HOLOMORPHY OF EISENSTEIN SERIES AT SPECIAL POINTS AND COHOMOLOGY OF ARITHMETIC SUBGROUPS OF $SL_n(\mathbb{Q})$

by

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§ 0 Introduction

Given an arithmetic torsionfree subgroup Γ of the group G of real points of a connected semisimple algebraic group over $\mathbb{Q}$ the cohomology $H^*(\Gamma\backslash X;\mathbb{C})$ of Γ can be defined as the cohomology of the complex $\Omega^*(\Gamma\backslash X;\mathbb{C})$ of $\mathbb{C}$-valued smooth differential forms on $\Gamma\backslash X$, where X denotes the associated symmetric space. This paper continues in the case of an arithmetic subgroup $\Gamma \subset SL_n(\mathbb{Z})$ the general discussion in [27], I how Eisenstein series can be used to construct harmonic forms on $\Gamma\backslash X$ which represent non-trivial cohomology classes in $H^*(\Gamma\backslash X;\mathbb{C})$.

Starting with the Borel-Serre compactification $\Gamma\backslash\overline{X}$ of $\Gamma\backslash X$ whose boundary $\partial(\Gamma\backslash\overline{X})$ is a union of finitely many faces $e'(P)$ associated to the proper parabolic $\mathbb{Q}$-subgroups of G modulo Γ-conjugation one studies the various restrictions

$$(1) \qquad r_P^* : H^*(\Gamma\backslash X;\mathbb{C}) = H^*(\Gamma\backslash\overline{X};\mathbb{C}) \to H^*(e'(P),\mathbb{C})$$

of the cohomology of Γ onto the cohomology of a face. Via Eisenstein series one tries to construct classes in $H^*(\Gamma\backslash\overline{X};\mathbb{C})$ with a non-zero restriction to $\partial(\Gamma\backslash\overline{X})$ and to get hold of cross-sections to (suitable families of) the restrictions in (1) or, ultimately, the restriction $r^* : H^*(\Gamma\backslash\overline{X},\mathbb{C}) \to H^*(\partial(\Gamma\backslash\overline{X}),\mathbb{C})$ in this way. For motivation and background we refer to [12], [13], [27].

This approach was initiated by Harder ([12], [13]). If the $\mathbb{Q}$-rank of G is one, he has shown the existence of a subspace $H^*_{Eis}(\Gamma\backslash X;\mathbb{C})$ in $H^*(\Gamma\backslash\overline{X};\mathbb{C})$ which restricts isomorphically onto $\text{Im } r$ and whose elements are obtained either by evaluating suitable Eisenstein series at special points or by taking residues of such at simple poles. Since there is almost no information concerning the behaviour of Eisenstein series at certain values which are of interest here the result of Harder has to be seen as an answer up to the existence of poles. It can be made more precise in the case SL_2/k defined over an algebraic number field k where one gets out of this a complete description of $\text{Im } r^*$ (cf. [12], [14]).

For groups of higher rank the situation is not investigated thoroughly. However, as a first step, there is a general result (cf. [27], § 4) describing in which way an Eisenstein series $E(\phi,\Lambda)$ which is associated to a cuspidal differential form on a face $e'(P)$ and depends on a complex parameter Λ provides us with a closed harmonic form on $\Gamma\backslash X$ and with a non-trivial class in $H^*(\Gamma\backslash X;\mathbb{C})$ if $E(\phi,\Lambda)$ is holomorphic at a special point Λ_o uniquely determined by ϕ. As examples in [12], [27] show $E(\phi,\Lambda)$ may very well have poles at such points; therefore it makes sense in dealing with the condition of holomorphy to limit ourselves to special cases.

Given an arithmetic subgroup Γ of $SL_n(\mathbb{Z})$, $n > 2$, this paper discusses the questions mentioned in the case of Eisenstein series associated to cuspidal forms on faces $e'(P)$ corresponding to a Γ-conjugacy class of maximal parabolic $\mathbb{Q}$-subgroups of $G = SL_n(\mathbb{R})$ (i.e. faces of minimal codimension in $\partial(\Gamma\backslash\bar{X})$). Let $C(P)$ denote the class of maximal parabolic $\mathbb{Q}$-subgroups associated to P (If $n = 2m$ is even we have to assume that P is not of type m in the sense of 3.1.). As a main result, we construct a subspace $H^*_{C(P)}(\Gamma\backslash\bar{X},\mathbb{C})$ in $H^*(\Gamma\backslash\bar{X},\mathbb{C})$ generated by <u>regular</u> Eisenstein cohomology classes which describes <u>completely</u> that part of the cohomology at infinity of Γ contributed by the cusp cohomology spaces $H^*_{cusp}(e'(Q),\mathbb{C})$

where Q runs through a set of representatives for the elements of $C(P)$ modulo conugation by Γ . By a regular Eisenstein class we mean a class which is represented by a harmonic differential form $E(\phi,\Lambda_o)$ on $\Gamma\backslash X$ obtained as a value of an Eisenstein series $E(\phi,\Lambda)$ at a special point Λ_o where $E(\phi,\Lambda)$ is holomorphic in Λ_o . We note that we are not forced to use residues of Eisenstein series in the description of $H^*_{C(P)}(\Gamma\backslash\bar{X},\mathbb{C})$. It maps isomorphically under the natural restriction onto the image of

$$(2) \qquad r^*_{C(P),cusp} : H^*(\Gamma\backslash\bar{X},\mathbb{C}) \to \bigoplus H^*_{cusp}(e'(Q),\mathbb{C})$$

and its image is of dimension equal to one half the dimension of the right hand side in (2) for a congruence subgroup $\Gamma = \Gamma(k)$.

We note that for different associate classes $C(P)$ the corresponding subspaces $H^*_{C(P)}(\Gamma\backslash\bar{X},\mathbb{C})$ in $H^*(\Gamma\backslash\bar{X},\mathbb{C})$ are linearly independent.

This result is in fact a consequence of a more detailed study of the behaviour of Eisenstein series as above at special points, of the corresponding cohomology classes and its images under the various restrictions. Section 2 reviews briefly, in a form convenient for us, the ingredients of the construction of Eisenstein cohomology classes and some of the results in [27], I we will need lateron. As a first step towards the proof we show that every cuspidal cohomology class $[\phi]$ in $H^*_{cusp}(e'(P),\mathbb{C})$ has to be of "tempered type" (4.3.). By some additional arguments this is a consequence of determining (originally done by Casselman) the irreducible unitary representations of $SL_n(\mathbb{R})$ which occur with non-zero multiplicity in the cuspidal spectrum $L^2_o(\Gamma\backslash SL_n(\mathbb{R}))$ and have non-zero relative Liealgebra cohomology. Since there is no reference for this we have included a proof in section 3. This implies in particular a strong vanishing result for the cusp cohomology $H^*_{cusp}(\Gamma\backslash X,\mathbb{C}) \subset H^*(\Gamma\backslash X,\mathbb{C})$ of Γ outside a range $[C_u(n),C_o(n)]$ of length rk $SL_n(\mathbb{R})$ - rk $SO(n)$ centered around the middle dimension $(1/2)\cdot \dim X$. The bounds $C_u(n)$, $C_o(n)$ are explicitly given in terms of n in 3.5. It turns out, that the "cuspidal cohomology dimension" $C_o(n)$ is much smaller than the cohomological dimension $cd(\Gamma)$ of Γ.

Since a class $[\phi]$ in $H^*_{cusp}(e'(P),\mathbb{C})$ is of "tempered type", the question of holomorphy of the associated Eisenstein series $E(\phi,\Lambda)$ can be attacked by the methods developed in [27], § 6. We obtain that $E(\phi,\Lambda)$ is holomorphic at the special point Λ_o under a certain condition on the degree of $[\phi]$ (cf. 4.4). This provides us with a non-trivial cohomology class $[E(\phi,\Lambda_o)]$ in $H^*(\Gamma\backslash\overline{X},\mathbb{C})$. The image of this class under the various restrictions r^*_R in (1) is determined in 4.4.(2)-(4). As a special case it turns out that for a given class $[\phi]$ of a degree greater or equal to the cuspidal cohomological dimension $C_o(n)$ the class $[E(\phi,\Lambda_o)]$ in $H^*(\Gamma\backslash\overline{X},\mathbb{C})$ restricts non-trivial to the class $[\phi]$ we started with and trivial to all other faces $e'(R) \neq e'(P)$. This is obtained by a study of the constant term of $E(\phi,\Lambda_o)$ along R and the intertwining operators involved.

Given the maximal parabolic $\mathbb{Q}$-subgroup (not of type m if $n = 2m$) Theorem 4.7. deals then with the subspace $H^*_{C(P)}(\Gamma\backslash\overline{X},\mathbb{C})$ which is generated by the regular Eisenstein cohomology classes constructed as above. Some partial results in the case P of type m are discussed in 4.8., 4.9., and we indicate briefly in 4.10. some consequences out of our results so far for the structure of the cohomology of a congruence subgroup $\Gamma(k)$ as a module under the finite group $SL_n(\mathbb{Z}/k\mathbb{Z})$. We conclude section 4 with some examples and remarks concerning cusp cohomology classes in $H^*_{cusp}(\Gamma\backslash\overline{X},\mathbb{C})$ resp. $H^*_{cusp}(e'(P),\mathbb{C})$.

In section 5 we indicate briefly how unpublished results due to Langlands and Borel (cf. 5.1.(4) resp. 5.3.(5)) imply that the subspace $H^*_{C(P)}(\Gamma\backslash\bar{X},\mathbb{C})$ constructed above is as large as possible and describes completely that part of the cohomology at infinity of Γ contributed by the cusp cohomology spaces $\oplus H^*(e'(Q),\mathbb{C})$ (cf. 5.4. - 5.6.).

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Notation. Beside the usual conventions we fix the following ones:

(1) The algebraic groups considered here are linear and can be identified with algebraic subgroups of some $\underline{GL}_n(\mathbb{C})$. We follow the notations in [1]. If $\underline{G}$ is (Zariski)-connected $\mathbb{Q}$-group, we denote by $G = \underline{G}(\mathbb{R})$ the group of real points of $\underline{G}$. An arithmetic subgroup of $\underline{G}$ is a subgroup of $\underline{G}(\mathbb{Q})$ which is commensurable with $\phi(\underline{G}) \cap \underline{GL}_n(\mathbb{Z})$ for some injective morphism $\phi : \underline{G} \to \underline{GL}_n$ defined over $\mathbb{Q}$. For a connected $\mathbb{Q}$-group $\underline{G}$ we put ${}^{o}\underline{G} = \bigcap \ker \chi^2$ where χ runs through the group $X_{\mathbb{Q}}(\underline{G})$ of $\mathbb{Q}$-morphisms from $\underline{G}$ to $\underline{GL}_1$. The group ${}^{o}\underline{G}(\mathbb{R})$ contains each compact subgroup of $\underline{G}(\mathbb{R})$ and each arithmetic subgroup of $\underline{G}$ ([9], 1.2.).

(2) With respect to the theory of representations of a Lie group with finitely many connected components we use mainly the notations in [10].

§ 1. Preliminaries on cohomology of arithmetic groups

1.1. Let $\underline{G}$ be a connected reductive $\mathbb{Q}$-group with $\mathrm{rank}_{\mathbb{Q}}\, \underline{G} > 0$ and without non-trivial rational character defined over $\mathbb{Q}$. Let K be a maximal compact subgroup of the group $G = \underline{G}(\mathbb{R})$ of real points of $\underline{G}$, and denote by $X = G/K$ the associated symmetric space. Endowed with a G-invariant Riemannian metric the space X is a complete Riemannian manifold with negative curvature. Let (τ, E) be a finite-dimensional (complex) rational representation of G. We choose an admissible scalar product on E in the sense of Matsushima-Murakami (cf. [10], II, 2.2). Let $\Gamma \subset G$ be a torsion-free arithmetic subgroup of $\underline{G}$. The group G operates properly and freely on X, and G operates also on the space $\Omega^*(X;E)$ of smooth E-valued differential forms on X. The quotient space $\Gamma\backslash X$ is a non-compact $K(\Gamma,1)$-manifold of finite volume. Our object of concern is the Eilenberg-MacLane cohomology space $H^*(\Gamma;E)$ which is usually identified in a canonical way with the cohomology $H^*(\Gamma\backslash X;E)$ of the subcomplex $\Omega^*(X,E)^{\Gamma}$ of Γ-invariant elements in $\Omega^*(X,E)$, i.e. we have the identifications

$$(1) \qquad H^*(\Gamma,E) = H^*(\Gamma\backslash X,\tilde{E}) = H^*(\Gamma\backslash X,E)$$

where the middle term denotes singular cohomology of $\Gamma\backslash X$ with coefficients in the local system defined by (τ, E) (see, for example, [10], VII, 2).

1.2. Denote by $\underline{g}$ resp. $\underline{k}$ the Liealgebra of G resp. K, and let (π, V) be a $(\underline{g},K)$-module (cf. [10], 0, § 2.5). The relative Liealgebra cohomology of $\underline{g}$ mod K with coefficients in V is then defined as the cohomology of the complex $D^*(\underline{g},K;V) = \mathrm{Hom}_K(\Lambda^*(\underline{g}/\underline{k}),V)$ with the usual differential as in [10], I, § 1. Since the space $F_{(K)}$ of K-finite vectors in a differentiable G-module F is a $(\underline{g},K)$-module in a natural way the above notion makes also sense for F if we put then $D^*(\underline{g},K;F) = D^*(\underline{g},K;F_{(K)})$.

The space of smooth functions on $\Gamma\backslash G$ with values in $\mathbb{C}$ will be denoted by $C^\infty(\Gamma\backslash G)$. The lifting of forms via the projection $G \to G/K = X$ induces then an isomorphism of differential complexes

$$(1) \qquad \Omega^*(X,E)^\Gamma \tilde{\to} D^*(\underline{g},K;C^\infty(\Gamma\backslash G) \otimes E) ,$$

whence an isomorphism in cohomology (cf. [10], VII, 2.7.)

$$(2) \qquad H^*(\Gamma\backslash X,E) = H^*(\underline{g},K;C^\infty(\Gamma\backslash G) \otimes E) .$$

__1.3.__ In this identification one can replace $C^\infty(\Gamma\backslash G)$ by certain subspaces. Let $U(\underline{g})$ be the universal enveloping algebra of $\underline{g}$ over $\mathbb{C}$. We let $C^\infty_{mg}(\Gamma\backslash G)$ be the space of smooth functions on $\Gamma\backslash G$ which together with their $U(\underline{g})$-derivatives have moderate growth (cf. [2], 3.2). Moreover, we let $C^\infty_{umg}(\Gamma\backslash G)$ be the space of smooth functions on $\Gamma\backslash G$ of uniform moderate growth i.e. the exponent limiting the growth on a Siegel set can be chosen independently of the derivatives. Using 1.2.(1) we put

$$(1) \qquad \Omega^*_?(\Gamma\backslash X,E) = D^*(\underline{g},K;C^\infty_?(\Gamma\backslash G) \otimes E) \quad \text{with} \quad ? = mg \quad \text{resp.} \quad umg .$$

Then it was proved by Borel ([4], 3.2.) that the inclusions

$$(2) \qquad \Omega^*_{umg}(\Gamma\backslash X,E) \to \Omega^*_{mg}(\Gamma\backslash X,E) \to \Omega^*(\Gamma\backslash X,E)$$

induce isomorphisms in cohomology.

__1.4.__ Let $\Omega^*_c(\Gamma\backslash X,E)$ (resp. $\Omega^*_{fd}(\Gamma\backslash X,E)$) denote the complex of forms in $\Omega^*(\Gamma\backslash X,E)$ with compact support (resp. which together with their exterior differentials are fast decreasing ([2], 3.2)). Then the natural inclusion $\Omega_c^*(\Gamma\backslash X,E) \to \Omega_{fd}^*(\Gamma\backslash X,E)$ induces an isomorphism in cohomology ([2], 5.2.), i.e. one has via the de Rham theorem

$$H_c^*(\Gamma\backslash X;\tilde{E}) = H^*(\Omega_{fd}(\Gamma\backslash X,E)) =: H_c^*(\Gamma\backslash X,E)$$

where H_c^* refers to cohomology with compact support.

1.5. Cusp cohomology. Let $L^2(\Gamma\backslash G)$ be the space of complex valued square integrable functions on $\Gamma\backslash G$, viewed as usual as a unitary G-module via right translations. The space $L_o^2(\Gamma\backslash G)$ of square integrable cuspidal functions on $\Gamma\backslash G$ is a G-invariant subspace of $L^2(\Gamma\backslash G)$ and it decomposes into a direct Hilbert sum of closed irreducible subspaces H_π with finite multiplicities $m(\pi,\Gamma)$ (cf. [15] 1. § 2 or [11])

$$(1) \qquad L_o^2(\Gamma\backslash G) = \widetilde{\bigoplus_{\pi\in\hat{G}}} m(\pi,\Gamma)H_\pi \quad .$$

The inclusion $L_o^2(\Gamma\backslash G)^\infty$ of the subspace of C^∞-vectors in $L_o^2(\Gamma\backslash G)$ into $C^\infty(\Gamma\backslash G)$ induces a natural homomorphism in $(\mathfrak{g},K)$-cohomology

$$(2) \qquad H^*(\mathfrak{g},K,L_o^2(\Gamma\backslash G)^\infty \otimes E) \to H^*(\mathfrak{g},K,C^\infty(\Gamma\backslash G) \otimes E) = H^*(\Gamma\backslash X,E)$$

which is injective ([2], 5.5). By definition, the cusp cohomology $H_{cusp}^*(\Gamma\backslash X,E)$ of $\Gamma\backslash X$ with coefficients in E is the image of the homomorphism in (2). We remark that $H_{cusp}^*(\Gamma\backslash X,E)$ can be identified with the space $\underline{H}_{cusp}^*(\Gamma\backslash X,E)$ of harmonic forms in $D^*(\mathfrak{g},K;L_o^2(\Gamma\backslash G)^\infty \otimes E)$. This may also be interpreted in terms of differential forms on $\Gamma\backslash X$ (cf. [7], § 5). The direct sum decomposition (1) of $L_o^2(\Gamma\backslash G)$ yields then

$$(3) \qquad H_{cusp}^*(\Gamma\ X;E) = \bigoplus_{\pi\in\hat{G}} m(\pi,\Gamma)H(\mathfrak{g},K;H_\pi^\infty \otimes E)$$

where $\pi \in \hat{G}$ runs over the finite set of equivalence classes of all irreducible unitary representations of G whose infinitesimal character χ_π is equal to the infinitesimal character χ_{τ^*} of the representation (τ^*, E^*) contragredient to E (cf. [10], I, Thm. 5.3.).

§ 2 Eisenstein cohomology classes for arithmetic groups

In [27], I, the use of Eisenstein series to construct cohomology classes for an arithmetic subgroup Γ of G and to describe its cohomology "at infinity" is discussed. We have to (and will) assume some familiarity with it. But for the convenience of the reader and also in order to fix notations we review briefly in this section the main facts.

2.1. Preliminaries. Let $\underline{P}$ be a parabolic subgroup of $\underline{G}$ defined over $\mathbb{Q}$, $\underline{N}$ its unipotent radical and $\kappa : \underline{P} \to \underline{P}/\underline{N} = \underline{M}$ the canonical projection. Let $\underline{S}_P$ be the maximal central $\mathbb{Q}$-split torus of $\underline{P}/\underline{N}$, and denote the identity component of $\underline{S}_P(\mathbb{R})$ by S_P. A split component of $P = \underline{P}(\mathbb{R})$ is a subgroup A of a Levi subgroup of P such that A is mapped isomorphically via κ onto S_P. By A_P we denote the unique split component of P which is stable under the Cartan involution θ associated to K (cf. [9], 1.9.). We let then $M = Z_G(A_P)$ be the unique θ-stable Levi subgroup of P. The projection κ induces a canonical isomorphism $\mu : M \tilde{\to} P/N = \underline{M}(\mathbb{R})$, and we denote by ${}^{o}M$ the inverse image of ${}^{o}\underline{M}(\mathbb{R})$. We have then $P = M \cdot N$ as a semidirect product, $P = A_P \ltimes {}^{o}P$ and $M = {}^{o}M \times A_P$. Since M is θ-stable, one has $K \cap P = K \cap {}^{o}M$, and $K_P = K \cap P$ is a maximal compact subgroup of ${}^{o}M$, M and P.

Choose a minimal parabolic $\mathbb{Q}$-subgroup P_o of G with split component A_o. We assume that the Lie algebra $\underline{a}_o$ of A_o is orthogonal to $\underline{k}$ with respect to a symmetric non-degenerate bilinear form B on $\underline{g}$ chosen as in 1.3. [27]. (For $\underline{g}$ semisimple B is the Killing form.) Let $\underline{h}$ be a Cartan subalgebra of $\underline{g}$ containing $\underline{a}_o$. Let $\Phi = \Phi(\underline{g}_{\mathbb{C}}, \underline{h}_{\mathbb{C}})$ (resp. $\Phi_{\mathbb{R}} = \Phi(\underline{g}_{\mathbb{C}}, \underline{a}_{o_{\mathbb{C}}})$) be the set of roots of $\underline{g}_{\mathbb{C}}$ with respect to $\underline{h}_{\mathbb{C}}$ (resp. $\underline{a}_{o_{\mathbb{C}}}$). Its elements will also be viewed as roots of $G(\mathbb{C})$ with respect to

$H = Z_G(\underline{h})$ (resp. A_o). For a parabolic pair (P,A) (which is, by definition, a parabolic $\mathbb{Q}$-subgroup P of G with split component A) we denote the set of roots of P with respect to A by $\Phi(P,A)$, and $\Delta(P,A)$ is the set of simple roots of P with respect to A . For a fixed ordering on Φ we denote by Φ^+ (resp. Δ) the set of positive (resp. simple) roots of Φ , and analogously for the $\mathbb{R}$-roots $\Phi_{\mathbb{R}}$.

We fix an ordering on $\Phi = \Phi(\underline{g}_{\mathbb{C}},\underline{h}_{\mathbb{C}})$ which is compatible with the ordering on the real roots $\Phi_{\mathbb{R}}$ given by the choice of P_o with the unique Θ-stable split component $A_{P_o} = A_o$ via the condition $\Phi^+_{\mathbb{R}} = \Phi(P_o,A_{P_o})$.

Let (P,A) be a parabolic pair with $A \subset A_{P_o}$. Then $\underline{b} = \underline{h} \cap {}^o\underline{m}$ is a Cartan subalgebra of the Lie algebra ${}^o\underline{m}$ of ${}^oM$, and one has $\underline{h} = \underline{b} \oplus \underline{a}$. We set $\Phi_M = \Phi(\underline{m}_{\mathbb{C}},\underline{h}_{\mathbb{C}}) = \Phi({}^o\underline{m}_{\mathbb{C}},\underline{b}_{\mathbb{C}})$ resp. $\Delta_M = \Delta \cap \Phi_M$ as in [27], 1.7..

As usual, the value of a character α on $a \in A$ is denoted by $\alpha(a)$ or a^α . The element $\rho_P \in \underline{a}^*$ is defined by $\rho_P(a) = (\det \mathrm{Ad}\, a_{|\underline{n}})^{1/2}$, $a \in A$. If one puts $\rho = (1/2) \sum_{\alpha \in \Phi^+} \alpha$ and $\rho_{o_M} = (1/2) \sum_{\alpha \in \Phi^+_M} \alpha$ then $\rho_{|\underline{a}_o} = \rho_{P_o}$ and $\rho_{|\underline{b}} = \rho_{o_M}$.

2.2. The faces in the Borel-Serre compactification. As before, let Γ be a torsionfree arithmetic subgroup of G . The quotient $\Gamma\backslash X$ may be identified to the interior of a compact manifold $\Gamma\backslash\overline{X}$ with corners [9]; the inclusion $\Gamma\backslash X \to \Gamma\backslash\overline{X}$ is a homotopy equivalence. The boundary $\partial(\Gamma\backslash\overline{X})$ is a disjoint union of a finite number of faces $e'(Q)$, which correspond bijectively to the Γ-conjugacy classes of proper parabolic $\mathbb{Q}$-subgroups of G . Denote by $\underline{\underline{P}}$ the set of parabolic $\mathbb{Q}$-subgroups of G . For a given P in $\underline{\underline{P}}$, $P \neq G$, we denote the natural restriction of the cohomology of $\Gamma\backslash\overline{X}$ onto the cohomology of the corresponding face $e'(P)$ in $\partial(\Gamma\backslash\overline{X})$ by

$$(1) \qquad r^*_P \; : \; H^*(\Gamma\backslash\overline{X},\tilde{E}) \to H^*(e'(P),\tilde{E}) \quad .$$

Let P be a parabolic $\mathbb{Q}$-subgroup of G with θ-stable split component $A = A_P$, N its unipotent radical. The natural projection $\kappa : P \to P/N$ induces then an isomorphism $\mu : M \xrightarrow{\sim} P/N$, where $M = Z_G(A_P)$. We put $\Gamma_P = \Gamma \cap P$ resp. $\Gamma_N = \Gamma \cap N$. Then the projection $\Gamma_M = \kappa(\Gamma_P)$ is an arithmetic subgroup of P/N, and $K_M = \kappa(K \cap P)$ is a maximal compact subgroup of P/N which is canonically isomorphic to $K \cap P = K \cap M$ via μ. The quotient $Z_M = ({}^{o}P/N)/K_M$ is again a symmetric space which we can also view as ${}^{o}M/K \cap M$. Note that $\mu^{-1}(\Gamma_M)$ is an arithmetic subgroup of ${}^{o}M$ if M is defined over $\mathbb{Q}$. It contains $\Gamma \cap M$ as a subgroup of finite index. Then the face $e'(P)$ inherits a fiber bundle structure from

(2) $$\Gamma_N \backslash N \to e'(P) = \Gamma_P \backslash {}^{o}P/K \cap P \to \Gamma_M \backslash Z_M \quad .$$

We observe that the fibers are compact manifolds. In the sequel we will identify the base space $\Gamma_M \backslash Z_M$ of the fibration with $\mu^{-1}(\Gamma_M)\backslash {}^{o}M/K \cap M$.

<u>2.3.</u> <u>Cohomology of a face $e'(P)$</u>. First of all, the cohomology of the fiber $\Gamma_N \backslash N$ in 2.2.(2) can be identified with the cohomology of the Lie algebra $\underline{n}$ of N. Via this identification $H^*(\underline{n},E) = H^*(\Gamma_N \backslash N;E)$ (cf. § 2 [27]) the natural M-module structure on the Lie algebra cohomology $H^*(\underline{n},E)$ restricts to the action of Γ_M on the cohomology of the fiber, which inherits therefore by extension a natural M-module structure. If we put (cf. [10], III, 1.4.) (where $W = W(\mathfrak{g}_{\mathbb{C}},\underline{h}_{\mathbb{C}})$ is the Weylgroup of $\mathfrak{g}_{\mathbb{C}}$)

(1) $$W^P = \{w \in W \mid w^{-1}(\Delta_M) \subset \Phi^+\}$$

then there is an isomorphism of M-modules ([18], 5.13)

(2) $$H^q(\underline{n},E_\lambda) = \bigoplus_{\substack{w \in W^P \\ \ell(w) = q}} F_{w(\lambda+\rho)-\rho}$$

(we assume $E = E_\lambda$ has highest weight λ with $\lambda \in \underline{h}^*_{\mathbb{C}}$ dominant) where F_ν denotes an irreducible $M(\mathbb{C})$-module with highest weight ν, $\nu \in \underline{b}^*_{\mathbb{C}}$.

One can construct then a natural embedding $\eta : \Omega^*(\Gamma_M\backslash Z_M, H^*(\underline{n},E)) \to \Omega^*(\Gamma_P\backslash {}^oP/K_P,E)$ on the level of differential forms which induces (see [13], Thm. 2.8. resp. [27], Thm. 2.7.) an isomorphism in cohomology, i.e. the spectral sequence in cohomology associated to the fibration 2.2.(2) of $e'(P)$ degenerates at E_2. We have

(3) $$H^*(e'(P),E) \cong H^*(\Gamma_M\backslash Z_M, H^*(\underline{n},E)) \quad .$$

<u>2.4.</u> <u>Cusp cohomology of a face $e'(P)$.</u> The space $L^2_o(\Gamma_M\backslash {}^oM)$ of square integrable cuspidal functions on $\Gamma_M\backslash {}^oM$ decomposes into a direct Hilbert sum of closed irreducible ${}^oM$-invariant subspaces H_π with finite multiplicities $m(\pi,\Gamma_M)$. If V_π denotes the isotypic component of $\pi \in {}^o\hat{M}$ we may write

(1) $$L^2_o(\Gamma_M\backslash {}^oM) = \widetilde{\bigoplus_{\pi \in {}^o\hat{M}}} V_\pi \quad .$$

Using 2.3.(3) the cusp cohomology of the face $e'(P)$ is defined as

(2) $$H^*_{cusp}(e'(P),E) = H^*_{cusp}(\Gamma_M\backslash Z_M, H^*(\underline{n},E)) \quad .$$

By 1.5. it can naturally be viewed as the image of the injective homomorphism

(3) $$H^*({}^o\underline{m},K_M,L^2_o(\Gamma_M\backslash {}^oM)^\infty \otimes H^*(\underline{n},E)) \to H^*(\Gamma_M\backslash Z_M,H^*(\underline{n},E)) \quad .$$

Using (1) and 2.3. we have then a finite sum decomposition

(4) $$H^*_{cusp}(e'(P),E) = \bigoplus_{\pi \in {}^o\hat{M}} \bigoplus_{w \in W^P} H^*({}^o\underline{m},K_M,V_\pi \otimes F_{w(\lambda+\rho)-\rho}) \quad .$$

As in 3.2. [27] we call a non-trivial cohomology class $[\phi] \neq 0$ in

$H^*_{cusp}(e'(P),E)$ (represented by a cuspidal form $\phi \in \Omega^*(\Gamma_M\backslash Z_M, H^*(\underline{n},E))$) a <u>cuspidal class of type</u> (π,w) if there exist $\pi \in {}^o\hat{M}$ with $V_\pi \subset L^2_o(\Gamma_M\backslash {}^oM)$ and $w \in W^P$ such that $[\phi]$ is in the image of $H^*({}^o\underline{m},K_M,V_\pi \otimes F_{w(\lambda+\rho)-\rho})$.

Induced by the adjoint action the split component $A = A_P$ of P operates on $H^*(\underline{n},E)$ in a way which is also obtained by restriction from the action of $M = {}^oM\cdot A$. This yields a decomposition of $H^*(\underline{n},E)$ into weight spaces with respect to A_P which are already given by 2.3.(2) , i.e. each $F_{w(\lambda+\rho)-\rho}$ gives via restriction a multiple of an A_P-module of weight $w(\lambda+\rho)-\rho|_{\underline{a}_P}$. According to this decomposition we call an element $[\phi]$ in $H^*(e'(P),E)$ <u>a class of weight</u> $\mu \in \underline{a}_P^*$ if $[\phi] \in H^*(\Gamma_M\backslash Z_M;F_\mu)$.

<u>2.5.</u> <u>Construction of Eisenstein series</u>. Let $0 \neq [\phi] \in H^*_{cusp}(e'(P),E)$ be a non-trivial cuspidal cohomology class of type (π,w) $(\pi \in {}^o\hat{M}, w \in W^P)$ represented by a harmonic cusp form $\phi \in \Omega^*(e'(P),E)$. As explained in [27], § 4 we associate to ϕ via the differential form

(1) $$\phi_\Lambda = \phi\, a^{\Lambda+\rho} \quad \text{in} \quad \Omega^*(\Gamma_P\backslash X;E)$$

the Eisenstein series

(2) $$E(\phi,\Lambda) := \sum_{\gamma\in\Gamma_P\backslash\Gamma} \gamma\circ\phi_\Lambda \ .$$

This Eisenstein series is first defined for all Λ in

(3) $$(\underline{a}^*_{\mathbb{C}})^+ = \{\Lambda \in \underline{a}^*_{\mathbb{C}} \mid \operatorname{Re}\Lambda \in \rho_P + (\underline{a}^*)^+\}$$

where $(\underline{a}^*)^+ = \{\lambda \in \underline{a}^* \mid (\lambda,\alpha) > 0 \text{ for all } \alpha \in \Delta(P,A)\}$ and is holomorphic in that tube. Via analytic continuation it admits a meromorphic extension to all of $\underline{a}^*_{\mathbb{C}}$. We refer to [20], [15], [23] for the general theory of Eisenstein series. If $\Lambda_o \in \underline{a}^*_{\mathbb{C}}$ is fixed and $E(\phi,\Lambda)$ is holomorphic at this point, then $E(\phi,\Lambda_o)$ is an E-valued, Γ-invariant differential form on X , i.e. we have $E(\phi,\Lambda_o) \in \Omega^*(\Gamma\backslash X;E)$.

In the frame work of relative Lie algebra complexes this construction is described as follows: Attaching ϕ_Λ to ϕ is given by a map (defined in 3.6. [27])

$$(4) \qquad D^*({}^o\underline{m},K_M;V_{\pi,(K_M)} \otimes H^*(\underline{n},E_\lambda)) \to D^*(\underline{g},K;I_{P,\pi,\Lambda,(K)} \otimes E_\lambda) \quad .$$

Here V_π is the isotypic component occuring in $L^2_o(\Gamma_M\backslash {}^oM)$ of the unitary representation $\pi \in {}^o\hat{M}$, and we let $(i_{P,\pi,\Lambda}, I_{P,\pi,\Lambda})$ be the representation induced from $V_\pi \otimes \mathbb{C}_{\rho+\Lambda}$ (viewed by trivial extension to N as a P-module) in the sense of III, 2.2. [10] where $\mathbb{C}_\nu$ denotes the onedimensional A-module $\mathbb{C}$ on which A operates via $\nu \in \underline{a}^*_{\mathbb{C}}$. It is convenient to view the representations $I_{P,\pi,\Lambda}$ as a family realized on the fixed space $C^\infty(\Gamma_P NA\backslash G,V_\pi)$ of V_π-valued smooth functions on $\Gamma_P NA\backslash G$. Using the identification 1.2.(1) the Eisenstein form $E(\phi,\Lambda_o)$ is then obtained as the image of ϕ_{Λ_o} under the map

$$(5) \qquad Eis_{\Lambda_o} \ : \ D^*(\underline{g},K;I_{P,\pi,\Lambda_o,(K)} \otimes E) \to D^*(\underline{g},K;C^\infty(\Gamma\backslash G) \otimes E)$$

induced from the $(\underline{g},K)$-module homomorphism

$$(6) \qquad E(\ ,\Lambda_o) \ : \ I_{P,\pi,\Lambda_o,(K)} \to C^\infty(\Gamma\backslash G)$$

which is given by the usual Eisenstein summation and evaluating at the point Λ_o (cf. [27], 4.1. - 4.4.).

Recall the following result ([27], Thm. 4.11.) concerning the construction of Eisenstein cohomology classes.

<u>2.6.</u> THEOREM. - <u>Let</u> P <u>be a parabolic $\mathbb{Q}$-subgroup of</u> G <u>with θ-stable split component</u> A_P . <u>Let</u> $[\phi] \in H^*_{cusp}(e'(P),E_\lambda)$ <u>be a non-trivial cuspidal</u>

cohomology class of type (π,w) $(\pi \in {}^{o}\hat{M}, w \in W^{P})$ represented by a harmonic cuspidal form $\phi \in \Omega^*(e'(P),E_\lambda)$. If the Eisenstein series $E(\phi,\Lambda)$ assigned to $[\phi]$ (in 2.5.) is holomorphic at the point

(1) $$\Lambda_o = -w(\lambda+\rho)|_{\underline{a}_P}$$

(which is real and uniquely determined by $[\phi]$) then $E(\phi,\Lambda_o)$ is a closed harmonic differential form on $\Gamma\backslash X$ and represents a non-trivial class $[E(\phi,\Lambda_o)]$ in $H^*(\Gamma\backslash X;E_\lambda)$.

We call such a class $[E(\phi,\Lambda_o)]$ a regular Eisenstein cohomology class.

2.7. The restriction of a regular Eisenstein cohomology class. The image of the regular Eisenstein cohomology class $[E(\phi,\Lambda_o)]$ as in 2.6. under the restriction $r_Q^* : H^*(\Gamma\backslash\bar{X},E) \to H^*(e'(Q),E)$ (Q a parabolic $\mathbb{Q}$-subgroup of G) is given as $[E(\phi,\Lambda_o)_Q]|_{e'(Q)}$ i.e. equal to the restriction to $e'(Q)$ of the class $[E(\phi,\Lambda_o)_Q]$ represented by the constant Fourier coefficient $E(\phi,\Lambda_o)_Q \in \Omega^*(\Gamma_Q\backslash X,E)$ of $E(\phi,\Lambda_o)$ along Q . The theory of the constant term implies then various results on $r_Q^*([E(\phi,\Lambda_o)])$ (cf. [27], 1.10 resp. 4.7.). We recall the following important case: If Q is associated to P then by definition the finite set $W(A_P,A_Q)$ of isomorphisms of A_P onto A_Q induced by inner automorphisms of G defining a $\mathbb{Q}$-isomorphism of $\underline{M}_P(\mathbb{R})$ onto $\underline{M}_Q(\mathbb{R})$ is not empty, and we have

(1) $$r_Q^*([E(\phi,\Lambda_o)]) = \sum_{s \in W(A_P,A_Q)} [\underline{\underline{c}}(s,\Lambda_o)_{s\Lambda_o}(\phi_{\Lambda_o})]|_{e'(Q)}$$

where $\underline{\underline{c}}(s,\Lambda_o)_{s\Lambda_o} : \Omega^*(\Gamma_P\backslash X,E) \to \Omega^*(\Gamma_Q\backslash X,E)$ is a certain "intertwining" operator precisely defined in [27], 4.10. We point out that if (P,A_P) resp. (Q,A_Q) are standard (but the argument extends easily to the general case) a summand $[\underline{\underline{c}}(s,\Lambda_o)_{s\Lambda_o}(\phi_{\Lambda_o})]|_{e'(Q)}$ in (1) is a cohomology class in

$H^*_{cusp}(e'(Q),E_\lambda)$ of weight $v_s(\rho+\lambda)-\rho|_{\underline{a}_Q}$ where v_s is a uniquely determined element in W^Q with (cf. [27], 4.10)

(2) $$v_s(\rho+\lambda)|_{\underline{a}_Q} + s\Lambda_o = 0 \quad \text{resp.} \quad \chi_{s_\pi} = \chi_{-v_s(\rho+\lambda)|_{\underline{b}_{Q,\mathbb{C}}}} .$$

Here we write ${}^s\pi \in {}^o\hat{M}_Q$ for the image of $\pi \in {}^o\hat{M}_P$ under the bijection of ${}^o\hat{M}_P$ onto ${}^o\hat{M}_Q$ induced by $s \in W(A_P,A_Q)$.

§ 3 Cusp cohomology of arithmetic subgroups of $SL_n(\mathbb{Q})$

This section is mainly devoted to prove a vanishing result outside a certain range for the cuspidal cohomology $H^*_{cusp}(\Gamma\backslash X;E)$ of an arithmetic subgroup Γ of $SL_n(\mathbb{Q})$ with arbitrary coefficients E. The proof of this result involves to show that an irreducible unitary representation (π_o, H_{π_o}) of $G = SL_n(\mathbb{R})$ which contributes non-trivially to $H^*_{cusp}(\Gamma\backslash X;E) = \bigoplus_{\pi\in\hat{G}} m(\pi,\Gamma)H^*(\underline{g},K;H^\infty_\pi \otimes E)$ (i.e. one has $m(\pi_o,\Gamma) \neq 0$ and $H^*(\underline{g},K;H^\infty_{\pi_o} \otimes E) \neq \{0\}$) is tempered. This fact will be used in § 4.

3.1. We consider now the case of the $\mathbb{Q}$-split algebraic $\mathbb{Q}$-group $\underline{G} = SL_n/\mathbb{Q}$. Let P_o be the minimal parabolic $\mathbb{Q}$-subgroup of $G = \underline{G}(\mathbb{R})$ consisting of the upper triangular matrices, and let T_o be the torus of diagonal matrices. An element in T_o is denoted by $\mathrm{diag}(t_i)$. We choose as maximal compact subgroup $K = SO(n)$. Then $A_o = \{\mathrm{diag}(t_i) \in T_o \mid t_i > 0\}$ is the split component of T_o which is stable with respect to the Cartan involution θ_K. We put $\Phi = \Phi(\underline{g}_{\mathbb{C}}, \underline{a}_{o_{\mathbb{C}}})$, and denote by Δ (resp. Φ^+) the set of simple (resp. positive) roots with respect to the chosen ordering, i.e. we have $\Delta = \{\alpha_i \mid i = 1,\dots,n-1\}$, where α_i denotes the usual mapping t_i/t_{i+1} on T_o. The Weyl group W of $\underline{g}_{\mathbb{C}}$ with respect to $\underline{a}_{o_{\mathbb{C}}}$ is generated by the simple reflections w_i associated to α_i. Since $\underline{G}$ is split over $\mathbb{Q}$ we may (and will) identify Φ and $\Phi_{\mathbb{R}}$.

The conjugacy classes of parabolic $\mathbb{Q}$-subgroups of G are parametrized by subsets J of Δ. In particular, if Q is a maximal parabolic $\mathbb{Q}$-subgroup of G, then it is conjugate to a standard maximal parabolic $\mathbb{Q}$-subgroup P_j given by

$$(1) \qquad P_j := P_{\Delta-\{\alpha_j\}} = \{(a_{ik}) \in G \mid a_{ik} = 0 \,, k \leq j < i\} \qquad j = 1,\dots,n-1 \,.$$

We say that Q <u>is of type</u> j . If we put

$$T_{\Delta-\{\alpha_j\}} = \Big(\bigcap_{\substack{\alpha_i\in\Delta\\ i\neq j}} \ker \alpha_i\Big)^{\circ}$$

we have $P_j = Z(T_{\Delta-\{\alpha_j\}})\cdot N_{P_j}$. The Θ_K-stable split component A_j of P_j is given by

(2) $$A_j = \left\{ \begin{pmatrix} a^{-1} & & & & \\ & \ddots & & & \\ & & a^{-1} & & \\ & & & b & \\ & & & & \ddots \\ & & & & & b \end{pmatrix} j \;\middle|\; b = a^{\frac{j}{n-j}},\ a > 0\ ,\ a \in \mathbb{R} \right\} .$$

We let $M_j := Z(A_j)$ and have the Langlands decomposition $P_j = {}^{O}M_j \cdot A_j \cdot N_j$ where we abbreviated $N_j = N_{P_j}$. Note that we have $\Delta_{M_j} = \{\alpha_i \in \Delta \mid \alpha_i \neq \alpha_j\}$.

We define the element w_o in $\underline{G}(\mathbb{Q})$ by

(3) $$w_o := \begin{pmatrix} 0 & & 1 \\ & 1 & \\ -1 & & 0 \end{pmatrix} .$$

Since w_o conjugates A_j into A_{n-j} we see that P_j is associated to P_{n-j} , but not to any P_i , $i \neq n-j$ $(j = 1,\dots,n-1)$. Thus the class $C(P_j)$ of maximal parabolic $\mathbb{Q}$-subgroups Q associated to P_j consists out of groups Q of type j and $n-j$. If $n = 2m$ is even, there is exactly one associate class, namely $C(P_m)$ whose elements Q are conjugate to its opposite $\overline{Q} = Q^{opp}$. It follows that we have in this case

(4) $$W(A_m) = \{1, \mathrm{Int}\, w_o\} \qquad \text{for} \quad n = 2m ,$$

otherwise P_j is not conjugate to $\overline{P}_j$, and we have

(5) $$W(A_j) = \{1\} \qquad j = 1,\dots,n-1,\ \ j \neq m \ \text{for}\ n = 2m .$$

Note that there are always $[n/2]$ associate classes of maximal parabolic $\mathbb{Q}$-subgroups of $G = SL_n(\mathbb{R})$.

<u>3.2.</u> Let Γ be a torsionfree arithmetic subgroup of $\underline{G}(\mathbb{Q}) = SL_n(\mathbb{Q})$; accordingly we will denote $G = SL_n(\mathbb{R})$, $\underline{g} = sl_n(\mathbb{R})$, $K = SO(n)$ etc. By 1.5.(3) the cusp cohomology of Γ decomposes into a finite direct sum

$$H^*_{cusp}(\Gamma\backslash X;E) = H^*(\underline{g},K;L^2_o(\Gamma\backslash G) \otimes E) = \bigoplus_{\pi \in \hat{G}} m(\pi,\Gamma) H^*(\underline{g},K;H^\infty_\pi \otimes E)$$

where $\pi \in \hat{G}$ runs over the finite set of equivalence classes of irreducible unitary representations of G whose infinitesimal character χ_π is equal to the infinitesimal character χ_{τ^*} of the representation (τ^*,E^*) contragredient to the given finite-dimensional one (τ,E) of G . In order to establish a vanishing theorem for $H^*_{cusp}(\Gamma\backslash X;E)$ in certain degrees the following result is decisive.

<u>3.3.</u> THEOREM. - <u>Given an arithmetic subgroup</u> Γ <u>of</u> $SL_n(\mathbb{Q})$ <u>and a rational finite dimensional representation</u> (τ,E) <u>of</u> $G = SL_n(\mathbb{R})$ <u>there is (up to equivalence) at most one (resp. two for</u> n <u>even) irreducible unitary representation</u> (π_o,H_{π_o}) <u>of</u> G <u>such that</u> $H^*(\underline{g},K;H^\infty_{\pi_o} \otimes E) \neq \{0\}$ <u>and</u> (π_o,H_{π_o}) <u>occurs with non-zero multiplicity</u> $m(\pi_o,\Gamma)$ <u>in the cuspidal spectrum</u> $L^2_o(\Gamma\backslash G)$. <u>Such a representation is necessarily tempered</u>.

<u>Remarks</u>: (1) A precise description of such a representation (π_o,H_{π_o}) is given in the proof.

(2) Analogous results can be worked out for $GL_n(\mathbb{R})$, $SL^{\pm}_n(\mathbb{R})$ along the same lines of arguments.

(3) It seems that the last assertion is not correct in general for other groups than the above (cf. [34]).

<u>Proof of 3.3.</u> Let (π,H_π) be an irreducible unitary representation of $SL_n(\mathbb{R})$ which occurs with non-trivial multiplicity in the right regular

representation of $SL_n(\mathbb{R})$ on the space $L_o^2(\Gamma\backslash SL_n(\mathbb{R}))$ of cuspidal functions on $\Gamma\backslash SL_n(\mathbb{R})$. By [28] (π, H_π) admits a so-called Whittaker-model. Now Kostant ([19], Thm. E) has shown that the irreducible admissible representations of $SL_n(\mathbb{R})$ which admit a Whittaker-model are precisely the large representations of $SL_n(\mathbb{R})$ in the sense of Vogan ([30], § 6). (This result was also independently obtained by Casselman and Zuckerman.) Using the characterization given in Theorem 6.2. in [30] an irreducible (unitary) representation of $SL_n(\mathbb{R})$ which admits a Whittaker-model is therefore infinitesimally equivalent to a representation of the form

(1) $\quad \mathrm{Ind}_{P,\sigma,\nu}$

where $P = {}^oMAN$ is a cuspidal parabolic subgroup of $SL_n(\mathbb{R})$ containing the minimal parabolic subgroup P_o (cf. 3.1.), $\sigma \in {}^o\hat{M}$ a discrete series representation of ${}^oM$ and ν a character on A. In particular, up to equivalence the data can be given as

$$ {}^oM = \left\{ \begin{pmatrix} m_1 & & & 0 \\ & \ddots\, m_i & & \\ & & \ddots & \\ 0 & & & \pm 1 \end{pmatrix} \in SL_n(\mathbb{R}) \;\middle|\; \begin{array}{l} m_i \in SL_2^{\pm}(\mathbb{R}) \\ i = 1,\dots,[\frac{n}{2}] \end{array} \right\} \quad \text{for } n \text{ odd} $$

resp.

$$ {}^oM = \left\{ \begin{pmatrix} m_1 & & & 0 \\ & \ddots & & \\ & & m_i & \\ 0 & & & \ddots \end{pmatrix} \in SL_n(\mathbb{R}) \;\middle|\; \begin{array}{l} m_i \in SL_2^{\pm}(\mathbb{R}) \\ i = 1,\dots,[\frac{n}{2}] \end{array} \right\} \quad \text{for } n \text{ even} $$

and $\sigma = \otimes\, \sigma_i$ is (up to index 2 for n even) a tensorproduct of discrete series representations of the various copies of $SL_2^{\pm}(\mathbb{R})$. In order to see which ones of these representations (1) have non-trivial $(\underline{g},K)$-cohomology with coefficients in (τ,E) we use the results of Vogan-Zuckerman [33].

Depending on the highest weight λ of the given finite dimensional representation (τ, E) of $SL_n(\mathbb{R})$ they have exhibited a certain collection $\{A_{\mathfrak{q}}(\lambda)\}$ of irreducible representations of $SL_n(\mathbb{R})$ (more precisely, of $sl_n(\mathbb{C})$-modules which are parametrized by θ-stable parabolic subalgebras of $sl_n(\mathbb{C})$ in the sense of [33] § 2) such that for any irreducible unitary representation of $SL_n(\mathbb{R})$ having non-zero cohomology with coefficients in E the associated $(\mathfrak{g},K)$-module is contained in that collection (Thm. 5.6. [33]) and vice-versa as recently proved by Vogan [32]. In order to prove our assertion we check the representations in (1) versus the collection $\{A_{\mathfrak{q}}(\lambda)\}$ by means of their characterization in terms of Langlands-parameters given in [33], § 6.

For simplicity we treat the case of untwisted coefficients (i.e. $E \cong \mathbb{C}$) first. Theorem 3.3. in [10], III gives now necessary conditions on the data ν , χ_σ and λ which have to be satisfied if the Lie algebra cohomology of $\mathrm{Ind}_{P,\sigma,\nu}$ is non-trivial. Using this and Theorem 6.16 in [33] it turns out for $\mathrm{Ind}_{P,\sigma,\nu}$ as in (1) that in order to have non-zero cohomology $H^*(\mathfrak{g},K;\mathrm{Ind}_{P,\sigma,\nu} \otimes \mathbb{C})$ the character ν has to be the trivial character and σ_i the discrete series representation of $SL_2^{\pm}(\mathbb{R})$ of lowest $O(2)$-type $n-2i+2$ $(i = 1,\ldots,[\frac{n}{2}])$. Note that for n even ${}^{o}M$ has index two in the product of $SL_2^{\pm}(\mathbb{R})$'s , so there are two possible representations $\sigma \in {}^{o}\hat{M}$ in that case. Since $\mathrm{Ind}_{P,\sigma,\nu}$ is unitarily induced from a discrete series representation it is a tempered representation.

Now take E an arbitrary finite dimensional rational representation of highest weight λ . For most E there will be <u>no</u> irreducible unitary representations with non-trivial cohomology. By Theorem 5.3 [33] (in particular, by formula 5.1.(a)) a necessary condition in order to have $H^*(\mathfrak{g},K;H_\pi \otimes E_\lambda)$ non-zero is that if we set $\beta_i = e_i - e_{n-i}$ $(i = 1,\ldots,[\frac{n}{2}])$ then the projection of λ perpendicular to the span of the β_i must be purely imaginary. If this condition is satisfied the same theorem and the discussion above provide (up to equivalence) a unique (resp. two for n even) irreducible unitary representations (π_o, H_{π_o}) of $SL_n(\mathbb{R})$ with $H^*(\underline{g}, K; H^\infty_{\pi_o} \otimes E_\lambda) \neq \{0\}$ and a Whittaker-model. It is of the form $\mathrm{Ind}_{P,\sigma,o}$, P as above in (1), $\sigma = \otimes \sigma_i \in {}^o\hat{M}$ where σ_i $(i = 1,\ldots,[\frac{n}{2}])$ is the discrete series representation of $SL_2^\pm(\mathbb{R})$ of lowest $O(2)$-type $n-2i+2+n_i$ for some integers n_i (all of the same parity) depending on λ. We omit to give the precise formula since it will not be used lateron. In particular, (π_o, H_{π_o}) is a tempered representation.

3.4. As tempered ones these representations (π_o, H_{π_o}) in 3.3. have trivial $(\underline{g}, K)$-cohomology outside a certain range of length $\mathrm{rk}\, SL_n(\mathbb{R}) - \mathrm{rk}\, SO(n)$ according to Proposition III, 5.3. in [10]. We will determine the exact bounds of this range in terms of n.

As in [10], III, 4.3. we put for a Lie group L with finitely many connected components and with reductive Liealgebra $\mathrm{Lie}(L)$ and a maximal compact subgroup Q of L

(1) $$2q(L) = \dim L - \dim Q$$

(2) $$\ell_o(L) = \mathrm{rk}\, L - \mathrm{rk}\, Q$$

and we define then

(3) $$2q_o(L) = 2q(L) - \ell_o(L) \quad .$$

Note that $q_o(L)$ is an element in $\mathbb{Z}$.

If L is now of connected type with finite center (in the sense of [10], O, 3.1.) then we have for an irreducible tempered $(\mathrm{Lie}(L), Q)$-module V

and a rational finite dimensional representation (τ,E) of L that (cf. [10], III, 5.3.)

(4) $$H^q(\mathrm{Lie}(L),Q;V \otimes E) = 0 \quad \text{for} \quad q \notin [q_o(L),q_o(L) + \ell_o(L)]$$
$$q < rk_{\mathbb{R}} L, q > 2q(L) - rk_{\mathbb{R}} L$$

We apply this now to the case $L = SL_n(\mathbb{R})$ and $Q = SO(n)$. Then

(5) $$2q(SL_n(\mathbb{R})) = n^2-1 - \frac{n(n-1)}{2} = \dim(SL_n(\mathbb{R})/SO(n))$$
$$\ell_o(SL_n(\mathbb{R})) = n-1 - [\tfrac{n}{2}]$$

(where [ ] denotes the Gauss-bracket) and therefore we get

(6) $$q_o(SL_n(\mathbb{R})) = 1/2\ (\dim X - \ell_o(SL_n(\mathbb{R})) = \begin{cases} (1/4)n^2 & n = 2m \\ (1/4)(n^2-1) & n = 2m+1 \end{cases}$$

resp.

(7) $$q_o(SL_n(\mathbb{R})) + \ell_o(SL_n(\mathbb{R})) = \begin{cases} \frac{(n+1)^2-1}{4} - 1 & n = 2m \\ \frac{(n+1)^2}{4} - 1 & n = 2m+1 \end{cases}.$$

We abbreviate the values in the second equation of (6) by $C_u(n)$ and the values in (7) by $C_o(n)$, i.e. we put

(8) $$C_u(n) = q_o(SL_n(\mathbb{R})) \quad \text{resp.} \quad C_o(n) = q_o(SL_n(\mathbb{R})) + \ell_o(SL_n(\mathbb{R})).$$

One sees that the intervall $[C_u(n),C_o(n)]$ of length $rk\ SL_n(\mathbb{R}) - rk\ SO(n)$ is concentrated around the middle dimension $(1/2) \dim X$.

As a consequence of 3.3. we obtain now

<u>3.5.</u> PROPOSITION. - <u>Let</u> Γ <u>be an arithmetic subgroup of</u> $SL_n(\mathbb{Q})$, <u>and</u> (τ,E) <u>a rational finite dimensional representation of</u> $SL_n(\mathbb{R})$. <u>Then one has for the cusp cohomology</u> $H^*_{cusp}(\Gamma\backslash X;E)$ <u>of</u> Γ <u>with coefficients in</u> E

<u>the following vanishing result</u>

$$H^q_{cusp}(\Gamma\backslash X;E) = 0 \qquad \text{for} \qquad q \notin [c_u(n), c_o(n)]$$

<u>where</u>

$$c_u(n) = \begin{cases} \frac{n^2}{4} & n \ \underline{\text{even}} \\ \frac{n^2-1}{4} & n \ \underline{\text{odd}} \end{cases}$$

<u>resp.</u>

$$c_o(n) = \begin{cases} \frac{(n+1)^2-1}{4} - 1 & n \ \underline{\text{even}} \\ \frac{(n+1)^2}{4} - 1 & n \ \underline{\text{odd}} \end{cases}$$

§ 4 Holomorphy of certain Eisenstein series at special points and corresponding cohomology classes

Given an arithmetic subgroup $\Gamma \subset SL_n(\mathbb{Z})$, $n > 2$, we consider now Eisenstein series $E(\phi,\Lambda)$ associated to cuspidal forms ϕ on faces $e'(P)$ corresponding to a Γ-conjugacy class of maximal parabolic $\mathbb{Q}$-subgroups of $G = SL_n(\mathbb{R})$. We discuss the question of holomorphy of $E(\phi,\Lambda)$ at special points, the construction of corresponding cohomology classes and its behaviour under the various restrictions to the cohomology of the faces in $\partial(\Gamma\backslash\overline{X})$. This study allows us to construct the subspaces $H^*_{C(P)}(\Gamma\backslash X,\mathbb{C})$ in $H^*(\Gamma\backslash X;\mathbb{C})$ and to obtain the results on them described in § 0.

4.1. Let P be a maximal parabolic $\mathbb{Q}$-subgroup of $G = SL_n(\mathbb{R})$ with split component $A_P = A$ and Langlands decomposition $P = {}^oMAN$. Without loss of generality we may (and will) assume that (P,A_P) is standard i.e. is the group of real points of a standard maximal parabolic $\mathbb{Q}$-subgroup $\underline{P}_{\Delta-\{\alpha\}}$ of $SL_n(\mathbb{Q})$, $\alpha \in \Delta$ and $A_P \subset A_o$. Note that $\dim A = 1$ and that we may identify $\underline{a}^*_{\mathbb{C}}$ with $\mathbb{C}$ in a natural way by the condition $\rho_P = 1$. Let $[\phi] \neq 0$ be a cuspidal cohomology class in $H^*_{cusp}(e'(P),\mathbb{C})$ of type (π,w), $\pi \in {}^o\hat{M}$, $w \in W^P$. We consider the Eisenstein series $E(\phi,\Lambda)$ associated to $[\phi]$ by 2.5.. It is first defined for Λ in the region

$$\begin{aligned}(\underline{a}^*_{\mathbb{C}})^+ &= \{\Lambda \in \underline{a}^*_{\mathbb{C}} \mid \operatorname{Re}\Lambda \in \rho_P + (\underline{a}^*)^+\} \\ &= \{\Lambda \in \underline{a}^*_{\mathbb{C}} \mid (\operatorname{Re}\Lambda,\alpha) > (\rho_P,\alpha),\ \alpha \in \Delta(P,A)\}\end{aligned}$$

but it admits a meromorphic extension to all of $\underline{a}^*_{\mathbb{C}}$. By a general result (cf. [15] IV, § 7, Thm 7 resp. Lemmata 98, 99) the possible poles of $E(\phi,\Lambda)$ for arbitrary Λ with $(\operatorname{Re}\Lambda,\alpha) > 0$ can only occur in the real intervall

$$I = \{\Lambda \in \underline{a}^*_{\mathbb{C}} \mid \operatorname{Im} \Lambda = 0 , (\rho_P,\alpha) \geq (\operatorname{Re} \Lambda,\alpha) \geq 0\} ,$$

are simple, and there are only finitely many poles in I. The other possible poles of $E(\phi,\Lambda)$ lie in the region $\{\Lambda \in \underline{a}^*_{\mathbb{C}} \mid (\operatorname{Re} \Lambda,\alpha) < 0 , \alpha \in \Delta(P,A)\}$. Since we are interested by 2.6. in evaluating $E(\phi,\Lambda)$ at the point $\Lambda_o = -w(\rho)|_{\underline{a}}$ the following lemma by which Λ_o is expressed as a multiple of α is useful.

4.2. LEMMA. - Let $P = P_{\Delta-\{\alpha\}}$ be a standard maximal parabolic $\mathbb{Q}$-subgroup of $SL_n(\mathbb{R})$ with split component A. For $w \in W^P$, one has

(1) $$\Lambda_o = -w(\rho)|_{\underline{a}} = (-\frac{\dim N}{2} + \ell(w)) \cdot \alpha|_{\underline{a}} .$$

The Lie algebra cohomology $H^*(n,\mathbb{C})$ is an ${}^oM{\cdot}A$-module and as such can be decomposed according to the weights with respect to A. The A-weights are determined by a theorem of Kostant recalled in 2.3., 2.4.. For $w \in W^P$, the weight $w(\rho)-\rho|_{\underline{a}}$ occurs in $H^{\ell(w)}(\underline{n},\mathbb{C})$ where $\ell(w)$ denotes the length of w. Since $H^*(\underline{n},\mathbb{C})$ is naturally embedded as a $({}^o\underline{m} \oplus \underline{a})$-stable summand of $\Lambda^*\underline{n}^* \otimes \mathbb{C}$ ([18], 5.7.) $w(\rho)-\rho|_{\underline{a}}$ is among the weights for $\Lambda^{\ell(w)}\underline{n}^* \otimes \mathbb{C}$ under the natural action of A. The simple root α in $\Delta(P,A)$ occurs in each rootspace of N with multiplicity one, hence we have

(2) $$w(\rho)-\rho|_{\underline{a}} = -\ell(w)\alpha|_{\underline{a}}$$

resp.

(3) $$\rho|_{\underline{a}} = (1/2) \sum_{\beta \in \phi^+} \beta|_{\underline{a}} = (1/2)\dim N \cdot \alpha|_{\underline{a}}$$

and formula (1) follows.

Remark: Since we know already by 2.6. that $\Lambda_o = -w(\rho)|_{\underline{a}}$ is real for a given cuspidal class of type (π,w), $\pi \in {}^o\hat{M}$, $w \in W^P$, 4.2. shows that

Λ_o varies between ρ_P and $-\rho_P$ on the "real axis" and that Λ_o lies in the real intervall I if $\ell(w) \geq (1/2) \cdot \dim N$.

One of the main steps in dealing with the question of holomorphy of $E(\phi,\Lambda)$ at the special point Λ_o is the following result which says, roughly spoken, that each unitary representation of ${}^oM$ which contributes non-trivially to the cusp cohomology $H^*_{cusp}(e'(P),\mathbb{C})$ of the face $e'(P)$ has to be tempered.

4.3. PROPOSITION. - *Let* $P = {}^oMAN$ *be a maximal parabolic* $\mathbb{Q}$-*subgroup of* $SL_n(\mathbb{R})$.

(1) *If* $\pi_o \in {}^o\hat{M}$ *is an irreducible unitary representation of* ${}^oM$ *such that there exists a non-trivial cuspidal cohomology class* $[\phi] \in H^*_{cusp}(e'(P),\mathbb{C}) = \bigoplus_{\pi \in {}^o\hat{M}} \bigoplus_{w \in W^P} H^*({}^o\underline{m},K_M;V_\pi \otimes F_{w(\rho)-\rho})$ *of type* (π_o,w) *for some* $w \in W^P$, *then* π_o *is a tempered representation.*

(2) *The following vanishing result for the cusp cohomology of* $e'(P)$ *holds*

$$H^q_{cusp}(e'(P),\mathbb{C}) = 0 \quad \text{for} \quad q \notin [q_o({}^oM), q_o({}^oM) + \ell_o({}^oM) + \dim N]$$

with $q_o(\ )$ *resp.* $\ell_o(\ )$ *as defined in* 3.4..

The proof reduces more or less to the arguments given in the proof of 3.3.. We may assume that P is a standard maximal parabolic $\mathbb{Q}$-subgroup of type j . Then ${}^oM$ is a subgroup of index 2 in the direct product $\tilde{M} = SL^{\pm}_j(\mathbb{R}) \times SL^{\pm}_{n-j}(\mathbb{R})$. Given an irreducible unitarv $({}^o\underline{m},K_M)$-module (σ,H_σ) there exists an irreducible unitary $({}^om,K_{\tilde{M}})$-module $(\tilde{\sigma},H_{\tilde{\sigma}})$ such that $\tilde{\sigma}$ viewed as $({}^o\underline{m},K_M)$-module is isomorphic to σ or decomposes into a direct sum $\sigma \oplus \sigma'$. One can describe $\tilde{\sigma}$ with the help of the induced module $\mathrm{Ind}_{(\underline{m},K_M)}^{(\underline{m},K_{\tilde{M}})}(\sigma,H_\sigma) =: \mathrm{Ind}\,\sigma$ (cf. 0.3. 25. in [31]), and the first (resp. second) case corresponds to the fact that the induced module $\mathrm{Ind}\,\sigma$ is reducible (resp. irreducible). If now (σ,H_σ) has non-zero cohomology with coefficients F then also $(\tilde{\sigma},H_{\tilde{\sigma}})$ has non-zero $(\underline{m},K_{\tilde{M}})$-cohomology. This follows from the results of Vogan-Zuckerman [33] and Vogan [31], [32]. Depending on F they have constructed a certain collection $\{A_{\underline{q}}(F)\}$ of irreducible unitary $({}^{o}\underline{m},K_M)$-modules such that each irreducible unitary $({}^{o}\underline{m},K_M)$-module with non-zero cohomology with coefficients F is contained in that list. Using this description of (σ,H_σ) (resp. $(\tilde{\sigma},H_{\tilde{\sigma}})$), the alternative characterization of the $A_{\underline{q}}(F)$ by means of the Zuckerman-functor ([31], 6.3) our claim follows from the computation of the cohomology of the modules with coefficients in F in [31] 6.3.4. (cf. also 5.5 in [33]).

This discussion applies to the irreducible unitary $({}^{o}\underline{m},K_M)$-module corresponding to the irreducible unitary $\pi_o \in {}^{o}\hat{M}$ with non-trivial multiplicity $m(\pi_o,\Gamma_M)$ in $L_o^2(\Gamma_M\backslash{}^{o}M)$ and non-zero cohomology $H^*({}^{o}\underline{m},K_M;V_{\pi_o} \otimes F_{w(\rho)-\rho})$. But now one has that $\tilde{\pi}_o$ has to have a Whittaker model; therefore the same line of arguments as in the proof of 3.3. shows that the representation $\tilde{\pi}_o$ of $\tilde{M} = SL_j^{\pm}(\mathbb{R}) \times SL_{n-j}^{\pm}(\mathbb{R})$ which has non-zero cohomology and admits a Whittaker model is unitarily induced from a discrete series representation, and hence also π_o is tempered. This proves (1).

We recall $H^*_{cusp}(e'(P),\mathbb{C}) = \bigoplus_{\pi \in {}^{o}\hat{M}} H^*({}^{o}\underline{m},K_M;V_\pi \otimes H^*(\underline{n},\mathbb{C}))$. Then assertion (2) follows from 3.4.(4) and (1).

<u>Remarks</u>: (1) In the same way one sees, that for a fixed $w \in W^P$ one has

$$H^q_{cusp}(\Gamma_M\backslash Z_M,F_{w(\rho)-\rho}) = 0 \quad \text{for} \quad q \notin [q_o({}^{o}M),q_o({}^{o}M) + \ell_o({}^{o}M)]$$

(2) An analogue for an arbitrary parabolic $\mathbb{Q}$-subgroup of $G = SL_n(\mathbb{R})$ is proved by similar arguments.

4.4. THEOREM. - Let $\Gamma \subset SL_n(\mathbb{Z})$ be a torsionfree subgroup of finite index, $n > 2$; let P be a maximal parabolic $\mathbb{Q}$-subgroup of $SL_n(\mathbb{R})$ with Langlands decomposition $P = {}^oMAN$ as in 2.1., and suppose that P is not of type m if $n = 2m$. Let

$$[\phi] \in H^*_{cusp}(e'(P),\mathbb{C}) = \bigoplus_{\pi \in {}^o\hat{M}} \bigoplus_{w \in W^P} H^*({}^o\underline{m},K_M;V_\pi \otimes F_{w(\rho)-\rho})$$

be a non-trivial cuspidal cohomology class of type (π,w), $\pi \in {}^o\hat{M}$, $w \in W^P$ and $\deg[\phi] = p = q+\ell(w)$. We have

(1) If $\ell(w) \geq 1/2(\dim N)$, then the Eisenstein series $E(\phi,\Lambda)$, $\Lambda \in \underline{a}^*_{\mathbb{C}}$, associated to $[\phi]$ is holomorphic at $\Lambda_o = -w(\rho)|_{\underline{a}}$. The Eisenstein form $E(\phi,\Lambda_o) \in \Omega^p(\Gamma\backslash X;\mathbb{C})$ is closed and harmonic and represents a non-trivial cohomology class in $H^p(\Gamma\backslash X;\mathbb{C})$ (called regular Eisenstein cohomology class).

Let Q be an arbitrary parabolic $\mathbb{Q}$-subgroup of $SL_n(\mathbb{R})$, $Q \neq SL_n(\mathbb{R})$, and denote by r^*_Q the restriction of $H^*(\Gamma\backslash\bar{X};\mathbb{C})$ on the cohomology $H^*(e'(Q),\mathbb{C})$ of the corresponding face $e'(Q)$ in the boundary of $\Gamma\backslash\bar{X}$. Then we have if $\ell(w) \geq 1/2(\dim N)$ (where " $\underset{H}{\sim}$ " denotes conjugation by $H \subset SL_n(\mathbb{R})$).

(2) for $Q \underset{\Gamma}{\sim} P$ $\quad r^*_Q([E(\phi,\Lambda_o)]) = [\phi]$

(3) for $Q \underset{\Gamma}{\not\sim} P$ and $Q \underset{SL_n(\mathbb{Q})}{\not\sim} P^{opp}$ $\quad r^*_Q([E(\phi,\Lambda_o)]) = 0$

(4) for $Q \underset{SL_n(\mathbb{Q})}{\sim} P^{opp}$ $\quad r^*_Q([E(\phi,\Lambda_o)]) = 0$

if $\deg[\phi] = q+\ell(w) > q_o({}^oM)+\ell_o({}^oM)+(\dim N - \ell(w))$

and $\quad r^*_Q([E(\phi,\Lambda_o)]) = [\underline{c}(s,\Lambda_o)_{s\Lambda_o}(\phi_{\Lambda_o})]|_{e'(Q)}$

otherwise where $s \in W(A_P,A_Q)$ (cf. 2.7.).

<u>Proof of 4.4.</u> <u>ad</u> (1): Let $0 \neq [\phi] \in H^P_{cusp}(e'(P),\mathbb{C})$ of type (π,w), $\pi \in {}^{o}\hat{M}$, $w \in W^P$. The representation $\pi \in {}^{o}\hat{M}$ is tempered by 4.3.(1), Lemma 4.2. and the assumption $\ell(w) \geq 1/2(\dim N)$ imply that the point $\Lambda_o = -w(\rho)|_{\underline{a}} = (-(1/2)\dim N + \ell(w))\alpha|_{\underline{a}}$ lies in the real intervall

$$I = \{\Lambda \in \underline{a}^*_{\mathbb{C}} \mid \operatorname{Im} \Lambda = 0 , (\rho_P,\alpha) \geq (\operatorname{Re} \Lambda,\alpha) \geq 0\}$$

where the Eisenstein series $E(\phi,\Lambda)$ has only finitely many possible poles and these are simple. But in fact, since P is not symmetric (in the sense of [17] Thm. 7) and π is tempered $E(\phi,\Lambda)$ has no poles in I as shown in 6.4.(1) [27] under these assumptions. The argument given in [27] relies on the fact that the Langlands quotient $J(P,\pi,\Lambda_o)$ associated to the given data (P,π,Λ_o) is not unitarizable as $(\underline{g},K)$-module. It follows that $E(\phi,\Lambda)$ is holomorphic in $\Lambda_o = -w(\rho)|_{\underline{a}}$. The other assertions in (1) are given by 2.6. .

<u>ad</u> (2): We recall (cf. 2.7. or [27], 1.10.) that the restriction of the class $[E(\phi,\Lambda_o)]$ under r^*_Q is equal to the restriction to $e'(Q)$ of the class $[E(\phi,\Lambda_o)_Q]$ represented by the constant Fourier coefficient $E(\phi,\Lambda_o)_Q \in \Omega^*(\Gamma_Q\backslash X,\mathbb{C})$ of $E(\phi.\Lambda_o)$ with respect to Q, i.e. we have

$$r^*_Q([E(\phi,\Lambda_o)]) = [E(\phi,\Lambda_o)_Q]|_{e'(Q)} \quad .$$

If we consider now the case that Q is not associated to P and take into account that $\operatorname{prk}(Q) \geq \operatorname{prk}(P) = 1$ then $r^*_Q([E(\phi,\Lambda_o)] = 0$ because the constant Fourier coefficient $E(\phi,\Lambda_o)_Q$ vanishes identically (cf. [27] 4.11.(2) resp. Corollary 2 to Lemma 33 in [15]). If we assume now that Q is associated to P we know (cf. 2.7.)

$$(5) \qquad r^P_Q([E(\phi,\Lambda_o)]) = \sum_{s \in W(A_P,A_Q)} [\underline{\underline{c}}(s,\Lambda_o)_{s\Lambda_o}(\phi_{\Lambda_o})]|_{e'(Q)} \quad .$$

In dealing with the terms on the right hand side we have to distinguish three cases:

(i) Q <u>is</u> Γ-<u>conjugate to</u> P : This implies $e'(Q) = e'(P)$ (by 7.7.(1) in [9]) and we can assume $P = Q$. Since $W(A_P) = \{1\}$ by 3.1.(5) the sum on the right hand side of (5) reduces to the term

$$r_Q^P([E(\phi,\Lambda_o)]) = [\underline{c}(1,\Lambda_o)_{\Lambda_o}(\phi_{\Lambda_o})]|_{e'(Q)}$$

$$= [\phi_{\Lambda_o}]|_{e'(Q)}$$

$$= [\phi]$$

by 4.9. and 4.11.(6) in [27].

(ii) Q <u>is</u> $SL_n(\mathbb{Q})$-<u>conjugate</u>, <u>but not</u> Γ-<u>conjugate to</u> P : Therefore there is an element $g \in SL_n(\mathbb{Q})$ with $P^g = Q$, and $A_Q = A_P^g$ is a split component of Q . Since P is not of type m if $n = 2m$ we know that $W(A_P) = \{1\}$. Hence we have

$$W(A_P,A_Q) = gW(A_P) = \{\mathrm{Int}\ g_{|A_P}\}$$

i.e. the only element $s \in W(A_P,A_Q)$ is induced by an element $g \in SL_n(\mathbb{Q})$ with $P^g = Q$. As explained in 4.8. in [27] the intertwining operator $\underline{c}(s,\Lambda_o)_{s\Lambda_o}$ associated to s is given as a sum over terms which are parametrized by the set $\Gamma(s) = \Gamma \cap Pg^{-1}Q$. But under the assumptions made in this case the set $\Gamma(s)$ is empty. Indeed, let γ be an element in $\Gamma(s)$. Then $\gamma = pg^{-1}q$ with $p \in P$, $q \in Q$, and we would have

$$\gamma^{-1}P\gamma = q^{-1}gp^{-1}Ppg^{-1}q = Q ,$$

contradicting the fact that Q is not Γ-conjugate to P . It follows that

$r^*_Q([E(\phi,\Lambda_o)]) = 0$ also in this case. Observe that we have now proved assertions (2) and (3) completely.

(iii) Q <u>is not</u> $SL_n(\mathbb{Q})$-<u>conjugate to</u> P , <u>but is</u> $SL_n(\mathbb{Q})$-<u>conjugate to</u> P^{opp} : For simplicity we assume first $P = P_i$ and $Q = P_{n-i}$ for some i , $i = 1,\dots,n-1$. The element $w_o \in SL_n(\mathbb{Q})$ defined in 3.1.(3) satisfies $w_o A_i w_o^{-1} = A_{n-i}$, and we have (cf. 3.1.)

$$W(A_i,A_{n-i}) = \{\mathrm{Int}\ w_o|_{A_i}\} .$$

We will write s for the only element in $W(A_i,A_{n-i}\}$. As before, the restriction of $[E(\phi,\Lambda_o)]$ on $H^p(e'(P_{n-i}),\mathbb{C})$ is given by

$$(6) \qquad r^p_{P_{n-i}}([E(\phi,\Lambda_o)]) = [\underline{\underline{c}}(s,\Lambda_o)_{s\Lambda_o}(\phi_{\Lambda_o})]|_{e'(P_{n-i})} .$$

This is a cohomology class in $H^p_{cusp}(e'(P_{n-i}),\mathbb{C})$ of weight $v_s(\rho)-\rho|_{\underline{a}_{n-i}}$ where v_s is a uniquely determined element in $W^{P_{n-i}}$ with (cf. 2.7.)

$$(7) \qquad v_s(\rho)|_{\underline{a}_{n-i}} + s\Lambda_o = 0$$

and

$$(8) \qquad \chi_{s_\pi} = \chi_{-v_s(\rho)}|_{\underline{b}_{n-i,\mathbb{C}}}$$

We claim now that for $\Lambda_o = -w(\rho)|_{\underline{a}_i}$

$$(9) \qquad s\Lambda_o|_{\underline{a}_{n-i}} = -(\ell(w) - \frac{\dim N}{2})\alpha_{n-i}|_{\underline{a}_{n-i}}$$

Since $\alpha_i(w_o a w_o^{-1}) = -\alpha_{n-i}(a)$ for $a \in \underline{a}_{n-i}$ and $\Lambda_o = (\ell(w) - (1/2)\dim N_i)\alpha_i|_{\underline{a}_i}$ by 4.2. formula (9) is easily seen. This allows us to determine the weight $v_s(\rho)-\rho|_{\underline{a}_{n-i}}$. By condition (7) and formula (9) for $s\Lambda_o$ we get

$$(10) \qquad v_s(\rho)|_{\underline{a}_{n-i}} = -(-\ell(w) + (1/2)\dim N_i)\alpha_{n-i}|_{\underline{a}_{n-i}}$$

Using the identities $\rho|_{\underline{a}_{n-i}} = ((1/2)\dim N_{n-i})\cdot\alpha_{n-i}$ and $\dim N_i = \dim N_{n-i}$ we see

$$(11) \qquad v_s(\rho)-\rho|_{\underline{a}_{n-i}} = (\ell(w) - \dim N_{n-i})\alpha_{n-i}|_{\underline{a}_{n-i}}$$

Now we recall that as $({}^oM_{n-i}\cdot A_{n-i})$-module

$$H^t(\underline{n}_{n-i},\mathbb{C}) = \bigoplus_{\substack{v\in W^{P_{n-i}} \\ \ell(v) = t}} F_{v(\rho)-\rho}$$

and (cf. proof of 4.2.)

$$v(\rho)-\rho|_{\underline{a}_{n-i}} = \rho_{P_{n-i}} - \ell(v)\alpha_{n-i} - \rho_{P_{n-i}}$$

$$= -\ell(v)\alpha_{n-i}|_{\underline{a}_{n-i}} \quad .$$

Comparing this with (11) we see that v_s is an element in $W^{P_{n-i}}$ of length $(\dim N_{n-i}-\ell(w))$, i.e.

$$(12) \qquad \ell(v_s) = \dim N_{n-i} - \ell(w) \quad .$$

(We note that this last condition alone does not uniquely determine v_s as an example in the case SL_3 already shows.)

We assume now for the degree of $[\phi]$ that

$$(13) \qquad \deg[\phi] = p = q+\ell(w) > q_o({}^oM_i) + \ell_o({}^oM_i) + (\dim N_i-\ell(w)) \quad .$$

We observe that $q_o({}^oM_i) = q_o({}^oM_{n-i})$ resp. $\ell_o({}^oM_i) = \ell_o({}^oM_{n-i})$. Thus the degree of the restricted class $r^P_{P_{n-i}}([E(\phi,\Lambda_o)]) = [\underline{c}(s,\Lambda_o)_{s\Lambda_o}(\phi_{\Lambda_o})]|_{e'(P_{n-i})}$

is greater then $q_o({}^oM_{n-i}) + \ell_o({}^oM_{n-i}) + \ell(v_s)$ by (12). On the other hand we know that this class (indeed, the representing differential form $\underline{\underline{c}}(s,\Lambda_o)_{s\Lambda_o}(\phi_{\Lambda_o})$) is of weight $v_s(\rho)-\rho|_{\underline{a}_{n-i}}$ with $\ell(v_s) = \dim N_{n-i}-\ell(w)$ which can only possibly occur in $H^*_{cusp}(e'(P_{n-i}),\mathbb{C})$ in a range up to the degree $q_o({}^oM_{n-i}) + \ell_o({}^oM_{n-i}) + \ell(v_s)$ by the remark (1) after 4.3.. This implies that we have $r^P([E(\phi,\Lambda_o)]) = 0$.

For simplicity we have assumed $Q = P_{n-i}$; otherwise $Q = P^g_{n-i}$, $g \in SL_n(\mathbb{Q})$, $g \neq 1$ and $('Q_o,'A_o) := (P^g_o,A^g_o)$ is a minimal parabolic $\mathbb{Q}$-subgroup of G with respect to which (Q,A^g_i) is standard. Then $W(A_i,A_Q) = \{\mathrm{Int}\, g|_{A_{n-i}} \circ w_o\}$, and the argument runs exactly along the same lines by considering the weight of $[\underline{\underline{c}}(s,\Lambda_o)_{s\Lambda_o}(\phi_{\Lambda_o})]|_{e'(Q)}$. One only has to use the analogues $'W$, $'W^Q$ of W , $W^{P_{n-i}}$ given by fixing a new minimal parabolic $\mathbb{Q}$-subgroup of $SL_n(\mathbb{R})$.

4.5. The assumption made in assertion 4.4.(4) is rather technical. However, we can rephrase the statement in a slightly weaker but more convenient form. Since the cohomology classes $[\phi] \in H^*_{cusp}(e'(P),\mathbb{C})$ of type (π,w) considered in 4.4. satisfy the inequality $\ell(w) \geq (1/2)\dim N$ we can weaken the assumption in 4.4.(4) to

(1) $$\deg[\phi] = q+\ell(w) \geq q_o({}^oM) + \ell_o({}^oM) + [(1/2)\dim N] + 1 \quad .$$

The lower bound on the right hand side is independent of the chosen P and can be related to another bound associated to the cusp cohomology of Γ . This is implied by the following

LEMMA. - Let P be a maximal parabolic $\mathbb{Q}$-subgroup of $SL_n(\mathbb{R})$ with Langlands decomposition $P = {}^oMAN$. Then we have (cf. 3.4.(2), (3), (8) for notation)

(2) $$q_o({}^oM) + \ell_o({}^oM) + [\frac{\dim N}{2}] + 1 = C_o(n)$$

where $C_o(n)$ is the highest degree in which there is possibly a non-vanishing cusp cohomology class in $H^*_{cusp}(\Gamma\backslash X;\mathbb{C})$.

Let P be of type j ; then we have the following formulas

(3) $$2q({}^oM) = ((j^2-1)+(n-j)^2-1) - (\frac{j(j-1)}{2} + \frac{(n-j)(n-j-1)}{2})$$

(4) $$\ell_o({}^oM) = (j-1) + (n-j-1) - ([\frac{j}{2}] + [\frac{n-j}{2}])$$

and we obtain

(5) $$q_o({}^oM) = 2q({}^oM) - \ell_o({}^oM)$$

$$= 1/4(j^2+(n-j)^2-n+2([\frac{j}{2}] + [\frac{n-j}{2}])$$

One checks for $n = 2m+1$

(6) $$[\frac{j}{2}] + [\frac{n-j}{2}] = [\frac{n}{2}]$$

and for $n = 2m$

(7) $$[\frac{j}{2}] + [\frac{n-j}{2}] = \begin{cases} \frac{n}{2} & j \text{ even} \\ \frac{n}{2} - 1 & j \text{ odd} \end{cases}$$

Since one has for $n = 2m+1$, $m \geq 1$, that $\dim N$ is even and $[\frac{\dim N}{2}] = \frac{j(n-j)}{2}$ holds formulas (4), (5) and (6) imply

$$q_o({}^oM) + \ell_o({}^oM) + [\frac{\dim N}{2}] + 1 = (1/4)(n+1)^2 - 1$$

which is equal to $C_o(n)$ by 3.5..

The cases $n = 2m$, j even resp. j odd are similarly checked.

In the following we retain the notation and assumptions in Theorem 4.4. and 4.5., in particular we have the maximal parabolic $\mathbb{Q}$-subgroup P is not of type m if $n = 2m$.

4.6. COROLLARY. - Let $[\phi] \in H^*_{cusp}(e'(P),\mathbb{C})$ be a non-trivial cohomology class of type (π,w) , $\pi \in {}^o\hat{M}$, $w \in W^P$, and $\deg[\phi] = p \geq C_o(n)$, then the associated Eisenstein series evaluated at $\Lambda_o = -w(\rho)|_{\mathfrak{a}}$ represents a non-trivial cohomology class in $H^p(\Gamma\backslash X;\mathbb{C})$ whose restriction to a face $e'(Q)$ in $\partial(\Gamma\backslash\overline{X})$ is given by

$$(1) \qquad r_Q^P([E(\phi,\Lambda_o)]) = \begin{cases} [\phi] & \text{for } Q \underset{\Gamma}{\sim} P \\ 0 & \text{otherwise} \end{cases}$$

The non-trivial class $[\phi] \in H^q({}^o\underline{m},K_M;V_\pi \otimes F_{w(\rho)-\rho})$ of type (π,w) has degree

$$p = q+\ell(w) \geq C_o(n) = q_o({}^oM)+\ell_o({}^oM) + [(1/2)\dim N] + 1$$

by 4.5.. This implies that $\ell(w) \geq (1/2)\dim N$. Indeed, if $\ell(w) < (1/2)\dim N$, we would have $q > q_o({}^oM) + \ell_o({}^oM)$. But in this degree there is no non-trivial cusp cohomology with coefficients in $F_{w(\rho)-\rho}$ by remark (1) after 4.3..

4.7. THEOREM. - Let $\Gamma \subset SL_n(\mathbb{Z})$ be a torsionfree subgroup of finite index, $n > 2$; let P be a maximal parabolic $\mathbb{Q}$-subgroup of $SL_n(\mathbb{R})$, and denote its associate class of parabolic $\mathbb{Q}$-subgroups of $SL_n(\mathbb{R})$ by $C(P)$. We assume that P is not of type m if $n = 2m$. Let $H^*_{C(P)}(\Gamma\backslash X;\mathbb{C})$ be the subspace in $H^*(\Gamma\backslash X;\mathbb{C})$ which is generated by the regular Eisenstein cohomology classes $[E(\phi,\Lambda_o)]$ constructed by 4.4. for all Q in a set of representatives of $\Gamma\backslash C(P)$ and all non-trivial classes $[\phi] \in H^*_{cusp}(e'(Q),\mathbb{C})$

<u>of type</u> (π,w) <u>with</u> $\pi \in {}^{o}\hat{M}_Q$, $w \in W^Q$ <u>and</u> $\ell(w) \geq (1/2)\dim N_Q$. <u>Then</u>

(1) $$\dim H^*_{C(P)}(\Gamma\backslash X;\mathbb{C}) \geq (1/2)\dim \bigoplus_{Q \in \Gamma\backslash C(P)} H^*_{cusp}(e'(Q),\mathbb{C})$$

(2) <u>Under the restriction</u>

$$r^*_{C(P)} \;:\; H^*(\Gamma\backslash\bar{X},\mathbb{C}) \to \bigoplus_{Q \in \Gamma\backslash C(P)} H^*(e'(Q),\mathbb{C})$$

<u>the space</u> $H^q_{C(P)}(\Gamma\backslash X;\mathbb{C})$ <u>is mapped isomorphically onto</u> $\bigoplus_{Q \in \Gamma\backslash C(P)} H^q_{cusp}(e'(Q),\mathbb{C})$ <u>for</u> $q \geq C_o(n)$ (<u>we refer to</u> 3.4. <u>for the definition of</u> $C_o(n)$ <u>resp.</u> 4.5., 4.6.).

The space $H^q_{C(P)}(\Gamma\backslash X;\mathbb{C})$ is generated for $q \geq C_o(n)$ by regular classes $[E(\phi,\Lambda_o)]$, $[\phi] \in H^q_{cusp}(e'(Q),\mathbb{C})$, whose restrictions to the cohomology of a face $e'(R)$ are given by 4.6. as

(3) $$r^q_R([E(\phi,\Lambda_o)]) = \begin{cases} [\phi] & R \text{ is } \Gamma\text{-conjugate to } Q \\ 0 & \text{otherwise} \end{cases}$$

This implies (2).

If we consider now a class $[\phi] \neq 0$ in $H^q_{cusp}(e'(Q),\mathbb{C})$ with $q_o({}^{o}M_Q) + (1/2)\dim N_Q \leq q \leq C_o(n)$ the information on the image of $[E(\phi,\Lambda_o)]$ under the various restrictions is not as good as above in (3). Indeed, in general we only know that for a given class $[\phi]$ of type (π,w) with $\ell(w) \geq (1/2)\dim N_Q$

(4) $$r^q_R([E(\phi,\Lambda_o)]) = \begin{cases} [\phi] & R \text{ is } \Gamma\text{-conjugate to } Q \\ [\underline{\underline{c}}(s,\Lambda_o)_{s\Lambda_o}(\phi_{\Lambda_o})]|_{e'(R)} & R \text{ is } SL_n(\mathbb{Q})\text{-conjugate to } Q^{opp},\ s \in W(A_Q,A_R) \\ 0 & \text{otherwise} \end{cases}$$

holds. But recall that in the second case in (4) $[\underline{\underline{c}}(s,\Lambda_o)_{s\Lambda_o}(\phi_{\Lambda_o})]|_{e'(R)}$

is a class in $H^*_{cusp}(e'(R),\mathbb{C})$ of type $({}^s\pi, v_s)$ with uniquely determined ${}^s\pi \in {}^o\hat{M}_R$ and $v_s \in W^R$ with $\ell(v_s) = \dim N_R - \ell(w)$ (cf. 4.4.(iii)). If we assume now that the given class $[\phi]$ of type (π,w) satisfies the strict inequality $\ell(w) > (1/2)\dim N_Q$ then it follows by using $\dim N_R = \dim N_Q$ that the classes $r^q_R([E(\phi,\Lambda_o)])$, R is $SL_n(\mathbb{Q})$-conjugate to Q^{opp} , are not contained in the sum

$$\bigoplus_{S\in\Gamma\backslash C(P)} \bigoplus_{\substack{p+r=q \\ r>(1/2)\dim N_S}} H^p_{cusp}(\Gamma_{M_S}\backslash Z_S, H^r(\underline{n}_S)) \quad .$$

This implies that the dimension of $H^q_{C(P)}(\Gamma X;\mathbb{C})$ is at least as large as the one of this space, i.e.

$$\text{(5)} \qquad \dim H^q_{C(P)}(\Gamma\backslash X;\mathbb{C}) \geq \dim \bigoplus_{S\in\Gamma\backslash C(P)} \bigoplus_{\substack{p+r=q \\ r>(1/2)\dim N_S}} H^p_{cusp}(\Gamma_{M_S}\backslash Z_S, H^r(\underline{n}_S)) \ .$$

Besides the classes just described there are also possibly regular Eisenstein classes $[E(\phi,\Lambda_o)]$ in $H^*_{C(P)}(\Gamma\backslash X;\mathbb{C})$ which are built up by a non-trivial $[\phi] \in H^*_{cusp}(e'(Q),\mathbb{C})$, $Q \in \Gamma\backslash C(P)$, of type (π,w) with $\ell(w) = (1/2)\dim N_Q$. Observe that we have then $\Lambda_o = 0$ by 4.2. in this case and that $[\underline{c}(s,0)_o(\phi_o)]|_{e'(R)}$ is a class of type $({}^s\pi, v_s)$ with $\ell(v_s) = (1/2)\dim N_R = \ell(w)$ for an R which is $SL_n(\mathbb{Q})$-conjugate to Q^{opp}. Assertion (1) follows now directly from these considerations and the following argument. For a fixed maximal parabolic $\mathbb{Q}$-subgroup Q of $SL_n(\mathbb{R})$ there is a natural isomorphism given by the usual $*$-operator between the two spaces of harmonic cusp forms (cf. 1.5.)

$$\text{(6)} \qquad \underline{H}^p_{cusp}(\Gamma_{M_Q}\backslash Z_Q, H^r(\underline{n}_Q)) \xrightarrow{\sim} \underline{H}^{D-p}(\Gamma_{M_Q}\backslash Z_Q, H^{d-r}(\underline{n}_Q))$$

where $Z_Q = {}^{o}M_Q/K_{M_Q}$, $D = \dim Z_Q$ and $d = \dim N_Q$. We note that a cuspidal differential form ψ in $\underline{H}^p_{cusp}(\Gamma_{M_Q}\backslash Z_Q, H^r(\underline{n}_Q))$ of weight $w(\rho)-\rho$ $(= -\ell(w)\alpha|_{\underline{a}}$ if $Q = P_{\Delta-\{\alpha\}})$ with respect to the action of A_Q on $H^r(\underline{n}_Q)$ is transformed under the $*$-operator into a cuspidal harmonic form of weight $-2\rho - (w(\rho)-\rho)$ $(= -(\dim N_Q - \ell(w))\alpha|_{\underline{a}}$ if $Q = P_{\Delta-\{\alpha\}})$. In particular, (6) induces an isomorphism

$$(7) \quad \bigoplus_{\substack{p,r \\ r<(1/2)\dim N_Q}} H^p_{cusp}(\Gamma_{M_Q}\backslash Z_Q, H^r(\underline{n}_Q)) \xrightarrow{\sim} \bigoplus_{\substack{p,r \\ r>(1/2)\dim N_Q}} H^p_{cusp}(\Gamma_{M_Q}\backslash Z_Q, H^r(\underline{n}_Q))$$

or, of course, more general, (6) induces an isomorphism

$$(8) \qquad H^q_{cusp}(e'(Q),\mathbb{C}) \xrightarrow{\sim} H^{T-q}_{cusp}(e'(Q),\mathbb{C})$$

where $T = \dim e'(Q) = D+d$. Assertion (1) is now implied by (6), (7) if we take into account the additional regular Eisenstein classes $[E(\phi,0)]$ described above. Since we don't know if the restriction $r^*_R([E(\phi,0)])$ (for R conjugate to Q^{opp} by $SL_n(\mathbb{Q})$) vanishes or not we are forced to allow inequality in (1).

<u>Remark</u>: The proof of 4.7. brings a somewhat more precise but also more elaborate statement than (2) namely the restriction of $H^q(\Gamma\backslash\overline{X},\mathbb{C})$ onto

$$\bigoplus_{Q\in\Gamma\backslash C(P)} \bigoplus_{\substack{p+r=q \\ r>(1/2)\dim N_Q}} H^p_{cusp}(\Gamma_{M_Q}\backslash Z_Q, H^r(\underline{n}_Q))$$

is surjective for each q (where the right hand side lives at all i.e. $q \geq q_o({}^{o}M_Q) + 1/2(\dim N_Q)$).

4.8. A remark in the case of a parabolic subgroup of type m if $n = 2m$. The final argument given in 4.4. to show that the Eisenstein series $E(\phi,\Lambda)$ associated to a given cuspidal class $[\phi] \in H^*_{cusp}(e'(P),\mathbb{C})$ of type (π,w), $\pi \in {}^{o}\hat{M}$, $w \in W^P$ is holomorphic at $\Lambda_o = -w(\rho)|_{\underline{a}}$ relied on the fact that the Langlands quotient $J(P,\pi,\Lambda_o)$ corresponding to the given data (P,π,Λ_o) is not unitarizable. Since this question doesn't have such a simple answer for $P = P_m$ if $n = 2m$ we had to exclude this case from our discussion (cf. also [17], Thm. 7). Indeed, fix one of the irreducible unitary representations π of ${}^{o}M_{P_m}$ which have Whittaker model and non-trivial cohomology $H^*({}^{o}\underline{m}_{P_m}, K_{M_{P_m}}; H_\pi)$. Write $\Lambda_q = q\cdot\alpha_m$ with $q \in \mathbb{R}$, $q \geq 0$ and note $\rho_{P_m} = \frac{1}{2}\cdot m^2\cdot\alpha_m$. Then the Langlands quotient $J(P_m,\pi,\Lambda_q)$ corresponding to the data (P_m,π,Λ_q) is unitary for $q \leq (1/2)\cdot m$, and, for example, one has for $m = 2$ exactly that $J(P_m,\pi,\Lambda_q)$ is unitarizable if and only if $0 \leq q \leq 1$. By giving estimates for a bound (smaller than $(1/2)m^2$) up to which $J(P,\pi,\Lambda_q)$ can be unitary similar results can be obtained for $m > 2$ (cf. 4.9. for a first step in this direction).

Therefore the discussion of possible poles of $E(\phi,\Lambda)$ in the case of a maximal parabolic $\mathbb{Q}$-subgroup of type m if $n = 2m$ needs some additional information which we don't know completely up to now. It is helpful to deal with this question in an adelic setting. However, it has been shown for suitable $\Gamma \subset SL_n(\mathbb{Z})$ of finite index and a suitable function ψ that the associated Eisenstein series $E(\psi,\Lambda)$ has a simple pole at $\Lambda_{q_o} = (1/2)\rho_{P_m}$ i.e. $q = (1/4)m^2$ (cf. [29] 3.4.1. resp. [16]).

Nevertheless, the following is true (The analogue for P not of type m if $n = 2m$ is already contained in 4.4.):

4.9. PROPOSITION. - Let $n = 2m$, $n > 2$, $\Gamma \subset SL_n(\mathbb{Z})$ a torsionfree subgroup of finite index and let P be a maximal parabolic $\mathbb{Q}$-subgroup of $SL_n(\mathbb{R})$ of type m . Let $[\phi] \in H^*_{cusp}(e'(P),\mathbb{C})$ be a non-trivial cuspidal cohomology class of type (π, w_P) , $\pi \in {}^o\hat{M}$ and w_P the longest element in W^P . Then $E(\phi,\Lambda)$ is holomorphic at $\Lambda_o = -w_P(\rho)|_{\underline{a}}$, and the Eisenstein form $E(\phi,\Lambda_o)$ represents a non-trivial cohomology class in $H^*(\Gamma\backslash X;\mathbb{C})$. The restrictions of $[E(\phi,\Lambda_o)]$ under r^*_Q , Q an arbitrary parabolic $\mathbb{Q}$-subgroup of $SL_n(\mathbb{R})$, can be described analogously as in 4.4. resp. 4.6..

Since π is a tempered representation by 4.3.(1) the assertion reduces to 6.4.(2) in [27].

4.10. On $H^*(\Gamma(k)\backslash X;\mathbb{C})$ as $SL_n(\mathbb{Z}/k\mathbb{Z})$-module. We conclude this section with an application of 4.4., 4.6. and 4.9. to the natural structure of the cohomology $H^*(\Gamma(k)\backslash X;\mathbb{C})$ of a full congruence subgroup of level k in $SL_n(\mathbb{Z})$ as a module of $SL_n(\mathbb{Z})/\Gamma(k) = SL_n(\mathbb{Z}/k\mathbb{Z})$. As a particular case we show that it contains a submodule which is related via induction of representations of finite groups in a certain way to the cusp cohomology $H^*_{cusp}(\Gamma'(k)\backslash X';\mathbb{C})$ of the full congruence subgroup $\Gamma'(k)$ of level k in $SL_{n-1}(\mathbb{Z})$. This relation reflects in a simple, but striking manner the inductive procedure to build up at least part of the Eisenstein cohomology of $\Gamma(k)$ out of the cusp cohomology of the $\Gamma(k)$'s in the various $SL_j(\mathbb{Z})$, $j = 2,\dots,n-1$.

For brevity we only sketch the main steps.

We consider the full congruence subgroup $\Gamma(k)$ of $SL_n(\mathbb{Z})$ of level k , $k \geq 3$. We fix k once and for all, and write (a little bit careless)

(1) $$_f SL := \Gamma(k)\backslash SL_n(\mathbb{Z}) = SL_n(\mathbb{Z}/k\mathbb{Z}) \quad .$$

In a natural way the group $SL_n(\mathbb{Z})$ operates on the Borel-Serre compactification $\Gamma(k)\backslash\overline{X}$, and induces an action of ${}_fSL$ on $H^*(\Gamma(k)\backslash\overline{X}),\mathbb{C})$ resp. $H^*(\partial(\Gamma(k)\backslash\overline{X}),\mathbb{C})$. Let P_J be the standard parabolic $\mathbb{Q}$-subgroup of $SL_n(\mathbb{R})$ of type $J \subset \Delta$ (cf. 3.1.) and define with respect to the fixed $\Gamma(k)$ the subgroup ${}_fP_J$ of ${}_fSL$ by

(2) $$ {}_fP_J := (\Gamma(k) \cap P_J)\backslash(SL_n(\mathbb{Z}) \cap P_J) \quad . $$

Then one can organize all faces $e'(P)$ in $\partial(\Gamma(k)\backslash\overline{X})$ which correspond to a parabolic $\mathbb{Q}$-subgroup P of type J as an induced bundle

(3) $$ S_J = {}_fSL \underset{{}_fP_J}{\times} e'(P_J) $$

which is a disjoint union of copies of $e'(P_J)$; it has a natural action of ${}_fSL$ extending the previous one of ${}_fP_J$ on $e'(P_J)$ (cf. for this construction [21], § 3). Its cohomology as ${}_fSL$-module is given by

(4) $$ H^*(S_J,\mathbb{C}) = \mathrm{Ind}_{{}_fP_J}^{{}_fSL} [H^*(e'(P_J);\mathbb{C})] \quad , $$

where $\mathrm{Ind}_{{}_fP_J}^{{}_fSL}$ denotes the induced representation.

We restrict now to the case of a standard maximal parabolic $\mathbb{Q}$-subgroup P_i of type $\Delta - \{\alpha_i\}$, $i = 1,\dots,n-1$; then $H^*(e'(P_i),\mathbb{C}) = H^*(\Gamma_{M_i}\backslash Z_{M_i},H^*(\underline{n}_i,\mathbb{C}))$ in the notation of 2.3.. Let

$$ {}_fM_i := (\Gamma(k) \cap {}^oM_i)\backslash(SL_n(\mathbb{Z}) \cap {}^oM_i) \quad \text{resp.} \quad {}_fN_i := (\Gamma(k) \cap N_i)\backslash(SL_n(\mathbb{Z}) \cap N_i) $$

Then one has an exact sequence

(5) $$ 1 \to {}_fN_i \to {}_fP_i \xrightarrow{\pi_i} {}_fM_i \to 1 \quad , $$

and ${}_fP_i$ is a split group extension of ${}_fM_i$ by ${}_fN_i$. The ${}_fP_i$-module structure on $H^*(e'(P_i),\mathbb{C})$ is the pullback under π_i of the ${}_fM_i$-module structure of $H^*(\Gamma_{M_i}\backslash Z_{M_i}, H^*(\underline{n}_i,\mathbb{C}))$ induced by the action of $SL_n(\mathbb{Z}) \cap {}^oM_i$ on $\Gamma_{M_i}\backslash Z_{M_i}$ resp. $H^*(\underline{n}_i,\mathbb{C})$.

If we consider now, for example, only cuspidal cohomology classes $[\phi]$ in $H^*_{cusp}(e'(P_i),\mathbb{C})$ of type (π, w_{P_i}) , $\pi \in {}^o\hat{M}_i$, and degree $q + \ell(w_{P_i})$ where w_{P_i} denotes the longest element in W^{P_i} then the corresponding Eisenstein series $E(\phi,\Lambda)$ is holomorphic at $\Lambda_o = -w_{P_i}(\rho)|_{\underline{a}_i} = \rho_{P_i}$ by 4.4., 4.6. and 4.9.. The harmonic form $E(\phi,\Lambda_o)$ represents a non-trivial cohomology class in $H^*(\Gamma(k)\backslash X;\mathbb{C})$ of degree $q+\ell(w_{P_i}) = q+\dim N_i$ whose restrictions are given by

$$(6)\qquad r_Q^{q+\ell(w_{P_i})}{}^{P_i}([E(\phi,\Lambda_o)]) = \begin{cases} [\phi] & \text{for } Q \underset{\Gamma(k)}{\sim} P_i \\ 0 & \text{otherwise} \end{cases} .$$

This follows by 4.6., 4.9. from the fact that $q+\ell(w_{P_i}) \geq q_o({}^oM_i)+\ell(w_{P_i}) \geq$ $\geq C_o(n)$. Since the restriction

$$r^*_{C(P_i)} \;:\; H^*(\Gamma(k)\backslash\overline{X},\mathbb{C}) \to \bigoplus_{P\in\Gamma\backslash C(P_i)} H^*(e'(P),\mathbb{C})$$

is compatible with the natural action of ${}_fSL$ on both sides (cf. [9] 7.6. : $g\cdot e'(P) = e'(P^g)$ for $g \in SL_n(\mathbb{Q})$) the result above shows that

$$(7)\qquad \mathrm{Ind}_{{}_fP_i}^{{}_fSL}\,[H^*_{cusp}(\Gamma_{M_i}\backslash Z_{M_i}, H^{\dim N_i}(\underline{n}_i,\mathbb{C}))]$$

<u>is a ${}_fSL$-submodule of</u> $H^{*+\dim N_i}(\Gamma(k)\backslash\overline{X},\mathbb{C})$.

Similar results can be worked out by means of cuspidal classes $[\phi]$ of other types with $\ell(w) \geq (1/2)\dim N$.

In particular, fixing $i = 1$ (or $i = n-1$) we have ${}^oM_1 = SL^{\pm}_{n-1}(\mathbb{R})$ and $\ell(w_{P_1}) = \dim N_1 = n-1$. If we denote by $\Gamma'(k)$ (resp. X') the congruence subgroup of level k of $SL_{n-1}(\mathbb{Z})$ (resp. $SL_{n-1}(\mathbb{R})/SO(n-1)$)

we see that as $\mathbb{C}$-vector spaces

(8) $$H^*_{cusp}(\Gamma_{M_1}\backslash Z_{M_1}, H^{\dim N_1}(\underline{n}_1,\mathbb{C})) = H^*_{cusp}(\Gamma'(k)\backslash X',\mathbb{C})$$

and the ${}_fM_1$-module structure on the left hand side restricted to $SL_{n-1}(\mathbb{Z}/k\mathbb{Z}) \subset {}_fM_1$ coincides with the natural action of $SL_{n-1}(\mathbb{Z}/k\mathbb{Z})$ on $H^*_{cusp}(\Gamma'(k)\backslash X';\mathbb{C})$. Together with (7) this illustrates our remark at the beginning of this paragraph.

<u>4.11.</u> <u>Remarks and examples</u>. (1) For $n = 4$ let $\Gamma(k) \subset SL_4(\mathbb{Z})$ be a congruence subgroup with $k \geq 3$. The cohomological dimension $cd(\Gamma(k))$ of $\Gamma(k)$ is 6, $H^q_{cusp}(\Gamma(k),\mathbb{C}) = 0$, $q \neq 4,5$ and, in particular, $C_o(4) = 5$. Let P be a maximal parabolic $\mathbb{Q}$-subgroup of $SL_4(\mathbb{R})$ of type 1 or 3 ; then $\dim N_P = 3$ and $H^q_{cusp}(e'(P),\mathbb{C}) = \oplus H^p_{cusp}(\Gamma_M\backslash Z_M, H^r(\underline{n},\mathbb{C}))$ in the notation of 2.4.. Since $\Gamma_M = \Gamma'(k) \subset SL_3(\mathbb{Z})$ the right hand side vanishes for $p \neq 2,3$. By 4.6., 4.7. there is a subspace in $H^q_{C(P)}(\Gamma(k)\backslash X;\mathbb{C})$ $q = 5$ resp. $q = 6$ which is mapped isomorphically onto $H^2_{cusp}(\Gamma_M\backslash Z_M, H^3(\underline{n},\mathbb{C}))$ resp. $H^3_{cusp}(\Gamma_M\backslash Z_M, H^3(\underline{n},\mathbb{C}))$. Using 4.10.(8) the dimension of each of these spaces is at least $k(k+1)$ for k a prime with $k \equiv 3 \bmod 8$ and $k \equiv -1 \bmod 3$ by the result in [22].

(2) This and other examples (cf. 9.11. [27]) show that the subspace $H^*_{C(P)}(\Gamma\backslash X;\mathbb{C})$ of $H^*(\Gamma\backslash X;\mathbb{C})$ obtains his life from non-vanishing results for $H^*_{cusp}(e'(P),\mathbb{C})$ which are closely related to non-vanishing results for the cusp cohomology of $\Gamma \subset SL_r(\mathbb{Z})$. Unfortunately, there is not too much known about this except that there is a widely believed conjecture which says (adopting the framework of § 1):

(*) Given an arithmetic subgroup Γ of $\underline{G}(\mathbb{Q})$ there exists a subgroup Γ' of Γ of finite index such that $H^*_{cusp}(\Gamma'\backslash X,E)$ doesn't vanish.

In the case $\underline{G}/\mathbb{Q} = SL_n/\mathbb{Q}$ one knows that (*) is correct for $n = 2$ as a consequence of the Eichler-Shimura isomorphism and for $n = 3$ by [22]. For other n it is an open question

(3) We have limited ourselves in this paper to the case $SL_n/\mathbb{Q}$ and P a maximal parabolic $\mathbb{Q}$-subgroup where one gets rather complete results. For other parabolics the Eisenstein series $E(\phi,\Lambda)$ in question may very well have poles at special points Λ_o (cf. 2.6.). In this case, we have to take residues of Eisenstein series in order to describe the situation. We refer to [12], [14], [26], [27] for some examples in an adelic setting. However, the techniques of this paper work also, for example, in the case $G = Sp_n(\mathbb{R})$ and some special choices of maximal parabolic $\mathbb{Q}$-subgroups of G. But, in general, an analogue of 4.3. will possibly not be true as also the results in [34] indicate.

(4) We take this opportunity to correct an error in [27]: The computation in the proof of 5.6. (also used in 6.7.) is incorrect, and the counterexample is implicit already given by 8.4.(1). Therefore, 5.6. and Cor. 6.7. in [27] have to be cancelled in this form. It is not necessary in remark 8.5.(2) [27] to refer to 5.6. resp. 6.7. if one wants to work out the case of non-trivial coefficients as indicated there.

§ 5 On the cohomological contribution by the cusp cohomology of the faces in $\partial(\Gamma\backslash\bar{X})$ of minimal codimension

In this section we indicate briefly how unpublished results due to R. P. Langlands and A. Borel imply that the subspace $H^*_{C(P)}(\Gamma\backslash X;\mathbb{C})$ of Eisenstein cohomology classes in $H^*(\Gamma\backslash X;\mathbb{C})$ is as large as possible and describes completely that part of the cohomology at infinity of $H^*(\Gamma\backslash X,\mathbb{C})$ contributed by the cusp cohomology spaces $\oplus H^*_{cusp}(e'(Q),\mathbb{C})$, $Q \in \Gamma\backslash C(P)$. Indeed, it will follow that $H^*_{C(P)}(\Gamma\backslash X,\mathbb{C})$ generated by regular values of Eisenstein series (cf. 4.7.) maps isomorphically onto the image of the restriction

$$r^*_{C(P),cusp} \; : \; H^*(\Gamma\backslash X;\mathbb{C}) \to \bigoplus_{Q \in \Gamma\backslash C(P)} H^*_{cusp}(e'(Q),\mathbb{C})$$

if P is not of type m in the case $n = 2m$.

I learned the result I need (5.3.(5)) from A. Borel and I thank him very much for allowing me to sketch the main steps in the argument in 5.2., 5.3., which is of a general nature.

We retain the general notation of § 1.

5.1. We have recalled in 1.3. that the cohomology of Γ can already be computed by using the complex of differential forms whose coefficient functions are of uniform moderate growth, i.e. we have that the inclusion

$$(1) \qquad \Omega^*_{umg}(\Gamma\backslash X,E) \to \Omega^*(\Gamma\backslash X,E)$$

induces an isomorphism in cohomology. By an unpublished result of Langlands [5], [4] one can decompose the left hand side into subspaces parametrized by the classes of associated parabolic $\mathbb{Q}$-subgroups of G and obtains in this way an analoguous decomposition of $H^*(\Gamma\backslash X;E)$. We describe it in more detail.

Let $P = {}^{O}MAN$ be a parabolic $\mathbb{Q}$-subgroup of G. If $f \in C^{\infty}_{umg}(\Gamma\backslash G)$ is a smooth function on $\Gamma\backslash G$ of uniform moderate growth (cf. 1.3.), then also $f_P(\ ,k) \in C^{\infty}_{umg}(\Gamma_M\backslash M)$, uniformly in k, where

$$(2) \qquad f_P(m,k) \quad := \int_{\Gamma_N\backslash N} f(n\ m\ k)\ dn$$

is the constant Fourier coefficient of f with respect to P. We say that a function f in $C^{\infty}_{umg}(\Gamma\backslash G)$ is <u>neglegible with respect to</u> P (denoted by $f \perp P$) if $f_P(ma,k)$, $m \in {}^{O}M$, is orthogonal to all cuspidal functions on $\Gamma_M\backslash {}^{O}M$ for all $a \in A$ and $k \in K$ (cf. Lemma 31 in [15]).

For a given class $C(Q)$ of associate parabolic $\mathbb{Q}$-subgroups of G one defines

$$(3) \qquad V(\Gamma\backslash G;C(Q)) \quad := \left\{ f \in C^{\infty}_{umg}(\Gamma\backslash G) \;\middle|\; f \perp P \text{ for all } P \notin C(Q) \right\} .$$

One has then due to Langlands that the space $C^{\infty}_{umg}(\Gamma\backslash G)$ has a decomposition as a direct sum

$$(4) \qquad C^{\infty}_{umg}(\Gamma\backslash G) \quad = \quad \bigoplus V(\Gamma\backslash G;C(Q))$$

where $C(Q)$ runs through the finitely many classes of associate parabolic $\mathbb{Q}$-subgroups of G. If one combines (4) with (1) one obtains a decomposition in cohomology

$$(5) \qquad H^*(\Gamma\backslash X;E) \quad = \quad H^*(\underline{g},K;C^{\infty}_{umg}(\Gamma\backslash G) \otimes E) \qquad \text{(cf. 1.3.)}$$

$$= \bigoplus_{C(Q)} H^*(\underline{g},K;V(\Gamma\backslash G,C(Q)) \otimes E) \quad .$$

A summand $H^*(\underline{g},K;V(\Gamma\backslash G;C(Q)) \otimes E)$ will also be denoted by $H^*(\Gamma\backslash X,E)_{C(Q)}$. For $Q = G$ we have

$$(6) \qquad H^*(\Gamma\backslash X;E)_{C(G)} \quad = \quad H^*_{cusp}(\Gamma\backslash X;E)$$

i.e. the corresponding summand coincides with the cusp cohomology.

5.2. As recalled in 1.3., 1.4. the natural inclusions

$$\Omega^*_{mg}(\Gamma\backslash X;\mathbb{C}) \to \Omega^*(\Gamma\backslash X;\mathbb{C})$$

resp.

$$\Omega^*_c(\Gamma\backslash X;\mathbb{C}) \to \Omega^*_{fd}(\Gamma\backslash X;\mathbb{C})$$

(for notation we refer to 1.3., 1.4.) induce isomorphisms in cohomology

(1) $$H^*(\Omega_{mg}(\Gamma\backslash X;\mathbb{C})) \xrightarrow{\sim} H^*(\Gamma\backslash X;\mathbb{C})$$

resp.

(2) $$H^*_c(\Gamma\backslash X;\mathbb{C}) \xrightarrow{\sim} H^*(\Omega_{fd}(\Gamma\backslash X;\mathbb{C}))$$

where H^*_c refers to cohomology with compact supports. With $N = \dim \Gamma\backslash X$ there is the usual pairing

(3) $$\Omega^q_c(\Gamma\backslash X;\mathbb{C}) \times \Omega^{N-q}(\Gamma\backslash X;\mathbb{C}) \to \Omega^N(\Gamma\backslash X;\mathbb{C}) \quad .$$

Since the product $\alpha \wedge \beta$ of a fast decreasing form α with a form β of moderate growth is again fast decreasing we have also a pairing

(4) $$\Omega^q_{fd}(\Gamma\backslash X;\mathbb{C}) \times \Omega^{N-q}_{mg}(\Gamma\backslash X;\mathbb{C}) \to \Omega^N_{fd}(\Gamma\backslash X;\mathbb{C}) \quad .$$

Since $H^N_c(\Gamma\backslash X;\mathbb{C}) \cong \mathbb{C}$ given by integration the isomorphism (2) gives also an isomorphism $H^N(\Omega_{fd}(\Gamma\backslash X;\mathbb{C})) \cong \mathbb{C}$ which is again defined by integration. Thus (3), (4) yield a commutative diagram of sesquilinear pairings

(5) $$\begin{array}{ccccc} H^q(\Omega_{fd}(\Gamma\backslash X;\mathbb{C})) & \times & H^{N-q}(\Omega_{mg}(\Gamma\backslash X;\mathbb{C})) & \to & \mathbb{C} \\ \uparrow\wr & & \downarrow\wr & & \\ H^q_c(\Gamma\backslash X;\mathbb{C}) & \times & H^{N-q}(\Gamma\backslash X;\mathbb{C}) & \to & \mathbb{C} \end{array} \quad ;$$

they will be denoted by $\langle\ ,\ \rangle$.

On the other hand, we let $\underline{H}^*_{cusp}(\Gamma\backslash X;\mathbb{C})$ denote the space of harmonic cuspidal $\mathbb{C}$-valued differential forms on $\Gamma\backslash X$. Since a cuspidal form is fast decreasing these harmonic forms belong to $\Omega^*_{fd}(\Gamma\backslash X;\mathbb{C})$. We recall that the cusp cohomology $H^*_{cusp}(\Gamma\backslash X;\mathbb{C})$ of Γ can be identified with $\underline{H}^*_{cusp}(\Gamma\backslash X;\mathbb{C})$ in a natural way ([2], 5.5). Then the product

$$(\alpha,\beta) \to \int_{\Gamma\backslash X} \alpha\wedge *\beta \quad , \quad \alpha \in \Omega^q_{fd}(\Gamma\backslash X;\mathbb{C}) \quad , \quad \beta \in \Omega^q_{mg}(\Gamma\backslash X;\mathbb{C})$$

induces a pairing, denoted by $(\ ,\ )$,

$$(6) \qquad \underline{H}^q_{cusp}(\Gamma\backslash X;\mathbb{C}) \times H^q(\Omega_{mg}(\Gamma\backslash X;\mathbb{C})) \to \mathbb{C}$$

which is positive non-degenerate on $\underline{H}^q_{cusp}(\Gamma\backslash X;\mathbb{C})$. One obtains an orthogonal decomposition

$$(7) \qquad H^q(\Gamma\backslash X;\mathbb{C}) = \underline{H}^q_{cusp}(\Gamma\backslash X;\mathbb{C}) \oplus (\underline{H}^q_{cusp}(\Gamma\backslash X;\mathbb{C}))^{\perp}$$

with respect to $(\ ,\ )$ and a natural complement to the cusp cohomology in $H^q(\Gamma\backslash X;\mathbb{C})$ in this way. Observe that $(\underline{H}^q_{cusp}(\Gamma\backslash X;\mathbb{C}))^{\perp}$ is also the orthogonal complement to $\underline{H}^{N-q}_{cusp}(\Gamma\backslash X;\mathbb{C})$ with respect to the pairing $\langle\ ,\ \rangle$ defined above.

These considerations apply also to the cusp cohomology $H^*_{cusp}(e'(P),\mathbb{C})$ of a face $e'(P)$ in $\Gamma\backslash\overline{X}$.

5.3. Let P be a maximal parabolic $\mathbb{Q}$-subgroup of G ; then the associated face $e'(P)$ is open in the boundary $\partial(\Gamma\backslash\overline{X})$ of the Borel-Serre compactification. By extending a fast decreasing form on $e'(P)$ by zero to one on $\partial(\Gamma\backslash\overline{X})$ one obtains a map

$$(1) \qquad i^* : H^*_c(e'(P),\mathbb{C}) \to H^*(\partial(\Gamma\backslash\overline{X});\mathbb{C}) \quad .$$

Using the pairing defined above (and its analogue on $H^*(\partial(\Gamma\backslash\bar{X}),\mathbb{C})$ one sees then that the map

(2) $$\bigoplus_{Q\in\Gamma\backslash C(P)} \underline{H}^*_{cusp}(e'(Q),\mathbb{C}) \rightarrow H^*(\partial(\Gamma\backslash\bar{X});\mathbb{C})$$

induced by i^* is injective. Of course, this is also true if we sum over all Γ-conjugacy classes of maximal parabolic $\mathbb{Q}$-subgroups of G .

We consider now the total restriction

(3) $$r^* \ : \ H^*(\Gamma\backslash\bar{X},\mathbb{C}) \rightarrow H^*(\partial(\Gamma\backslash\bar{X}),\mathbb{C})$$

of the cohomology of Γ to the cohomology of the boundary resp. the various restrictions

(4) $$r^*_{C(P)} \ : \ H^*(\Gamma\backslash\bar{X},\mathbb{C}) \rightarrow \bigoplus_{Q\in\Gamma\backslash C(P)} H^*(e'(Q),\mathbb{C}) \quad .$$

With respect to $<\ ,\ >$ the space $\mathrm{Im}\, r^q \cap H^q(\partial(\Gamma\backslash\bar{X}),\mathbb{C})$ is orthogonal to $\mathrm{Im}\, r^{s-q} \cap H^{s-q}(\partial(\Gamma\backslash\bar{X}),\mathbb{C})$ where $s = \dim(\partial(\Gamma\backslash\bar{X}))$, and we claim that we also have

(5) $$\bigoplus_{Q\in\Gamma\backslash C(P)} \underline{H}^q_{cusp}(e'(P)) \cap \mathrm{Im}\, r^q_{C(P)}$$ <u>is orthogonal with respect</u>

<u>to</u> $<\ ,\ >$ <u>to</u> $$\bigoplus_{Q\in\Gamma\backslash C(P)} \underline{H}^{s-q}_{cusp}(e'(Q),\mathbb{C}) \cap \mathrm{Im}\, r^{s-q}_{C(P)} \quad .$$

This is proved if we can find for a given ϕ in $\bigoplus_{Q\in\Gamma\backslash C(P)} \underline{H}^q_{cusp}(e'(Q),\mathbb{C}) \cap \mathrm{Im}\, r^q_{C(P)}$ an element $[\omega]$ in $H^q(\Gamma\backslash\bar{X};\mathbb{C})$ such that $r^q([\omega]) = i^q(\phi)$ in $H^q(\partial(\Gamma\backslash\bar{X});\mathbb{C})$. But the existence of such an element $[\omega]$ follows immediately from the direct sum decomposition 5.1.(5)

(6) $$H^q(\Gamma\backslash\bar{X};\mathbb{C}) = \bigoplus_{R\in\underline{P}} H^q(\Gamma\backslash\bar{X},\mathbb{C})_{C(R)} \quad ,$$

the defining properties of the elements in each summand $H^q(\Gamma\backslash\bar{X};\mathbb{C})_{C(R)}$ and the interpretation of $r^*_{C(P)} = \bigoplus_{Q \in \Gamma\backslash C(P)} r^*_Q$ in terms of taking the constant Fourier coefficient along Q (cf. 1.9. in [27] resp. 2.2.(4) and 2.7.)

From assertion (5) one obtains now the following upper bound for the dimension of the image of the restriction

$$(7) \qquad r^*_{C(P),cusp} \;:\; H^*(\Gamma\backslash\bar{X};\mathbb{C}) \to \bigoplus_{Q \in \Gamma\backslash C(P)} H^*_{cusp}(e'(Q),\mathbb{C})$$

given by $r^*_{C(P)}$ composed with the projection to $H^*_{cusp}(e'(Q),\mathbb{C})$ in each summand $H^*(e'(Q),\mathbb{C})$ in the right hand side of (4),

$$(8) \qquad \dim \operatorname{Im} r^*_{C(P),cusp} \le (1/2)\dim\Big(\bigoplus_{Q \in \Gamma\backslash C(P)} H^*_{cusp}(e'(Q),\mathbb{C})\Big) \quad .$$

5.4. We come back to the setting of § 4. Let P be a maximal parabolic $\mathbb{Q}$-subgroup of $G = SL_n(\mathbb{R})$ which is of type i with $i \neq m$ if $n = 2m$. We compare the results on the subspace $H^*_{C(P)}(\Gamma\backslash X,\mathbb{C})$ of $H^*(\Gamma\backslash X;\mathbb{C})$ obtained in § 4 with the general estimate on $\dim \operatorname{Im} r^*_{C(P),cusp}$ given by 5.3.(8).

By the proof of 4.7. (cf. remark following 4.7.) there is a subspace in $H^q_{C(P)}(\Gamma\backslash X,\mathbb{C})$ generated by the regular Eisenstein cohomology classes $[E(\phi,\Lambda_o)]$ corresponding to $[\phi] \in H^q_{cusp}(e'(Q),\mathbb{C})$, $Q \in \Gamma\backslash C(P)$, of type (π,w), $\pi \in {}^oM$, $w \in W^P$ with $\ell(w) > (1/2)\dim N_Q$ which is mapped isomorphically under the natural restriction onto

$$(1) \qquad \bigoplus_{Q \in \Gamma\backslash C(P)} \; \bigoplus_{\substack{p+r=q \\ r>(1/2)\dim N_Q}} H^p_{cusp}(\Gamma_{M_Q}\backslash Z_Q, H^r(\underline{n}_Q,\mathbb{C})) \quad .$$

Therefore the dimension of $H^q_{C(P)}(\Gamma\backslash X;\mathbb{C})$ is greater than the one of the space in (1).

In the discussion of classes $[\phi]$ in $H^*_{cusp}(e'(Q),\mathbb{C})$ of type (π,w) with $\dim N_Q$ even and $\ell(w) = (1/2)\dim N_Q$ we assume for simplicity that

$\Gamma = \Gamma(k)$ is a full congruence subgroup of level $k \geq 3$. The associate class $C(P)$ of P contains the maximal parabolic $\mathbb{Q}$-subgroups of $SL_n(\mathbb{R})$ of type i and $n-i$ (cf. 3.1.). The number $p_{max,i}(k)$ of $\Gamma(k)$-conjugacy classes of maximal parabolic $\mathbb{Q}$-subgroups of $SL_n(\mathbb{R})$ is then given by (cf. 4.10.)

(2) $$p_{max,i}(k) = |{}_f SL/{}_f P_i| ,$$

and we have

(3) $$p_{max,i}(k) = p_{max,n-i}(k) .$$

If we start now with linearly independent classes $[\phi] \neq 0$ in $H^*_{cusp}(e'(Q),\mathbb{C})$, $Q \in \Gamma\backslash C(P)$ <u>and</u> Q <u>is of type</u> i , of type (π,w) with $\ell(w) = (1/2)\dim N_Q$ the corresponding Eisenstein cohomology classes $E(\phi,0)$ are all linearly independent because we have for the restriction of such a class

$$r^*_R([E(\phi,0)]) = \begin{cases} [\phi] & R \text{ is } \Gamma\text{-conjugate to } Q \\ 0 & R \text{ of type } i \text{, but not } \Gamma\text{-conjugate to } Q \\ [\underline{c}(s,o)_o(\phi_o)]|_{e'(R)} & R \text{ of type } n-i \text{ and } s \in W(A_Q,A_R) \end{cases}$$

where R runs through a set of representatives of $\Gamma\backslash C(P)$.

But if we start now with a non-trivial class $[\psi] \neq 0$ in $H^*_{cusp}(e'(Q),\mathbb{C})$, $Q \in \Gamma\backslash C(P)$ <u>and</u> Q <u>is of type</u> $n-i$, of type (π,w) with $\ell(w) = (1/2)\dim N_Q$ the corresponding class $[E(\psi,0)]$ is linear dependent on the classes $[E(\phi,0)]$ constructed just before. This follows from the fact that otherwise in view of 4.7.(5), (7) and the discussion above $\dim \operatorname{Im} r^*_{C(P),cusp}$ would exceed $(1/2)\dim \bigoplus_{Q\in\Gamma\backslash C(P)} H^*_{cusp}(e'(Q),\mathbb{C})$ which contradicts the estimate 5.3.(8).

Indeed, the relation between the classes $[E(\phi,0)]$ (corresponding to type i parabolics) and $[E(\psi,0)]$ (corresponding to type (n-i)-parabolics) can be derived from the functional equation for the intertwining operator $c(s,\Lambda_o)$ defined in 2.7.(5), which is proved, for example, in [15], V, § 2 or [20], 6.1.. Details are not of interest here and left to the reader.

However, by 4.4., 4.7. and this discussion we obtain as a final result:

5.5. THEOREM. - Let $\Gamma = \Gamma(k)$ be the congruence subgroup of level k, $k \geq 3$, of $SL_n(\mathbb{Z})$, $n > 2$ and let P be a maximal parabolic $\mathbb{Q}$-subgroup of $SL_n(\mathbb{R})$. We assume that P is not of type m if $n = 2m$. Then the subspace $H^*_{C(P)}(\Gamma\backslash X;\mathbb{C})$ in $H^*(\Gamma\backslash X;\mathbb{C})$ generated by the regular Eisenstein cohomology classes $[E(\phi,\Lambda_o)]$ for all $Q \in \Gamma\backslash C(P)$, all non-trivial classes $[\phi]$ in $H^*_{cusp}(e'(Q),\mathbb{C})$ of type (π,w) with $\pi \in {}^o\hat{M}$, $w \in W^Q$ and $\ell(w) \geq (1/2)\dim N_Q$ is mapped under the restriction

$$r^*_{C(P)} \quad : \quad H^*(\Gamma\backslash\overline{X},\mathbb{C}) \to \bigoplus_{Q \in \Gamma\backslash C(P)} H^*(e'(Q),\mathbb{C})$$

isomorphically onto the image $\operatorname{Im} r^*_{C(P),cusp}$ of $r^*_{C(P),cusp}$ (defined in 5.3.(7)), and we have

$$\dim H^*_{C(P)}(\Gamma\backslash X;\mathbb{C}) = (1/2)\dim\Big(\bigoplus_{Q \in \Gamma\backslash C(P)} H^*_{cusp}(e'(Q),\mathbb{C})\Big) \quad .$$

For a parabolic $\mathbb{Q}$-subgroup R of $SL_n(\mathbb{R})$, which is not maximal we have $r^*_{C(R)}(H^*_{C(P)}(\Gamma\backslash X,\mathbb{C})) = 0$.

An easy consequence of this result is that we have now a complete description of that part of the cohomology at infinity of an arithmetic subgroup of $SL_n(\mathbb{Z})$, n odd, which corresponds to the cuspidal cohomology of the faces in the boundary of the Borel-Serre compactification of minimal codimension. The main point is here, that we are not forced to use residues of Eisenstein series as, for example, in other cases described in [27], Thm. 9.11. or [14]. We have here that each class in $H^*(\Gamma\backslash\bar{X};\mathbb{C})$ which restricts non-trivially to $\oplus H^*_{cusp}(e'(Q),\mathbb{C})$, $Q \in \Gamma\backslash \underline{P}$ a maximal parabolic $\mathbb{Q}$-subgroup, can be written as a linear combination of Eisenstein cohomology classes represented by regular values of Eisenstein series.

5.6. COROLLARY. - Let $\Gamma = \Gamma(k)$ be a congruence subgroup of level k, $k \geq 3$, of $SL_n(\mathbb{Z})$, n odd. Then the subspace

$$\bigoplus H^*_{C(P)}(\Gamma\backslash X;\mathbb{C}) \quad ,$$

where P runs through a set of representatives for the set of associate classes of maximal parabolic $\mathbb{Q}$-subgroups of $SL_n(\mathbb{R})$ is mapped isomorphically under the natural restriction onto $\oplus \operatorname{Im} r^*_{C(P),cusp}$.

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