A special case. Let Ξ be a set of objects of an abelian category C_X closed under finite coproducts. We set $\widetilde{\mathfrak{W}}_\Xi = \{[M] \mid M \in \Xi\}$ and denote by $\mathfrak{W}_\Xi$ the set consisting of C_X, 0, and of intersections of arbitrary subfamilies of $\widetilde{\mathfrak{W}}_\Xi$. The set of topologizing subcategories $\mathfrak{W}_\Xi$ satisfies the conditions of 2.5.1, and the topology it defines coincides with the topology τ_Ξ introduced in 2.3.

2.5.3. Monoids of topologizing subcategories and associated topologies on the spectrum. Let $\mathfrak{W}$ be a set of topologizing subcategories of C_X containing C_X and the the zero subcategory $\mathbf{0}$ and closed under the Gabriel multiplication and arbitrary intersections. Then $\mathfrak{W}$ satisfies the conditions of 2.5.1, because for any pair of topologizing subcategories $\mathcal{T}_1$, $\mathcal{T}_2$, their Gabriel product $\mathcal{T}_1 \bullet \mathcal{T}_2$ contains $\mathcal{T}_1 \cup \mathcal{T}_2$ and is contained in the Serre subcategory $(\mathcal{T}_1 \cup \mathcal{T}_2)^-$ generated by $\mathcal{T}_1 \cup \mathcal{T}_2$.

Taking as $\mathfrak{W}$ the monoid $\mathfrak{T}(X)$ of all topologizing subcategories of C_X, we recover the topology $\tau^0_{\mathfrak{t}}$ on $\mathbf{Spec}(X)$ associated with the *specialization* preorder $\supseteq$: the closure of a subset of the spectrum consists of all specializations of the elements of this subset. This is the finest among the reasonable topologies on the spectrum.

The map $\mathfrak{G} \longmapsto \tau_{\mathfrak{G}}$ is a surjective map from the family of full monoidal subcategories of $(\mathfrak{T}(X), \bullet)$ closed under arbitrary intersections onto the set of topologies on $\mathbf{Spec}^0_{\mathfrak{t}}(X)$ which are coarser than the topology $\tau^0_{\mathfrak{t}}$ corresponding to $\mathfrak{T}(X)$.

2.5.4. The coarse Zariski topology. Recall that a full subcategory $\mathbb{T}$ of C_X is *reflective* if the inclusion functor $\mathbb{T} \hookrightarrow C_X$ has a left adjoint. Suppose that the category C_X has supremums of sets of subobjects (for instance, C_X has infinite coproducts). Then, by [R, III.6.2.2], the intersection of any set of reflective topologizing subcategories is a reflective topologizing subcategory. Taking as $\mathfrak{W}$ the subcategory $\mathfrak{T}^{\mathfrak{c}}(X)$ of reflective topologizing subcategories, we obtain the *coarse Zariski* topology on $\mathbf{Spec}(X)$ which we denote by $\tau_{\mathfrak{Z}}$.

2.5.4.1. Proposition. *Suppose* C_X *has the property (sup) and a generator of finite type. Then the topological space* $(\mathbf{Spec}(X), \tau_{\mathfrak{Z}})$ *is quasi-compact.*

Proof. See [R, III.6.5.2.1]. ∎

2.5.4.2. Example: the coarse Zariski topology on an affine noncommutative scheme. Let C_X be the category $R-mod$ of left modules over an associative unital ring R. For every two-sided ideal α in R, let $\mathbb{T}_\alpha$ denote the full subcategory of $R-mod$ whose objects are modules annihilated by the ideal α. By [R, III.6.4.1], the map $\alpha \longmapsto \mathbb{T}_\alpha$ is an isomorphism of the preorder $(I(R), \supseteq)$ of two-sided ideals of the ring R onto $(\mathfrak{T}^{\mathfrak{c}}(X), \subseteq)$. Moreover, $\mathbb{T}_\alpha \bullet \mathbb{T}_\beta = \mathbb{T}_{\alpha\beta}$ for any pair of two-sided ideals α, β. This means that the map $\alpha \longmapsto \mathbb{T}_\alpha$ is an isomorphism of monoidal categories (preorders), where the monoidal structure on $I(R)$ is the multiplication of ideals. Note by passing that it follows from this description that every reflective topologizing subcategory of $C_X = R-mod$ is coreflective.

One of the consequences of 2.5.4.1 is that the topological space $(\mathbf{Spec}(X), \tau_3)$ is quasi-compact. This fact is a special case of a more precise assertion: an open subset $\mathcal{U}$ of the space $(\mathbf{Spec}(X), \tau_3)$ is quasi-compact iff $\mathcal{U} = U(\mathbb{T}_\alpha) = \mathbf{Spec}(X) - V(\mathbb{T}_\alpha)$ for a finitely generated two-sided ideal α of the ring R. Two different proofs of this theorem can be found in [R]: I.5.6 and III.6.5.3.1. One of its consequences is that quasi-compact open sets form a base of the Zariski topology on $\mathbf{Spec}(X)$. In fact, every two-sided ideal α is the supremum of a set $\{\alpha_i \mid i \in J\}$ of its two-sided subideals, so that $U(\mathbb{T}_\alpha) = U(sup(\mathbb{T}_{\alpha_i} \mid i \in J)) = \bigcup_{i\in J} U(\mathbb{T}_{\alpha_i})$.

2.5.4.3. Digression: the coarse Zariski topology and the Levitski spectrum. By definition, the *Levitski spectrum* $LSpec(R)$ of an associative unital ring R consists of all prime ideals p of R such that the quotient ring R/p has no non-trivial locally nilpotent ideals. If the ring R is left noetherian, then the Levitski spectrum coincides with the prime spectrum. We endow the Levitski spectrum $LSpec(R)$ with the *Zariski* topology induced from the prime spectrum: its closed sets are $\mathcal{V}(\alpha) = \{p \in LSpec(R) \mid \alpha \subseteq p\}$, where α runs through the set of all two-sided ideals of the ring R.

Let C_X be the category $R-mod$ of left modules over an associative ring R. The map which assigns to every R-module M the annihilator $Ann(M)$ of M reverses the preorder $\succ$; that is if $[N] \subseteq [M]$, then $Ann(N) \subseteq Ann(M)$. This implies that the map

$$[M] \longmapsto Ann(M)$$

is well defined and if $[M]$ is an object of $\mathbf{Spec}(X)$, then $Ann(M) = Ann(L)$ for any nonzero subobject of M. One can see that the latter implies that $Ann(M)$ is a prime ideal, if $[M]$ is an element of $\mathbf{Spec}(X)$. More precise statement is as follows:

Theorem ([R, I.5.3]) *The image of the map*

$$\mathbf{Spec}(X) \longrightarrow Spec(R), \quad [M] \longmapsto Ann(M),$$

coincides with the Levitski spectrum $LSpec(R)$ *of the ring* R *and the map is a quasi-homeomorphism of* $(\mathbf{Spec}(X), \tau_3)$ *onto the Levitski spectrum* $LSpec(R)$.

2.5.4.4. Note. Let $C_X = R - mod$ for an associative ring R. Then the sets $Supp(R/\mathfrak{n}) = \mathcal{V}([R/\mathfrak{n}])$, where $\mathfrak{n}$ runs through the set $I_\ell(R)$ of all left ideals of the ring R, form a base of the Zariski closed sets on $\mathbf{Spec}(X)$. By 2.5.4.2, closed sets of the *coarse* Zariski topology are precisely the sets $Supp(R/\alpha)$, where α runs through the set $I(R)$ of all two-sided ideals of R. This shows that the coarse Zariski topology is, indeed, coarser than the Zariski topology on the spectrum, and these topologies coincide if the ring R is commutative. One can show that they coincide if R is a PI ring.

Unfortunately, the coarse Zariski topology is trivial, or too coarse in many important examples of noncommutative affine schemes. Thus, the coarse Zariski on the spectrum of $X = \mathbf{Sp}(R)$ is trivial iff R is a simple ring (i.e. it does not have non-trivial two-sided ideals). In particular, it is trivial if C_X is the category of D-modules on the affine space $\mathbb{A}^n$, because the algebra A_n of differential operators on $\mathbb{A}^n$ is simple. The coarse Zariski topology on $\mathbf{Spec}(X)$ is non-trivial, but not sufficiently rich, when C_X is the category of representations of a semisimple Lie algebra over a field of characteristic zero.

3. Local 'spaces' and $\mathbf{Spec}^-(-)$.

3.1. Local 'spaces'. A 'space' X and the representing it abelian category C_X are called *local* if C_X has the smallest nonzero topologizing subcategory, $C_{X_{\mathfrak{t}}}$.

One can see that $C_{X_{\mathfrak{t}}}$ is the only closed point of $\mathbf{Spec}(X)$. It follows from definitions that a 'space' X is local iff $\mathbf{Spec}(X)$ has only one closed point which belongs to support of any nonzero object of the category C_X.

3.1.1. Proposition. *Let X be local, and let the category C_X have simple objects. Then all simple objects of C_X are isomorphic to each other, and every nonzero object of $C_{X_{\mathfrak{t}}}$ is a finite coproduct of copies of a simple object.*

Proof. In fact, if M is a simple object in C_X, then $[M]$ is a closed point of $\mathbf{Spec}(X)$. If X is local, this closed point is unique. Therefore, objects of $C_{X_{\mathfrak{t}}}$ are finite coproducts of copies of M (see the argument of 2.1). ∎

3.1.2. The residue 'space' of a local 'space'. Let X be local 'space' and $C_{X_{\mathfrak{t}}}$ the smallest non-trivial topologizing subcategory of the category C_X. We regard the inclusion functor $C_{X_{\mathfrak{t}}} \hookrightarrow C_X$ as an inverse image functor of a morphism of 'spaces' $X \longrightarrow X_{\mathfrak{t}}$ and call $X_{\mathfrak{t}}$ the *residue* 'space' of X.

3.1.3. The residue skew field of a local 'space'. Suppose that X is a local 'space' such that the category C_X has a simple object, M. We denote by k_X the ring $C_X(M, M)^o$ opposite to the ring of endomorphisms of the object M. Since M is simple, k_X is a skew field which we call the *residue skew field* of the local 'space' X. It follows from 3.1.1 that the residue skew field of X (if any) is defined uniquely up to isomorphism.

It follows that the *residue* category $C_{X_{\mathfrak{t}}}$ of the 'space' X is naturally equivalent to the category of finite dimensional k_X-vector spaces.

3.2. $\mathbf{Spec}^-(X)$. By definition, $\mathbf{Spec}^-(X)$ is formed by all Serre subcategories $\mathcal{P}$ of C_X such that $X/\mathcal{P}$ is a local 'space'. It is endowed with the *specialization* preorder $\supseteq$.

3.2.1. Support in $\mathbf{Spec}^-(X)$. We define the support of an object M of C_X in $\mathbf{Spec}^-(X)$ as the set $Supp^-(M)$ of all $\mathcal{P} \in \mathbf{Spec}^-(X)$ which do not contain M, or, equivalently, the localization of M at $\mathcal{P}$ is nonzero. We leave as an exercise proving the analogue of 2.2.1 for $Supp^-(-)$.

3.2.2. Zariski topology. We introduce the topology, $\tau_{\mathfrak{f}}^{-}$, on $\mathbf{Spec}^{-}(X)$ the same way as the topology $\tau_{\mathfrak{f}}$ on the spectrum $\mathbf{Spec}(X)$: supports of objects of finite type form a base of its closed sets. If the category C_X has enough objects of finite type, then we call $\tau_{\mathfrak{f}}^{-}$ the *Zariski* topology on $\mathbf{Spec}^{-}(X)$.

3.2.3. Remark on topologies on $\mathbf{Spec}^{-}(X)$. For any (topologizing) subcategory $\mathcal{T}$ of C_X, we set $\mathcal{V}^{-}(\mathcal{T}) = \{\mathcal{P} \in \mathbf{Spec}^{-}(X) \mid \mathcal{T} \not\subseteq \mathcal{P}\}$. If $\mathcal{T} = [M]$ for an object M, then the set $\mathcal{V}^{-}(\mathcal{T})$ coincides with the support $Supp^{-}(M)$ of the object M. There is an analogue of 2.5.1 for the sets $\mathcal{V}^{-}(\mathcal{T})$. In particular, any submonoid $\mathfrak{M}$ of the monoid $(\mathfrak{T}(X), \bullet)$ of topologizing subcategories of C_X which is closed under arbitrary intersections determines a topology $\tau_{\mathfrak{M}}^{-}$ on $\mathbf{Spec}^{-}(X)$ whose closed sets are $\mathcal{V}^{-}(\mathcal{T})$, where $\mathcal{T}$ runs through $\mathfrak{M}$. And all "reasonable" topologies on $\mathbf{Spec}^{-}(X)$ are of this form (see 2.5.3). In particular, taking the submonoid $\mathfrak{T}^{\mathfrak{c}}(X)$ of reflective topologizing subcategories of C_X (and assuming that C_X has supremums of sets of subobjects), we obtain the *coarse Zariski* topology $\tau_{\mathfrak{Z}}^{-}$ on $\mathbf{Spec}^{-}(X)$ (similar to 2.5.4). Details are left to the reader.

3.2.4. Indecomposable injectives and $\mathbf{Spec}^{-}(-)$. If C_X is a Grothendieck category with Gabriel-Krull dimension (say, C_X is locally noetherian), then the elements of $\mathbf{Spec}^{-}(X)$ are in bijective correspondence with the set of isomorphism classes of indecomposable injectives of the category C_X. The bijective correspondence is given by the map which assigns to every indecomposable injective E of C_X its *left orthogonal* – the full subcategory ${}^{\perp}E$ generated by all objects M of C_X such that $C_X(M, E) = 0$.

In other words, $\mathbf{Spec}^{-}(X)$ is isomorphic to the Gabriel spectrum of the category C_X.

An advantage of the spectrum $\mathbf{Spec}^{-}(X)$ is that it makes sense for all abelian categories, even those which do not have indecomposable injectives at all. For instance, if C_X is the category of coherent sheaves on a noetherian scheme, then its Gabriel spectrum is empty, while $\mathbf{Spec}^{-}(X)$ coincides with $\mathbf{Spec}(X)$ and is homeomorphic to the the underlying topological space of the scheme.

3.3. The spectra $\mathbf{Spec}(X)$, $\mathbf{Spec}_{\mathfrak{t}}^{1,1}(X)$, and $\mathbf{Spec}^{-}(X)$. Let C_X be a svelte abelian category. For any subcategory $\mathcal{P}$ of the category C_X, we denote by $\mathcal{P}^{\mathfrak{t}}$ the intersection of all topologizing subcategories of C_X properly containing $\mathcal{P}$.

The elements of the spectrum $\mathbf{Spec}_{\mathfrak{t}}^{1,1}(X)$ are all Serre subcategories $\mathcal{P}$ of C_X such that $\mathcal{P}^{\mathfrak{t}} \neq \mathcal{P}$. We endow $\mathbf{Spec}_{\mathfrak{t}}^{1,1}(X)$ with the *specialization* preorder $\supseteq$.

3.3.1. Proposition. *The spectrum* $\mathbf{Spec}_{\mathfrak{t}}^{1,1}(X)$ *consists of all topologizing subcategories* $\mathcal{P}$ *of the category* C_X *such that* $\mathcal{P}_{\mathfrak{t}} \stackrel{\mathrm{def}}{=} \mathcal{P}^{\mathfrak{t}} \bigcap \mathcal{P}^{\perp}$ *is nonzero.*

Proof. If $\mathcal{P} \in \mathbf{Spec}_{\mathfrak{t}}^{1,1}(X)$, i.e. $\mathcal{P}$ is a Serre subcategory of C_X which is properly contained in $\mathcal{P}^{\mathfrak{t}}$, then it follows from 1.4.4(b) that $\mathcal{P}_{\mathfrak{t}} \neq 0$.

Suppose now that $\mathcal{P}$ is a topologizing subcategory of C_X such that $\mathcal{P}_{\mathfrak{t}} \neq 0$. We claim that then $\mathcal{P}$ is a Serre subcategory, i.e. $\mathcal{P} = \mathcal{P}^{-}$.

In fact, let $\mathcal{S}$ be a topologizing subcategory of C_X which is not contained in $\mathcal{P}$. Then $\mathcal{P} \bullet \mathcal{S}$ contains $\mathcal{P}^{\mathfrak{t}}$ properly and $(\mathcal{P} \bullet \mathcal{S}) \bigcap \mathcal{P}^{\perp} \subseteq \mathcal{S}$. In particular, $\mathcal{P}_{\mathfrak{t}} \subseteq \mathcal{S}$. Since $\mathcal{P}_{\mathfrak{t}} \neq 0$, this implies that $\mathcal{S}$ is not contained in $\mathcal{P}^{-}$. This (and 1.4.2) shows that $\mathcal{P} = \mathcal{P}^{-}$. ■

For any subcategory $\mathcal{Q}$ of the category C_X, we denote by $\widehat{\mathcal{Q}}$ the union of all topologizing subcategories of C_X which do not contain $\mathcal{Q}$. It is easy to see, that for a pair $\mathcal{Q}_1$, $\mathcal{Q}_2$ topologizing subcategories, $\mathcal{Q}_1 \subseteq \mathcal{Q}_2$ iff $\widehat{\mathcal{Q}}_1 \subseteq \widehat{\mathcal{Q}}_2$.

If $\mathcal{Q}$ has one object, L, the subcategory $\widehat{\mathcal{Q}}$ is the union of all topologizing subcategories of C_X which do not contain L. We shall write $\langle L \rangle$ instead of $\widehat{\mathcal{Q}}$.

3.3.2. Proposition. *(a)* $\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X) \subseteq \mathbf{Spec}^-(X)$.

(b) For any $\mathcal{Q} \in \mathbf{Spec}(X)$*, the subcategory* $\widehat{\mathcal{Q}}$ *is an element of* $\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X)$ *and the map*

$$\mathbf{Spec}(X) \longrightarrow \mathbf{Spec}^{1,1}_{\mathfrak{t}}(X), \quad \mathcal{Q} \longmapsto \widehat{\mathcal{Q}},$$

is an isomorphism of preorders.

Proof. (a) If $\mathcal{P} \in \mathbf{Spec}^{1,1}_{\mathfrak{t}}(X)$, then $\mathcal{P}^{\mathfrak{t}}/\mathcal{P}$ is contained in and equivalent to the smallest nonzero topologizing subcategory of $C_X/\mathcal{P}$.

(b1) If $\mathcal{Q} \in \mathbf{Spec}(X)$, then $\widehat{\mathcal{Q}}$ is a Serre subcategory.

In fact, suppose that $\widehat{\mathcal{Q}} \neq \widehat{\mathcal{Q}}^-$, and let M be an object of $\widehat{\mathcal{Q}}^-$ which does not belong to its subcategory $\widehat{\mathcal{Q}}$. The latter means that $\mathcal{Q} \subseteq [M]$. Let $\mathcal{Q} = [L]$ for some $L \in Spec(X)$ (cf. 2). The inclusion $\mathcal{Q} \subseteq [M]$ means that L is a subquotient of a coproduct of a finite number, $M^{\oplus n}$, of copies of M. Since $M^{\oplus n}$ is an object of $\widehat{\mathcal{Q}}^-$, the object L has a nonzero subobject N which belongs to $\widehat{\mathcal{Q}}$; i.e. $\mathcal{Q} \nsubseteq [N]$. But, since $L \in Spec(X)$, the subcategories $[N]$ and $[L] = \mathcal{Q}$ coincide. Contradiction.

(b2) It follows from the definition of $\widehat{\mathcal{Q}}$ that, for any subcategory $\mathcal{Q}$, the subcategory $\widehat{\mathcal{Q}}^{\mathfrak{t}}$ coincides with the intersection of all topologizing subcategories of C_X containing $\widehat{\mathcal{Q}} \bigcup \mathcal{Q}$. In particular, $\widehat{\mathcal{Q}}$ belongs to $\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X)$ whenever $\widehat{\mathcal{Q}}$ is a Serre subcategory. Together with (b1), this shows that the assignment $\mathcal{Q} \longmapsto \widehat{\mathcal{Q}}$ induces a map $\mathbf{Spec}(X) \longrightarrow \mathbf{Spec}^{1,1}_{\mathfrak{t}}(X)$.

(b3) Let $\mathcal{P} \in \mathbf{Spec}^{1,1}_{\mathfrak{t}}(X)$. It follows from 3.3.1 that $\mathcal{P}_{\mathfrak{t}} \neq 0$. Moreover, by the argument of 3.3.1, if $\mathbb{T}$ is a topologizing subcategory of C_X such that $\mathbb{T} \nsubseteq \mathcal{P}$, then $\mathcal{P}_{\mathfrak{t}} = \mathcal{P}^{\mathfrak{t}} \bigcap \mathcal{P}^{\perp} \subseteq \mathbb{T}$.

(c) Let $\mathcal{P} \in \mathbf{Spec}^{1,1}_{\mathfrak{t}}(X)$. Every nonzero object of $\mathcal{P}_{\mathfrak{t}} = \mathcal{P}^{\mathfrak{t}} \bigcap \mathcal{P}^{\perp}$ belongs to $Spec(X)$.

Let L be a nonzero object of $\mathcal{P}_{\mathfrak{t}}$ and L_1 its nonzero subobject of, hence $[L_1] \subseteq [L]$. If $[L_1] \nsubseteq [L]$, then it follows from (b3) above that $[L_1] \subseteq \mathcal{P}$, or, equivalently, $L_1 \in Ob\mathcal{P}$. This contradicts to the assumption that the object L is $\mathcal{P}$-torsion free.

(d) Let $\mathcal{P} \in \mathbf{Spec}^{1,1}_{\mathfrak{t}}(X)$. Then $\mathcal{P} = \langle L \rangle$ for any nonzero object of $\mathcal{P}_{\mathfrak{t}} = \mathcal{P}^{\mathfrak{t}} \bigcap \mathcal{P}^{\perp}$.

Let L be a nonzero object of $\mathcal{P}_{\mathfrak{t}}$. Since L does not belong to the Serre subcategory $\langle L \rangle$, by (b3), we have the inclusion $\langle L \rangle \subseteq \mathcal{P}$. On the other hand, if $\langle L \rangle \subsetneqq \mathcal{P}$, then $L \in Ob\mathcal{P}$ which is not the case. Therefore $\mathcal{P} = \langle L \rangle$.

(e) The topologizing subcategory $[\mathcal{P}_{\mathfrak{t}}]$ coincides with the subcategory $[L]$ for any nonzero object L of $\mathcal{P}_{\mathfrak{t}}$.

Clearly $[L] \subseteq [\mathcal{P}_{\mathfrak{t}}]$ for any $L \in Ob\mathcal{P}_{\mathfrak{t}}$. By (b3), if $\mathcal{P}_{\mathfrak{t}} \nsubseteq [L]$, then $[L] \subseteq \mathcal{P}$, hence $L = 0$.

Since, by (c), every nonzero object of $\mathcal{P}_{\mathfrak{t}}$ belongs to $Spec(X)$, this shows that $[\mathcal{P}_{\mathfrak{t}}]$ is an element of $\mathbf{Spec}(X)$.

(f) It follows from the argument above that the map

$$\mathbf{Spec}(X) \longrightarrow \mathbf{Spec}^{1,1}_{\mathfrak{t}}(X), \quad \mathcal{Q} \longmapsto \widehat{\mathcal{Q}},$$

is inverse to the map $\mathbf{Spec}_{\mathfrak{t}}^{1,1}(X) \longrightarrow \mathbf{Spec}(X)$ which assigns to every $\mathcal{P}$ the topologizing subcategory $[\mathcal{P}_{\mathfrak{t}}]$. ■

3.3.2.1. Note. Let $\mathfrak{W}$ be a submonoid of the monoid $(\mathfrak{T}(X), \bullet)$ of topologizing subcategories of the category C_X which is closed under arbitrary intersections. Let $\tau_{\mathfrak{W}}^{1,1}$ denote the topology on $\mathbf{Spec}_{\mathfrak{t}}^{1,1}(X)$ induced by the topology $\tau_{\mathfrak{W}}^{-}$ on $\mathbf{Spec}^{-}(X)$ (cf. 3.2.2). Then the map $\mathbf{Spec}(X) \longrightarrow \mathbf{Spec}_{\mathfrak{t}}^{1,1}(X)$ of 3.3.2 is an isomorphism from the topological space $(\mathbf{Spec}(X), \tau_{\mathfrak{W}})$ (defined in 2.5.1) and $(\mathbf{Spec}_{\mathfrak{t}}^{1,1}(X), \tau_{\mathfrak{W}}^{1,1})$.

3.3.3. The difference between $\mathbf{Spec}_{\mathfrak{t}}^{1,1}(X)$ and $\mathbf{Spec}^{-}(X)$. Let C_X be the category $R-mod$ of left modules over a commutative associative unital ring R. If the ring R is noetherian, then the map which assigns to each prime ideal p of R the isomorphism class of the injective hull of the quotient module R/p is an isomorphism between the Gabriel spectrum of C_X (which is, in this case, naturally isomorphic to $\mathbf{Spec}^{-}(X)$) and the prime spectrum of the ring R [Gab, Ch.VI]. In this case, $\mathbf{Spec}_{\mathfrak{t}}^{1,1}(X) = \mathbf{Spec}^{-}(X)$, i.e. the map $\mathcal{Q} \longmapsto \widehat{\mathcal{Q}}$ is an isomorphism between $\mathbf{Spec}(X)$ and $\mathbf{Spec}^{-}(X)$.

If a commutative ring R is not noetherian, the spectrum $\mathbf{Spec}^{-}(X)$ might be much bigger than (the image of) the prime spectrum $Spec(R)$ of the ring R, while $\mathbf{Spec}(X)$ (hence $\mathbf{Spec}_{\mathfrak{t}}^{1,1}(X)$) is naturally isomorphic to $Spec(R)$: the isomorphism is given by the map which assigns to a prime ideal p the topologizing subcategory $[R/p]$; the inverse map assigns to every element $\mathcal{Q} = [M]$ of the spectrum $\mathbf{Spec}(X)$ the annihilator of the module M.

3.4. Zariski topology on the spectra of a scheme.

3.4.1. Proposition. *Let $X \xrightarrow{f} S$ be an affine morphism. Suppose that the category C_S has the property (sup) (cf. 1.4.5) and enough objects of finite type. Then the category C_X has enough objects of finite type.*

Proof. Let M be an arbitrary nontrivial object. Since the direct image functor f_* of f is conservative, the object f_* is nontrivial. By assumption, there is a nonzero morphism $L \longrightarrow f_*(M)$, where L is an object of finite type. But, then the adjoint morphism $f^*(L) \longrightarrow M$ is nonzero. It remains to show that the inverse image functor f^* maps objects of finite type to objects of finite type.

In fact, let $L \in ObC_S$ be an object of finite type, and let $\{N_i \mid i \in J\}$ be a filtered system of subobjects of $f^*(L)$ such that $sup_{i\in J} N_i \longrightarrow f^*(L)$ is an isomorphism. Since f_* is exact and has a right adjoint, it preserves monomorphisms and colimits; in particular, $sup_{i\in J} f_*(N_i) \longrightarrow f_*f^*(L)$ is an isomorphism. Since the object L is of finite type and the system of subobjects $\{f_*(N_i) \mid i \in J\}$ of $f_*f^*(L)$ is filtered, and the category C_S has the property (sup), the adjunction morphism $L \longrightarrow f_*f^*(L)$ factors through the morphism $f_*(N_i) \longrightarrow f_*f^*(L)$ for some $i \in J$. This implies that the identical morphism $f^*(L) \longrightarrow f^*(L)$ factors through the monomorphism $N_i \longrightarrow f^*(L)$; hence $N_i \longrightarrow f^*(L)$ is an isomorphism. ■

3.4.2. The Zariski topology on $\mathbf{Spec}^{-}(X)$. Let $X \xrightarrow{f} S$ be a scheme over S (see I.3.4.1). We assume that C_X and C_S are abelian categories, inverse image functors of f are additive, and the category C_S has the property (sup) and enough objects of finite type.

We call a subset $\mathcal{V}$ of the spectrum $\mathbf{Spec}^-(X)$ *Zariski open* if for any open immersion $U \xrightarrow{\mathfrak{u}} X$ (cf. I.3.5) which is affine over S (that is $f \circ \mathfrak{u}$ is affine), the intersection of (the image of) $\mathbf{Spec}^-(U)$ and the subset $\mathcal{V}$ is a Zariski open subset of $\mathbf{Spec}^-(U)$.

We use here the fact that, by 3.4.1, the category C_U has enough objects of finite type; so that Zariski topology on $\mathbf{Spec}^-(U)$ is well defined (in 2.4).

We denote the Zariski topology on $\mathbf{Spec}^-(X)$ by $\tau_{\mathfrak{z}}^-$.

A standard argument shows that a subset $\mathcal{V}$ of $\mathbf{Spec}^-(X)$ is open iff the intersection of $\mathcal{V}$ with $\mathbf{Spec}^-(U_i)$ is open for some affine cover $\{U_i \xrightarrow{\mathfrak{u}_i} X \mid i \in J\}$ of $X \xrightarrow{f} S$.

3.4.3. Zariski topology on other spectra. Zariski topology on the spectrum $\mathbf{Spec}_{\mathfrak{t}}^{1,1}(X)$ is induced by the Zariski topology on $\mathbf{Spec}^-(X)$ via the embedding

$$\mathbf{Spec}_{\mathfrak{t}}^{1,1}(X) \hookrightarrow \mathbf{Spec}^-(X).$$

The Zariski topology on $\mathbf{Spec}(X)$ is determined by the requirement that the canonical bijection $\mathbf{Spec}(X) \xrightarrow{\sim} \mathbf{Spec}_{\mathfrak{t}}^{1,1}(X)$ is a homeomorphism with respect to Zariski topologies.

4. 'Locality' theorems.

4.1. Proposition. *Let $\{\mathcal{T}_i \mid i \in J\}$ be a finite set of thick subcategories of an abelian category C_X such that $\bigcap_{i \in J} \mathcal{T}_i = 0$; and let u_i^* be the localization functor $C_X \longrightarrow C_X/\mathcal{T}_i$. The following conditions on a nonzero coreflective topologizing subcategory $\mathcal{Q}$ of C_X are equivalent:*

(a) $\mathcal{Q} \in \mathbf{Spec}(X)$,

(b) $[u_i^*(\mathcal{Q})] \in \mathbf{Spec}(X/\mathcal{T}_i)$ for every $i \in J$ such that $\mathcal{Q} \not\subseteq \mathcal{T}_i$.

Here $[u_i^*(\mathcal{Q})]$ denote the topologizing subcategory of $C_X/\mathcal{T}_i$ spanned by $u_i^*(\mathcal{Q})$.

Proof. The assertion follows from [R4, 9.6.1]. ■

4.1.1. Note. The condition (b) of 4.1 can be reformulated as follows:

(b') For any $i \in J$, either $u_i^*(\mathcal{Q}) = 0$, or $[u_i^*(\mathcal{Q})] \in \mathbf{Spec}(X/\mathcal{T}_i)$.

4.2. Proposition. *Let C_X be an abelian category and $\mathfrak{U} = \{U_i \xrightarrow{u_i} X \mid i \in J\}$ a finite set of continous morphisms such that $\{C_X \xrightarrow{u_i^*} C_{U_i} \mid i \in J\}$ is a conservative family of exact localizations.*

(a) The morphisms $U_{ij} = U_i \cap U_j \xrightarrow{u_{ij}} U_i$ are continuous for all $i, j \in J$.

(b) Let L_i be an object of $Spec(U_i)$; i.e. $[L_i]_{\mathfrak{c}} \in \mathbf{Spec}(U_i)$ and L_i is $\langle L_i \rangle$-torsion free. The following conditions are equivalent:

(i) $L_i \simeq u_i^*(L)$ for some $L \in Spec(X)$;

(ii) for any $j \in J$ such that $u_{ij}^*(L_i) \neq 0$, the object $u_{ji*}u_{ij}^*(L_i)$ of C_{U_j} has an associated point; i.e. it has a subobject L_{ij} which belongs to $Spec(U_j)$.

Proof. The assertion follows from 4.1 (see [R4, 9.7.1]). ■

4.3. Examples. (a) If C_X is the category of quasi-coherent sheaves on a quasi-compact quasi-separated scheme $\mathcal{X}$ and each U_i is the category of quasi-coherent sheaves on an open subscheme of $\mathcal{X}$, then the glueing conditions of 4.2 hold for any $L_i \in Spec(U_i)$; i.e. the spectrum $\mathbf{Spec}(X)$ is naturally identified with $\bigcup_{i\in J} \mathbf{Spec}(U_i)$.

(b) Similarly, if C_X is the category of holonomic D-modules over a sheaf of twisted differential operators on a smooth quasi-compact scheme $\mathcal{X}$, and $\{U_i \xrightarrow{u_i} X \mid i \in J\}$ is a cover of X corresponding to an open Zariski cover of $\mathcal{X}$, then $\mathbf{Spec}(X) = \bigcup_{i\in J} \mathbf{Spec}(U_i)$.

This is due to the functoriality of sheaves of holonomic modules with respect to direct and inverse image functors of open immersions and the fact that holonomic modules are of finite length (hence they have associated closed points).

4.4. Proposition. *Let C_X be an abelian category and $\mathfrak{U} = \{U_i \xrightarrow{u_i} X \mid i \in J\}$ a finite set of morphisms of 'spaces' whose inverse image functors, $\{C_X \xrightarrow{u_i^*} C_{U_i} \mid i \in J\}$, form a conservative family of exact localizations, and $Ker(u_i^*)$ is a coreflective subcategory for every $i \in J$. Then $\mathbf{Spec}^-(X) = \bigcup_{i\in J} \mathbf{Spec}^-(U_i)$.*

Proof. The equality is proven in [R4, 9.5]. ∎

5. Noncommutative k-schemes.

5.1. k-'Spaces'. Let k be an associative unital ring. A *k-'space'* is a continuous morphism $X \xrightarrow{f} \mathbf{Sp}(k)$, i.e. its inverse image functor, $k - mod \xrightarrow{f^*} C_X$ has a right adjoint, f_* – the direct image functor of f. The object $\mathcal{O} = f^*(k)$ plays the role of the structure sheaf on X. One can show that the object $\mathcal{O}$ together with the composition $k^o \xrightarrow{\varphi_f} C_X(\mathcal{O},\mathcal{O})$ of the isomorphism $k^o \xrightarrow{\sim} Hom_k(k,k)$ (where k^o is a ring opposite to k) and a ring morphism $Hom_k(k,k) \longrightarrow C_X(f^*(k), f^*(k)) = C_X(\mathcal{O},\mathcal{O})$ (due to the fact that f^* is an additive functor) determine the functor f^* uniquely up to isomorphism; hence the pair $(\mathcal{O}, k^o \xrightarrow{\varphi_f} C_X(\mathcal{O},\mathcal{O})$ determines the morphism $X \xrightarrow{f} \mathbf{Sp}(k)$.

In particular, every $\mathbb{Z}$-'space' $X \xrightarrow{f} \mathbf{Sp}(\mathbb{Z})$ is uniquely determined by the object $\mathcal{O} = f^*(\mathbb{Z})$. There is a bijective correspondence between isomorphism classes of objects $\mathcal{O}$ such that 'small' coproducts of copies of $\mathcal{O}$ exist and $\mathbb{Z}$-'space' structures $X \longrightarrow \mathbf{Sp}(\mathbb{Z})$.

5.2. Affine k-'spaces'. The k-space $X \xrightarrow{f} \mathbf{Sp}(k)$ is *affine* if f is affine, i.e. the direct image functor f_* is conservative (– reflects isomorphisms) and has a right adjoint, $f^!$. This means, precisely, that the object $\mathcal{O}$ is a projective generator of finite type.

5.3. Noncommutative k-schemes. Consider a k-'space' $(X, X \xrightarrow{f} \mathbf{Sp}(k))$ for which there exists a family $\{\mathcal{T}_i \mid i \in J\}$ of Serre subcategories of C_X such that $\bigcap_{i\in J} \mathcal{T}_i = 0$ and the compositions of $X/\mathcal{T}_i \longrightarrow X$ and $X \xrightarrow{f} \mathbf{Sp}(k)$ are affine for all $i \in J$; i.e. (X, f) is a *weak scheme* in the sense of I.2.2.

This weak scheme is a *scheme*, if for every $i \in J$, the set

$$\mathcal{V}^-(\mathcal{T}_i) \stackrel{\text{def}}{=} \{\mathcal{P} \in \mathbf{Spec}^-(X) \mid \mathcal{T}_i \not\subseteq \mathcal{P}\}$$

is closed in *Zariski* topology.

6. Zariski geometric center and reconstruction of schemes.

6.1. The center of a category and localizations. Recall that the *center*, $\mathfrak{z}(C_Y)$, of a svelte additive category C_Y is the ring of endomorphisms of its identical functor. If C_Y is a category of left modules over a ring R, then the center of C_Y is naturally isomorphic to the center of the ring R.

6.1.1. Proposition. *Let C_X be an abelian category and τ a topology on $\mathbf{Spec}(X)$. The map $\widetilde{\mathcal{O}}_{X,\tau}$ which assigns to every open subset W of $\mathbf{Spec}(X)$ the center of the quotient category $C_X/\mathcal{S}_W$, where $\mathcal{S}_W = \bigcap_{\mathcal{Q}\in W} \widehat{\mathcal{Q}}$ is a presheaf of commutative rings on $(\mathbf{Spec}(X), \tau)$.*

Proof. This follows from a general (and easily verified) fact that the map which assigns to a svelte category its center is functorial with respect to localization functors. ■

6.2. Zariski geometric center. Given a topology τ on $\mathbf{Spec}(X)$, we denote by $\mathcal{O}_{X,\tau}$ the sheaf associated with the presheaf $\widetilde{\mathcal{O}}_{X,\tau}$. The ringed space $((\mathbf{Spec}(X), \tau), \mathcal{O}_{X,\tau})$ is called the *geometric center* of (X, τ). If τ is the Zariski topology, then we write simply $(\mathbf{Spec}(X), \mathcal{O}_X)$ and call this ringed space the *Zariski geometric center* of X.

6.2.1. Proposition. *Suppose that the category C_X has enough objects of finite type. Then the Zariski geometric center of X is a locally ringed topological space.*

Proof. Under the conditions, one can show that the stalk of the sheaf $\mathcal{O}_X$ at a point $\mathcal{Q}$ of the spectrum is isomorphic to the center of the local category $C_X/\widehat{\mathcal{Q}}$. On the other hand, the center of any local category (in particular, the center of $C_X/\widehat{\mathcal{Q}}$) is a local ring (see [R, Ch. III]). ■

6.3. Commutative schemes which can be reconstructed from their categories of quasi-coherent or coherent sheaves. Let $\mathbf{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a ringed topological space and $\mathbf{U} = (\mathcal{U}, \mathcal{O}_{\mathcal{U}}) \xrightarrow{\mathfrak{j}} (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ an open immersion. Then the morphism $\mathfrak{j}$ has an exact inverse image functor $\mathfrak{j}^*$ and a fully faithful direct image functor $\mathfrak{j}_*$. This implies that $Ker(\mathfrak{j}^*)$ is a Serre subcategory of the category $\mathcal{O}_X - \mathcal{M}od$ of sheaves of $\mathcal{O}_X$-modules and the unique functor

$$\mathcal{O}_X - \mathcal{M}od/Ker(\mathfrak{j}^*) \longrightarrow \mathcal{O}_{\mathcal{U}} - \mathcal{M}od$$

induced by $\mathfrak{j}^*$ is an equivalence of categories [Gab, III.5].

Suppose now that $\mathbf{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is a scheme and $Qcoh_{\mathbf{X}}$ the category of quasi-coherent sheaves on $\mathbf{X}$. The inverse image functor $\mathfrak{j}^*$ of the immersion $\mathfrak{j}$ maps quasi-coherent sheaves to quasi-coherent sheaves. Let u^* denote the functor $Qcoh_{\mathbf{X}} \longrightarrow Qcoh_{\mathbf{U}}$ induced by $\mathfrak{j}^*$. The funtor u^*, being the composition of the exact full embedding of $Qcoh_{\mathbf{X}}$ into $\mathcal{O}_X - \mathcal{M}od$ and the exact functor $\mathfrak{j}^*$, is exact; hence it is represented as the composition of an exact localization $Qcoh_{\mathbf{X}} \longrightarrow Qcoh_{\mathbf{X}}/Ker(u^*)$ and a uniquely defined exact functor $Qcoh_{\mathbf{X}}/Ker(u^*) \longrightarrow Qcoh_{\mathbf{U}}$. If the direct image functor $\mathfrak{j}_*$ of the immersion $\mathfrak{j}$ maps quasi-coherent sheaves to quasi-coherent sheaves, then it induces a fully faithful functor

$Qcoh_{\mathbf{U}} \xrightarrow{u_*} Qcoh_{\mathbf{X}}$ which is a right adjoint to u^*. In particular, the canonical functor $Qcoh_{\mathbf{X}}/Ker(u^*) \longrightarrow Qcoh_{\mathbf{U}}$ is an equivalence of categories.

The reconstruction of a scheme $\mathbf{X}$ from the category $Qcoh_{\mathbf{X}}$ of quasi-coherent sheaves on $\mathbf{X}$ is based on the existence of an affine cover $\{\mathbf{U}_i \xrightarrow{u_i} \mathbf{X} \mid i \in J\}$ such that the canonical functors $Qcoh_{\mathbf{X}}/Ker(u_i^*) \longrightarrow Qcoh_{\mathbf{U}_i}$, $i \in J$, are category equivalences. It follows from the discussion above (or from [GZ, I.2.5.2]) that this is garanteed if the inverse image functor $Qcoh_{\mathbf{X}} \xrightarrow{u_i} Qcoh_{\mathbf{U}_i}$ has a fully faithful right adjoint.

On the other hand, one can deduce from 4.1 (and the equality $\bigcap_{i\in\mathfrak{J}} \mathcal{T}_i^- = \big(\bigcap_{i\in\mathfrak{J}} \mathcal{T}_i\big)^-$ for any finite set $\{\mathcal{T}_i \mid i \in \mathfrak{J}\}$ of topologizing subcategories of C_X; see [R4, 4.1]) that if there exists an affine cover $\{\mathbf{U}_i \xrightarrow{u_i} \mathbf{X} \mid i \in J\}$ such that the canonical functors

$$Qcoh_{\mathbf{X}}/Ker(u_i^*) \longrightarrow Qcoh_{\mathbf{U}_i},\ i \in J,$$

are category equivalences, then $Ker(u_i^*)$ is a Serre subcategory for all $i \in J$, which implies (in these circumstances) the existence of a right adjoint u_{i_*} to u_i^* for each $i \in J$.

6.4. Proposition. *Let* $\mathbf{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ *be a quasi-compact quasi-separated commutative scheme. Then the scheme* $\mathbf{X}$ *is isomorphic to the Zariski geometric center* $(\mathbf{Spec}(X), \mathcal{O}_X)$ *of the 'space'* X *represented by the category* $Qcoh_{\mathbf{X}}$ *of quasi-coherent sheaves on* $\mathbf{X}$.

Proof. The assertion follows from the 'locality' theorem 4.1. See details in [R4, 9.8]. ■

6.4.1. Note. If the C_X is the category of quasi-coherent sheaves on a quasi-compact quasi-separated scheme, then Zariski topology on $\mathbf{Spec}(X)$ coincides with the *coarse Zariski topology* (see 2.5.4).

6.5. The reduced geometric center of a 'space'. Fix a 'space' X and a topology τ on $\mathbf{Spec}(X)$. The map which assigns to each open subset $\mathcal{U}$ of $\mathbf{Spec}(X)$ the prime spectrum of the ring $\mathcal{O}_{X,\tau}(\mathcal{U})$ of global sections of the sheaf $\mathcal{O}_{X,\tau}$ over $\mathcal{U}$ is a functor from the category of $Open(\tau)$ open sets of the topology τ to the category of topological spaces. We denote its colimit by $Spec(\mathcal{O}_{X,\tau})$.

6.5.1. Proposition. *There is a canonical morphism*

$$(\mathbf{Spec}(X), \tau) \xrightarrow{\mathfrak{p}_X} Spec(\mathcal{O}_{X,\tau}) \tag{1}$$

of topological spaces.

Proof. Fix an element $\mathcal{Q}$ of $\mathbf{Spec}(X)$. For every open subset $\mathcal{U}$ of $\mathbf{Spec}(X)$ containing $\mathcal{Q}$, there is a ring morphism from $\mathcal{O}_{X,\tau}(\mathcal{U})$ to the center $\mathfrak{z}(C_X/\widehat{\mathcal{Q}})$ of the local category $C_X/\widehat{\mathcal{Q}}$. The ring $\mathfrak{z}(C_X/\widehat{\mathcal{Q}})$ is local [R, Ch.III]. The preimage of its unique maximal ideal in $\mathcal{O}_{X,\tau}(\mathcal{U})$ is a prime ideal of $\mathcal{O}_{X,\tau}(\mathcal{U})$. The image of this prime ideal in $Spec(\mathcal{O}_{X,\tau})$ does not depend on the choice of $\mathcal{U}$. This defines the map (1). ■

The map (1) is rarely injective. We denote by $\mathcal{O}^{\mathfrak{r}}_{X,\tau}$ the direct image $\mathfrak{p}_{X*}(\mathcal{O}_{X,\tau})$ of the sheaf $\mathcal{O}_{X,\tau}$ and call the ringed topological space $(Spec(\mathcal{O}_{X,\tau}), \mathcal{O}^{\mathfrak{r}}_{X,\tau})$ the *reduced geometric* *center* of (X,τ). If τ is the Zariski topology, then we call the reduced geometric center of (X,τ) simply the *reduced geometric center* of X.

6.6. The reduced geometric center of a noncommutative scheme. One can show that the noncommutative scheme structure on $X \xrightarrow{f} \mathbf{Sp}(k)$ induces a scheme structure on the reduced geometric center of X. If C_X is the category of quasi-coherent sheaves on a commutative quasi-compact, quasi-separated scheme $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$, then the reduced geometric center is naturally isomorphic to the geometric center of X. In particular, it is isomorphic to the scheme.

6.7. Complement: a geometric realization of an abelian category. Let C_X be an abelian category. We have a contravariant pseudo-functor from the category of the Zariski open sets of the spectrum $\mathbf{Spec}(X)$ to *Cat* which assigns to each open set $\mathcal{U}$ of $\mathbf{Spec}(X)$ the quotient category $C_X/\mathcal{S}_{\mathcal{U}}$, where $\mathcal{S}_{\mathcal{U}} = \bigcap_{\mathcal{Q}\in\mathcal{U}} \widehat{\mathcal{Q}}$, and to each embedding $\mathcal{U} \hookrightarrow \mathcal{V}$ the corresponding localization functor. To this psedo-functor, there corresponds (by a standard formalism) a fibered category over the Zariski topology of $\mathbf{Spec}(X)$. The associated stack, $\mathfrak{F}^{\mathfrak{z}}_X$, is a stack of local categories: its stalk at each point $\mathcal{Q}$ of $\mathbf{Spec}(X)$ is equivalent to the local category $C_X/\widehat{\mathcal{Q}}$.

We regard the stack $\mathfrak{F}^{\mathfrak{z}}_X$ as a *geometric realization* of the abelian category C_X.

If X is a (noncommutative) scheme, then the stack $\mathfrak{F}^{\mathfrak{z}}_X$ is *locally affine.*

6.7.1. Note. Taking the center of each fiber of the stack $\mathfrak{F}^{\mathfrak{z}}_X$, we recover the presheaf of commutative rings $\widetilde{\mathcal{O}}_X$, hence the geometric center of the 'space' X.

7. The spectrum $\mathbf{Spec}^0_{\mathfrak{c}}(X)$ and "big" schemes.

7.1. The spectrum $\mathbf{Spec}^0_{\mathfrak{c}}(X)$. If C_X is the category of quasi-coherent sheaves on a non-quasi-compact scheme, like, for instance, the flag variety of a Kac-Moody Lie algebra, or a noncommutative scheme which does not have a finite affine cover (say, the quantum flag variety of a Kac-Moody Lie algebra, or the corresponding quantum D-scheme), then the spectrum $\mathbf{Spec}(X)$ is not sufficient. It should be replaced by the spectrum $\mathbf{Spec}^0_{\mathfrak{c}}(X)$ whose elements are coreflective topologizing subcategories of C_X of the form $[M]_{\mathfrak{c}}$ (i.e. generated by the object M) such that if L is a nonzero subobject of M, then $[L]_{\mathfrak{c}} = [M]_{\mathfrak{c}}$.

There is a natural map $\mathbf{Spec}(X) \longrightarrow \mathbf{Spec}^0_{\mathfrak{c}}(X)$ which assigns to every $\mathcal{Q} \in \mathbf{Spec}(X)$ the smallest coreflective subactegory $[\mathcal{Q}]_{\mathfrak{c}}$ containing $\mathcal{Q}$.

If the category C_X has enough objects of finite type, this canonical map is a bijection.

7.2. Supports, topologies. The *support* of an object M of the category C_X in the spectrum $\mathbf{Spec}^0_{\mathfrak{c}}(X)$ is the set $Supp_{\mathfrak{c}}(M)$ of all $\mathcal{Q} \in \mathbf{Spec}^0_{\mathfrak{c}}(X)$ which are contained in the coreflective subcategory $[M]_{\mathfrak{c}}$ generated by M.

The topologies on $\mathbf{Spec}^0_{\mathfrak{c}}(X)$ are defined via the same pattern as the topologies on $\mathbf{Spec}(X)$ – either closed sets obtained as supports of certain family of objects, or as supports of a family of topologizing subcategories (cf. 2.3 and 2.5).

7.2.1. The Zariski topology and the coarse Zariski topology. In particular, the coarse *Zariski* topology on $\mathbf{Spec}^0_{\mathfrak{c}}(X)$ is defined similarly to the coarse Zariski topology on $\mathbf{Spec}(X)$, using the monoid (under the Gabriel multiplication) of *bireflective* (that is reflective and coreflective) topologizing subcategories. The Zariski topology on $\mathbf{Spec}^0_{\mathfrak{c}}(X)$ is defined the same way as the Zariski topology on $\mathbf{Spec}(X)$ under condition that X is a scheme over a 'space' S and the category C_S has property (sup) and enough objects of finite type (see 3.4).

7.3. The locality theorem. The 'locality' theorem for the spectrum $\mathbf{Spec}^0_{\mathfrak{c}}(X)$ is as follows:

7.3.1. Proposition. *Let $\{\mathcal{T}_i \mid i \in J\}$ be a set of coreflective thick subcategories of an abelian category C_X such that $\bigcap_{i\in J} \mathcal{T}_i = 0$; and let u_i^* denote the localization functor $C_X \longrightarrow C_X/\mathcal{T}_i$. The following conditions on a nonzero coreflective topologizing subcategory $\mathcal{Q}$ of C_X are equivalent:*

(a) $\mathcal{Q} \in \mathbf{Spec}^0_{\mathfrak{c}}(X)$,

(b) $[u_i^*(\mathcal{Q})]_{\mathfrak{c}} \in \mathbf{Spec}^0_{\mathfrak{c}}(X/\mathcal{T}_i)$ for every $i \in J$ such that $\mathcal{Q} \not\subseteq \mathcal{T}_i$.

Proof. See [R4, 10.4.3]. ■

7.4. The reconstruction of commutative schemes. The reconstruction theorem for non-quasi-compact commutative schemes looks as follows.

7.4.1. Proposition. *Let C_X be the category of quasi-coherent sheaves on a commutative scheme $\mathbf{X} = (\mathcal{X}, \mathcal{O})$. Suppose that there is an affine cover $\{\mathcal{U}_i \hookrightarrow \mathcal{X} \mid i \in J\}$ of the scheme $\mathbf{X}$ such that all immersions $\mathcal{U}_i \hookrightarrow \mathcal{X}$, $i \in J$, have a direct image functor. Then the geometric center $(\mathbf{Spec}^0_{\mathfrak{c}}(X), \mathcal{O}_X)$ corresponding to the (coarse) Zariski topology is isomorphic to the scheme $\mathbf{X}$.*

Proof. See [R4, 10.7.1]. ■

7.4.2. Note. If the $\mathbf{X} = (\mathcal{X}, \mathcal{O})$ is a quasi-compact and quasi-separated scheme, then the category C_X of quasi-coherent sheaves on $\mathbf{X}$ has enough objects of finite type, hence the spectrum $\mathbf{Spec}^0_{\mathfrak{c}}(X)$ coincides with $\mathbf{Spec}(X)$. Thus, the reconstruction theorem 6.4 is a special case of 7.4.1.

Lecture 3. Pseudo-geometry and geometry of 'spaces' represented by triangulated categories.

This lecture is sketches the beginning of one of the simplest forms of derived noncommutative geometry. Here 'spaces' are represented by svelte triangulated categories (we call them *t-'spaces'*) and morphisms by isomorphism classes of *triangle* functors. We start with pseudo-geometry following pattern of Lecture 1, that is we consider *continuous* morphisms and look for a triangulated version of Beck's theorem (which plays a central role for studying 'spaces' represented by ordinary categories, incorporating both affine schemes and, in the dual context, descent theory). The triangulated picture, turns to be much easier: the triangulated version of Beck's theorem on descent side states that every continuous morphism is the composition of a comonadic morphism and a continuous localization. In particular, any *faithfully flat* (in triangle sense) morphism is comonadic.

The geometric picture looks even better. There are two spectra, $\mathbf{Spec}^{1,1}_{\mathfrak{L}}(\mathfrak{X})$ and $\mathbf{Spec}^{1/2}_{\mathfrak{L}}(\mathfrak{X})$ which are triangulated analogs of the spectra respectively $\mathbf{Spec}^{1,1}_{\mathfrak{t}}(\mathfrak{X})$ and $\mathbf{Spec}(\mathfrak{X})$. There is a natural bijective map $\mathbf{Spec}^{1/2}_{\mathfrak{L}}(\mathfrak{X}) \xrightarrow{\sim} \mathbf{Spec}^{1,1}_{\mathfrak{L}}(\mathfrak{X})$. But, unlike the bijection $\mathbf{Spec}(\mathfrak{X}) \xrightarrow{\sim} \mathbf{Spec}^{1,1}_{\mathfrak{t}}(\mathfrak{X})$ of II.3.3.2, this map does not preserve the specialization preorder $\supseteq$. Notice that the specialization preorder on $\mathbf{Spec}^{1,1}_{\mathfrak{L}}(\mathfrak{X})$ is what we expect from specialization. So that the preorder $(\mathbf{Spec}^{1,1}_{\mathfrak{L}}(\mathfrak{X}), \supseteq)$ is regarded as the "principal" spectrum of the t-'space' $\mathfrak{X}$. On the other hand, the points of the spectrum $\mathbf{Spec}^{1/2}_{\mathfrak{L}}(\mathfrak{X})$ are closed with respect to the topology determined by the specialization preorder, or a natural version of Zariski topology on $\mathbf{Spec}^{1/2}_{\mathfrak{L}}(\mathfrak{X})$. This gives certain technical advantages (not used here) and curious interpretations.

1. 'Spaces' represented by triangulated categories. Recall that a $\mathbb{Z}$-*category* is a category endowed with an action of $\mathbb{Z}$, where $\mathbb{Z}$ is regarded as a monoidal category: objects are elements and the tensor product is given by addition. In other words, a $\mathbb{Z}$-category is a category $C_{\mathfrak{X}}$ with an auto-equivalence $\theta_{\mathfrak{X}}$ and an associativity isomorphism $\theta_{\mathfrak{X}} \circ (\theta_{\mathfrak{X}} \circ \theta_{\mathfrak{X}}) \xrightarrow{\sim} (\theta_{\mathfrak{X}} \circ \theta_{\mathfrak{X}}) \circ \theta_{\mathfrak{X}}$ satisfying the usual cocycle conditions.

1.1. The category of triangulated categories. Triangulated k-linear categories are triples $(C_{\mathfrak{X}}, \theta_{\mathfrak{X}}; \mathfrak{Tr}_{\mathfrak{X}})$, where $(C_{\mathfrak{X}}, \theta_{\mathfrak{X}})$ is an additive k-linear $\mathbb{Z}$-category, and $\mathfrak{Tr}_{\mathfrak{X}}$ a full subcategory of the category of diagrams of the form

$$\mathcal{L} \longrightarrow \mathcal{M} \longrightarrow \mathcal{N} \longrightarrow \theta_{\mathfrak{X}}(\mathcal{L}).$$

The objects of the subcategory $\mathfrak{Tr}_{\mathfrak{X}}$ are called *triangles.* They satisfy to well known axioms due to Verdier [Ve1]. We denote a triangulated category $(C_{\mathfrak{X}}, \theta_{\mathfrak{X}}; \mathfrak{Tr}_{\mathfrak{X}})$ by $\mathcal{CT}_{\mathfrak{X}}$.

A *triangle* k-linear functor from a triangulated k-linear category $\mathcal{CT}_{\mathfrak{X}} = (C_{\mathfrak{X}}, \theta_{\mathfrak{X}}; \mathfrak{Tr}_{\mathfrak{X}})$ to a triangulated k-linear category $\mathcal{CT}_{\mathfrak{Y}} = (C_{\mathfrak{Y}}, \theta_{\mathfrak{Y}}; \mathfrak{Tr}_{\mathfrak{Y}})$ is a pair (F, ϕ), where F is a k-linear functor $C_{\mathfrak{X}} \longrightarrow C_{\mathfrak{Y}}$ and ϕ a functor isomorphism $\theta_{\mathfrak{Y}} \circ F \xrightarrow{\sim} F \circ \theta_{\mathfrak{X}}$ such that for any triangle $\mathcal{L} \longrightarrow \mathcal{M} \longrightarrow \mathcal{N} \longrightarrow \theta_{\mathfrak{X}}(\mathcal{L})$ of $\mathcal{CT}_{\mathfrak{X}}$, the diagram

$$F(\mathcal{L}) \longrightarrow F(\mathcal{M}) \longrightarrow F(\mathcal{N}) \longrightarrow \theta_{\mathfrak{Y}}(F(\mathcal{L})),$$

where $F(\mathcal{N}) \longrightarrow \theta_{\mathfrak{Y}}(F(\mathcal{L}))$ is the composition of $F(\mathcal{N} \longrightarrow \theta_{\mathfrak{X}}(\mathcal{L}))$ and the isomorphism $F\theta_{\mathfrak{X}}(\mathcal{L}) \xrightarrow{\phi(\mathcal{L})} \theta_{\mathfrak{Y}}(F(\mathcal{L}))$, is a triangle of the triangulated category $\mathcal{C}\mathfrak{T}_{\mathfrak{Y}}$.

We denote by $\mathfrak{TrCat}_k$ the category whose objects are svelte triangulated categories and morphisms are triangle functors between them.

1.2. The category of t-'spaces'. If $\mathcal{CT}_{\mathfrak{X}} = (C_{\mathfrak{X}}, \theta_{\mathfrak{X}}; \mathfrak{Tr}_{\mathfrak{X}})$ is a svelte Karoubian (that is the category $C_{\mathfrak{X}}$ is Karoubian) k-linear triangulated category, we say that it represents a *t-'space'* $\mathfrak{X}$. A morphism $\mathfrak{X} \xrightarrow{\mathfrak{f}} \mathfrak{Y}$ from a t-'space' $\mathfrak{X}$ to a t-'space' $\mathfrak{Y}$ is an isomorphism class of triangle functors from $\mathcal{CT}_{\mathfrak{Y}}$ to $\mathcal{CT}_{\mathfrak{X}}$. A representative of a morphism $\mathfrak{f}$ will be called an *inverse image* functor of $\mathfrak{f}$ and denoted, usually, by $\mathfrak{f}^*$. The composition $f \circ g$ is, by definition, the isomorphism class of the composition $g^* \circ f^*$ of inverse image functors of respectively g and f. This defines the category $\mathfrak{Esp}_{\mathfrak{Tr}}$ of t-'spaces'.

2. Triangulated categories and Frobenius $\mathbb{Z}$-categories. We need some facts about *abelianization* of triangulated categories which are discussed in more general setting and in bigger detail in Lecture 4.

For any k-linear category $C_{\mathfrak{X}}$, we denote by $\mathcal{M}_k(\mathfrak{X})$ the abelian category of presheaves of k-modules on $C_{\mathfrak{X}}$ and by $C_{\mathfrak{X}_\mathfrak{a}}$ the full subcategory of $\mathcal{M}_k(\mathfrak{X})$ generated by all presheaves of k-modules which a left resolution formed by representable presheaves. Since $C_{\mathfrak{X}_\mathfrak{a}}$ contains all representable presheaves, the Yoneda functor $C_{\mathfrak{X}} \xrightarrow{\mathfrak{h}_{\mathfrak{X}}} \mathcal{M}_k(\mathfrak{X})$ factors through the embedding $C_{\mathfrak{X}_\mathfrak{a}} \longrightarrow \mathcal{M}_k(\mathfrak{X})$. We denote the corestriction $C_{\mathfrak{X}} \longrightarrow C_{\mathfrak{X}_\mathfrak{a}}$ of the Yoneda functor by $\mathfrak{H}_{\mathfrak{X}}$. Every k-linear functor $C_X \xrightarrow{F} C_Y$ induces a right exact functor $C_{X_\mathfrak{a}} \xrightarrow{F_\mathfrak{a}} C_{Y_\mathfrak{a}}$

$$\begin{array}{ccc} C_X & \xrightarrow{F} & C_Y \\ {\scriptstyle h_X}\big\downarrow & & \big\downarrow{\scriptstyle h_Y} \\ C_{X_\mathfrak{a}} & \xrightarrow{F_\mathfrak{a}} & C_{Y_\mathfrak{a}} \end{array} \qquad (1)$$

commutes. The functor $F_\mathfrak{a}$ is determined uniquely up to isomorphism.

If $C_{\mathfrak{X}}$ is a $\mathbb{Z}$-category, then the categories $\mathcal{M}_k(\mathfrak{X})$ and $C_{\mathfrak{X}_\mathfrak{a}}$ inherit a $\mathbb{Z}$-action such that the functors $\mathfrak{h}_{\mathfrak{X}}$ and $\mathfrak{H}_{\mathfrak{X}}$ become $\mathbb{Z}$-functors. It follows that for every $\mathbb{Z}$-functor $C_X \xrightarrow{F} C_Y$, the functor $C_{X_\mathfrak{a}} \xrightarrow{F_\mathfrak{a}} C_{Y_\mathfrak{a}}$ is a $\mathbb{Z}$-functor.

2.1. Frobenius abelian $\mathbb{Z}$-categories. An exact k-linear $\mathbb{Z}$-category is called a *Frobenius* category if it has enough projectives and injectives and its projectives and injectives coincide. In this lecture, we are interested only in abelian Frobenius categories. We denote by $\mathfrak{F}_{\mathbb{Z}}\mathfrak{Cat}_k$ the category whose objects are svelte Frobenius k-linear abelian $\mathbb{Z}$-categories and morphisms are exact k-linear functors which map projectives to projectives.

2.2. Theorem. *(a) For any triangulated k-linear category $\mathcal{C}\mathfrak{T}_{\mathfrak{X}} = (C_{\mathfrak{X}}, \theta_{\mathfrak{X}}; \mathfrak{Tr}_{\mathfrak{X}})$, the category $C_{\mathfrak{X}_\mathfrak{a}}$ is a Frobenius abelian k-linear $\mathbb{Z}$-category. If the category $C_{\mathfrak{X}}$ is Karoubian, then the canonical functor $C_{\mathfrak{X}} \xrightarrow{\mathfrak{H}_{\mathfrak{X}}} C_{\mathfrak{X}_\mathfrak{a}}$ induces an equivalence between the category $C_{\mathfrak{X}}$ and the full subcategory of $C_{\mathfrak{X}_\mathfrak{a}}$ formed by its projectives.*

(b) The correspondence $\mathcal{C}\mathfrak{T}_{\mathfrak{X}} \longmapsto C_{\mathfrak{X}_\mathfrak{a}}$ extends to a fully faithful functor from the category $\mathfrak{TrCat}_k$ to the category $\mathfrak{F}_{\mathbb{Z}}\mathfrak{Cat}_k$.

Proof. The assertion follows from Proposition IV.8.7.4. It is equivalent to a part of Theorem 3.2.1 in [Ve2] (see Remark IV.8.7.5). ∎

3. Localizations, continuous morphisms, and (co)monadic morphisms.

3.1. Localizations. Let $\mathfrak{X} \xrightarrow{\mathfrak{f}} \mathfrak{Y}$ be a morphism of t-'spaces'. Its inverse image functor $C_{\mathfrak{Y}} \xrightarrow{\mathfrak{f}^*} C_{\mathfrak{X}}$ is a composition of the localization at the thick subcategory $Ker(\mathfrak{f}^*)$ and a faithful triangle functor. In other words, we have a canonical decomposition $\mathfrak{f} = \mathfrak{p}_{\mathfrak{f}} \circ \mathfrak{f}_{\mathfrak{c}}$, where $\mathfrak{p}_{\mathfrak{f}}^*$ is the localization functor $C_{\mathfrak{Y}} \longrightarrow C_{\mathfrak{Y}}/Ker(\mathfrak{f}^*)$ and $\mathfrak{f}_{\mathfrak{c}}^*$ is a faithful triangle functor determined (uniquely once $\mathfrak{f}^*$ is fixed, hence) uniquely up to isomorphism.

We call a morphism of t-'spaces' $\mathfrak{X} \xrightarrow{\mathfrak{f}} \mathfrak{Y}$ a *localization* if $\mathfrak{f}_{\mathfrak{c}}$ is an isomorphism, or, equivalently, if its inverse image functor is a category equivalence.

3.2. Continuous morphisms. We call a morphism $\mathfrak{X} \xrightarrow{\mathfrak{f}} \mathfrak{Y}$ *continuous* if its inverse image functor, $\mathfrak{f}^*$ has a right adjoint, $\mathfrak{f}_*$, and this right adjoint is a triangle functor.

3.3. Monads and comonads in triangulated categories. Let $\mathfrak{T}\mathcal{C}_{\mathfrak{X}} = (\mathcal{C}_{\mathfrak{X}}, \theta_{\mathfrak{X}}, \mathfrak{Tr}_{\mathfrak{X}})$ be a triangulated category. A monad on $\mathfrak{T}\mathcal{C}_{\mathfrak{X}}$ (or a monad on the corresponding t-'space' $\mathfrak{X}$) is a monad $\mathcal{F} = (F, \mu)$ on the category $\mathcal{C}_{\mathfrak{X}}$ such that F is a triangle functor and $F^2 \xrightarrow{\mu} F$ is a morphism of triangle functors. Dually, a comonad on $\mathfrak{T}\mathcal{C}_{\mathfrak{X}}$ (or $\mathfrak{X}$) is a comonad $\mathcal{G} = (G, \delta)$ such that G is a triangle functor on $\mathfrak{T}\mathcal{C}_{\mathfrak{X}}$ and $G \xrightarrow{\delta} G^2$ is a morphism of triangle functors.

The category $(\mathcal{F}/\mathfrak{X}) - mod$ of $\mathcal{F}$-modules has a structure of triangulated category induced by the forgetful functor $(\mathcal{F}/\mathfrak{X}) - mod \xrightarrow{\mathfrak{f}_*} \mathcal{C}_{\mathfrak{X}}$.

The following assertion is the triangulated version of Beck's theorem.

3.4. Proposition. *Let $\mathfrak{T}\mathcal{C}_{\mathfrak{X}}$ and $\mathfrak{T}\mathcal{C}_{\mathfrak{Y}}$ be Karoubian triangulated categories and*

$$\mathfrak{T}\mathcal{C}_{\mathfrak{Y}} \xrightarrow{f^*} \mathfrak{T}\mathcal{C}_{\mathfrak{X}} \xrightarrow{f_*} \mathfrak{T}\mathcal{C}_{Y}$$

a pair of adjoint triangle functors with adjunction morphisms

$$f^*f_* \xrightarrow{\epsilon_f} Id_{\mathfrak{T}\mathcal{C}_{\mathfrak{X}}} \quad and \quad Id_{\mathfrak{T}\mathcal{C}_{\mathfrak{Y}}} \xrightarrow{\eta_f} f_*f^*.$$

(a) The canonical functor

$$\mathfrak{T}\mathcal{C}_{\mathfrak{Y}} \xrightarrow{\widetilde{\mathfrak{f}^*}} \mathcal{G}_f - Comod = (\mathfrak{X}\backslash\mathcal{G}_f) - Comod, \quad M \longmapsto (f^*(M), f^*\eta_f(M)), \qquad (1)$$

is a localization functor. It is a category equivalence iff the functor f^* is faithful.

(b) Dually, the canonical functor

$$\mathfrak{T}\mathcal{C}_{\mathfrak{X}} \xrightarrow{\widetilde{\mathfrak{f}_*}} \mathcal{F}_f - mod = (\mathcal{F}_f/\mathfrak{Y}) - mod, \quad L \longmapsto (f_*(L), f_*\epsilon_f(L)), \qquad (2)$$

is a localization. It is a category equivalence iff the functor f_* is faithful.

Here $\mathcal{G}_f = (G_f, \delta_f) = (f^* f_*, f^* \eta_f f_*)$ and $\mathcal{F}_f = (F_f, \mu_f) = (f_* f^*, f_* \epsilon_f f^*)$ are respectively the comonad and the monad associated with the pair of adjoint functors f^*, f_*.

Proof. It suffices to prove (a), because the two assertions are dual to each other.

Let $\mathcal{C}_{\mathfrak{X}_\mathfrak{a}}$ denote the *abelianization* of the triangulated category $\mathfrak{T}\mathcal{C}_{\mathfrak{X}}$. Any triangle functor $\mathfrak{T}\mathcal{C}_{\mathfrak{X}} \xrightarrow{F} \mathfrak{T}\mathcal{C}_{\mathfrak{Y}}$ gives rise to an exact $\mathbb{Z}$-functor $\mathcal{C}_{\mathfrak{X}_\mathfrak{a}} \xrightarrow{F_\mathfrak{a}} \mathcal{C}_{\mathfrak{Y}_\mathfrak{a}}$ between the corresponding abelian $\mathbb{Z}$-categories which maps injective objects to injective objects. In particular, we have a pair of adjoint exact $\mathbb{Z}$-functors

$$\mathcal{C}_{\mathfrak{Y}_\mathfrak{a}} \xrightarrow{f_\mathfrak{a}^*} \mathcal{C}_{\mathfrak{X}_\mathfrak{a}} \xrightarrow{f_{\mathfrak{a}*}} \mathcal{C}_{\mathfrak{Y}_\mathfrak{a}}$$

which map injectives to injectives. Thus, we have the canonical functor

$$\mathcal{C}_{\mathfrak{Y}_\mathfrak{a}} \xrightarrow{\widetilde{\mathfrak{f}_\mathfrak{a}^*}} (\mathfrak{X}_\mathfrak{a} \backslash \mathcal{G}_{f_\mathfrak{a}}) - Comod.$$

Since both adjoint functors, $f_\mathfrak{a}^*$ and $f_{\mathfrak{a}*}$ are exact functors between abelian categories, it follows from Beck's theorem that $\widetilde{\mathfrak{f}_\mathfrak{a}^*}$ is a localization functor. If the functor f^* is faithful, then the functor $f_\mathfrak{a}^*$ is faithful. This follows from the fact that every object M of the category $\mathcal{C}_{\mathfrak{X}_\mathfrak{a}}$ is a quotient object of an object N of $\mathcal{C}_{\mathfrak{X}}$ and a subobject of an object L of $\mathfrak{T}\mathcal{C}_{\mathfrak{X}}$. Therefore, the composition $N \xrightarrow{\beta} L$ of the epimorphism $N \longrightarrow M$ and a monomorphism $M \longrightarrow L$ is nonzero iff M is nonzero. Since the functor f^* is faithful, $f^*(\beta) \neq 0$ whenever $M \neq 0$, which, in turn, implies that $f_\mathfrak{a}^*(M) \neq 0$ if $M \neq 0$.

Since the functor $\mathcal{C}_{\mathfrak{Y}_\mathfrak{a}} \xrightarrow{f_\mathfrak{a}^*} \mathcal{C}_{\mathfrak{X}_\mathfrak{a}}$ is exact and the category $\mathcal{C}_{\mathfrak{Y}_\mathfrak{a}}$ is abelian, the faithfulness of $f_\mathfrak{a}^*$ is equivalent to its conservativeness. The fact that $f_\mathfrak{a}^*$ is conservative implies that $\widetilde{\mathfrak{f}_\mathfrak{a}^*}$ is conservative too. Therefore, being a localization functor, $\widetilde{\mathfrak{f}_\mathfrak{a}^*}$ is a category equivalence. ■

3.5. Continuous and (co)monadic morphisms of t-spaces. Let $\mathfrak{X} \xrightarrow{\mathfrak{f}} \mathfrak{Y}$ be a continuous morphism. We call it *comonadic* (resp. *monadic*) if its inverse (resp. direct) image functor is faithful. By 3.4, a continuous morphism $\mathfrak{X} \xrightarrow{\mathfrak{f}} \mathfrak{Y}$ is comonadic iff the canonical functor

$$\mathfrak{T}\mathcal{C}_{\mathfrak{Y}} \xrightarrow{\widetilde{\mathfrak{f}^*}} \mathcal{G}_f - Comod = (\mathfrak{X} \backslash \mathcal{G}_f) - Comod, \quad M \longmapsto (f^*(M), f^*\eta_f(M)), \qquad (1)$$

is a category equivalence. Dually, $\mathfrak{f}$ is a monadic morphism iff

$$\mathfrak{T}\mathcal{C}_{\mathfrak{X}} \xrightarrow{\widetilde{\mathfrak{f}_*}} \mathcal{F}_f - mod = (\mathcal{F}_f / \mathfrak{Y}) - mod, \quad L \longmapsto (f_*(L), f_*\epsilon_f(L)), \qquad (2)$$

is a category equivalence.

One can show that if $\mathfrak{X} \xrightarrow{\mathfrak{f}} \mathfrak{Y}$ is a continuous morphism, then both localization $\mathfrak{p}_\mathfrak{f}$ and the 'faithful' component $\mathfrak{f}_\mathfrak{c}$ in the decomposition $\mathfrak{f} = \mathfrak{p}_\mathfrak{f} \circ \mathfrak{f}_\mathfrak{c}$ (see 3.1) are continuous morphisms. Thus, every continuous morphism of t-'spaces' is a unique composition of a continuous localization and a comonadic morphism.

4. Presite of localizations.

4.1. Proposition. *Let $\mathcal{TC}_{\mathfrak{X}}$ be a svelte triangulated category and $\{T_i \mid i \in J\}$ a finite family of thick triangulated subcategories of $\mathcal{TC}_{\mathfrak{X}}$. Then* $\big(\bigcap_{i\in J} T_i\big) \sqcup S = \bigcap_{i\in J}(T_i \sqcup S)$ *for any thick triangulated subcategory S.*

Proof. Let $\mathcal{C}_{\mathfrak{X}_{\mathfrak{a}}}$ denote the abelianization of the triangulated category $\mathcal{TC}_{\mathfrak{X}}$. For a triangulated subcategory $\mathcal{T}$ of $\mathcal{TC}_{\mathfrak{X}}$, let $\mathcal{T}^{\mathfrak{a}}$ denote the smallest thick subcategory of $\mathcal{C}_{\mathfrak{X}_{\mathfrak{a}}}$ generated by the image of $\mathcal{T}$ in $\mathcal{C}_{\mathfrak{X}_{\mathfrak{a}}}$.

(a) If $\mathcal{T}$ is a thick triangulated subcategory of $\mathcal{TC}_{\mathfrak{X}}$, then $\mathcal{T} = \mathcal{T}^{\mathfrak{a}} \bigcap \mathcal{C}_{\mathfrak{X}}$.

In fact, objects of the subcategory $\mathcal{T}^{\mathfrak{a}}$ are arbitrary subquotients of objects of $\mathcal{T}$. Let M be an object of $\mathcal{C}_{\mathfrak{X}}$ which is a subquotient of an object N of $\mathcal{C}_{\mathfrak{X}_{\mathfrak{a}}}$, i.e. there exists a diagram $N \xrightarrow{\mathfrak{j}} K \xrightarrow{\mathfrak{e}} M$ in which $\mathfrak{j}$ is a monomorphism and $\mathfrak{e}$ is an epimorphism. Since M is a projective object, the epimorphism $\mathfrak{e}$ splits, i.e. there exists a morphism $M \xrightarrow{h} K$ such that $\mathfrak{e} \circ h = id_M$. Since M is an injective object of $\mathcal{C}_{\mathfrak{X}_{\mathfrak{a}}}$, the monomorphism $\mathfrak{j} \circ h$ splits. If the object N belongs to the subcategory $\mathcal{T}$, then M is also an object of $\mathcal{T}$, because thick subcategories contain all direct summands of all their objects.

(b) The equality $(\mathcal{S} \sqcup \mathcal{T})^{\mathfrak{a}} = \mathcal{S}^{\mathfrak{a}} \sqcup \mathcal{T}^{\mathfrak{a}}$ holds for any pair $\mathcal{S}$, $\mathcal{T}$ of thick triangulated subcategories of $\mathcal{TC}_{\mathfrak{X}}$.

In fact, the squares

$$\begin{array}{ccc} C_{\mathfrak{X}} & \longrightarrow & C_{\mathfrak{X}}/\mathcal{T} \\ \downarrow & & \downarrow \\ C_X/\mathcal{S} & \longrightarrow & C_X/(\mathcal{S} \sqcup \mathcal{T}) \end{array} \quad \text{and} \quad \begin{array}{ccc} C_{\mathfrak{X}_{\mathfrak{a}}} & \longrightarrow & C_{\mathfrak{X}_{\mathfrak{a}}}/\mathcal{T}^{\mathfrak{a}} \\ \downarrow & & \downarrow \\ C_{\mathfrak{X}_{\mathfrak{a}}}/\mathcal{S}^{\mathfrak{a}} & \longrightarrow & C_{\mathfrak{X}_{\mathfrak{a}}}/(\mathcal{S}^{\mathfrak{a}} \sqcup \mathcal{T}^{\mathfrak{a}}) \end{array}$$

are cocartesian and the abelianization functor transforms cocartesian squares into cocartesian squares, which implies that the unique functor

$$C_{\mathfrak{X}_{\mathfrak{a}}}/(\mathcal{S}^{\mathfrak{a}} \sqcup \mathcal{T}^{\mathfrak{a}}) \longrightarrow C_{\mathfrak{X}_{\mathfrak{a}}}/(\mathcal{S} \sqcup \mathcal{T})^{\mathfrak{a}}$$

is a category equivalence.

(c) The equality $\bigcap_{i\in J} \mathbb{T}_i^{\mathfrak{a}} = \big(\bigcap_{i\in J} \mathbb{T}_i\big)^{\mathfrak{a}}$ holds for any finite family $\{\mathbb{T}_i \mid i \in J\}$ of thick triangulated subcategories of $\mathcal{TC}_{\mathfrak{X}}$.

Replacing $\mathcal{TC}_{\mathfrak{X}}$ by $\mathcal{TC}_{\mathfrak{X}}/\mathcal{T}$ and $\mathcal{T}_i$ by $\mathcal{T}_i/\mathcal{T}$, where $\mathcal{T} = \bigcap_{i\in J} \mathbb{T}_i$, we reduce to the case $\bigcap_{i\in J} \mathbb{T}_i = 0$. In this case, the claim is $\bigcap_{i\in J} \mathbb{T}_i^{\mathfrak{a}} = 0$. The equality $\bigcap_{i\in J} \mathbb{T}_i = 0$ means precisely that the triangle functor

$$\mathcal{TC}_{\mathfrak{X}} \longrightarrow \prod_{i\in J} \mathcal{TC}_{\mathfrak{X}}/\mathcal{T}_i$$

induced by the localization functors $\{\mathcal{TC}_{\mathfrak{X}} \longrightarrow \mathcal{TC}_{\mathfrak{X}}/\mathcal{T}_i \mid i \in J\}$ is faithful. But, then its abelianization,

$$\mathcal{C}_{\mathfrak{X}_{\mathfrak{a}}} \longrightarrow \prod_{i\in J} \mathcal{C}_{\mathfrak{X}_{\mathfrak{a}}}/\mathcal{T}_i^{\mathfrak{a}},$$

is a faithful functor, i.e. its kernel, the intersection $\bigcap_{i\in J} \mathbb{T}_i^{\mathfrak{a}}$, equals to zero.

(d) It follows from (a), (b) and (c) that

$$(\bigcap_{i\in J} \mathcal{T}_i)\sqcup\mathcal{S} = ((\bigcap_{i\in J} \mathcal{T}_i^{\mathfrak{a}})\sqcup\mathcal{S}^{\mathfrak{a}})\bigcap \mathcal{C}_{\mathfrak{X}} = (\bigcap_{i\in J}(\mathcal{T}_i^{\mathfrak{a}}\sqcup\mathcal{S}^{\mathfrak{a}}))\bigcap\mathcal{C}_{\mathfrak{X}} =$$
$$\bigcap_{i\in J}((\mathcal{T}_i^{\mathfrak{a}}\sqcup\mathcal{S}^{\mathfrak{a}})\bigcap\mathcal{C}_{\mathfrak{X}}) = \bigcap_{i\in J}(\mathcal{T}_i\sqcup\mathcal{S}).$$

for any finite family $\{\mathcal{S},\ \mathcal{T}_i \mid i\in J\}$ of thick triangulated subcategories of $\mathcal{TC}_{\mathfrak{X}}$. ∎

4.2. Presite of exact localizations. Let $\mathfrak{Esp}_{\mathfrak{Tr}}^{\mathfrak{L}}$ denote the subcategory of the category $\mathfrak{Esp}_{\mathfrak{Tr}}$ of t-'spaces' whose objects are t-'spaces' and morphisms are localizations (i.e. their inverse image functors are compositions of localization functors and category equivalences). We call a set $\{\mathfrak{U}_i \stackrel{\mathfrak{u}_i}{\to} \mathfrak{X} \mid i\in J\}$ of morphisms of $\mathfrak{Esp}_{\mathfrak{Tr}}^{\mathfrak{L}}$ a *cover* of the t-'space' $\mathfrak{X}$ if there is a finite subset $\mathfrak{J}$ of J such that the family of inverse image functors $\{\mathfrak{TC}_{\mathfrak{X}} \xrightarrow{\mathfrak{u}_i^*} \mathfrak{TC}_{\mathfrak{U}_i} \mid i\in\mathfrak{J}\}$ is conservative. We denote the set of all such covers of $\mathfrak{X}$ by $\mathfrak{T}_{\mathfrak{f}}(\mathfrak{X})$.

4.2.1. Proposition. *The covers defined above form a pretopology, $\mathfrak{T}_{\mathfrak{f}}$, on $\mathfrak{Esp}_{\mathfrak{Tr}}^{\mathfrak{L}}$.*

Proof. The morphisms of the subcategory $\mathfrak{Esp}_{\mathfrak{Tr}}^{\mathfrak{L}}$ are determined, uniquely up to isomorphism, by the kernel of their inverse image functors. A family of inverse image functors $\{\mathfrak{TC}_{\mathfrak{X}} \xrightarrow{\mathfrak{u}_i^*} \mathfrak{TC}_{\mathfrak{U}_i} \mid i\in\mathfrak{J}\}$ is conservative iff the intersection of kernels of these inverse image functors is zero. The assertion follows now from 4.1. ∎

5. The spectra of t-'spaces'.

Fix a svelte triangulated category $\mathcal{C}\mathfrak{T}_{\mathfrak{X}} = (\mathcal{C}_{\mathfrak{X}},\theta_{\mathfrak{X}},\mathfrak{Tr}_{\mathfrak{X}})$. Here $\theta_{\mathfrak{X}}$ denote its *suspension* functor and $\mathfrak{Tr}_{\mathfrak{X}}$ its category of *triangles* (otherwise called *admissible triangles*). We denote by $\mathfrak{Tht}(\mathfrak{X})$ the preorder (with respect to the inclusion) of all thick triangulated subcategories of $\mathcal{C}\mathfrak{T}_{\mathfrak{X}}$. Recall that a full triangulated subcategory of $\mathcal{C}\mathfrak{T}_{\mathfrak{X}}$ is called *thick* if it contains all direct summands of its objects.

5.1. $\mathbf{Spec}^1_{\mathfrak{L}}(\mathfrak{X})$ and its decompositions. For any triangulated subcategory $\mathcal{T}$ of $\mathcal{C}\mathfrak{T}_{\mathfrak{X}}$, let $\mathcal{T}^\star$ denote the intersection of all triangulated subcategories of $\mathcal{C}\mathfrak{T}_{\mathfrak{X}}$ which contain $\mathcal{T}$ properly. And let $\mathcal{T}_\star$ be the intersection of $\mathcal{T}^\star$ and $\mathcal{T}^\perp$ – the right orthogonal to $\mathcal{T}$. Recall that $\mathcal{T}^\perp$ is the full subcategory of $\mathcal{C}\mathfrak{T}_{\mathfrak{X}}$ generated by all objects N such that $\mathcal{C}\mathfrak{T}_{\mathfrak{X}}(N,M)=0$ for all $M\in Ob\mathcal{T}$. It follows that $\mathcal{T}^\perp$ is triangulated (for any subcategory $\mathcal{T}$ which is stable by the translation functor).

We denote by $\mathbf{Spec}^1_{\mathfrak{L}}(\mathfrak{X})$ the subpreorder of $\mathfrak{Tht}(\mathfrak{X})$ formed by all thick triangulated subcategories $\mathcal{P}$ for which $\mathcal{P}^\star\neq\mathcal{P}$. We have a decomposition

$$\mathbf{Spec}^1_{\mathfrak{L}}(\mathfrak{X}) = \mathbf{Spec}^{1,1}_{\mathfrak{L}}(\mathfrak{X})\coprod\mathbf{Spec}^{1,0}_{\mathfrak{L}}(\mathfrak{X})$$

of $\mathbf{Spec}^1_{\mathfrak{L}}(\mathfrak{X})$ into a disjoint union of

$$\mathbf{Spec}^{1,1}_{\mathfrak{L}}(\mathfrak{X}) = \{\mathcal{P}\in\mathfrak{Tht}(\mathfrak{X}) \mid \mathcal{P}_\star\neq 0\} \quad\text{and}$$
$$\mathbf{Spec}^{1,0}_{\mathfrak{L}}(\mathfrak{X}) = \{\mathcal{P}\in\mathbf{Spec}^1_{\mathfrak{L}}(\mathfrak{X}) \mid \mathcal{P}_\star = 0\}.$$

5.2. $\mathfrak{L}$-Local triangulated categories and $\mathbf{Spec}^{1,1}_{\mathfrak{L}}(\mathfrak{X})$. We call a triangulated category $C\mathfrak{T}_{\mathfrak{Y}}$ $\mathfrak{L}$-*local* if it has the smallest nonzero thick triangulated subcategory.

5.2.1. Proposition. *Let $\mathcal{P} \in \mathbf{Spec}^{1,1}_{\mathfrak{L}}(\mathfrak{X})$. Then*
(a) $\mathcal{P} = {}^{\perp}\mathcal{P}_{*}$.
(b) The triangualted category $\mathcal{P}^{\perp}$ is $\mathfrak{L}$-local and $\mathcal{P}_{\star}$ is its smallest nonzero thick triangulated subcategory.

Proof. See [R7, 12.7.1]. ■

5.2.2. Proposition. *Suppose that infinite coproducts or products exist in $C\mathcal{T}_{\mathfrak{X}}$. Let $\mathcal{P}$ be a thick triangulated subcategory of $C\mathcal{T}_{\mathfrak{X}}$. Then the following properties of are equivalent:*
(i) $\mathcal{P}_{\star} = \mathcal{P}^{\perp} \cap \mathcal{P}^{\star}$ is nonzero, i.e. $\mathcal{P} \in \mathbf{Spec}^{1,1}_{\mathfrak{L}}(\mathfrak{X})$;
(ii) $\mathcal{P}$ belongs to $\mathbf{Spec}^{1}_{\mathfrak{L}}(\mathfrak{X})$ and the composition of the inclusion $\mathcal{P}_{\star} \hookrightarrow C\mathcal{T}_{\mathfrak{X}}$ and the localization functor $C\mathcal{T}_{\mathfrak{X}} \xrightarrow{q^{*}_{\mathcal{P}}} C\mathcal{T}_{\mathfrak{X}}/\mathcal{P}$ induces an equivalence of triangulated categories $\mathcal{P}_{\star} \xrightarrow{\sim} \mathcal{P}^{\star}/\mathcal{P}$.
(iii) $\mathcal{P}$ belongs to $\mathbf{Spec}^{1}_{\mathfrak{L}}(\mathfrak{X})$ and the inclusion functor $\mathcal{P} \hookrightarrow \mathcal{P}^{\star}$ has a right adjoint.
(iv) $\mathcal{P}$ belongs to $\mathbf{Spec}^{1}_{\mathfrak{L}}(\mathfrak{X})$ and $\mathcal{P}^{\perp}$ is nonzero.

Proof. See [R7, 12.7.3]. ■

5.2.3. Corollary. *Suppose that infinite coproducts or products exist in $C\mathcal{T}_{\mathfrak{X}}$. Then $\mathbf{Spec}^{1,0}_{\mathfrak{L}}(\mathfrak{X})$ consists of all $\mathcal{P} \in \mathbf{Spec}^{1}_{\mathfrak{L}}(\mathfrak{X})$ such that $\mathcal{P}^{*} \neq \mathcal{P}$ and $\mathcal{P}^{\perp} = 0$.*

5.2.4. Remark. Loosely, 5.2.3 says that the elements of $\mathbf{Spec}^{1,0}_{\mathfrak{L}}(\mathfrak{X})$ can be regarded as "fat" points – they generate (in a weak sense) the whole category $C\mathfrak{T}_{\mathfrak{X}}$.

5.2.5. Proposition. *(a) Let $\{\mathcal{T}_i \mid i \in J\}$ be a finite set of thick subcategories of a triangulated category $\mathcal{T}C_{\mathfrak{X}}$ such that $\bigcap_{i\in J} \mathcal{T}_i = 0$. Then*

$$\mathbf{Spec}^{1}_{\mathfrak{L}}(\mathfrak{X}) = \bigcup_{i\in J} \mathbf{Spec}^{1}_{\mathfrak{L}}(\mathfrak{X}/\mathcal{T}_i) \qquad (1)$$

(b) Suppose that ${}^{\perp}(\mathcal{T}_i^{\perp}) = \mathcal{T}_i$ for all $i \in J$. Then

$$\mathbf{Spec}^{1,1}_{\mathfrak{L}}(\mathfrak{X}) = \bigcup_{i\in J} \mathbf{Spec}^{1,1}_{\mathfrak{L}}(\mathfrak{X}/\mathcal{T}_i) \qquad (2)$$

Proof. (a) The inclusion $\bigcup_{i\in J} \mathbf{Spec}^{1}_{\mathfrak{L}}(\mathfrak{X}/\mathcal{T}_i) \subseteq \mathbf{Spec}^{1}_{\mathfrak{L}}(\mathfrak{X})$ follows from the functoriality of $\mathbf{Spec}^{1}_{\mathfrak{L}}(-)$ with respect to localizations. Let $\mathcal{P} \in \mathbf{Spec}^{1}_{\mathfrak{L}}(\mathfrak{X})$. By 4.1,

$$\mathcal{P} = \big(\bigcap_{i\in J} T_i\big) \sqcup \mathcal{P} = \bigcap_{i\in J} (T_i \sqcup \mathcal{P}) \qquad (3)$$

which implies that $\mathcal{T}_i \subseteq \mathcal{P}$ for some $i \in J$. In fact, if $\mathcal{T}_i \nsubseteq \mathcal{P}$ for all $i \in J$, then $T_i \sqcup \mathcal{P}$ contains properly $\mathcal{P}_i$ for all $i \in J$, hence the intersection $\bigcap_{i \in J}(T_i \sqcup \mathcal{P})$ contains properly $\mathcal{P}$, which contradicts to (3). This proves the inverse inclusion, that is the equality (1).

(b) The inclusion $\bigcup_{i \in J} \mathbf{Spec}^{1,1}_{\mathfrak{L}}(\mathfrak{X}/\mathcal{T}_i) \subseteq \mathbf{Spec}^{1,1}_{\mathfrak{L}}(\mathfrak{X})$ follows from the functoriality of $\mathbf{Spec}^{1,1}_{\mathfrak{L}}(-)$ with respect to localizations at thick subcategories $\mathcal{T}$ such that ${}^{\perp}(\mathcal{T}^{\perp}) = \mathcal{T}$. The inverse inclusion follows from (a). ■

5.3. The spectrum $\mathbf{Spec}^{1/2}_{\mathfrak{L}}(\mathfrak{X})$. Let $\mathbf{Spec}^{1/2}_{\mathfrak{L}}(\mathfrak{X})$ denote the full subpreorder of $\mathfrak{Th}\mathfrak{t}(\mathfrak{X})$ whose objects are thick triangulated subcategories $\mathcal{Q}$ such that ${}^{\perp}\mathcal{Q}$ belongs to $\mathbf{Spec}^{1}_{\mathfrak{L}}(\mathfrak{X})$ and every thick triangulated subcategory of $\mathcal{CT}_{\mathfrak{X}}$ properly containing ${}^{\perp}\mathcal{Q}$ contains $\mathcal{Q}$; i.e. ${}^{\perp}\mathcal{Q} \vee \mathcal{Q}$ is the smallest thick triangulated subcategory of $\mathcal{CT}_{\mathfrak{X}}$ properly containing ${}^{\perp}\mathcal{Q}$.

5.3.1. Proposition. *(a) The map $\mathcal{Q} \longmapsto {}^{\perp}\mathcal{Q}$ induces a bijective map*

$$\mathbf{Spec}^{1/2}_{\mathfrak{L}}(\mathfrak{X}) \xrightarrow{\sim} \mathbf{Spec}^{1,1}_{\mathfrak{L}}(\mathfrak{X}). \tag{1}$$

(b) If $\mathcal{Q}$ is an object of $\mathbf{Spec}^{1/2}_{\mathfrak{L}}(\mathfrak{X})$, then $\mathcal{Q}$ is a minimal nonzero thick triangulated subcategory of $\mathcal{CT}_{\mathfrak{X}}$.

(c) Suppose that $\mathcal{CT}_{\mathfrak{X}}$ has infinite coproducts or products. Then the following properties of a thick triangulated subcategory $\mathcal{Q}$ are equivalent:

(i) $\mathcal{Q}$ belongs to $\mathbf{Spec}^{1/2}_{\mathfrak{L}}(\mathfrak{X})$;

(ii) $\mathcal{Q}$ is a minimal nonzero thick triangulated subcategory of $\mathcal{CT}_{\mathfrak{X}}$ such that ${}^{\perp}\mathcal{Q}$ belongs to $\mathbf{Spec}^{1}_{\mathfrak{L}}(\mathfrak{X})$.

Proof. See [R7, 12.8.1]. ■

5.4. Flat spectra. Recall the following assertion which is due to Verdier:

5.4.1. Proposition [Ve1, 10–5]. *Let $\mathbb{T}$ be a thick triangulated subcategory of $\mathcal{CT}_{\mathfrak{X}}$, and let*

$$\mathbb{T} \xrightarrow{\iota^*_{\mathbb{T}}} \mathcal{CT}_{\mathfrak{X}} \xrightarrow{q^*_{\mathbb{T}}} \mathcal{CT}_{\mathfrak{X}}/\mathbb{T}$$

be the inclusion and localization functors. The following properties are equivalent:

(a) The functor $\iota^*_{\mathbb{T}}$ has a right adjoint.

(b) The functor $q^*_{\mathbb{T}}$ has a right adjoint.

Let $\mathfrak{Se}(\mathfrak{X})$ denote the family of all thick triangulated subcategories of the triangulated category $\mathcal{CT}_{\mathfrak{X}}$ which satisfy equivalent conditions of 12.10.3. We define the *complete flat spectrum* of $\mathfrak{X}$, $\mathbf{Spec}^{1}_{\mathfrak{fL}}(\mathfrak{X})$, by setting

$$\mathbf{Spec}^{1}_{\mathfrak{fL}}(\mathfrak{X}) = \mathbf{Spec}^{1}_{\mathfrak{L}}(\mathfrak{X}) \bigcap \mathfrak{Se}(\mathfrak{X}). \tag{1}$$

We define the *flat spectrum of* $\mathfrak{X}$ as a full subpreorder, $\mathbf{Spec}^{0}_{\mathfrak{fL}}(\mathfrak{X})$, of $\mathfrak{Th}\mathfrak{t}(\mathfrak{X})$ whose objects are all $\mathcal{P}$ such that $\widehat{\mathcal{P}} \in \mathbf{Spec}^{1}_{\mathfrak{fL}}(\mathfrak{X})\}$.

It follows from these definitions that the map $\mathcal{P} \longmapsto \widehat{\mathcal{P}}$ defines an injective morphism

$$\mathbf{Spec}^0_{\mathfrak{f}\mathfrak{L}}(\mathfrak{X}) \longrightarrow \mathbf{Spec}^1_{\mathfrak{f}\mathfrak{L}}(\mathfrak{X}). \tag{2}$$

Let $\mathbf{Spec}^{1/2}_{\mathfrak{f}\mathfrak{L}}(\mathfrak{X})$ denote the full subpreorder of $\mathbf{Spec}^{1/2}_{\mathfrak{L}}(\mathfrak{X})$ whose objects are all $\mathcal{Q}$ such that ${}^\perp\mathcal{Q}$ belongs to $\mathfrak{Se}(\mathfrak{X})$.

5.4.2. Proposition. *(a) The map*

$$\mathfrak{Tht}(\mathfrak{X}) \longrightarrow \mathfrak{Tht}(\mathfrak{X}), \quad \mathcal{Q} \longmapsto {}^\perp\mathcal{Q},$$

induces an isomorphism

$$\mathbf{Spec}^{1/2}_{\mathfrak{f}\mathfrak{L}}(\mathfrak{X}) \xrightarrow{\sim} \mathbf{Spec}^1_{\mathfrak{f}\mathfrak{L}}(\mathfrak{X}). \tag{3}$$

(b) $\mathbf{Spec}^0_{\mathfrak{f}\mathfrak{L}}(\mathfrak{X}) = \mathbf{Spec}^0_{\mathfrak{L}}(\mathfrak{X}) \bigcap \mathbf{Spec}^{1/2}_{\mathfrak{f}\mathfrak{L}}(\mathfrak{X})$. *The canonical morphism (2) is the composition of the inclusion* $\mathbf{Spec}^0_{\mathfrak{f}\mathfrak{L}}(\mathfrak{X}) \hookrightarrow \mathbf{Spec}^{1/2}_{\mathfrak{f}\mathfrak{L}}(\mathfrak{X})$ *and the isomorphism (3).*

Proof. See [R7, 12.10.1].

5.5. Supports and Zariski topology.

5.5.1. Supports. For any object M of the category C_X, the support of M in $\mathbf{Spec}^1_{\mathfrak{L}}(\mathfrak{X})$ is defined by $Supp^1_{\mathfrak{L}}(M) = \{\mathcal{P} \in \mathbf{Spec}^1_{\mathfrak{L}}(X) \mid M \notin Ob\mathcal{P}\}$. It follows that $Supp^1_{\mathfrak{L}}(\mathcal{L} \oplus \mathcal{M}) = Supp^1_{\mathfrak{L}}(\mathcal{L}) \bigcup Supp^1_{\mathfrak{L}}(\mathcal{M})$.

5.5.2. Topologies on $\mathbf{Spec}^1_{\mathfrak{L}}(X)$ and $\mathbf{Spec}^{1,1}_{\mathfrak{L}}(X)$. We follow the pattern of 2.4. Let Ξ be a class of objects of C_X closed under finite coproducts. For any set E of objects of Xi, let $\mathcal{V}^1_{\mathfrak{L}}(E)$ denote the intersection $\bigcap_{M \in E} Supp^1_{\mathfrak{L}}(M)$. Then, for any family $\{E_i \mid i \in \mathfrak{I}\}$ of such sets, we have, evidently,

$$\mathcal{V}(\bigcup_{i \in J} E_i) = \bigcap_{i \in J} \mathcal{V}(E_i).$$

It follows from the equality $Supp^1_{\mathfrak{L}}(M \oplus N) = Supp^1_{\mathfrak{L}}(M) \bigcup Supp^1_{\mathfrak{L}}(N)$ (see 2.2.1(a)) that $\mathcal{V}^1_{\mathfrak{L}}(E \oplus \widetilde{E}) = \mathcal{V}^1_{\mathfrak{L}}(E) \bigcup \mathcal{V}^1_{\mathfrak{L}}(\widetilde{E})$. Here $E \oplus \widetilde{E} \stackrel{\mathrm{def}}{=} \{M \oplus N \mid M \in E,\ N \in \widetilde{E}\}$.

This shows that the subsets $\mathcal{V}^1_{\mathfrak{L}}(E)$ of $\mathbf{Spec}^1_{\mathfrak{L}}(\mathfrak{X})$, where E runs through subsets of Ξ, are all closed sets of a topology, τ^1_{Ξ}, on the spectrum $\mathbf{Spec}^1_{\mathfrak{L}}(\mathfrak{X})$.

We denote by $\tau^{1,1}_{\Xi}$ the induced topology on $\mathbf{Spec}^{1,1}_{\mathfrak{L}}(\mathfrak{X})$.

5.5.3. Compact topology. The class $\Xi_{\mathfrak{c}}(X)$ of compact objects of the category $C_{\mathfrak{X}}$ is closed under finite coproducts, hence it defines a topology on $\mathbf{Spec}^1_{\mathfrak{L}}(\mathfrak{X})$ which we denote by $\tau_{\mathfrak{c}}$ and call the *compact* topology.

Restricting the compact topology to $\mathbf{Spec}^{1,1}_{\mathfrak{L}}(\mathfrak{X})$ or to $\mathbf{Spec}^1_{\mathfrak{f}\mathfrak{L}}(\mathfrak{X})$, we obtain the compact topology on these spectra.

5.5.4. Zariski topology on $\mathbf{Spec}^{1,1}_{\mathfrak{L}}(\mathfrak{X})$. We define the *Zariski* topology on the spectrum $\mathbf{Spec}^{1,1}_{\mathfrak{L}}(\mathfrak{X})$ by taking as a base of closed sets the supports of compact objects and closures (i.e. the sets of all specializations) of points of $\mathbf{Spec}^{1,1}_{\mathfrak{L}}(\mathfrak{X})$.

If the category $\mathcal{C}_{\mathfrak{X}}$ is generated by compact objects, then the Zariski topology coincides with the compact topology $\tau_{\mathfrak{c}}$.

5.5.5. Zariski topology on $\mathbf{Spec}^{1/2}_{\mathfrak{L}}(\mathfrak{X})$. It is important to realize that the topologies we define are determined in the first place by the choice of a preorder on the set of thick subcategories (or topologizing subcategories in the case of abelian categories). And so far, the preorder was always the inverse inclusion.

Following these pattern, for any object M of a svelte triangulated category $\mathcal{C}\mathfrak{T}_{\mathfrak{X}}$, we define the *support* of M in $\mathbf{Spec}^{1/2}_{\mathfrak{L}}(\mathfrak{X})$ as the set of all $\mathcal{Q} \in \mathbf{Spec}^{1/2}_{\mathfrak{L}}(\mathfrak{X})$ such that the smallest thick tringulated subcategory $[M]_{\mathfrak{tr}}$ containing M contains also $\mathcal{Q}$.

We define the *Zariski* topology on $\mathbf{Spec}^{1/2}_{\mathfrak{L}}(\mathfrak{X})$ by taking supports of compact objects and the finite subsets of $\mathbf{Spec}^{1/2}_{\mathfrak{L}}(\mathfrak{X})$ as a base of its closed sets.

It follows from this definition of Zariski topology and 5.2.1(b) that all points of the spectrum $\mathbf{Spec}^{1/2}_{\mathfrak{L}}(\mathfrak{X})$ are closed; that is Zariski topology on $\mathbf{Spec}^{1/2}_{\mathfrak{L}}(\mathfrak{X})$ is a T_1*-topology.* The bijective map

$$\mathbf{Spec}^{1/2}_{\mathfrak{L}}(\mathfrak{X}) \xrightarrow{\sim} \mathbf{Spec}^{1,1}_{\mathfrak{L}}(\mathfrak{X}) \qquad (4)$$

is continuous, but, usually, not a homeomorphism.

5.5.6. Remark. Suppose that C_X is the heart of a t-structure on $\mathcal{C}\mathfrak{T}_{\mathfrak{X}}$. Then we have a commutative diagram

$$\begin{array}{ccc} \mathbf{Spec}(X) & \longrightarrow & \mathbf{Spec}^{1/2}_{\mathfrak{L}}(\mathfrak{X}) \\ \wr\downarrow & & \downarrow\wr \\ \mathbf{Spec}^{1,1}_{\mathfrak{t}}(X) & \longrightarrow & \mathbf{Spec}^{1,1}_{\mathfrak{L}}(\mathfrak{X}) \end{array} \qquad (5)$$

where horizontal arrows are embeddings and vertical arrows are canonical bijections. Thus, the Zariski topology on $\mathbf{Spec}^{1/2}_{\mathfrak{L}}(\mathfrak{X})$ induces a T_1-topology on the spectrum $\mathbf{Spec}(X)$ of the 'space' represented by the abelian category C_X, which, obviously, differs from Zariski topology on $\mathbf{Spec}(X)$, unless $\mathbf{Spec}(X)$ is of zero Krull dimension.

5.6. A geometric realization of a triangulated category. We follow an obvious modification of the pattern of 6.7. Namely, we assign to a Karoubian triangulated category $\mathfrak{T}\mathcal{C}_{\mathfrak{X}}$ having a set of compact generators the contravariant pseudo-functor from the category of Zariski open subsets of the spectrum $\mathbf{Spec}^{1,1}_{\mathfrak{L}}(\mathfrak{X})$ to the category of svelte triangulated categories. The associated stack is the stack of local triangulated categories.

5.7. The geometric center. We define the *center* of a svelte triangulated category $\mathfrak{T}\mathcal{C}_{\mathfrak{Y}} = (\mathcal{C}_{\mathfrak{Y}}, \theta_{\mathfrak{Y}}, \mathfrak{Tr}_{\mathfrak{Y}})$ as the subring $\mathcal{O}^{\mathfrak{T}}(\mathfrak{Y})$ of the center $\mathfrak{z}(\mathcal{C}_{\mathfrak{X}})$ of the category $\mathcal{C}_{\mathfrak{Y}}$ formed by $\theta_{\mathfrak{Y}}$-invariant endomorphisms of the identical functor of $\mathcal{C}_{\mathfrak{Y}}$. One can show that the ring $\mathcal{O}^{\mathfrak{T}}(\mathfrak{Y})$ is local if the triangulated category $\mathfrak{T}\mathcal{C}_{\mathfrak{Y}}$ is local.

Let $\mathfrak{T}\mathcal{C}_{\mathfrak{X}}$ be a Karoubian triangulated category with a set of compact generators and $\mathfrak{T}\mathfrak{F}^{\mathfrak{Z}}_{\mathfrak{X}}$ the corresponding stack of local triangulated categories (cf. 5.6). Assigning to each fiber of the stack $\mathfrak{T}\mathfrak{F}^{\mathfrak{Z}}_{\mathfrak{X}}$ its center, we obtain a presheaf of commutative rings on the spectrum $\mathbf{Spec}^{1,1}_{\mathfrak{L}}(\mathfrak{X})$ endowed with the Zariski topology. The associated sheaf, $\mathcal{O}^{\mathfrak{T}}_{\mathfrak{X}}$, is a sheaf of local rings. We call the locally ringed topological space $(\mathbf{Spec}^{1,1}_{\mathfrak{L}}(\mathfrak{X}), \mathcal{O}^{\mathfrak{T}}_{\mathfrak{X}})$ the *geometric centrum* of the triangulated category $\mathfrak{T}\mathcal{C}_{\mathfrak{X}}$.

5.7.1. Note. Similarly to the abelian case, one can define the *reduced geometric centrum* of $\mathfrak{T}\mathcal{C}_{\mathfrak{X}}$. Details of this construction are left to the reader.

5.8. On the spectra of a monoidal triangulated category.

5.8.1. A remark on spectral cuisine. There are certain rather simple general pattern of producing spectra starting from a preorder (they are outlined in [R6]). Here, in Section 8, these pattern are applied to the preorder $\mathfrak{Tht}(\mathfrak{X})$ of thick triangulated subcategories of the triangulated category $\mathcal{C}\mathfrak{T}_{\mathfrak{X}}$.

5.8.2. Application to monoidal triangulated categories. Suppose that a triangulated category $\mathfrak{T}\mathcal{C}_{\mathfrak{X}}$ has a structure of a monoidal category. Then, replacing the preorder of thick subcategories with the preorder of those thick subcategories which are ideals of $\mathfrak{T}\mathcal{C}_{\mathfrak{X}}$ and mimiking the definitions of $\mathbf{Spec}^{1}_{\mathfrak{L}}(\mathfrak{X})$ and $\mathbf{Spec}^{1,1}_{\mathfrak{L}}(\mathfrak{X})$, we obtain the spectra respectively $\mathbf{Spec}^{1}_{\mathfrak{L},\otimes}(\mathfrak{X})$ and $\mathbf{Spec}^{1,1}_{\mathfrak{L},\otimes}(\mathfrak{X})$. If the monoidal category $\mathfrak{T}\mathcal{C}_{\mathfrak{X}}$ is symmetric, then $\mathbf{Spec}^{1}_{\mathfrak{L},\otimes}(\mathfrak{X})$ coincides with the spectrum introduced by P. Balmer in different terms, as a straightforward imitation of the notion of a prime ideal of a commutative ring. This spectrum has nice properties. Unfortunately, triangulated categories associated with noncommutative 'spaces' of interest do not have any symmetric monoidal structure.

It is not clear at the moment what might be the role of the spectra $\mathbf{Spec}^{1}_{\mathfrak{L},\otimes}(\mathfrak{X})$ and $\mathbf{Spec}^{1,1}_{\mathfrak{L},\otimes}(\mathfrak{X})$ (if any) in the case of a non-symmetric monoidal category.

Lecture 4. Non-abelian homological algebra.

The preliminaries are dedicated to kernels of arrows of arbitrary categories with initial objects. They are complemented in Appendix. In the treatment of non-abelian homological algebra, we adopt here an intermediate level of generality – right or left exact (instead of fibred or cofibred) categories, which turns the main body of this text into an exercise on satellites along the lines of [Gr], in which abelian categories are replaced by right (or left) exact categories with initial (resp. final) objects. An analysis of obtained facts leads to the notions of stable category of a left exact category and to the notions of quasi-suspended and quasi-triangulated categories.

1. Preliminaries: kernels and cokernels of morphisms.

Let C_X be a category with an initial object, x. For a morphism $M \xrightarrow{f} N$ we define the *kernel of* f as the upper horizontal arrow in a cartesian square

$$\begin{array}{ccc} Ker(f) & \xrightarrow{\mathfrak{k}(f)} & M \\ f' \downarrow & cart & \downarrow f \\ x & \longrightarrow & N \end{array}$$

when the latter exists.

Cokernels of morphisms are defined dually, via a cocartesian square

$$\begin{array}{ccc} N & \xrightarrow{\mathfrak{c}(f)} & Cok(f) \\ f \uparrow & cocart & \uparrow f' \\ M & \longrightarrow & y \end{array}$$

where y is a final object of C_X.

If C_X is a pointed category (i.e. its initial objects are final), then the notion of the kernel is equivalent to the usual one: the diagram $Ker(f) \xrightarrow{\mathfrak{k}(f)} M \overset{f}{\underset{0}{\rightrightarrows}} N$ is exact.

Dually, the cokernel of f makes the diagram $M \overset{f}{\underset{0}{\rightrightarrows}} N \xrightarrow{\mathfrak{c}(f)} Cok(f)$ exact.

1.1. Lemma. *Let C_X be a category with an initial object x.*

(a) Let a morphism $M \xrightarrow{f} N$ of C_X have a kernel. The canonical morphism $Ker(f) \xrightarrow{\mathfrak{k}(f)} M$ is a monomorphism, if the unique arrow $x \xrightarrow{i_N} N$ is a monomorphism.

(b) If $M \xrightarrow{f} N$ is a monomorphism, then $x \xrightarrow{i_M} M$ is the kernel of f.

Proof. The pull-backs of monomorphisms are monomorphisms. ■

1.2. Corollary. *Let C_X be a category with an initial object x. The following conditions are equivalent:*

(a) If $M \xrightarrow{f} N$ *has a kernel, then the canonical arrow* $Ker(f) \xrightarrow{\mathfrak{k}(f)} M$ *is a monomorphism.*

(b) The unique arrow $x \xrightarrow{i_M} M$ *is a monomorphism for any* $M \in ObC_X$*.*

Proof. $(a) \Rightarrow (b)$, because, by 1.1(b), the unique morphism $x \xrightarrow{i_M} M$ is the kernel of the identical morphism $M \longrightarrow M$. The implication $(b) \Rightarrow (a)$ follows from 1.1(a). ■

1.3. Note. The converse assertion is not true in general: a morphism might have a trivial kernel without being a monomorphism. It is easy to produce an example in the category of pointed sets.

1.4. Examples.

1.4.1. Kernels of morphisms of unital k-algebras. Let C_X be the category Alg_k of associative unital k-algebras. The category C_X has an initial object – the k-algebra k. For any k-algebra morphism $A \xrightarrow{\varphi} B$, we have a commutative square

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ {\scriptstyle \mathfrak{k}(\varphi)}\uparrow & & \uparrow \\ k \oplus K(\varphi) & \xrightarrow{\epsilon(\varphi)} & k \end{array}$$

where $K(\varphi)$ denote the kernel of the morphism φ in the category of non-unital k-algebras and the morphism $\mathfrak{k}(\varphi)$ is determined by the inclusion $K(\varphi) \longrightarrow A$ and the k-algebra structure $k \longrightarrow A$. This square is cartesian. In fact, if

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ {\scriptstyle \gamma}\uparrow & & \uparrow \\ C & \xrightarrow{\psi} & k \end{array}$$

is a commutative square of k-algebra morphisms, then C is an augmented algebra: $C = k \oplus K(\psi)$. Since the restriction of $\varphi \circ \gamma$ to $K(\psi)$ is zero, it factors uniquely through $K(\varphi)$. Therefore, there is a unique k-algebra morphism $C = k \oplus K(\psi) \xrightarrow{\beta} Ker(\varphi) = k \oplus K(\varphi)$ such that $\gamma = \mathfrak{k}(\varphi) \circ \beta$ and $\psi = \epsilon(\varphi) \circ \beta$.

This shows that each (unital) k-algebra morphism $A \xrightarrow{\varphi} B$ has a canonical kernel $Ker(\varphi)$ equal to the augmented k-algebra corresponding to the ideal $K(\varphi)$.

It follows from the description of the kernel $Ker(\varphi) \xrightarrow{\mathfrak{k}(\varphi)} A$ that it is a monomorphism iff the k-algebra structure $k \longrightarrow A$ is a monomorphism.

Notice that cokernels of morphisms are not defined in Alg_k, because this category does not have final objects.

1.4.2. Kernels and cokernels of maps of sets. Since the only initial object of the category *Sets* is the empty set $\emptyset$ and there are no morphisms from a non-empty set to $\emptyset$, the kernel of any map $X \longrightarrow Y$ is $\emptyset \longrightarrow X$. The cokernel of a map $X \xrightarrow{f} Y$ is the projection $Y \xrightarrow{\mathfrak{c}(f)} Y/f(X)$, where $Y/f(X)$ is the set obtained from Y by the contraction of $f(X)$ into a point. So that $\mathfrak{c}(f)$ is an isomorphism iff either $X = \emptyset$, or $f(X)$ is a one-point set.

1.4.3. Presheaves of sets. Let C_X be a svelte category and $C_X^\wedge$ the category of non-trivial presheaves of sets on C_X (that is we exclude the *trivial* presheaf which assigns to every object of C_X the empty set). The category $C_X^\wedge$ has a final object which is the constant presheaf with values in a one-element set. If C_X has a final object, y, then $\widehat{y} = C_X(-, y)$ is a final object of the category $C_X^\wedge$. Since $C_X^\wedge$ has small colimits, it has cokernels of arbitrary morphisms which are computed object-wise, that is using 1.4.2.

If the category C_X has an initial object, x, then the presheaf $\widehat{x} = C_X(-, x)$ is an initial object of the category $C_X^\wedge$. In this case, the category $C_X^\wedge$ has kernels of all its morphisms (because $C_X^\wedge$ has limits) and the Yoneda functor $C_X \xrightarrow{h} C_X^\wedge$ preserves kernels.

Notice that the initial object of $C_X^\wedge$ is not isomorphic to its final object unless the category C_X is pointed, i.e. initial objects of C_X are its final objects.

1.5. Some properties of kernels. See Appendix.

2. Right exact categories and (right) 'exact' functors.

We define a *right exact* category as a pair $(C_X, \mathfrak{E}_X)$, where C_X is a category and $\mathfrak{E}_X$ is a pretopology on C_X whose covers are *strict epimorphisms*; that is for any element $M \longrightarrow L$ of $\mathfrak{E}$ (– a cover), the diagram $M \times_L M \rightrightarrows M \longrightarrow L$ is exact. This requirement means precisely that the pretopology $\mathfrak{E}_X$ is *subcanonical*; i.e. every representable presheaf of sets on C_X is a sheaf. We call the elements of $\mathfrak{E}_X$ *deflations* and assume that all isomorphisms are deflations.

2.1. The coarsest and the finest right exact structures. The coarsest right exact structure on a category C_X is the discrete pretopology: the class of deflations coincides with the class $Iso(C_X)$ of all isomorphisms of the category C_X.

Let $\mathfrak{E}_X^{\mathfrak{s}}$ denote the class of all *universally strict* epimorphisms of C_X; i.e. elements of $\mathfrak{E}_X^{\mathfrak{s}}$ are strict epimorphisms $M \xrightarrow{\mathfrak{e}} N$ such that for any morphism $\widetilde{N} \xrightarrow{f} N$, there exists a cartesian square

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{\widetilde{f}} & M \\ {\scriptstyle \widetilde{\mathfrak{e}}}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{e}} \\ \widetilde{N} & \xrightarrow{f} & N \end{array}$$

whose left vertical arrow is a strict epimorphism. It follows that $\mathfrak{E}_X^{\mathfrak{s}}$ is the finest right exact structure on the category C_X. We call this structure *canonical*.

If C_X is an abelian category or a topos, then $\mathfrak{E}_X^{\mathfrak{s}}$ consists of all epimorphisms.

If C_X is a quasi-abelian category, then $\mathfrak{E}_X^{\mathfrak{s}}$ consists of all strict epimorphisms.

2.2. Right 'exact' and 'exact' functors. Let $(C_X, \mathfrak{E}_X)$ and $(C_Y, \mathfrak{E}_Y)$ be right exact categories. A functor $C_X \xrightarrow{F} C_Y$ will be called *right 'exact'* (resp. *'exact'*) if it maps deflations to deflations and for any deflation $M \xrightarrow{\mathfrak{e}} N$ of $\mathfrak{E}_X$ and any morphism

$\widetilde{N} \xrightarrow{f} N$, the canonical arrow $F(\widetilde{N} \times_N M) \longrightarrow F(\widetilde{N}) \times_{F(N)} F(M)$ is a deflation (resp. an isomorphism).

In other words, the functor F is 'exact' if it maps deflations to deflations and preserves pull-backs of deflations.

2.3. Weakly right 'exact' and weakly 'exact' functors. A functor $C_X \xrightarrow{F} C_Y$ is called *weakly right 'exact'* (resp. *weakly 'exact'*) if it maps deflations to deflations and for any arrow $M \longrightarrow N$ of $\mathfrak{E}_X$, the canonical morphism $F(M \times_N M) \longrightarrow F(M) \times_{F(N)} F(M)$ is a deflation (resp. an isomorphism). In particular, weakly 'exact' functors are weakly right 'exact'.

2.4. Note. Of cause, 'exact' (resp. right 'exact') functors are weakly 'exact' (resp. weakly right 'exact'). In the additive (actually a more general) case, weakly 'exact' functors are 'exact' (see 2.7 and 2.7.2).

2.5. Interpretation: 'spaces' represented by right exact categories. Weakly right 'exact' functors will be interpreted as inverse image functors of morphisms between 'spaces' represented by right exact categories. We consider the category $\mathfrak{Esp}_{\mathfrak{r}}^{\mathfrak{w}}$ whose objects are pairs $(X, \mathfrak{E}_X)$, where $(C_X, \mathfrak{E}_X)$ is a svelte right exact category. A morphism from $(X, \mathfrak{E}_X)$ to $(Y, \mathfrak{E}_Y)$ is a morphism of 'spaces' $X \xrightarrow{\varphi} Y$ whose inverse image functor $C_Y \xrightarrow{\varphi^*} C_X$ is a weakly right 'exact' functor from $(C_Y, \mathfrak{E}_Y)$ to $(C_X, \mathfrak{E}_X)$. The map which assigns to every 'space' X the pair $(X, Iso(C_X))$ is a full embedding of the category $|Cat|^o$ of 'spaces' into the category $\mathfrak{Esp}_{\mathfrak{r}}^{\mathfrak{w}}$. This full embedding is a right adjoint functor to the forgetful functor

$$\mathfrak{Esp}_{\mathfrak{r}}^{\mathfrak{w}} \longrightarrow |Cat|^o, \quad (X, \mathfrak{E}_X) \longmapsto X.$$

2.5.1. Proposition. *Let $(C_X, \mathfrak{E}_X)$ and $(C_Y, \mathfrak{E}_Y)$ be additive right exact categories and $C_X \xrightarrow{F} C_Y$ an additive functor. Then*

(a) The functor F is weakly right 'exact' iff it maps deflations to deflations and the sequence

$$F(Ker(\mathfrak{e})) \longrightarrow F(M) \xrightarrow{F(\mathfrak{e})} F(N) \longrightarrow 0$$

is exact for any deflation $M \xrightarrow{\mathfrak{e}} N$.

(b) The functor F is weakly 'exact' iff it maps deflations to deflations and the sequence

$$0 \longrightarrow F(Ker(\mathfrak{e})) \longrightarrow F(M) \xrightarrow{F(\mathfrak{e})} F(N) \longrightarrow 0$$

is 'exact' for any deflation $M \xrightarrow{\mathfrak{e}} N$.

Proof. See A.2(b). ■

2.6. Conflations and fully exact subcategories of a right exact category. Fix a right exact category $(C_X, \mathfrak{E}_X)$ with an initial object x. We denote by $\mathcal{E}_X$ the class of all sequences of the form $K \xrightarrow{\mathfrak{k}} M \xrightarrow{\mathfrak{e}} N$, where $\mathfrak{e} \in \mathfrak{E}_X$ and $K \xrightarrow{\mathfrak{k}} M$ is a kernel of $\mathfrak{e}$. Expanding the terminology of exact additive categories, we call such sequences *conflations.*

2.6.1. Fully exact subcategories of a right exact category. We call a full subcategory $\mathcal{B}$ of C_X a *fully exact* subcategory of the right exact category $(C_X, \mathfrak{E}_X)$, if $\mathcal{B}$ contains the initial object x and is *closed under extensions*; i.e. if objects K and N in a conflation $K \xrightarrow{\mathfrak{k}} M \xrightarrow{\mathfrak{e}} N$ belong to $\mathcal{B}$, then M is an object of $\mathcal{B}$.

In particular, fully exact subcategories of $(C_X, \mathfrak{E}_X)$ are strictly full subcategories.

2.6.2. Proposition. *Let $(C_X, \mathfrak{E}_X)$ be a right exact category with an initial object x and $\mathcal{B}$ its fully exact subcategory. Then the class $\mathfrak{E}_{X,\mathcal{B}}$ of all deflations $M \xrightarrow{\mathfrak{e}} N$ such that M, N, and $Ker(\mathfrak{e})$ are objects of $\mathcal{B}$ is a structure of a right exact category on $\mathcal{B}$ such that the inclusion functor $\mathcal{B} \longrightarrow C_X$ is an 'exact' functor $(\mathcal{B}, \mathfrak{E}_{X,\mathcal{B}}) \longrightarrow (C_X, \mathfrak{E}_X)$.*

Proof. The argument is an application of facts of Appendix. ∎

2.6.3. Remark. Let $(C_X, \mathfrak{E}_X)$ be a right exact category with an initial object x and $\mathcal{B}$ its strictly full subcategory containing x. Let $\mathfrak{E}$ be a right exact structure on $\mathcal{B}$ such that the inclusion functor $\mathcal{B} \xrightarrow{\mathfrak{J}} C_X$ maps deflations to deflations and preserves kernels of deflations. Then $\mathfrak{E}$ is contained in $\mathfrak{E}_{X,\mathcal{B}}$. In particular, $\mathfrak{E}$ is contained in $\mathfrak{E}_{X,\mathcal{B}}$ if the inclusion functor is an 'exact' functor from $(\mathcal{B}, \mathfrak{E})$ to $(C_X, \mathfrak{E}_X)$. This shows that if $\mathcal{B}$ is a fully exact subcategory of $(C_X, \mathfrak{E}_X)$, then $\mathfrak{E}_{X,\mathcal{B}}$ is the finest right exact structure on $\mathcal{B}$ such that the inclusion functor $\mathcal{B} \longrightarrow C_X$ is an exact functor from $(\mathcal{B}, \mathfrak{E}_{X,\mathcal{B}})$ to $(C_X, \mathfrak{E}_X)$.

2.7. Proposition. *Let $(C_X, \mathfrak{E}_X)$ and $(C_Y, \mathfrak{E}_Y)$ be right exact categories and F a functor $C_X \longrightarrow C_Y$ which maps conflations to conflations. Suppose that the category C_Y is additive. Then the functor F is 'exact'.*

2.7.1. Corollary. *Let $(C_X, \mathfrak{E}_X)$ and $(C_Y, \mathfrak{E}_Y)$ be additive k-linear right exact categories and F an additive functor $C_X \longrightarrow C_Y$. Then the functor F is weakly 'exact' iff it is 'exact'.*

Proof. By 2.5.1, a k-linear functor $C_X \xrightarrow{F} C_Y$ is a weakly 'exact' iff it maps conflations to conflations. The assertion follows now from 2.7. ∎

2.7.2. The property (†). In Proposition 2.7, the assumption that the category C_Y is additive is used only at the end of the proof (part (b)). Moreover, additivity appears there only because it garantees the following property:

(†) if the rows of a commutative diagram

$$\begin{array}{ccccc} \widetilde{L} & \longrightarrow & \widetilde{M} & \longrightarrow & \widetilde{N} \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ L & \longrightarrow & M & \longrightarrow & N \end{array}$$

are conflations and its right and left vertical arrows are isomorphisms, then the middle arrow is an isomorphism.

So that the additivity of C_Y in 2.7 can be replaced by the property (†) for $(C_Y, \mathfrak{E}_Y)$.

2.7.3. An observation. The following obvious observation helps to establish the property (†) for many non-additive right exact categories:

If $(C_X, \mathfrak{E}_X)$ and $(C_Y, \mathfrak{E}_Y)$ are right exact categories and $C_X \xrightarrow{F} C_Y$ is a conservative functor which maps conflations to conflations, then the property (†) holds in $(C_X, \mathfrak{E}_X)$ provided it holds in $(C_Y, \mathfrak{E}_Y)$.

2.7.3.1. Example. Let $(C_Y, \mathfrak{E}_Y)$ be right exact k-linear category, $(C_X, \mathfrak{E}_X)$ a right exact category, and $C_X \xrightarrow{F} C_Y$ is a conservative functor which maps conflations to conflations. Then the property (†) holds in $(C_X, \mathfrak{E}_X)$.

For instance, the property (†) holds for the right exact category $(Alg_k, \mathfrak{E}^{\mathfrak{s}})$ of associative unital k-algebras with strict epimorphisms as deflations, because the forgetful functor $Alg_k \xrightarrow{\mathfrak{f}_*} k-mod$ is conservative, maps deflations to deflations (that is to epimorphisms) and is left exact. Therefore, it maps conflations to conflations.

2.8. Proposition. *(a) Let $(C_X, \mathfrak{E}_X)$ be a svelte right exact category. The Yoneda embedding induces an 'exact' fully faithful functor $(C_X, \mathfrak{E}_X) \xrightarrow{\mathfrak{j}_X^*} (C_{X_\mathfrak{E}}, \mathfrak{E}^{\mathfrak{s}}_{X_\mathfrak{E}})$, where $C_{X_\mathfrak{E}}$ is the category of sheaves of sets on the presite $(C_X, \mathfrak{E}_X)$ and $\mathfrak{E}^{\mathfrak{s}}_{X_\mathfrak{E}}$ the family of all universally strict epimorphisms of $C_{X_\mathfrak{E}}$ (– the canonical structure of a right exact category).*

(b) Let $(C_X, \mathfrak{E}_X)$ and $(C_Y, \mathfrak{E}_Y)$ be right exact categories and $(C_X, \mathfrak{E}_X) \xrightarrow{\varphi^*} (C_Y, \mathfrak{E}_Y)$ a weakly right 'exact' functor. There exists a functor $C_{X_\mathfrak{E}} \xrightarrow{\widetilde{\varphi}^*} C_{Y_\mathfrak{E}}$ such that the diagram

$$\begin{array}{ccc} C_X & \xrightarrow{\varphi^*} & C_Y \\ \mathfrak{j}_X^* \downarrow & & \downarrow \mathfrak{j}_Y^* \\ C_{X_\mathfrak{E}} & \xrightarrow{\widetilde{\varphi}^*} & C_{Y_\mathfrak{E}} \end{array} \qquad (1)$$

quasi commutes, i.e. $\widetilde{\varphi}^* \mathfrak{j}_X^* \simeq \mathfrak{j}_Y^* \varphi^*$. The functor $\widetilde{\varphi}^*$ is defined uniquely up to isomorphism and has a right adjoint, $\widetilde{\varphi}_*$.

Proof. (a) Since the right exact structure $\mathfrak{E}_X$ of C_X is a subcanonical pretopology, the Yoneda embedding takes values in the category $C_{X_\mathfrak{E}}$ of sheaves on $(C_X, \mathfrak{E}_X)$, hence it induces a full embedding of C_X into $C_{X_\mathfrak{E}}$ which preserves all small limits and maps deflations to deflations. In particular it is an 'exact' functor from $(C_X, \mathfrak{E}_X)$ to $(C_{X_\mathfrak{E}}, \mathfrak{E}^{\mathfrak{s}}_{X_\mathfrak{E}})$.

(b) Every weakly right exact functor $(C_X, \mathfrak{E}_X) \longrightarrow (C_Y, \mathfrak{E}_Y)$ determines a continuous (i.e. having a right adjoint) functor between the categories of presheaves of sets, which is compatible with the sheafification functor, hence determines uniquely a continuous functor between the corresponding categories of sheaves making commute the diagram (1). ■

2.9. Application: right exact additive categories and exact categories.

2.9.1. Proposition. *Let $(C_X, \mathfrak{E}_X)$ be an additive k-linear right exact category. Then there exists an exact category $(C_{X_\mathfrak{e}}, \mathfrak{E}_{X_\mathfrak{e}})$ and a fully faithful k-linear 'exact' functor $(C_X, \mathfrak{E}_X) \xrightarrow{\gamma_X^*} (C_{X_\mathfrak{e}}, \mathfrak{E}_{X_\mathfrak{e}})$ which is universal; that is any 'exact' k-linear functor from $(C_X, \mathcal{E}_X)$ to an exact k-linear category factorizes uniquely through γ_X^*.*

Proof. We take as $C_{X_\mathfrak{e}}$ the smallest fully exact subcategory of the category $C_{X_\mathfrak{E}}$ of sheaves of k-modules on $(C_X, \mathfrak{E}_X)$ containing all representable sheaves. Objects of the category $C_{X_{\mathfrak{e}}}$ are sheaves $\mathcal{F}$ such that there exists a finite filtration

$$0 = \mathcal{F}_0 \longrightarrow \mathcal{F}_1 \longrightarrow \ \dots \ \longrightarrow \mathcal{F}_n = \mathcal{F}$$

such that $\mathcal{F}_m/\mathcal{F}_{m-1}$ is representable for $1 \leq m \leq n$. The subcategory $C_{X_{\mathfrak{e}}}$, being a fully exact subcategory of an abelian category, is exact. The remaining details are left as an exercise. ■

3. Satellites in right exact categories.

3.1. Preliminaries: trivial morphisms, pointed objects, and complexes. Let C_X be a category with initial objects. We call a morphism of C_X *trivial* if it factors through an initial object. It follows that an object M is initial iff id_M is a trivial morphism. If C_X is a pointed category, then the trivial morphisms are usually called *zero morphisms*.

3.1.1. Trivial compositions and pointed objects. If the composition of arrows $L \xrightarrow{f} M \xrightarrow{g} N$ is trivial, i.e. there is a commutative square

$$\begin{array}{ccc} L & \xrightarrow{f} & M \\ {\scriptstyle\xi}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ x & \xrightarrow{i_N} & N \end{array}$$

where x is an initial object, and the morphism g has a kernel, then f is the composition of the canonical arrow $Ker(g) \xrightarrow{\mathfrak{k}(g)} M$ and a morphism $L \xrightarrow{f_g} Ker(g)$ uniquely determined by f and ξ. If the arrow $x \xrightarrow{i_N} N$ is a monomorphism, then the morphism ξ is uniquely determined by f and g ; therefore in this case, the arrow f_g does not depend on ξ.

3.1.1.1. Pointed objects. In particular, f_g does not depend on ξ, if N is a *pointed* object. The latter means that therre exists an arrow $N \longrightarrow x$.

3.1.2. Complexes. A sequence of arrows

$$\dots \xrightarrow{f_{n+1}} M_{n+1} \xrightarrow{f_n} M_n \xrightarrow{f_{n-1}} M_{n-1} \xrightarrow{f_{n-2}} \dots \qquad (1)$$

is called a *complex* if each its arrow has a kernel and the next arrow factors *uniquely* through this kernel.

3.1.3. Lemma. *Let each arrow in the sequence*

$$\dots \xrightarrow{f_3} M_3 \xrightarrow{f_2} M_2 \xrightarrow{f_1} M_1 \xrightarrow{f_0} M_0 \qquad (2)$$

of arrows have a kernel and the composition of any two consecutive arrows is trivial. Then

$$\dots \xrightarrow{f_4} M_4 \xrightarrow{f_3} M_3 \xrightarrow{f_2} M_2 \qquad (3)$$

is a complex. If M_0 is a pointed object, then (2) is a complex.

Proof. The objects M_i are pointed for $i \geq 2$, which implies that $(Ker(f_i) \xrightarrow{\mathfrak{k}(f_i)} M_{i+1}$ are monomorphisms for all $i \geq 2$, hence) (3) is a complex (see 3.1.1). ■

3.1.4. Corollary. *A sequence of morphisms*

$$\dots \xrightarrow{f_{n+1}} M_{n+1} \xrightarrow{f_n} M_n \xrightarrow{f_{n-1}} M_{n-1} \xrightarrow{f_{n-2}} \dots$$

unbounded on the right is a complex iff the composition of any pair of its consecutive arrows is trivial and for every i, there exists a kernel of the morphism f_i.

3.1.5. Example. Let C_X be the category Alg_k of unital associative k-algebras. The algebra k is its initial object, and every morphism of k-algebras has a kernel. Pointed objects of C_X which have a morphism to initial object are precisely augmented k-algebras. If the composition of pairs of consecutive arrows in the sequence

$$\dots \xrightarrow{f_3} A_3 \xrightarrow{f_2} A_2 \xrightarrow{f_1} A_1 \xrightarrow{f_0} A_0$$

is trivial, then it follows from the argument of 3.1.2 that A_i is an augmented k-algebra for all $i \geq 2$. And any unbounded on the right sequence of algebras with trivial compositions of pairs of consecutive arrows is formed by augmented algebras.

3.1.6. 'Exact' complexes. Let $(C_X, \mathcal{E}_X)$ be a right exact category with an initial object. We call a sequence of two arrows $L \xrightarrow{f} M \xrightarrow{g} N$ in C_X 'exact' if the arrow g has a kernel, and f is the composition of $Ker(g) \xrightarrow{\mathfrak{k}(g)} M$ and a deflation $L \xrightarrow{f_g} Ker(g)$. A complex is called 'exact' if any pair of its consecutive arrows forms an 'exact' sequence.

3.2. ∂^*-functors. Fix a right exact category $(C_X, \mathfrak{E}_X)$ with an initial object x and a category C_Y with an initial object. A *∂^*-functor* from $(C_X, \mathcal{E}_X)$ to C_Y is a system of functors $C_X \xrightarrow{T_i} C_Y$, $i \geq 0$, together with a functorial assignment to every conflation $E = (N \xrightarrow{\mathfrak{j}} M \xrightarrow{\mathfrak{e}} L)$ and every $i \geq 0$ a morphism $T_{i+1}(L) \xrightarrow{\mathfrak{d}_i(E)} T_i(N)$ which depends functorially on the conflation E and such that the sequence of arrows

$$\dots \xrightarrow{T_2(\mathfrak{e})} T_2(L) \xrightarrow{\mathfrak{d}_1(E)} T_1(N) \xrightarrow{T_1(\mathfrak{j})} T_1(M) \xrightarrow{T_1(e)} T_1(L) \xrightarrow{\mathfrak{d}_0(E)} T_0(N) \xrightarrow{T_0(\mathfrak{j})} T_0(M)$$

is a complex. Taking the trivial conflation $x \longrightarrow x \longrightarrow x$, we obtain that $T_i(x) \xrightarrow{id_{T_i(x)}} T_i(x)$ is a trivial morphism, or, equivalently, $T_i(x)$ is an initial object, for every $i \geq 1$.

Let $T = (T_i, \mathfrak{d}_i |\ i \geq 0)$ and $T' = (T'_i, \mathfrak{d}'_i |\ i \geq 0)$ be a pair of ∂^*-functors from $(C_X, \mathcal{E}_X)$ to C_Y. A morphism from T to T' is a family $f = (T_i \xrightarrow{f_i} T'_i \mid i \geq 0)$ of functor morphisms such that for any conflation $E = (N \xrightarrow{\mathfrak{j}} M \xrightarrow{\mathfrak{e}} L)$ of the exact category C_X and every $i \geq 0$, the diagram

$$\begin{array}{ccc} T_{i+1}(L) & \xrightarrow{\mathfrak{d}_i(E)} & T_i(N) \\ {\scriptstyle f_{i+1}(L)}\downarrow & & \downarrow{\scriptstyle f_i(N)} \\ T'_{i+1}(L) & \xrightarrow{\mathfrak{d}'_i(E)} & T'_i(N) \end{array}$$

commutes. The composition of morphisms is naturally defined. Thus, we have the category $\mathcal{H}om^*((C_X, \mathfrak{E}_X), C_Y)$ of ∂^*-functors from $(C_X, \mathcal{E}_X)$ to C_Y.

3.2.1. Trivial ∂^*-functors. We call a ∂^*-functor $T = (T_i, \mathfrak{d}_i|\ i \geq 0)$ *trivial* if all T_i are functors with values in initial objects. One can see that trivial ∂^*-functors are precisely initial objects of the category $\mathcal{H}om^*((C_X, \mathfrak{E}_X), C_Y)$. Once an initial object y of the category C_Y is fixed, we have a canonical trivial functor whose components equal to the constant functor with value in y – it maps all arrows of C_X to id_y.

3.2.2. Some natural functorialities. Let $(C_X, \mathfrak{E}_X)$ be a right exact category with an initial object and C_Y a category with initial object. If C_Z is another category with an initial object and $C_Y \xrightarrow{F} C_Z$ a functor which maps initial objects to initial objects, then for any ∂^*-functor $T = (T_i, \mathfrak{d}_i|\ i \geq 0)$, the composition $F \circ T = (F \circ T_i, F\mathfrak{d}_i|\ i \geq 0)$ of T with F is a ∂^*-functor. The map $(F, T) \longmapsto F \circ T$ is functorial in both variables; i.e. it extends to a functor

$$Cat_*(C_Y, C_Z) \times \mathcal{H}om^*((C_X, \mathfrak{E}_X), C_Y) \longrightarrow \mathcal{H}om^*((C_X, \mathfrak{E}_X), C_Z). \tag{1}$$

Here Cat_* denotes the subcategory of Cat whose objects are categories with initial objects and morphisms are functors which map initial objects to initial objects.

On the other hand, let $(C_{\mathfrak{X}}, \mathfrak{E}_{\mathfrak{X}})$ be another right exact category with an initial object and Φ a functor $C_{\mathfrak{X}} \longrightarrow C_X$ which maps conflations to conflations. In particular, it maps initial objects to initial objects (because if x is an initial object of $C_{\mathfrak{X}}$, then $x \longrightarrow M \xrightarrow{id_M} M$ is a conflation; and $\Phi(x \longrightarrow M \xrightarrow{id_M} M)$ being a conflation implies that $\Phi(x)$ is an initial object). For any ∂^*-functor $T = (T_i, \mathfrak{d}_i|\ i \geq 0)$ from $(C_X, \mathfrak{E}_X)$ to C_Y, the composition $T \circ \Phi = (T_i \circ \Phi, \mathfrak{d}_i\Phi|\ i \geq 0)$ is a ∂^*-functor from $(C_{\mathfrak{X}}, \mathfrak{E}_{\mathfrak{X}})$ to C_Y. The map $(T, \Phi) \longmapsto T \circ \Phi$ extends to a functor

$$\mathcal{H}om^*((C_X, \mathfrak{E}_X), C_Y) \times \mathcal{E}x_*((C_{\mathfrak{X}}, \mathfrak{E}_{\mathfrak{X}}), (C_X, \mathfrak{E}_X)) \longrightarrow \mathcal{H}om^*((C_{\mathfrak{X}}, \mathfrak{E}_{\mathfrak{X}}), C_Y), \tag{2}$$

where $\mathcal{E}x_*((C_{\mathfrak{X}}, \mathfrak{E}_{\mathfrak{X}}), (C_X, \mathfrak{E}_X))$ denotes the full subcategory of $\mathcal{H}om(C_{\mathfrak{X}}, C_X)$ whose objects are preserving conflations functors $C_{\mathfrak{X}} \longrightarrow C_X$.

3.3. Universal ∂^*-functors. Fix a right exact category $(C_X, \mathfrak{E}_X)$ with an initial object x and a category C_Y with an initial object y.

A ∂^*-functor $T = (T_i, \mathfrak{d}_i|\ i \geq 0)$ from $(C_X, \mathcal{E}_X)$ to C_Y is called *universal* if for every ∂^*-functor $T' = (T'_i, \mathfrak{d}'_i|\ i \geq 0)$ from $(C_X, \mathcal{E}_X)$ to C_Y and every functor morphism $T'_0 \xrightarrow{g} T_0$, there exists a unique morphism $f = (T'_i \xrightarrow{f_i} T_i \mid i \geq 0)$ from T' to T such that $f_0 = g$.

3.3.1. Interpretation. Consider the functor

$$\mathcal{H}om^*((C_X, \mathfrak{E}_X), C_Y) \xrightarrow{\Psi^*} \mathcal{H}om(C_X, C_Y) \tag{3}$$

which assigns to every ∂^*-functor (resp. every morphism of ∂^*-functors) its zero component. For any functor $C_X \xrightarrow{F} C_Y$, we have a presheaf of sets $\mathcal{H}om(\Psi^*(-), F)$ on the category $\mathcal{H}om^*((C_X, \mathfrak{E}_X), C_Y)$. Suppose that this presheaf is representable by an object (i.e. a ∂^*-functor) $\Psi_*(F)$. Then $\Psi_*(F)$ is a universal ∂^*-functor.

Conversely, if $T = (T_i, \mathfrak{d}_i |\ i \geq 0)$ is a universal ∂^*-functor, then $T \simeq \Psi_*(T_0)$.

3.3.2. Proposition. *Let $(C_X, \mathfrak{E}_X)$ be a right exact category with an initial object x; and let C_Y be a category with initial objects, kernels of morphisms, and limits of filtered systems. Then, for any functor $C_X \xrightarrow{F} C_Y$, there exists a unique up to isomorphism universal ∂^*-functor $T = (T_i, \mathfrak{d}_i |\ i \geq 0)$ such that $T_0 = F$.*

In other words, the functor

$$\mathcal{H}om^*((C_X, \mathfrak{E}_X), C_Y) \xrightarrow{\Psi^*} \mathcal{H}om(C_X, C_Y) \qquad (3)$$

which assigns to each morphism of ∂^*-functors its zero component has a right adjoint, Ψ_*.

Proof. For an arbitrary functor $C_X \xrightarrow{F} C_Y$, we set

$$S_-(F)(L) = \lim Ker(F(\mathfrak{k}(\mathfrak{e}))),$$

where the limit is taken by the (filtered) system of all deflations $M \xrightarrow{\mathfrak{e}} L$. Since deflations form a pretopology, the map $L \longmapsto S_-(F)(L)$ extends naturally to a functor $C_X \xrightarrow{S_-(F)} C_Y$. By the definition of $S_-(F)$, for any conflation $E = (N \xrightarrow{\mathfrak{j}} M \xrightarrow{\mathfrak{e}} L)$, there exists a unique morphism $S_-(F)(L) \xrightarrow{\widetilde{\mathfrak{d}}^0_F(E)} Ker(F(\mathfrak{j}))$. We denote by $\mathfrak{d}^0_F(E)$ the composition of $\widetilde{\mathfrak{d}}^0_F(E)$ and the canonical morphism $Ker(F(j)) \longrightarrow F(N)$.

Notice that the correspondence $F \longmapsto (S_-(F), \mathfrak{d}^0_F)$ is functorial. Applying the iterations of the functor S_- to F, we obtain a ∂^*-functor $S_-^{\bullet}(F) = (S_-^i(F)|i \geq 0)$. This ∂^*-functor is universal. ∎

3.3.3. Remark. Let the assumptions of 3.3.2 hold. Then we have a pair of adjoint functors

$$\mathcal{H}om^*((C_X, \mathfrak{E}_X), C_Y) \xrightarrow{\Psi^*} \mathcal{H}om(C_X, C_Y) \xrightarrow{\Psi_*} \mathcal{H}om^*((C_X, \mathfrak{E}_X), C_Y)$$

By 3.3.2, the adjunction morphism $\Psi^*\Psi_* \longrightarrow Id$ is an isomorphism which means that Ψ_* is a fully faithful functor and Ψ^* is a localization functor at a left multiplicative system.

3.3.4. Proposition. *Let $(C_X, \mathfrak{E}_X)$ be a right exact category with an initial object and $T = (T_i, \mathfrak{d}_i \mid i \geq 0)$ a ∂^*-functor from $(C_X, \mathcal{E}_X)$ to C_Y. Let C_Z be another category with an initial object and F a functor from C_Y to C_Z which preserves initial objects, kernels of morphisms and limits of filtered systems. Then*

(a) If T is a universal ∂^*-functor, then $F \circ T = (F \circ T_i, F\mathfrak{d}_i |\ i \geq 0)$ is universal.

(b) If, in addition, the functor F is fully faithful, then the ∂^*-functor $F \circ T$ is universal iff T is universal.

Proof. (a) Since the functor F preserves kernels of morphisms and filtered limits (that is all types of limits which appear in the construction of $S_-(G)(L)$), the natural morphism

$$F \circ S_-(G)(L) \longrightarrow S_-(F \circ G)(L)$$

is an isomorphism for any functor $C_X \xrightarrow{G} C_Y$ such that $S_-(G)(L) = \lim Ker(G(\mathfrak{k}(\mathfrak{e})))$ exists. Moreover, $\mathfrak{d}_0^{F\circ G}$ is naturally isomorphic to $F\mathfrak{d}_0^G$. Here *naturally isomorphic* means that for any conflation $E = (N \xrightarrow{\mathfrak{j}} M \xrightarrow{\mathfrak{e}} L)$, there is a commutative diagram

$$\begin{array}{ccc} F\circ S_-(G)(L) & \xrightarrow{F\mathfrak{d}_0^G(E)} & F\circ G(N) \\ \wr\downarrow & & \downarrow id \\ S_-(F\circ G)(L) & \xrightarrow{\mathfrak{d}_0^{FG}(E)} & F\circ G(N) \end{array}$$

commutes. Therefore, the natural morphisms $F\circ S_-^i(T_0) \xrightarrow{\varphi_i} S_-^i(F\circ T_0)$ are are isomorphisms for all $i\geq 0$ and $\varphi = (\varphi_i \mid i\geq 0)$ is an isomorphism of ∂^*-functors

$$(F\circ S_-^i(T_0), F\mathfrak{d}_i^{T_0} \mid i\geq 0) \xrightarrow{\sim} (S_-^i(F\circ T_0), \mathfrak{d}_i^{F\circ T_0} \mid i\geq 0).$$

(b) By (a), we have a functor isomorphism $F\circ T_{i+1} \xrightarrow{\sim} F\circ S_-(T_i)$ for all $i\geq 0$. Since the functor F is fully faithful, this isomorphism is the image of a uniquely determined isomorphism $T_{i+1} \xrightarrow{\sim} S_-(T_i)$. The assertion follows now from (the argument of) 3.3.2. Details are left as an exercise. ∎

3.3.5. An application. Let $(C_X, \mathfrak{E}_X)$ be a right exact category and C_Y a category. We assume that both categories, C_X and C_Y have initial objects. Consider the Yoneda embedding

$$C_Y \xrightarrow{h_Y} C_Y^\wedge, \quad M \longmapsto \widehat{M} = C_Y(-, M).$$

of the category C_Y into the category $C_Y^\wedge$ of presheaves of sets on C_Y. The functor h_Y is fully faithful and preserves all limits. In particular, it satisfies the conditions of 3.3.4(b). Therefore, a ∂^*-functor $T = (T_i, \mathfrak{d}_i \mid i\geq 0)$ from $(C_X, \mathcal{E}_X)$ to C_Y is universal iff the ∂^*-functor $\widehat{T} \stackrel{\mathrm{def}}{=} h_Y\circ T = (\widehat{T}_i, \widehat{\mathfrak{d}}_i \mid i\geq 0)$ from $(C_X, \mathcal{E}_X)$ to $C_Y^\wedge$ is universal.

Since the category $C_Y^\wedge$ has all limits (and colimits), it follows from 3.3.2 that, for any functor $C_X \xrightarrow{G} C_Y^\wedge$, there exists a unique up to isomorphism universal ∂^*-functor $T = (T_i, \mathfrak{d}_i | \ i\geq 0) = \Psi_*(G)$ whose zero component coincides with G. In particular, for every functor $C_X \xrightarrow{F} C_Y$, there exists a unique up to isomorphism universal ∂^*-functor $T = (T_i, \mathfrak{d}_i \mid i\geq 0)$ from $(C_X, \mathfrak{E}_X)$ such that $T_0 = h_Y\circ F = \widetilde{F}$. It follows from 3.3.4(b) that a universal ∂^*-functor whose zero component coincides with F exists if and only if for all $L\in ObC_X$ and all $i\geq 1$, the presheaves of sets $T_i(L)$ are representable.

3.3.6. Remark. Let $(C_X, \mathfrak{E}_X)$ be a svelte right exact category with an initial object x and C_Y a category with an initial object y and limits. Then, by the argument of 3.3.2, we have an endofunctor S_- of the category $\mathcal{H}om(C_X, C_Y)$ of functors from C_X to C_Y, together with a cone $S_- \xrightarrow{\lambda} \mathfrak{y}$, where $\mathfrak{y}$ is the constant functor with the values in the initial object y of the category C_Y. For any conflation $E = (N \xrightarrow{\mathfrak{j}} M \xrightarrow{\mathfrak{e}} L)$ of $(C_X, \mathfrak{E}_X)$

and any functor $C_X \xrightarrow{F} C_Y$, we have a commutative diagram

$$\begin{array}{ccccc} S_-F(L) & \xrightarrow{\lambda(L)} & y & & \\ {\scriptstyle \mathfrak{d}_0(E)} \downarrow & & \downarrow & & \\ F(N) & \xrightarrow{F\mathfrak{j}} & F(M) & \xrightarrow{F\mathfrak{e}} & F(L) \end{array}$$

3.4. The dual picture: ∂-functors and universal ∂-functors. Let $(C_X, \mathfrak{I}_X)$ be a left exact category, which means by definition that $(C_X^{op}, \mathfrak{I}_X^{op})$ is a right exact category. A ∂-functor on $(C_X, \mathfrak{I}_X)$ is the data which becomes a ∂^*-functor in the dual right exact category. A ∂-functor on $(C_X, \mathfrak{I}_X)$ is *universal* if its dualization is a universal ∂^*-functor. We leave to the reader the reformulation in the context of ∂-functors of all notions and facts about ∂^*-functors.

3.5. Universal ∂^*-functors and 'exactness'.

3.5.1. The properties $(CE5)$ and $(CE5^*)$. Let $(C_X, \mathfrak{E}_X)$ be a right exact category. We say that it satisfies $(CE5^*)$ (resp. $(CE5)$) if the limit of a filtered system (resp. the colimit of a cofiltered system) of conflations in $(C_Y, \mathfrak{E}_Y)$ exists and is a conflation.

In particular, if $(C_X, \mathfrak{E}_X)$ satisfies $(CE5^*)$ (resp. $(CE5)$), then the limit of any filtered system (resp. the colimit of any cofiltered system) of deflations is a deflation.

The properties $(CE5)$ and $(CE5^*)$ make sense for left exact categories as well. Notice that a right exact category satisfies $(CE5^*)$ (resp. $(CE5)$) iff the dual left exact category satisfies $(CE5)$ (resp. $(CE5^*)$).

3.5.2. Note. If $(C_X, \mathfrak{E}_X)$ is an abelian category with the canonical exact structure, then the property $(CE5)$ for $(C_X, \mathfrak{E}_X)$ is equivalent to the Grothendieck's property (AB5) and, therefore, the property $(CE5^*)$ is equivalent to $(AB5^*)$ (see [Gr, 1.5]).

The property $(CE5)$ holds for Grothendieck toposes.

In what follows, we use $(CE5^*)$ for right exact categories and the dual property $(CE5)$ for left exact categories.

3.5.3. Proposition. *Let $(C_X, \mathfrak{E}_X)$, $(C_Y, \mathfrak{E}_Y)$ be right exact categories, and $(C_Y, \mathfrak{E}_Y)$ satisfy $(CE5^*)$. Let F be a weakly right 'exact' functor $(C_X, \mathfrak{E}_X) \longrightarrow (C_Y, \mathfrak{E}_Y)$ such that $S_-(F)$ exists. Then for any conflation $E = (N \xrightarrow{\mathfrak{j}} M \xrightarrow{\mathfrak{e}} L)$ in $(C_X, \mathfrak{E}_X)$, the sequence*

$$S_-(F)(N) \xrightarrow{S_-(F)(\mathfrak{j})} S_-(F)(M) \xrightarrow{S_-(F)(\mathfrak{e})} S_-(F)(L) \xrightarrow{\mathfrak{d}_0(E)} F(N) \xrightarrow{F(\mathfrak{j})} F(M) \qquad (1)$$

is 'exact'. The functor $S_-(F)$ is a weakly right 'exact' functor from $(C_X, \mathfrak{E}_X)$ to $(C_Y, \mathfrak{E}_Y)$.

3.5.4. 'Exact' ∂^*-functors and universal ∂^*-functors. Fix right exact categories $(C_X, \mathfrak{E}_X)$ and $(C_Y, \mathfrak{E}_Y)$, both with initial objects. A ∂^*-functor $T = (T_i, \mathfrak{d}_i |\ i \geq 0)$ from $(C_X, \mathfrak{E}_X)$ to C_Y is called *'exact'* if for every conflation $E = (N \xrightarrow{\mathfrak{j}} M \xrightarrow{\mathfrak{e}} L)$ in $(C_X, \mathfrak{E}_X)$, the complex

$$\ldots \xrightarrow{T_2(\mathfrak{e})} T_2(L) \xrightarrow{\mathfrak{d}_1(E)} T_1(N) \xrightarrow{T_1(\mathfrak{j})} T_i(M) \xrightarrow{T_1(e)} T_1(L) \xrightarrow{\mathfrak{d}_0(E)} T_0(N) \xrightarrow{T_0(\mathfrak{j})} T_0(M)$$

is 'exact'.

3.5.4.1. Proposition. *Let* $(C_X, \mathfrak{E}_X)$, $(C_Y, \mathfrak{E}_Y)$ *be right exact categories. Suppose that* $(C_Y, \mathfrak{E}_Y)$ *satisfies* $(CE5^*)$. *Let* $T = (T_i|\ i \geq 0)$ *be a universal* ∂^**-functor from* $(C_X, \mathfrak{E}_X)$ *to* $(C_Y, \mathfrak{E}_Y)$. *If the functor* T_0 *is right 'exact', then the universal* ∂^**-functor* T *is 'exact'.*

Proof. If T_0 is right 'exact', then, by 3.5.3, the functor $T_1 \simeq S_-(T_0)$ is right 'exact' and for any conflation $E = (N \xrightarrow{\mathfrak{j}} M \xrightarrow{\mathfrak{e}} L)$, the sequence

$$T_1(N) \xrightarrow{T_1(\mathfrak{j})} T_1(M) \xrightarrow{T_1(e)} T_1(L) \xrightarrow{\mathfrak{d}_0(E)} T_0(N) \xrightarrow{T_0(\mathfrak{j})} T_0(M)$$

is 'exact'. Since $T_{n+1} = S_-(T_n)$, the assertion follows from 3.5.3 by induction. ■

3.5.4.2. Corollary. *Let* $(C_X, \mathfrak{E}_X)$ *be a right exact category. For each object* L *of* C_X, *the* ∂*-functor* $Ext^\bullet_X(L) = (Ext^i_X(L)\ |\ i \geq 0)$ *is 'exact'.*

Suppose that the category C_X *is* k*-linear. Then for each* $L \in ObC_X$, *the* ∂*-functor* $\mathcal{E}xt^\bullet_X(L) = (\mathcal{E}xt^i_X(L)\ |\ i \geq 0)$ *is 'exact'.*

Proof. In fact, the ∂-functor $Ext^\bullet_X(L)$ is universal by definition (see 3.4.1), and the functor $Ext^0_X(L) = C_X(-, L)$ is left exact. In particular, it is left 'exact'.

If C_X is a k-linear category, then the universal ∂-functors $\mathcal{E}xt^\bullet_X(L)$, $L \in ObC_X$, with the values in the category of k-modules (defined in 3.4.2) are 'exact' by a similar reason. ■

4. Coeffaceable functors, universal ∂^*-functors, and pointed projectives.

4.1. Projectives and projective deflations. Fix a right exact category $(C_X, \mathfrak{E}_X)$. We call an object P of C_X *projective* if every deflation $M \longrightarrow P$ splits. Equivalently, any morphism $P \xrightarrow{f} N$ factors through any deflation $M \xrightarrow{\mathfrak{e}} N$.

We denote by $\mathcal{P}_{\mathfrak{E}_X}$ the full subcategory of C_X generated by projective objects.

4.1.1. Example. Let $(C_X, \mathfrak{E}_X)$ be a right exact category whose deflations split. Then every object of C_X is a projective object of $(C_X, \mathfrak{E}_X)$.

A deflation $M \longrightarrow L$ is called *projective* if it factors through any deflation of L.

Any deflation $P \longrightarrow L$ with P projective is a projective deflation. On the other hand, an object P is projective iff the identical morphism $P \longrightarrow P$ is a projective deflation.

4.2. Coeffaceble functors and projectives. Let $(C_X, \mathfrak{E}_X)$ be a right exact category and C_Y a category with an initial object. We call a functor $C_X \xrightarrow{F} C_Y$ *coeffaceable*, or $\mathfrak{E}_X$*-coeffaceable*, if for any object L of C_X, there exists a deflation $M \xrightarrow{\mathfrak{t}} L$ such that $F(\mathfrak{t})$ is a trivial morphism.

It follows that if a functor $C_X \xrightarrow{F} C_Y$ is $\mathfrak{E}_X$-coeffaceable, then it maps all projectives to initial objects and all projective deflations to trivial arrows.

So that if the right exact category $(C_X, \mathfrak{E}_X)$ has enough projective deflations (resp. enough projectives), then a functor $C_X \xrightarrow{F} C_Y$ is $\mathfrak{E}_X$-coeffaceable iff $F(\mathfrak{e})$ is trivial for any projective deflation $\mathfrak{e}$ (resp. $F(M)$ is an initial object for every projective object M).

4.3. Proposition. *Let* $(C_X, \mathfrak{E}_X)$ *be a right exact category with initial objects and* $T = (T_i, \mathfrak{d}_i \mid i \geq 0)$ *a universal* ∂^**-functor from* $(C_X, \mathfrak{E}_X)$ *to* C_Y*. Then* $T_i(P)$ *is an initial object for any pointed projective object* P *and for all* $i \geq 1$*.*

4.3.1. Corollary. *Let* $(C_X, \mathfrak{E}_X)$ *be a right exact category with initial objects and* $T = (T_i, \mathfrak{d}_i \mid i \geq 0)$ *a universal* ∂^**-functor from* $(C_X, \mathfrak{E}_X)$ *to* C_Y*. Suppose that* $(C_X, \mathfrak{E}_X)$ *has enough projectives and projectives of* $(C_X, \mathfrak{E}_X)$ *are pointed objects. Then the functors* T_i *are coeffaceable for all* $i \geq 1$*.*

Proof. The assertion follows from 4.3 and 4.2. ∎

4.4. Proposition. *Let* $(C_X, \mathfrak{E}_X)$ *and* $(C_Y, \mathfrak{E}_Y)$ *be right exact categories with initial objects; and let* $T = (T_i, \mathfrak{d}_i | \ i \geq 0)$ *be an 'exact'* ∂^**-functor from* $(C_X, \mathcal{E}_X)$ *to* $(C_Y, \mathfrak{E}_Y)$*.*

If the functors T_i *are* $\mathfrak{E}_X$*-coeffaceable for* $i \geq 1$*, then* T *is a universal* ∂^**-functor.*

Proof. The argument is similar to the proof in [Gr] of the corresponding assertion for abelian categories. ∎

4.5. Proposition. *Let* $(C_X, \mathfrak{E}_X)$*,* $(C_Y, \mathfrak{E}_Y)$*, and* $(C_Z, \mathfrak{E}_Z)$ *be right exact categories. Suppose that* $(C_X, \mathfrak{E}_X)$ *has enough projectives and* C_Y *has kernels of all morphisms. If* $T = (T_i | \ i \geq 0)$ *is a universal, 'exact'* ∂^**-functor from* $(C_X, \mathfrak{E}_X)$ *to* $(C_Y, \mathfrak{E}_Y)$ *and* F *a functor from* $(C_Y, \mathfrak{E}_Y)$ *to* $(C_Z, \mathfrak{E}_Z)$ *which respects conflations, then the composition* $F \circ T = (F \circ T_i | \ i \geq 0)$ *is a universal 'exact'* ∂^**-functor.*

Proof. Under the conditions of the proposition, the composition $F \circ T$ is an 'exact' functor such that the functors $F \circ T_i$, $i \geq 1$, map pointed projectives of $(C_X, \mathfrak{E}_X)$ to trivial objects (because T_i map pointed projectives to trivial objects by 4.3 and F maps trivial objects to trivial objects). Since there are enough pointed projectives (hence all projectives are pointed), this implies that the functors $F \circ T_i$ are coeffaceable for $i \geq 1$. Therefore, by 4.4, $F \circ T$ is a universal ∂^*-functor. ∎

4.6. Sufficient conditions for having enough pointed projectives.

4.6.1. Proposition. *Let* $(C_X, \mathfrak{E}_X)$ *and* $(C_Z, \mathfrak{E}_Z)$ *be right exact categories and* $C_Z \xrightarrow{f^*} C_X$ *a functor having a right adjoint* f_**. Suppose that* f^* *maps deflations of the form* $N \longrightarrow f_*(M)$ *to deflations and the adjunction arrow* $f^* f_*(M) \xrightarrow{\epsilon(M)} M$ *is a deflation for all* M *(which is the case if any morphism* $\mathfrak{t}$ *of* C_X *such that* $f_*(\mathfrak{t})$ *is a split epimorphism belongs to* $\mathfrak{E}_X$*). Let* $(C_Z, \mathfrak{E}_Z)$ *have enough projectives, and all projectives are pointed objects. Then each projective of* $(C_X, \mathfrak{E}_X)$ *is a pointed object.*

If, in addition, f_* *maps deflations to deflations, then* $(C_X, \mathfrak{E}_X)$ *has enough projectives.*

4.6.2. Note. The conditions of 4.6.1 can be replaced by the requirement that if $N \longrightarrow f_*(M)$ is a deflation, then the corresponding morphism $f^*(N) \longrightarrow M$ is a deflation. This requirement follows from the conditions of 4.6.1, because the morphism $f^*(N) \longrightarrow M$ corresponding to $N \xrightarrow{\mathfrak{t}} f_*(M)$ is the composition of $f^*(\mathfrak{t})$ and the adjunction arrow $f^* f_*(M) \xrightarrow{\epsilon(M)} M$.

4.6.3. Example. Let $(C_X, \mathfrak{E}_X)$ be the category Alg_k of associative k-algebras endowed with the canonical (that is the finest) right exact structure. This means that class

$\mathfrak{E}_X$ of deflations coincides with the class of all are strict epimorphisms of k-algebras. Let $(C_Y, \mathfrak{E}_Y)$ be the category of k-modules with the canonical exact structure, and f_* the forgetful functor $Alg_k \longrightarrow k - mod$. Its left adjoint, f^* preserves strict epimorphisms, and the functor f_* preserves and reflects deflations; i.e. a k-algebra morphism t is a strict epimorphism iff $f_*(t)$ is an epimorphism. In particular, the adjunction arrow $f^*f_*(A) \longrightarrow A$ is a strict epimorphism for all A. By 4.6.1, $(C_X, \mathfrak{E}_X)$ has enough projectives and each projective has a morphism to the initial object k; that is each projective has a structure of an augmented k-algebra.

4.7. Acyclic objects and the universality of ∂^*-functors. Given a ∂^*-functor $T = (T_i|\ i \geq 0)$ from a right exact category $(C_X, \mathfrak{E}_X)$ to a category C_Y, we call an object M of C_X *T-acyclic* if $T_i(M)$ is a trivial object for all $i \geq 1$.

4.7.1. Proposition. *Let $(C_{\mathfrak{A}}, \mathfrak{E}_{\mathfrak{A}})$ and $(C_X, \mathfrak{E}_X)$ be right exact categories with initial objects and $C_{\mathfrak{A}} \xrightarrow{G} C_X$ a functor which preserves conflations. Let $T = (T_i|\ i \geq 0)$ be an 'exact' ∂^*-functor from $(C_X, \mathfrak{E}_X)$ to a category $C_{\mathcal{Z}}$ with initial objects. If there are enough objects M of C_X such that $G(M)$ is a T-acyclic object, then $T \circ G$ is a universal ∂^*-functor.*

Proof. Since the functor G maps conflation to conflations, and the ∂^*-functor T is 'exact', its composition $T \circ G$ is an 'exact' ∂^*-functor. Since there are *enough* objects in $C_{\mathfrak{A}}$ which the functor G maps to acyclic objects (i.e. for each object L of $C_{\mathfrak{A}}$, there is a deflation $M \longrightarrow L$ such that $G(M)$ is T-acyclic), the functor $T_i \circ G$ is effaceable for all $i \geq 1$. Therefore, by 4.6, the composition $T \circ G$ is a universal ∂^*-functor. ∎

5. Universal problems for universal ∂^*- and ∂-functors.

5.1. The categories of universal ∂^*- and ∂-functors. Fix a right exact svelte category $(C_X, \mathfrak{E}_X)$ with an initial object. Let $\partial^*\mathfrak{Un}(X, \mathfrak{E}_X)$ denote the category whose objects are universal ∂^*-functors from $(C_X, \mathfrak{E}_X)$ to categories C_Y (with initial objects). Let T be a universal ∂^*- functor from $(C_X, \mathfrak{E}_X)$ to C_Y and $\widetilde{T}$ a universal ∂^*- functor from $(C_X, \mathfrak{E}_X)$ to C_Z. A morphism from T to T' is a pair (F, ϕ), where F is a functor from C_Y to C_Z and ϕ is a ∂^*-functor isomorphism $F \circ T \xrightarrow{\sim} T'$. If (F', ϕ') is a morphism from T' to T'', then the composition of (F, ϕ) and (F', ϕ') is defined by $(F', \phi') \circ (F, \phi) = (F' \circ F, \phi' \circ F'\phi)$.

Dually, for a left exact category $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ with a final object, we denote by $\partial\mathfrak{Un}(\mathfrak{X}, \mathfrak{I}_{\mathfrak{X}})$ the category whose objects are universal ∂-functors from $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ to categories with final object. Given two universal ∂-functors T and T' from $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ to respectively C_Y and C_Z, a morphism from T to T' is a pair (F, ψ), where F is a functor from C_Y to C_Z and ψ is a functor isomorphism $T' \xrightarrow{\sim} F \circ T$. The composition is defined by $(F', \psi') \circ (F, \psi) = (F' \circ F, F'\psi \circ \psi')$.

5.2. Universal problems for universal ∂-functors with values in complete categories and ∂^*-functors with values in cocomplete categories.

Let $(C_X, \mathfrak{E}_X)$ be a svelte right exact category. We denote by $\partial^*\mathfrak{Un}_{\mathfrak{c}}(X, \mathfrak{E}_X)$ the subcategory of $\partial^*\mathfrak{Un}(X, \mathfrak{E}_X)$ whose objects are universal ∂^*-functors from $(C_X, \mathfrak{E}_X)$ to *complete* (i.e. having limits of small diagrams) categories C_Y and morphisms between these universal ∂^*-functors are pairs (F, ϕ), where the functor F preserves limits.

For a svelte left exact category $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$, we denote by $\partial\mathfrak{Un}^{\mathfrak{c}}(\mathfrak{X}, \mathfrak{I}_{\mathfrak{X}})$ the subcategory of $\partial\mathfrak{Un}(\mathfrak{X}, \mathfrak{I}_{\mathfrak{X}})$ whose objects are ∂-functors with values in cocomplete categories and morphisms are pairs (F, ψ) such that the functor F preserves small colimits.

5.2.1. Proposition. *Let $(C_X, \mathfrak{E}_X)$ be a svelte right exact category with initial objects and $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ a svelte left exact category with final objects. The categories $\partial^*\mathfrak{Un}_{\mathfrak{c}}(X, \mathfrak{E}_X)$ and $\partial\mathfrak{Un}^{\mathfrak{c}}(\mathfrak{X}, \mathfrak{I}_{\mathfrak{X}})$ have initial objects.*

Proof. It suffices to prove the assertion about $\partial\mathfrak{Un}^{\mathfrak{c}}(\mathfrak{X}, \mathfrak{I}_{\mathfrak{X}})$, because the assertion about ∂^*-functors is obtained via dualization.

Consider the Yoneda embedding

$$C_{\mathfrak{X}} \xrightarrow{\ h_{\mathfrak{X}}\ } C_X^{\wedge}, \quad M \longmapsto C_{\mathfrak{X}}(-, M).$$

We denote by $Ext^{\bullet}_{\mathfrak{X},\mathfrak{I}_{\mathfrak{X}}}$ the universal ∂-functor from $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ to $C^{\wedge}_{\mathfrak{X}}$ such that $Ext^0_{\mathfrak{X},\mathfrak{I}_{\mathfrak{X}}} = h_{\mathfrak{X}}$. The claim is that $Ext^{\bullet}_{\mathfrak{X},\mathfrak{I}_{\mathfrak{X}}}$ is an initial object of the category $\partial\mathfrak{Un}^{\mathfrak{c}}(\mathfrak{X}, \mathfrak{I}_{\mathfrak{X}})$.

In fact, let C_Y be a cocomplete category. By [GZ, II.1.3], the composition with the Yoneda embedding $C_{\mathfrak{X}} \xrightarrow{h_{\mathfrak{X}}} C^{\wedge}_{\mathfrak{X}}$ is an equivalence between the category $\mathcal{H}om_{\mathfrak{c}}(C^{\wedge}_{\mathfrak{X}}, C_Y)$ of *continuous* (that is having a right adjoint, or, equivalently, preserving colimits) functors $C^{\wedge}_{\mathfrak{X}} \longrightarrow C_Y$ and the category $\mathcal{H}om(C_{\mathfrak{X}}, C_Y)$ of functors from $C_{\mathfrak{X}}$ to C_Y. Let $C_{\mathfrak{X}} \xrightarrow{F} C_Y$ be an arbitrary functor and $C^{\wedge}_{\mathfrak{X}} \xrightarrow{F_{\mathfrak{c}}} C_Y$ the corresponding continuous functor. By definition, $S_+F(L) = \mathrm{colim}(Cok(F(M \longrightarrow Cok(\mathfrak{j}))$, where $L \xrightarrow{\mathfrak{j}} M$ runs through inflations of L. Since $F_{\mathfrak{c}}$ preserves colimits, it follows from (the dual version of) 3.3.4(a) that $F_{\mathfrak{c}} \circ Ext^{\bullet}_{\mathfrak{X},\mathfrak{I}_{\mathfrak{X}}}$ is a universal ∂-functor whose zero component is $F_{\mathfrak{c}} \circ Ext^0_{\mathfrak{X},\mathfrak{I}_{\mathfrak{X}}} = F_{\mathfrak{c}} \circ h_{\mathfrak{X}} = F$. Therefore, by (the dual version of the argument of) 3.3.2, the universal ∂-functor $F_{\mathfrak{c}} \circ Ext^{\bullet}_{\mathfrak{X},\mathfrak{I}_{\mathfrak{X}}}$ is isomorphic to the right satellite $S^{\bullet}_+F$ of the functor F. This shows that $Ext^{\bullet}_{\mathfrak{X},\mathfrak{I}_{\mathfrak{X}}}$ is an initial object of the category $\partial\mathfrak{Un}^{\mathfrak{c}}(\mathfrak{X}, \mathfrak{I}_{\mathfrak{X}})$. ■

5.3. The universal problem for arbitrary universal ∂- and ∂^*-functors. Let $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ be a svelte left exact category with final objects. Let $C_{\mathfrak{X}_{\mathfrak{s}}}$ denote the smallest strictly full subcategory of the category $C^{\wedge}_{\mathfrak{X}}$ containing all presheaves $Ext^n_{\mathfrak{X},\mathfrak{I}_{\mathfrak{X}}}(L)$, $L \in ObC_{\mathfrak{X}}$, $n \geq 0$. Let $\mathfrak{Ext}^{\bullet}_{\mathfrak{X},\mathfrak{I}_{\mathfrak{X}}}$ denote the corestriction of the ∂-functor $Ext^{\bullet}_{\mathfrak{X},\mathfrak{I}_{\mathfrak{X}}}$ to the subcategory $C_{\mathfrak{X}_{\mathfrak{s}}}$. Thus, $Ext^{\bullet}_{\mathfrak{X},\mathfrak{I}_{\mathfrak{X}}}$ is the composition of the ∂-functor $\mathfrak{Ext}^{\bullet}_{\mathfrak{X},\mathfrak{I}_{\mathfrak{X}}}$ and the inclusion functor $C_{\mathfrak{X}_{\mathfrak{s}}} \xrightarrow{\mathfrak{J}_{\mathfrak{X}}} C^{\wedge}_{\mathfrak{X}}$. It follows that $\mathfrak{Ext}^{\bullet}_{\mathfrak{X},\mathfrak{I}_{\mathfrak{X}}}$ is a universal ∂-functor.

5.3.1. Proposition. *Let $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ a svelte left exact category with final objects. For any universal ∂-functor $T = (T_i, \mathfrak{d}_i \mid i \geq 0)$ from $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ to a category C_Y (with final objects), there exists a unique (up to isomorphism) functor $C_{\mathfrak{X}_{\mathfrak{s}}} \xrightarrow{T^{\mathfrak{s}}} C_Y$ such that $T = T^{\mathfrak{s}} \circ \mathfrak{Ext}^{\bullet}_{\mathfrak{X},\mathfrak{I}_{\mathfrak{X}}}$ and the diagram*

$$\begin{array}{ccc}
C^{\wedge}_{\mathfrak{X}} & \xrightarrow{\ T_0^*\ } & C_Y^{\vee^{op}} \\
{\scriptstyle \mathfrak{J}_{\mathfrak{X}}}\uparrow & & \uparrow{\scriptstyle h^o_Y} \\
C_{\mathfrak{X}_{\mathfrak{s}}} & \xrightarrow{\ T^{\mathfrak{s}}\ } & C_Y
\end{array}$$

commutes. Here $C_Y^{\vee^{op}}$ denote the category of presheaves of sets on C_Y^{op} (i.e. functors $C_Y \longrightarrow Sets$) and h_Y^o the (dual) Yoneda functor $C_Y \longrightarrow C_Y^{\vee^{op}}$, $L \longmapsto C_Y(L, -)$.; and T_0^* is a unique continuous (i.e. having a right adjoint) functor such that $T_0^* \circ h_{\mathfrak{X}} = h_Y^o \circ T_0$.

Proof. The category $C_Y^{\vee^{op}}$ is cocomplete (and complete) and the functor h_Y^o preserves colimits. Therefore, by 3.3.4, the composition $h_Y^o \circ T$ is a universal ∂-functor from $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ to $C_Y^{\vee^{op}}$. By 5.2.1, the ∂-functor $h_Y^o \circ T$ is the composition of the universal ∂-functor $Ext^\bullet_{\mathfrak{X},\mathfrak{I}_{\mathfrak{X}}}$ from $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ to $C_{\mathfrak{X}}^\wedge$ and the unique continuous functor $C_{\mathfrak{X}}^\wedge \xrightarrow{T_0^*} C_Y^{\vee^{op}}$ such that $T_0^* \circ h_{\mathfrak{X}} = h_Y^o \circ T_0$. Since the functor h_Y^o is fully faithful, this implies that the universal ∂-functor $T = (T_i, \mathfrak{d}_i \mid i \geq 0)$ is isomorphic to the composition of the corestriction of $Ext^\bullet_{\mathfrak{X},\mathfrak{I}_{\mathfrak{X}}}$ to the subcategory $C_{\mathfrak{X}_\mathfrak{s}}$ and a unique functor $C_{\mathfrak{X}_\mathfrak{s}} \xrightarrow{T^\mathfrak{s}} C_Y$ such that the composition $h_Y^o \circ T^\mathfrak{s}$ coincides with the restriction of the functor T_0^* to the subcategory $C_{\mathfrak{X}_\mathfrak{s}}$. ■

5.3.2. Note. The formulation of the dual assertion about the universal ∂^*-functors is left to the reader.

5.4. The k-linear version. Fix a right exact svelte k-linear additive category $(C_X, \mathfrak{E}_X)$. Let $\partial_k^*\mathfrak{Un}(X, \mathfrak{E}_X)$ denote the category whose objects are universal k-linear ∂^*-functors from $(C_X, \mathfrak{E}_X)$ to k-linear additive categories C_Y. Let T be a universal k-linear ∂^*-functor from $(C_X, \mathfrak{E}_X)$ to C_Y and $\widetilde{T}$ a universal k-linear ∂^*- functor from $(C_X, \mathfrak{E}_X)$ to C_Z. A morphism from T to T' is a pair (F, ϕ), where F is a k-linear functor from C_Y to C_Z and ϕ is a ∂^*-functor isomorphism $F \circ T \xrightarrow{\sim} T'$. If (F', ϕ') is a morphism from T' to T'', then the composition of (F, ϕ) and (F', ϕ') is defined by $(F', \phi') \circ (F, \phi) = (F' \circ F, \phi' \circ F'\phi)$.

We denote by $\partial_k^*\mathfrak{Un}^\mathfrak{c}(X, \mathfrak{E}_X)$ the subcategory of $\partial_k^*\mathfrak{Un}(X, \mathfrak{E}_X)$ whose objects are k-linear ∂^*-functors with values in complete categories and morphisms are pairs (F, ϕ) such that the functor F preserves small limits.

Dually, for a left exact svelte k-linear additive category $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$, we denote by $\partial_k\mathfrak{Un}(\mathfrak{X}, \mathfrak{I}_{\mathfrak{X}})$ the category whose objects are universal k-linear ∂-functors from $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ to additive k-linear categories. Given two universal ∂-functors T and T' from $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ to respectively C_Y and C_Z, a morphism from T to T' is a pair (F, ψ), where F is a k-linear functor from C_Y to C_Z and ψ a functor isomorphism $T' \xrightarrow{\sim} F \circ T$. The composition is defined by $(F', \psi') \circ (F, \psi) = (F' \circ F, F'\psi \circ \psi')$.

We denote by $\partial_k\mathfrak{Un}^\mathfrak{c}(\mathfrak{X}, \mathfrak{I}_{\mathfrak{X}})$ the subcategory of $\partial_k\mathfrak{Un}(\mathfrak{X}, \mathfrak{I}_{\mathfrak{X}})$ whose objects are k-linear ∂-functors with values in cocomplete categories and morphisms are pairs (F, ψ) such that the functor F preserves small colimits.

5.4.1. Proposition. *Let $(C_X, \mathfrak{E}_X)$ (resp. $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$) be a svelte right (resp. left) exact additive k-linear category. The categories $\partial_k^*\mathfrak{Un}_\mathfrak{c}(X, \mathfrak{E}_X)$ and $\partial_k\mathfrak{Un}^\mathfrak{c}(\mathfrak{X}, \mathfrak{I}_{\mathfrak{X}})$ have initial objects.*

Proof. By duality, it suffices to prove that the category $\partial_k\mathfrak{Un}^\mathfrak{c}(\mathfrak{X}, \mathfrak{I}_{\mathfrak{X}})$ has an initial object. The argument is similar to the argument of 5.2.1, except for the category $C_{\mathfrak{X}}^\wedge$ of presheaves of sets on the category $C_{\mathfrak{X}}$ is replaced by the category $\mathcal{M}_k(\mathfrak{X})$ of presheaves of k-modules on $C_{\mathfrak{X}}$. The initial object of the category $\partial_k\mathfrak{Un}^\mathfrak{c}(\mathfrak{X}, \mathfrak{I}_{\mathfrak{X}})$ is the universal k-linear ∂-functor $\mathcal{E}xt^\bullet_{(\mathfrak{X},\mathfrak{I}_{\mathfrak{X}})}$ from $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ to the category $\mathcal{M}_k(\mathfrak{X})$ whose zero component is the (k-linear) Yoneda embedding $C_{\mathfrak{X}} \longrightarrow \mathcal{M}_k(\mathfrak{X}), \quad L \longmapsto C_{\mathfrak{X}}(-, L)$. ■

Let $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ be a svelte k-linear additive left exact category. Let $\mathcal{M}_k^{\mathfrak{s}}(\mathfrak{X}$ denote the smallest additive strictly full subcategory of the category $\mathcal{M}_k(\mathfrak{X})$ containing all presheaves $\mathcal{E}xt^n_{(\mathfrak{X},\mathfrak{I}_{\mathfrak{X}})}(L)$, $L \in ObC_{\mathfrak{X}}$, $n \geq 0$. Let $\mathcal{E}\mathfrak{xt}^{\bullet}_{(\mathfrak{X},\mathfrak{I}_{\mathfrak{X}})}$ denote the corestriction of the ∂-functor $\mathcal{E}xt^{\bullet}_{(\mathfrak{X},\mathfrak{I}_{\mathfrak{X}})}$ to the subcategory $\mathcal{M}_k^{\mathfrak{s}}(\mathfrak{X}$. Thus, $\mathcal{E}xt^{\bullet}_{(\mathfrak{X},\mathfrak{I}_{\mathfrak{X}})}$ is the composition of the k-linear ∂-functor $\mathcal{E}\mathfrak{xt}^{\bullet}_{\mathfrak{X},\mathfrak{I}_{\mathfrak{X}}}$ and the inclusion functor

$$\mathcal{M}_k^{\mathfrak{s}}(\mathfrak{X}) \xrightarrow{\mathfrak{I}_{\mathfrak{X}}} \mathcal{M}_k(\mathfrak{X}).$$

It follows that $\mathfrak{E}\mathfrak{xt}^{\bullet}_{\mathfrak{X},\mathfrak{I}_{\mathfrak{X}}}$ is a universal ∂-functor.

5.4.2. Proposition. *Let $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ be a svelte left exact category with final objects. For any universal ∂-functor $T = (T_i, \mathfrak{d}_i \mid i \geq 0)$ from $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ to a category C_Y (with final objects), there exists a unique (up to isomorphism) functor $C_{\mathfrak{X}_{\mathfrak{s}}} \xrightarrow{T^{\mathfrak{s}}} C_Y$ such that $T = T^{\mathfrak{s}} \circ \mathfrak{E}\mathfrak{xt}^{\bullet}_{\mathfrak{X},\mathfrak{I}_{\mathfrak{X}}}$ and the diagram*

$$\begin{array}{ccc} C_{\mathfrak{X}}^{\wedge} & \xrightarrow{T_0^*} & C_Y^{\vee^{op}} \\ {\scriptstyle \mathfrak{I}_{\mathfrak{X}}}\uparrow & & \uparrow {\scriptstyle h_Y^o} \\ C_{\mathfrak{X}_{\mathfrak{s}}} & \xrightarrow{T^{\mathfrak{s}}} & C_Y \end{array}$$

commutes. Here $C_Y^{\vee^{op}}$ denote the category of presheaves of sets on C_Y^{op} (i.e. functors $C_Y \longrightarrow Sets$) and h_Y^o the (dual) Yoneda functor $C_Y \longrightarrow C_Y^{\vee^{op}}$, $L \longmapsto C_Y(L,-)$.; and T_0^* is a unique continuous (i.e. having a right adjoint) functor such that $T_0^* \circ h_{\mathfrak{X}} = h_Y^o \circ T_0$.

Proof. The argument is similar to that of 5.3.1. ∎

6. The stable category of a left exact category.

6.1. Reformulations. Fix a svelte left exact category $(C_X, \mathfrak{I}_X)$. Let $\widehat{\Theta}_X^*$ denote the continuous (that is having a right adjoint) functor $C_X^{\wedge} \longrightarrow C_X^{\wedge}$ determined (uniquely up to isomorphism) by the equality $Ext_X^1 = \widehat{\Theta}_X^* \circ h_X$. To any conflation $N \xrightarrow{\mathfrak{j}} M \xrightarrow{\mathfrak{e}} L$, corresponds the diagram

$$\begin{array}{ccccc} \widehat{N} & \xrightarrow{\widehat{\mathfrak{j}}} & \widehat{M} & \xrightarrow{\widehat{\mathfrak{e}}} & \widehat{L} \\ & & \downarrow & & \downarrow {\scriptstyle \mathfrak{d}_0(E)} \\ & & \widehat{x} & \xrightarrow{\lambda(\widehat{N})} & \widehat{\Theta}_X^*(\widehat{N}) \end{array} \qquad (1)$$

where $\widehat{L} = h_X(L)$ and $\widehat{x}$ is the final object of the category $C_X^{\wedge}$ of presheaves on C_X.

Due to the universality of $Ext_X^{\bullet}$, all the information about universal ∂-functors from the left exact category $(C_X, \mathfrak{I}_X)$, is encoded in the diagrams (1), where $N \xrightarrow{\mathfrak{j}} M \xrightarrow{\mathfrak{e}} L$ runs through the class of conflations of $(C_X, \mathfrak{I}_X)$.

In fact, it follows from the (argument of) 3.3.4(a) that the universal ∂-functor $Ext_X^{\bullet}$ is isomorphic to the ∂-functor of the form $(\widehat{\Theta}_X^{*n} \circ h_X, \widehat{\Theta}_X^{*n}(\mathfrak{d}_0) \mid n \geq 0)$; and for any functor

F from C_X to a category C_Y with colimits and final objects, the universal ∂-functor $(T_i, \mathfrak{d}_i \mid i \geq 0)$ from $(C_X, \mathfrak{I}_X)$ to C_Y with $T_0 = F$ is isomorphic to

$$F^* \circ Ext^{\bullet}_X = (F^* \widehat{\Theta}^{*n}_X \circ h_X, F^* \widehat{\Theta}^{*n}_X(\mathfrak{d}_0) \mid n \geq 0). \tag{2}$$

6.2. Note. If C_X is a pointed category, then the presheaf $\widehat{x} = C_X(-, x)$ is both a final and an initial object of the category $C_X^{\wedge}$. In particular, the morphism $\widehat{x} \xrightarrow{\lambda(\widehat{N})} \widehat{\Theta}^*_X(\widehat{N})$ in (1) is uniquely defined, hence is not a part of the data. In this case, the data consists of the diagrams

$$\widehat{N} \xrightarrow{\ \widehat{\mathfrak{j}}\ } \widehat{M} \xrightarrow{\ \widehat{\mathfrak{e}}\ } \widehat{L} \xrightarrow{\ \mathfrak{d}_0(E)\ } \widehat{\Theta}^*_X(\widehat{N}),$$

where $E = (N \xrightarrow{\mathfrak{j}} M \xrightarrow{\mathfrak{e}} L)$ runs through conflations of $(C_X, \mathfrak{I}_X)$.

6.3. Stable category of $(C_X, \mathfrak{I}_X)$. Consider the full subcategory $C_{X_\mathfrak{s}}$ of the category $C_X^{\wedge}$ whose objects are $\widehat{\Theta}^{*n}_X(\mathcal{M})$, where $\mathcal{M}$ runs through representable presheaves and n through nonnegative integers. We denote by $\theta_{X_\mathfrak{s}}$ the endofunctor $C_{X_\mathfrak{s}} \longrightarrow C_{X_\mathfrak{s}}$ induced by $\widehat{\Theta}^*_X$. It follows that $C_{X_\mathfrak{s}}$ is the smallest $\widehat{\Theta}^*_X$-stable strictly full subcategory of the category $C_X^{\wedge}$ containing all presheaves $\widehat{M} = C_X(-, M)$, $M \in ObC_X$.

6.3.1. Triangles. We call the diagram

$$\widehat{N} \xrightarrow{\ \widehat{\mathfrak{j}}\ } \widehat{M} \xrightarrow{\ \widehat{\mathfrak{e}}\ } \widehat{L} \xrightarrow{\ \mathfrak{d}_0(E)\ } \widehat{\Theta}^*_X(\widehat{N}), \tag{1}$$

where $E = (N \xrightarrow{\mathfrak{j}} M \xrightarrow{\mathfrak{e}} L)$ is a conflation in $(C_X, \mathfrak{I}_X)$, a *standard triangle*.

A *triangle* is any diagram in $C_{X_\mathfrak{s}}$ of the form

$$\mathcal{N} \xrightarrow{\mathfrak{j}} \mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L} \xrightarrow{\mathfrak{d}} \theta_{X_\mathfrak{s}}(\mathcal{N}), \tag{2}$$

which is isomorphic to a standard triangle. It follows that for every triangle, the diagram

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\ \mathfrak{e}\ } & \mathcal{L} \\ \downarrow & & \downarrow \mathfrak{d} \\ \widehat{x} & \xrightarrow{\ \lambda(\mathcal{N})\ } & \widehat{\Theta}^*_X(\mathcal{N}) \end{array}$$

commutes. Triangles form a category $\mathfrak{Tr}_{X_\mathfrak{s}}$: morphisms from

$$\mathcal{N} \xrightarrow{\ \mathfrak{j}\ } \mathcal{M} \xrightarrow{\ \mathfrak{e}\ } \mathcal{L} \xrightarrow{\ \mathfrak{d}\ } \theta_{\mathfrak{X}}(\mathcal{N})$$

to

$$\mathcal{N}' \xrightarrow{\ \mathfrak{j}'\ } \mathcal{M}' \xrightarrow{\ \mathfrak{e}'\ } \mathcal{L}' \xrightarrow{\ \mathfrak{d}'\ } \theta_{\mathfrak{X}}(\mathcal{N}')$$

are given by commutative diagrams

$$\begin{array}{ccccccc}
\mathcal{N} & \xrightarrow{\mathfrak{j}} & \mathcal{M} & \xrightarrow{\mathfrak{e}} & \mathcal{L} & \xrightarrow{\mathfrak{d}} & \theta_{\mathfrak{X}}(\mathcal{N}) \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \theta_{\mathfrak{X}}(f)} \\
\mathcal{N}' & \xrightarrow{\mathfrak{j}'} & \mathcal{M}' & \xrightarrow{\mathfrak{e}'} & \mathcal{L}' & \xrightarrow{\mathfrak{d}'} & \theta_{\mathfrak{X}}(\mathcal{N}')
\end{array}$$

The composition is obvious.

6.3.2. The prestable category of a left exact category. Thus, we have obtained a data $(C_{X_\mathfrak{s}}, (\theta_{X_\mathfrak{s}}, \lambda), \mathfrak{Tr}_{X_\mathfrak{s}})$. We call this data the *prestable* category of the left exact category $(C_X, \mathfrak{I}_X)$.

6.3.3. The stable category of a left exact category with final objects. Let $(C_X, \mathfrak{I}_X)$ be a left exact category with a final object x and $(C_{X_\mathfrak{s}}, \theta_{X_\mathfrak{s}}, \lambda; \mathfrak{Tr}_{X_\mathfrak{s}})$ the associated with $(C_X, \mathfrak{I}_X)$ presuspended category. Let $\Sigma = \Sigma_{\theta_{X_\mathfrak{s}}}$ be the class of all arrows $\mathfrak{t}$ of $C_{X_\mathfrak{s}}$ such that $\theta_{X_\mathfrak{s}}(\mathfrak{t})$ is an isomorphism.

We call the quotient category $C_{X_\mathfrak{s}} = \Sigma^{-1}C_{X_\mathfrak{s}}$ the *stable* category of the left exact category $(C_X, \mathfrak{I}_X)$. The endofunctor $\theta_{X_\mathfrak{s}}$ determines a conservative endofunctor $\theta_{X_\mathfrak{s}}$ of the stable category $C_{X_\mathfrak{s}}$. The localization functor $C_{X_\mathfrak{s}} \xrightarrow{\mathfrak{q}^*_\Sigma} C_{X_\mathfrak{s}}$ maps final objects to final objects. Let $\lambda_\mathfrak{s}$ denote the image $\widetilde{x} = \mathfrak{q}^*_\Sigma(\widehat{x}) \longrightarrow \theta_{X_\mathfrak{s}}$ of the cone $\widehat{x} \xrightarrow{\lambda} \theta_{X_\mathfrak{s}}$.

Finally, we denote by $\mathfrak{Tr}_{X_\mathfrak{s}}$ the strictly full subcategory of the category of diagrams of the form $\mathcal{N} \longrightarrow \mathcal{M} \longrightarrow \mathcal{L} \longrightarrow \theta_{X_\mathfrak{s}}(\mathcal{N})$ generated by $\mathfrak{q}^*_\Sigma(\mathfrak{Tr}_{X_\mathfrak{s}})$.

The data $(C_{X_\mathfrak{s}}, \theta_{X_\mathfrak{s}}, \lambda_\mathfrak{s}; \mathfrak{Tr}_{X_\mathfrak{s}})$ will be called the *stable* category of $(C_X, \mathfrak{I}_X)$.

6.4. Dual notions. If $(C_\mathfrak{X}, \mathfrak{E}_\mathfrak{X})$ is a right exact category with an initial object, one obtains, dualizing the definitions of 6.3.2 and 6.3.3, the notions of the *precostable* and *costable* category of $(C_\mathfrak{X}, \mathfrak{E}_\mathfrak{X})$.

6.5. The k-linear version. Let $(C_X, \mathfrak{I}_X)$ be a k-linear additive svelte left exact category. Replacing the category of $C_X^\wedge$ of presheaves of sets by the category $\mathcal{M}_k(X)$ of presheaves of k-modules on C_X and the functor $Ext^1_{(X,\mathfrak{I}_X)}$ by its k-linear version, $\mathcal{E}xt^1_{(X,\mathfrak{I}_X)}$, we obtain the k-linear versions of prestable and stable categories of the left exact category $(C_X, \mathfrak{I}_X)$.

6.5.1. Note. If $(C_X, \mathfrak{I}_X)$ is a k-linear exact category (that is $\mathfrak{I}_X$ happen to be the class of inflations of a k-linear exact category) with enough injectives, than its stable category defined above is equivalent to the conventional stable category of $(C_X, \mathfrak{I}_X)$. Recall that the latter has the same objects as C_X and its morphisms are *homotopy* classes of morphisms of C_X: two morphisms $M \overset{f}{\underset{g}{\rightrightarrows}} N$ are *homotopy equivalent* to each other if their difference $f - g$ factors through an injective object.

Notice that our construction of stable category of $(C, \mathfrak{I}_X)$ does not require any additional hypothesis. In particular, it extends the notion of the stable category to arbitrary exact categories.

7. Complement: presuspended and quasi-suspended categories.

It is tempting to follow Keller's example [Ke1], [KeV] and turn essential properties of prestable and stable categories of a left exact category into axioms. We call the corresponding notions respectively *presuspended* and *quasi-suspended* categories.

7.1. Presuspended and quasi-suspended categories. Fix a category $C_{\mathfrak{X}}$ with a final object x and a functor $C_{\mathfrak{X}} \xrightarrow{\widetilde{\theta}_{\mathfrak{X}}} x\backslash C_{\mathfrak{X}}$, or, what is the same, a pair $(\theta_{\mathfrak{X}}, \lambda)$, where $\theta_{\mathfrak{X}}$ is an endofunctor $C_{\mathfrak{X}} \longrightarrow C_{\mathfrak{X}}$ and λ is a cone $x \longrightarrow \theta_{\mathfrak{X}}$. We denote by $\widetilde{\mathfrak{Tr}}_{\mathfrak{X}}$ the category whose objects are all diagrams of the form

$$\mathcal{N} \xrightarrow{\mathfrak{j}} \mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L} \xrightarrow{\mathfrak{d}} \theta_{\mathfrak{X}}(\mathcal{N})$$

such that the square

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\mathfrak{e}} & \mathcal{L} \\ \big\downarrow & & \big\downarrow \mathfrak{d} \\ x & \xrightarrow{\lambda(\mathcal{N})} & \theta_{\mathfrak{X}}(\mathcal{N}) \end{array}$$

commutes. Morphisms from

$$\mathcal{N} \xrightarrow{\mathfrak{j}} \mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L} \xrightarrow{\mathfrak{d}} \theta_{\mathfrak{X}}(\mathcal{N})$$

to

$$\mathcal{N}' \xrightarrow{\mathfrak{j}'} \mathcal{M}' \xrightarrow{\mathfrak{e}'} \mathcal{L}' \xrightarrow{\mathfrak{d}'} \theta_{\mathfrak{X}}(\mathcal{N}')$$

are triples $(\mathcal{N} \xrightarrow{f} \mathcal{N}', \mathcal{M} \xrightarrow{g} \mathcal{M}', \mathcal{L} \xrightarrow{h} \mathcal{L}')$ such that the diagram

$$\begin{array}{ccccccc} \mathcal{N} & \xrightarrow{\mathfrak{j}} & \mathcal{M} & \xrightarrow{\mathfrak{e}} & \mathcal{L} & \xrightarrow{\mathfrak{d}} & \theta_{\mathfrak{X}}(\mathcal{N}) \\ f \big\downarrow & & \big\downarrow g & & \big\downarrow h & & \big\downarrow \theta_{\mathfrak{X}}(f) \\ \mathcal{N}' & \xrightarrow{\mathfrak{j}'} & \mathcal{M}' & \xrightarrow{\mathfrak{e}'} & \mathcal{L}' & \xrightarrow{\mathfrak{d}'} & \theta_{\mathfrak{X}}(\mathcal{N}') \end{array}$$

commutes. The composition of morphisms is natural.

7.1.1. Presuspended categories. A *presuspended* category is a triple $(C_{\mathfrak{X}}, \widetilde{\theta}_{\mathfrak{X}}, \mathfrak{Tr}_{\mathfrak{X}})$, where $C_{\mathfrak{X}}$ and $\widetilde{\theta}_{\mathfrak{X}} = (\theta_{\mathfrak{X}}, \lambda)$ are as above and $\mathfrak{Tr}_{\mathfrak{X}}$ is a strictly full subcategory of the category $\widetilde{\mathfrak{Tr}}_{\mathfrak{X}}$ whose objects are called *triangles*, which satisfies the following conditions:

(PS1) Let $C_{\mathfrak{X}_0}$ denote the full subcategory of $C_{\mathfrak{X}}$ generated by objects $\mathcal{N}$ such that there exists a triangle $\mathcal{N} \xrightarrow{\mathfrak{j}} \mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L} \xrightarrow{\mathfrak{d}} \theta_{\mathfrak{X}}(\mathcal{N})$. For every $\mathcal{N} \in ObC_{\mathfrak{X}_0}$, the diagram

$$\mathcal{N} \xrightarrow{id_{\mathcal{N}}} \mathcal{N} \longrightarrow x \xrightarrow{\lambda(\mathcal{N})} \theta_{\mathfrak{X}}(\mathcal{N})$$

is a triangle.

(PS2) For any triangle $\mathcal{N} \xrightarrow{\mathfrak{j}} \mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L} \xrightarrow{\mathfrak{d}} \theta_{\mathfrak{X}}(\mathcal{N})$ and any morphism $\mathcal{N} \xrightarrow{f} \mathcal{N}'$ with $\mathcal{N}' \in ObC_{\mathfrak{X}_0}$, there is a triangle $\mathcal{N}' \xrightarrow{\mathfrak{j}'} \mathcal{M}' \xrightarrow{\mathfrak{e}'} \mathcal{L} \xrightarrow{\mathfrak{d}'} \theta_{\mathfrak{X}}(\mathcal{N}')$ such that f extends to a morphism of triangles

$$(\mathcal{N} \xrightarrow{\mathfrak{j}} \mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L} \xrightarrow{\mathfrak{d}} \theta_{\mathfrak{X}}(\mathcal{N})) \xrightarrow{(f,g,id_{\mathcal{L}})} (\mathcal{N}' \xrightarrow{\mathfrak{j}'} \mathcal{M}' \xrightarrow{\mathfrak{e}'} \mathcal{L} \xrightarrow{\mathfrak{d}'} \theta_{\mathfrak{X}}(\mathcal{N}')).$$

(PS3) For any pair of triangles

$$\mathcal{N} \xrightarrow{\mathfrak{j}} \mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L} \xrightarrow{\mathfrak{d}} \theta_{\mathfrak{X}}(\mathcal{N}) \quad \text{and} \quad \mathcal{N}' \xrightarrow{\mathfrak{j}'} \mathcal{M}' \xrightarrow{\mathfrak{e}'} \mathcal{L}' \xrightarrow{\mathfrak{d}'} \theta_{\mathfrak{X}}(\mathcal{N}')$$

and any commutative square

$$\begin{array}{ccc} \mathcal{N} & \xrightarrow{\mathfrak{j}} & \mathcal{M} \\ f\downarrow & & \downarrow g \\ \mathcal{N}' & \xrightarrow{\mathfrak{j}'} & \mathcal{M}' \end{array}$$

there exists a morphism $\mathcal{L} \xrightarrow{h} \mathcal{L}'$ such that (f, g, h) is a morphism of triangles, i.e. the diagram

$$\begin{array}{ccccccc} \mathcal{N} & \xrightarrow{\mathfrak{j}} & \mathcal{M} & \xrightarrow{\mathfrak{e}} & \mathcal{L} & \xrightarrow{\mathfrak{d}} & \theta_{\mathfrak{X}}(\mathcal{N}) \\ f\downarrow & & \downarrow g & & \downarrow h & & \downarrow \theta_{\mathfrak{X}}(f) \\ \mathcal{N}' & \xrightarrow{\mathfrak{j}'} & \mathcal{M}' & \xrightarrow{\mathfrak{e}'} & \mathcal{L}' & \xrightarrow{\mathfrak{d}'} & \theta_{\mathfrak{X}}(\mathcal{N}') \end{array}$$

commutes.

(PS4) For any pair of triangles

$$\mathcal{N} \xrightarrow{u} \mathcal{M} \xrightarrow{v} \mathcal{L} \xrightarrow{w} \theta_{\mathfrak{X}}(\mathcal{N}) \quad \text{and} \quad \mathcal{M} \xrightarrow{x} \mathcal{M}' \xrightarrow{s} \widetilde{\mathcal{M}} \xrightarrow{r} \theta_{\mathfrak{X}}(M),$$

there exists a commutative diagram

$$\begin{array}{ccccccc} \mathcal{N} & \xrightarrow{u} & \mathcal{M} & \xrightarrow{v} & \mathcal{L} & \xrightarrow{w} & \theta_{\mathfrak{X}}(\mathcal{N}) \\ id\downarrow & & x\downarrow & & \downarrow y & & \downarrow id \\ \mathcal{N} & \xrightarrow{u'} & \mathcal{M}' & \xrightarrow{v'} & \mathcal{L}' & \xrightarrow{w'} & \theta_{\mathfrak{X}}(\mathcal{N}) \\ & & s\downarrow & & \downarrow t & & \downarrow \theta_{\mathfrak{X}}(u) \\ & & \widetilde{\mathcal{M}} & \xrightarrow{id} & \widetilde{\mathcal{M}} & \xrightarrow{r} & \theta_{\mathfrak{X}}(\mathcal{M}) \\ & & r\downarrow & & \downarrow r' & & \\ & & \theta_{\mathfrak{X}}(\mathcal{M}) & \xrightarrow{\theta_{\mathfrak{X}}(v)} & \theta_{\mathfrak{X}}(\mathcal{L}) & & \end{array} \tag{2}$$

whose two upper rows and two central columns are triangles.

(PS5) For every triangle $\mathcal{N} \xrightarrow{\mathfrak{j}} \mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L} \xrightarrow{\mathfrak{d}} \theta_{\mathfrak{X}}(\mathcal{N})$, the sequence

$$\ldots \longrightarrow C_{\mathfrak{X}}(\theta_{\mathfrak{X}}(\mathcal{N}), -) \longrightarrow C_{\mathfrak{X}}(\mathcal{L}, -) \longrightarrow C_{\mathfrak{X}}(\mathcal{M}, -) \longrightarrow C_{\mathfrak{X}}(\mathcal{N}, -)$$

is exact.

7.1.2. Remarks. (a) If $C_{\mathfrak{X}}$ is an additive category, then three of the axioms above coincide with the corresponding Verdier's axioms of triangulated category (under condition that $C_{\mathfrak{X}_0} = C_{\mathfrak{X}}$). Namely, (PS1) coincides with the first half of the axiom (TRI), the axiom (PS3) coincides with the axiom (TRIII), and (PS4) with (TRIV) (see [Ve2, Ch.II]).

(b) It follows from (PS4) that if $\mathcal{N} \longrightarrow \mathcal{M} \longrightarrow \mathcal{L} \longrightarrow \theta_{\mathfrak{X}}(\mathcal{N})$ is a triangle, then all three objects, $\mathcal{N}$, $\mathcal{M}$, and $\mathcal{L}$, belong to the subcategory $C_{\mathfrak{X}_0}$.

(c) The axiom (PS2) can be regarded as a base change property, and axiom (PS4) expresses the stability of triangles under composition. So that the axioms (PS1), (PS2) and (PS4) say that triangles form a 'pretopology' on the subcategory $C_{\mathfrak{X}_0}$. The axiom (PS5) says that this pretopology is *subcanonical*: the representable presheaves are sheaves.

These interpretations (as well as the axioms themselves) come from the main examples: prestable and stable categories of a left exact category.

7.1.3. Quasi-suspended categories. A presuspended category $(C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, \lambda; \mathfrak{Tr}_{\mathfrak{X}})$ will be called *quasi-suspended* if the functor $\theta_{\mathfrak{X}}$ is conservative. We denote by $\mathfrak{SCat}$ the full subcategory of the category $\mathfrak{PCat}$ of presuspended categories whose objects are conservative presuspended svelte categories.

7.2. Examples.

7.2.1. The presuspended category of presheaves of sets on a left exact category. Fix a left exact category $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$. Let $\widehat{\Theta}^*_{\mathfrak{X}}$ be a continuous endofunctor of $C^{\wedge}_{\mathfrak{X}} = C_{\mathfrak{X}^{\wedge}}$ determined uniquely up to isomorphism by the equality $Ext^1_{\mathfrak{X},\mathfrak{I}} = \widehat{\Theta}^*_{\mathfrak{X}} \circ h_{\mathfrak{X}}$. We call a *standard* triangle all diagrams of the form

$$\widehat{N} \xrightarrow{\widehat{\mathfrak{j}}} \widehat{M} \xrightarrow{\widehat{\mathfrak{e}}} \widehat{L} \xrightarrow{\mathfrak{d}_0(E)} \widehat{\Theta}^*_X(\widehat{N}), \qquad (1)$$

where $E = (N \xrightarrow{\mathfrak{j}} M \xrightarrow{\mathfrak{e}} L)$ is any conflation in $(C_X, \mathfrak{I}_X)$. *Triangle* is an object of the category $\widetilde{\mathfrak{Tr}}_{\mathfrak{X}^{\wedge}}$ which is isomorphic to a standard triangle. We denote by $\widehat{\mathfrak{Tr}}_{\mathfrak{X}}$ the full subcategory of the category $\widetilde{\mathfrak{Tr}}_{\mathfrak{X}^{\wedge}}$ whose objects are triangles. One can see that $\mathfrak{T}\mathcal{C}_{\mathfrak{X}^{\wedge}} = (C_{\mathfrak{X}^{\wedge}}, \widehat{\Theta}^*_{\mathfrak{X}}, \lambda_{\mathfrak{X}}; \widehat{\mathfrak{Tr}}_{\mathfrak{X}})$ is a presuspended category.

In fact, $C_{\mathfrak{X}_0}$ is the full subcategory of $C^{\wedge}_{\mathfrak{X}}$ generated by all representable functors. The property (PS1) holds, because $N \xrightarrow{id_N} N \longrightarrow x$ is a conflation for any object N of $C_{\mathfrak{X}}$. The property (PS2) holds, because for any conflation $E = (N \xrightarrow{\mathfrak{j}} M \xrightarrow{\mathfrak{e}} L)$ and any morphism $N \xrightarrow{f} N'$, we have a commutative diagram

$$\begin{array}{ccccc}
N & \xrightarrow{\mathfrak{j}} & M & \xrightarrow{\mathfrak{e}} & L \\
{\scriptstyle f}\downarrow & cocart & \downarrow{\scriptstyle \widetilde{f}} & & \downarrow{\scriptstyle id_L} \\
N' & \xrightarrow{\mathfrak{j}'} & M' & \xrightarrow{\mathfrak{e}'} & L
\end{array}$$

whose rows are conflations and left square is cocartesian. The property (PS3) holds, because for any commutative diagram

$$\begin{array}{ccccc} N & \xrightarrow{\mathfrak{j}} & M & \xrightarrow{\mathfrak{e}} & L \\ f\downarrow & & \downarrow g & & \\ N' & \xrightarrow{\mathfrak{j}'} & M' & \xrightarrow{\mathfrak{e}'} & L' \end{array}$$

whose rows are conflations, there exists a unique arrow $L \xrightarrow{h} L'$ which makes the diagram

$$\begin{array}{ccccc} N & \xrightarrow{\mathfrak{j}} & M & \xrightarrow{\mathfrak{e}} & L \\ f\downarrow & & \downarrow g & & \downarrow h \\ N' & \xrightarrow{\mathfrak{j}'} & M' & \xrightarrow{\mathfrak{e}'} & L' \end{array}$$

commute, i.e. (f, g, h) is a morphism of conflations. Since $Ext^\bullet_{\mathfrak{X}}$ is a ∂-functor, this implies the commutativity of the diagram

$$\begin{array}{ccccccc} N & \xrightarrow{\mathfrak{j}} & M & \xrightarrow{\mathfrak{e}} & L & \xrightarrow{\mathfrak{d}_0(E)} & \widehat{\Theta}^*_{\mathfrak{X}}(N) \\ f\downarrow & & \downarrow g & & \downarrow h & & \downarrow \widehat{\Theta}^*_{\mathfrak{X}}(f) \\ N' & \xrightarrow{\mathfrak{j}'} & M' & \xrightarrow{\mathfrak{e}'} & L' & \xrightarrow{\mathfrak{d}_0(E')} & \widehat{\Theta}^*_{\mathfrak{X}}(N') \end{array}$$

where E' denotes the conflation $N' \xrightarrow{\mathfrak{j}'} M' \xrightarrow{\mathfrak{e}'} L'$.

For any conflation $E = (N \xrightarrow{\mathfrak{j}} M \xrightarrow{\mathfrak{e}} L)$, the sequence

$$\widehat{N} \xrightarrow{\widehat{\mathfrak{j}}} \widehat{M} \xrightarrow{\widehat{\mathfrak{e}}} \widehat{L} \xrightarrow{\mathfrak{d}_0(E)} \widehat{\Theta}^*_{\mathfrak{X}}(\widehat{N}) \xrightarrow{\widehat{\Theta}^*_{\mathfrak{X}}(\mathfrak{j})} \ldots$$

is exact, because $Ext^\bullet_{\mathfrak{X}}$ is an 'exact' ∂-functor. This implies the property (PS5), that is the exactness of

$$\ldots \longrightarrow C_{\mathfrak{X}}(\widehat{\Theta}^*_{\mathfrak{X}}(N), -) \longrightarrow C_{\mathfrak{X}}(L, -) \longrightarrow C_{\mathfrak{X}}(M, -) \longrightarrow C_{\mathfrak{X}}(N, -).$$

7.2.2. The associated quasi-suspended category. It is obtained via localization of the presuspended category $\mathfrak{TC}_{\mathfrak{X}^\wedge} = (C_{\mathfrak{X}^\wedge}, \widehat{\Theta}^*_{\mathfrak{X}}, \lambda_{\mathfrak{X}}; \widehat{\mathfrak{Tr}_{\mathfrak{X}}})$ (see 7.2.1) at the class of arrows $\Sigma_{\widehat{\Theta}^*_{\mathfrak{X}}} = \{s \in HomC_{\mathfrak{X}^\wedge} \mid \widehat{\Theta}^*_{\mathfrak{X}}(s)$ is an isomorphism$\}$. Since $\widehat{\Theta}^*_{\mathfrak{X}}$ is a continuous functor, the localization $\mathfrak{q}^*_{\mathfrak{X}}$ at $\Sigma_{\widehat{\Theta}^*_{\mathfrak{X}}}$ is a continuous (that is having a right adjoint) functor too. In particular, the functor $\mathfrak{q}^*_{\mathfrak{X}}$ preserves colimits of small diagrams. The fact that $\mathfrak{q}^*_{\mathfrak{X}}$ is right exact implies that the category $\mathfrak{TC}_{\mathfrak{X}_\mathfrak{e}}$ obtained by applying the localization functor $\mathfrak{q}^*_{\mathfrak{X}}$ to $\mathfrak{TC}_{\mathfrak{X}^\wedge}$ inherits all the properties of $\mathfrak{TC}_{\mathfrak{X}^\wedge}$, including the exactness of the sequence

$$\mathfrak{q}^*_{\mathfrak{X}}\big(\widehat{N} \xrightarrow{\widehat{\mathfrak{j}}} \widehat{M} \xrightarrow{\widehat{\mathfrak{e}}} \widehat{L} \xrightarrow{\mathfrak{d}_0(E)} \widehat{\Theta}^*_{\mathfrak{X}}(\widehat{N}) \xrightarrow{\widehat{\Theta}^*_{\mathfrak{X}}(\mathfrak{j})} \ldots\big).$$

By construction, the *suspension* functor $\theta_{\mathfrak{X}_\mathfrak{c}}$ induced by $\widehat{\Theta}^*_{\mathfrak{X}}$ on the quotient category $\mathcal{C}_{\mathfrak{X}_\mathfrak{c}} = \Sigma^{-1}_{\widehat{\Theta}^*_{\mathfrak{X}}} C_{\mathfrak{X}^\wedge}$ (it is uniquely determined by the equality $\theta_{\mathfrak{X}_\mathfrak{c}} \circ \mathfrak{q}^*_{\mathfrak{X}} = \mathfrak{q}^*_{\mathfrak{X}} \circ \widehat{\Theta}^*_{\mathfrak{X}}$) is conservative; i.e. $\mathfrak{T}\mathcal{C}_{\mathfrak{X}_\mathfrak{c}} = (\mathcal{C}_{\mathfrak{X}_\mathfrak{c}}, \theta_{\mathfrak{X}_\mathfrak{c}}, \lambda_{\mathfrak{X}_\mathfrak{c}}; \mathfrak{Tr}_{\mathfrak{X}_\mathfrak{c}})$ is a quasi-suspended category.

Notice that the category $\mathcal{C}_{\mathfrak{X}_\mathfrak{c}}$ is cocomplete and complete. This follows from the corresponding properties of the category $C_{\mathfrak{X}^\wedge} = C^\wedge_{\mathfrak{X}}$ of presheaves of sets on $C_{\mathfrak{X}}$ and the fact that the localization functor $\mathfrak{q}^*_{\mathfrak{X}}$ has a right adjoint [GZ, I.1].

7.2.3. Reduced presuspended categories. Let $\mathfrak{T}\mathcal{C}_{\mathfrak{X}} = (\mathcal{C}_{\mathfrak{X}}, \theta_{\mathfrak{X}}, \lambda_{\mathfrak{X}}; Tr_{\mathfrak{X}})$ be a presuspended category and $\mathcal{C}_{\mathfrak{X}_0}$ the full subcategory of $\mathcal{C}_{\mathfrak{X}}$ generated by objects $\mathcal{N}$ such that there exists a triangle $\mathcal{N} \xrightarrow{\mathfrak{j}} \mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L} \xrightarrow{\mathfrak{d}} \theta_{\mathfrak{X}}(\mathcal{N})$. Let $\mathcal{C}_{\mathfrak{X}_1}$ be the smallest $\theta_{\mathfrak{X}}$-stable strictly full subcategory of $\mathcal{C}_{\mathfrak{X}}$ containing the subcategory $\mathcal{C}_{\mathfrak{X}_0}$ and $\theta_{\mathfrak{X}_1}$ the endofunctor of $\mathcal{C}_{\mathfrak{X}_1}$ induced by $\theta_{\mathfrak{X}}$. One can see that $\mathfrak{T}\mathcal{C}_{\mathfrak{X}_1} = (\mathcal{C}_{\mathfrak{X}_1}, \theta_{\mathfrak{X}_1}, \lambda_{\mathfrak{X}_1}; Tr_{\mathfrak{X}})$ is a presuspended category. It is quasi-suspended if $\mathfrak{T}\mathcal{C}_{\mathfrak{X}}$ is quasi-suspended.

We call $\mathfrak{T}\mathcal{C}_{\mathfrak{X}_1}$ the *reduced* presuspended category associated with $\mathfrak{T}\mathcal{C}_{\mathfrak{X}}$. In particular, we call the presuspended category $\mathfrak{T}\mathcal{C}_{\mathfrak{X}}$ *reduced* if it coincides with $\mathfrak{T}\mathcal{C}_{\mathfrak{X}_1}$.

7.2.4. Prestable and stable categories of a left exact category. The reduced presuspended category associated with the presuspended category $\mathfrak{T}\mathcal{C}_{\mathfrak{X}^\wedge}$ of presheaves of sets on a left exact category $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ (see 7.2.1) coincides with the *prestable* category of $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ defined in 6.3.2.

The reduced presuspended (actually, quasi-suspended) category associated with the quasi-suspended category associated with $\mathfrak{T}\mathcal{C}_{\mathfrak{X}^\wedge}$ (see 7.2.2) is naturally equivalent to the stable category of $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ introduced in 6.3.3.

7.3. The category of presuspended categories. Let $\mathfrak{T}C_{\mathfrak{X}} = (C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, \lambda_{\mathfrak{X}}; Tr_{\mathfrak{X}})$ and $\mathfrak{T}C_{\mathfrak{Y}} = (C_{\mathfrak{Y}}, \theta_{\mathfrak{Y}}, \lambda_{\mathfrak{Y}}; Tr_{\mathfrak{Y}})$ be presuspended categories. A *triangle* functor from $\mathfrak{T}C_{\mathfrak{X}}$ to $\mathfrak{T}C_{\mathfrak{Y}}$ is a pair (F, ϕ), where F is a functor $C_{\mathfrak{X}} \longrightarrow C_{\mathfrak{Y}}$ which maps initial object to an initial object and ϕ is a functor isomorphism $F \circ \theta_{\mathfrak{X}} \longrightarrow \theta_{\mathfrak{Y}} \circ F$ such that for every triangle $\mathcal{N} \xrightarrow{u} \mathcal{M} \xrightarrow{v} \mathcal{L} \xrightarrow{w} \theta_{\mathfrak{X}}(\mathcal{N})$ of $\mathfrak{T}C_{\mathfrak{X}}$, the sequence

$$F(\mathcal{N}) \xrightarrow{F(u)} F(\mathcal{M}) \xrightarrow{F(v)} F(\mathcal{L}) \xrightarrow{\phi(\mathcal{N})F(w)} \theta_{\mathfrak{Y}}(F(\mathcal{N}))$$

is a triangle of $\mathfrak{T}C_{\mathfrak{Y}}$. The composition of triangle functors is defined naturally:

$$(G, \psi) \circ (F, \phi) = (G \circ F, \psi F \circ G\phi).$$

Let (F, ϕ) and (F', ϕ') be triangle functors from $\mathfrak{T}_- C_{\mathfrak{X}}$ to $\mathfrak{T}_- C_{\mathfrak{Y}}$. A morphism from (F, ϕ) to (F', ϕ') is given by a functor morphism $F \xrightarrow{\lambda} F'$ such that the diagram

$$\begin{array}{ccc} \theta_{\mathfrak{Y}} \circ F & \xrightarrow{\phi} & F \circ \theta_{\mathfrak{X}} \\ {\scriptstyle \theta_{\mathfrak{Y}}\lambda} \downarrow & & \downarrow {\scriptstyle \lambda\theta_{\mathfrak{X}}} \\ \theta_{\mathfrak{Y}} \circ F' & \xrightarrow{\phi'} & F' \circ \theta_{\mathfrak{X}} \end{array}$$

commutes. The composition is the compsition of the functor morphisms.

Altogether gives the definition of a bicategory $\mathfrak{PCat}$ formed by svelte presuspended categories, triangle functors as 1-morphisms and morphisms between them as 2-morphisms.

As usual, the term "category $\mathfrak{PCat}$" means that we forget about 2-morphisms.

Dualizing (i.e. inverting all arrows in the constructions above), we obtain the bicategory $\mathfrak{P}^o\mathfrak{Cat}$ formed by *precosuspended* svelte categories as objects, triangular functors as 1-morphisms, and morphisms between them as 2-morphisms.

7.3.1. The subcategory of quasi-suspended categories. We denote by $\mathfrak{SCat}$ the full subcategory of the category $\mathfrak{PCat}$ of presuspended svelte categories whose objects are quasi-suspended categories.

7.3.2. From presuspended categories to quasi-suspended categories. Let $(C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, \lambda; \mathfrak{Tr}_{\mathfrak{X}})$ be a presuspended category and $\Sigma = \Sigma_{\theta_{\mathfrak{X}}}$ the class of all arrows s of the category $C_{\mathfrak{X}}$ such that $\theta_{\mathfrak{X}}(s)$ is an isomorphism. Let $\Theta_{\mathfrak{X}}$ denote the endofunctor of the quotient category $\Sigma^{-1}C_{\mathfrak{X}}$ uniquely determined by the equality $\Theta_{\mathfrak{X}} \circ \mathfrak{q}^*_{\Sigma} = \mathfrak{q}^*_{\Sigma} \circ \theta_{\mathfrak{X}}$, where $\mathfrak{q}^*_{\Sigma}$ denotes the localization functor $C_{\mathfrak{X}} \longrightarrow \Sigma^{-1}C_{\mathfrak{X}}$. Notice that the functor $\mathfrak{q}^*_{\Sigma}$ maps final objects to final objects. Let $\widetilde{\lambda}$ denote the morphism $\mathfrak{q}^*_{\Sigma}(x) \longrightarrow \Theta_{\mathfrak{X}}$ induced by $x \xrightarrow{\lambda} \theta_{\mathfrak{X}}$ (that is by $\mathfrak{q}^*_{\Sigma}(x) \xrightarrow{\mathfrak{q}^*_{\Sigma}(\lambda)} \mathfrak{q}^*_{\Sigma} \circ \theta_{\mathfrak{X}} = \Theta_{\mathfrak{X}} \circ \mathfrak{q}^*_{\Sigma}$) and $\widetilde{\mathfrak{Tr}}_{\mathfrak{X}}$ the essential image of $\mathfrak{Tr}_{\mathfrak{X}}$. Then the data $(\Sigma^{-1}C_{\mathfrak{X}}, \Theta_{\mathfrak{X}}, \widetilde{\lambda}; \widetilde{\mathfrak{Tr}}_{\mathfrak{X}})$ is a quasi-suspended category.

The constructed above map $(C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, \lambda; \mathfrak{Tr}_{\mathfrak{X}}) \longmapsto (\Sigma^{-1}C_{\mathfrak{X}}, \Theta_{\mathfrak{X}}, \widetilde{\lambda}; \widetilde{\mathfrak{Tr}}_{\mathfrak{X}})$ extends to a functor $\mathfrak{PCat} \xrightarrow{\mathfrak{J}^*} \mathfrak{SCat}$ which is a left adjoint to the inclusion functor $\mathfrak{SCat} \xrightarrow{\mathfrak{J}_*} \mathfrak{PCat}$. The natural triangle (localization) functors $(C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, \lambda; \mathfrak{Tr}_{\mathfrak{X}}) \xrightarrow{\mathfrak{q}^*_{\Sigma}} (\Sigma^{-1}C_{\mathfrak{X}}, \Theta_{\mathfrak{X}}, \widetilde{\lambda}; \widetilde{\mathfrak{Tr}}_{\mathfrak{X}})$ form an adjunction arrow $Id_{\mathfrak{PCat}} \longrightarrow \mathfrak{J}_*\mathfrak{J}^*$. The other adjunction arrow is identical.

7.4. Quasi-triangulated categories. Let $(C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, \lambda; \mathfrak{Tr}_{\mathfrak{X}})$ be a presuspended category. We call it *quasi-triangulated*, if the endofunctor $\theta_{\mathfrak{X}}$ is an auto-equivalence.

In particular, every quasi-triangulated category is quasi-suspended. Let $\mathfrak{QTr}$ denote the full subcategory of $\mathfrak{PCat}$ (or $\mathfrak{SCat}$) whose objects are quasi-triangulated subcategories. We call a quasi-triangulated category *strict* if θ_X is an isomorphism of categories.

7.4.1. Proposition. *The inclusion functor $\mathfrak{QTr} \longrightarrow \mathfrak{PCat}$ has a left adjoint. More precisely, for each prestable category, $\mathfrak{TC}_{\mathfrak{X}} = (C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, \lambda; \mathfrak{Tr}_{\mathfrak{X}})$, there is a triangle functor from $\mathfrak{TC}_{\mathfrak{X}}$ to a strict quasi-triangulated category such that any triangle functor to a quasi-triangulated category factors uniquely through this functor.*

Proof. The argument is a standard procedure of inverting a functor, which was originated, probably, in Grothendieck's work on derivators. One can mimik the argument of the similar theorem (from suspended to strict triangulated categories) from [KeV]. ∎

7.5. The k-linear version. It is obtained by restricting to k-linear additive categories and k-linear functors. Otherwise all axioms and facts look similarly. Details are left to the reader.

7.5.1. Remark. Notice that the notion of a *quasi-suspended* k-linear category presented here differs from the notion of *suspended* category proposed by Keller and Vossieck [KeV1]. In particular, the notion of a quasi-triangulated k-linear category is different from the notion of a triangulated k-linear category.

7.6. Dual notions. Dualizing the notion of a presuspended category, we obtain the notion of a *precosuspended* category. The corresponding, dual, axioms will be denoted by (PS1*), ... , (PS5*). A precosuspended category $\mathfrak{T}C_{\mathfrak{X}} = (C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, \lambda; \mathfrak{Tr}_{\mathfrak{X}})$ will be called *quasi-cosuspended* if the functor $\theta_{\mathfrak{X}}$ is conservative and *quasi-cotriangulated* if $\theta_{\mathfrak{X}}$ is an auto-equivalence.

8. Complement: cohomological and homological functors.

8.1. Cohomological functors.

Fix a svelte presuspended category $\mathfrak{T}C_{\mathfrak{X}} = (\mathcal{C}_{\mathfrak{X}}, \theta_{\mathfrak{X}}, \lambda; \mathfrak{Tr}_{\mathfrak{X}})$. Let $(C_Y, \mathfrak{I}_Y)$ be a left exact category. We say that a functor $C_{\mathfrak{X}} \xrightarrow{F} C_Y$ is a *cohomological* functor from $\mathfrak{T}C_{\mathfrak{X}}$ to $(C_Y, \mathfrak{I}_Y)$, if for any triangle $\mathcal{N} \xrightarrow{u} \mathcal{M} \xrightarrow{v} \mathcal{L} \xrightarrow{w} \theta_{\mathfrak{X}}(\mathcal{N})$ of $\mathfrak{T}C_{\mathfrak{X}}$, the sequence

$$F(\mathcal{N}) \xrightarrow{F(u)} F(\mathcal{M}) \xrightarrow{F(v)} F(\mathcal{L}) \xrightarrow{F(w)} F\theta_{\mathfrak{X}}(\mathcal{N}) \xrightarrow{F\theta_{\mathfrak{X}}(u)} \dots$$

is 'exact'. We denote by $\mathfrak{CF}(\mathfrak{X})$ the category whose objects are cohomological functors from the presuspended category $\mathfrak{T}C_{\mathfrak{X}}$ to svelte left exact categories. Morphisms from a cohomological functor $\mathfrak{T}C_{\mathfrak{X}} \xrightarrow{\mathcal{H}} (C_Y, \mathfrak{I}_Y)$ to a cohomological functor $\mathfrak{T}C_{\mathfrak{X}} \xrightarrow{\mathcal{G}} (C_Z, \mathfrak{I}_Z)$ is a pair (F, ϕ), where F is a functor $C_Y \longrightarrow C_Z$ and ϕ a functor isomorphism $F \circ \mathcal{H} \xrightarrow{\sim} \mathcal{G}$. The composition is defined in a standard way.

8.1.1. Note. The axiom (PS5) says that the (dual) Yoneda functor

$$C_{\mathfrak{X}} \xrightarrow{h^o_{\mathfrak{X}}} (C^{\vee}_{\mathfrak{X}})^{op}, \quad \mathcal{M} \longmapsto \mathcal{C}_{\mathfrak{X}}(\mathcal{M}, -),$$

is cohomological. Equivalently, all representable functors, $C_{\mathfrak{X}}(-, \mathcal{V})$, are cohomological functors from $\mathfrak{T}C_{\mathfrak{X}}$ to $Sets^{op}$, or *homological* functors from the precosuspended category $\mathfrak{T}C^{op}_{\mathfrak{X}} = \mathfrak{T}C_{\mathfrak{X}^o}$ to $Sets$.

8.2. Universal 'exact' ∂-functors and cohomological functors. Fix a svelte left exact category $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ and consider the category $\partial\mathfrak{Un}^{\mathfrak{c}}(\mathfrak{X}, \mathfrak{I})$ of ∂-functors from $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ to cocomplete categories (cf. 5.2 and 5.1). By 5.2.1, every universal ∂-functor $T = (T_i, \mathfrak{d}_i \mid i \geq 0)$ from $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ to a cocomplete category C_Y is the composition of the universal ∂-functor $Ext^{\bullet}_{\mathfrak{X},\mathfrak{I}}$ from $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ to $C^{\wedge}_{\mathfrak{X}} = C_{\mathfrak{X}^{\wedge}}$ and a continuous (i.e. having a right adjoint) functor $C_{\mathfrak{X}^{\wedge}} \xrightarrow{T^*_0} C_Y$ which is determined uniquely up to isomorphism by the equality $T_0 = T^*_0 \circ h_{\mathfrak{X}}$, where $h_{\mathfrak{X}}$ is the Yoneda embedding $C_{\mathfrak{X}} \longrightarrow C_{\mathfrak{X}^{\wedge}} = C^{\wedge}_{\mathfrak{X}}$.

Suppose now that T is an 'exact' ∂-functor from $(\mathcal{C}_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ to $(C_Y, \mathfrak{I}_Y)$ for some left exact structure $\mathfrak{I}_Y$ on the category C_Y. Then the functor T^*_0 maps the exact sequence

$$\widehat{N} \xrightarrow{\widehat{\mathfrak{j}}} \widehat{M} \xrightarrow{\widehat{\mathfrak{e}}} \widehat{L} \xrightarrow{\mathfrak{d}_0(E)} \widehat{\Theta}^*_{\mathfrak{X}}(\widehat{N}) \xrightarrow{\widehat{\Theta}^*_{\mathfrak{X}}(\mathfrak{j})} \dots$$

to an 'exact' sequence, i.e. T^*_0 is a cohomological functor from the presuspended category $\mathfrak{T}C_{\mathfrak{X}^{\wedge}} = (C_{\mathfrak{X}^{\wedge}}, \widehat{\Theta}^*_{\mathfrak{X}}, \lambda_{\mathfrak{X}}; \widehat{\mathfrak{Tr}_{\mathfrak{X}}})$ to the left exact category $(C_Y, \mathfrak{I}_Y)$.

Set $T^+ \stackrel{\mathrm{def}}{=} (T_i, \mathfrak{d}_i \mid i \geq 1)$. It follows that T^+ is the composition of

$$Ext^+_{\mathfrak{X},\mathfrak{I}} = (\widehat{\Theta}_{\mathfrak{X}}^{*^i}, \widehat{\Theta}_{\mathfrak{X}}^{*^i}\mathfrak{d}_0 \mid i \geq 1) \circ h_{\mathfrak{X}} = \widehat{\Theta}_{\mathfrak{X}}^* \circ Ext^{\bullet}_{\mathfrak{X},\mathfrak{I}} = \widehat{\Theta}_{\mathfrak{X}}^* \circ (\widehat{\Theta}_{\mathfrak{X}}^{*^i}, \widehat{\Theta}_{\mathfrak{X}}^{*^i}\mathfrak{d}_0 \mid i \geq 0) \circ h_{\mathfrak{X}}$$

and the continuous functor T_0^*. Thus,

$$Ext^+_{\mathfrak{X},\mathfrak{I}} \simeq (\theta^i_{\mathfrak{X}_\mathfrak{c}}, \theta^i_{\mathfrak{X}_\mathfrak{c}}\widetilde{\mathfrak{d}}_0 \mid i \geq 1) \circ (\mathfrak{q}^*_{\mathfrak{X}} \circ h_{\mathfrak{X}}) = \theta_{\mathfrak{X}_\mathfrak{c}} \circ (\theta^i_{\mathfrak{X}_\mathfrak{c}}, \theta^i_{\mathfrak{X}_\mathfrak{c}}\widetilde{\mathfrak{d}}_0 \mid i \geq 0) \circ (\mathfrak{q}^*_{\mathfrak{X}} \circ h_{\mathfrak{X}}),$$

where $\theta_{\mathfrak{X}_\mathfrak{c}}$ is the suspension endofunctor of the category $\mathcal{C}_{\mathfrak{X}_\mathfrak{c}} = \Sigma^{-1}_{\widehat{\Theta}^*_{\mathfrak{X}}} C_{\mathfrak{X}^\wedge}$ (cf. 7.2.2) which is uniquely determined by the equality $\theta_{\mathfrak{X}_\mathfrak{c}} \circ \mathfrak{q}^*_{\mathfrak{X}} = \mathfrak{q}^*_{\mathfrak{X}} \circ \widehat{\Theta}^*_{\mathfrak{X}}$ and $\widetilde{\mathfrak{d}}_0$ is uniquely determined by the equality $\widetilde{\mathfrak{d}}_0 \mathfrak{q}^*_{\mathfrak{X}} = \mathfrak{q}^*_{\mathfrak{X}} \mathfrak{d}_0$. This shows that T^+ determines uniquely up to isomorphism (and is determined by) a continuous cohomological functor $T_0^* \circ \theta_{\mathfrak{X}_\mathfrak{c}}$ from the quasi-suspended category $\mathfrak{T}\mathcal{C}_{\mathfrak{X}_\mathfrak{c}} = (\mathcal{C}_{\mathfrak{X}_\mathfrak{c}}, \theta_{\mathfrak{X}_\mathfrak{c}}, \lambda_{\mathfrak{X}_\mathfrak{c}}; \mathfrak{Tr}_{\mathfrak{X}_\mathfrak{c}})$ associated with the presuspended category $\mathfrak{T}\mathcal{C}_{\mathfrak{X}^\wedge}$ of presheaves of sets on the left exact category $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ (see 7.2.2).

It follows from 7.2.3 and 7.2.4 that T^+ determines a cohomological functor $\mathcal{H}_T$ from the stable category $\mathcal{C}_{\mathfrak{X}_\mathfrak{s}}$ of the left exact category of $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$. The functor $\mathcal{H}_T$ is the restriction of the functor $T_0^* \circ \theta_{\mathfrak{X}_\mathfrak{c}}$ to the stable category $\mathcal{C}_{\mathfrak{X}_\mathfrak{s}}$, which is a subcategory of the category $\mathcal{C}_{\mathfrak{X}_\mathfrak{c}}$.

8.3. Homological functors. Homological functors from a precosuspended category $\mathfrak{T}\mathcal{C}_{\mathfrak{X}}$ are defined dually. We denote by $\mathfrak{H}\mathfrak{F}(\mathfrak{X})$ the category of homological functors from $\mathfrak{T}\mathcal{C}_{\mathfrak{X}}$ to svelte right exact categories with initial objects.

8.4. Homological functors to cocomplete right exact categories.

8.4.1. Cocomplete right exact categories. We call a right exact category $(C_Y, \mathfrak{E}_Y)$ *cocomplete* if C_Y has colimits and initial objects, and $\mathfrak{E}_Y$ consists of all strict epimorphisms (in particular, $\mathfrak{E}_Y$ is the finest right exact structure on C_Y).

8.4.2. Examples. (a) Any Grothendieck topos endowed with the canonical pretopology is a cocomplete right exact category.

(b) If $(C_Y, \mathfrak{E}_Y)$ is an abelian (or a quasi-abelian) category with the canonical right exact structure, then it is cocomplete iff the category C_Y has small coproducts.

8.4.3. The category $\mathfrak{H}\mathfrak{F}_\mathfrak{c}(\mathfrak{X})$. Let $\mathfrak{T}\mathcal{C}_{\mathfrak{X}} = (\mathcal{C}_{\mathfrak{X}}, \theta_{\mathfrak{X}}, \lambda; \mathfrak{Tr}_{\mathfrak{X}})$ be a svelte precosuspended category. We denote by $\mathfrak{H}\mathfrak{F}_\mathfrak{c}(\mathfrak{X})$ the subcategory of the category $\mathfrak{H}\mathfrak{F}(\mathfrak{X})$ of homological functors from $\mathfrak{T}\mathcal{C}_{\mathfrak{X}}$ (cf. 8.3) whose objects are homological functors from $\mathfrak{T}\mathcal{C}_{\mathfrak{X}}$ to svelte cocomplete right exact categories and morphisms are morphisms (F, ϕ) of $\mathfrak{H}\mathfrak{F}(\mathfrak{X})$ such that F is a continuous (i.e. having a right adjoint) functor.

8.4.4. Proposition. *The category $\mathfrak{H}\mathfrak{F}_\mathfrak{c}(\mathfrak{X})$ has an initial object.*

Proof. By the axiom (PS5*) (dual to (PS5)), the Yoneda functor

$$C_{\mathfrak{X}} \xrightarrow{\ h_{\mathfrak{X}}\ } C^{\wedge}_{\mathfrak{X}}, \quad \mathcal{M} \longmapsto \mathcal{C}_{\mathfrak{X}}(-, \mathcal{M}),$$

is a homological functor from $\mathfrak{T}\mathcal{C}_{\mathfrak{X}}$ to the category $C_{\mathfrak{X}^\wedge}$ endowed with the canonical pretopology. The claim is that the cohomological functor $\mathfrak{T}\mathcal{C}_{\mathfrak{X}} \xrightarrow{h_{\mathfrak{X}}} C^{\wedge}_{\mathfrak{X}}$ is *universal*; i.e. it is an initial object of the category $\mathfrak{H}\mathfrak{F}_\mathfrak{c}(\mathfrak{X})$.

In fact, for any functor $C_{\mathfrak{X}} \xrightarrow{F} C_Y$, where C_Y is a cocomplete category, there is a continuous functor $C^{\wedge}_{\mathfrak{X}} \xrightarrow{F^*} C_Y$ which is determined uniquely up to isomorphism by the equality $F^* \circ h^o_{\mathfrak{X}} = F$. By [GZ, II.1.3], the map $F \longmapsto F_*$ extends to an equivalence between the category $\mathcal{H}om(\mathcal{C}_{\mathfrak{X}}, C_Y)$ of functors from $\mathcal{C}_{\mathfrak{X}}$ to C_Y and the category $\mathfrak{Hom}^{\mathfrak{c}}(\mathcal{C}^{\wedge}_{\mathfrak{X}}, C_Y)$ of continuous functors from $C^{\wedge}_{\mathfrak{X}}$ to C_Y. ∎

8.5. Homological functors to fully right exact categories.

8.5.1. Fully right exact and fully left exact categories. We call a right exact category $(C_Y, \mathfrak{E}_Y)$ *fully right exact* if the Yoneda embedding of C_Y into the category $C_{Y_{\mathfrak{E}}}$ of sheaves of sets on $(C_Y, \mathfrak{E}_Y)$ establishes an equivalence between $(C_Y, \mathfrak{E}_Y)$ and a fully exact subcategory of the category $C_{Y_{\mathfrak{E}}}$.

A *fully left exact* category is defined dually.

8.5.2. Note. The additive version of these notions coincides with the notion of an exact category, because, in additive case, any fully right (or left) exact category is a fully exact subcategory of an abelian category.

8.5.3. Proposition. *(a) Any Grothendieck topos is a fully right exact category.*

(b) Any fully exact (that is full and closed under extensions) subcategory of a fully right exact category is a fully right exact category.

Proof. (a) A right exact category $(C_X, \mathfrak{E}_X)$ is a Grothendieck topos iff the canonical functor $(C_X, \mathfrak{E}_X) \xrightarrow{\mathfrak{j}^*_X} (C_{X_{\mathfrak{E}}}, \mathfrak{E}_{X_{\mathfrak{E}}})$ is an equivalence of right exact categories.

(b) The argument is left to the reader. ∎

8.5.4. The category of homological functors to fully right exact categories. Let $\mathfrak{HF}_{\mathfrak{r}}(\mathfrak{X})$ denote the subcategory of the category $\mathfrak{HF}(\mathfrak{X})$ of homological functors whose objects are homological functors with values in fully right exact svelte categories and morphisms from a homological functor $\mathfrak{T}\mathcal{C}_{\mathfrak{X}} \xrightarrow{\mathcal{H}} (C_Y, \mathfrak{E}_Y)$ to a homological functor $\mathfrak{T}\mathcal{C}_{\mathfrak{X}} \xrightarrow{\mathcal{G}} (C_Z, \mathfrak{E}_Z)$ is a pair (F, ϕ) such that the functor $C_Y \xrightarrow{F} C_Z$ maps deflations to deflations and conflations to 'exact' sequences.

8.5.5. Proposition. *The category $\mathfrak{HF}_{\mathfrak{r}}(\mathfrak{X})$ has an initial object.*

Proof. Let $\mathcal{C}_{\mathfrak{X}_{\mathfrak{e}}}$ denote the smallest $\theta_{\mathfrak{X}}$-stable fully exact subcategory of the category $\mathcal{C}^{\wedge}_{\mathfrak{X}}$ (endowed with the canonical right exact structure) containing all representable presheaves. The Yoneda embedding induces a homological functor $\mathcal{C}_{\mathfrak{X}} \xrightarrow{\mathfrak{h}_{\mathfrak{X}}} \mathcal{C}_{\mathfrak{X}_{\mathfrak{e}}}$. The claim is that this homological functor is an initial object of the category $\mathfrak{HF}_{\mathfrak{r}}(\mathfrak{X})$.

In fact, let $(C_Y, \mathfrak{E}_Y)$ be a right exact category and $\mathcal{G}$ a homological functor from $\mathfrak{T}\mathcal{C}_{\mathfrak{X}}$ to $(C_Y, \mathfrak{E}_Y)$. The composition of $\mathcal{G}$ with the Yoneda embedding $(C_Y, \mathfrak{E}_Y) \xrightarrow{\mathfrak{j}^*_Y} (C_{Y_{\mathfrak{E}}}, \mathfrak{E}^{\mathfrak{s}}_{Y_{\mathfrak{E}}})$ (cf. 2.8) is a homological functor. Since the category of sheaves $C_{Y_{\mathfrak{E}}}$ is cocomplete, there is a quasi-commutative diagram

$$\begin{array}{ccc} \mathcal{C}_{\mathfrak{X}} & \xrightarrow{\mathcal{G}} & C_Y \\ {\scriptstyle h_{\mathfrak{X}}}\downarrow & & \downarrow{\scriptstyle \mathfrak{j}^*_Y} \\ \mathcal{C}^{\wedge}_{\mathfrak{X}} & \xrightarrow{\mathcal{G}^*} & C_{Y_{\mathfrak{E}}} \end{array} \qquad (1)$$

where $\mathcal{G}^*$ is a continuous functor determined uniquely up to isomorphism by the quasi-commutativity of the diagram (1). Since the functor $\mathcal{G}^*$ is right exact, the preimage ${\mathcal{G}^*}^{-1}(C_{\mathcal{Z}})$ of any fully exact subcategory $C_{\mathcal{Z}}$ of $C_{Y_{\mathfrak{E}}}$, is a fully exact subcategory of $C^{\wedge}_{\mathfrak{X}}$. In particular, the preimage of C_Y is a fully exact subcategory of $\mathcal{C}^{\wedge}_{\mathfrak{X}}$. By hypothesis, C_Y is a fully exact category, i.e. the full subcategory $\widetilde{C}_Y$ of $C_{Y_{\mathfrak{E}}}$ generated by all representable functors is a fully exact subcategory of $C^{\wedge}_{\mathfrak{X}}$. Therefore, ${\mathcal{G}^*}^{-1}(\widetilde{C}_Y)$ is a fully exact subcategory of $\mathcal{C}^{\wedge}_{\mathfrak{X}}$ containing all representable functors; so that it contains the subcategory $\mathcal{C}_{\mathfrak{X}_{\mathfrak{e}}}$; i.e. the restriction of the functor $\mathcal{G}^*$ to the subcategory $\mathcal{C}_{\mathfrak{X}_{\mathfrak{e}}}$ takes values in the subcategory $\widetilde{C}_Y$. Therefore, $\mathcal{G}^*$ the composition of $\mathcal{G}^*$ and the inclusion functor $\mathcal{C}_{\mathfrak{X}_{\mathfrak{e}}} \hookrightarrow \mathcal{C}^{\wedge}_{\mathfrak{X}}$ is isomorphic to the composition $\mathfrak{j}^*_Y \circ \mathcal{G}^*_{\mathfrak{e}}$, where the functor $\mathcal{C}_{\mathfrak{X}_{\mathfrak{e}}} \xrightarrow{\mathcal{G}^*_{\mathfrak{e}}} C_Y$ is determined uniquely up to isomorphism. ∎

8.6. Universal cohomological functors. For the reader's convenience, we sketch some details of the dual picture. Fix a svelte presuspended category $\mathfrak{T}\mathcal{C}_{\mathfrak{X}} = (\mathcal{C}_{\mathfrak{X}}, \theta_{\mathfrak{X}}, \lambda; \mathfrak{Tr}_{\mathfrak{X}})$. We denote by $\mathcal{CH}^{\mathfrak{c}}(\mathfrak{X})$ the category whose objects are cohomological functors from $\mathfrak{T}\mathcal{C}_{\mathfrak{X}}$ to svelte complete categories. Morphisms from a cohomological functor $\mathcal{C}_{\mathfrak{X}} \xrightarrow{\mathcal{H}} C_Y$ to a cohomological functor $\mathcal{C}_{\mathfrak{X}} \xrightarrow{\mathcal{G}} C_Z$ is a pair (F, ϕ), where F is a cocontinuous (that is having a left adjoint) functor $C_Y \longrightarrow C_Z$ and ϕ a functor isomorphism $F \circ \mathcal{H} \xrightarrow{\sim} \mathcal{G}$. The composition is defined in a standard way.

8.6.1. Proposition. *The category $\mathcal{CH}^{\mathfrak{c}}(\mathfrak{X})$ has an initial object.*

Proof. By the axiom (PS5), the dual Yoneda functor

$$C_{\mathfrak{X}} \xrightarrow{h^o_{\mathfrak{X}}} (C^{\vee}_{\mathfrak{X}})^{op}, \quad \mathcal{M} \longmapsto \mathcal{C}_{\mathfrak{X}}(\mathcal{M}, -),$$

is cohomological. The claim is that the cohomological functor $h^o_{\mathfrak{X}}$ is *universal*; i.e. it is an initial object of the category $\mathcal{CH}^{\mathfrak{c}}(\mathfrak{X})$.

In fact, for any functor $C_{\mathfrak{X}} \xrightarrow{F} C_Y$, where C_Y is a complete category, there is a cocontinuous functor $(C^{\vee}_{\mathfrak{X}})^{op} \xrightarrow{F_*} C_Y$ which is determined uniquely up to isomorphism by the equality $F_* \circ h^o_{\mathfrak{X}} = F$. By (the dual version of) [GZ, II.1.3], the map $F \longmapsto F_*$ extends to an equivalence between the category $\mathcal{H}om(\mathcal{C}_{\mathfrak{X}}, C_Y)$ of functors from $\mathcal{C}_{\mathfrak{X}}$ to C_Y and the category $\mathfrak{Hom}^{\mathfrak{c}}((\mathcal{C}^{\vee}_{\mathfrak{X}})^{op}, C_Y)$ of cocontinuous functors from $(C^{\vee}_{\mathfrak{X}})^{op}$ to C_Y. ∎

8.6.2. Cohomological functors to fully left exact categories. Let $\mathfrak{CHF}_{\ell}(\mathfrak{X})$ denote the subcategory of the category $\mathfrak{CHF}(\mathfrak{X})$ of cohomological functors whose objects are cohomological functors with values in fully left exact svelte categories and morphisms from a cohomological functor $\mathfrak{T}\mathcal{C}_{\mathfrak{X}} \xrightarrow{\mathcal{H}} (C_Y, \mathfrak{E}_Y)$ to a cohomological functor $\mathfrak{T}\mathcal{C}_{\mathfrak{X}} \xrightarrow{\mathcal{G}} (C_Z, \mathfrak{E}_Z)$ is a morphism (F, ϕ) of $\mathfrak{CHF}(\mathfrak{X})$ such that the functor $C_Y \xrightarrow{F} C_Z$ maps inflations to inflations and conflations to 'exact' sequences.

8.6.3. Proposition. *The category $\mathfrak{CHF}_{\ell}(\mathfrak{X})$ has an initial object.*

Proof. The assertion follows from 8.5.5 by duality. ∎

8.7. Universal homological and cohomological functors in k-linear case. Let $\mathfrak{T}\mathcal{C}_{\mathfrak{X}} = (\mathcal{C}_{\mathfrak{X}}, \theta_{\mathfrak{X}}, \lambda; \mathfrak{T}\mathfrak{r}_{\mathfrak{X}})$ be a k-linear precosuspended category. We denote by $\mathcal{H}\mathfrak{F}_k(\mathfrak{X})$ the category whose objects are k-linear homological functors from $\mathfrak{T}\mathcal{C}_{\mathfrak{X}}$ to svelte exact k-linear categories and morphisms are morphisms (F, ϕ) of homological functors (that is morphisms of the category $\mathfrak{H}\mathfrak{F}_{\mathfrak{r}}(\mathfrak{X})$) such that the (right 'exact') functor F is k-linear.

8.7.1. Proposition. *The category $\mathfrak{H}\mathfrak{F}_k(\mathfrak{X})$ has an initial object.*

Proof. (a) Let $C_{\mathfrak{X}_k}$ denote the smallest $\theta_{\mathfrak{X}}$-stable fully exact subcategory of the category $\mathcal{M}_k(\mathfrak{X})$ of presheaves of k-modules on $\mathcal{C}_{\mathfrak{X}}$ containing all representable functors. The Yoneda embedding induces a k-linear homological functor $\mathfrak{T}\mathcal{C}_{\mathfrak{X}} \xrightarrow{\mathfrak{h}_{\mathfrak{X}_k}} (C_{\mathfrak{X}_k}, \mathfrak{E}_{\mathfrak{X}_k})$, where $\mathfrak{E}_{\mathfrak{X}_k}$ is the (right) exact structure induced by the canonical right exact structure on $\mathcal{M}_k(\mathfrak{X})$. The claim is that the homological functor $\mathfrak{h}_{\mathfrak{X}_k}$ is an initial object of the category $\mathfrak{H}\mathfrak{F}_k(\mathfrak{X})$. The argument follows is similar to the argument of 8.5.5. ∎

8.7.2. The k-linear additive categories and exact categories with enough projectives. For any k-linear additive category C_Y, let $C_{Y_{\mathfrak{a}}}$ denote the full subcategory of the category $\mathcal{M}_k(Y)$ of presheaves of k-modules on C_Y whose objects are those presheaves of k-modules which have a left resolution formed by representable presheaves. One can deduce from [Ba, I.6.7] that $C_{Y_{\mathfrak{a}}}$ is a fully exact subcategory of the abelian k-linear category $\mathcal{M}_k(Y)$. Since every representable functor is a projective object of the abelian category $\mathcal{M}_k(Y)$ and every deflation is a strict epimorphism, it follows that representable functors are projectives of the exact category $C_{Y_{\mathfrak{a}}}$. It follows from the definition of $C_{Y_{\mathfrak{a}}}$ that it has enough projectives with respect to the exact structure induced from $\mathcal{M}_k(Y)$.

8.7.2.1. Proposition. *The correspondence $C_Y \longmapsto C_{Y_{\mathfrak{a}}}$ is a functor from the category Add_k of svelte k-linear additive categories and k-linear functors to the category $\mathfrak{P}_{\mathfrak{r}}\mathfrak{E}\mathfrak{x}_k$ of svelte exact categories with enough projectives and right exact functors which map projectives to projectives.*

Proof. In fact, any k-linear functor $C_Y \xrightarrow{\varphi} C_Z$ extends uniquely up to isomorphism to a continuous k-linear functor $\mathcal{M}_k(Y) \xrightarrow{\varphi^*} \mathcal{M}_k(Z)$ such that the diagram

$$\begin{array}{ccc} C_Y & \xrightarrow{\varphi} & C_Z \\ {\scriptstyle h_Y}\downarrow & & \downarrow{\scriptstyle h_Z} \\ \mathcal{M}_k(Y) & \xrightarrow{\varphi^*} & \mathcal{M}_k(Z) \end{array}$$

commutes. Since the functor φ^* is right exact and maps representable functors to representable functors, it induces a functor $C_{Y_{\mathfrak{a}}} \xrightarrow{\varphi_{\mathfrak{a}}} C_{Z_{\mathfrak{a}}}$ which is right 'exact'. ∎

8.7.2.2. Remarks. (a) Let C_Y be a k-linear additive svelte category. Each projective of the associated exact category $C_{Y_{\mathfrak{a}}}$ is a direct summand of a representable functor. Therefore, if the category C_Y is Karoubian, then the canonical embedding $C_Y \xrightarrow{\mathfrak{h}_{Y_{\mathfrak{a}}}} C_{Y_{\mathfrak{a}}}$ induces an equivalence of the category C_Y and the full subcategory of $C_{Y_{\mathfrak{a}}}$ generated by all projectives of $(C_{Y_{\mathfrak{a}}}, \mathfrak{E}_{Y_{\mathfrak{a}}})$.

(b) Suppose that C_Y is a k-linear additive category endowed with an action

$$\mathfrak{M} \times C_Y \xrightarrow{\widetilde{\Phi}} C_Y$$

of a monoidal category $\widetilde{\mathfrak{M}} = (\mathfrak{M}, \odot, \mathbb{I})$. Then the category $C_{Y_\mathfrak{a}}$ is endowed with a natural action of $\widetilde{\mathfrak{M}}$ by right exact endofunctors which preserve projectives.

In fact, the action $\mathfrak{M} \times C_Y \longrightarrow C_Y$ extends uniquely up to isomorphism to a *continuous* action (i.e. an action by continuous endofunctors) of $\widetilde{\mathfrak{M}}$ on the category $\mathcal{M}_k(Y)$ of sheaves of k-modules on C_Y which is compatible with the Yoneda embedding. This action induces an action $\widetilde{\Phi}_\mathfrak{a}$ of $\widetilde{\mathfrak{M}}$ on $C_{Y_\mathfrak{a}}$ such that the diagram

$$\begin{array}{ccc} \mathfrak{M} \times C_Y & \xrightarrow{Id_{\mathfrak{M}} \times \mathfrak{h}_{Y_\mathfrak{a}}} & \mathfrak{M} \times C_{Y_\mathfrak{a}} \\ {\scriptstyle \widetilde{\Phi}} \downarrow & & \downarrow {\scriptstyle \widetilde{\Phi}_\mathfrak{a}} \\ C_Y & \xrightarrow{\mathfrak{h}_{Y_\mathfrak{a}}} & C_{Y_\mathfrak{a}} \end{array}$$

quasi-commutes. Notice that the action $\widetilde{\Phi}_\mathfrak{a}$ preserves projectives.

(b1) It follows from (a) above that if the category C_Y is Karoubian, then the functor $C_Y \xrightarrow{\mathfrak{h}_{Y_\mathfrak{a}}} C_{Y_\mathfrak{a}}$ induces an equivalence of $\widetilde{\mathfrak{M}}$-category $(C_Y, \widetilde{\Phi})$ and the full $\widetilde{\mathfrak{M}}$-subcategory of the exact $\widetilde{\mathfrak{M}}$-category $(C_{Y_\mathfrak{a}}, \widetilde{\Phi}_\mathfrak{a})$ generated by all its projectives.

(c) Let $\mathfrak{T}\mathcal{C}_\mathfrak{X} = (\mathcal{C}_\mathfrak{X}, \theta_\mathfrak{X}, \lambda; \mathfrak{Tr}_\mathfrak{X})$ be a svelte additive k-linear precosuspended category. It follows from (b) that $\mathcal{C}_{\mathfrak{X}_\mathfrak{a}}$ is a svelte exact k-linear $\mathbb{Z}_+$-category, which has enough projectives. It follows that the canonical embedding $\mathcal{C}_\mathfrak{X} \xrightarrow{\mathfrak{h}_{\mathfrak{X}_\mathfrak{a}}} \mathcal{C}_{\mathfrak{X}_\mathfrak{a}}$ is a k-linear homological functor from $\mathfrak{T}\mathcal{C}_\mathfrak{X}$ to the exact k-linear category $\mathcal{C}_{\mathfrak{X}_\mathfrak{a}}$.

By (b1), if the category $\mathcal{C}_\mathfrak{X}$ is Karoubian, then it is equivalent to the full $\mathbb{Z}_+$-subcategory of $\mathcal{C}_{\mathfrak{X}_\mathfrak{a}}$ generated by all projectives of the right exact category $(\mathcal{C}_{\mathfrak{X}_\mathfrak{a}}, \mathfrak{E}_{\mathfrak{X}_\mathfrak{a}})$.

8.7.3. The subcategory $\mathfrak{H}\mathfrak{F}_k^\mathfrak{e}(\mathfrak{X})$ of homological functors. For a k-linear precosuspended category $\mathfrak{T}\mathcal{C}_\mathfrak{X}$, we denote by $\mathfrak{H}\mathfrak{F}_k^\mathfrak{e}(\mathfrak{X})$ the subcategory of the category $\mathfrak{H}\mathfrak{F}_k(\mathfrak{X})$ whose objects are k-linear homological functors $\mathfrak{T}\mathcal{C}_\mathfrak{X} \xrightarrow{\mathcal{G}} (C_\mathcal{Z}, \mathfrak{E}_\mathcal{Z})$ such that for any arrow f of the category $\mathcal{C}_\mathfrak{X}$, there exists a cokernel of $\mathcal{G}(f)$. Morphisms of $\mathfrak{H}\mathfrak{F}_k^\mathfrak{e}(\mathfrak{X})$ are morphisms (F, ϕ) such that the k-linear functor F is 'exact'.

8.7.4. Proposition. *(a) Let $\mathfrak{T}\mathcal{C}_\mathfrak{X} = (\mathcal{C}_\mathfrak{X}, \theta_\mathfrak{X}, \lambda; \mathfrak{Tr}_\mathfrak{X})$ be a precosuspended category having the following property: any morphism $M \xrightarrow{f} L$ of $\mathcal{C}_\mathfrak{X}$ extends to a triangle $\theta_\mathfrak{X}(L) \xrightarrow{\mathfrak{d}} N \xrightarrow{g} M \xrightarrow{f} L$. Then the homological functor $\mathcal{C}_\mathfrak{X} \xrightarrow{\mathfrak{h}_{\mathfrak{X}_\mathfrak{a}}} \mathcal{C}_{\mathfrak{X}_\mathfrak{a}}$ (see 8.7.2.2(c)) is an initial object of the category $\mathfrak{H}\mathfrak{F}_k^\mathfrak{e}(\mathfrak{X})$.*

(b) Suppose that $\mathfrak{T}\mathcal{C}_\mathfrak{X}$ is a quasi-triangulated k-linear category satisfying the condition of (a). Then the exact category $\mathcal{C}_{\mathfrak{X}_\mathfrak{a}}$ is abelian.

Proof. (a) Suppose that $\mathfrak{T}\mathcal{C}_\mathfrak{X} = (\mathcal{C}_\mathfrak{X}, \theta_\mathfrak{X}, \lambda; \mathfrak{Tr}_\mathfrak{X})$ is a k-linear precosuspended category satisfying the condition of (a). Then the corestriction $\mathcal{C}_\mathfrak{X} \xrightarrow{\mathfrak{h}_{\mathfrak{X}_\mathfrak{a}}} \mathcal{C}_{\mathfrak{X}_\mathfrak{a}}$ of the Yoneda embedding to the subcategory $\mathcal{C}_{\mathfrak{X}_\mathfrak{a}}$ is an initial object of the category $\mathfrak{H}\mathfrak{F}_k^\mathfrak{e}(\mathfrak{X})$.

The proof of this fact is the same as the argument of [R8, C4.3.4].

(b) If the translation functor $\theta_{\mathfrak{X}}$ is an auto-equivalence, then the category $\mathcal{C}_{\mathfrak{X}_\mathfrak{z}}$ is abelian. The proof of this fact follows the arguments of [R8, C4.4]. ∎

8.7.5. Remark. If $\mathfrak{T}\mathcal{C}_{\mathfrak{X}}$ is a triangulated category, then, by 8.7.4(b), the category $\mathcal{C}_{\mathfrak{X}_\mathfrak{a}}$ is abelian. In this case, the universal homological functor $\mathfrak{T}\mathcal{C}_{\mathfrak{X}} \xrightarrow{\mathfrak{h}_{\mathfrak{X}_\mathfrak{a}}} \mathcal{C}_{\mathfrak{X}_\mathfrak{a}}$ is equivalent to the 'abelianization' functor of Verdier [Ve2, II.3]. The latter follows from the fact that the Verdier's *abelianization* functor is universal among the homological functors to abelian categories. More precisely, it is an initial object of the full subcategory of the category $\mathfrak{H}\mathfrak{F}_k^\mathfrak{e}(\mathfrak{X})$ whose objects are homological functors from the triangulated category $\mathfrak{T}\mathcal{C}_{\mathfrak{X}}$ to abelian categories.

8.7.6. Cohomological functors. The formulation of the corresponding facts about k-linear cohomological functors is left to the reader.

Appendix: some properties of kernels.

A.1. Proposition. *Let $M \xrightarrow{f} N$ be a morphism of C_X which has a kernel pair, $M \times_N M \overset{p_1}{\underset{p_2}{\rightrightarrows}} M$. Then the morphism f has a kernel iff p_1 has a kernel, and these two kernels are naturally isomorphic to each other.*

Proof. Suppose that f has a kernel, i.e. there is a cartesian square

$$\begin{array}{ccc} Ker(f) & \xrightarrow{\mathfrak{k}(f)} & M \\ f' \downarrow & & \downarrow f \\ x & \xrightarrow{i_N} & N \end{array} \qquad (1)$$

Then we have the commutative diagram

$$\begin{array}{ccccc} Ker(f) & \xrightarrow{\gamma} & M \times_N M & \xrightarrow{p_2} & M \\ f' \downarrow & & p_1 \downarrow & & \downarrow f \\ x & \xrightarrow{i_M} & M & \xrightarrow{f} & N \end{array} \qquad (2)$$

which is due to the commutativity of (1) and the fact that the unique morphism $x \xrightarrow{i_N} N$ factors through the morphism $M \xrightarrow{f} N$. The morphism γ is uniquely determined by the equality $p_2 \circ \gamma = \mathfrak{k}(f)$. The fact that the square (1) is cartesian and the equalities $p_2 \circ \gamma = \mathfrak{k}(f)$ and $i_N = f \circ i_M$ imply that the left square of the diagram (2) is cartesian, i.e. $Ker(f) \xrightarrow{\gamma} M \times_N M$ is the kernel of the morphism p_1.

Conversely, if p_1 has a kernel, then we have a diagram

$$\begin{array}{ccccc} Ker(p_1) & \xrightarrow{\mathfrak{k}(p_1)} & M \times_N M & \xrightarrow{p_2} & M \\ p_1' \downarrow & cart & p_1 \downarrow & cart & \downarrow f \\ x & \xrightarrow{i_M} & M & \xrightarrow{f} & N \end{array}$$

which consists of two cartesian squares. Therefore the square

$$\begin{array}{ccc} Ker(p_1) & \xrightarrow{\mathfrak{k}(f)} & M \\ {\scriptstyle p_1'}\downarrow & cart & \downarrow {\scriptstyle f} \\ x & \xrightarrow{i_N} & N \end{array}$$

with $\mathfrak{k}(f) = p_2 \circ \mathfrak{k}(p_1)$ is cartesian. ■

A.2. Remarks. (a) Needless to say that the picture obtained in (the argument of) A.1 is symmetric, i.e. there is an isomorphism $Ker(p_1) \xrightarrow{\tau_f'} Ker(p_2)$ which is an arrow in the commutative diagram

$$\begin{array}{ccccc} Ker(p_1) & \xrightarrow{\mathfrak{k}(p_1)} & M \times_N M & \xrightarrow{p_1} & M \\ {\scriptstyle \tau_f'}\downarrow\wr & & {\scriptstyle \tau_f}\downarrow\wr & & \downarrow {\scriptstyle id_M} \\ Ker(p_2) & \xrightarrow{\mathfrak{k}(p_2)} & M \times_N M & \xrightarrow{p_2} & M \end{array}$$

(b) Let a morphism $M \xrightarrow{f} N$ have a kernel pair, $M \times_N M \overset{p_1}{\underset{p_2}{\rightrightarrows}} M$, and a kernel. Then, by A.1, there exists a kernel of p_1, so that we have a morphism $Ker(p_1) \xrightarrow{\mathfrak{k}(p_1)} M \times_N M$ and the diagonal morphism $M \xrightarrow{\Delta_M} M \times_N M$. Since the left square of the commutative diagram

$$\begin{array}{ccccc} x & \longrightarrow & Ker(p_1) & \xrightarrow{p_1'} & x \\ \downarrow & cart & {\scriptstyle \mathfrak{c}(p_1)}\downarrow & & \downarrow \\ M & \xrightarrow{\Delta_M} & M \times_N M & \xrightarrow{p_1} & M \end{array}$$

is cartesian and compositions of the horizontal arrows are identical morphisms, it follows that its left square is cartesian too. Loosely, one can say that the intersection of $Ker(p_1)$ with the diagonal of $M \times_N M$ is zero. If there exists a coproduct $Ker(p_1) \coprod M$, then the pair of morphisms $Ker(p_1) \xrightarrow{\mathfrak{k}(p_1)} M \times_N M \xleftarrow{\Delta_M} M$ determine a morphism

$$Ker(p_1) \coprod M \longrightarrow M \times_N M.$$

If the category C_X is additive, then this morphism is an isomorphism, or, what is the same, $Ker(f) \coprod M \simeq M \times_N M$. In general, it is rarely the case, as the reader can find out looking at the examples of 1.4.

A.3. Proposition. *Let*

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{\widetilde{f}} & \widetilde{N} \\ {\scriptstyle \widetilde{g}}\downarrow & cart & \downarrow {\scriptstyle g} \\ M & \xrightarrow{f} & N \end{array} \qquad (3)$$

be a cartesian square. Then $Ker(f)$ exists iff $Ker(\widetilde{f})$ exists, and they are naturally isomorphic to each other.

A.4. The kernel of a composition and related facts. Fix a category C_X with an initial object x.

A.4.1. The kernel of a composition. Let $L \xrightarrow{f} M$ and $M \xrightarrow{g} N$ be morphisms such that there exist kernels of g and $g \circ f$. Then the argument similar to that of A.3 shows that we have a commutative diagram

$$\begin{array}{ccccc} Ker(gf) & \xrightarrow{\widetilde{f}} & Ker(g) & \xrightarrow{g'} & x \\ \mathfrak{k}(gf) \downarrow & cart & \downarrow \mathfrak{k}(g) & cart & \downarrow i_N \\ L & \xrightarrow{f} & M & \xrightarrow{g} & N \end{array} \qquad (1)$$

whose both squares are cartesian and all morphisms are uniquely determined by f, g and the (unique up to isomorphism) choice of the objects $Ker(g)$ and $Ker(gf)$.

Conversely, if there is a commutative diagram

$$\begin{array}{ccccc} K & \xrightarrow{u} & Ker(g) & \xrightarrow{g'} & x \\ t \downarrow & cart & \downarrow \mathfrak{k}(g) & & \downarrow i_N \\ L & \xrightarrow{f} & M & \xrightarrow{g} & N \end{array}$$

whose left square is cartesian, then its left vertical arrow, $K \xrightarrow{t} L$, is the kernel of the composition $L \xrightarrow{g \circ f} N$.

A.4.2. Remarks. (a) It follows from A.3 that the kernel of $L \xrightarrow{f} M$ exists iff the kernel of $Ker(gf) \xrightarrow{\widetilde{f}} Ker(g)$ exists and they are isomorphic to each other. More precisely, we have a commutative diagram

$$\begin{array}{ccccccc} Ker(\widetilde{f}) & \xrightarrow{\mathfrak{k}(\widetilde{f})} & Ker(gf) & \xrightarrow{\widetilde{f}} & Ker(g) & \xrightarrow{g'} & x \\ \wr\downarrow & & \mathfrak{k}(gf) \downarrow & cart & \downarrow \mathfrak{k}(g) & cart & \downarrow i_N \\ Ker(f) & \xrightarrow{\mathfrak{k}(f)} & L & \xrightarrow{f} & M & \xrightarrow{g} & N \end{array}$$

whose left vertical arrow is an isomorphism.

(b) Suppose that $(C_X, \mathfrak{E}_X)$ is a right exact category (with an initial object x). If the morphism f above is a deflation, then it follows from this observation that the canonical morphism $Ker(gf) \xrightarrow{\widetilde{f}} Ker(g)$ is a deflation too. In this case, $Ker(f)$ exists, and we have a commutative diagram

$$\begin{array}{ccccc} Ker(\widetilde{f}) & \xrightarrow{\mathfrak{k}(\widetilde{f})} & Ker(gf) & \xrightarrow{\widetilde{f}} & Ker(g) \\ \wr\downarrow & & \mathfrak{k}(gf) \downarrow & cart & \downarrow \mathfrak{k}(g) \\ Ker(f) & \xrightarrow{\mathfrak{k}(f)} & L & \xrightarrow{f} & M \end{array}$$

whose rows are conflations.

The following observations is useful (and are used) for analysing diagrams.

A.4.3. Proposition. *(a) Let $M \xrightarrow{g} N$ be a morphism with a trivial kernel. Then a morphism $L \xrightarrow{f} M$ has a kernel iff the composition $g \circ f$ has a kernel, and these two kernels are naturally isomorphic one to another.*

(b) Let

$$\begin{array}{ccc} L & \xrightarrow{f} & M \\ \gamma \downarrow & & \downarrow g \\ \widetilde{M} & \xrightarrow{\phi} & N \end{array}$$

be a commutative square such that the kernels of the arrows f and ϕ exist and the kernel of g is trivial. Then the kernel of the composition $\phi \circ \gamma$ is isomorphic to the kernel of the morphism f, and the left square of the commutative diagram

$$\begin{array}{ccccccc} Ker(f) & \xrightarrow{\sim} & Ker(\phi\gamma) & \xrightarrow{\mathfrak{k}(f)} & L & \xrightarrow{f} & M \\ & & \tilde{\gamma} \downarrow & cart & \gamma \downarrow & & \downarrow g \\ & & Ker(\phi) & \xrightarrow{\mathfrak{k}(\phi)} & \widetilde{M} & \xrightarrow{\phi} & N \end{array}$$

is cartesian.

Proof. (a) Since the kernel of g is trivial, the diagram A.4.1(1) specializes to the diagram

$$\begin{array}{ccccc} Ker(gf) & \xrightarrow{\widetilde{f}} & x & \xrightarrow{id_x} & x \\ \mathfrak{k}(gf) \downarrow & cart & \downarrow \mathfrak{k}(g) & & \downarrow i_N \\ L & \xrightarrow{f} & M & \xrightarrow{g} & N \end{array}$$

with cartesian squares. The left cartesian square of this diagram is the definition of $Ker(f)$. The assertion follows from A.4.1.

(b) Since the kernel of g is trivial, it follows from (a) that $Ker(f)$ is naturally isomorphic to the kernel of $g \circ f = \phi \circ \gamma$. The assertion follows now from A.4.1. ■

A.4.4. Corollary. *Let C_X be a category with an initial object x. Let $L \xrightarrow{f} M$ be a strict epimorphism and $M \xrightarrow{g} N$ a morphism such that $Ker(g) \xrightarrow{\mathfrak{k}(g)} M$ exists and is a monomorphism. Then the composition $g \circ f$ is a trivial morphism iff g is trivial.*

A.4.4.1. Note. The following example shows that the requirement "$Ker(g) \longrightarrow M$ is a monomorphism" in A.4.4 cannot be omitted.

Let C_X be the category Alg_k of associative unital k-algebras, and let $\mathfrak{m}$ be an ideal of the ring k such that the epimorphism $k \longrightarrow k/\mathfrak{m}$ does not split. Then the identical morphism $k/\mathfrak{m} \longrightarrow k/\mathfrak{m}$ is non-trivial, while its composition with the projection $k \longrightarrow k/\mathfrak{m}$ is a trivial morphism.

A.5. The coimage of a morphism. Let $M \xrightarrow{f} N$ be an arrow which has a kernel, i.e. we have a cartesian square

$$\begin{array}{ccc} Ker(f) & \xrightarrow{\mathfrak{k}(f)} & M \\ {\scriptstyle f'}\downarrow & cart & \downarrow{\scriptstyle f} \\ x & \xrightarrow{i_N} & N \end{array}$$

with which one can associate a pair of arrows $Ker(f) \overset{\mathfrak{k}(f)}{\underset{0_f}{\rightrightarrows}} M$, where 0_f is the composition of the projection f' and the unique morphism $x \xrightarrow{i_M} M$. Since $i_N = f \circ i_M$, the morphism f equalizes the pair $Ker(f) \overset{\mathfrak{k}(f)}{\underset{0_f}{\rightrightarrows}} M$. If the cokernel of this pair of arrows exists, it will be called the *coimage of* f and denoted by $Coim(f)$, or. loosely, $M/Ker(f)$.

Let $M \xrightarrow{f} N$ be a morphism such that $Ker(f)$ and $Coim(f)$ exist. Then f is the composition of the canonical strict epimorphism $M \xrightarrow{p_f} Coim(f)$ and a uniquely defined morphism $Coim(f) \xrightarrow{j_f} N$.

A.5.1. Lemma. *Let $M \xrightarrow{f} N$ be a morphism such that $Ker(f)$ and $Coim(f)$ exist. There is a natural isomorphism $Ker(f) \xrightarrow{\sim} Ker(p_f)$.*

Proof. The outer square of the commutative diagram

$$\begin{array}{ccccc} Ker(f) & \xrightarrow{f'} & x & \longrightarrow & x \\ {\scriptstyle \mathfrak{k}(f)}\downarrow & cart & \downarrow & & \downarrow \\ M & \xrightarrow{p_f} & Coim(f) & \xrightarrow{j_f} & L \end{array} \qquad (1)$$

is cartesian. Therefore, its left square is cartesian which implies, by A.3, that $Ker(p_f)$ is isomorphic to $Ker(f')$. But, $Ker(f') \simeq Ker(f)$. ■

A.5.2. Note. By A.4.1, all squares of the commutative diagram

$$\begin{array}{ccccc} Ker(f) & \xrightarrow{f'} & x & & \\ {\scriptstyle id}\downarrow & cart & \downarrow & & \\ Ker(j_f p_f) & \xrightarrow{\widetilde{p}_f} & Ker(j_f) & \longrightarrow & x \\ {\scriptstyle \mathfrak{k}(f)}\downarrow & cart & \downarrow & cart & \downarrow \\ M & \xrightarrow{p_f} & Coim(f) & \xrightarrow{j_f} & L \end{array} \qquad (2)$$

are cartesian.

If C_X is an additive category and $M \xrightarrow{f} L$ is an arrow of C_X having a kernel and a coimage, then the canonical morphism $Coim(f) \xrightarrow{j_f} L$ is a monomorphism. Quite a few non-additive categories have this property.

A.5.3. Example. Let C_X be the category Alg_k of associative unital k-algebras. Since cokernels of pairs of arrows exist in Alg_k, any algebra morphism has a coimage. It follows from 1.4.1 that the coimage of an algebra morphism $A \xrightarrow{\varphi} B$ is $A/K(\varphi)$, where $K(\varphi)$ is the kernel of ϕ in the usual sense (i.e. in the category of non-unital algebras). The canonical decomposition $\varphi = j_\varphi \circ p_\varphi$ coincides with the standard presentation of φ as the composition of the projection $A \longrightarrow A/K(\varphi)$ and the monomorphism $A/K(\varphi) \longrightarrow B$. In particular, φ is strict epimorphism of k-algebras iff it is isomorphic to the associated coimage map $A \xrightarrow{p_\varphi} Coim(\varphi) = A/K(\varphi)$.

Lecture 5. Universal K-functors.

1. Preliminaries: left exact categories of right exact 'spaces'.

We start with left exact stuctures formed by localizations of 'spaces' represented by svelte categories. Then the obtained facts are used to define natural left exact structures on the category of 'spaces' represented by right exact categories.

The following proposition is a refinement of [R3, 1.4.1].

1.1. Proposition. *Let $Z \xleftarrow{f} X \xrightarrow{q} Y$ be morphisms of 'spaces' such that q (i.e. its inverse image functor $C_Y \xrightarrow{q^*} C_X$) is a localization. Then*

(a) The canonical morphism $Z \xrightarrow{\widetilde{q}} Z \coprod_{f,q} Y$ is a localization.

(b) If q is a continuous localization, then $\widetilde{q}$ is a continuous localization.

(c) If $\Sigma_{q^*} = \{s \in HomC_Y \mid q^*(s)$ is invertible$\}$ is a left (resp. right) multiplicative system, then $\Sigma_{\widetilde{q}^*}$ has the same property.

1.2. Corollary. *Let $Z \xleftarrow{f} X \xrightarrow{q} Y$ be morphisms of 'spaces' such that q is a localization, and let $Z \xrightarrow{\widetilde{q}} Z \coprod_{f,q} Y$ be a canonical morphism. Suppose that the category C_Y has finite limits (resp. finite colimits). Then $\widetilde{q}^*$ is a left (resp. right) exact localization, if the localization q^* is left (resp. right) exact.*

Proof. By 1.1(a), $\widetilde{q}^*$ is a localization functor.

Suppose that the category C_Y has finite limits and the localization functor $C_Y \xrightarrow{q^*} C_X$ is left exact. Then it follows from [GZ, I.3.4] that $\Sigma_{q^*} = \{s \in HomC_Y \mid q^*(s)$ is invertible$\}$ is a right multiplicative system. The latter implies, by 1.1(c), that $\Sigma_{\widetilde{q}^*}$ is a right multiplicative system. Therefore, by [GZ, I.3.1], the localization functor $\widetilde{q}^*$ is left exact. ∎

The following assertion is a refinement of [R3, 1.4.2].

1.3. Proposition. *Let $X \xleftarrow{p} Z \xrightarrow{q} Y$ be morphisms of 'spaces' such that p^* and q^* are localization functors. Then the square*

$$\begin{array}{ccc} Z & \xrightarrow{q} & Y \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p_1} \\ X & \xrightarrow{q_1} & X \coprod_{p,q} Y \end{array}$$

is cartesian.

1.4. Left exact structures on the category of 'spaces'. Let $\mathfrak{L}$ denote the class of all localizations of 'spaces' (i.e. morphisms whose inverse image functors are localizations). We denote by $\mathfrak{L}_\ell$ (resp. $\mathfrak{L}_\mathfrak{r}$) the class of localizations $X \xrightarrow{q} Y$ of 'spaces' such that $\Sigma_{q^*} = \{s \in HomC_Y \mid q^*(s)$ is invertible$\}$ is a left (resp. right) multiplicative system. We denote by $\mathfrak{L}_{\mathfrak{e}}$ the intersection of $\mathfrak{L}_\ell$ and $\mathfrak{L}_{\mathfrak{r}}$ (i.e. the class of localizations q such that Σ_{q^*} is a multiplicative system) and by $\mathfrak{L}^{\mathfrak{c}}$ the class of continuous (i.e. having a direct image functor) localizations of 'spaces'. Finally, we set $\mathfrak{L}^{\mathfrak{c}}_{\mathfrak{e}} = \mathfrak{L}^{\mathfrak{c}} \cap \mathfrak{L}_{\mathfrak{e}}$; i.e. $\mathfrak{L}^{\mathfrak{c}}_{\mathfrak{e}}$ is the class of continuous localizations $X \xrightarrow{q} Y$ such that Σ_{q^*} is a multiplicative system.

1.4.1. Proposition. *Each of the classes of morphisms $\mathfrak{L}$, $\mathfrak{L}_\ell$, $\mathfrak{L}_{\mathfrak{r}}$, $\mathfrak{L}_{\mathfrak{e}}$, $\mathfrak{L}^{\mathfrak{c}}$, and $\mathfrak{L}^{\mathfrak{c}}_{\mathfrak{e}}$ are structures of a left exact category on the category $|Cat|^o$ of 'spaces'.*

Proof. It is immediate that each of these classes is closed under composition and contains all isomorphisms of the category $|Cat|^o$. It follows from 1.1 that each of the classes is stable under cobase change. In other words, the arrows of each class can be regarded as cocovers of a copretopology. It remains to show that these copretopologies are subcanonical. Since $\mathfrak{L}$ is the finest copretopology, it suffices to show that $\mathfrak{L}$ is subcanonical.

The copretopology $\mathfrak{L}$ being subcanonical means precisely that for any localization $X \xrightarrow{q} Y$, the square

$$\begin{array}{ccc} X & \xrightarrow{q} & Y \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q_1} \\ Y & \xrightarrow{q_2} & Y \coprod_{q,q} Y \end{array}$$

is cartesian. But, this follows from 1.3. ■

1.5. Observation. Each object of the left exact category $(|Cat|^o, \mathfrak{L}^{\mathfrak{c}})$ is injective.

In fact, a 'space' X is an injective object of $(|Cat|^o, \mathfrak{L}^{\mathfrak{c}})$ iff each inflation $X \xrightarrow{q} Y$ is split; i.e. there is a morphism $Y \xrightarrow{\mathfrak{t}} X$ such that $\mathfrak{t} \circ q = id_X$. Since the morphism q is continuous, it has a direct image functor, q_*, which is fully faithful, because q^* is a localization functor. The latter means precisely that the adjunction arrow $q^* q_* \longrightarrow Id_{C_X}$ is an isomorphism; i.e. the morphism $Y \xrightarrow{\mathfrak{t}} X$ whose inverse image functor coincides with q_* satisfies the equality $\mathfrak{t} \circ q = id_X$.

1.6. Left exact structures on the category of right (or left) exact 'spaces'. A *right exact 'space'* is a pair $(X, \mathfrak{E}_X)$, where X is a 'space' and $\mathfrak{E}_X$ is a right exact structure on the category C_X. We denote by $\mathfrak{Esp}_{\mathfrak{r}}$ the category whose objects are right exact 'spaces' $(X, \mathfrak{E}_X)$ and morphisms from $(X, \mathfrak{E}_X)$ to $(Y, \mathfrak{E}_Y)$ are given by morphisms $X \xrightarrow{f} Y$ of 'spaces' whose inverse image functor, f^*, is *'exact'*; i.e. f^* maps deflations to deflations and preserves pull-backs of deflations.

Dually, a *left exact 'space'* is a pair $(Y, \mathfrak{I}_Y)$, where $(C_Y, \mathfrak{I}_Y)$ is a left exact category. We denote by $\mathfrak{Esp}_\ell$ the category whose objects are left exact 'spaces' $(Y, \mathfrak{I}_Y)$ and morphisms $(Y, \mathfrak{I}_Y) \longrightarrow (Z, \mathfrak{I}_Z)$ are given by morphisms $Y \longrightarrow Z$ whose inverse image functors are *'coexact'*, which means that they preserve inflations and their push-forwards.

1.6.1. Note. The categories $\mathfrak{Esp}_{\mathfrak{r}}$ and $\mathfrak{Esp}_\ell$ are naturally isomorphic to each other: the isomorphism is given by the dualization functor $(X, \mathfrak{E}_X) \longmapsto (X^o, \mathfrak{E}^{op}_X)$. Therefore, every assertion about the category $\mathfrak{Esp}_{\mathfrak{r}}$ of right exact 'spaces' translates into an assertion about the category $\mathfrak{Esp}_\ell$ of left exact 'spaces' and vice versa.

1.6.2. Proposition. *The category $\mathfrak{Esp}_{\mathfrak{r}}$ has fibered coproducts.*

1.6.3. Canonical left exact structures on the category $\mathfrak{Esp}_\mathfrak{r}$. Let $\mathfrak{L}_{\mathfrak{es}}$ denote the class of all morphisms $(X, \mathfrak{E}_X) \xrightarrow{q} (Y, \mathfrak{E}_Y)$ of right exact 'spaces' such that q^* is a localization functor and each arrow of $\mathfrak{E}_X$ is isomorphic to an arrow $q^*(\mathfrak{e})$ for some $\mathfrak{e} \in \mathfrak{E}_Y$.

If Σ_{q^*} is a left or right multiplicative system, then this condition means that $\mathfrak{E}_X$ is the smallest right exact structure containing $q^*(\mathfrak{E}_Y)$.

1.6.3.1. Proposition. *The class $\mathfrak{L}_{\mathfrak{es}}$ is a left exact structure on the category $\mathfrak{Esp}_\mathfrak{r}$ of right exact 'spaces'.*

1.6.3.2. Corollary. *Each of the classes of morphisms of 'spaces' $\mathfrak{L}_\ell$, $\mathfrak{L}_\mathfrak{r}$, $\mathfrak{L}_\mathfrak{e}$, $\mathfrak{L}^\mathfrak{c}$, and $\mathfrak{L}^\mathfrak{c}_\mathfrak{e}$ (cf. 1.4, 1.4.1) induces a structure of a left exact category on the category $\mathfrak{Esp}_\mathfrak{r}$ of right exact 'spaces'.*

Proof. The class $\mathfrak{L}_\ell$ induces the class $\mathfrak{L}^{\mathfrak{es}}_\ell$ of morphisms of the category $\mathfrak{Esp}_\mathfrak{r}$ formed by all arrows $(X, \mathfrak{E}_X) \xrightarrow{q} (Y, \mathfrak{E}_Y)$ from $\mathfrak{L}_{\mathfrak{es}}$ such that the morphism of 'spaces' $X \xrightarrow{q} Y$ belongs to $\mathfrak{L}_\ell$. Similarly, we define the classes $\mathfrak{L}^{\mathfrak{es}}_\ell$, $\mathfrak{L}^{\mathfrak{es}}_\mathfrak{r}$, $\mathfrak{L}^{\mathfrak{c}}_{\mathfrak{es}}$, and $\mathfrak{L}^{\mathfrak{e};\mathfrak{c}}_{\mathfrak{es}}$. ∎

1.6.3.3. The left exact structure $\mathfrak{L}^{\mathfrak{es}}_{\mathfrak{sq}}$. For a right exact 'space' $(X, \mathfrak{E}_X)$, let $Sq(X, \mathfrak{E}_X)$ denote the class of all cartesian squares in the category C_X some of the arrows of which (at least two) belong to $\mathfrak{E}_X$.

The class $\mathfrak{L}^{\mathfrak{es}}_{\mathfrak{sq}}$ consists of all morphisms $(X, \mathfrak{E}_X) \xrightarrow{\mathfrak{q}} (Y, \mathfrak{E}_Y)$ of right exact 'spaces' such that its inverse image functor, $\mathfrak{q}^*$, is equivalent to a localization functor and each square of $Sq(X, \mathfrak{E}_X)$ is isomorphic to some square of $\mathfrak{q}^*(Sq(Y, \mathfrak{E}_Y))$.

1.6.3.4. Proposition. *The class $\mathfrak{L}^{\mathfrak{es}}_{\mathfrak{sq}}$ is a left exact structure on the category $\mathfrak{Esp}_\mathfrak{r}$ of right exact 'spaces' which is coarser than $\mathfrak{L}_{\mathfrak{es}}$ and finer than $\mathfrak{L}^{\mathfrak{es}}_\mathfrak{r}$.*

Proof. The argument is left to the reader. ∎

1.7. Relative right exact 'spaces'. The category $\mathfrak{Esp}_\mathfrak{r}$ of right exact 'spaces' has initial objects and no final object. Final objects appear if we fix a right exact 'space' $\mathcal{S} = (S, \mathfrak{E}_\mathcal{S})$ and consider the category $\mathfrak{Esp}_\mathfrak{r}/\mathcal{S}$ instead of $\mathfrak{Esp}_\mathfrak{r}$. The category $\mathfrak{Esp}_\mathfrak{r}/\mathcal{S}$ has a natural final object and cokernels of all morphisms. It also inherits left exact structures from $\mathfrak{Esp}_\mathfrak{r}$, in particular those defined above (see 1.6.3.2). Therefore, our theory of derived functors (satellites) can be applied to functors from $\mathfrak{Esp}_\mathfrak{r}/\mathcal{S}$.

1.8. The category of right exact k-'spaces'. For a commutative unital ring k, we denote by $\mathfrak{Esp}^\mathfrak{r}_k$ the category whose objects are right exact 'spaces' $(X, \mathfrak{E}_X)$ such that C_X is a k-linear additive category and morphisms are morphisms of right exact 'spaces' whose inverse image functors are k-linear.

Each of the left exact structures $\mathfrak{L}_{\mathfrak{es}}$, $\mathfrak{L}^{\mathfrak{es}}_\ell$, $\mathfrak{L}^{\mathfrak{es}}_\mathfrak{r}$, $\mathfrak{L}^{\mathfrak{es}}_\mathfrak{r}$, $\mathfrak{L}^{\mathfrak{c}}_{\mathfrak{es}}$, and $\mathfrak{L}^{\mathfrak{e};\mathfrak{c}}_{\mathfrak{es}}$ induces a left exact structure on the category $\mathfrak{Esp}^\mathfrak{r}_k$ of right exact k-'spaces'. We denote them by respectively $\mathfrak{L}_{\mathfrak{es}}(k)$, $\mathfrak{L}^{\mathfrak{es}}_\ell(k)$, $\mathfrak{L}^{\mathfrak{es}}_\mathfrak{r}(k)$, $\mathfrak{L}^{\mathfrak{es}}_\mathfrak{r}(k)$, $\mathfrak{L}^{\mathfrak{c}}_{\mathfrak{es}}(k)$, and $\mathfrak{L}^{\mathfrak{e},\mathfrak{c}}_{\mathfrak{es}}(k)$.

2. The group K_0 of a right (or left) exact 'space'.

2.1. The group $\mathbb{Z}_0|C_X|$. For a svelte category C_X, we denote by $|C_X|$ the set of isomorphism classes of objects of C_X, by $\mathbb{Z}|C_X|$ the free abelian group generated by $|C_X|$, and by $\mathbb{Z}_0(C_X)$ the subgroup of $\mathbb{Z}|C_X|$ generated by differences $[M] - [N]$ for all arrows $M \longrightarrow N$ of the category C_X. Here $[M]$ denotes the isomorphism class of an object M.

2.2. Proposition. *(a) The maps $X \longmapsto \mathbb{Z}|C_X|$ and $X \longmapsto \mathbb{Z}_0(C_X)$ extend naturally to presheaves of $\mathbb{Z}$-modules on the category of 'spaces' $|Cat|^o$ (i.e. to functors from $(|Cat|^o)^{op}$ to $\mathbb{Z} - mod$).*

(b) If the category C_X has an initial (resp. final) object x, then $\mathbb{Z}_0(C_X)$ is the subgroup of $\mathbb{Z}|C_X|$ generated by differences $[M] - [x]$, where $[M]$ runs through the set $|C_X|$ of isomorphism classes of objects of C_X.

Proof. The argument is left to the reader. ■

2.3. Note. Evidently, $\mathbb{Z}|C_X| \simeq \mathbb{Z}|C_X^{op}|$ and $\mathbb{Z}_0(C_X) \simeq \mathbb{Z}_0(C_X^{op})$.

2.4. The group K_0 of a right exact 'space'. Let $(X, \mathfrak{E}_X)$ be a right exact 'space'. We denote by $K_0(X, \mathfrak{E}_X)$ the quotient of the group $\mathbb{Z}_0|C_X|$ by the subgroup generated by the expressions $[M'] - [L'] + [L] - [M]$ for all cartesian squares

$$\begin{array}{ccc} M' & \xrightarrow{\ \widetilde{f}\ } & M \\ {\scriptstyle \mathfrak{e}'}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{e}} \\ L' & \xrightarrow{\ f\ } & L \end{array}$$

whose vertical arrows are deflations.

We call $K_0(X, \mathfrak{E}_X)$ the *group* K_0 of the right exact 'space' $(X, \mathfrak{E}_X)$.

2.4.1. Example: the group K_0 of a 'space'. Any 'space' X is identified with the *trivial* right exact 'space' $(X, Iso(C_X))$. We set $K_0(X) = K_0(X, Iso(C_X))$. That is $K_0(X)$ coincides with the group $\mathbb{Z}_0(C_X)$.

2.4.2. Proposition. *Let $(X, \mathfrak{E}_X)$ be a right exact 'space' such that the category C_X has initial objects. Then $K_0(X, \mathfrak{E}_X)$ is isomorphic to the quotient of the group $\mathbb{Z}_0(X)$ by the subgroup generated by the expressions $[M] - [L] - [N]$ for all conflations $N \longrightarrow M \longrightarrow L$.*

2.5. Proposition. *(a) The map $(X, \mathfrak{E}_X) \longmapsto K_0(X, \mathfrak{E}_X)$ extends to a contravariant functor, K_0, from the category $\mathfrak{Esp}_{\mathfrak{r}}$ of right exact 'spaces' to the category $\mathbb{Z} - mod$.*

(b) Let $(X, \mathfrak{E}_X) \xrightarrow{f} (Y, \mathfrak{E}_Y)$ be a morphism of $\mathfrak{Esp}_{\mathfrak{r}}$ such that every object of the category C_X is isomorphic to the inverse image of an object of C_Y. Then the map $K_0(Y, \mathfrak{E}_Y) \xrightarrow{K_0(f)} K_0(X, \mathfrak{E}_X)$ is a group epimorphism.

In particular, the functor K_0 maps 'exact' localizations to epimorphisms.

3. Higher K-groups of right exact 'spaces'.

3.1. The relative functors K_0 and their derived functors. Fix a right exact 'space' $\mathcal{Y} = (Y, \mathfrak{E}_{\mathcal{Y}})$. The functor $(\mathfrak{Esp}_{\mathfrak{r}})^{op} \xrightarrow{K_0} \mathbb{Z} - mod$ induces a functor

$$(\mathfrak{Esp}_{\mathfrak{r}}/\mathcal{Y})^{op} \xrightarrow{K_0^{\mathcal{Y}}} \mathbb{Z} - mod$$

defined by

$$K_0^{\mathcal{Y}}(\mathcal{X},\xi) = K_0^{\mathcal{Y}}(\mathcal{X},\mathcal{X} \xrightarrow{\xi} \mathcal{Y}) = Cok(K_0(\mathcal{Y}) \xrightarrow{K_0(\xi)} K_0(\mathcal{X}))$$

and acting correspondingly on morphisms.

The main advantage of the functor $K_0^{\mathcal{Y}}$ is that its domain, the category $\mathfrak{Esp}_{\mathfrak{r}}/\mathcal{Y}$ has a final object, cokernels of morphisms, and natural left exact structures induced by left exact structures on $\mathfrak{Esp}_{\mathfrak{r}}$. Fix a left exact structure $\mathfrak{I}$ on $\mathfrak{Esp}_{\mathfrak{r}}$ (say, one of those defined in 6.8.3.2) and denote by $\mathfrak{I}_{\mathcal{Y}}$ the left exact structure on $\mathfrak{Esp}_{\mathfrak{r}}/\mathcal{Y}$ induced by $\mathfrak{I}$. Notice that, since the category $\mathbb{Z}-mod$ is complete (and cocomplete), there is a well defined satellite endofunctor of $\mathcal{H}om((\mathfrak{Esp}_{\mathfrak{r}}/\mathcal{Y})^{op}, \mathbb{Z}-mod)$, $F \longmapsto \mathcal{S}_{\mathfrak{I}_{\mathcal{Y}}}F$. So that for every functor F from $(\mathfrak{Esp}_{\mathfrak{r}}/\mathcal{Y})^{op}$ to $\mathbb{Z}-mod$, there is a unique up to isomorphism universal ∂^*-functor $(\mathcal{S}^i_{\mathfrak{I}_{\mathcal{Y}}}F, \mathfrak{d}_i \mid i \geq 0)$.

In particular, there is a universal contravariant ∂^*-functor $K_\bullet^{\mathcal{Y},\mathfrak{I}} = (K_i^{\mathcal{Y},\mathfrak{I}}, \mathfrak{d}_i \mid i \geq 0)$ from the right exact category $(\mathfrak{Esp}_{\mathfrak{r}}/\mathcal{Y}, \mathfrak{I}_{\mathcal{Y}})$ of right exact 'spaces' over $\mathcal{Y}$ to the category $\mathbb{Z}-mod$ of abelian groups; that is $K_i^{\mathcal{Y},\mathfrak{I}} = \mathcal{S}^i_{\mathfrak{I}_{\mathcal{Y}}}K_0^{\mathcal{Y},\mathfrak{I}}$ for all $i \geq 0$.

We call the groups $K_i^{\mathcal{Y},\mathfrak{I}}(\mathcal{X},\xi)$ *universal* K-groups of the right exact 'space' $(\mathcal{X},\xi)$ over $\mathcal{Y}$ with respect to the left exact structure $\mathfrak{I}$.

3.2. 'Exactness' properties. In general, the ∂^*-functor $K_\bullet^{\mathcal{Y},\mathfrak{I}}$ is not 'exact'. The purpose of this section is to find some natural left exact structures $\mathfrak{I}$ on the category $\mathfrak{Esp}_{\mathfrak{r}}/\mathcal{Y}$ of right exact 'spaces' over $\mathcal{Y}$ and some of its subcategories for which the ∂^*-functor $K_\bullet^{\mathcal{Y},\mathfrak{I}}$ is 'exact'.

3.2.1. Proposition. *Let $(X,\xi) \xrightarrow{q} (X',\xi')$ be a morphism of the category $\mathfrak{Esp}_{\mathfrak{r}}/\mathcal{Y}$ such that $X \xrightarrow{q} X'$ belongs to $\mathfrak{L}_{\mathfrak{es}}$ (cf. 6.8.3) and has the following property:*

(#) if $M \xrightarrow{\mathfrak{s}} L$ is a morphism of $C_{X'}$ such that $q^*(\mathfrak{s})$ is invertible, then the element $[M]-[L]$ of the group $K_0(X')$ belongs to the image of the map $K_0(X'') \xrightarrow{K_0(\mathfrak{c}_q)} K_0(X')$, where $(X',\xi') \xrightarrow{\mathfrak{c}_q} (X'',\xi'')$ is the cokernel of the morphism $(X,\xi) \xrightarrow{q} (X',\xi')$.

Suppose, in addition, that one of the following two conditions holds:

(i) the category $C_{X'}$ has an initial object;

(ii) for any pair of arrows $N \xrightarrow{f} L \xleftarrow{\mathfrak{s}} M$, of the category $C_{X'}$ such that $\mathfrak{q}^*(\mathfrak{s})$ is invertible, there exists a commutative square

$$\begin{array}{ccc} \widetilde{N} & \xrightarrow{\widetilde{f}} & M \\ {\scriptstyle \mathfrak{t}}\downarrow & & \downarrow{\scriptstyle \mathfrak{s}} \\ N & \xrightarrow{f} & L \end{array}$$

such that $\mathfrak{q}^*(\mathfrak{t})$ is invertible.

Then for every conflation $(X,\xi) \xrightarrow{\mathfrak{q}} (X',\xi') \xrightarrow{\mathfrak{c}_{\mathfrak{q}}} (X'',\xi'')$ of the left exact category $(\mathfrak{Esp}_{\mathfrak{r}}/\mathcal{Y}, \mathfrak{I}_{\mathcal{Y}})$ the sequence

$$K_0^{\mathcal{Y}}(X'',\xi'') \xrightarrow{K_0^{\mathcal{Y}}(\mathfrak{c}_{\mathfrak{q}})} K_0^{\mathcal{Y}}(X',\xi') \xrightarrow{K_0^{\mathcal{Y}}(\mathfrak{q})} K_0^{\mathcal{Y}}(X,\xi) \longrightarrow 0$$

of morphisms of abelian groups is exact.

3.2.2. Proposition. *The class $\mathfrak{L}_{\mathfrak{es}}^{\mathcal{Y}}$ of all morphisms $(X,\xi) \xrightarrow{q} (X',\xi')$ of $\mathfrak{Esp}_{\mathfrak{r}}/\mathcal{Y}$ such that $X \xrightarrow{q} X'$ belongs to $\mathfrak{L}_{\mathfrak{es}}$ and satisfies the condition (#) of 3.2.1, is a left exact structure on the category $\mathfrak{Esp}_{\mathfrak{r}}/\mathcal{Y}$.*

3.2.2.1. Proposition. *The class $\mathfrak{L}_{\mathfrak{es},\mathfrak{r}}^{\mathcal{Y}}$ of all morphisms $(X,\xi) \xrightarrow{q} (X',\xi')$ og $\mathfrak{L}_{\mathfrak{es}}^{\mathcal{Y}}$ such that the functor $C_{X'} \xrightarrow{q^*} C_X$ satisfies the condition (ii) of 3.2.1, is a left exact structure on the category $\mathfrak{Esp}_{\mathfrak{r}}/\mathcal{Y}$.*

3.2.3. Proposition. *Let $\mathcal{Y} = (Y, \mathfrak{E}_Y)$ be a right exact 'space', and let $\mathfrak{I}$ be a left exact structure on the category $\mathfrak{Esp}_{\mathfrak{r}}/\mathcal{Y}$ which is coarser than $\mathfrak{L}_{\mathfrak{es},\mathfrak{r}}^{\mathcal{Y}}$ (cf. 3.2.2). Then the universal ∂^*-functor $K_\bullet^{\mathcal{Y}} = (K_i^{\mathcal{Y}}, \mathfrak{d}_i \mid i \geq 0)$ from the left exact category $(\mathfrak{Esp}_{\mathfrak{r}}/\mathcal{Y}, \mathfrak{I}_{\mathcal{Y}})$ to the category $\mathbb{Z}-mod$ of abelian groups is 'exact'; i.e. for any conflation $(X,\xi) \xrightarrow{\mathfrak{q}} (X',\xi') \xrightarrow{\mathfrak{c}_\mathfrak{q}} (X'',\xi'')$, the associated long sequence*

$$\ldots \xrightarrow{K_1^{\mathcal{Y}}(\mathfrak{q})} K_1^{\mathcal{Y}}(X,\xi) \xrightarrow{\mathfrak{d}_0} K_0^{\mathcal{Y}}(X'',\xi'') \xrightarrow{K_0^{\mathcal{Y}}(\mathfrak{c}_\mathfrak{q})} K_0^{\mathcal{Y}}(X',\xi') \xrightarrow{K_0^{\mathcal{Y}}(\mathfrak{q})} K_0^{\mathcal{Y}}(X,\xi) \longrightarrow 0$$

is exact.

Proof. Since the left exact structure $\mathfrak{I}_{\mathcal{Y}}$ is coarser than $\mathfrak{L}_{\mathfrak{es}}^{\mathcal{Y}}$, it satisfies the condition (#) of 3.2.1. Therefore, by 3.2.1, for any conflation $(X,\xi) \xrightarrow{\mathfrak{q}} (X',\xi') \xrightarrow{\mathfrak{c}_\mathfrak{q}} (X'',\xi'')$ of the left exact category $(\mathfrak{Esp}_{\mathfrak{r}}^*/\mathcal{Y}, \mathfrak{I}_{\mathcal{Y}})$, the sequence

$$K_0^{\mathcal{Y}}(X'',\xi'') \xrightarrow{K_0^{\mathcal{Y}}(\mathfrak{c}_\mathfrak{q})} K_0^{\mathcal{Y}}(X',\xi') \xrightarrow{K_0^{\mathcal{Y}}(\mathfrak{q})} K_0^{\mathcal{Y}}(X,\xi) \longrightarrow 0$$

of $\mathbb{Z}$-modules is exact. Therefore, by [Lecture III, 3.5.4.1], the universal ∂^*-functor $K_\bullet^{\mathcal{Y}} = (K_i^{\mathcal{Y}}, \mathfrak{d}_i \mid i \geq 0)$ from $(\mathfrak{Esp}_{\mathfrak{r}}^*/\mathcal{Y}, \mathfrak{I}_{\mathcal{Y}})^{op}$ to $\mathbb{Z} - mod$ is 'exact'. ∎

The following proposition can be regarded as a machine for producing universal 'exact' K-functors.

3.2.4. Proposition. *Let $\mathcal{Y} = (Y, \mathfrak{E}_Y)$ be a right exact 'space', $(\mathcal{C}_\mathfrak{S}, \mathfrak{I}_\mathfrak{S})$ a left exact category with final objects, and $\mathfrak{F}$ a functor $C_\mathfrak{S} \longrightarrow \mathfrak{Esp}_{\mathfrak{r}}/\mathcal{Y}$ which maps conflations of $(\mathcal{C}_\mathfrak{S}, \mathfrak{I}_\mathfrak{S})$ to conflations of the left exact category $(\mathfrak{Esp}_{\mathfrak{r}}/\mathcal{Y}, \mathfrak{L}_{\mathfrak{es},\mathfrak{r}}^{\mathcal{Y}})$. Then there exists a (unique up to isomorphism) universal ∂^*-functor $K_\bullet^{\mathfrak{S},\mathfrak{F}} = (K_i^{\mathfrak{S},\mathfrak{F}}, \mathfrak{d}_i \mid i \geq 0)$ from the right exact category $(\mathcal{C}_\mathfrak{S}, \mathfrak{I}_\mathfrak{S})^{op}$ to $\mathbb{Z} - mod$ whose zero component, $K_0^{\mathfrak{S},\mathfrak{F}}$, is the composition of the functor $\mathcal{C}_\mathfrak{S}^{op} \xrightarrow{\mathfrak{F}^{op}} \mathfrak{Esp}_{\mathfrak{r}}/\mathcal{Y}^{op}$ and the functor $K_0^{\mathcal{Y}}$.*

The ∂^*-functor $K_\bullet^{\mathfrak{S},\mathfrak{F}}$ is 'exact'.

Proof. The existence of the ∂^*-functor $K_\bullet^{\mathfrak{S},\mathfrak{F}}$ follows, by [Lecture III, 3.3.2], from the completeness (– existence of limits of small diagrams) of the category $\mathbb{Z} - mod$ of abelian groups. The main thrust of the proposition is the 'exactness' of $K_\bullet^{\mathfrak{S},\mathfrak{F}}$.

By hypothesis, the functor $\mathfrak{F}$ maps conflations to conflations. Therefore, it follows from 3.2.1 that for any conflation $\mathfrak{X} \longrightarrow \mathfrak{X}' \longrightarrow \mathfrak{X}''$ of the left exact category $(\mathcal{C}_\mathfrak{S}, \mathfrak{I}_\mathfrak{S})$, the sequence of abelian groups $K_0^{\mathfrak{S},\mathfrak{F}}(\mathfrak{X}'') \longrightarrow K_0^{\mathfrak{S},\mathfrak{F}}(\mathfrak{X}') \longrightarrow K_0^{\mathfrak{S},\mathfrak{F}}(\mathfrak{X}) \longrightarrow 0$ is exact. By [Lecture III, 3.5.4.1], this implies the 'exactness' of the ∂^*-functor $K_\bullet^{\mathcal{Y}}$. ■

3.3. The 'absolute' case. Let $|Cat_*|^o$ denote the subcategory of the category $|Cat|^o$ of 'spaces' whose objects are 'spaces' represented by categories with initial objects and morphisms are those morphisms of 'spaces' whose inverse image functors map initial objects to initial objects. The category $|Cat_*|^o$ is pointed: it has a canonical zero (that is both initial and final) object, x, which is represented by the category with one (identical) morphism. Thus, the initial objects of the category $|Cat|^o$ of all 'spaces' are zero objects of the subcategory $|Cat_*|^o$.

Each morphism $X \xrightarrow{\mathfrak{f}} Y$ of the category $|Cat_*|^o$ has a cokernel, $Y \xrightarrow{\mathfrak{c}_\mathfrak{f}} \mathcal{C}(\mathfrak{f})$, where the category $C_{\mathcal{C}(\mathfrak{f})}$ representing the 'space' $\mathcal{C}(\mathfrak{f})$ is the kernel $Ker(\mathfrak{f}^*)$ of the functor $\mathfrak{f}^*$. By definition, $Ker(\mathfrak{f}^*)$ is the full subcategory of the category C_Y generated by all objects of C_Y which the functor $\mathfrak{f}^*$ maps to initial objects. The inverse image functor $\mathfrak{c}_\mathfrak{f}^*$ of the canonical morphism $\mathfrak{c}_\mathfrak{f}$ is the natural embedding $Ker(\mathfrak{f}^*) \longrightarrow C_Y$.

Let $\mathfrak{Esp}_\mathfrak{r}^*$ denote the category formed by right exact 'spaces' with initial objects and those morphisms of right exact 'spaces' whose inverse image functor is 'exact' and maps initial objects to initial objects. The category $\mathfrak{Esp}_\mathfrak{r}^*$ is pointed and the forgetfull functor

$$\mathfrak{Esp}_\mathfrak{r}^* \xrightarrow{\ \mathfrak{J}^*\ } |Cat_*|^o, \quad (X, \mathfrak{E}_X) \longmapsto X,$$

is a left adjoint to the canonical full embedding $|Cat_*|^o \xrightarrow{\mathfrak{J}_*} \mathfrak{Esp}_\mathfrak{r}^*$ which assigns to every 'space' X the right exact category $(X, Iso(C_X))$. Both functors, $\mathfrak{J}^*$ and $\mathfrak{J}_*$, map zero objects to zero objects.

Let x be a zero object of the category $\mathfrak{Esp}_\mathfrak{r}^*$. Then $\mathfrak{Esp}_\mathfrak{r}^*/x$ is naturally isomorphic to $\mathfrak{Esp}_\mathfrak{r}^*$ and the relative K_0-functor K_0^x coincides with the functor K_0.

3.3.1. The left exact structure $\mathfrak{L}_{\mathfrak{es}}^*$. We denote by $\mathfrak{L}_{\mathfrak{es}}^*$ the canonical left exact structure $\mathfrak{L}_{\mathfrak{es}}^x$; it does not depend on the choice of the zero object x. It follows from the definitions above that $\mathfrak{L}_{\mathfrak{es}}^*$ consists of all morphisms $(X, \mathfrak{E}_X) \xrightarrow{\mathfrak{q}} (Y, \mathfrak{E}_Y)$ having the following properties:

(a) $C_Y \xrightarrow{\mathfrak{q}^*} C_X$ is a localization functor (which is 'exact'), and every arrow of $\mathfrak{E}_X$ is isomorphic to an arrow of $\mathfrak{q}^*(\mathfrak{E}_Y)$.

(b) If $M \xrightarrow{s} M'$ is an arrow of C_Y such that $\mathfrak{q}^*(s)$ is an isomorphism, then $[M]-[M']$ is an element of $KerK_0(\mathfrak{q})|$.

3.3.2. Proposition. *Let $(\mathcal{C}_\mathfrak{S}, \mathfrak{I}_\mathfrak{S})$ be a left exact category, and $C_\mathfrak{S} \xrightarrow{\mathfrak{F}} \mathfrak{Esp}_\mathfrak{r}^*$ a functor which maps conflations of $(\mathcal{C}_\mathfrak{S}, \mathfrak{I}_\mathfrak{S})$ to conflations of the left exact category $(\mathfrak{Esp}_\mathfrak{r}^*, \mathfrak{L}_{\mathfrak{es}}^*)$. Then there exists a (unique up to isomorphism) universal ∂^*-functor $K_\bullet^{\mathfrak{S},\mathfrak{F}} = (K_i^{\mathfrak{S},\mathfrak{F}}, \widetilde{\mathfrak{d}}_i \mid i \geq 0)$ from $(\mathcal{C}_\mathfrak{S}, \mathfrak{I}_\mathfrak{S})^{op}$ to $\mathbb{Z}-mod$ whose zero component, $K_0^{\mathfrak{S},\mathfrak{F}}$, is the composition of the functor $\mathcal{C}_\mathfrak{S}^{op} \xrightarrow{\mathfrak{F}^{op}} (\mathfrak{Esp}_\mathfrak{r}^*)^{op}$ and the functor K_0.*

The ∂^*-functor $K_\bullet^{\mathfrak{S},\mathfrak{F}}$ is 'exact'. In particular, the ∂^*-functor $K_\bullet = (K_i, \mathfrak{d}_i| \ i \geq 0)$ from $(\mathfrak{Esp}_\mathfrak{r}^*, \mathfrak{L}_{\mathfrak{es}}^*)$ to $\mathbb{Z}-mod$ is 'exact'.

Proof. The assertion is a special case of 3.2.4. ∎

3.4. Universal K-theory of abelian categories. Let $\mathfrak{Esp}_k^{\mathfrak{a}}$ denote the category whose objects are 'spaces' X represented by k-linear abelian categories and morphisms $X \xrightarrow{f} Y$ are represented by k-linear exact functors.

There is a natural functor

$$\mathfrak{Esp}_k^{\mathfrak{a}} \xrightarrow{\mathfrak{F}} \mathfrak{Esp}_{\mathfrak{r}}^{*} \tag{1}$$

which assigns to each object X of the category $\mathfrak{Esp}_k^{\mathfrak{a}}$ the right exact (actually, exact) 'space' $(X, \mathfrak{E}_X^{st})$, where $\mathfrak{E}_X^{st}$ is the *standard* (i.e. the finest) right exact structure on the category C_X, and maps each morphism $X \xrightarrow{f} Y$ to the morphism $(X, \mathfrak{E}_X^{st}) \xrightarrow{f} (Y, \mathfrak{E}_Y^{st})$ of right exact 'spaces'. One can see that the functor $\mathfrak{F}$ maps the zero object of the category $\mathfrak{Esp}_k^{\mathfrak{a}}$ (represented by the zero category) to a zero object of the category $\mathfrak{Esp}_{\mathfrak{r}}^{*}$.

3.4.1. Proposition. *Let C_X and C_Y be k-linear abelian categories endowed with the standard exact structure. Any exact localization functor $C_Y \xrightarrow{\mathfrak{q}^*} C_X$ satisfies the conditions (a) and (b) of 3.3.1.*

Proof. In fact, each morphism $\mathfrak{q}^*(M) \xrightarrow{\widetilde{h}} \mathfrak{q}^*(N)$ is of the form $\mathfrak{q}^*(h)\mathfrak{q}^*(s)^{-1}$ for some morphisms $M' \xrightarrow{h} N$ and $M' \xrightarrow{s} M$ such that $\mathfrak{q}^*(s)$ is invertible. The morphism h is a (unique) composition $\mathfrak{j} \circ \mathfrak{e}$, where $\mathfrak{j}$ is a monomorphism and $\mathfrak{e}$ is an epimorphism. Since the functor $\mathfrak{q}^*$ is exact, $\mathfrak{q}^*(\mathfrak{j})$ is a monomorphism and $\mathfrak{q}^*(\mathfrak{e})$ is an epimorphism. Therefore, $\widetilde{h}$ is an epimorphism iff $\mathfrak{q}^*(\mathfrak{j})$ is an isomorphism. This shows that the condition (a) holds.

Let $M \xrightarrow{s} M'$ be a morphism and

$$0 \longrightarrow Ker(s) \longrightarrow M \xrightarrow{s} M' \longrightarrow Cok(s) \longrightarrow 0$$

the associated with s exact sequence. Representing s as the composition, $\mathfrak{j}\circ\mathfrak{e}$, of a monomorphism $\mathfrak{j}$ and an epimorphism $\mathfrak{e}$, we obtain two short exact sequences,

$$0 \longrightarrow Ker(s) \longrightarrow M \xrightarrow{\mathfrak{e}} N \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow N \xrightarrow{\mathfrak{j}} M' \longrightarrow Cok(s) \longrightarrow 0,$$

hence $[M] = [Ker(s)] + [N]$ and $[M'] = [N] + [Cok(s)]$, or $[M'] = [M] + [Ker(s)] - [Cok(s)]$ in $K_0(Y)$. It follows from the exactness of the functor $\mathfrak{q}^*$ that the morphism $\mathfrak{q}^*(s)$ is an isomorphism iff $Ker(s)$ and $Cok(s)$ are objects of the category $Ker(\mathfrak{q}^*)$. Therefore, in this case, it follows that $[M'] = [M]$ modulo $\mathbb{Z}|Ker(\mathfrak{q}^*)|$ in $K_0(Y)$. ∎

3.4.2. Proposition. *(a) The class $\mathfrak{L}^{\mathfrak{a}}$ of all morphisms $X \xrightarrow{q} Y$ of the category $\mathfrak{Esp}_k^{\mathfrak{a}}$ such that $C_Y \xrightarrow{q^*} C_X$ is a localization functor, is a left exact structure on $\mathfrak{Esp}_k^{\mathfrak{a}}$.*

(b) The functor $\mathfrak{Esp}_k^{\mathfrak{a}} \xrightarrow{\mathfrak{F}} \mathfrak{Esp}_{\mathfrak{r}}^{*}$ is an 'exact' functor from the left exact category $(\mathfrak{Esp}_k^{\mathfrak{a}}, \mathfrak{L}^{\mathfrak{a}})$ to the left exact category $(\mathfrak{Esp}_{\mathfrak{r}}^{*}, \mathfrak{L}_{\mathfrak{es}}^{*})$. Moreover, $\mathfrak{L}^{\mathfrak{a}} = \mathfrak{F}^{-1}(\mathfrak{L}_{\mathfrak{es}}^{*})$, that is the left exact structure $\mathfrak{L}^{\mathfrak{a}}$ is induced by the left exact structure $\mathfrak{L}_{\mathfrak{es}}^{*}$ via the functor $\mathfrak{F}$.

3.4.3. The universal Grothendieck K-functor. The composition $K_0^{\mathfrak{a}}$ of the functor

$$(\mathfrak{Esp}_k^{\mathfrak{a}})^{op} \xrightarrow{\mathfrak{F}^{op}} (\mathfrak{Esp}_{\mathfrak{r}}^{*})^{op}$$

and the functor $(\mathfrak{Esp}^*_{\mathfrak{r}})^{op} \xrightarrow{K_0^*} \mathbb{Z}-mod$ assigns to each object X of the category $\mathfrak{Esp}^{\mathfrak{a}}_k$ the abelian group $K_0^*(X, \mathfrak{E}^{st}_X)$. It follows from 2.4.2 that the group $K_0^*(X, \mathfrak{E}^{st}_X)$ coincides with the Grothendieck group of the abelian category C_X. Therefore, we call $K_0^{\mathfrak{a}}$ the *Grothendieck K_0-functor.*

3.4.4. Proposition. *There exists a universal ∂^*-functor $K^{\mathfrak{a}}_\bullet = (K^{\mathfrak{a}}_i, \mathfrak{d}^{\mathfrak{a}}_i \mid i \geq 0)$ from the right exact category $(\mathfrak{Esp}^{\mathfrak{a}}_k, \mathfrak{L}^{\mathfrak{a}})^{op}$ to the category $\mathbb{Z}-mod$ whose zero component is the Grothendieck functor K_0. The universal ∂^*-functor $K^{\mathfrak{a}}_\bullet$ is 'exact'; that is for any exact localization $X \xrightarrow{\mathfrak{q}} X'$, the canonical long sequence*

$$\dots \xrightarrow{K_1^{\mathfrak{a}}(\mathfrak{q})} K_1^{\mathfrak{a}}(X) \xrightarrow{\mathfrak{d}_0^{\mathfrak{a}}(\mathfrak{q})} K_0^{\mathfrak{a}}(X'') \xrightarrow{K_0^{\mathfrak{a}}(\mathfrak{e}_{\mathfrak{q}})} K_0^{\mathfrak{a}}(X') \xrightarrow{K_0^{\mathfrak{a}}(\mathfrak{q})} K_0^{\mathfrak{a}}(X) \longrightarrow 0 \qquad (3)$$

is exact.

Proof. By 3.4.2(b), the functor $\mathfrak{Esp}^{\mathfrak{a}}_k \xrightarrow{\mathfrak{F}} \mathfrak{Esp}^*_{\mathfrak{r}}$ is an 'exact' functor from the left exact category $(\mathfrak{Esp}^{\mathfrak{a}}_k, \mathfrak{L}^{\mathfrak{a}})$ to the left exact category $(\mathfrak{Esp}^*_{\mathfrak{r}}, \mathfrak{L}^*_{\mathfrak{es}})$ which maps the zero object of the category $\mathfrak{Esp}^{\mathfrak{a}}_k$ (– the 'space' represented by the zero category) to a zero object of the category $\mathfrak{Esp}^*_{\mathfrak{r}}$. Therefore, $\mathfrak{F}$ maps conflations to conflations.

The assertion follows now from 3.3.2.1 applied to the functor $\mathfrak{F}$. ∎

3.4.5. The universal ∂^*-functor $K^{\mathfrak{a}}_\bullet$ and the Quillen's K-theory. For a 'space' X represented by a svelte k-linear abelian category C_X, we denote by $K^{\mathfrak{Q}}_i(X)$ the i-th Quillen's K-group of the category C_X. For each $i \geq 0$, the map $X \longmapsto K^{\mathfrak{Q}}_i(X)$ extends naturally to a functor

$$(\mathfrak{Esp}^{\mathfrak{a}}_k)^{op} \xrightarrow{K_i^{\mathfrak{Q}}} \mathbb{Z}-mod$$

It follows from the Quillen's localization theorem [Q, 5.5] that for any exact localization $X \xrightarrow{\mathfrak{q}} X'$ and each $i \geq 0$, there exists a *connecting morphism* $K^{\mathfrak{Q}}_{i+1}(X) \xrightarrow{\mathfrak{d}_i^{\mathfrak{Q}}(\mathfrak{q})} K^{\mathfrak{Q}}_0(X'')$, where $C_{X''} = Ker(\mathfrak{q}^*)$, such that the sequence

$$\dots \xrightarrow{K_1^{\mathfrak{Q}}(\mathfrak{q})} K_1^{\mathfrak{Q}}(X) \xrightarrow{\mathfrak{d}_0^{\mathfrak{Q}}(\mathfrak{q})} K_0^{\mathfrak{Q}}(X'') \xrightarrow{K_0^{\mathfrak{Q}}(\mathfrak{e}_{\mathfrak{q}})} K_0^{\mathfrak{Q}}(X') \xrightarrow{K_0^{\mathfrak{Q}}(\mathfrak{q})} K_0^{\mathfrak{Q}}(X) \longrightarrow 0 \qquad (4)$$

is exact. It follows (from the proof of the Quillen's localization theorem) that the connecting morphisms $\mathfrak{d}^{\mathfrak{Q}}_i(\mathfrak{q})$, $i \geq 0$, depend functorially on the localization morphism $\mathfrak{q}$. In other words, $K^{\mathfrak{Q}}_\bullet = (K^{\mathfrak{Q}}_i, \mathfrak{d}^{\mathfrak{Q}}_i \mid i \geq 0)$ is an 'exact' ∂^*-functor from the left exact category $(\mathfrak{Esp}^{\mathfrak{a}}_k, \mathfrak{L}^{\mathfrak{a}})^{op}$ to the category $\mathbb{Z}-mod$ of abelian groups.

Naturally, we call the ∂^*-functor $K^{\mathfrak{Q}}_\bullet$ the *Quillen's K-functor.*

Since $K^{\mathfrak{a}}_\bullet = (K^{\mathfrak{a}}_i, \mathfrak{d}^{\mathfrak{a}}_i \mid i \geq 0)$ is a universal ∂^*-functor from $(\mathfrak{Esp}^{\mathfrak{a}}_k, \mathfrak{L}^{\mathfrak{a}})^{op}$ to $\mathbb{Z}-mod$, the identical isomorphism $K^{\mathfrak{Q}}_0 \longrightarrow K^{\mathfrak{a}}_0$ extends uniquely to a ∂^*-functor morphism

$$K^{\mathfrak{Q}}_\bullet \xrightarrow{\varphi^{\mathfrak{Q}}_\bullet} K^{\mathfrak{a}}_\bullet. \qquad (5)$$

4. The universal K-theory of exact categories. Let $\mathfrak{Esp}^{\mathfrak{e}}_k$ denote the subcategory of the category $\mathfrak{Esp}^*_{\mathfrak{r}}$ whose objects are 'spaces' represented by svelte exact k-linear categories and inverse image of morphisms are k-linear functors.

There is a natural functor

$$\mathfrak{Esp}_k^{\mathfrak{e}} \xrightarrow{\mathfrak{F}_{\mathfrak{r}}} \mathfrak{Esp}_{\mathfrak{r}}^* \tag{1}$$

which maps objects and morphisms of the category $\mathfrak{Esp}_k^{\mathfrak{e}}$ to the corresponding objects and morphisms of the category $\mathfrak{Esp}_{\mathfrak{r}}^*$.

4.1. Proposition. *The functor $\mathfrak{Esp}_k^{\mathfrak{e}} \xrightarrow{\mathfrak{F}_{\mathfrak{r}}} \mathfrak{Esp}_{\mathfrak{r}}^*$ preserves cocartesian squares and maps the zero object of the category $\mathfrak{Esp}_k^{\mathfrak{e}}$ to the zero object of the category $\mathfrak{Esp}_{\mathfrak{r}}^*$.*

Proof. The argument is similar to that of 7.5.2(b). Details are left to the reader. ■

4.2. Corollary. *The class of morphisms $\mathfrak{L}_k^{\mathfrak{e}} = \mathfrak{F}_{\mathfrak{r}}^{-1}(\mathfrak{L}_{\mathfrak{es}}^*)$ is a left exact structure on the category $\mathfrak{Esp}_k^{\mathfrak{e}}$ and $\mathfrak{F}_r$ is an 'exact' functor from the left exact category $(\mathfrak{Esp}_k^{\mathfrak{e}}, \mathfrak{L}_k^{\mathfrak{e}})$ to the left exact category $(\mathfrak{Esp}_{\mathfrak{r}}^*, \mathfrak{L}_{\mathfrak{es}}^*)$.*

The composition $K_0^{\mathfrak{e}}$ of the inclusion functor

$$(\mathfrak{Esp}_k^{\mathfrak{a}})^{op} \xrightarrow{\mathfrak{F}^{op}} (\mathfrak{Esp}_{\mathfrak{r}}^*)^{op}$$

and the functor $(\mathfrak{Esp}_{\mathfrak{r}}^*)^{op} \xrightarrow{K_0^*} \mathbb{Z} - mod$ assigns to each object X of the category $\mathfrak{Esp}_k^{\mathfrak{a}}$ the abelian group $K_0^*(X, \mathfrak{E}_X^{st})$ which coincides with the Quillen's group K_0 of the exact category $(C_X, \mathfrak{E}_X)$.

4.3. Proposition. *There exists a universal ∂^*-functor $K_\bullet^{\mathfrak{e}} = (K_i^{\mathfrak{e}}, \mathfrak{d}_i^{\mathfrak{e}} \mid i \geq 0)$ from the right exact category $(\mathfrak{Esp}_k^{\mathfrak{e}}, \mathfrak{L}^{\mathfrak{e}})^{op}$ to the category $\mathbb{Z} - mod$ whose zero component is the functor $K_0^{\mathfrak{e}}$. The universal ∂^*-functor $K_\bullet^{\mathfrak{e}}$ is 'exact'; that is for any exact localization $(X, \mathfrak{E}_X) \xrightarrow{\mathfrak{q}} (X', \mathfrak{E}_{X'})$ which belongs to $\mathfrak{L}^{\mathfrak{e}}$, the canonical long sequence*

$$\begin{array}{ccccccc}
K_1^{\mathfrak{e}}(X.\mathfrak{E}_X) & \xleftarrow{K_1^{\mathfrak{e}}(\mathfrak{q})} & K_1^{\mathfrak{e}}(X', \mathfrak{E}_{X'}) & \xleftarrow{K_1^{\mathfrak{e}}(\mathfrak{c}_{\mathfrak{q}})} & K_0^{\mathfrak{e}}(X'', \mathfrak{E}_{X''}) & \xleftarrow{\mathfrak{d}_1^{\mathfrak{e}}(\mathfrak{q})} & \dots \\
\mathfrak{d}_0^{\mathfrak{e}}(\mathfrak{q}) \downarrow & & & & & & \\
K_0^{\mathfrak{e}}(X'', \mathfrak{E}_{X''}) & \xrightarrow{K_0^{\mathfrak{e}}(\mathfrak{c}_{\mathfrak{q}})} & K_0^{\mathfrak{e}}(X', \mathfrak{E}_{X'}) & \xrightarrow{K_0^{\mathfrak{e}}(\mathfrak{q})} & K_0^{\mathfrak{e}}(X, \mathfrak{E}_X) & \longrightarrow & 0
\end{array} \tag{4}$$

is exact.

Proof. The functor $\mathfrak{Esp}_k^{\mathfrak{e}} \xrightarrow{\mathfrak{F}} \mathfrak{Esp}_{\mathfrak{r}}^*$ is an 'exact' functor from the left exact category $(\mathfrak{Esp}_k^{\mathfrak{e}}, \mathfrak{L}^{\mathfrak{e}})$ to the left exact category $(\mathfrak{Esp}_{\mathfrak{r}}^*, \mathfrak{L}_{\mathfrak{es}}^*)$ which maps the zero object of the category $\mathfrak{Esp}_k^{\mathfrak{e}}$ (– the 'space' represented by the zero category) to a zero object of the category $\mathfrak{Esp}_{\mathfrak{r}}^*$. Therefore, $\mathfrak{F}$ maps conflations to conflations. It remains to apply 3.3.2. ■

Lecture 6. Comparison theorems for higher K-theory: reduction by resolution, additivity, devissage. Towards some applications.

In the first four sections, we fix a left exact subcategory of the left exact category $(\mathfrak{Esp}^*_{\mathfrak{r}}, \mathfrak{L}^*_{\mathfrak{es}})$ of right exact 'spaces', or a left exact subcategory of the left exact category $\mathfrak{Esp}^{\mathfrak{r}}_k$ of k-linear right exact 'spaces', endowed with the induced left exact structure. The higher K-functors are computed as satellites of the restriction of the functor K_0 to this left exact subcategory.

1. Reduction by resolution.

1.1. Proposition. *Let $(C_X, \mathfrak{E}_X)$ be a right exact category with initial objects and C_Y its fully exact subcategory such that*

(a) If $M' \longrightarrow M \longrightarrow M''$ is a conflation with $M \in ObC_Y$, then $M' \in ObC_Y$.

(b) For any $M'' \in ObC_X$, there exists a deflation $M \longrightarrow M''$ with $M \in ObC_Y$.

Then the morphism $K_\bullet(Y, \mathfrak{E}_Y) \longrightarrow K_\bullet(X, \mathfrak{E}_X)$ is an isomorphism.

Proof. The first part of the argument of 1.1 shows that if C_Y is a fully exact subcategory of a right exact category $(C_X, \mathfrak{E}_X)$ satisfying the condition (b) and F_0 is a functor from $\mathfrak{Esp}^{op}_{\mathfrak{r}}$ to a category with filtered limits such that $F_0(Y, \mathfrak{E}_Y) \longrightarrow F_0(X, \mathfrak{E}_X)$ is an isomorphism, then $S^n_- F_0(Y, \mathfrak{E}_Y) \longrightarrow S^n_- F_0(X, \mathfrak{E}_X)$ is an isomorphism for all $n \geq 0$.

The condition (a) is used only in the proof that $K_0(Y, \mathfrak{E}_Y) \longrightarrow K_0(X, \mathfrak{E}_X)$ is an isomorphism. ∎

1.2. Proposition. *Let $(C_X, \mathfrak{E}_X)$ and $(C_{\mathfrak{Z}}, \mathfrak{E}_{\mathfrak{Z}})$ be right exact categories with initial objects and $T = (T_i, \mathfrak{d}_i \mid i \geq 0)$ an 'exact' ∂^*-functor from $(C_X, \mathfrak{E}_X)$ to $(C_{\mathfrak{Z}}, \mathfrak{E}_{\mathfrak{Z}})$. Let C_Y be the full subcategory of C_X generated by T-acyclic objects (that is objects V such that $T_i(V)$ is an initial object of $C_{\mathfrak{Z}}$ for $i \geq 1$). Assume that for every $M \in ObC_X$, there is a deflation $P \longrightarrow M$ with $P \in ObC_Y$, and that $T_n(M)$ is an initial object of $C_{\mathfrak{Z}}$ for n sufficiently large. Then the natural map $K_\bullet(Y, \mathfrak{E}_Y) \longrightarrow K_\bullet(X, \mathfrak{E}_X)$ is an isomorphism.*

Proof. The assertion is deduced from 1.1 in the usual way (see [Q]). ∎

1.3. Proposition. *Let $(C_X, \mathfrak{E}_X)$ be a right exact category with initial objects; and let*

$$\begin{array}{ccccc}
Ker(f') & \xrightarrow{\beta_1'} & Ker(f) & \xrightarrow{\alpha_1'} & Ker(f'') \\
{\scriptstyle \mathfrak{k}'}\downarrow & & {\scriptstyle \mathfrak{k}}\downarrow & & \downarrow{\scriptstyle \mathfrak{k}''} \\
Ker(\alpha_1) & \xrightarrow{\beta_1} & A_1 & \xrightarrow{\alpha_1} & A_1'' \\
{\scriptstyle f'}\downarrow & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f''} \\
Ker(\alpha_2) & \xrightarrow{\beta_2} & A_2 & \xrightarrow{\alpha_2} & A_2''
\end{array} \qquad (3)$$

be a commutative diagram (determined by its lower right square) such that $Ker(\mathfrak{k}'')$ and $Ker(\beta_2)$ are trivial. Then

(a) The upper row of (3) is 'exact', and the morphism β_1' is the kernel of α_1'.

(b) Suppose, in addition, that the arrows f', α_1 and α_2 in (3) are deflations and $(C_X, \mathfrak{E}_X)$ has the following property:

(#) If $M \xrightarrow{\mathfrak{e}} N$ is a deflation and $M \xrightarrow{p} M$ an idempotent morphism (i.e. $p^2 = p$) which has a kernel and such that the composition $\mathfrak{e} \circ p$ is a trivial morphism, then the composition of the canonical morphism $Ker(p) \xrightarrow{\mathfrak{k}(p)} M$ and $M \xrightarrow{\mathfrak{e}} N$ is a deflation.

Then the upper row of (3) is a conflation.

1.4. Proposition. *Let $(C_X, \mathfrak{E}_X)$ be a right exact category with initial objects having the property (#) of 1.3. Let C_Y be a fully exact subcategory of a right exact category $(C_X, \mathfrak{E}_X)$ which has the following properties:*

(a) If $N \longrightarrow M \longrightarrow L$ is a conflation in $(C_X, \mathfrak{E}_X)$ and N, M are objects of C_Y, then L belongs to C_Y too.

(b) For any deflation $M \longrightarrow \mathcal{L}$ with $\mathcal{L} \in ObC_Y$, there exist a deflation $\mathcal{M} \longrightarrow \mathcal{L}$ with $\mathcal{M} \in ObC_Y$ and a morphism $\mathcal{M} \longrightarrow M$ such that the diagram

$$\begin{array}{ccc} & \mathcal{M} & \\ \swarrow & & \searrow \\ M & \longrightarrow & \mathcal{L} \end{array}$$

commutes.

(c) If P, $\mathcal{M}$ are objects of C_Y and $P \longrightarrow x$ is a morphism to initial object, then $P \coprod \mathcal{M}$ exists (in C_X) and the sequence $P \longrightarrow P \coprod \mathcal{M} \longrightarrow \mathcal{M}$ (where the left arrow is the canonical coprojection and the right arrow corresponds to the $\mathcal{M} \xrightarrow{id} \mathcal{M}$ and the composition of $P \longrightarrow x \longrightarrow \mathcal{M}$) is a conflation.

Let C_{Y_n} be the full subcategory of C_X generated by all objects L having a C_Y-resolution of the length $\leq n$, and $C_{Y_\infty} = \bigcup_{n \geq 0} C_{Y_n}$. Then C_{Y_n} is a fully exact subcategory of $(C_X, \mathfrak{E}_X)$ for all $n \leq \infty$ and the natural morphisms

$$K_\bullet(Y, \mathfrak{E}_Y) \xrightarrow{\sim} K_\bullet(Y_1, \mathfrak{E}_{Y_1}) \xrightarrow{\sim} \ldots \xrightarrow{\sim} K_\bullet(Y_n, \mathfrak{E}_{Y_n}) \xrightarrow{\sim} K_\bullet(Y_\infty, \mathfrak{E}_{Y_\infty})$$

are isomorphisms for all $n \geq 0$.

1.5. Proposition. *Let $(C_X, \mathfrak{E}_X)$ be a right exact category with initial objects having the property (#) of 1.3. Let C_Y be a fully exact subcategory of a right exact category $(C_X, \mathfrak{E}_X)$ satisfying the conditions (a) and (c) of 1.4. Let $M' \longrightarrow M \longrightarrow M''$ be a conflation in $(C_X, \mathfrak{E}_X)$, and let $\mathcal{P}' \longrightarrow M'$, $\mathcal{P}'' \longrightarrow M''$ be C_Y-resolutions of the length $n \geq 1$. Suppose that resolution $\mathcal{P}'' \longrightarrow M'$ is projective. Then there exists a C_Y-resolution $\mathcal{P} \longrightarrow M$ of the length n such that $\mathcal{P}_i = \mathcal{P}'_i \coprod \mathcal{P}''_i$ for all $i \geq 1$ and the splitting 'exact' sequence $\mathcal{P}' \longrightarrow \mathcal{P} \longrightarrow \mathcal{P}''$ is an 'exact' sequence of complexes.*

2. Additivity of 'characteristic' filtrations.

2.1. Characteristic 'exact' filtrations and sequences.

2.1.1. The right exact 'spaces' $(X_n, \mathfrak{E}_{X_n})$. For a right exact exact 'space' $(X, \mathfrak{E}_X)$, let C_{X_n} be the category whose objects are sequences $M_n \longrightarrow M_{n-1} \longrightarrow \ldots \longrightarrow M_0$ of n morphisms of $\mathfrak{E}_X$, $n \geq 1$, and morphisms between sequences are commutative diagrams

$$\begin{array}{ccccccc} M_n & \longrightarrow & M_{n-1} & \longrightarrow & \ldots & \longrightarrow & M_0 \\ {\scriptstyle f_n}\downarrow & & {\scriptstyle f_{n-1}}\downarrow & & \ldots & & \downarrow{\scriptstyle f_0} \\ M'_n & \longrightarrow & M'_{n-1} & \longrightarrow & \ldots & \longrightarrow & M'_0 \end{array}$$

Notice that if x is an initial object of the category C_X, then $x \longrightarrow \ldots \longrightarrow x$ is an initial object of C_{X_n}.

We denote by $\mathfrak{E}_{X_n}$ the class of all morphisms (f_i) of the category C_{X_n} such that $f_i \in \mathfrak{E}_X$ for all $0 \le i \le n$.

2.1.1.1. Proposition. *(a) The pair $(C_{X_n}, \mathfrak{E}_{X_n})$ is a right exact category.*

(b) The map which assigns to each right exact 'space' $(X, \mathfrak{E}_X)$ the right exact 'space' $(X_n, \mathfrak{E}_{X_n})$ extends naturally to an 'exact' endofunctor of the left exact category $(\mathfrak{Esp}_r, \mathfrak{L}_{\mathfrak{es}})$ of right 'exact' 'spaces' which induces an 'exact' endofunctor $\mathcal{P}_n$ of its exact subcategory $(\mathfrak{Esp}^*_r, \mathfrak{L}^*_{\mathfrak{es}})$.

Proof. The argument is left to the reader. ■

2.1.2. Proposition. (Additivity of 'characteristic' filtrations) *Let $(C_X, \mathfrak{E}_X)$ and $(C_Y, \mathfrak{E}_Y)$ be right exact categories with initial objects and $f^*_n \xrightarrow{\mathfrak{t}_n} f^*_{n-1} \xrightarrow{\mathfrak{t}_{n-1}} \ldots \xrightarrow{\mathfrak{t}_1} f^*_0$ a sequence of deflations of 'exact' functors from $(C_X, \mathfrak{E}_X)$ to $(C_Y, \mathfrak{E}_Y)$ such that the functors $\mathfrak{k}^*_i = Ker(\mathfrak{t}^*_i)$ are 'exact' for all $1 \le i \le n$. Then $K_\bullet(f_n) = K_\bullet(f_0) + \sum_{1 \le i \le n} K_\bullet(\mathfrak{k}_i)$.*

Proof. The argument uses facts on kernels (see Appendix A to Lecture 4). ■

2.1.3. Corollary. *Let $(C_X, \mathfrak{E}_X)$ and $(C_Y, \mathfrak{E}_Y)$ be right exact categories with initial objects and $g^* \longrightarrow f^* \longrightarrow h^*$ a conflation of 'exact' functors from $(C_X, \mathfrak{E}_X)$ to $(C_Y, \mathfrak{E}_Y)$. Then $K_\bullet(f) = K_\bullet(g) + K_\bullet(h)$.*

2.1.4. Corollary. (Additivity for 'characteristic' 'exact' sequences) *Let*

$$\mathfrak{f}^*_n \longrightarrow \mathfrak{f}^*_{n-1} \longrightarrow \ldots \longrightarrow \mathfrak{f}^*_1 \longrightarrow \mathfrak{f}^*_0$$

be an 'exact' sequence of 'exact' functors from $(C_X, \mathfrak{E}_X)$ to $(C_Y, \mathfrak{E}_Y)$ which map initial objects to initial objects. Suppose that $\mathfrak{f}^*_1 \longrightarrow \mathfrak{f}^*_0$ is a deflation and $\mathfrak{f}^*_n \longrightarrow \mathfrak{f}^*_{n-1}$ is the kernel of $\mathfrak{f}^*_{n-1} \longrightarrow \mathfrak{f}^*_{n-2}$. Then the morphism $\sum_{0 \le i \le n} (-1)^i K_\bullet(\mathfrak{f}_i)$ from $K_\bullet(X, \mathfrak{E}_X)$ to $K_\bullet(Y, \mathfrak{E}_Y)$ is equal to zero.

Proof. The assertion follows from 2.1.3 by induction.

A more conceptual proof goes along the lines of the argument of 2.1.2. Namely, we assign to each right exact category $(C_Y, \mathfrak{E}_Y)$ the right exact category $(C_{Y^{\mathfrak{c}}_n}, \mathfrak{E}_{Y^{\mathfrak{c}}_n})$ whose objects are 'exact' sequences $\mathcal{L} = (L_n \longrightarrow L_{n-1} \longrightarrow \ldots \longrightarrow L_1 \longrightarrow L_0)$, where $L_1 \longrightarrow L_0$ is a deflation and $L_n \longrightarrow L_{n-1}$ is the kernel of $L_{n-1} \longrightarrow L_{n-2}$. This assignment defines an endofunctor $\mathfrak{P}^{\mathfrak{c}}_n$ of the category $\mathfrak{Esp}^*_{\mathfrak{r}}$ of right exact 'spaces' with initial objects, and maps $\mathcal{L} \longmapsto L_i$ determine morphisms $\mathfrak{P}^{\mathfrak{c}}_n \longrightarrow Id_{\mathfrak{Esp}^*_{\mathfrak{r}}}$. The rest of the argument is left to the reader. ■

3. Infinitesimal 'spaces'. Devissage.

3.1. The Gabriel multiplication in right exact categories. Fix a right exact category $(C_X, \mathfrak{E}_X)$ with initial objects. Let $\mathbb{T}$ and $\mathbb{S}$ be subcategories of the category C_X.

The *Gabriel product* $\mathbb{S} \bullet \mathbb{T}$ is the full subcategory of C_X whose objects M fit into a conflation $L \xrightarrow{g} M \xrightarrow{h} N$ such that $L \in Ob\mathbb{S}$ and $N \in Ob\mathbb{T}$.

3.1.1. Proposition. *Let $(C_X, \mathcal{E}_X)$ be a right exact category with initial objects. For any subcategories $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{D}$ of the category C_X, there is the inclusion*

$$\mathcal{A} \bullet (\mathcal{B} \bullet \mathcal{D}) \subseteq (\mathcal{A} \bullet \mathcal{B}) \bullet \mathcal{D}.$$

Proof. An exercise on kernels and cartesian squares. ■

3.1.2. Corollary. *Let $(C_X, \mathfrak{E}_X)$ be an exact category. Then the Gabriel multiplication is associative.*

Proof. Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{D}$ be subcategories of C_X. By 3.1.1, we have the inclusion $\mathcal{A} \bullet (\mathcal{B} \bullet \mathcal{D}) \subseteq (\mathcal{A} \bullet \mathcal{B}) \bullet \mathcal{D}$. The opposite inclusion holds by duality, because $(\mathcal{A} \bullet \mathcal{B})^{op} = \mathcal{B}^{op} \bullet \mathcal{A}^{op}$. ■

3.2. The infinitesimal neighborhoods of a subcategory. Let $(C_X, \mathfrak{E}_X)$ be a right exact category with initial objects. We denote by $\mathbb{O}_X$ the full subcategory of C_X generated by all initial objects of C_X. For any subcategory $\mathcal{B}$ of C_X, we define subcategories $\mathcal{B}^{(n)}$ and $\mathcal{B}_{(n)}$, $0 \le n \le \infty$, by setting $\mathcal{B}^{(0)} = \mathbb{O}_X = \mathcal{B}_{(0)}$, $\mathcal{B}^{(1)} = \mathcal{B} = \mathcal{B}_{(1)}$, and

$$\mathcal{B}^{(n)} = \mathcal{B}^{(n-1)} \bullet \mathcal{B} \quad \text{for} \quad 2 \le n < \infty; \quad \text{and} \quad \mathcal{B}^{(\infty)} = \bigcup_{n \ge 1} \mathcal{B}^{(n)};$$

$$\mathcal{B}_{(n)} = \mathcal{B} \bullet \mathcal{B}_{(n-1)} \quad \text{for} \quad 2 \le n < \infty; \quad \text{and} \quad \mathcal{B}_{(\infty)} = \bigcup_{n \ge 1} \mathcal{B}_{(n)}$$

It follows that $\mathcal{B}^{(n)} = \mathcal{B}_{(n)}$ for $n \le 2$ and, by 3.1.1, $\mathcal{B}_{(n)} \subseteq \mathcal{B}^{(n)}$ for $3 \le n \le \infty$.

We call the subcategory $\mathcal{B}^{(n+1)}$ the *upper n^{th} infinitesimal neighborhood* of $\mathcal{B}$ and the subcategory $\mathcal{B}_{(n+1)}$ the *lower n^{th} infinitesimal neighborhood* of $\mathcal{B}$. It follows that $\mathcal{B}^{(n+1)}$ is the strictly full subcategory of C_X generated by all $M \in ObC_X$ such that there exists a sequence of arrows

$$M_0 \xrightarrow{\mathfrak{j}_1} M_1 \xrightarrow{\mathfrak{j}_2} \dots \xrightarrow{\mathfrak{j}_n} M_n = M$$

with the property: $M_0 \in Ob\mathcal{B}$, and for each $n \ge i \ge 1$, there exists a deflation $M_i \xrightarrow{\mathfrak{e}_i} N_i$ with $N_i \in Ob\mathcal{B}$ such that $M_{i-1} \xrightarrow{\mathfrak{j}_i} M_i \xrightarrow{\mathfrak{e}_i} N_i$ is a conflation.

Similarly, $\mathcal{B}_{(n+1)}$ is a strictly full subcategory of C_X generated by all $M \in ObC_X$ such that there exists a sequence of deflations

$$M = M_n \xrightarrow{\mathfrak{e}_n} \dots \xrightarrow{\mathfrak{e}_2} M_1 \xrightarrow{\mathfrak{e}_1} M_0$$

such that M_0 and $Ker(\mathfrak{e}_i)$ are objects of $\mathcal{B}$ for $1 \le i \le n$.

3.2.1. Note. It follows that $\mathcal{B}^{(n)} \subseteq \mathcal{B}^{(n+1)}$ for all $n \ge 0$, if $\mathcal{B}$ contains an initial object of the category C_X.

3.3. Fully exact subcategories of a right exact category. Fix a right exact category $(C_X, \mathcal{E}_X)$. A subcategory $\mathcal{A}$ of C_X is a *fully exact* subcategory of $(C_X, \mathcal{E}_X)$ if $\mathcal{A} \bullet \mathcal{A} = \mathcal{A}$.

3.3.1. Proposition. *Let $(C_X, \mathcal{E}_X)$ be a right exact category with initial objects. For any subcategory $\mathcal{B}$ of C_X, the subcategory $\mathcal{B}^{(\infty)}$ is the smallest fully exact subcategory of $(C_X, \mathcal{E}_X)$ containing $\mathcal{B}$.*

Proof. Let $\mathcal{A}$ be a fully exact subcategory of the right exact category $(C_X, \mathfrak{E}_X)$, i.e. $\mathcal{A} = \mathcal{A} \bullet \mathcal{A}$. Then $\mathcal{B}^{(\infty)} \subseteq \mathcal{A}$, iff $\mathcal{B}$ is a subcategory of $\mathcal{A}$.

On the other hand, it follows from 3.1.1 and the definition of the subcategories $\mathcal{B}^{(n)}$ (see 3.2) that $\mathcal{B}^{(n)} \bullet \mathcal{B}^{(m)} \subseteq \mathcal{B}^{(m+n)}$ for any nonnegative integers n and m. In particular, $\mathcal{B}^{(\infty)} = \mathcal{B}^{(\infty)} \bullet \mathcal{B}^{(\infty)}$, that is $\mathcal{B}^{(\infty)}$ is a fully exact subcategory of $(C_X, \mathcal{E}_X)$ containing $\mathcal{B}$. ■

3.4. Cofiltrations. Fix a right exact category $(C_X, \mathfrak{E}_X)$ with initial objects. A *cofiltration of the length* n+1 of an object M is a sequence of deflations

$$M = M_n \xrightarrow{\mathfrak{e}_n} \dots \xrightarrow{\mathfrak{e}_2} M_1 \xrightarrow{\mathfrak{e}_1} M_0. \qquad (1)$$

The cofiltration (1) is said to be *equivalent* to a cofiltration

$$M = \widetilde{M}_m \xrightarrow{\widetilde{\mathfrak{e}}_n} \dots \xrightarrow{\widetilde{\mathfrak{e}}_2} \widetilde{M}_1 \xrightarrow{\widetilde{\mathfrak{e}}_1} \widetilde{M}_0$$

if $m = n$ and there exists a permutation σ of $\{0, \dots, n\}$ such that $Ker(\mathfrak{e}_i) \simeq Ker(\widetilde{\mathfrak{e}}_{\sigma(i)})$ for $1 \leq i \leq n$ and $M_0 \simeq \widetilde{M}_0$.

The following assertion is a version (and a generalization) of Zassenhouse's lemma.

3.4.1. Proposition. *Let $(C_X, \mathfrak{E}_X)$ have the following property:*

($\ddagger$) *for any pair of deflations $M_1 \xleftarrow{\mathfrak{t}_1} M \xrightarrow{\mathfrak{t}_2} M_2$, there is a commutative square*

$$\begin{array}{ccc} M & \xrightarrow{\mathfrak{t}_1} & M_1 \\ {\scriptstyle \mathfrak{t}_2}\downarrow & & \downarrow{\scriptstyle \mathfrak{p}_2} \\ M_2 & \xrightarrow{\mathfrak{p}_1} & M_3 \end{array}$$

of deflations such that the unique morphism $M \longrightarrow M_1 \times_{M_3} M_2$ is a deflation.

Then any two cofiltrations of an object M have equivalent refinements.

3.5. Devissage.

3.5.1. Proposition. (Devissage for K_0.) *Let $((X, \mathfrak{E}_X), Y)$ be an infinitesimal 'space' such that $(X, \mathfrak{E}_X)$ has the following property (which appeared in 3.4.1):*

($\ddagger$) *for any pair of deflations $M_1 \xleftarrow{\mathfrak{t}_1} M \xrightarrow{\mathfrak{t}_2} M_2$, there is a commutative square*

$$\begin{array}{ccc} M & \xrightarrow{\mathfrak{t}_1} & M_1 \\ {\scriptstyle \mathfrak{t}_2}\downarrow & & \downarrow{\scriptstyle \mathfrak{p}_2} \\ M_2 & \xrightarrow{\mathfrak{p}_1} & M_3 \end{array}$$

of deflations such that the unique morphism $M \longrightarrow M_1 \times_{M_3} M_2$ is a deflation.

Then the natural morphism

$$K_0(Y, \mathfrak{E}_Y) \longrightarrow K_0(X, \mathfrak{E}_X) \tag{1}$$

is an isomorphism.

3.5.2. The ∂^*-functor $K_\bullet^{\mathfrak{sq}}$. Let $\mathfrak{L}^{\mathfrak{es}}_{\rtimes}$ denote the left exact structure on the category $\mathfrak{Esp}^{\rtimes}$ of $\mathfrak{Esp}_\mathfrak{r}$ (cf. 3.9.4) induced by the (defined in 6.8.3.3) left exact structure $\mathfrak{L}^{\mathfrak{es}}_{\mathfrak{sq}}$ on the category $\mathfrak{Esp}_\mathfrak{r}$ of right exact 'spaces'. Let $K_i^{\mathfrak{sq}}(X, \mathfrak{E}_X)$ denote the i-th satellite of the functor K_0 with respect to the left exact structure $\mathfrak{L}^{\mathfrak{es}}_{\rtimes}$.

3.5.3. Proposition. *Let $((X, \mathfrak{E}_X), Y)$ be an infinitesimal 'space' such that the right exact 'space' $(X, \mathfrak{E}_X)$ has the property (‡) of 3.4.1, the category C_X has final objects, and all morphisms to final objects are deflations. Then the natural morphism*

$$K_i^{\mathfrak{sq}}(Y, \mathfrak{E}_Y) \longrightarrow K_i^{\mathfrak{sq}}(X, \mathfrak{E}_X) \tag{8}$$

is an isomorphism for all $i \geq 0$.

Proof. The assertion follows from a general devissage theorem for universal ∂^*-functors whose zero component satisfy devissage property (like K_0, by 3.5.1). ■

4. An application: K-groups of 'spaces' with Gabriel-Krull dimension.

4.1. Gabriel-Krull filtration. We recall the notion of the *Gabriel filtration* of an abelian category as it is presented in [R, 6.6]. Let C_X be an abelian category. The *Gabriel filtration of* X assigns to every cardinal α a Serre subcategory C_{X_α} of C_X which is constructed as folows:

Set $C_{X_0} = \mathbb{O}$.

If α is not a limit cardinal, then C_{X_α} is the smallest Serre subcategory of C_X containing all objects M such that the localization $q^*_{\alpha-1}(M)$ of M at $C_{X_{\alpha-1}}$ has a finite length.

If β is a limit cardinal, then C_{X_β} is the smallest Serre subcategory containing all subcategories C_{X_α} for $\alpha < \beta$.

Let C_{X_ω} denote the smallest Serre subcategory containing all the subcategories C_{X_α}. Clearly the quotient category C_X/C_{X_ω} has no simple objects.

An object M is said to have the *Gabriel-Krull dimension* β, if β is the smallest cardinal such that M belongs to C_{X_β}.

The 'space' X has a Gabriel-Krull dimension if $X = X_\omega$.

Every locally noetherian abelian category (e.g. the category of quasi-coherent sheaves on a noetherian scheme, or the category of left modules over a left noetherian associative algebra) has a Gabriel-Krull dimension.

It follows that for any limit ordinal β, we have $K_\bullet(X_\beta) = \bigcup_{\alpha<\beta} K_\bullet(X_\alpha)$. Therefore, $K_\bullet(X_\omega) = \bigcup_{\alpha \in \mathfrak{Or}_\mathfrak{n}} K_\bullet(X_\alpha)$, where $\mathfrak{Or}_\mathfrak{n}$ denotes the set of non-limit ordinals.

4.2. Reduction via localization. If α is a non-limit ordinal, we have the exact localization $C_{X_\alpha} \xrightarrow{q^*_{\alpha-1}} C_{X_\alpha}/C_{X_{\alpha-1}} = C_{X^q_\alpha}$, hence the corresponding long exact sequence

$$\ldots \longrightarrow K_{n+1}(X^q_\alpha) \xrightarrow{\mathfrak{d}^\alpha_n} K_n(X_{\alpha-1}) \longrightarrow K_n(X_\alpha) \longrightarrow K_n(X^q_\alpha) \longrightarrow \ \ldots \ \longrightarrow K_0(X^q_\alpha) \quad (1)$$

of K-groups.

4.3. Reduction by devissage. Suppose that the category C_X is *noetherian*, i.e. all objects of C_X are noetherian. Then the quotient category $C_{X^q_\alpha} = C_{X_\alpha}/C_{X_{\alpha-1}}$ is noetherian. Notice that the Krull dimension of X^q_α equals to zero; hence all objects of the category $C_{X^q_\alpha}$ have a finite length. Let $C_{X^q_{\alpha,\mathfrak{s}}}$ denote the full subcategory of $C_{X^q_\alpha}$ generated by semisimple objects. By devissage, the natural morphism $K_\bullet(X^q_{\alpha,\mathfrak{s}}) \longrightarrow K_\bullet(X^q_\alpha)$ is an isomorphism. If C_Y is a svelte abelian category whose objects are semisimple of finite length, then $K_\bullet(Y) = \coprod_{\mathcal{Q}\in\mathbf{Spec}(Y)} K_\bullet(\mathbf{Sp}(D_\mathcal{Q}))$, where $D_\mathcal{Q}$ is the *residue* skew field of the point $\mathcal{Q}$ of the spectrum of Y, which is the skew field $C_Y(M,M)^o$ of the endomorphisms of the simple object M such that $\mathcal{Q} = [M]$. In particular,

$$K_\bullet(X^q_\alpha) = \coprod_{\mathcal{Q}\in\mathbf{Spec}(X^q_\alpha)} K_\bullet(\mathbf{Sp}(D_\mathcal{Q}))$$

for every non-limit ordinal α.

5. First definitions of K-theory and G-theory of noncommutative schemes.

The purpose of this section is to sketch the first notions which allow extension of K-theory and G-theory to noncommutative schemes and more general locally affine 'spaces'. We consider here only the class of so-called *semiseparated* locally affine 'spaces' and schemes which includes the main examples of noncommutative schemes and locally affine 'spaces', starting from quantum flag varieties and noncommutative Grassmannians. Commutative semiseparated schemes are schemes $\mathcal{X}$ whose diagonal moprhism $\mathcal{X} \xrightarrow{\Delta_X} \mathcal{X} \times \mathcal{X}$ is affine. In particular, every separated scheme is semiseparated.

Semiseparated noncommutative (in particular, commutative) schemes and locally affine 'spaces' over an affine scheme are particularly convenient, because the category of quasi-coherent sheaves on them is described by a linear algebra data provided by *flat descent*.

5.1. Semiseparated schemes. Flat descent. We shall consider semiseparated schemes and more general locally affine 'spaces' over an affine scheme, $\mathcal{S} = \mathbf{Sp}(R)$. These are pairs (X, f), where X is a 'space' and f a continuous morphism $X \longrightarrow \mathcal{S}$ for which there exists a finite affine cover $\{U_i \xrightarrow{u_i} X \mid i \in J\}$ such that every morphism u_i is flat and affine. In this case, the corresponding morphism

$$U_J = \coprod_{i\in J} U_i \xrightarrow{\mathfrak{u}} X$$

is fflat (=faithfully flat) and affine. By the (dual version of) Beck's theorem (Lecture 1, 2.3.1), there is a commutative diagram

$$\begin{array}{ccc} C_X & \longrightarrow & \mathcal{G}_{\mathfrak{u}} - Comod \\ \mathfrak{u}^* \searrow & & \swarrow \widehat{\mathfrak{u}}^* \\ & R_{f\mathfrak{u}} - mod & \end{array}$$

where the horizontal arrow is a category equivalence. Here we identify the category C_{U_J} with the category of $R_{f\mathfrak{u}}$-modules for a ring $R_{f\mathfrak{u}}$ over R corresponding (by Beck's theorem) to the affine morphism $U_J \xrightarrow{f\mathfrak{u}} \mathbf{Sp}(R)$ – the monad $\mathcal{F}_{f\mathfrak{u}}$ on $R - mod$ is isomorphic to the monad $R_{f\mathfrak{u}} \otimes_R -$ (see Lecture 1). Since the morphism $\mathfrak{u}$ is affine, the associated comonad $\mathcal{G}_{\mathfrak{u}} = (G_{\mathfrak{u}}, \delta_{\mathfrak{u}})$, that is the functor $G_{\mathfrak{u}} = \mathfrak{u}^*\mathfrak{u}_*$, is continuous: the composition $\mathfrak{u}^!\mathfrak{u}_*$ is its right adjoint. Therefore, $G_{\mathfrak{u}}$ is isomorphic to the tensoring $\mathcal{M}_{\mathfrak{u}} \otimes_{R_{f\mathfrak{u}}} -$ by an $R_{f\mathfrak{u}}$-bimodule $\mathcal{M}_{\mathfrak{u}}$ determined uniquely up to isomorphism. The comonad structure $\delta_{\mathfrak{u}}$ induces a map $\mathcal{M} \xrightarrow{\widetilde{\delta}_{\mathfrak{u}}} \mathcal{M}_{\mathfrak{u}} \otimes_{R_{f\mathfrak{u}}} \mathcal{M}_{\mathfrak{u}}$ which turnes $\mathcal{M}_{\mathfrak{u}}$ into a coalgebra in the monoidal category of $R_{f\mathfrak{u}}$-bimodules. Thus, the category C_X is naturally equivalent to the category $(\mathcal{M}_{\mathfrak{u}}, \widetilde{\delta}_{\mathfrak{u}}) - Comod$ of $(\mathcal{M}_{\mathfrak{u}}, \widetilde{\delta}_{\mathfrak{u}})$. Its objects are pairs $(V, V \xrightarrow{\zeta} \mathcal{M}_{\mathfrak{u}} \otimes_{R_{f\mathfrak{u}}} V)$, where V is a left $R_{f\mathfrak{u}}$-module, which satisfy the usual comodule conditions. The structure morphism $X \xrightarrow{f} \mathbf{Sp}(R)$ is encoded in the structure object $\mathcal{O} = f^*(R)$, or, what is the same, a comodule structure $R_{f\mathfrak{u}} \xrightarrow{\zeta_{f\mathfrak{u}}} \mathcal{M}_{\mathfrak{u}} \otimes_{R_{f\mathfrak{u}}} R_{f\mathfrak{u}}$ on the left module $R_{f\mathfrak{u}}$, which we can replace, thanks to an isomorphism $\mathcal{M}_{\mathfrak{u}} \otimes_{R_{f\mathfrak{u}}} R_{f\mathfrak{u}} \simeq \mathcal{M}_{\mathfrak{u}}$, by a morphism $R_{f\mathfrak{u}} \xrightarrow{\zeta_{f\mathfrak{u}}} \mathcal{M}_{\mathfrak{u}}$ satisfying the natural associativity condition and whose composition with counit $\mathcal{M} \xrightarrow{\varepsilon_{\mathfrak{u}}} R_{f\mathfrak{u}}$ of the coalgebra $(\mathcal{M}_{\mathfrak{u}}, \widetilde{\delta}_{\mathfrak{u}})$ is the identical morphism.

Thus, Beck's theorem provides a description of the category of quasi-coherent sheaves on a semiseparated noncommutative (that is not necessarily commutative) scheme in terms of linear algebra.

5.2. The category of vector bundles. Fix a locally affine 'space' (X, f). We call an object $\mathcal{M}$ of the category C_X a *vector bundle* if its inverse image, $u_J^*(\mathcal{M})$ is a projective ΓU_J-module of finite type, or, equivalently, $u_i^*(\mathcal{M})$ is a projective ΓU_i-module of finite type for each $i \in J$. We denote by $\mathcal{P}(X)$ the full subcategory of the category C_X whose objects are vector bundles on X.

5.3. The category of coherent objects. We call an object $\mathcal{M}$ of the category C_X *coherent* if $u_i^*(\mathcal{M})$ is coherent for each $i \in J$. We denote by $Coh(X)$ the full subcategory of C_X generated by coherent objects.

5.3.1. Proposition. *(a) The notions of a projective and coherent objects are well defined.*

(b) $Coh(X)$ is a thick subcategory of C_X. In particular, it is an abelian category.

(c) $\mathcal{P}(X)$ an fully exact (i.e. closed under extensions) subcategory of C_X. In particular, $\mathcal{P}(X)$ is an exact category.

Proof. (a) Semiseparated finite covers form a filtered system: if $U_J \xrightarrow{u_J} X \xleftarrow{u_I} \widetilde{\mathcal{U}}_I$ are fflat and affine, then all arrows in the cartesian square

$$\begin{array}{ccc} U_J \times_X \widetilde{\mathcal{U}}_I & \longrightarrow & \widetilde{\mathcal{U}}_I \\ \big\downarrow & cart & \big\downarrow \\ U_J & \longrightarrow & X \end{array}$$

are fflat and affine. This follows from the categorical description of the cartesian product corresponding to direct image functors of $U_J \longrightarrow X$ and $\widetilde{\mathcal{U}} \longrightarrow X$.

(b) & (c). An exercise for the reader. ■

5.4. The category of locally affine semiseparated 'spaces'. Let $\mathfrak{Laff}_{\mathcal{S}}$ denote the subdiagram of the category $|Cat|^o_{\mathcal{S}}$ of $\mathcal{S}$-'spaces' whose objects are locally affine quasi-compact semiseparated $\mathcal{S}$-'spaces' and morphisms are those morphisms $X \xrightarrow{f} Y$ of $\mathcal{S}$-'spaces' which can be lifted to a morphism of semiseparated covers. More precisely, for any morphism $X \xrightarrow{f} Y$ of $\mathfrak{Laff}_{\mathcal{S}}$ and any affine cover $U_Y \xrightarrow{\pi_Y} Y$, there is a commutative diagram

$$\begin{array}{ccc} U_X & \xrightarrow{\widetilde{f}} & U_Y \\ {\scriptstyle \pi_X}\big\downarrow & & \big\downarrow{\scriptstyle \pi_Y} \\ X & \xrightarrow{f} & Y \end{array}$$

where the left vertical arrow is an affine cover of X.

One can see that $\mathfrak{Laff}_{\mathcal{S}}$ is a subcategory of $|Cat|^o_{\mathcal{S}}$.

For each object (X, f) of $\mathfrak{Laff}_{\mathcal{S}}$, let $X_{\mathcal{P}}$ denote the 'space' defined by $C_{X_{\mathcal{P}}} = \mathcal{P}(X)$ and $X_{\mathfrak{C}}$ the 'space' defined by $C_{X_{\mathfrak{C}}} = Coh(X, f)$.

5.5. Proposition. *The map $(X, f) \longmapsto X_{\mathcal{P}}$ is a functor from $\mathfrak{Laff}_{\mathcal{S}}$ to the category $\mathfrak{Esp}_{\mathfrak{r}}$ whose objects are 'spaces' represented by exact categories and whose morphisms have 'exact' inverse image functors.*

Proof. In fact, restricted to the affine schemes, the functor takes values in the category $\mathfrak{Ex}$, because an inverse image of (automatically affine) morphism between affine $\mathcal{S}$-'spaces' maps conflations to conflations. The general case follows from the affine case via affine covers, because the inverse image functors of the covers are fflat. ■

5.6. The functor $\mathcal{K}_\bullet$. We define the K-theory functor $\mathcal{K}_\bullet$ as the universal ∂-functor from the category $\mathfrak{Laff}_{\mathcal{S}}$ semiseparated locally affine 'spaces' endowed with left exact structure induced by the functor from $\mathfrak{Laff}_{\mathcal{S}}$ to the category of right exact 'spaces' which assigns to every locally affine semiseparated 'space' the right exact 'space' represented by the category of vector bundles.

5.7. The category $\mathfrak{Laff}^{\mathfrak{fl}}_{\mathcal{S}}$. We denote this way the subcategory of the category $\mathfrak{Laff}_{\mathcal{S}}$ of locally affine 'spaces' formed by flat morphisms.

5.7.1. Proposition. *The map $(X, f) \longmapsto X_{\mathcal{P}}$ is a functor from $\mathfrak{Laff}^{\mathfrak{fl}}_{\mathcal{S}}$ to the category $\mathfrak{Esp}_{\mathfrak{a}}$ whose objects are 'spaces' represented by abelian categories and whose morphisms have exact inverse image functors.*

Proof. An exercise for the reader. ∎

5.8. The functor $G_\bullet$. We endow the category $\mathfrak{Laff}_{\mathcal{S}}^{\mathfrak{fl}}$ with the left exact structure $\widetilde{\mathfrak{I}}_{\mathfrak{S}}$) induced by the standard left exact structure on $\mathfrak{Esp}_{\mathfrak{a}}$ (inverse image functors of inflations are exact localizations) via the functor of 5.7.1. We define the ∂-functor $G_\bullet$ as the universal ∂-functor from the left exact category $(\mathfrak{Laff}_{\mathcal{S}}^{\mathfrak{fl}}, \widetilde{\mathfrak{I}}_{\mathfrak{S}})$, whose zero component assigns to every locally affine semiseparated 'space' (X, f) the K_0-group of the 'space' represented by the category of coherent sheaves on (X, f).

5.9. Proposition. *Let $i \longmapsto (X_i, f_i)$ be a filtered projective system of locally affine $\mathcal{S}$-'spaces' such that the transition morphisms $(X_i, f_i) \longrightarrow (X_j, f_j)$ are affine, and let $(X, f) = \lim(X_i, f_i)$. Then*

$$\mathcal{K}_\bullet(X, f) \simeq \operatorname{colim}(\mathcal{K}_\bullet(X_i, f_i)). \tag{2}$$

If in addition the transition morphisms are flat, then

$$G_\bullet(X, f) \simeq \operatorname{colim}(G_\bullet(X_i, f_i)). \tag{2'}$$

Proof. It follows from the assumptions that a filtered projective system of locally affine $\mathcal{S}$-'spaces' and affine morphisms induces a filtered inductive system of the exact categories $\mathcal{P}(X_i, f_i)$ of vector-bundles. Its colimit, $\mathcal{P}(X, f)$ is an exact category whose conflations are images of conflations of $\mathcal{P}(X_i, f_i)$. Whence the isomorphism (2).

If, in addition, the transition morphisms are flat, then the inverse image functors of the transition functors induce exact functors between categories of coherent objects. This implies the isomorphism (2'). ∎

5.10. Regular locally affine 'spaces'. For a locally affine $\mathcal{S}$-'space' (X, f), we denote by $\mathbb{H}(X, f)$ the full subcategory of the category $Coh(X, f)$ which have a $\mathcal{P}(X)$-resolution.

5.10.1. Proposition. *(a) $\mathbb{H}(X, f)$ is a fully exact subcategory of the category $Coh(X, f)$. In particular, it is an exact category.*

(b) Set $\mathbb{H}(X, f) = C_{X_{\mathbb{H}}}$. The embedding of categories $\mathcal{P}(X, f) \hookrightarrow \mathbb{H}(X, f)$ induces an isomorphism $\mathcal{K}_\bullet(X, f) \stackrel{\mathrm{def}}{=} K_\bullet(X_{\mathcal{P}}) \xrightarrow{\sim} K_\bullet(X_{\mathbb{H}})$.

Proof. (a) By a standard argument.

(b) The fact is a consequence of the Resolution Theorem. ∎

5.10.2. Definition. A locally affine 'space' is called *regular* if $\mathbb{H}(X, f) = Coh(X, f)$.

Thus, if (X, f) is a regular locally affine 'space', then $\mathcal{K}_\bullet(X, f) = G_\bullet(X, f)$.

5.10.3. Remark. If (X, f) is an affine $\mathcal{S}$-'space', then the regularity coincides with the usual notion of regularity of rings ($\mathcal{S}$ is assumed to be affine). Similarly, if (X, f) is an $\mathcal{S}$-'space' corresponding to a commutative scheme.

The notion of $\mathbb{H}(X, f)$ is *local* in the following sense:

5.10.4. Proposition. *Let (X, f) be a locally affine $\mathcal{S}$-'space'. The following conditions on an object $\mathcal{M}$ of C_X are equivalent:*

(a) $\mathcal{M}$ belongs to $\mathbb{H}(X, f)$;

(b) $u_i^*(\mathcal{M})$ belongs to $\mathbb{H}(U_i, fu_i)$ for some finite cover $\{U_i \xrightarrow{u_i} X \mid i \in J\}$ of (X, f) and for all $i \in J$;

(c) $u_i^*(\mathcal{M})$ belongs to $\mathbb{H}(U_i, fu_i)$ for some finite cover $\{U_i \xrightarrow{u_i} X \mid i \in J\}$ of (X, f) and for all $i \in J$.

Proof. Obviously, $(c) \Rightarrow (b)$. In the rest of the argument, one can assume that the covers consist of one fflat affine morphism. The assertion follows from the fact that such covers form a filtered system. Details are left as an exercise. ■

5.10.5. Examples. The quantum flag varieties and the corresponding twisted quantum D-schemes [LR] are examples of regular schemes. Noncommutative Grassmannians [KR1], [KR3] are examples of regular locally affine 'spaces' which are not schemes.

6. Remarks on K-theory and quantized enveloping algebras.

In a sense, the standard K-theory based on the category of vector bundles, or G-theory based on the category of all coherent sheaves, do not give much valuable information from the point of view of representation theory. For instance, if $\mathfrak{g}$ is a finite-dimensional Lie algebra over a field k, then $K_\bullet(U(\mathfrak{g})) \simeq K_\bullet(k)$ and, similarly, $K_\bullet(A_n(k)) \simeq K_\bullet(k)$, where $A_n(k)$ is the n-th Weyl algebra over k. This indicates that one should study K-theory of other subcategories of the category of $U(\mathfrak{g})$-modules. The subcategory which received most of attention in seventies and the beginning of eighties was the category $\mathcal{O} = \mathcal{O}(\mathfrak{g})$ of representations of a semi-simple (or reductive) Lie algebra $\mathfrak{g}$ introduced by I.M. Gelfand and his collaborators. The highlight of its study was Kazhdan-Lusztig conjecture and, the most important, its prove, which led to the reformulation of the representation theory of reductive algebraic groups in terms of D-modules and D-schemes making it a part of noncommutative algebraic geometry, even before this branch of mathematics emerged.

The main basic fact which allowed to reduce the problems of representation theory to the study of D-modules on flag varieties is the Beilinson-Bernstein localization theorem which says that the global section functor induces an equivalence between the category of D-modules on the flag variety of a reductive Lie algebra $\mathfrak{g}$ over a field of zero characteristic and the category of $U(\mathfrak{g})$-modules with trivial central character (and its twisted version). Harish-Chandra modules and their different generalizations turned out to be *holonomic* D-modules. As a result, holonomic modules on flag varieties became the main object of study in representation theory of reductive algebraic groups.

On the other hand, the notions of quantum flag variety and the appropriate categories of twisted D-modules were introduced in [LR]. And it was established a quantum version of Beilinson-Bernstein localization theorem [LR], [T], which reduces the study of representations of the quantized enveloping algebra $U_q(\mathfrak{g})$ to the study of twisted D-modules on quantum flag variety, like in the classical case. The notion of a holonomic D-module is extended to the setting of noncommutative algebraic geometry [R5]. In particular, there exists a notion of a *quantum* holonomic D-module.

All initial ingredients are present and the area of research is wide open.

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