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REMARKS ON AUTOMORPHISM AND COHOMOLOGY OF CYCLIC COVERINGS

RENJIE LYU AND XUANYU PAN

Abstract. We show that the automorphism group of a smooth cyclic covering acts on its cohomology faithfully with a few well known exceptions. Firstly, we prove the faithfulness of the action in characteristic zero. The main ingredients of the proof are the equivariant deformation theory and the decomposition of the sheaf of differential forms due to Esnault and Viehweg. In positive characteristic, we use a lifting criterion of automorphisms to reduce to characteristic zero. To use this criterion, we prove the degeneration of Hodge-to-deRham spectral sequences and the infinitesimal Torelli theorem for cyclic coverings in positive characteristic.

Contents

1. Introduction

The Torelli theorem says that:
An isomorphism $\varphi : \mathrm{H}^*(X) \simeq \mathrm{H}^*(X')$ between the cohomology groups which preserves some algebraic structures (e.g. Hodge structures) is induced by an isomorphism $\psi : X \simeq X'$ between the varieties.
It is natural to ask whether the map ψ which satisfies $\psi^* = \varphi$ is (up to a sign) unique.
It is equivalent to ask

Question 1.1. *Does the automorphism group* $\mathrm{Aut}(X)$ *act on the cohomology group* $\mathrm{H}^*(X)$ *faithfully?*

Recently, Javanpeykar and Loughran [JL15a] relate this fundamental question to the Lang-Vojta conjecture and the Shafarevich conjecture. The positive answer to the question for hypersurfaces shows that the stack of hypersurfaces is uniformisable by a smooth affine scheme ([JL15b]). On the other hand, the second author [Pan15] gives a positive answer to this question for smooth complex cubic fourfolds,

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and use it to relate the symmetry of the defining equation of a cubic fourfold to its middle Picard number.

Historically speaking, this fundamental question is explored for varieties of low dimension. For example, a positive answer to algebraic curves of genus at least 2 is confirmed in [DM69]. Later, Burns, Rapoport, Shafarevich and Ogus confirm this question for $K3$ surfaces over an algebraically closed field, see [Huy, Chapter 15]. But few higher-dimensional cases are confirmed, see [CPZ15] and [JL15b]. In this paper, we confirm Question 1.1 for cyclic coverings only with a few exceptions. Our main theorems are Theorem 4.9 and 6.5.
The proof of Theorem 4.9 depends on the equivariant deformation theory (Theorem 3.9), the infinitesimal Torelli theorem of cyclic coverings proved by Wehler, and the finiteness of the automorphism groups of cyclic coverings (Theorem 4.5). Theorem 6.5 is the positive characteristic version of Theorem 4.9. We use a lifting criterion of automorphisms to reduce Theorem 6.5 to Theorem 4.9. To apply this criterion, we need the degeneration of the Hodge-to-de Rham spectral sequences and the infinitesimal Torelli theorem for the cyclic coverings of projective spaces in positive characteristic. We use the logarithmic differential forms together with Deligne's method in [DK73, Exp XI] to show the degeneration of the Hodge-to-de Rham spectral sequences (Theorem 5.8). To show the infinitesmial Torelli theorem (Theorem 5.10), we use a version of Flenner's criterion in positive charactersitic (cf. Theorem 5.9) which is developed by the second author with X. Chen and D. Zhang in the paper [CPZ15].
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2. Finite Cyclic Coverings

In this section, we review some basic facts of the cyclic coverings of smooth projective varieties.

Definition 2.1. *Let Z be a smooth projective variety over an algebraically closed field K, and let $\mathcal{L}$ be an invertible sheaf on Z. Assume that k is an integer number such that $\mathcal{L}^k$ has a nontrivial section $s \in H^0(Z, \mathcal{L}^k)$ whose zero divisor $D = Z(s)$ is smooth. There is a natural $\mathcal{O}_Z$-algebra*

$$\mathcal{A} := \bigoplus_{i=0}^{k-1} \mathcal{L}^{-i},$$

where the multiplication structure is given by the section $s^\vee : \mathcal{L}^{-k} \longrightarrow \mathcal{O}_Z$. We define the affine morphism associated to the invertible sheaf $\mathcal{L}$

$$f : X := \underline{\mathrm{Spec}}(\mathcal{A}) \to Z \tag{2.1.1}$$

to be the k-fold cyclic covering of Z branched along D.

In the following, some geometric results about cyclic coverings are shown through Lemma 2.2, 2.3 and Proposition 2.4.
We denote by $\mathbb{V}(L) := \underline{\mathrm{Spec}}(\mathrm{Sym}^{\bullet}\mathcal{L}^{\vee})$ the total space of the invertible sheaf $\mathcal{L}$, and let $\pi_L : \mathbb{V}(L) \to Z$ be the natural projection. If $t \in \Gamma(\mathbb{V}(L), \pi_L^*\mathcal{L})$ is the tautological section, then the k-fold cyclic covering X is exactly the zero divisor of the equation

$$t^k - \pi_L^* s$$

in $\mathbb{V}(L)$. In particular, let $\{U_\alpha\}$ be an affine open cover of Z such that $\mathcal{L}|_{U_\alpha}$ is trivial. Assume that D is defined by the equation $\Phi_\alpha(\underline{z}) = 0$ on U_α. Then X is locally defined by the equation

$$\omega_\alpha^k - \Phi_\alpha(\underline{z}) = 0, \tag{2.1.2}$$

where $(\underline{z}, \omega_\alpha)$ are the local coordinates on $\mathbb{V}(L)|_{U_\alpha} = U_\alpha \times \mathbb{A}^1$. If k is not divided by $\mathrm{char}(K)$, then it follows from the equation (2.1.2) that X is smooth.
Let $\mathcal{E}$ be the locally free sheaf $\underline{\mathcal{O}}_Z \oplus \mathcal{L}^{-1}$. Denote by $\hat{L}$ be the relative projective bundle $\mathbb{P}(\mathcal{E})$ over Z. The cyclic covering X is a divisor in $\hat{L}$ naturally as follows

$$\begin{array}{ccccc} X & \overset{g}{\hookrightarrow} & \mathbb{V}(L) & \overset{i}{\hookrightarrow} & \hat{L} \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle \pi_L} & \swarrow{\scriptstyle \pi} & \\ & & Z & & \end{array} \tag{2.1.3}$$

,

where i is the natural inclusion. Let σ be the section of the projection $\pi : \hat{L} \to Z$ induced by the canonical map $\mathcal{E} \to \underline{\mathcal{O}}_Z$,. Denote by $C := \sigma(Z)$ the image of the section σ. In fact, C is the zero locus of the tautological section $\tau \in \Gamma(\hat{L}, \pi^*\mathcal{L})$.

Lemma 2.2. *With the notations as above, we have that:*

(i) $\mathrm{Pic}(\hat{L}) \simeq \mathrm{Pic}(Z) \oplus \mathbb{Z}[\underline{\mathcal{O}}_{\mathbb{P}(\mathcal{E})}(1)]$*;*

(ii) the invertible sheaf $\underline{\mathcal{O}}_{\hat{L}}(C)$ *associated to the divisor* C *on* $\hat{L}$ *is isomorphic to* $\pi^*\mathcal{L} \otimes \underline{\mathcal{O}}_{\mathbb{P}(\mathcal{E})}(1)$*;*

(iii) the invertible sheaf $\underline{\mathcal{O}}_{\hat{L}}(X)$ *associated to the divisor* X *on* $\hat{L}$ *is isomorphic to* $\pi^*\mathcal{L}^k \otimes \underline{\mathcal{O}}_{\mathbb{P}(\mathcal{E})}(k)$*;*

(iv) the line bundle $\underline{\mathcal{O}}_{\mathbb{P}(\mathcal{E})}(1)|_X$ *is trivial.*

Proof. The conclusions $(i), (ii)$ follows from the standard results for projective bundles (cf. [Har77, Proposition 2.3 and 2.6, page 370-371])
For the third assertion, we may assume that $\underline{\mathcal{O}}_{\hat{L}}(X)$ can be written as

$$\underline{\mathcal{O}}_{\mathbb{P}(\mathcal{E})}(d) + \pi^*\mathscr{M}, \tag{2.2.1}$$

where $\mathscr{M} \in \mathrm{Pic}(Z)$. In the following, we first determine the value of d. Suppose that ξ is a fiber of the projection π. It follows from the defining equation (2.1.2) of X that the intersection number $[X] \cdot [\xi]$ is equal to k. On the other hand, we have $c_1(\underline{\mathcal{O}}_{\mathbb{P}(\mathcal{E})}(d)) \cdot [\xi] = d$ and $\pi^* c_1(\mathscr{M}) \cdot [\xi] = 0$. Hence, we obtain $d = k$.
Then we show $\mathscr{M} = \underline{\mathcal{O}}_Z(D)$. Recall that $\mathbb{V}(L)$ is the line bundle associated to $\mathcal{L}$ with local coordinates $(w_\alpha, \underline{z})$. The image C is locally defined by the equation $\omega_\alpha = 0$. We claim that X and C intersect transversely in $\hat{L}$ and the push-forward class $\pi_*([X] \cdot [C])$ is equal to the class of the branched lous $[D]$ and the first chern class $c_1((M))$ of $\mathscr{M}$ in $\mathrm{Pic}(Z)$. Indeed, if p is a point of $X \cap C$ such that

$$p = (0, \underline{z}) \text{ and } \Phi_\alpha(\underline{z}) = 0.$$

It is easy to see that the vector $\frac{\partial}{\partial w_\alpha}$ lies in the tangent space T_pX of X at p. Then the transversality condition

$$T_pX + T_pC = T_p\mathbb{V}(L).$$

is satisfied. Furthermore, the scheme-theoretic intersection of X and C is the reduced scheme $f^{-1}(D)_{\mathrm{red}}$ associated to $f^{-1}(D)$. Note that the ramified divisor $f^{-1}(D)_{\mathrm{red}}$ is isomorphic to the branched locus D via π. Therefore, we have

$$[D] = \pi_*[f^{-1}(D)_{\mathrm{red}}] = \pi_*([X]\cdot[C]).$$

Moreover, since C is the image of the section σ induced by $\mathcal{E} \to \underline{\mathcal{O}}_Z$, we have $\sigma^*\underline{\mathcal{O}}_{\mathbb{P}(\mathcal{E})}(1) = \underline{\mathcal{O}}_Z$. It implies that the intersection $c_1(\underline{\mathcal{O}}_{\mathbb{P}(\mathcal{E})}(1))\cdot[C]$ is zero. Therefore, through (2.2.1), we obtain the equalities as follows,

$$[D] = \pi_*([X]\cdot[C]) = \pi_*(c_1(\underline{\mathcal{O}}_{\hat{L}}(X))\cdot[C]) = \pi_*(\pi^*c_1(\mathscr{M})\cdot[C]) = c_1(\mathscr{M}).$$

It follows that

$$\underline{\mathcal{O}}_{\hat{L}}(X) = \underline{\mathcal{O}}_{\mathbb{P}(\mathcal{E})}(k)\otimes\pi^*\mathscr{M} = \underline{\mathcal{O}}_{\mathbb{P}(\mathcal{E})}(k)\otimes\pi^*\underline{\mathcal{O}}_Z(D) = \underline{\mathcal{O}}_{\mathbb{P}(\mathcal{E})}(k)\otimes\pi^*\mathcal{L}^k.$$

We prove our claim as well as the third assertion.
By the second and third assertions, we obtain $\underline{\mathcal{O}}_{\hat{L}}(X) = \underline{\mathcal{O}}_{\hat{L}}(C)^{\otimes k}$. It follows that

$$c_1(\underline{\mathcal{O}}_{\mathbb{P}(\mathcal{E})}(1))\cdot[X] = c_1(\underline{\mathcal{O}}_{\mathbb{P}(\mathcal{E})}(1))\cdot k[C] = 0,$$

in other words, we have $\underline{\mathcal{O}}_{\mathbb{P}(\mathcal{E})}(1)|_X = \underline{\mathcal{O}}_X$.

□

Lemma 2.3. *With the notations as above, we have*

$$\underline{\mathcal{O}}_X(f^{-1}(D)_{red}) = \underline{\mathcal{O}}_{\hat{L}}(C)|_X = f^*\mathcal{L}.$$

Proof. It follows from the results and the proof of the Lemma 2.2 immediately. □

Proposition 2.4. *With the notations as above, we have that:*

(i) $g^*\Omega^1_{\hat{L}/Z} = f^*\mathcal{L}^{-1}$*;*
(ii) the normal sheaf $N_{X/\hat{L}}$ *of* X *in* $\hat{L}$ *is isomorphic to* $f^*\mathcal{L}^k$*;*
(iii) the canonical sheaf κ_X *of* X *is isomorphic to* $f^*(\kappa_Z\otimes\mathcal{L}^{k-1})$*, where* κ_Z *is the canonical sheaf of* Z*.*

Proof. (i) Consider the Euler sequence of sheaves

$$0 \to \Omega^1_{\hat{L}/Z} \to \pi^*\mathcal{E}\otimes\underline{\mathcal{O}}_{\mathbb{P}(\mathcal{E})}(-1) \to \underline{\mathcal{O}}_{\mathbb{P}(\mathcal{E})} \to 0. \tag{2.4.1}$$

It follows that $\Omega^1_{\hat{L}/Z} = \wedge^2(\pi^*\mathcal{E}\otimes\underline{\mathcal{O}}_{\mathbb{P}(\mathcal{E})}(-1)) = \pi^*\mathcal{L}^{-1}\otimes\underline{\mathcal{O}}_{\mathbb{P}(\mathcal{E})}(-2)$. From Lemma 2.2 (iv), we see that $\underline{\mathcal{O}}_{\mathbb{P}(\mathcal{E})}(2)|_X$ is trivial. Therefore, It implies that $g^*\Omega^1_{\hat{L}/Z} = f^*\mathcal{L}^{-1}$.

(ii) The same argument gives $N_{X/\hat{L}} = \underline{\mathcal{O}}_{\hat{L}}(-X)|_X = f^*\mathcal{L}^k$.

(iii) By the adjunction formula, we have

$$\kappa_X = \kappa_{\hat{L}}|_X\otimes N_{X/\hat{L}},$$

where $\kappa_{\hat{L}}$ is the canonical bundle of $\hat{L}$. Again it follows from the short exact sequence (2.4.1) that $\kappa_{\hat{L}}|_X = f^*(\kappa_Z\otimes\mathcal{L}^{-1})$. Hence, it follows that

$$\kappa_X = f^*(\kappa_Z\otimes\mathcal{L}^{k-1}).$$

□

3. Deformation Theory and Infinitesimal Torelli Theorem

In this section, we show that the deformations of some automorphisms of a cyclic coverings are unobstructed, cf. Theorem 3.9. We start this section by recalling the theory of equivariant deformations.
Let X be a smooth and proper scheme over a field k. Assume that G is a finite subgroup of the automorphism group $\mathrm{Aut}_k(X)$ with the natural inclusion

$$\iota : G \hookrightarrow \mathrm{Aut}_k(X).$$

Denote by $\mathscr{C}_k$ the category of Artinian local k-algebras with residue field k. An infinitesimal deformation of (X, ι) over an Artinian local k-algebra A is a triple $(\mathcal{X}, \tilde{\iota}, \psi)$ consisting of a scheme $\mathcal{X}$ which is flat and proper over A, an injective group homomorphism

$$\tilde{\iota} : G \hookrightarrow \mathrm{Aut}_A(\mathcal{X})$$

and an isomorphism

$$\psi : \mathcal{X} \times_{\mathrm{Spec}(A)} \mathrm{Spec}(k) \to X$$

of schemes over k such that $\tilde{\iota}_{|X} = \iota$ via the natural restriction $\mathrm{Aut}_A(\mathcal{X}) \to \mathrm{Aut}_k(X)$ induced by ψ. Two infinitesimal deformations $(\mathcal{X}, \tilde{\iota}, \psi)$ and $(\mathcal{X}', \tilde{\iota}', \psi')$ are isomorphic if there exists an isomorphism

$$\Phi : \mathcal{X} \to \mathcal{X}'$$

over A which induces the identity on the closed fiber X and $\Phi \circ \tilde{\iota}(\sigma) = \tilde{\iota}'(\sigma) \circ \Phi$ for any $\sigma \in G$.

Definition 3.1. *With the notations as above, the equivairant deformation functor*

$$\mathrm{Def}_X^G : \mathscr{C}_k \to \textbf{\textit{Sets}}$$

assigns each $A \in \mathscr{C}_k$ to the set $\mathrm{Def}_X^G(A)$ consisting of isomorphism classes of infinitesimal deformations of (X, ι) over A.

Definition 3.2. *Suppose that F and H are covariant functors from $\mathscr{C}_k$ to **Sets**. A morphism $F \to H$ is called smooth if for any surjection $B \to A$ in $\mathscr{C}_k$, the map*

$$F(B) \to F(A) \times_{H(A)} H(B)$$

is surjective.

The covariant functors from $\mathscr{C}_k$ to **Sets** are called *functors of Artin rings* in [Sch68]. We refer to the following proposition which is used to prove the smoothness of morphisms of *functors of Artin rings.*

Proposition 3.3. *[Ser06, Proposition 2.3.6] Let $\mathscr{C}_k$ be the category of Artinian local k-algebras. Suppose that F (resp. H) is the functor of Artin rings having a semiuniversal formal element and an obstruction space $obs(F)$ (resp. $obs(H)$). Let $k[\epsilon] \in \mathscr{C}_k$ be the dual number and $t_F := F(k[\epsilon])$ be the space of first-order deformations. If a morphism $h : F \longrightarrow H$ satisfies the following two conditions:*

1. *the tangent map $dh : t_F \to t_H$ is surjective;*
2. *the obstruction map $\delta : obs(F) \to obs(H)$ is injective,*

then h is smooth.

Recall that G is a finite subgroup of $\mathrm{Aut}_k(X)$. It is known that the space of first-order equivariant deformations $\mathrm{Def}^G_X(k[\epsilon])$ (resp. obstruction space $obs(\mathrm{Def}^G_X)$) is isomorphic to the G-invariant part of $H^1(X, \Theta_X)$ (resp. $H^2(X, \Theta_X)$), where Θ_X is the tangent sheaf of X. We denotes them by $H^1(X, \Theta_X)^G$ (resp. $H^2(X, \Theta_X)^G$). For the details, we refer to [BM00, Proposition 3.2.1 and 3.2.3], in which the results are built on curves but also hold for higher-dimensional smooth projective varieties. Let h be the forgetful functor

$$h : \mathrm{Def}^G_X \to \mathrm{Def}_X \tag{3.3.1}$$

which associates to an infinitesimal deformation $(\mathcal{X}, \tilde{\iota}, \psi)$ over A, the underlying infinitesimal deformation $\mathcal{X}$ over A. Then the associated tangent map

$$\mathrm{Def}^G_X(k[\epsilon]) = H^1(X, \Theta_X)^G \xrightarrow{dh} H^1(X, \Theta_X) = \mathrm{Def}_X(k[\epsilon])$$

and the obstruction map

$$obs(\mathrm{Def}^G_X) = H^2(X, \Theta_X)^G \xrightarrow{\delta} H^2(X, \Theta_X) = obs(\mathrm{Def}_X)$$

are both natural inclusions. In the following, we assume that the field k is the field of complex numbers $\mathbb{C}$.

Proposition 3.4. *Use the same notations as above. Suppose that n is the dimension of X and the cup product*

$$\lambda_p : H^1(X, \Theta_X) \to \mathrm{Hom}(H^{n-p}(X, \Omega^p_X), H^{n-p+1}(X, \Omega^{p-1}_X)) \tag{3.4.1}$$

is injective for some p. If the group G acts trivially on $H^n(X, \mathbb{C})$, then the forgetful functor h in (3.3.1) is smooth.

Proof. We specialize Proposition 3.3 to our case for $F = \mathrm{Def}^G_X$ and $H = \mathrm{Def}_X$. The condition (2) of Proposition 3.3 is automatically satisfied. In order to verify the condition (1), we use the following lemma. □

Lemma 3.5. *Let X be a smooth and proper scheme over $\mathbb{C}$ of dimension n , and let G be a finite group of automorphisms. Assume that the group G acts trivially on $H^n(X, \mathbb{C})$, and for some integer p, the cup product map λ_p (3.4.1) is injective. Then the cohomology $H^1(X, \Theta_X)$ is G-invariant.*

Proof. Note that the map λ_p is G-equivariant. It gives rise to the following diagram

$$\begin{array}{ccc} H^1(X, \Theta_X)^G & \longrightarrow & \mathrm{Hom}(H^{n-p}(X, \Omega^p_X), H^{n-p+1}(X, \Omega^{p-1}_X))^G \\ \downarrow & & \| \\ H^1(X, \Theta_X) & \longrightarrow & \mathrm{Hom}(H^{n-p}(X, \Omega^p_X), H^{n-p+1}(X, \Omega^{p-1}_X)). \end{array} \tag{3.5.1}$$

The right vertical identity follows from the assumption that the action of G on $H^n(X, \mathbb{C})$ is trivial. Therefore, the injectivity of λ_p implies that

$$H^1(X, \Theta_X)^G = H^1(X, \Theta_X).$$

□

The following infinitesimal Torelli theorem of cyclic coverings is due to Wehler.

Theorem 3.6. *([Weh86, Theorem 4.8]) Let X be a smooth cyclic covering of $\mathbb{P}^n_{\mathbb{C}}$ of dimension $n \geq 2$. Then the cup product*

$$\lambda_p : H^1(X, \Theta_X) \to \mathrm{Hom}(H^{n-p}(X, \Omega^p_X), H^{n-p+1}(X, \Omega^{p-1}_X))$$

is injective for some p with the only exceptions

- *X is a 3-fold covering of $\mathbb{P}^2$ branched along a cubic curve;*
- *X is a 2-fold covering of $\mathbb{P}^2$ branched along a quartic curve.*

In the following, we introduce some notions for stating Theorem 3.7.
Let $f : X \to Z$ be a morphism between schemes over an algebraically closed field k, and let A be an artinian local k-algebra. An infinitesimal deformation $(\mathcal{X}, F)$ over A of the morphism f is cartesian diagrams

$$\begin{array}{ccc} X & \longrightarrow & \mathcal{X} \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle F} \\ Z & \longrightarrow & Z \times \mathrm{Spec}(A) \\ \downarrow & & \downarrow{\scriptstyle p} \\ \mathrm{Spec}(k) & \longrightarrow & \mathrm{Spec}(A) \end{array} \tag{3.6.1}$$

such that F is flat. Two deformations $(\mathcal{X}, F)$ and $(\mathcal{Y}, G)$ are isomorphic if there exists an isomorphism $\psi : \mathcal{X} \simeq \mathcal{Y}$ such that $G \circ \psi = F$ and the restriction of ψ to the closed fiber X gives the identity Id_X. Then it defines a functor of Artin rings $\mathrm{Def}_{X/Z}$ by setting

$$\mathrm{Def}_{X/Z}(A) = \{\text{isomorphic classes of infinitesimal deformations of } f \text{ over } A\}.$$

Naturally, the forgetful map $\varrho : \mathrm{Def}_{X/Z} \to \mathrm{Def}_X$ assigns a deformation of the form (3.6.1) to the deformation $p \circ F : \mathcal{X} \to \mathrm{Spec}(A)$ of X over A. The functor $\mathrm{Def}_{X/Z}$ is called the local Hilbert functor H^X_Z if f and F in the diagram (3.6.1) are closed immersions. In this case, we denote the forgetful functor by $\delta : H^X_Z \to \mathrm{Def}_X$.

Theorem 3.7. *([Weh86, Theorem 3.9]) With the notations as in Proposition 2.4, we assume that Z is the projective space $\mathbb{P}^n_{\mathbb{C}}$ for $n \geq 2$. If X is not a $K3$-surface, then the forgetful maps $\varrho : \mathrm{Def}_{X/\mathbb{P}^n} \to \mathrm{Def}_X$ and $\delta : H^X_{\hat{L}} \to \mathrm{Def}_X$ are both smooth.*

Proposition 3.8. *Let X be a smooth k-fold cyclic covering of $\mathbb{P}^n_{\mathbb{C}}$. The deformation functor Def_X is smooth.*

Proof. If $n = 1$ then it is obvious that Def_X is smooth. Therefore, we can assume that n is at least 2. If X is not a $K3$-surface, we can apply Theorem 3.7 and claim that the local Hilbert functor $H^X_{\hat{L}}$ is unobstructed. Then it follows from the smoothness of the forgetful map δ that Def_X is also unobstructed. Indeed, by Proposition 2.4 (iii) the obstruction space of the local Hilbert functor $H^X_{\hat{L}}$ is

$$H^1(X, N_{X/\hat{L}}) = H^1(X, f^*\mathcal{L}^k).$$

Moreover, we have

$$H^1(X, f^*\mathcal{L}^k) = H^1(\mathbb{P}^n_{\mathbb{C}}, \mathcal{L}^k \otimes f_*\underline{\mathcal{O}}_X) = \bigoplus_{i=1}^{k} H^1(\mathbb{P}^n_{\mathbb{C}}, \mathcal{L}^i) = 0$$

by the projection formula and Definition 2.1. Thus we prove our claim. On the other hand, it is well known that the deformation functor Def_X of a K3-surface is unobstructed. Therefore our proposition follows. □

We state our main theorem of this section.

Theorem 3.9. *Suppose that X is a smooth k-fold cyclic covering of $\mathbb{P}^n_{\mathbb{C}}$. Let G be a finite subgroup of the automorphisms $\mathrm{Aut}(X)$. If G acts on $H^n(X, \mathbb{C})$ trivially, then the equivariant deformations of X with respect to the action induced by G are unobstructed, i.e., the functor Def^G_X is smooth and $\mathrm{Def}^G_X = \mathrm{Def}_X$ with the only exceptions*

- *X is a 3-fold covering of $\mathbb{P}^2$ branched along a cubic curve;*
- *X is a 2-fold covering of $\mathbb{P}^2$ branched along a quartic curve;*

Proof. By Proposition 3.4 and Theorem 3.6, we conclude that the forgetful functor $h : \mathrm{Def}^G_X \to \mathrm{Def}_X$ is smooth in our case. We prove the deformation functor Def_X is smooth in Proposition 3.8, then it follows that Def^G_X is also smooth. By Lemma 3.5, the differential map

$$\mathrm{Def}^G_X(\mathbb{C}[\epsilon]) = H^1(X, \Theta_X)^G \xrightarrow{dh} H^1(X, \Theta_X) = \mathrm{Def}_X(\mathbb{C}[\epsilon])$$

is an identity. It implies that $\mathrm{Def}^G_X = \mathrm{Def}_X$. □

Remark 3.10. *The theorem is equivalent to say that an automorphism in G can be deformed to an automorphism of any small deformation of X. Instead of using the equivariant deformation theory, the second author provides an alternative view point to prove this theorem from the variational Hodge conjectures for graph cycles, cf. [Pan16, Corollary 3.3].*

4. Automorphisms of Cyclic Coverings

In this section, we assume that X is a smooth k-fold cyclic covering of $\mathbb{P}^n$ over an algebraically closed field K. Let $g : X \to X$ be the automorphism that associates a point $(\omega_\alpha, \underline{z})$ in X (cf. (2.1.2)) to the point $(\varrho\omega_\alpha, \underline{z})$ where ϱ is a primitive k-th root of unity. We denotes $\mathrm{Cov}(X/\mathbb{P}^n)$ the group of covering transformations generated by g. In other words, we have

$$\mathrm{Cov}(X/\mathbb{P}^n) = \langle g \rangle = \mathbb{Z}/k\mathbb{Z}.$$

In the following, we show the finiteness of the automorphism group $\mathrm{Aut}(X)$, see Theorem 4.5. We start with a lemma.

Lemma 4.1. *Let $f : X \to \mathbb{P}^n$ be a smooth k-fold cyclic covering, and let σ be an automorphism of X. If σ satisfies the following two conditions:*

- *$\sigma^* f^* \underline{\mathcal{O}}_{\mathbb{P}^n}(1)$ is isomorphic to the line bundle $f^* \underline{\mathcal{O}}_{\mathbb{P}^n}(1)$;*
- *$\dim H^0(X, f^* \underline{\mathcal{O}}_{\mathbb{P}^n}(1)) = n + 1$,*

then the automorphism σ induces a unique automorphism μ of $\mathbb{P}^n$, which fits into the following commutative diagram

(4.1.1)
$$\begin{array}{ccc} X & \xrightarrow{\sigma} & X \\ \downarrow f & & \downarrow f \\ \mathbb{P}^n & \xrightarrow{\mu} & \mathbb{P}^n. \end{array}$$

Proof. The morphism f gives rise to global sections $s_i = f^*x_i$ of $f^*\underline{\mathcal{O}}_{\mathbb{P}^n}(1)$ for $i = \{0, 1, \cdots, n\}$. If $\dim H^0(X, f^*\underline{\mathcal{O}}_{\mathbb{P}^n}(1)) = n+1$, then the set $\{s_0, \cdots, s_n\}$ forms a basis of the complete linear system $|f^*\underline{\mathcal{O}}_{\mathbb{P}^n}(1)|$. If $\sigma^* f^*\underline{\mathcal{O}}_{\mathbb{P}^n}(1)$ is isomorphic to the line bundle $f^*\underline{\mathcal{O}}_{\mathbb{P}^n}(1)$, then it follows that $\sigma^* s_i = \sum_{j=0}^{n} \alpha_{ij} s_j$. Hence, the matrix $(\alpha_{ij})_{0\le i,j\le n}$ gives the desired automorphism

$$\mu\big([X_0 : X_1 :, \cdots, X_n]\big) = \big[\sum_{i=0}^{n} \alpha_{0i} X_i :, \cdots, \sum_{i=0}^{n} \alpha_{ni} X_i\big]$$

in $\mathrm{Aut}(\mathbb{P}^n)$.

□

Lemma 4.2. *Let X be smooth k-fold cyclic covering $f : X \to \mathbb{P}^n$ branched along a smooth hypersurface D. If X is not a hypersurfaces in $\mathbb{P}^{n+1}$, then*

$$\dim\ H^0(X, f^*\underline{\mathcal{O}}_{\mathbb{P}^n}(1)) = n+1.$$

Proof. Use the notations as in Definition 2.1 and Proposition 4.6. Assume that $\mathcal{L} = \underline{\mathcal{O}}_{\mathbb{P}^n}(m)$ such that $\mathcal{L}^k = \underline{\mathcal{O}}_{\mathbb{P}^n}(D)$. The hypothesis in the lemma is equivalent to say that m is strictly great than 1. Since f is a finite morphism, we have

$$\begin{aligned} H^0(X, f^*\underline{\mathcal{O}}_{\mathbb{P}^n}(1)) &= H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1) \otimes f_*\mathcal{O}_X) \\ &= \bigoplus_{i=0}^{k-1} H^0(\mathbb{P}^n, \underline{\mathcal{O}}_{\mathbb{P}^n}(1) \otimes \mathcal{L}^{-i}). \end{aligned}$$

Therefore we obtain $\dim H^0(X, \underline{\mathcal{O}}_X(1)) = n+1$ when $m > 1$. □

Proposition 4.3. *With the notations as above, we assume that the two assumptions in Lemma 4.1 hold for X and every auotmorphism $\sigma \in \mathrm{Aut}(X)$. Then we obtain a short exact sequence*

$$1 \longrightarrow \mathrm{Cov}(X/\mathbb{P}^n) \longrightarrow \mathrm{Aut}(X) \longrightarrow \mathrm{Aut}_L(D) \longrightarrow 1. \tag{4.3.1}$$

Here the group $\mathrm{Aut}_L(D)$ consists of the linear automorphisms of $D(\subseteq \mathbb{P}^n)$.

Proof. The linear automorphism μ associated to the automorphism σ in the diagram (4.1.1) preserves the ramified divisor D. Therefore, we obtain a homomorphism

$$\begin{aligned} \mathrm{Aut}(X) &\to \mathrm{Aut}_L(D) \\ \sigma &\to \mu|_D \end{aligned}$$

It is easy to see that the homomorphism $\mathrm{Aut}(X) \to \mathrm{Aut}_L(D)$ is surjective with kernel $\mathrm{Cov}(X/\mathbb{P}^n)$. □

In the following, we take $K = \mathbb{C}$. Note that Lemma 4.2 had shown the second condition in Lemma 4.1 holds for a smooth cyclic covering X if it is not a hypersurface. In the following proposition, we investigate the smooth cyclic coverings who satisfy the first condition.

Proposition 4.4. *Let X be a smooth k-fold covering $f : X \to \mathbb{P}^n$ branched along a smooth hypersurface D. If one of the following conditions hold:*

(1) $\dim X \ge 4$;

(2) $\dim X = 3$ *and the branch locus* D *is a smooth surface in* $\mathbb{P}^3$ *with* $\deg(D) \geq 4$,

then we have $\mathrm{Pic}(X) = \mathbb{Z} = \mathbb{Z}\langle f^*\underline{\mathcal{O}}_{\mathbb{P}^n}(1)\rangle$. *In particular,* $\sigma^* f^* \underline{\mathcal{O}}_{\mathbb{P}^n}(1)$ *is isomorphic to the line bundle* $f^*\underline{\mathcal{O}}_{\mathbb{P}^n}(1)$ *for any automorphism* σ *of* X.

Proof. Denote by B the reduced scheme $[f^{-1}(D)]_{red}$ associated to the scheme $f^{-1}(D)$. It is known that $f^*\mathcal{L} = \underline{\mathcal{O}}_X(B)$, see Lemma 2.3. Hence, the invertible sheaf $\underline{\mathcal{O}}_X(B)$ is ample.

(i) Suppose that the dimension of X is at least 4. Since $\underline{\mathcal{O}}_X(B)$ is ample, the Lefschetz hyperplane theorem gives the isomorphism

$$\mu^* : \mathrm{Pic}(X) \simeq \mathrm{Pic}(B).$$

induced by the inclusion $\mu : B \hookrightarrow X$. Moreover, we have the following natural commutative diagram

$$\begin{array}{ccc} \mathrm{Pic}(\mathbb{P}^n) & \xrightarrow{\nu^*} & \mathrm{Pic}(D) \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f|_B^*} \\ \mathrm{Pic}(X) & \xrightarrow{\mu^*} & \mathrm{Pic}(B), \end{array} \tag{4.4.1}$$

where ν^* is induced by the inclusion $\nu : D \hookrightarrow \mathbb{P}^n$. Again by the Lefschetz hyperplane theorem the restriction map ν^* is an isomorphism. Note that B is isomorphic to D via f, then it follows that

$$\mathbb{Z}\langle\underline{\mathcal{O}}_{\mathbb{P}^n}(1)\rangle = \mathrm{Pic}(\mathbb{P}^n) \overset{f^*}{\simeq} \mathrm{Pic}(X).$$

So we have $\mathrm{Pic}(X) = \mathbb{Z}\langle f^*\underline{\mathcal{O}}_{\mathbb{P}^n}(1)\rangle$.

(ii) Suppose that the dimension of X is 3. By the projection formula and Definition 2.1, the first and second cohomology group of the structure sheaf $\underline{\mathcal{O}}_X$ vanishes

$$H^j(X, \underline{\mathcal{O}}_X) \simeq H^j(\mathbb{P}^n, f_*\underline{\mathcal{O}}_X) = \bigoplus_{i=0}^{k-1} H^j(\mathbb{P}^3, \mathcal{L}^{-i}) = 0 \text{ for } j = 1, 2.$$

It follows that the cycle class map $c_1 : \mathrm{Pic}(X) \to H^2(X, \mathbb{Z})$ is an isomorphism for any smooth cyclic covering X of $\mathbb{P}^3$. In the following, we first show that the induced map $f^* : \mathrm{Pic}(\mathbb{P}^3) \to \mathrm{Pic}(X)$ is an isomorphism for a very general cyclic covering X with $\deg(D) \geq 4$. Note that the second cohomolgy group $H^2(-, \mathbb{Z})$ is a deformation invariant and the cycle class c_1 is an isomorphism as shown above, we conclude that $f^* : \mathrm{Pic}(\mathbb{P}^3) \to \mathrm{Pic}(X)$ is an isomorphism for any smooth cyclic covering X with $\deg(D) \geq 4$.

In fact, if D is a very general smooth surface in $\mathbb{P}^3$ with $\deg(D) \geq 4$, the Noether-Lefschetz Theorem yields an isomorphism $\nu^* : \mathrm{Pic}(\mathbb{P}^3) \to \mathrm{Pic}(D)$. Therefore, the induced map $\mu^* : \mathrm{Pic}(X) \to \mathrm{Pic}(B)$ is surjective since it has an inverse section $f^* \circ \nu^{*-1} \circ f^*|_B^{-1}$ (cf. the diagram (4.4.1)).

On the other hand, the induced map $H^2(X, \mathbb{Z}) \to H^2(B, \mathbb{Z})$ is injective by the Lefschetz hyperplane theorem, which implies that the induced map $\mu^* : \mathrm{Pic}(X) \to \mathrm{Pic}(B)$ is injective. Therefore, we have

$$\mathrm{Pic}(X) \simeq \mathrm{Pic}(B) \simeq \mathrm{Pic}(D) \simeq \mathrm{Pic}(\mathbb{P}^3) = \mathbb{Z}\langle\underline{\mathcal{O}}_{\mathbb{P}^n}(1)\rangle.$$

for a very general cyclic covering X of $\mathbb{P}^3$ branched along a smooth surface D with $\deg D \geq 4$. Then the assertion follows.

□

Theorem 4.5 (Finiteness of Automorphisms)**.** *Let X be a smooth cyclic covering of $\mathbb{P}^n_{\mathbb{C}}$ branched along a smooth hypersurface D of degree d. Assume that X is not a quadric hypersurface and the dimension of X is at least 3. Then the automorphism group* $\mathrm{Aut}(X)$ *is finite. Moreover, if X is very general, the automorphism group* $\mathrm{Aut}(X) = \mathrm{Cov}(X/\mathbb{P}^n)$.

Proof. Note that X is a quadric hypersurface if $d = 2$. In the rest we may assume that d is at least 3.

- X is a smooth hypersurface in $\mathbb{P}^{n+1}$ with $n \geq 3$. Poonen proves that for a smooth hypersurface $Y \subset \mathbb{P}^m$ of degree l, the linear automorphism group $\mathrm{Aut}_L(Y)$ is finite if $m \geq 2$ and $l \geq 3$ ([Poo05, Theorem 1.3]). Moreover, $\mathrm{Aut}(Y) = \mathrm{Aut}_L(Y)$ if $m \neq 2, l \neq 3$ or $m \neq 3, l \neq 4$([Poo05, Theorem 1.1]). Therefore, in our case it follows that $\mathrm{Aut}(X) = \mathrm{Aut}_L(X)$. In particular, the automorphism group $\mathrm{Aut}(X)$ is finite.
- X is not a hypersurface. Then Lemma 4.2 and Proposition 4.4 verify the assumptions in Lemma 4.1. Therefore, we can apply Proposition 4.3 to conclude that $\mathrm{Aut}(X)$ is finite if $\dim X \geq 4$ or if $\dim X = 3$ and $\deg D \geq 4$. Note that X is a smooth cubic 3-fold if $\dim X = 3$ and $\deg D = 3$, which is included in the above situation.

□

Proposition 4.6. *Let $f : X \to \mathbb{P}^n$ be a smooth k-fold cyclic covering branched along a smooth hypersurface D. Then the natural representation*

$$\psi : \mathrm{Cov}(X/\mathbb{P}^n) \to GL(\mathrm{H}^n(X, \mathbb{C})) \tag{4.6.1}$$

is faithful.

Proof. Let $\mathcal{L}$ be the line bundle on $\mathbb{P}^n$ such that $\mathcal{L}^k = \underline{\mathcal{O}}_{\mathbb{P}^n}(D)$. There is a decomposition of the sheaf of differential forms [EV92, Lemma 3.16]

$$f_*\Omega^q_X = \Omega^q_{\mathbb{P}^n} \oplus \bigoplus_{i=1}^{k-1} \Omega^q_{\mathbb{P}^n}(\log D) \otimes \mathcal{L}^{-i}. \tag{4.6.2}$$

If g is the generator of the group of covering transformations, for any integer m the transformation g^m acts on $\Omega^q_{\mathbb{P}^r}(\log D) \otimes \mathcal{L}^{-i}$ by multiplying ϱ^{mi}. It implies that the Hodge cohomology group $H^p(X, \Omega^q_X)$ splits into ϱ^{mi}-eigenvalue subspaces

$$H^p(X, \Omega^q_X) = H^p(\mathbb{P}^n, \Omega^q_{\mathbb{P}^n}) \oplus \bigoplus_{i=1}^{k-1} H^p(\mathbb{P}^n, \Omega^q_{\mathbb{P}^n}(\log D) \otimes \mathcal{L}^{-i}). \tag{4.6.3}$$

In particular, if $\psi(g^m)$ is $\mathrm{Id}_{H^n(X,\mathbb{C})}$, then m is equal to 0 modulo k. We thus prove the proposition. □

In the following, we show that $\mathrm{Aut}(X)$ acts faithfully on $H^2(X, \mathbb{C})$ separately for $\dim X = 2$. In higher dimensions, the proof of faithfulness use the results developed above, see Theorem 4.9.

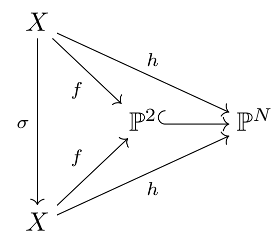
Recall that for a k-fold cyclic covering of $\mathbb{P}^2$, there is

$$\kappa_X = f^*(\kappa_{\mathbb{P}^2} \otimes \mathcal{L}^{k-1}), \tag{4.6.4}$$

see Proposition 2.4. We prove Proposition 4.8 with respect to the different type of κ_X. Let us start with a lemma.

Lemma 4.7. *Let X be a smooth cyclic covering $f: X \to \mathbb{P}^2$ of $\mathbb{P}^2$. Suppose that X is of general type. Then the action of $\mathrm{Aut}(X)$ on $H^2(X, \mathbb{C})$ is faithful.*

Proof. Since X is of general type, the canonical bundle κ_X is isomorphic to $f^*\underline{\mathcal{O}}_{\mathbb{P}^2}(d)$ for some positive integer d by the formula (4.6.4). The morphism f gives rise to global sections $\{s_i = f^*x_i, \text{ for } i = 0, 1, 2\}$, which generate the line bundle $f^*\underline{\mathcal{O}}_{\mathbb{P}^2}(1)$. Let $N = \binom{d+2}{d} - 1$, then the $d-$symmetric products $\{s_0^d, \dots, s_2^d\}$, as a linear system of the complete linear system $|\kappa_X|$, gives a map $h : X \to \mathbb{P}^N$ that factors as $X \xrightarrow{f} \mathbb{P}^2 \hookrightarrow \mathbb{P}^N$, where the later is the d-uple embedding. It is follows that the pullback $h^*(\underline{\mathcal{O}}_{\mathbb{P}^N}(1))$ is equal to the canonical bundle κ_X.
Assume that an automorphism $\sigma \in \mathrm{Aut}(X)$ acts trivially on $H^2(X, \mathbb{C})$. It is clear that $\sigma^* = \mathrm{Id}\,|_{H^0(X,\kappa_X)}$ and $\sigma^*h^*\underline{\mathcal{O}}_{\mathbb{P}^N}(1) = \sigma^*\kappa_X = h^*\underline{\mathcal{O}}_{\mathbb{P}^N}(1)$. Therefore, we have $h \circ \sigma = h$. It induces the following diagram



Therefore, the automorphism σ lies in $\mathrm{Cov}(X/\mathbb{P}^2)$. By Proposition 4.6, we obtain $g = \mathrm{Id}_X$. We prove the lemma. □

Proposition 4.8. *Let X be a smooth cyclic covering $f: X \to \mathbb{P}^2$ of $\mathbb{P}^2$. If X is not a quadric surface, then the action of $\mathrm{Aut}(X)$ on $H^2(X, \mathbb{C})$ is faithful.*

Proof. Assume that X is a k-fold covering and the $\mathcal{L}$ is the line bundle $\underline{\mathcal{O}}_{\mathbb{P}^2}(m)$ as in (4.6.4). The canonical bundle κ_X is either ample or trivial or anti-ample.
If κ_X is ample, then our proposition follows from Lemma 4.7.
If κ_X is trivial, then X is a $K3$ surface. In this case, the conclusion is well known.
If κ_X is anti-ample, i.e., X is a Fano surface, the possible cases are

- $(m, k) = (1, 2)$, X is a quadric surface in $\mathbb{P}^3$;
- $(m, k) = (2, 2)$, X is a 2-fold covering branched over a quartic curve;
- $(m, k) = (1, 3)$, X is a cubic surface in $\mathbb{P}^3$.

The last two cases are del Pezzo surfaces with degree 2 and 3 respectively. Hence, they are blowups of projective planes along 7 and 6 points in general position respectively. Denote the blowup by $\mathrm{Bl} : X \to \mathbb{P}^2$. If $\sigma^* = \mathrm{Id}$ on $H^2(X, \mathbb{C})$, then σ fixes the all the exceptional divisors. Hence, in both cases, the automrophism σ yields an automorphism $\rho \in \mathrm{Aut}(\mathbb{P}^2)$ with $\mathrm{Bl} \circ \sigma = \rho \circ \mathrm{Bl}$ and ρ fixes more than 4 points in general position. Then It follows that $\rho = \mathrm{Id}_{\mathbb{P}^2}$ and $\sigma = \mathrm{Id}_X$. □

Now we are able to give the answer to the question in the introduction.

Theorem 4.9. *Let $f : X \to \mathbb{P}^n_{\mathbb{C}}$ be a smooth k-fold cyclic covering over the complex numbers with $n \geq 2$. Suppose that X is not a quadric hypersurface. Then the natural representation*

$$\varphi : \mathrm{Aut}(X) \longrightarrow \mathrm{GL}(\mathrm{H}^n(X, \mathbb{C}))$$

is faithful.

Proof. If the dimension of X is 2, then the theorem follows from Proposition 4.8. We assume that dim$X$ is at least 3.
Let g be an automorphism of X such that $\varphi(g) = \mathrm{Id}$, and let G be the cyclic group generated by g. It follows from Theorem 4.5 that G is a finite group. Suppose that Y is a small deformation of X. It follows from Theorem 3.9 that the natural group action $G \times X \to X$ can be extended to a group action $G \times Y \to Y$ with $G \in \mathrm{Aut}(Y)$.
On the other hand, it follows from Theorem 3.7 that the small deformation Y remains a smooth cyclic covering of $\mathbb{P}^n$ branched along a smooth hypersurface D_Y. Note that the degree of D_Y is at least 3 since X is not a quadric hypersurface in our hypothesis, then the linear automorphism group $\mathrm{Aut}_L(D_Y)$ of a general smooth hypersurface D_Y is trivial, see [MM64]. Therefore, it follows from Proposition 4.3 that $\mathrm{Aut}(Y) = \mathrm{Cov}(Y/\mathbb{P}^n)$ for a general small deformation Y. By Proposition 4.6, we conclude that the group G is trivial. Therefore, the automorphism g of X can be deformed to an identity $g_Y = \mathrm{Id}_Y$ of Y. It implies that $g = \mathrm{Id}_X$ by specialization. □

5. Hodge Decomposition for Finite Cyclic Coverings

Let X be a smooth cyclic covering of a projective space $\mathbb{P}^n$ branched along a smooth hypersurface D over an algebraically closed field K. The algebraic de Rham cohomology of X is defined to be the hypercohomology of the algebraic de Rham complex

$$H^m_{DR}(X/K) := \mathbb{H}^m(X, \Omega^\bullet_{X/K}).$$

The Hodge-de Rham spectral sequence is given by

$$E_1^{p,q} = H^q(X, \Omega^p_{X/K}) \Rightarrow \mathbb{H}^{p+q}(X, \Omega^\bullet_{X/K})\,. \tag{5.0.1}$$

If $K = \mathbb{C}$, the classical Hodge theory shows that the spectral sequence (5.0.1) of X degenerates at the level E_1. It follows the Hodge decomposition

$$\bigoplus_{i+j=m} H^i(X, \Omega^j_{X/\mathbb{C}}) = H^m_{DR}(X/\mathbb{C}).$$

In this section, our goal is to show that the relative Hodge-de Rham spectral sequence of X degenerates at the level E_1 (see Theorem 5.8).
Deligne use Theorem 5.1 to show that the relative Hodge-de Rham spectral sequence of a projective bundle $\mathbb{P}(\mathcal{E})$ over a scheme S

$$E_1^{j,i} = R^i p_* \Omega^j_{\mathbb{P}(\mathcal{E})/S} \Rightarrow \mathbb{R}^{i+j} p_*(\Omega^\bullet_{\mathbb{P}(\mathcal{E})/S})$$

degenerates at the level E_1.

Theorem 5.1. *[DK73, Exposé XI, Theorem 1.1]*
Let $\mathcal{E}$ be a locally free sheaf of rank $r+1$ over a scheme S, and let $\mathbb{P}(\mathcal{E})$ be the associated projective bundle $p : \mathbb{P}(\mathcal{E}) \to S$ with the first chern class $\eta \in H^0(S, R^1 p_* \Omega^1_{\mathbb{P}(\mathcal{E})/S})$ of the invertible sheaf $\underline{\mathcal{O}}_{\mathbb{P}(\mathcal{E})}(1)$. Then we have:

1. *The sheaves $R^i p_* \Omega^j_{\mathbb{P}(\mathcal{E})/S}(n)$ are locally free and compatible with base change;*

(2) *For $0 \le i \le r$, the coherent sheaf $R^i p_* \Omega^i_{\mathbb{P}(\mathcal{E})/S}$ has rank one with the generator $\eta^i \in H^0(S, R^i p_* \Omega^i_{\mathbb{P}(\mathcal{E})/S})$. Furthermore,*

$$R^i p_* \Omega^j_{\mathbb{P}(\mathcal{E})/S} = 0 \quad \text{for } i \neq j \text{ or } i \geq r;$$

(3) *If $n \neq 0$, then $R^i p_* \Omega^j_{\mathbb{P}(\mathcal{E})/S}(n)$ are zero with the only exceptions*
   (a) *$i = 0$ and $n \geq j$,*
   (b) *$i = r$ and $n \leq j - r$.*

Deligne shows that the degeneration of the Hodge-de Rham spectral sequence holds for a smooth family of complete intersections by using the following proposition.

Proposition 5.2. *[DK73, Exposé XI, Proposition 1.3] Let $f : X \to S$ be a smooth and proper morphism over a noetherian scheme S, and let $\mathscr{F}$ be a coherent sheaf of X. Suppose that there is an integer $d \geq 0$ and W is a locally free sheaf of rank c over X together with a section $s : W \to \underline{\mathcal{O}}_X$ of $W^\vee$ such that*

(1) *the subscheme H of X defined by the zero locus of the section s is smooth over S,*
(2) *locally on X, the coordinates of the section s form a regular sequence with respect to $\underline{\mathcal{O}}_X$ and $\mathscr{F}$,*
(3) *for any nonzero integers k_i, we have*

$$R^i f_*(\otimes_i \wedge^{k_i} W \otimes \Omega^j_{X/S} \otimes \mathscr{F}) = 0 \text{ for all } i + j < d.$$

Then we have
(a) *$R^i f_*(\Omega^j_{X/S} \otimes \mathscr{F}) \xrightarrow{\sim} R^i f_*(\Omega^j_{H/S} \otimes \mathscr{F})$ for $i + j < d - c$,*
(b) *$R^i f_*(\Omega^j_{X/S} \otimes \mathscr{F}) \hookrightarrow R^i f_*(\Omega^j_{H/S} \otimes \mathscr{F})$ for $i + j = d - c$.*

In the following, we prove similar results as Proposition 5.2 for a smooth family of cyclic coverings(see Proposition 5.6). We start with a definition.

Definition 5.3. *Let S be a noetherian scheme and $p : \mathbb{P}^n_S \to S$ be a relative projective bundle. Denote by $\mathcal{L}$ the invertible sheaf $\underline{\mathcal{O}}_{\mathbb{P}^n_S}(l)$ where l is a positive integer. Assume that s is a section of $\mathcal{L}^k$ for some positive integer k such that the restriction of s to each fiber $\mathbb{P}^n_t (t \in S)$ defines a smooth hypersurface $D_t (\subseteq \mathbb{P}^n_t)$ of degree kl. As in Definition 2.1, the section s defines an $\underline{\mathcal{O}}_{\mathbb{P}^n_S}$-algebra*

$$\mathcal{A} = (\bigoplus_{i=0}^{k-1} \mathcal{L}^{-i}).$$

Let

$$f : \mathcal{X} := \underline{\mathrm{Spec}}(\mathcal{A}) \to \mathbb{P}^n_S$$

be the associated affine morphism. Denote by $D := Z(s)$ the zero locus of the section s. Naturally, it gives a family of k-fold cyclic coverings of $\mathbb{P}^n_S$ over S branched along D as follows

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\quad f \quad} & \mathbb{P}^n_S \\ & {\scriptstyle \pi}\searrow \quad \swarrow{\scriptstyle p} & \\ & S. & \end{array} \tag{5.3.1}$$

It is clear that D is a flat family over S, see [Mil80, Chapter I Proposition 2.5]. Through the rest of this section, we assume that the morphism π is smooth. In particular, the integer k is not divided by $\mathrm{char}(\kappa(t))$ for all $t \in S$, where $\kappa(t)$ is the residue field of the point t.
For any smooth morphism $h : X \to Y$ and a relative normal crossing divisor D of X over Y, the notion of logarithmic de Rham complex

$$\Omega^{\bullet}_{X/Y}(\log D)$$

is well defined (see [BDIP96, Section 7]). Then we have the following lemma.

Lemma 5.4. *With the same notations in Definition 5.3. We have that*

$$R^i f_* \Omega^j_{\mathcal{X}/S} = 0,\ \ i \neq 0; \tag{5.4.1}$$

$$f_*\Omega^j_{\mathcal{X}/S} = \Omega^j_{\mathbb{P}^n_S/S} \oplus \bigoplus_{\mu=1}^{k-1} \Omega^j_{\mathbb{P}^n_S/S}(\log D) \otimes \mathcal{L}^{-\mu}; \tag{5.4.2}$$

$$R^i\pi_*\Omega^j_{\mathcal{X}/S} = R^i p_*(\Omega^j_{\mathbb{P}^n_S/S} \oplus \bigoplus_{\mu=1}^{k-1} \Omega^j_{\mathbb{P}^n_S/S}(\log D) \otimes \mathcal{L}^{-\mu}). \tag{5.4.3}$$

Proof. By the construction of cyclic coverings, the morphism f is finite. Hence, the first assertion follows.
The absolute version of the decomposition (5.4.2) has been proved, see [EV92, Lemma 3.16 d)]. For the sake of completeness, we show the proof can even be carried out in the relative version.
Let $\mathbb{A}^n_S$ be an affine open subset of $\mathbb{P}^n_S$, and let $U \subset \mathcal{X}$ be the inverse image $f^{-1}(\mathbb{A}^n_S)$. Denote by s' the local defining equation of the branched locus $D \cap \mathbb{A}^n_S$ on $\mathbb{A}^n_S$. We may assume that the tuple $\{s', x_1, \cdots, x_{n-1}\}$ is a local coordinate system of the smooth morphism $p : \mathbb{A}^n_S \to S$, which induces a basis $\{ds', dx_1, \cdots, dx_{n-1}\}$ of the locally free sheaf $\Omega^1_{\mathbb{A}^n_S/S}$. Then the $\underline{\mathcal{O}}_{\mathbb{A}^n_S}$-module $\Omega^1_{\mathbb{A}^n_S/S}(\log D)$ is locally free of finite type with a basis $\{\frac{ds'}{s'}, dx_1, \cdots, dx_{n-1}\}$. Similarly, we have a local coordinate system $\{t', f^*x_1, \cdots, f^*x_{n-1}\}$ on U, where t' is the restriction of the tautological section $t \in H^0(\mathcal{X}, f^*\mathcal{L})$ to U. Denote by B the zero locus of the section t' in U. Then the associated $\underline{\mathcal{O}}_U$-module $\Omega^1_{U/S}(\log B)$ is locally free of finite type with a basis $\{\frac{dt'}{t'}, f^*dx_1, \cdots, f^*dx_{n-1}\}$.
Firstly, we show the relative Hurwitz's formula

$$f^*\Omega^j_{\mathbb{A}^n_S/S}(\log D) = \Omega^j_{U/S}(\log B).$$

Recall the Definition 5.3 that we have an equation $t'^k - f^*(s') = 0$ on U, cf. (2.1.2), which implies $f^*\frac{ds'}{s'} = \frac{dt'^k}{t'^k} = k \cdot \frac{dt'}{t'}$. In fact, we can invert the integer k in the $\Gamma(S, \underline{\mathcal{O}}_S)$-module $\Gamma(U, \underline{\mathcal{O}}_U)$ since k is not divided by the characteristic of the residue field of any point of S. Therefore, the differential form $f^*\frac{ds'}{s'}$ and sheaf $f^*\Omega^1_{\mathbb{A}^n_S/S}$ generate $\Omega^1_{U/S}(\log B)$. We obtain the relative Hurwitz's formula by exterior products.
By the relative Hurwitz's formula and the projection formula, we have a natural inclusion

$$f_*\Omega^j_{U/S} \subset f_*\Omega^j_{U/S}(\log B) = \Omega^j_{\mathbb{A}^n_S/S}(\log D) \otimes f_*\underline{\mathcal{O}}_U = \bigoplus_{i=0}^{k-1} \Omega^j_{\mathbb{A}^n_S/S}(\log D) \otimes \mathcal{L}^{-i}.$$

We claim that, indeed, the subsheaf

$$\Omega^j_{\mathbb{A}^n_S/S} \oplus \bigoplus_{i=1}^{k-1} \Omega^j_{\mathbb{A}^n_S/S}(\log D) \otimes \mathcal{L}^{-i} \subseteq \bigoplus_{i=0}^{k-1} \Omega^j_{\mathbb{A}^n_S/S}(\log D) \otimes \mathcal{L}^{-i}$$

is $f_*\Omega^j_{U/S}$. Let σ be a local section of $\Omega^j_{\mathbb{A}^n_S/S}(\log D) \otimes \mathcal{L}^{-i}$ written as

$$\sigma = \psi \cdot s'^i$$

for some local section ψ of $\Omega^j_{\mathbb{A}^n_S/S}(\log D)$ and the local generator s' of $\mathcal{L}^{-1}$. Moreover, the section ψ is of the form

$$\omega \wedge \frac{ds'}{s'} \text{ or } \omega$$

where the local section ω has no pole along D. Therefore, the pullback of the section σ is given by

$$\begin{aligned} f^*\sigma &= k \cdot f^*\omega \wedge \frac{dt'}{t'} \cdot f^*s'^i = k \cdot f^*\omega \wedge \frac{dt'}{t'} \cdot t'^i \text{ or} \\ f^*\sigma &= f^*\omega \cdot f^*s'^i = f^*\omega \cdot t'^i. \end{aligned}$$

Note that σ lies in $f_*\Omega^j_{U/S}$ if and only if the differential form $f^*\sigma$ has no pole along the divisor B. Therefore, the local section σ lies in $f_*\Omega^j_{U/S}$ if and only if $i \geq 1$ or $i = 0, \psi = \omega$. We prove the second assertion.

Using the diagram (5.3.1), we obtain the Leray spectral sequence

$$E_2^{a,b} = R^a p_* R^b f_* \Omega^j_{\mathcal{X}/S} \Longrightarrow R^i \pi_* \Omega^j_{\mathcal{X}/S},\ a + b = i. \tag{5.4.4}$$

By the first assertion, we have $E_2^{a,b} = 0$ unless $b = 0$. Therefore, the spectral sequence (5.4.4) degenerates and it follows from the second assertion that

$$R^i \pi_* \Omega^j_{\mathcal{X}/S} = E_\infty^{i,0} = E_2^{i,0} = R^i p_* (\Omega^j_{\mathbb{P}^n_S/S} \oplus \bigoplus_{\mu=1}^{k-1} \Omega^j_{\mathbb{P}^n_S/S}(\log D) \otimes \mathcal{L}^{-\mu}).$$

□

Proposition 5.5. *With the notations as in Definition 5.3. Then we have*

$$R^i \pi_* (\Omega^j_{\mathcal{X}/S} \otimes f^* \mathcal{L}^{-m}) = 0,\ i + j < n, m \geq 1.$$

Proof. A similar arguement as in the proof of Lemma 5.4 (5.4.4) gives

$$R^i \pi_* (\Omega^j_{X/S} \otimes f^* \mathcal{L}^{-m}) = R^i p_* (\Omega^j_{\mathbb{P}^n_S/S} \otimes \mathcal{L}^{-m} \oplus \bigoplus_{\mu=1}^{k-1} \Omega^j_{\mathbb{P}^n_S/S}(\log D) \otimes \mathcal{L}^{-\mu} \otimes \mathcal{L}^{-m}).$$

Note that $R^i p_* (\Omega^j_{\mathbb{P}^n_S/S} \otimes \mathcal{L}^{-m}) = 0$, see Theorem 5.1. Therefore, it suffices to prove

$$R^i p_* (\Omega^j_{\mathbb{P}^n_S/S}(\log D) \otimes \underline{\mathcal{O}}_{\mathbb{P}^n_S}(l)) = 0 \text{ for } i + j < n \text{ and } l < 0. \tag{5.5.1}$$

For simplicity, we denote by $\mathcal{A}^\vee$ the line bundle $\underline{\mathcal{O}}_{\mathbb{P}^n_S}(l)$. Note that there is a short exact sequence of residue map

$$0 \to \Omega^j_{\mathbb{P}^n_S/S} \to \Omega^j_{\mathbb{P}^n_S/S}(\log D) \xrightarrow{res} \iota_* \Omega^{j-1}_{D/S} \to 0 \tag{5.5.2}$$

where $\iota : D \hookrightarrow \mathbb{P}^n_S$ is the natural inclusion. In order to prove (5.5.1), it suffices to show

$$R^i p_*(\Omega^j_{\mathbb{P}^n_S/S} \otimes \mathcal{A}^\vee) = 0 \text{ for } i+j<n$$

and

$$R^q p_*(\iota_* \Omega^p_{D/S} \otimes \mathcal{A}^\vee) = 0 \text{ for } q+p<\dim D.$$

By Theorem 5.1 again, we have $R^i p_*(\Omega^j_{\mathbb{P}^n_S/S} \otimes \mathcal{A}^\vee) = 0$ for $i+j<n$ since the invertible sheaf $\mathcal{A}^\vee$ is anti-ample. Hence, in the following, we show that $R^q p_*(\iota_* \Omega^p_{D/S} \otimes \mathcal{A}^\vee) = 0$ for $q+p<\dim D$

Let d be the degree of the smooth divisor D. Then there is a natural resolution

$$0 \to \underline{\mathcal{O}}_D(-p\cdot d) \to \Omega^1_{\mathbb{P}^n_S/S}(-(p-1)\cdot d)|_D \to \cdots \to \Omega^p_{\mathbb{P}^n_S/S}|_D \to \Omega^p_{D/S} \to 0.$$

of the sheaf of relative Kähler differentials $\Omega^p_{D/S}$. Tensoring the resolution with $\iota^*\mathcal{A}^\vee$, we obtain a complex $\mathcal{K}^\bullet$ whose a-th term K^a is $(\Omega^a_{\mathbb{P}^n_S/S}(-(p-a)\cdot d)\otimes\mathcal{A}^\vee)|_D$. Then the hypercohomology spectral sequence for the complex $\mathcal{K}^\bullet$ is

$$E_1^{a,b} = R^b p_*(\Omega^a_{\mathbb{P}^n_S/S}(-(p-a)\cdot d)\otimes \mathcal{A}^\vee|_D)$$

that abuts to $\mathbb{R}^{a+b}p_*(\mathcal{K}^\bullet) = R^{a+b-p}p_*(\Omega^p_{D/S}\otimes\mathcal{A}^\vee|_D))$. We claim that

$$R^b p_*(\Omega^a_{\mathbb{P}^n_S/S}(-(p-a)\cdot d)\otimes\mathcal{A}^\vee|_D) = 0 \text{ for } a+b<\dim D. \tag{5.5.3}$$

To see this, we consider the short exact sequence

$$0 \to \Omega^a_{\mathbb{P}^n_S/S}(-d)\otimes\mathscr{L} \to \Omega^a_{\mathbb{P}^n_S/S}\otimes\mathscr{L} \to \Omega^a_{\mathbb{P}^n_S/S}\otimes\mathscr{L}|_D \to 0,$$

where the invertible sheaf $\mathscr{L}$ is the anti-ample invertible sheaf

$$\underline{\mathcal{O}}_{\mathbb{P}^n_S/S}(-(p-a)\cdot d)\otimes\mathcal{A}^\vee.$$

By Theorem 5.1 (3),we have

$$R^b p_*(\Omega^a_{\mathbb{P}^n_S/S}\otimes\mathscr{L}) = 0 \text{ for } a+b<n-1$$
$$R^{b+1} p_*(\Omega^a_{\mathbb{P}^n_S/S}(-d)\otimes\mathscr{L}) = 0 \text{ for } a+b<n-1.$$

Note that $\dim D = n-1$, our claim (5.5.2) follows and we are done.

□

Recall the notations in Definition 5.3. Let $\pi_L : \mathbb{V}(L) \to \mathbb{P}^n_S$ be the line bundle over $\mathbb{P}^n_S$ associated to the invertible sheaf $\mathcal{L}$. The k-fold cyclic covering $\mathcal{X}$ is the zero locus of the equation $t^k - \pi_L^*(s) = 0$ in $\mathbb{V}(L)$, where $t \in H^0(\mathbb{V}(L), \pi_L^*\mathcal{L})$ is the tautological section in. Let $i : \mathcal{X} \hookrightarrow \mathbb{V}(L)$ be the natural inclusion, and let $\mathcal{B}$ be the zero locus of the section $i^*(t)$, i.e., $\mathcal{B}$ is defined by the equations

$$t^k - \pi_L^*(s) = 0 \text{ and } t = 0.$$

on $\mathbb{V}(L)$. Therefore, the restriction map $f|_{\mathcal{B}} : \mathcal{B} \to D(:= Z(s))$ is an isomorphism, cf. Lemma 2.2.

Proposition 5.6. *Use the notations as above. Let $g : \mathcal{B} \to S$ be the smooth family of divisors over S with the natural commutative diagram*

$$\begin{array}{ccc} \mathcal{B} & \hookrightarrow & \mathcal{X} \\ & {\scriptstyle g}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & S\,. & \end{array} \tag{5.6.1}$$

We have that

(1) $R^i\pi_*(\Omega^j_{\mathcal{X}/S}) \xrightarrow{\sim} R^i g_*(\Omega^j_{\mathcal{B}/S})$ *for* $i+j<n-1$.
(2) $R^i\pi_*(\Omega^j_{\mathcal{X}/S}) \hookrightarrow R^i g_*(\Omega^j_{\mathcal{B}/S})$ *for* $i+j=n-1$.

Proof. Note that $\mathcal{B}$ is defined by the section $i^*(t) \in H^0(\mathcal{X}, i^*\pi_L^*\mathcal{L}) = H^0(\mathcal{X}, f^*\mathcal{L})$. It follows that $\underline{\mathcal{O}}_{\mathcal{X}}(\mathcal{B})$ is isomorphic to $f^*\mathcal{L}$. We replace $X/S = \mathcal{X}/S, \mathscr{F} = \underline{\mathcal{O}}_{\mathcal{X}}, W = f^*\mathcal{L}^{-1}$ in Proposition 5.2. The condition (3) in Proposition 5.2 is verified by Proposition 5.5. Therefore, the assertions follow from Proposition 5.2. □

Use the above proposition and the decomposition of the coherent sheaf $R^i\pi_*\Omega^j_{\mathcal{X}/S}$ in the Lemma 5.4. We prove the following lemma that is analogous to the assertion (2) of the Theorem 5.1.

Lemma 5.7. *With the notations as in Definition 5.3. Let* $\eta \in H^0(S, R^1p_*\Omega^1_{\mathbb{P}(\mathcal{E})/S})$ *be the first Chern class of the twisting sheaf* $\underline{\mathcal{O}}_{\mathbb{P}(\mathcal{E})}(1)$. *We have that*

(1) $R^i\pi_*\Omega^j_{\mathcal{X}/S} = 0$, *if* $i \neq j$ *and* $i+j \neq n$.
(2) $R^i\pi_*\Omega^i_{\mathcal{X}/S}$ *is an invertible sheaf generated by* $f^*\eta^i$ *if* $2i \neq n$, *where* $\eta^i = c_1(\underline{\mathcal{O}}_{\mathbb{P}(\mathcal{E})}(i))$ *is the generator of the invertible sheaf* $R^ip_*\Omega^i_{\mathbb{P}^n_S/S}$.

Proof. We have a series of identities as follows,

$$R^i\pi_*\Omega^j_{\mathcal{X}/S} \simeq R^ig_*\Omega^j_{\mathcal{B}/S} \simeq R^ip_*\Omega^j_{D/S} \simeq R^ip_*\Omega^j_{\mathbb{P}^n_S/S} \text{ for } i+j<n-1.$$

The first identity is the result of the Proposition 5.6, the second is induced by the isomorphism $f|_{\mathcal{B}} : \mathcal{B} \xrightarrow{\sim} D$, and the third one is Proposition 5.2. In particular, the map $f : \mathcal{X} \to \mathbb{P}^n_S$ induces the isomorphism

$$f^* : R^ip_*\Omega^j_{\mathbb{P}^n_S/S} \to R^i\pi_*\Omega^j_{\mathcal{X}/S} \text{ for } i+j<n-1.$$

Moreover, by Lemma 5.4 we have

$$R^i\pi_*\Omega^j_{\mathcal{X}/S} = R^ip_*(\Omega^j_{\mathbb{P}^n_S/S} \oplus \bigoplus_{\mu=1}^{k-1} \Omega^j_{\mathbb{P}^n_S/S}(\log D) \otimes \mathcal{L}^{-\mu}).$$

It follows from the claim (5.5.1) in Proposition 5.5 that

$$R^ip_*(\Omega^j_{\mathbb{P}^n_S/S}(\log D) \otimes \mathcal{L}^{-\mu}) = 0, \text{ for } i+j \leq n-1.$$

Therefore, $f^* : R^ip_*\Omega^j_{\mathbb{P}^n_S/S} \to R^i\pi_*\Omega^j_{\mathcal{X}/S}$ is an isomorphism for $i+j \leq n-1$. Thus the lemma is proved for $i+j \leq n+1$.
For $i+j>n$, we show that the assertions follow from the Serre duality. Denoted by $Tr_1 : R^n\pi_*\Omega^n_{\mathcal{X}/S} \to \underline{\mathcal{O}}_S$ the trace map of the projective morphism $f : \mathcal{X} \to S$. The nondegenerate pairing

$$\wp : R^i\pi_*\Omega^j_{\mathcal{X}/S} \times R^{n-i}\pi_*\Omega^{n-j}_{\mathcal{X}/S} \to \underline{\mathcal{O}}_S.$$

via the trace map Tr_1 shows that $R^i\pi_*\Omega^j_{\mathcal{X}/S} = 0$ if $i \neq j$. We claim that the class $f^*\eta^i$ of the coherent sheaf $R^i\pi_*\Omega^i_{\mathcal{X}/S}$ is the dual class of $f^*\eta^{n-i}$ via the pairing $\wp$ for $2i>n$. It is suffices to prove that $\wp(\eta^i, \eta^{n-i}) = Tr_1(\eta^n) = 1$. Note that we have isomorphisms

$$h_0 : R^n\pi_*\Omega^n_{\mathcal{X}/S} \simeq R^n p_*(\Omega^n_{\mathbb{P}^n_S/S} \oplus \bigoplus_{\mu=1}^{k-1} \Omega^n_{\mathbb{P}^n_S/S}(\log D) \otimes \mathcal{L}^{-\mu}) \text{ and}$$

$$h_1 : R^n p_*(\Omega^n_{\mathbb{P}^n_S/S} \oplus \bigoplus_{\mu=1}^{k-1} \Omega^n_{\mathbb{P}^n_S/S}(\log D) \otimes \mathcal{L}^{-\mu}) \simeq \bigoplus_{\mu=0}^{k-1} R^n p_*(\Omega^n_{\mathbb{P}^n_S/S} \otimes \mathcal{L}^{\mu})$$

since $\Omega^n_{\mathbb{P}^n_S/S}(\log D) = \Omega^n_{\mathbb{P}^n_S/S} \otimes \underline{\mathcal{O}}_{\mathbb{P}^n_S/S}(D)$. Then it follows from Theorem 5.1 that

$$\bigoplus_{\mu=0}^{k-1} R^n p_*(\Omega^n_{\mathbb{P}^n_S/S} \otimes \mathcal{L}^{\mu}) = R^n p_*\Omega^n_{\mathbb{P}^n_S/S}.$$

The isomorphism $h_0 \circ h_1$ can be identified with the pullback

$$f^* : R^n p_*\Omega^n_{\mathbb{P}^n_S/S} \to R^n\pi_*\Omega^n_{\mathcal{X}/S}.$$

Suppose that $Tr_2 : R^n p_*\Omega^n_{\mathbb{P}^n_S/S} \to \underline{\mathcal{O}}_S$ is the trace map of the projective space $\mathbb{P}^n_S$. We have $Tr_1 \circ f^* = Tr_2$. Therefore, it gives rise to the identities

$$Tr_1(f^*\eta^n) = Tr_2(\eta^n) = Tr_2(c_1(\underline{\mathcal{O}}_{\mathbb{P}(\mathcal{E})}(n))) = 1.$$

For $2i > n$, it follows that the invertible sheaf $R^i\pi_*\Omega^i_{\mathcal{X}/S}$ is generated by $f^*\eta^i$. We prove the lemma. ☐

Theorem 5.8. *With the notations as in Definition 5.3, we have that*

(1) the coherent sheaves $R^i\pi_*\Omega^j_{\mathcal{X}/S}$ and $R^m\pi_*\Omega^\bullet_{\mathcal{X}/S}$ are locally free and compatible with base change,

(2) and the Hodge-de Rham spectral sequence

$$E_1^{j,i} = R^i\pi_*\Omega^j_{\mathcal{X}/S} \Longrightarrow R^{i+j}\pi_*(\Omega^\bullet_{\mathcal{X}/S}) \tag{5.8.1}$$

degenerates at the level E_1.

Proof. Note that there exists a scheme $\tilde{S}$ which is smooth and of finite type over $\operatorname{Spec}\mathbb{Z}$ and a smooth family of k-fold cyclic coverings $\tilde{f} : \tilde{\mathcal{X}} \to \tilde{S}$ with a cartesian diagram

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & \tilde{\mathcal{X}} \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \tilde{\pi}} \\ S & \xrightarrow{\ \mu\ } & \tilde{S}. \end{array} \tag{5.8.2}$$

By Lemma 5.7, the coherent sheaf $R^i\tilde{\pi}_*\Omega^j_{\tilde{X}/\tilde{S}}$ is locally free if $i + j \neq n$. On the other hand, the function of Euler characteristic of $\Omega^j_{\tilde{\mathcal{X}}_s/\tilde{S}_s}$

$$s \mapsto \chi(\Omega^j_{\tilde{\mathcal{X}}_s/\tilde{S}_s})$$

is locally constant on $\tilde{S}$. Hence, for a fixed integer j, the upper semi-continuous function

$$s \mapsto \dim H^i(\tilde{\mathcal{X}}_s, \Omega^j_{\tilde{\mathcal{X}}_s})$$

is locally constant on $\tilde{S}$. In particular, since the scheme $\tilde{S}$ is reduced, the coherent sheaf $R^i\tilde{\pi}_*\Omega^j_{\tilde{X}/\tilde{S}}$ is locally free for $i + j = n$. Moreover, locally free sheaves are preserved by base change, see [BDIP96, Proposition 6.6 (c)]. Therefore we prove that $R^i\pi_*\Omega^j_{\mathcal{X}/S}$ are locally free for any i,j.

It suffices to prove our theorem in the "universal" case. More precisely, we assume that the coherent sheaf $R^n\tilde{\pi}_*\Omega^\bullet_{\tilde{\mathcal{X}}/\tilde{S}}$ is locally free and the Hodge-de Rham spectral sequence

$$\tilde{E}_1^{j,i} := R^i\tilde{\pi}_*\Omega^j_{\tilde{X}/\tilde{S}} \Longrightarrow R^{i+j}\tilde{\pi}_*\Omega^\bullet_{\tilde{\mathcal{X}}/\tilde{S}} \tag{5.8.3}$$

degenerates at the level $\tilde{E}_1$. It follows from [BDIP96, Proposition 6.6 (d)] that the base change map

$$\mu^*R^m\tilde{\pi}_*\Omega^\bullet_{\tilde{\mathcal{X}}/\tilde{S}} \to R^m\pi_*\Omega^\bullet_{\mathcal{X}/S}$$

is an isomorphism since $R^m\tilde{\pi}_*\Omega^\bullet_{\tilde{\mathcal{X}}/\tilde{S}}$ is locally free. Therefore, the coherent sheaf $R^n\pi_*\Omega^\bullet_{\mathcal{X}/S}$ is locally free. Moreover, by the degeneration of the spectral sequence (5.8.3), we have the identities

$$R^{2i}\pi_*\Omega^\bullet_{\mathcal{X}/S} \simeq \mu^*R^{2i}\tilde{\pi}_*\Omega^\bullet_{\tilde{\mathcal{X}}/\tilde{S}} \simeq \mu^*R^i\tilde{\pi}_*\Omega^i_{\tilde{X}/\tilde{S}} \simeq R^i\pi_*\Omega^i_{\mathcal{X}/S} \text{ for } 2i \neq n.$$

It follows that $R^{2i}\pi_*\Omega^\bullet_{\mathcal{X}/S}$ is an invertible sheaf for $2i \neq 0$. We claim that the differential map $d_r : E_r^{j,i} \to E_r^{j+r,i-r+1}$ is zero for the Hodge-de Rham spectral sequence

$$E_1^{j,i} := R^i\pi_*\Omega^j_{\mathcal{X}/S} \Longrightarrow R^{i+j}\pi_*\Omega^\bullet_{\mathcal{X}/S}. \tag{5.8.4}$$

In fact, the Hodge numbers are known as

- $h^{i,j}$= rank($R^i\pi_*\Omega^j_{\mathcal{X}/S}$)=0 if $i+j \neq n$ and $i \neq j$,
- $h^{i,i}$= rank($R^i\pi_*\Omega^i_{\mathcal{X}/S}$)=1 if $2i \neq n$.

It follows that the only possible nonzero differential maps are

$$d_r : E_r^{i,i} \to E_r^{i+r,i-r+1} \text{ for } 2i = n-1$$

and

$$d'_r : E_r^{j-r,j+r-1} \to E_r^{j,j} \text{ for } 2j = n+1.$$

To check the degeneration of the spectral sequence (5.8.4) is a local property on S. We may shrink S to an open affine subset Spec(A) such that all the locally free sheaves $R^i\pi_*\Omega^j_{\mathcal{X}/S}$ are presented by free A-modules.

If $d'_r : E_r^{j-r,j+r-1} \to E_r^{j,j}$ is the first nozero differential map, the E_{r+1}-term $E_{r+1}^{j,j}$ is isomorphic to a nontrivial quotient of A which contradicts to the fact that $R^{2j}\pi_*\Omega^\bullet_{\mathcal{X}/S}$ is isomorphic to A. Similarly, the differential map d_r is trivial too. Therefore, we prove that the Hodge-de Rham spectral sequence (5.8.4) degenerates conditionally.

Now we verify our assumption. In fact, we can shrink $\tilde{S}$ to be an affine scheme Spec(B), where B is a Noethrian domain. Let K be the fraction field of B, and let $\tilde{\mathcal{X}_K}$ be the induced scheme by base change. The associated spectral sequence

$$\tilde{E}_1^{j,i} := R^i\tilde{\pi}_*\Omega^j_{\tilde{\mathcal{X}_K}/K} \Longrightarrow R^{i+j}\tilde{\pi}_*\Omega^\bullet_{\tilde{\mathcal{X}}_K/K} \tag{5.8.5}$$

degenerates as an application of the classical result of Deligne and Illusie [DI87]. Then the degeneration of (5.8.3) follows from (5.8.5). We prove the theorem.

□

5.1. **The Infinitesimal Torelli Theorem.** In section 3, we discussed the infinitesimal Torelli theorem of a cyclic covering X of $\mathbb{P}^n_{\mathbb{C}}$, see Theorem 3.6. It turns out that this conclusion can be generalized to an arbitrary field.

The approach to prove Theorem 3.6 is to verify Flenner's criterion of the infinitesimal Torelli theorem [Fle86, Theorem 1.1]. This criterion has been generalized to arbitrary fields in [CPZ15, Appendix A].

Theorem 5.9. *[CPZ15, Appendix A] Let X be a smooth proper scheme of dimension n over a field K. Assume the existence of a resolution of $\Omega^1_{X/K}$ by locally free sheaves*

$$0 \to \mathcal{G} \to \mathcal{F} \to \Omega^1_{X/K} \to 0.$$

Let $D_r\mathcal{G}$ be the divided power $\mathrm{Sym}^r(\mathcal{G}^\vee)^\vee$, and let κ_X be the canonical sheaf of X. If following two conditions:

1. *$H^{j+1}(X, \mathrm{Sym}^j \mathcal{G} \otimes \Lambda^{n-j-1}\mathcal{F} \otimes \kappa_X^{-1}) = 0$ for $0 \le j \le n-2$;*
2. *the pairing*

$$H^0(X, D_{n-p}(\mathcal{G}^\vee) \otimes \kappa_X) \otimes H^0(X, D_{p-1}(\mathcal{G}^\vee) \otimes \kappa_X) \to H^0(X, D_{n-1}(\mathcal{G}^\vee) \otimes \kappa_X^2)$$

is surjective for a suitable positive integer p no larger than n

are satisfied, then the cup product map

$$\lambda_p : H^1(X, \Theta_X) \to \mathrm{Hom}(H^{n-p}(X, \Omega^p_{X/K}), H^{n+1-p}(X, \Omega^{p-1}_{X/K})) \tag{5.9.1}$$

is injective.

Theorem 5.10. *Let X be a smooth k-fold cyclic covering of $\mathbb{P}^n_K$ branched along the smooth divisor D over a field K. Suppose that n is at least 2 and k is prime to $\mathrm{char}(K)$. Then the cup product (5.9.1) is injective for X with the only exceptions*

- *X is a 3-fold covering of $\mathbb{P}^2_K$ branched along a cubic curve;*
- *X is a 2-fold covering of $\mathbb{P}^2_K$ branched along a quartic curve;*

Proof. Suppose that X is a hypersurface of degree k in $\mathbb{P}^{n+1}_K$. If X is not a cubic surface, the infinitesimal Torelli theorem of X had been proved in [CPZ15, Proposition A.9.]. Therefore, we may assume that X is not a hypersurface. We use the notations in the diagram (2.1.3) with $Z = \mathbb{P}^n_{\mathbb{C}}$. Let us apply Theorem 5.9 to the following natural resolution

$$0 \to f^*\mathscr{L}_D^{-1} \to g^*\Omega^1_{\hat{L}} \to \Omega^1_{X/K} \to 0, \tag{5.10.1}$$

where $\mathscr{L}_D = \mathcal{L}^k = \underline{\mathcal{O}}_{\mathbb{P}^n}(mk)$, for some $m > 0$. The calculation in the proof of [Weh86, Theorem 4.8] works well in our contents except two cases in which the characteristic is taken into account. In the following, we prove that the two cases are not ruled out.

• $(n, m, k) = (3, 2, 2)$. The condition (2) in Theorem 5.9 is independent of the characteristic. By the calculation in [Weh86, Theorem 4.8], the condition (1) in Theorem 5.9 is equivalent to

$$H^1(X, g^*\Omega^2_{\hat{L}} \otimes \kappa_X^{-1}) = 0.$$

We calculate the cohomology group $H^1(X, g^*\Omega^2_{\hat{L}} \otimes \kappa_X^{-1})$ by using the natural short exact sequence

$$0 \to f^*\Omega^2_{\mathbb{P}^3} \to g^*\Omega^2_{\hat{L}} \to f^*(\Omega^1_{\mathbb{P}^3} \otimes \mathcal{L}^{-1}) \to 0.$$

induced by (5.10.1). Note that $\kappa_X = f^*(\kappa_{\mathbb{P}^3} \otimes \mathcal{L})$ by Proposition 2.4 (iv), it follows that

$$\begin{aligned} H^1(X, f^*(\Omega^1_{\mathbb{P}^3} \otimes \mathcal{L}^{-1}) \otimes \kappa_X^{-1}) &= H^1(\mathbb{P}^3, \Omega^1_{\mathbb{P}^3} \otimes \kappa_{\mathbb{P}^3}^{-1} \otimes \mathcal{L}^{-2} \otimes f_*\underline{\mathcal{O}}_X) \\ &= \bigoplus_{l=0}^{1} H^1(\mathbb{P}^3, \Omega^1_{\mathbb{P}^3}(-2l)) = H^1(\mathbb{P}^3, \Omega^1_{\mathbb{P}^3}). \end{aligned}$$

and

$$\begin{aligned} H^2(X, f^*\Omega^2_{\mathbb{P}^3} \otimes \kappa_X^{-1}) &= H^2(\mathbb{P}^3, \Omega^2_{\mathbb{P}^3}(2) \otimes f_*\underline{\mathcal{O}}_X) \\ &= \bigoplus_{l=0}^{1} H^2(\mathbb{P}^3, \Omega^2_{\mathbb{P}^3}(2-2l)) = H^2(\mathbb{P}^3, \Omega^2_{\mathbb{P}^3}). \end{aligned}$$

Therefore, we have the exact sequence

$$0 \to H^1(X, g^*\Omega^2_{\hat{L}} \otimes \kappa_X^{-1}) \to H^1(\mathbb{P}^3, \Omega^1_{\mathbb{P}^3}) \xrightarrow{\delta} H^2(\mathbb{P}^3, \Omega^2_{\mathbb{P}^3}).$$

The connecting morphism δ is the cup product with the frist Chern class $c_1(\mathscr{L}_D)$, see [Weh86, Page 470]. Note that the degree of D is 4. The connecting morphism is injective if $\mathrm{char}(K) \neq 2$. Recall that the smoothness of X implies that k is prime to $\mathrm{char}(K)$. Therefore, $\mathrm{char}(K) \neq 2$ in our case and we obtain $H^1(X, g^*\Omega^2_{\hat{L}} \otimes \kappa_X^{-1}) = 0$. Hence the condition (1) is satisfied for $(n, m, k) = (3, 2, 2)$.

• $(n, m, k) = (2, 2, 3)$. In this case, the canonical sheaf $\kappa_X = f^*\underline{\mathcal{O}}_{\mathbb{P}^2}(1)$ is ample. We refer to a criterion in [LWP77, Theorem 1′] characterizing the cup product

$$\lambda_2 : H^1(X, \Theta_X) \to \mathrm{Hom}(H^0(X, \Omega^2_X), H^1(X, \Omega^1_X))$$

is injective. We note that the proof of [LWP77, Theorem 1′] is algebraic and holds for any characteristic though the statement is for a complex compact Kähler manifold. Moreover, as it emphasizes, the first two assumptions in [LWP77, Theorem 1] imply the hypothesis in [LWP77, Theorem 1′]. In our case, it suffices to verify the second assumption of [LWP77, Theorem 1], which is equivalent to verify $H^0(X, \Omega^1_X \otimes \underline{\mathcal{O}}_X(1)) = 0$.
By the projection formula and Lemma 5.4, we have

$$\begin{aligned} H^0(X, \Omega^1_X \otimes \underline{\mathcal{O}}_X(1)) &= H^0(\mathbb{P}^2, \underline{\mathcal{O}}_{\mathbb{P}^2}(1) \otimes f_*\Omega^1_X) \\ &= H^0(\mathbb{P}^2, \Omega^1_{\mathbb{P}^2}(1)) \oplus \bigoplus_{i=1}^{2} H^0(\mathbb{P}^2, \Omega^1_{\mathbb{P}^2}(\log D) \otimes \underline{\mathcal{O}}_{\mathbb{P}^2}(1) \otimes \mathcal{L}^{-i}) \\ &= \bigoplus_{i=1}^{2} H^0(\mathbb{P}^2, \Omega^1_{\mathbb{P}^2}(\log D) \otimes \underline{\mathcal{O}}_{\mathbb{P}^2}(1) \otimes \mathcal{L}^{-i}). \end{aligned}$$

We claim that

$$H^0(\mathbb{P}^2, \Omega^1_{\mathbb{P}^2}(\log D) \otimes \underline{\mathcal{O}}_{\mathbb{P}^2}(1) \otimes \mathcal{L}^{-i}) = 0, \ 1 \leq i \leq 2.$$

In fact, since the invertible sheaf $\underline{\mathcal{O}}_{\mathbb{P}^2}(1) \otimes \mathcal{L}^{-i}$ is negative. It is just a special case of what we proved in Proposition 5.5, cf. (5.5.1). Therefore we prove our claim and the infinitesimal Torelli theorem holds for this case. □

6. Automorphisms in Positive Characteristic

Let K be an algebraically closed field of positive characteristic. We recall the main theorem of the paper [Pan16].

Theorem 6.1. *[Pan16, Theorem 1.7] Let $\bar{X}$ be a smooth projective scheme over the Witt ring $W := W(K)$ of K, and let X be the special fiber over $\mathrm{Spec}(K)$. Assume that the Hodge-de Rham spectral sequences of $\bar{X}/W$ degenerates at E_1 and the terms are locally free. Let g_0 be an automorphism of X such that the map*

$$\mathrm{H}^i_{\mathrm{cris}}(g_0) : \mathrm{H}^n_{\mathrm{cris}}(X/W) \to \mathrm{H}^n_{\mathrm{cris}}(X/W)$$

preserves the Hodge filtrations under the natural identification

$$\mathrm{H}^n_{\mathrm{cris}}(X/W) \cong \mathrm{H}^n_{\mathrm{DR}}(\bar{X}/W).$$

If the cup product

$$\mathrm{H}^1(X, T_X) \to \bigoplus_{p+q=n} \mathrm{Hom}(\mathrm{H}^q(X, \Omega^p_{X/K}), \mathrm{H}^{q+1}(X, \Omega^{p-1}_{X/K}))$$

is injective, then one can lift g_0 to an automorphism $g : \bar{X} \to \bar{X}$ of $\bar{X}$ over W.

We are able to show the main theorem of this paper.

Lemma 6.2. *Let X be a smooth k-cyclic covering of $\mathbb{P}^n_K$ branched along a smooth hypersurface D. Suppose that k is not divided by $\mathrm{char}(K)$ and $n \geq 2$. Then $H^1(X, \underline{\mathcal{O}}_X) = H^0(X, \Omega^1_X) = 0$. Moreover, if $n = 2$ and the canonical bundle κ_X is ample or trivial, then $H^0(X, T_X) = 0$.*

Proof. It follows from Lemma 5.4 and (5.5.1) that $H^1(X, \underline{\mathcal{O}}_X)$ and $H^0(X, \Omega^1_X)$ are zero. Furthermore, we have

$$H^0(X, T_X) = H^2(X, \Omega^1_{X/K} \otimes \kappa_X)$$

Note that X can be lift to the Witt ring $W(K)$ of K as a smooth cyclic covering of $\mathbb{P}^n_W$. It follows from [EV92, Corollary 11.3] that

$$H^0(X, T_X) = H^2(X, \Omega^1_{X/K} \otimes \kappa_X) = 0$$

if κ_X is ample and $n = 2$. If κ_X is trivial and $n = 2$, then X is a $K3$ surface. It is well known that $H^0(X, T_X) = 0$ □

Lemma 6.3. *Let X be a smooth k-cyclic covering over $\mathbb{P}^n_K$ branched along $D(\subseteq \mathbb{P}^n_K)$. Suppose that k is not divided by char(K) and $n \geq 2$. Then the Néron-Severi group $\mathrm{NS}(X)$ is torsion free.*

Proof. In fact, by the universal coefficient theorem of crystalline cohomology, we have an short exact sequence

$$0 \to \mathrm{H}^1_{\mathrm{cris}}(X/W) \otimes_W K \to \mathrm{H}^1_{\mathrm{DR}}(X/K) \to \mathrm{Tor}^W_1(\mathrm{H}^2_{\mathrm{cris}}(X/W), K) \to 0 \tag{6.3.1}$$

where W is the Witt ring of K.
It follows from Lemma 6.2 that $\mathrm{Tor}^W_1(\mathrm{H}^2_{\mathrm{cris}}(X/W), K) = 0$, in other words, the crystalline cohomology $\mathrm{H}^2_{\mathrm{cris}}(X/W)$ is p-torsion free. By a theorem of Illusie and Deligne, see [Del81, Remark 3.5] and [Ill79], we have an injection

$$\mathrm{NS}(X) \otimes \mathbb{Z}_p \hookrightarrow \mathrm{H}^2_{\mathrm{cris}}(X/W).$$

We conclude that $\mathrm{NS}(X)$ is p-torsion-free.

On the other hand, we have the short exact sequence [Mil80, Chapter V, Remark 3.29 (d)]

$$0 \to \mathrm{NS}(X) \otimes \mathbb{Z}_l \to \mathrm{H}^2_{\text{ét}}(X, \mathbb{Z}_l(1)) \to T_l(\mathrm{Br}(X)) \to 0. \tag{6.3.2}$$

We claim that $\mathrm{H}^2_{\text{ét}}(X, \mathbb{Z}_l(1))$ is torsion free. Therefore, the group $\mathrm{NS}(X)$ is torsion-free.

In fact, we denote the natural lifting of X over W by $\overline{X}$. Choose an embedding $W \to \mathbb{C}$. We have the variety $\overline{X}_{\mathbb{C}}$ which is a k-cyclic covering over $\mathbb{P}^n_{\mathbb{C}}$. Since a cyclic covering over a projective space is a hypersurface in a weighted projective space, it is simply connected by [Dol82, Theorem 3.2.4 (ii)'].

By the universal coefficient theorem, we have

$$\mathrm{H}^2_{sing}(\overline{X}_{\mathbb{C}}, \mathbb{Z}_l) = \mathrm{Hom}(\mathrm{H}_2(\overline{X}_{\mathbb{C}}, \mathbb{Z}), \mathbb{Z}_l) = \varprojlim_n \mathrm{Hom}(\mathrm{H}_2(\overline{X}_{\mathbb{C}}, \mathbb{Z}), \mathbb{Z}/l^n\mathbb{Z})$$

$$= \varprojlim_n \mathrm{H}^2_{\text{ét}}(\overline{X}_{\mathbb{C}}, \mathbb{Z}/l^n\mathbb{Z}) = \mathrm{H}^2_{\text{ét}}(\overline{X}_{\mathbb{C}}, \mathbb{Z}_l) = \mathrm{H}^2_{\text{ét}}(X, \mathbb{Z}_l).$$

Since $\mathrm{H}^2_{sing}(\overline{X}_{\mathbb{C}}, \mathbb{Z}_l)$ is torsion free, the group $\mathrm{H}^2_{\text{ét}}(X, \mathbb{Z}_l)$ is torsion-free. The claim holds for X over K.

□

Denote by $\mathrm{Aut}(X)_{tr}$ the kernel of $\mathrm{Aut}(X) \to \mathrm{H}^n_{\text{ét}}(X, \mathbb{Q}_l)$ ($l \neq char(K)$).

Lemma 6.4. *Let X be a smooth k-cyclic covering over $\mathbb{P}^n_K$ branched along D. Suppose that k is not divided by* $\mathrm{char}(K)$ *and n is at least 2. If one of the following conditions holds:*

(1) the degree of D in $\mathbb{P}^n$ is at least 3 and $n \geq 3$;
(2) $n = 2$ and the canonical bundle κ_X is ample or trivial,

then $\mathrm{Aut}(X)_{tr}$ *is finite.*

Proof. By Lemma 6.2 and Lemma 6.3, we conclude that $\mathrm{NS}(X) = \mathrm{Pic}(X)$ is torsion free. Therefore, the following map

$$c_1 : \mathrm{Pic}(X) \to \mathrm{Pic}(X) \otimes \mathbb{Z}_l \to \mathrm{H}^2_{\text{ét}}(X, \mathbb{Z}_l(1)) \tag{6.4.1}$$

is injective by (6.3.2) where l is a prime different from $\mathrm{char}(K)$.

- For $n \geq 3$. We claim $\mathrm{Pic}(X) = \mathbb{Z}$. In fact, we can lift X to a cyclic covering $\overline{X}$ over complex numbers $\mathbb{C}$ with $\mathrm{H}^2_{\text{ét}}(X, \mathbb{Q}_l) = \mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Q}_l)$. By Proposition 4.4, we have $\mathrm{H}^2_{\text{ét}}(X, \mathbb{Q}_l) = \mathbb{Q}_l$. The claim follows from Lemma 6.3. (If the degree of D is 3, then X is a cubic hypersurface of dimension at least 3 and the statement still holds by the Grothendieck-Lefschetz theorem). The claim implies that every automorphism preserves the ample line bundle $f^*\underline{\mathcal{O}}_{\mathbb{P}^n}(1)$. Note that $\mathrm{Aut}_L(D)$ is finite if $\deg(D) \geq 3$ ([Poo05, Theorem 1.3]). It follows from Lemma 4.1, Lemma 4.2, and Proposition 4.3 that $\mathrm{Aut}(X)$ is finite.
- For $n = 2$. We choose a very ample line bundle L on X such that the complete linear system

  $$|L| : X \to \mathbb{P}^N$$

  induces an embedding. It follows from the injectivity of the map c_1 (6.4.1) and the torsion-freeness of $\mathrm{Pic}(X)$ that every automorphism $f \in \mathrm{Aut}_{tr}(X)$

fixes the line bundle L, i.e., $f^*L = L$. Therefore, we have a linear automorphism g inducing the following diagram

$$\begin{array}{ccc} X & \xrightarrow{|L|} & \mathbb{P}^N \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{|L|} & \mathbb{P}^N \end{array}.$$

Let $G = \{h \in \mathrm{PGL_{N+1}} | h(X) = X\}$. We have $\mathrm{Aut}_{tr}(X) \subseteq G$. We claim G is an subalgebraic group of $\mathrm{PGL_{N+1}}$. In fact, we consider the Hilbert scheme $\mathrm{Hilb}(\mathbb{P}^N)$ parametrizing X. There is a natural action of $\mathrm{PGL_{N+1}}$ on $\mathrm{Hilb}(\mathbb{P}^N)$. The stabilizer of this action of the point $[X]$ parametrizing X is G. Therefore, G is algebraic. On the other hand, the infinitesimal automorphism of X is trivial, i.e., $\mathrm{H}^0(X, T_X) = 0$ (cf. Lemma 6.2). It follows that G is a subgroup scheme of $\mathrm{PGL_{N+1}}$ with $\dim(G) = 0$, hence, it is finite. We conclude that $\mathrm{Aut}_{tr}(X)$ is finite.

□

Theorem 6.5. *Let K be an algebraically closed field of positive characteristic, and let X be a smooth cyclic covering of $\mathbb{P}^n_K$ with $n \geq 2$. If X is not a quadric hypersurface, then the action of the automorphism group $\mathrm{Aut}(X)$ on $\mathrm{H}^n_{ét}(X, \mathbb{Q}_l)$ ($l \neq char(K)$) is faithful, i.e., the natural map*

$$\mathrm{Aut}(X) \to \mathrm{Aut}(\mathrm{H}^n_{ét}(X, \mathbb{Q}_l))$$

is injective.

Proof. Suppose that g_0 is an automorphism of X and in

$$\mathrm{Aut}(X)_{tr} := \mathrm{Ker}(\mathrm{Aut}(X) \to \mathrm{H}^n_{\text{ét}}(X, \mathbb{Q}_l)).$$

By Proposition 2.4, we have the canonical bundle formula

$$\kappa_X = f^*(\kappa_{\mathbb{P}^n} \otimes \mathcal{L}^{k-1}) = f^*\underline{\mathcal{O}}_{\mathbb{P}^n}(m) \text{ for some } m.$$

Therefore, the canonical bundle κ_X is ample, or trivial or anti-ample.

- Assume that X satisfies two conditions in Lemma 6.4, i.e., one of the following condition is satisfied
  - the dimension $\dim X$ is at least 3,
  - $\dim X = 2$ and the canonical bundle κ_X is ample or trivial.

  It follows from Lemma 6.4 that g_0 is of finite order. Let W be the Witt ring of K. Note that

  $$\det(\mathrm{Id} - g_0^* t, \mathrm{H}^n_{\text{cris}}(X/W)_{W[\frac{1}{p}]}) = \det(\mathrm{Id} - g_0^* t, \mathrm{H}^n_{\text{ét}}(X, \mathbb{Q}_l)),$$

  see [KM74, Theorem 2] and [Ill75, 3.7.3 and 3.10]. The finiteness of the order of g_0 implies that $\mathrm{H}^n_{\text{ét}}(g_0, \mathbb{Q}_l) = \mathrm{Id}$ if and only if $\mathrm{H}^n_{\text{cris}}(g_0)_{W[\frac{1}{p}]} = \mathrm{Id}$ since both $\mathrm{H}^n_{\text{ét}}(g_0, \mathbb{Q}_l)$ and $\mathrm{H}^n_{\text{cris}}(g_0)_{W[\frac{1}{p}]}$ are diagonalizable. Note that X can be lift to W as a smooth cyclic covering of $\mathbb{P}^n_W$. Let $\overline{X}$ be a such lifting of the X over W. It follows from Theorem 5.8 that

  $$\mathrm{H}^n_{\text{cris}}(X/W) = \mathrm{H}^n_{\text{DR}}(\overline{X}/W)$$

  is a finite free W-module. Therefore, we have $\mathrm{H}^n_{\text{cris}}(g_0) = \mathrm{Id}$.

  By our assumption, we conclude that X is neither

– a 3-fold covering of $\mathbb{P}^2_K$ branched along a cubic curve,
– nor a 2-fold covering of $\mathbb{P}^2_K$ branched along a quartic curve.

Therefore, the assumptions of Proposition 6.1 hold for g_0 by Theorem 5.8 and Theorem 5.10. By Proposition 6.1, one can lift the automorphism g_0 to an automorphism g of $\overline{X}/W$. Therefore, the theorem follows from Theorem 4.9.

- If κ_X is anti-ample and $\dim X = 2$, i.e., X is a Fano surface. The possible types of X are listed in the proof of Proposition 4.8. Since every Fano surface is a blowup of the projective plane, we can use the same argument as in Proposition 4.8 to prove the theorem.

□

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