ISOPERIMETRIC PROBLEMS HAVING

CONTINUA OF SOLUTIONS

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The *isoperimetric problem* asks for the hypersurface of smallest area which bounds a given volume v_0 . In Euclidean $\mathbb{R}^{n+1}$, the unique solution - up to translations - is the sphere of the appropriate radius. The isometry group of $\mathbb{R}^{n+1}$ generates an (n+1)-parameter family of solutions, all of which are congruent to each other. A similar situation occurs in any symmetric space.

We are interested in finding a continuum, or family depending continuously on one or more real variables, of embedded stationary solutions which are *not congruent*. An embedded compact hypersurface Σ will be stationary for the isoperimetric problem in a Riemannian manifold M^{n+1} if and only if it bounds the prescribed volume v_0 and has *constant mean curvature*; this follows from the first-variation formula (equation (3) below). According to a theorem of Aleksandrov, when M is the Euclidean space (or, by analogous means, the hyperbolic space) Σ must be a standard sphere ([1]). The family we seek can therefore exist only for less well-understood Riemannian metrics.

Theorem. *There exists a Riemannian manifold M of any dimension $n+1 \geq 2$, in which the compact embedded hypersurfaces bounding volume $v_0 = 1$ and having constant mean curvature include a one-parameter family $\{\Sigma_t\}$ with distinct mean curvatures.*

Note that the hypersurfaces found in the theorem cannot be congruent, since their mean curvatures are different. This latter property also has consequences for the so-called Lagrange multiplier method. Namely, one may find hypersurfaces of constant mean curvature H as stationary solutions of a second variational problem, in which the functional $A(\Sigma) + n\,H\,V(D(\Sigma))$ is considered. Here $D(\Sigma)$ is a domain with boundary Σ ; V denotes $(n+1)$-dimensional volume; and A is the n-dimensional measure, which we also call "area". In this problem, no constraint is assumed, and the constant H is prescribed in advance. By contrast, in the isoperimetric problem, the resulting value $H = H(v_0)$ of constant mean curvature may be difficult to determine in advance. The two problems may be seen to be equivalent once it is known that the function $H(v_0)$ is strictly monotone. But according to our theorem, this function is no better than a relation which may assume an entire interval of values for a single number v_0. In particular, our example may be relevant to attempts to generalize Gerhardt's construction of constant-mean-curvature foliations of certain Lorentzian manifolds ([2]).

This paper was stimulated by our recent collaboration with Stefan Hildebrandt ([3]), in which a variety of boundary-value problems were considered. In each case, an example was found having an interesting continuum of solutions. In particular, we have exploited the idea which is implicit in sections 3 and 4 of [3], that one should first find a plausible family of submanifolds, and then construct the problem of which each is a solution.

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1. Deformation of metrics according to a foliation

It is easy to show that any codimension-one foliation has leaves of constant mean curvature in some metric. For our purpose, however, we need more control over the resulting metric. The following lemma will be applied with the Euclidean metric playing the role of ds^2 .

Lemma 1. *Let* $\{\Sigma_t : t_2 < t < t_3\}$ *be an oriented codimension-one foliation of the Riemannian manifold* (M^{n+1}, ds^2) *, with unit normal vector field* ν *. Suppose a second Riemannian metric* $d\tilde{s}^2$ *is introduced, so that* ν *remains normal to the leaves* Σ_t *. Write* $\psi(x)$ *for the* $d\tilde{s}^2$*-length of* ν *, and write the* $d\tilde{s}^2$*-area form of* Σ_t *as* $d\tilde{A}_{\Sigma_t} = (\varphi(x))^n dA_{\Sigma_t}$ *. Then the* $d\tilde{s}^2$*-mean curvature* $\tilde{H}$ *of* Σ_t *satisfies*

(1) $\psi\tilde{H} = H - d(\log \varphi)(\nu)$.

Proof: We shall compute the mean curvature of the leaf Σ_0 by variational means. Choose a test function $\eta : \Sigma_0 \longrightarrow \mathbb{R}$ having compact support, and let a family of hypersurfaces F_t be chosen starting from $F_0 = \Sigma_0$, such that the distance from F_0 to F_t along the integral curves of ν through x equals $t\eta(x) + 0(t^2)$, asymptotically as $t \longrightarrow 0$.

We first show that

(2) $$d\tilde{A}_{F_t} = [(\varphi(x))^n + 0(t^2)]\, dA_{F_t} \quad .$$

Let $\Theta(x,t)$ be the ds^2-angle between ν and the unit normal vector to F_t . Observe that $\Theta(x,t) = 0(t)$ as $t \longrightarrow 0$. If $\{e_1,\ldots\ldots,e_n\}$ is a local ds^2-orthonormal basis for $T\Sigma_t$ such that $e_2,\ldots\ldots,e_n$ are also tangent to F_t , then

$$\{e_1 \cos\Theta + \nu \sin\Theta,\ e_2,\ldots\ldots,e_n\}$$

is a ds^2-orthonormal basis for TF_t . Since $d\tilde{A}_{\Sigma_t} = \varphi^n\, dA_{\Sigma_t}$ by hypothesis, we have $\widetilde{\| e_1\wedge\ldots\ldots\wedge e_n \|} = \varphi^n$. It follows that

$$\widetilde{\| (e_1 \cos\Theta + \nu \sin\Theta)\wedge e_2\wedge\ldots\ldots\wedge e_n \|}^2 =$$

$$= \cos^2\Theta\, \widetilde{\| e_1\wedge\ldots\ldots\wedge e_n \|}^2 + \sin^2\Theta\, \widetilde{\| \nu\wedge e_2\wedge\ldots\ldots\wedge e_n \|}^2 ,$$

since ν is $d\tilde{s}^2$-orthogonal to Σ_t . But this expression equals φ^{2n} plus terms of order $O(t^2)$, and formula (2) follows.

Now the first-variation formula for area is

$$(3) \qquad \left.\frac{d}{dt}\right|_{t=0} \tilde{A}(F_t) = - \int_{F_0} n\, \tilde{H}(F_0,x)\tilde{\eta}(x)\,d\tilde{A}(x) \ ,$$

where $\tilde{\eta} = \psi\eta$ is the $d\tilde{s}^2$-length of the variation vector field $\eta\nu$. Thus using equation (2), we see that

$$\int_{F_0} n\, \tilde{H}\, \psi\, \eta\, \varphi^n dA = \int_{F_0} n\, \tilde{H}\, \tilde{\eta}\, d\tilde{A} =$$

$$= - \left.\frac{d}{dt}\right|_{t=0} \int_{F_0} (\varphi^n + O(t^2))\,dA = \int_{F_0} n[d(\log \varphi)(\nu) - H]\eta\varphi^n dA \ .$$

Since η is arbitrary, formula (1) follows. q.e.d.

<u>Remark 1</u>. It is also possible, although more difficult, to give a direct derivation of the formula (1) for the mean curvature using moving frames.

It should be apparent from formula (1) that the mean curvature of the foliation $\{\Sigma_t\}$ is most conveniently controlled by choosing φ to be a constant. In particular, this choice yields stronger results than, for example, a conformal deformation $(\varphi = \psi)$.

Corollary 1. Suppose that the family $\{\Sigma_t\}$ of hypersurfaces of a Riemannian manifold (M, ds^2) covers a subset Ω of M, and forms a foliation of $\Omega \setminus \Gamma$ for some closed subset Γ of M. Assume that Σ_t has constant mean curvature $H(\Sigma_t, x) = h(t)$ for all x in a neighborhood U of Γ, and that $H(\Sigma_t, x) > 0$ everywhere. Let the metric $d\tilde{s}^2$ be defined on $\Omega \cup U$ by

$$d\tilde{s}^2 := ds^2\Big|_{T\Sigma_t} + \psi^2\, ds^2\Big|_{N\Sigma_t}$$

where $\psi(x) := H(\Sigma_t, x)/h(t)$. Then in the metric $d\tilde{s}^2$, Σ_t has constant mean curvature $h(t)$.

Proof: We apply formula (1) with $\varphi \equiv 1$. On U, we have $\psi \equiv 1$, so that $d\tilde{s}^2 = ds^2$ and $\tilde{H}(\Sigma_t, x) = H(\Sigma_t, x) = h(t)$. Thus ψ is well defined, smooth and positive on $\Omega \cup U$. The conclusion now follows from (1). q.e.d.

2. Construction of the example

The proof of our theorem requires us to construct hypersurfaces which bound constant volume. However, the method of Lemma 1 allows only direct control of area. Fortunately, for hypersurfaces of constant mean curvature, constant volume and constant area are equivalent.

Lemma 2. _Suppose that_ $\{\Sigma_t\}$ _is a family of immersed compact hypersurfaces of a Riemannian manifold_ M , _and that_ Σ_t _is the boundary of a region_ D_t _with multiplicities_. _Suppose that each_ Σ_t _has constant mean curvature_ $h(t) \neq 0$, _and that its area_ $A(\Sigma_t)$ _is constant_. _Then the volume_ $V(D_t)$ _is constant_.

Proof: If $\eta : \Sigma_t \longrightarrow \mathbb{R}$ is the normal component of the variation vector field, then the first variation of volume is

$$\frac{d}{dt} V(D_t) = \int_{\Sigma_t} \eta \, dA .$$

Meanwhile, as in equation (3), the first variation of area is

$$\frac{d}{dt} A(\Sigma_t) = - \int_{\Sigma_t} n\, H(\Sigma_t, x)\, \eta(x)\, dA(x) = - n\, h(t) \frac{d}{dt} V(D_t) .$$

Since area is assumed constant and $h(t) \neq 0$, we conclude that volume is constant. q.e.d.

Remark 2. The hypothesis $h(t) \neq 0$ is necessary, as may be seen from examples in which M is locally the Riemannian product $\Sigma_0 \times [-1, 1]$.

In order to construct our example, we shall find a smooth family of strictly convex closed hypersurfaces Σ_t in $\mathbb{R}^{n+1}$, whose Euclidean area is constant. We may observe that such a family cannot be a foliation, while Lemma 1 is valid _only_ for foliations. Therefore, as in Corollary 1, we shall choose $d\tilde{s}^2 = ds^2$ on a neighborhood U of the singular set Γ of $\{\Sigma_t\}$. This requires that each hypersurface Σ_t must have constant Euclidean mean curvature on U . We shall choose Σ_t to be a piece of a sphere near Γ .

Choose values $r > \varepsilon > 0$, and let Γ be the $(n-1)$-sphere $\{(x^0,x') \in \mathbb{R} \times \mathbb{R}^n = \mathbb{R}^{n+1} : x^0 = -\varepsilon , |x'| = r\}$.

<u>Lemma 3</u>. <u>There is a smooth family $\{\Sigma_t\}$ of hypersurfaces passing through Γ , each having constant Euclidean mean curvature $h(t)$ on a neighborhood U of Γ , forming a foliation except at Γ , and with constant Euclidean area</u>.

<u>Proof</u>: Without loss of generality, we may choose ε to be small, by translating the hypersurfaces Σ_t .

For each $t > 0$ we shall first choose Σ_t so that it agrees in the half-space $\{x^0 \leq 0\}$ with the sphere passing through Γ and centered at $(t,0)$. The radius $R = R(t)$ of this sphere satisfies

$$(4) \qquad R(t)^2 = r^2 + (t+\varepsilon)^2 ,$$

and its Euclidean mean curvature is $h(t) := 1/R(t)$. Note that $dR/dt > 0$.

We next extend the above spherical cap to form a closed, convex, Lipschitz-continuous hypersurface Σ_t having area independent of t. For simplicity, we may choose Σ_t to agree in the half-space $\{x^0 \geq 0\}$ with the sphere of radius $\rho(t)$ and centered at $(\gamma(t),0)$, where $\gamma > 0$ and

$$(5) \qquad \rho(t)^2 - \gamma(t)^2 = \rho_1(t)^2 := r^2 + 2\varepsilon t + \varepsilon^2 .$$

Note that Σ_t meets $\{x^0 = 0\}$ in the (n-1)-sphere of radius $\rho_1(t)$. Write α_{n-1} for the measure of the unit (n-1)-sphere in $\mathbb{R}^n$. We may compute

$$(6) \quad \frac{1}{\alpha_{n-1}} A(\Sigma_t) = \int_0^{\rho_1} (R^2-s^2)^{1/2} Rs^{n-1} ds + \left(\int_{\rho_1}^{\rho} + \int_0^{\rho}\right)(\rho^2-s^2)^{1/2} \rho s^{n-1} ds .$$

Observe that for fixed s, the integrand of the first integral is a decreasing function of R. More precisely, we find

$$(7) \qquad \frac{1}{\alpha_{n-1}} \frac{d}{dt} A(\Sigma_t \cap \{x^0 \leq 0\}) \leq -\delta_n\, t\, r^{n-2} ,$$

where $\delta_n > 0$ is independent of $\varepsilon \in (0,r)$ and uniform for $0 < b_0 \leq t/r \leq b_1$. Meanwhile, the last term of equation (6) is an increasing function of $\rho > \rho_1$, if ρ_1 is held constant. Now choose t_0 satisfying $rb_0 < t_0 < rb_1$, and choose a value for $\rho(t_0)$ with $\rho_1(t_0) < \rho(t_0) < R(t_0)$. These choices determine Σ_{t_0}; let a_0 be its area. For all other $t \in (rb_0, rb_1)$ we define $\rho(t)$ by the condition that

$$(8) \qquad A(\Sigma_t) = a_0 ,$$

and let $\gamma(t) > 0$ be determined by equation (5).

We shall now show that $\{\Sigma_t : rb_0 \leq t \leq rb_1\}$ is a Lipschitz foliation in the half-space $\{x^0 > -\varepsilon\}$, for small values of ε. As t tends to a value t_1, we may compute the derivative at $t = t_1$ of the distance to $\Sigma_t \cap \{x^0 \geq 0\}$ along various radial lines from $(\gamma(t_1), 0)$. This derivative assumes its minimum either along the x^0-axis, where it equals $d(\rho(t) + \gamma(t))/dt$; or at $\Sigma_t \cap \{x^0 = 0\}$, where it equals $(\rho_1/\rho)\, d\rho_1/dt = \varepsilon/\rho > 0$. That is, the foliation property will follow from positivity of

$$(9) \qquad \frac{d}{dt}(\rho(t) + \gamma(t)) = [(\gamma + \rho)d\rho/dt - \varepsilon]/\gamma ,$$

where we have used the derivative of equation (5). Now for $\varepsilon = 0$, we have $\rho_1(t) = r$ independent of t, and also

$$\frac{d}{dt} A(\Sigma_t \cap \{x^0 \leq 0\}) < 0$$

by inequality (7). Equation (8) now implies that $A(\Sigma_t \cap \{x^0 > 0\})$, which we know to be an increasing function of ρ, is also a strictly increasing function of t; therefore $d\rho/dt > 0$. But this implies that $d(\rho + \gamma)/dt > 0$ by equation (9), for $\varepsilon = 0$. By continuity, we have $d(\rho + \gamma)/dt > 0$ for sufficiently small $\varepsilon > 0$.

Finally, the inequality $\rho(t) < R(t)$ remains true for t in an appropriate closed interval about t_0. Geometrically, this means that Σ_t has an interior angle uniformly less than π at $\{x^0 = 0\}$. We now smooth Σ_t, in some canonical way, inside the region $\{-\varepsilon/2 < x^0 < \varepsilon/2\}$, so that its area is decreased by a constant value $\delta > 0$. For example, the smoothing may be done by convolution with one of a family of positive mollifiers on S^n, where Σ_t is represented as a graph in central projection. For δ sufficiently small, the resulting family of smooth hypersurfaces is a foliation except at Γ, and Σ_t has constant mean curvature $h(t)$ on $U := \{x^0 < -\varepsilon/2\}$.

q.e.d.

We may now summarize the proof of the theorem stated in the introduction. From Lemma 3, we find a smooth family (with uniform estimates) of strictly convex hypersurfaces Σ_t in $\mathbb{R}^{n+1}$, each containing Γ, forming a foliation of an open set $\Omega \setminus \Gamma$, having constant Euclidean mean curvature $h(t)$ on a neighborhood U of Γ, and such that $A(\Sigma_t)$ has a constant value a_1. Let D_t be the open set bounded by Σ_t. As in Corollary 1, we write $\psi(x) := H(\Sigma_t, x)/h(t)$ and define a new Riemannian metric on $\Omega \cup U$:

$$d\tilde{s}^2 = ds^2\Big|_{T\Sigma_t} + \psi^2\, ds^2\Big|_{N\Sigma_t} .$$

Then Σ_t has constant $d\tilde{s}^2$-mean curvature $\tilde{H}(\Sigma_t, x) = h(t)$ in the new metric. We may extend $d\tilde{s}^2$ arbitrarily to all of $\mathbb{R}^{n+1}$. Note that $\tilde{A}(\Sigma_t) \equiv a_1$. It now follows from Lemma 2, applied to

$\mathbb{R}^{n+1}$ with the metric $d\tilde{s}^2$, that the volume $\tilde{V}(D_t)$ is constant. By homothetic rescaling, we may assume $\tilde{V}(D_t) \equiv 1$. The theorem is proved.

Remark 3. It will be obvious to the reader that the family of hypersurfaces $\{\Sigma_t\}$ may be perturbed in many independent ways so that it still satisfies the conclusions of Lemma 3 for some closed set Γ. It appears likely that few of these families will lead to isometric Riemannian structures on $\mathbb{R}^{n+1}$. Intuitively, in other words, one expects that the class of metrics having the properties stated in the theorem has infinite dimension and infinite codimension among all Riemannian metrics.

Remark 4. It is not clear to us whether the extension of $d\tilde{s}^2$ to all of $\mathbb{R}^{n+1}$ may be chosen so that the specific family of hypersurfaces we have constructed will have minimum area among hypersurfaces bounding volume one. It may be noted, however, that the variation vector field has nodal domains which are consistent with a minimizing condition subject to one constraint.

Bibliography

1. Aleksandrov, A.D.: Uniqueness theorems for surfaces in the large, V. Vestnik, Leningrad Univ. 13, 19 5-8 (1958). Amer. Math. Soc. Transl. 21, 412-416.

2. Gerhardt, C.: H-surfaces in Lorentzian manifolds. Comm. Math. Phys. 89, 523-553 (1983).

3. Gulliver, R. and S. Hildebrandt: Boundary configurations spanning continua of minimal surfaces. Manuscripta Math. 54, 323-347 (1986).