DETERMINANT OF COMPLEXES AND HIGHER HESSIANS

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ABSTRACT. Let $X \subset \mathbb{P}^r$ be a smooth algebraic curve in projective space, over an algebraically closed field of characteristic zero. For each $m \in \mathbb{N}$, the m-flexes of X are defined as the points where the osculating hypersurface of degree m has higher contact than expected, and a hypersurface $H \subset \mathbb{P}^r$ is called a m-hessian if it cuts X along its m-flexes. When X is a complete intersection, we give an expression for a (rational) m-hessian as the determinant of a complex of graded free modules naturally associated to X.

Contents.

§1 Introduction.

Our purpose in this article is to extend a well known fact in the theory of algebraic curves, namely, that the Hessian of a plane curve intersects the curve in its flexes.

To recall the motivating situation, let $X \subset \mathbb{P}^2$ denote a smooth algebraic plane curve, given by an equation $F = 0$ of degree $d \geq 2$, over an algebraically closed field k of characteristic zero. The Hessian of F is defined as $H_1(F) = \det H(F)$, where

$$H(F) = \left(\frac{\partial^2 F}{\partial x_i \, \partial x_j} \right)_{0 \leq i,j \leq 2}$$

is the Hessian matrix of F, and the flexes of X (or 1-flexes, as they will be called below) are the points where the tangent line has contact three or more. The fact refered to before is that $(H_1(F) = 0) \cap X$ coincides with the set of flexes of X (and the multiplicity of intersection equals the weight of the flex, defined below).

At each point $x \in X$ there is a unique conic, the osculating conic, that has maximal contact with X. Also, there is a finite number of points, the 2-flexes of X, where the osculating conic has contact six or more. Cayley posed himself the question of finding a

2-Hessian, that is, a covariant curve $H_2(F)$ that cuts X along its 2-flexes. He found the expression

$$H_2(F) = (12d^2 - 54d + 57)\ h^2\ \mathrm{Jac}(F, h, \Omega_H) + (d-2)(12d-27)\ h^2\ \mathrm{Jac}(F, h, \Omega_F) \\ +40(d-2)^2\ h\ \mathrm{Jac}(F, h, \Psi)$$

where $h = H_1(F)$, $\Omega = (D^t H(F)^* D)h$, $\Psi = (Dh)^t H(F)^* (Dh)$, $D = (\partial/\partial x_0, \partial/\partial x_1, \partial/\partial x_2)$ and $H(F)^*$ is the adjoint matrix of 2×2 minors of $H(F)$. Cayley's notation Ω_H and Ω_F stands for certain first partial derivatives of Ω, see loc. cit. We remark that Cayley deals with sextatic points, that is, 2-flexes that are not 1-flexes; for this reason we multiplied his formula by h.

More generally, for each $m = 1, 2, 3, \ldots$ one may consider osculating curves of degree m and the corresponding m-flexes, thus obtaining a constellation of distinguished points on X. The problem of finding a covariant m-Hessian for $m > 2$ does not seem to have been solved in the classical literature.

Another direction of generalization is to curves in higher dimensional projective spaces. There is a fairly simple formula for a 1-Hessian for complete intersection curves in $\mathbb{P}^3$ due to Clebsch ([S], Art. 363). In this article we will obtain some information about m-Hessians for complete intersection curves in $\mathbb{P}^r$.

In order to consider the more general situation, we recall at this point some terminology regarding flexes. Let X be a smooth connected complete algebraic curve over k and L a line bundle over X. The different orders of vanishing at x of sections $s \in H^0(X, L)$ constitute a finite set with $n + 1 := \dim_k H^0(X, L)$ elements, and when written in increasing order

$$a_0^L(x) < \cdots < a_n^L(x)$$

it is called the vanishing sequence of L at x. The number $w^L(x) = \sum_{i=0}^n a_i^L(x) - i$ is called the weight of L at x, and x is said to be an L-flex if $w^L(x) > 0$ or, equivalently, if there exists a section $s \in H^0(X, L)$ vanishing at x to order at least $\dim_k H^0(X, L)$.

The divisor of L-flexes is defined as the divisor $\sum_{x \in X} w^L(x)\ x$; it may be given a determinantal expression as follows: let

$$\tau : H^0(X, L) \otimes_k \mathcal{O}_X \to \mathcal{P}^n_{X/k}(L)$$

denote the homomorphism obtained by composing the natural homomorphism $H^0(X, L) \otimes_k \mathcal{O}_X \to L$ and the universal differential operator $L \to \mathcal{P}^n_{X/k}(L)$ with values in the rank $n + 1$ bundle of principal parts of order n of L (see §2). It may be shown (see [P], [ACGH]) that the divisor of L-flexes is equal to the divisor of zeros of $\wedge^{n+1}\tau$. If we choose an isomorphism $\wedge^{n+1} H^0(X, L) \cong k$ then $\wedge^{n+1}\tau$ corresponds to a section $w_L \in H^0(X, \wedge^{n+1}\mathcal{P}^n_{X/k}(L))$ that will be called the Wronskian of L.

Now suppose that our curve X sits in a projective space $\mathbb{P}^r_k$. If $H \subset \mathbb{P}^r_k$ is a hypersurface, we say that H is a L-Hessian if H cuts X along the divisor of L-flexes. Notice that a necessary condition for an L-Hessian to exist is that $\wedge^{n+1}\mathcal{P}^n_{X/k}(L) \cong \mathcal{O}_X(N)$ for some $N \in \mathbb{N}$. Assuming that such an isomorphism exists and has been chosen, an L-Hessian is more precisely defined as a hypersurface H of degree N that restricts on X to w_L.

If $f : X \to S$ is a family of smooth curves in $\mathbb{P}^r$ and L is a line bundle on X such that $f_*(L)$ is locally free of rank $n+1$, we define a relative L-Hessian to be an effective Cartier divisor $\mathcal{H} \subset \mathbb{P}^r_S$ that restricts on X to the relative L-Wronskian $w_{f,L} \in H^0(X, \wedge^{n+1}\mathcal{P}^n_{X/S}(L) \otimes (\wedge^{n+1}f^*f_*(L))^{-1})$, obtained from the natural homomorphism

$$\tau_f : f^*f_*(L) \to \mathcal{P}^n_{X/S}(L)$$

Our general goal is then to write down an L-Hessian in case the family is the universal family parametrized by a Hilbert scheme, for a certain choice of L. The Hessian is also required to be invariant under the natural action of the projective group, a condition that implies that one has a Hessian on each family of curves of fixed degree and genus, compatible with pull-back.

We shall concentrate on the case of the universal family of complete intersection curves of multi-degree $(d_1, \ldots, d_{r-1})$ in $\mathbb{P}^r$, choosing $L = \mathcal{O}(m)$ for a certain $m \in \mathbb{N}$. The first remark about this case is that an m-Hessian for an individual curve X exists: since $\Omega^1_{X/k} \cong \mathcal{O}_X(d)$ for some $d \in \mathbb{N}$ it easily follows that the Wronskian w_L is a section of $\mathcal{O}_X(N)$ for some $N \in \mathbb{N}$, and since X is projectively normal w_L lifts to $\mathbb{P}^r_k$.

For this family, our goal amounts to the construction of a polynomial $\mathrm{GL}_{r+1}(k)$ equivariant map

$$H_m : \prod_{j=1}^{r-1} \mathrm{Sym}^{d_j}(k^{r+1}) \to \mathrm{Sym}^N(k^{r+1})$$

such that for each smooth complete intersection curve $X = (F_1 = \cdots = F_{r-1} = 0)$ the hypersurface $H_m(F_1, \ldots, F_{r-1})$ is an m-Hessian for X. In (5.11) we give a proof of existence and essential uniqueness of such a map H_m, also computing its degree in each variable. In (5.19) we give a constructive description of a rational m-Hessian.

The formulas of Hesse, Cayley and Clebsch are given as the determinant of an explicit *matrix* of differential operators applied to the F_i. What we shall do in this article is to express m-Hessians as the Div (see [KM]) of a *complex* of graded free modules. The differentials in this complex will be explicit matrices involving derivatives of the polynomials F_i.

The idea of using determinants of complexes for writing "almost explicit" formulas is borrowed from articles by I. Gelfand, M. Kapranov and A. Zelevinsky; these authors took such an approach to give an equation for the dual hypersurface of a projective variety.

To explain our construction, let us first observe that lifting the Wronskian w_L to projective space would be immediate if the homomorphism $\tau : H^0(X, L) \otimes_k \mathcal{O}_X \to \mathcal{P}^n_{X/k}(L)$ was the restriction to X of a homomorphism of bundles on projective space. Since this does not appear to be the case, we construct instead a homomorphism $\tilde{\tau}$ of complexes of graded free modules on projective space such that $\tilde{\tau}|_X$ is quasi-isomorphic to τ. The expression $\mathrm{Div}(\tilde{\tau})$ is then a determinantal description of a (rational, see (5.19)) Hessian.

Since our curve is a complete intersection, the Koszul complex provides a lifting of $H^0(X,L) \otimes_k \mathcal{O}_X$. On the other hand, the lifting of $\mathcal{P}^n_{X/k}(L)$ is done in two steps: first we use a resolution, described in §3, that involves the modules $\mathcal{P}^j_{\mathbb{P}^r/k}(\mathcal{O}(m))$; these modules are not graded free, and are then replaced by graded free modules by means of the Euler sequences (2.21) and a cone construction.

A related problem, considered in [S], [E] and [Dy], is the one of giving equations for Gaussian maps associated to osculating hypersurfaces of degree m. We plan to show in a future article how the present set up may be used to treat this question.

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§2 Preliminaries on differential operators.

For ease of reference, we first review some well known definitions and facts about differential operators. Next we consider the graded situation, and describe the structure of the modules of principal parts on projective spaces (see (2.25) and (2.31)). The algebra of differential operators on projective space is isomorphic to the the degree zero subring of the Weyl algebra modulo the two-sided ideal generated by the Euler operator, a known fact [BB] reproved in (2.23) (case $d = m = 0,\ n \to \infty$).

(2.1) Let k be a commutative unitary ring, A a commutative k-algebra, M and N two A-modules and n a natural number. A k-linear map $D : M \to N$ is called a differential operator of order $\leq n$ if for all $a \in A$ the commutator $[D, a] : M \to N$, defined by $[D, a](m) = D(am) - aD(m)$, is a differential operator of order $\leq n - 1$. Differential operators of order ≤ 0 are, by definition, the A-linear functions. The set of all the differential operators of order $\leq n$ from M to N will be denoted by

$$\mathrm{D}^n_{A/k}(M, N)$$

(2.2) The k-module $\mathrm{Hom}_k(M, N)$ has two commuting structures of A-module, defined by the formulas $(aL)(m) = aL(m)$ and $(La)(m) = L(am)$, i.e. it has a structure of (A, A)-bimodule. The subspaces $\mathrm{D}^n_{A/k}(M, N)$ are stable, and hence they are (A, A)-bimodules.

(2.3) Let $\mathrm{D}^\infty_{A/k}(M, N) = \bigcup_{n=0}^\infty \mathrm{D}^n_{A/k}(M, N) \subset \mathrm{Hom}_k(M, N)$ be the (A, A)-bimodule of differential operators of finite order. For three A-modules, composition induces a k-bilinear map

$$\mathrm{D}^n_{A/k}(M_1, M_2) \times \mathrm{D}^m_{A/k}(M_2, M_3) \to \mathrm{D}^{n+m}_{A/k}(M_1, M_3)$$

(this may be shown by induction, using the formula $[D_1 \circ D_2, a] = D_1 \circ [D_2, a] + D_2 \circ [D_1, a]$). In particular, composition defines on $\mathrm{D}^\infty_{A/k}(A, A)$ a ring structure, and hence a structure of (A, A)-bialgebra. It follows from the Jacobi formula $[[D_1, D_2], a] = [[D_1, a], D_2] - [[D_2, a], D_1]$ and induction that the commutator of two operators of respective orders $\leq n$ and $\leq m$, has order $\leq m + n - 1$, that is, the graded algebra associated to the filtration by order is commutative.

(2.4) To simplify the notation, we will sometimes abreviate $\mathrm{D}^n_{A/k}(A, A) = \mathrm{D}^n_{A/k}$.

(2.5) For fixed M and variable N, the covariant functor $\mathrm{D}^n_{A/k}(M, N)$ is representable: The two k-algebra homomorphisms $c, d : A \to A \otimes_k A$ defined by $c(a) = a \otimes 1$ and $d(a) = 1 \otimes a$ give $A \otimes_k A$ a structure of (A, A)-bialgebra. The kernel $I = I_{A/k}$ of the multiplication map $A \otimes_k A \to A$ is a bi-ideal and $A \otimes_k A/I^{n+1}$ has a natural structure of (A, A)-bialgebra. For an A-module M let us define the A-module

$$\mathrm{P}^n_{A/k}(M) = (A \otimes_k A/I^{n+1}) \otimes_A M$$

where, while taking $\otimes_A$ we view $A\otimes_k A/I^{n+1}$ as an A-module via d, and we provide $\mathrm{P}^n_{A/k}(M)$ with an A-module structure via c (i.e. tensor product in the category of bi-modules). Then

$$d^n_{A/k,M} : M \to \mathrm{P}^n_{A/k}(M)$$

defined by $m \mapsto (1 \otimes_k 1) \otimes_A m$ is a differential operator of order $\leq n$, and it is universal, that is, for any N the map

$$\mathrm{Hom}_A(\mathrm{P}^n_{A/k}(M), N) \to \mathrm{D}^n_{A/k}(M, N)$$

defined by composing with $d^n_{A/k,M}$, is bijective.

(2.6) In case $M = A$, instead of $\mathrm{P}^n_{A/k}(A)$ we shall sometimes write $\mathrm{P}^n_{A/k}$. The universal differential operator $d^n_{A/k} : A \to \mathrm{P}^n_{A/k}$ takes the form $d^n_{A/k}(a) = 1 \otimes_k a$.

(2.7) For a scheme X over a scheme S, with two sheaves of $\mathcal{O}_X$-modules M and N, we denote by $\mathcal{D}^n_{X/S}(M, N)$ the sheaf of local differential operators of order $\leq n$, and by

$$d^n_{X/S,M} : M \to \mathcal{P}^n_{X/S}(M)$$

the universal differential operator of order $\leq n$ with source M.

(2.8) If X/S is a relative local complete intersection then we have exact sequences of $\mathcal{O}_X$-Modules

$$0 \to \mathrm{Sym}^n(\Omega^1_{X/S}) \otimes_{\mathcal{O}_X} M \xrightarrow{\imath} \mathcal{P}^n_{X/S}(M) \xrightarrow{\pi} \mathcal{P}^{n-1}_{X/S}(M) \to 0$$

(2.9) We recall from [EGA IV] (16.7.3) that if M is quasi-coherent (resp. coherent) then $\mathcal{P}^n_{X/S}(M)$ is also quasi-coherent (resp. coherent). If X/S is smooth then $\mathcal{P}^n_{X/S}$ is locally free, hence flat, and the functor $M \mapsto \mathcal{P}^n_{X/S}(M)$ is exact. The natural map

$$\mathcal{H}om_{\mathcal{O}_X}(M, N) \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{P}^n_{X/S}(M), \mathcal{P}^n_{X/S}(N))$$

sending $\varphi \mapsto \mathcal{P}^n_{X/S}(\varphi)$, is a differential operator of order $\leq n$ (and, in fact, it is universal, see loc. cit. (16.7.8.2)).

(2.10) Suppose now that our k-algebra A is $\mathbb{Z}$-graded: $A = \bigoplus_{d\in\mathbb{Z}} A_d$ ($\mathbb{Z}$ could be replaced by a more general semigroup, but we will not follow this line here). We provide the k-algebra $A \otimes_k A$ with the total grading associated to its natural bi-grading. The multiplication map $A \otimes_k A \to A$ is homogeneous (of degree 0) and hence its kernel $I = I_{A/k}$ is a homogeneous ideal. It follows that for any graded A-module M, the A-module $\mathrm{P}^n_{A/k}(M)$ has a natural grading.

(2.11) If M and N are graded A-modules, we denote

$$\mathrm{Hom}_k(M,N)_d = \{L \in \mathrm{Hom}_k(M,N)/\ L(M_m) \subset N_{m+d},\ \forall m\}$$

and

$$\mathrm{D}^n_{A/k}(M,N)_d = \mathrm{D}^n_{A/k}(M,N) \cap \mathrm{Hom}_k(M,N)_d$$

If $D \in \mathrm{D}^n_{A/k}(M,N)_d$ then we say that D has order $\leq n$ and degree d.

(2.12) PROPOSITION: If A is a finitely generated k-algebra and M a finitely generated A-module then for any N

$$\mathrm{D}^n_{A/k}(M,N) = \bigoplus_{d\in\mathbf{Z}} \mathrm{D}^n_{A/k}(M,N)_d$$

that is, $\mathrm{D}^n_{A/k}(M,N)$ is naturally a graded A-module.

PROOF: Let us first observe that the natural isomorphism $\alpha : \mathrm{Hom}_A(\mathrm{P}^n_{A/k}(M),N) \to \mathrm{D}^n_{A/k}(M,N)$ takes $\mathrm{Hom}_A(\mathrm{P}^n_{A/k}(M),N)_d$ into $\mathrm{D}^n_{A/k}(M,N)_d$ for any d. Our hypothesis imply, by (2.8), that the A-module $\mathrm{P}^n_{A/k}(M)$ is finitely generated, and then [Bo,1] (§21, 6, Remark) says that the equality $\mathrm{Hom}_A(\mathrm{P}^n_{A/k}(M),N) = \bigoplus_{d\in\mathbf{Z}} \mathrm{Hom}_A(\mathrm{P}^n_{A/k}(M),N)_d$ holds true; the proposition then follows by applying α.

(2.13) EXAMPLE: for a graded A-module M, define the Euler operator $E_M \in \mathrm{Hom}_k(M,M)$ by $E_M(m) = d.m$ for $m \in M_d$. The equality $[E_M, a] = E_A(a)$ for $a \in A$, shows that E_M is a differential operator of order ≤ 1, and degree 0.

(2.14) Now suppose $A = k[x_0,\ldots,x_r]$ is a polynomial algebra, with the x_i of degree one. We assume that the ring k contains the rational numbers. It is well known ([EGA IV] (16.11) or [B]) that in this case $\mathrm{D}^\infty_{A/k}(A,A)$ is isomorphic to the Weyl algebra $k[x_0,\ldots,x_r,\partial_0,\ldots,\partial_r]$, with relations $[\partial_i, x_j] = \delta_{ij}$, $[\partial_i,\partial_j] = [x_i,x_j] = 0$. Every $D \in \mathrm{D}^\infty_{A/k}(A,A)$ may be written, in a unique way, as a finite sum

$$D = \sum_{\alpha\in\mathbf{N}^{r+1}} F_\alpha\, \partial^\alpha$$

where $F_\alpha \in A$ and ∂^α is the differential operator of order $\leq |\alpha|$ and degree $-|\alpha|$ defined by $\alpha!\ \partial^\alpha = \prod_{0\leq j\leq r} \partial_j^{\alpha_j}$. In the expression above, D is homogeneous of degree d (see (2.11)) iff all the F_α that appear are homogeneous and $\deg(F_\alpha) - |\alpha| = d$ for all α.

(2.15) One may reformulate (2.14) as follows ([EGA IV] (16.4.7)): The polynomial algebra $A[y] = A[y_0,\ldots,y_r] = k[x,y]$ has a structure of (A,A)-bialgebra defined by the ring homomorphisms $c'(F(x)) = F(x)$ and $d'(F(x)) = F(x+y)$ for $F(x) \in A = k[x]$. It is easy to check that

$$\sigma : A[y] \to A \otimes_k A$$

defined by $\sigma(x_i) = x_i \otimes 1$ and $\sigma(y_i) = 1 \otimes x_i - x_i \otimes 1$, is an isomorphism of (A, A)-bialgebras, and since $\sigma(y_0, \ldots, y_r) = I_{A/k}$, there are isomorphisms

$$\sigma : A[y]/(y)^{n+1} \to \mathrm{P}^n_{A/k}$$

It then follows from (2.5) that the differential operator of order $\leq n$

$$d'^n_{A/k} : A \to A[y]/(y)^{n+1}$$

defined by

$$d'^n_{A/k}(F(x)) = F(x+y) = \sum_\alpha \partial^\alpha(F(x))\, y^\alpha \quad \mathrm{mod}\ (y)^{n+1}$$

is universal.

(2.16) As an A-module via c', $A[y]/(y)^{n+1}$ is free with basis y^α for $|\alpha| \leq n$. The grading of $\mathrm{P}^n_{A/k}$ transports to the grading in $A[y]/(y)^{n+1}$ such that $\deg(y_i) = 1$, and the choice of basis defines an isomorphism of graded A-modules

$$A[y]/(y)^{n+1} \cong \bigoplus_{|\alpha| \leq n} A(-|\alpha|) \cong \bigoplus_{j=0}^{n} A(-j)^{\binom{n}{j}}$$

where $M(j)$ denotes the shift by $j \in \mathbb{Z}$ for a graded A-module M.

(2.17) According to (2.10), the elements $dx_i \in \Omega^1_{A/k}$ are homogeneous of degree one. Therefore there is an isomorphism of graded A-modules

$$\Omega^1_{A/k} \cong A(-1)^{r+1}$$

sending $\sum_i a_i dx_i \mapsto (a_i)$. The isomorphism (2.16) is then written more intrinsically as

$$\mathrm{P}^n_{A/k} \cong \bigoplus_{j=0}^{n} \mathrm{Sym}^j \Omega^1_{A/k}$$

In particular, the exact sequences (2.8) split in affine space.

(2.18) (jets and polars) For $F \in A = k[x]$ define its j-th polar

$$p^j(F) = \sum_{|\alpha|=j} \partial^\alpha(F)\ (dx)^\alpha \quad \in \mathrm{Sym}^j \Omega^1_{A/k}$$

It follows from (2.15) and (2.17) that the differential operator of order $\leq n$

$$p^n_{A/k} : A \to \bigoplus_{j=0}^{n} \mathrm{Sym}^j \Omega^1_{A/k}$$

$$p^n_{A/k}(F) = (p^0(F), \dots, p^n(F))$$

is universal. Notice that, according to (2.11), $p^n_{A/k}$ is homogeneous of degree zero.

(2.19) Applying $\mathrm{Hom}_A(-, A) = -^*$ in (2.17) we obtain an isomorphism of graded A-modules

$$\mathrm{D}^n_{A/k}(A, A) \cong \bigoplus_{j=0}^{n} \mathrm{Sym}^j \Omega^1_{A/k}{}^*$$

which corresponds to the unique representation of a differential operator described in (2.14).

(2.20) Now we consider $P = \mathbb{P}^r_k = \mathrm{Proj}(A)$. If M is a graded A-module, $M^\sim$ denotes the corresponding quasi-coherent sheaf of modules on projective space, $M(d)$ denotes the shifted graded module $M(d)_n = M_{n+d}$ and $\mathcal{O}_P(m) = \mathcal{O}(m) := A(m)^\sim$.

(2.21) PROPOSITION (Euler sequences): For each $n \in \mathbb{N}$ and $m, d \in \mathbb{Z}$ there is an exact sequence of $\mathcal{O}_P$-Modules

$$0 \to \mathrm{D}^{n-1}_{A/k}(A, A)(d)^\sim \xrightarrow{.(E-m)} \mathrm{D}^n_{A/k}(A, A)(d)^\sim \xrightarrow{\rho} \mathcal{D}^n_{P/k}(\mathcal{O}(m), \mathcal{O}(m+d)) \to 0$$

where $E = \sum_i x_i\, \partial_i$ is the Euler differential operator (2.13) and $.(E-m)$ is right multiplication by $(E-m) \in \mathrm{D}^1_{A/k}(A, A)_0$.

PROOF: To define ρ let us first remark that for each multiplicative set $S \subset A$ there is a natural map

$$\rho_S : \mathrm{D}^n_{A/k}(A, A) \to \mathrm{D}^n_{S^{-1}A/k}(S^{-1}A, S^{-1}A)$$

which may be defined first on derivations $\delta \in \mathrm{Der}_{A/k}(A, A)$ by the usual quotient rule, and then using the fact that since A is a polynomial algebra, $\mathrm{D}^n_{A/k}(A, A)$ is generated by the derivations. Notice that ρ_S preserves degree. Now consider $D \in \mathrm{D}^n_{A/k}(A, A)_d$, $f \in A_{m+k}$ and $g \in A_k$, so that f/g is a section of $\mathcal{O}_P(m)$ over the open set where $g \neq 0$. Then $\rho(D)(f/g)$ is a section of $\mathcal{O}_P(m+d)$ over the same open set. Since these open sets form a basis of open sets on P, this defines the sheaf homomorphism ρ.

It is clear from the definitions and the Euler formula that $\rho \circ .(E-m) = 0$, so that we have a complex, that we shall denote $(2.21)_n$. We will prove by induction on n that this complex is exact, starting with $(2.21)_0$ which is clearly exact ($\mathrm{D}^{-1}_{A/k} = 0$).

$(2.22)_n$ Combining (2.21) with the duals of (2.8), we obtain a commutative diagram with exact columns, where the maps denoted p are canonical projections

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{D}^{n-2}_{A/k}(d)^{\sim} & \xrightarrow{.(E-m)} & \mathrm{D}^{n-1}_{A/k}(d)^{\sim} & \longrightarrow & \mathcal{D}^{n-1}_{P/k}(\mathcal{O}(m),\mathcal{O}(m+d)) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{D}^{n-1}_{A/k}(d)^{\sim} & \xrightarrow{.(E-m)} & \mathrm{D}^{n}_{A/k}(d)^{\sim} & \longrightarrow & \mathcal{D}^{n}_{P/k}(\mathcal{O}(m),\mathcal{O}(m+d)) & \longrightarrow & 0 \\
 & & p\downarrow & & p\downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}(d)\otimes \mathrm{Sym}^{n-1}(\Omega^1_{A/k})^{*\sim} & \xrightarrow{e} & \mathcal{O}(d)\otimes \mathrm{Sym}^{n}(\Omega^1_{A/k})^{*\sim} & \longrightarrow & \mathcal{O}(d)\otimes \mathrm{Sym}^{n}(\Omega^1_{P/k})^{*} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Here e is defined as the composition

$$\mathcal{O}(d)\otimes\mathrm{Sym}^{n-1}(\Omega^1_{A/k})^{*\sim} \xrightarrow{.E} \mathcal{O}(d)\otimes\mathrm{Sym}^{n-1}(\Omega^1_{A/k})^{*\sim} \oplus \mathcal{O}(d)\otimes\mathrm{Sym}^{n}(\Omega^1_{A/k})^{*\sim} \xrightarrow{p} \mathcal{O}(d)\otimes\mathrm{Sym}^{n}(\Omega^1_{A/k})^{*\sim}$$

and is given by the formula (see (2.24))

$$e(\sum_{|\alpha|=n-1} F_\alpha\, \partial^\alpha) = \sum_{\alpha,i} x_i\, F_\alpha\, \partial_i\, \partial^\alpha$$

In other words, e is multiplication by E in the symmetric algebra of the module $(\Omega^1_{A/k}{}^*)^\sim$. The lower row of $(2.22)_1$ is dual to the usual Euler sequence

$$0 \longrightarrow \Omega^1_{P/k} \xrightarrow{\jmath} \Omega^1_{A/k}{}^{\sim} \xrightarrow{\varepsilon} \mathcal{O}_P \longrightarrow 0$$

which is known to be exact (ε is contraction with E and $\jmath$ is pull-back of 1-forms to affine space). Then for any n the lower row of $(2.22)_n$ is exact, since it is obtained from the usual Euler sequence by applying Sym^n. Now suppose $(2.21)_{n-1}$ is exact. Considering the long exact sequence of homology of $(2.22)_n$ it follows that $(2.21)_n$ is also exact, which finishes the proof.

(2.23) COROLLARY: there are exact sequences

$$0 \longrightarrow \mathrm{D}^{n-1}_{A/k}(A,A)_d \xrightarrow{H^0(.(E-m))} \mathrm{D}^{n}_{A/k}(A,A)_d \longrightarrow H^0(P, \mathcal{D}^n_{P/k}(\mathcal{O}(m),\mathcal{O}(m+d))) \longrightarrow 0$$

PROOF: we take cohomology in (2.21) and observe that (2.19) and the calculation of cohomology of line bundles on P imply that $H^0(P, \mathrm{D}^n_{A/k}(A,A)(d)^\sim) = \mathrm{D}^n_{A/k}(A,A)_d$ and $H^1(P, \mathrm{D}^n_{A/k}(A,A)(d)^\sim) = 0$ for all $n \in \mathbb{N}$ and $d \in \mathbb{Z}$.

(2.24) To make the map $H^0(.(E-m))$ explicit, in view of (2.19), we need to write down the standard form (2.14) for $D.(E-m)$ for each given $D = \sum_\alpha F_\alpha\, \partial^\alpha \in \mathrm{D}^{n-1}_{A/k}(A,A)_d$. This is achieved by combining $D.(E-m) = D.E - mD$ with the easily checked formula

$$D.E = \sum_\alpha \big(\sum_i x_i\, F_\alpha\, \partial_i\, \partial^\alpha\big) + |\alpha|\, F_\alpha\, \partial^\alpha$$

(2.25) Applying $\mathcal{H}om_{\mathcal{O}_P}(-, \mathcal{O}(m+d))$ in (2.21) we obtain the following exact sequence that describes the structure of $\mathcal{P}^n_{P/k}(\mathcal{O}(m))$

$$0 \to \mathcal{P}^n_{P/k}(\mathcal{O}(m)) \xrightarrow{\jmath} \mathrm{P}^n_{A/k}(A)(m)^\sim \xrightarrow{.(E-m)^*} \mathrm{P}^{n-1}_{A/k}(A)(m)^\sim \to 0$$

(2.26) Combining (2.25) with (2.17) and (2.18) we obtain a commutative diagram that describes the universal differential operator of order $\leq n$ for $\mathcal{O}(m)$

$$\begin{array}{ccccc}
0 \to \mathcal{P}^n_{P/k}(\mathcal{O}(m)) & \longrightarrow & \bigoplus_{j=0}^{n}(\mathrm{Sym}^j\Omega^1_{A/k})(m)^\sim & \xrightarrow{.(E-m)^*} & \bigoplus_{j=0}^{n-1}(\mathrm{Sym}^j\Omega^1_{A/k})(m)^\sim \to 0 \\
{\scriptstyle d^n_{P/k,\mathcal{O}(m)}}\uparrow & & {\scriptstyle p^n_{A/k}}\uparrow & & \\
\mathcal{O}(m) & = & \mathcal{O}(m) & &
\end{array}$$

(2.27) Let us make $.(E-m)^*$ explicit. It follows from (2.24) that $.(E-m)^*$ is given on basis elements by

$$.(E-m)^*((dx)^\beta) = c.(dx)^\beta + \sum_{\beta_i \geq 1} x_i (dx)^{\beta - e_i}$$

where $e_i \in \mathbb{N}^{r+1}$, $(e_i)_j = \delta_{ij}$ and $c = |\beta| - m$ for $|\beta| < n$, $c = 0$ for $|\beta| = n$.
In other words, $.(E-m)^*$ is the direct sum of maps

$$(c_j, \varepsilon) : (\mathrm{Sym}^j\Omega^1_{A/k})(m)^\sim \to (\mathrm{Sym}^j\Omega^1_{A/k})(m)^\sim \,\oplus\, (\mathrm{Sym}^{j-1}\Omega^1_{A/k})(m)^\sim$$

where c_j is multiplication by $j-m$ for $j < n$, $c_n = 0$, and ε is contraction with E.

Now we take a closer look at (2.25)-(2.27); we shall obtain a better description of the modules $\mathcal{P}^n_{P/k}(\mathcal{O}(m))$.

$(2.28)_n$ We start by writing down the dual of $(2.22)_n$

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{P}^{n-1}_{P/k}(\mathcal{O}(m)) & \xrightarrow{\jmath} & \mathrm{P}^{n-1}_{A/k}(m)^{\sim} & \xrightarrow{.(E-m)^{*}} & \mathrm{P}^{n-2}_{A/k}(m)^{\sim} & \longrightarrow & 0 \\
 & & \pi\uparrow & & \pi\uparrow & & \pi\uparrow & & \\
0 & \longrightarrow & \mathcal{P}^{n}_{P/k}(\mathcal{O}(m)) & \xrightarrow{\jmath} & \mathrm{P}^{n}_{A/k}(m)^{\sim} & \xrightarrow{.(E-m)^{*}} & \mathrm{P}^{n-1}_{A/k}(m)^{\sim} & \longrightarrow & 0 \\
 & & \imath\uparrow & & \imath\uparrow & & \imath\uparrow & & \\
0 & \longrightarrow & \mathrm{Sym}^n\Omega^1_{P/k}\otimes\mathcal{O}(m) & \xrightarrow{\jmath} & (\mathrm{Sym}^n\Omega^1_{A/k})(m)^{\sim} & \xrightarrow{\varepsilon} & (\mathrm{Sym}^{n-1}\Omega^1_{A/k})(m)^{\sim} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

(2.29) The composition of $\jmath$ with the projection $p : \mathrm{P}^n_{A/k}(m)^{\sim} \cong \bigoplus_{j=0}^{n}(\mathrm{Sym}^j\Omega^1_{A/k})(m)^{\sim} \to (\mathrm{Sym}^n\Omega^1_{A/k})(m)^{\sim}$ will be denoted

$$\varphi = \varphi_{m,n} : \mathcal{P}^n_{P/k}(\mathcal{O}(m)) \to (\mathrm{Sym}^n\Omega^1_{A/k})(m)^{\sim}$$

It is a homomorphism between bundles of the same rank. We are interested in the bundle on the left, and since the one on the right is understood (it is graded free), we shall analyse to what extent φ is an isomorphism. The answer depends on m and n and is given in the next Theorem.

(2.30) We need another piece of notation, namely

$$\lambda_j(m,n) := \prod_{i=n-j}^{n-1} (m-i)^{-1} \quad \in \mathbb{Q}$$

which is well defined as long as $i \neq m$ for $n-j \leq i \leq n-1$, that is, $m < n-j$ or $m \geq n$. When the values of m and n are understood, we will write λ_j instead of $\lambda_j(m,n)$.

(2.31) THEOREM: With the notation introduced above, the following is true

(a) If $m < 0$ or if $m \geq n$ then φ is an isomorphism. Its inverse is given by the formula

$$\varphi^{-1}(w_n) = \sum_{j=0}^{n} \lambda_j \; \varepsilon^j(w_n)$$

where

$$\varepsilon^j : \mathrm{Sym}^i\Omega^1_{A/k} \to \mathrm{Sym}^{i-j}\Omega^1_{A/k}$$

is iterated contraction with the Euler operator E.

(b) If $0 \leq m < n$ then there is an exact sequence

$$0 \longrightarrow \mathrm{Sym}^m \Omega^1_{A/k}(m)^\sim \xrightarrow{t} \mathcal{P}^n_{P/k}(\mathcal{O}(m)) \xrightarrow{\varphi} \mathrm{Sym}^n \Omega^1_{A/k}(m)^\sim \xrightarrow{\varepsilon^{n-m}} \mathrm{Sym}^m \Omega^1_{A/k}(m)^\sim \longrightarrow 0$$

where

$$t(w_m) = \sum_{i=0}^{m} \frac{\varepsilon^i(w_m)}{i!}$$

PROOF: A typical section w of $\bigoplus_{j=0}^n (\mathrm{Sym}^j \Omega^1_{A/k})(m)^\sim$ over an (unspecified) open set will be denoted $w = \sum_{j=0}^n w_j$ where w_j is a rational symmetric form of degree j on affine space; more precisely, $w_j = \sum_{|\alpha|=j} a_\alpha(x)\ (dx)^\alpha$ where the a_α are homogeneous rational functions of degree m. According to (2.25), such a w is a section of $\mathcal{P}^n_{P/k}(\mathcal{O}(m))$ if and only if $.(E-m)^*(w) = 0$. Using (2.27) we find

$$.(E-m)^*(w) = \sum_{j=0}^{n} .(E-m)^*(w_j) = \varepsilon(w_n) + \sum_{j=0}^{n-1} \varepsilon(w_j) + (j-m)w_j = \sum_{j=1}^{n} \varepsilon(w_j) + \sum_{j=0}^{n-1} (j-m)w_j$$

which equals zero if and only if

$$(*) \qquad (m-j)\ w_j = \varepsilon(w_{j+1}) \quad \text{for all } j = 0, 1, \ldots, n-1$$

Notice that with the present notation φ is defined by $\varphi(\sum_{j=0}^n w_j) = w_n$.

Now we consider our two cases.

(a) Here $m - j \neq 0$ for $0 \leq j \leq n-1$ and then $w_j = (m-j)^{-1}\ \varepsilon(w_{j+1})$. It follows that all the w_j are determined by w_n and it is easy to verify the proposed formula for φ^{-1}.

(b) Since the index $j = m$ now occurs, w_n does not determine all the other w_j and φ will have a non-zero kernel and cokernel. To determine these, we separate (*) into two groups of equations:

$$\begin{aligned} m\ w_0 &= \varepsilon(w_1) \\ (m-1)\ w_1 &= \varepsilon(w_2) \\ &\cdots \\ w_{m-1} &= \varepsilon(w_m) \end{aligned} \qquad \text{and} \qquad \begin{aligned} 0 = (m-m)\ w_m &= \varepsilon(w_{m+1}) \\ (-1)\ w_{m+1} &= \varepsilon(w_{m+2}) \\ &\cdots \\ (m-n+1)\ w_{n-1} &= \varepsilon(w_n) \end{aligned}$$

Notice that there are no conditions on w_m and that w_m determines w_j with $0 \leq j \leq m$. Suppose that $\varphi(w) = w_n = 0$. It follows from the second group of equations that

$w_{m+1}, \ldots, w_n$ are all zero, which implies that image(t) = kernel(φ).
On the other hand, the condition $\varepsilon(w_{m+1}) = 0$ is equivalent to $\varepsilon^{n-m}(w_n) = 0$, and it follows that image(φ) = kernel(ε^{n-m}).
Finally, the surjectivity of ε^{n-m} follows from the surjectivity of ε, proved in (2.22).

(2.32) If X is a scheme over a scheme S and $\varphi : M \to N$ is a homomorphism of $\mathcal{O}_X$-Modules, one has an induced homomorphism $\mathcal{P}^n_{X/S}(\varphi) : \mathcal{P}^n_{X/S}(M) \to \mathcal{P}^n_{X/S}(N)$ (as in (2.9)) that involves "derivatives of φ". With a view towards our applications, now we make this explicit, at least in the cases in which X is affine space and $M = N = \mathcal{O}_X$, and X is projective space and M and N are (direct sums of) line bundles.

(2.33) For a ring R and element $r \in R$ let us denote $.r : R \to R$ the operator of multiplication by r. Let A be a k-algebra as in (2.5) and $a \in A$. The operator $.a$ induces a map (2.9)

$$\mathrm{P}^n_{A/k}(.a) : \mathrm{P}^n_{A/k}(A) \to \mathrm{P}^n_{A/k}(A)$$

which is simply the one induced by $.(1 \otimes a) : A \otimes_k A \to A \otimes_k A$. In case A is a polynomial algebra, under the isomorphism σ of (2.15) $\mathrm{P}^n_{A/k}(.a)$ corresponds to multiplication by $a(x+y)$. Using the notation of polars as in (2.18), since $a(x+y) = \sum_{j=0}^n p^j(a)$, multiplication by $a(x+y)$ is a direct sum of multiplication operators in the symmetric algebra of $\Omega^1_{A/k}$

$$.p^{j-i}(a) : \mathrm{Sym}^i\Omega^1_{A/k} \to \mathrm{Sym}^j\Omega^1_{A/k}$$

for $0 \leq i \leq j \leq n$.

(2.34) We remark at this point that we also have a homomorphism

$$.a' : \mathrm{P}^{n-1}_{A/k}(A) \to \mathrm{P}^n_{A/k}(A)$$

(as in Example (3.8)) induced by multiplication by $a' = 1 \otimes a - a \otimes 1$. It follows easily from the definitions that its dual is the homomorphism

$$[\ ,a] : \mathrm{D}^n_{A/k}(A, A) \to \mathrm{D}^{n-1}_{A/k}(A, A)$$

sending $D \mapsto [D, a] = D.a - a.D$. In case A is a polynomial algebra $.a'$ corresponds, under the isomorphism σ of (2.15), to multiplication by $\Delta a = a(x + y) - a(x)$.

(2.35) Consider $A = k[x_0, \ldots, x_r]$ and let $F \in A_d$ be a homogeneous polynomial of degree d. Then F induces a homomorphism $.F : \mathcal{O}_P(m) \to \mathcal{O}_P(m + d)$ for each $m \in \mathbb{Z}$, and we wish to describe $\mathcal{P}^n_{P/k}(.F)$. Keeping track of gradings in (2.33), one has a homomorphism of degree zero

$$\mathrm{P}^n_{A/k}(.F) : \mathrm{P}^n_{A/k}(A)(m) \to \mathrm{P}^n_{A/k}(A)(m + d)$$

Taking (2.25) into account, $\mathcal{P}^n_{P/k}(.F)$ is described as the restriction of $\mathrm{P}^n_{A/k}(.F)^\sim$ to the submodule $\mathcal{P}^n_{P/k}(\mathcal{O}(m))$. Summarizing, $\mathcal{P}^n_{P/k}(.F)$ is explicitly given as a direct sum of maps which are multiplication by polars of F as in (2.33).

§3 Principal parts and closed subschemes.

In this section we study some resolutions (see (3.6) and (3.10)) of the modules of principal parts of a closed subscheme, constructed out of a given resolution of the structure sheaf of the subscheme.

We start by recalling [EGA IV] (16.4.20):

(3.1) PROPOSITION: Let Y be a scheme over a scheme S and $f : X \to Y$ a closed immersion corresponding to a quasi-coherent Ideal $J \subset \mathcal{O}_Y$. Then for any $n \in \mathbb{N}$ the natural map $f^*\mathcal{P}^n_{Y/S} \to \mathcal{P}^n_{X/S}$ is surjective, and its kernel is the Ideal of $f^*\mathcal{P}^n_{Y/S}$ generated by $f^*(\mathcal{O}_Y.d^n_{Y/S}(J))$.

(3.2) We remark that in the affine case where $S = \mathrm{Spec}(k)$, $Y = \mathrm{Spec}(A)$ and $X = \mathrm{Spec}(B)$ with $B = A/J$, (3.1) is equivalent to the fact that the natural map

$$A \otimes_k A/(I^{n+1}_{A/k} + J \otimes_k A + A \otimes_k J) \to B \otimes_k B/I^{n+1}_{B/k}$$

is an isomorphism.

(3.3) It follows from (3.1) that we have an exact sequence of $\mathcal{O}_X$-Modules

$$f^*\mathcal{P}^n_{Y/S}(J) \xrightarrow{f^*\mathcal{P}^n_{Y/S}(i)} f^*\mathcal{P}^n_{Y/S} \to \mathcal{P}^n_{X/S} \to 0$$

where i denotes the inclusion $J \to \mathcal{O}_Y$. Combining this with the exactness of the functor $\mathcal{P}^n_{Y/S}$ (now assuming Y/S smooth) and the right-exactness of f^*, we deduce that for each exact sequence of $\mathcal{O}_Y$-Modules

$$M \xrightarrow{\mu} \mathcal{O}_Y \to \mathcal{O}_X \to 0$$

the sequence

$$f^*\mathcal{P}^n_{Y/S}(M) \xrightarrow{f^*\mathcal{P}^n_{Y/S}(\mu)} f^*\mathcal{P}^n_{Y/S} \to \mathcal{P}^n_{X/S} \to 0$$

is exact.

(3.4) REMARK: In fact, $f^*\mathcal{P}^n_{Y/S}(\mu)$ factors through $f^*\mathcal{P}^{n-1}_{Y/S}(M)$. This is seen by combining the exact sequences (2.8) and the fact that $\mu(M) \subset J$, that is, $f^*(\mu) = 0$ (as in the proof of (3.10) below).

(3.5) Suppose that we are given a resolution (M, μ) of $\mathcal{O}_X$, that is, an exact complex of quasi-coherent $\mathcal{O}_Y$-Modules of the form

$$M_r \xrightarrow{\mu_r} \ldots \to M_1 \xrightarrow{\mu_1} M_0 = \mathcal{O}_Y \to \mathcal{O}_X \to 0$$

We are interested in functorially associating to (M,μ) a resolution of $\mathcal{P}^n_{X/S}$, continuing the first step (3.3). More precisely, we wish (in principle, see (3.11)) to construct a complex (M^n,μ^n) of $\mathcal{O}_Y$-Modules such that $f^*(M^n,\mu^n)$ is a resolution of $\mathcal{P}^n_{X/S}$ by $\mathcal{O}_X$-Modules.

(3.6) Let us consider the case in which $S = \mathrm{Spec}(k)$, $Y = \mathrm{Spec}(A)$ and $X = \mathrm{Spec}(B)$ with $B = A/J$, where the ideal J is generated by $a_1,\ldots,a_s \in A$. We denote $R = A\otimes_k A$, $I = I_{A/k}$ as in (2.5) and $a'_i = 1\otimes a_i - a_i\otimes 1$, and consider the Koszul complex $K(a')$ of the elements $a'_i \in R$. Writing $V = k^s$ this complex looks like

$$0 \longrightarrow \wedge^s V\otimes_k R \xrightarrow{\alpha_s} \ldots \longrightarrow \wedge^2 V\otimes_k R \xrightarrow{\alpha_2} V\otimes_k R \xrightarrow{\alpha_1} R$$

We observe that since $a'_i \in I$, the differentials α_j correspond to matrices with all the entries in I, that is, $\alpha_j(I^m.(\wedge^j V\otimes_k R)) \subset I^{m+1}.(\wedge^{j-1}V\otimes_k R)$ for any m. It follows that there is an induced complex (interpreting $I^j = R$ for $j \leq 0$)

$$0 \longrightarrow \wedge^s V\otimes_k R/I^{n-s+1} \longrightarrow \ldots \longrightarrow \wedge^2 V\otimes_k R/I^{n-1} \longrightarrow V\otimes_k R/I^n \longrightarrow R/I^{n+1}$$

that is, a complex

$$0 \longrightarrow \wedge^s V\otimes_k \mathrm{P}^{n-s}_{A/k} \xrightarrow{\alpha_s} \ldots \longrightarrow \wedge^2 V\otimes_k \mathrm{P}^{n-2}_{A/k} \xrightarrow{\alpha_2} V\otimes_k \mathrm{P}^{n-1}_{A/k} \xrightarrow{\alpha_1} \mathrm{P}^n_{A/k}$$

that we shall denote $\mathrm{P}^n_{A/k}(a_1,\ldots,a_s) = \mathrm{P}^n_{A/k}(a)$.

(3.7) PROPOSITION: Suppose that $a_1,\ldots,a_s \in A$ is a regular sequence and that A and B are smooth over k. Then $B\otimes_A \mathrm{P}^n_{A/k}(a)$ is a resolution of $\mathrm{P}^n_{B/k}$.

PROOF: Denote by $\mathcal{P}^n$ the augmented complex $B\otimes_A \mathrm{P}^n_{A/k}(a) \to \mathrm{P}^n_{B/k} \to 0$. We will show that $\mathcal{P}^n$ is exact by induction on n. The statement is obvious for $n = 0$. Assuming it for $n-1$, consider the natural surjective map $\mathcal{P}^n \to \mathcal{P}^{n-1}$ and let $\mathcal{Q}^n$ denote its kernel. The complex $\mathcal{Q}^n$ looks like

$$0 \to \wedge^s V\otimes_k I^{n-s}/I^{n-s+1} \to \cdots \to V\otimes_k I^{n-1}/I^n \to I^n/I^{n+1} \to I^n_B/I^{n+1}_B \to 0$$

and it has a natural map, call it σ, from the complex $\mathcal{Q}'^n$:

$$0 \to \wedge^s V\otimes_k \mathrm{Sym}^{n-s}(I/I^2) \to \cdots \to V\otimes_k \mathrm{Sym}^{n-1}(I/I^2) \to \mathrm{Sym}^n(I/I^2) \to \mathrm{Sym}^n(I_B/I_B^2) \to 0$$

Since A/k and B/k are smooth, all the A and B-modules in the complexes above are locally free. Therefore, the exact sequences $0 \to \mathcal{Q}^n \to \mathcal{P}^n \to \mathcal{P}^{n-1} \to 0$ remain exact after $B\otimes_A -$. Using the long exact sequence of homology, we are reduced to showing that $B\otimes_A \mathcal{Q}^n$ is exact for all $n \geq 1$. By the smoothness hypothesis again, σ is an isomorphism, and hence we only need to show that $B\otimes_A \mathcal{Q}'^n$ is exact. But this complex is exact because it is isomorphic to the n-th graded piece of the homomorphism of Koszul algebras

$$\wedge(J/J^2)\otimes_B \mathrm{Sym}(B\otimes_A \Omega^1_{A/k}) \to \mathrm{Sym}(\Omega^1_{B/k})$$

associated to the exact sequence

$$0 \to J/J^2 \to B \otimes_A \Omega^1_{A/k} \to \Omega^1_{B/k} \to 0$$

as in [I] (4.3.1.7). This finishes the proof.

(3.8) EXAMPLE: for $s = 1$, $B = A/A.a$ and the resolution in (3.6) is

$$0 \longrightarrow B \otimes_A \mathrm{P}^{n-1}_{A/k} \xrightarrow{.a'} B \otimes_A \mathrm{P}^n_{A/k} \to \mathrm{P}^n_{B/k} \longrightarrow 0$$

where $.a'$ is multiplication by $a' = 1 \otimes a - a \otimes 1$ (see (2.34)).

(3.9) EXAMPLE: In case $A = k[x]$ is a polynomial algebra, denoting $\Delta a = a(x+y) - a(x) \in A[y]$ for $a \in A$, the complex $\mathrm{P}^n_{A/k}(a_1, \ldots, a_s)$ of (3.6) is induced by the Koszul complex of $\Delta a_1, \ldots, \Delta a_s$, and is written

$$0 \longrightarrow \wedge^s V \otimes A[y]/(y)^{n-s+1} \xrightarrow{\Delta a} \ldots \longrightarrow \wedge^2 V \otimes A[y]/(y)^{n-1} \xrightarrow{\Delta a} V \otimes A[y]/(y)^n \xrightarrow{\Delta a} A[y]/(y)^{n+1}$$

Returning to the general problem in (3.5), and keeping the notation from there, we have the following proposition

(3.10) PROPOSITION: Assume Y/S is smooth and let (M, μ) be a complex of $\mathcal{O}_Y$-Modules such that $\mu_j(M_j) \subset J.M_{j-1}$ (i.e. $f^*(\mu_j) = 0$) for all j. Then for each $n \in \mathbb{N}$ there is a complex $\mathcal{P}^n(M, \mu)$ of $\mathcal{O}_X$-Modules, functorial in (M, μ), of the form

$$\ldots \longrightarrow f^*\mathcal{P}^{n-2}_{Y/S}(M_2) \longrightarrow f^*\mathcal{P}^{n-1}_{Y/S}(M_1) \longrightarrow f^*\mathcal{P}^n_{Y/S}(M_0)$$

where we use the convention $\mathcal{P}^m_{Y/S} = 0$ for $m < 0$.
PROOF: Let $\varphi : M \to N$ be a homomorphism of $\mathcal{O}_Y$-Modules such that $f^*(\varphi) = 0$ and consider the diagram below, obtained by applying (2.8) to M and N

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Sym}^n(\Omega^1_{Y/S}) \otimes M & \longrightarrow & \mathcal{P}^n_{Y/S}(M) & \longrightarrow & \mathcal{P}^{n-1}_{Y/S}(M) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 1\otimes\varphi} & & \downarrow{\scriptstyle \mathcal{P}^n_{Y/S}(\varphi)} & & \downarrow{\scriptstyle \mathcal{P}^{n-1}_{Y/S}(\varphi)} & & \\
0 & \longrightarrow & \mathrm{Sym}^n(\Omega^1_{Y/S}) \otimes N & \longrightarrow & \mathcal{P}^n_{Y/S}(N) & \longrightarrow & \mathcal{P}^{n-1}_{Y/S}(N) & \longrightarrow & 0
\end{array}$$

Now we apply f^* and observe that $f^*(1 \otimes \varphi) = 1 \otimes f^*(\varphi) = 0$. Therefore, $f^*\mathcal{P}^n_{Y/S}(\varphi)$ factors through $f^*\mathcal{P}^{n-1}_{Y/S}(M)$; denote

$$\mathcal{Q}^n(\varphi) : f^*\mathcal{P}^{n-1}_{Y/S}(M) \to f^*\mathcal{P}^n_{Y/S}(N)$$

the induced homomorphism.

If $\psi : L \to M$ is another homomorphism of $\mathcal{O}_Y$-Modules such that $f^*(\psi) = 0$ and such that $\varphi \circ \psi = 0$ then an easy diagram chase shows that $\mathcal{Q}^{n+1}(\varphi) \circ \mathcal{Q}^n(\psi) = 0$ for all $n \in \mathbb{N}$.

Now, given the complex (M, μ) and $n \in \mathbb{N}$, the desired complex $\mathcal{P}^n(M, \mu)$ is defined as

$$\cdots \xrightarrow{\mathcal{Q}^{n-2}(\mu_3)} f^*\mathcal{P}^{n-2}_{Y/S}(M_2) \xrightarrow{\mathcal{Q}^{n-1}(\mu_2)} f^*\mathcal{P}^{n-1}_{Y/S}(M_1) \xrightarrow{\mathcal{Q}^{n}(\mu_1)} f^*\mathcal{P}^{n}_{Y/S}(M_0)$$

(3.11) REMARK: An easy diagram chase shows that the natural homomorphisms $\tau_j : H^0(Y, M_j) \otimes \mathcal{O}_X \to f^*\mathcal{P}^{n-j}_{Y/S}(M_j)$ commute with the differentials and hence give a homomorphism of complexes

$$\bar{\tau} : H^0(Y, (M, \mu)) \otimes \mathcal{O}_X \to \mathcal{P}^n(M, \mu)$$

(3.12) REMARK: The complex $\mathcal{P}^n(M, \mu)$ is not necessarily f^* of a complex of $\mathcal{O}_Y$-Modules since the homomorphisms $\mathcal{Q}^i(\mu_j)$ are in principle defined only over X. However, in the special situation of (3.6) the two complexes considered coincide and are defined over Y.

§4 Resultants, duals and Div.

In this section we introduce some of the ideas of [GKZ] in order to motivate our construction in §5.

(4.1) Let us consider a smooth irreducible projective variety $X \subset \mathbb{P}(V)$ over an algebraically closed field k of characteristic zero. The dual variety $X^* \subset \mathbb{P}(V^*)$ is defined as the set of hyperplanes $H \subset \mathbb{P}(V)$ such that $X \cap H$ is singular.
We denote by j^1 the composition $V^* \to H^0(X, \mathcal{O}(1)) \to H^0(X, \mathcal{P}^1_{X/k}(\mathcal{O}(1)))$ where the second map is $H^0(d^1_{X/k,\mathcal{O}(1)})$ as in (2.7). Then X^* may also be described as the set of linear forms $s \in V^*$ such that their first jet $j^1(s)$ has a zero $x \in X$.
If X^* is a hypersurface in $\mathbb{P}(V^*)$ (which is the case for most X's) then we denote by δ_X a defining equation for X^*, so that $\delta_X(H) = 0$ iff $X \cap H$ is singular. For example [GKZ], if X is a Veronese variety then δ_X is a discriminant, and if X is a convenient Segre variety then δ_X is a hyperdeterminant.

(4.2) Let E be a vector bundle of rank r on a variety X of dimension n, and denote by $R_E \subset \mathbb{P}H^0(X, E)$ the subvariety of sections of E that have a zero $x \in X$. It is not hard to check that if $r = n+1$ and E is generated by its global sections then the resultant variety R_E is an irreducible hypersurface with degree equal to deg $c_n(E)$; let us denote by ρ_E an equation for R_E. As examples of resultants we may mention:
(a) If $X = \mathbb{P}^n$ and $E = \bigoplus_{j=0}^n \mathcal{O}(d_i)$ then R_E is the classical resultant hypersurface, that is, the collection of $n+1$-tuples of polynomials $F_0, \ldots, F_n$ of degrees $d_0, \ldots, d_n$ that have a common zero $x = (x_0, \ldots, x_n) \in \mathbb{P}^n$.
(b) If $X^n \subset \mathbb{P}^r$ and $E = \mathcal{O}_X(1)^{n+1}$ then R_E is closely related to the Chow form of X.
(c) If $X^n \subset \mathbb{P}^r$ and $E = \mathcal{P}^1_{X/k}(\mathcal{O}(1))$ then, according to the discussion in (4.1), we have

$$R_E \cap \mathbb{P}H^0(X, \mathcal{O}(1)) = X^*$$

via the natural inclusion $H^0(X, \mathcal{O}(1)) \hookrightarrow H^0(X, \mathcal{P}^1_{X/k}(\mathcal{O}(1)))$. In particular, dual varieties are linear sections of resultants, and we also see that if $\Omega^1_X(1)$, and hence E, is generated by global sections then R_E and X^* are hypersurfaces.

(4.3) It seems to be of some interest to write down explicit formulas for resultant polynomials ρ_E. In general this is not an easy task, already for instance in the classical case (4.2)(a) for $n > 1$. But I. Gelfand, M. Kapranov and A. Zelevinsky have shown that it is easy to give an expression for all resultants in terms of a single operation, the determinant functor.

(4.4) In the literature one finds at least two notions of determinant for complexes: the Whitehead torsion [Bu], [M], [R] and the Grothendieck determinant functor [Bu], [KM].

In one of its variations, the Whitehead torsion is a function τ_R that assigns to each homotopy equivalence $\varphi : E \to F$ between two finite complexes E and F of finitely generated projective modules over a ring R, an element $\tau_R(\varphi)$ in the group $K_1(R)$. The function τ_R satisfies some properties that make it, in the terminology of [Bu] (1.2), a theory of determinants with values in $K_1(R)$. The torsion of a homotopically trivial complex is then defined through the choice of a trivialization, see [R].

On the other hand, the Grothendieck determinant functor det : $\mathcal{C}^{\cdot}is_X \to \mathcal{P}is_X$ is a functor from the category of bounded complexes of finite locally free modules over a scheme X, with quasi-isomorphisms of complexes as morphisms, to the category of graded invertible modules on X, with module isomorphisms as morphisms. The functor det satisfies some properties that make it, in the terminology of [Bu] (1.6), a general theory of determinants with values in the Picard category $\mathcal{P}is_X$. The existence of det is proved in [Bu] for X affine and in [KM] for general X.

The relationship between theories of determinants with values in abelian groups and general theories of determinants with values in Picard categories is summarized in a question by Grothendieck [Bu], page 37.

In [KM] it is also proved that det extends to the category of perfect complexes, which means in particular that each perfect complex has associated to it a determinant line bundle, a fact that is sometimes used to define certain line bundles on moduli spaces (see [BGS] for example). The second ingredient of the functor det, namely its action on morphisms, has special interest for us since it leads to the Div construction: If $\varphi : E \to F$ is a morphism in $\mathcal{C}^{\cdot}is_X$ then there is an induced isomorphism of line bundles $\det(\varphi) : \det(E) \to \det(F)$, whereas if $\varphi : E \to F$ is a (good) homomorphism of complexes, not necessarily a quasi-isomorphism as above, then $\mathrm{Div}(\varphi)$ is a Cartier divisor supported on the closed subscheme where φ is not a quasi-isomorphism (see [KM] for definition and basic properties of Div).

(4.5) In more concrete terms, the determinant of an exact complex of finite dimensional vector spaces with chosen basis is obtained as an alternating product of determinants of certain well chosen minors of the matrices of the complex [GKZ,2]. A complex of graded free modules on projective space is determined by a collection of matrices of homogeneous polynomials; if it is generically exact then we may form its determinant D (over the function field) in the previous sense, and the divisor of zeros and poles of the rational function D is the Div of the complex.
One reason why this construction is only 'almost' explicit is that there are choices of minors. Another reason is that it gives a rational function even in cases, like (4.6) and (5.19), where one knows that the answer is actually a polynomial.

(4.6) Returning to (4.3), suppose that E is a vector bundle of rank $n+1$ on X^n, generated by global sections. For each $s \in H^0(X, E)$ denote by $K(E, s)$ the Koszul complex, obtained from multiplication by s in the exterior algebra of E

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{\wedge s} E \xrightarrow{\wedge s} \wedge^2 E \xrightarrow{\wedge s} \dots \xrightarrow{\wedge s} \wedge^{n+1} E \longrightarrow 0$$

Since a Koszul complex is exact iff it is exact at the 0-th spot, the resultant R_E is equal to the set of sections s such that $K(E, s)$ is not exact. For variable s, the complex above may be viewed as a complex $K(E)$ on $X \times \mathbb{P}H^0(X,E)$, and denoting by p the projection onto $\mathbb{P}H^0(X,E)$, we obtain a determinantal expression for the resultant

$$R_E = \text{Div } Rp_* K(E)$$

Since for a complex F and a line bundle L we have Div $Rp_*F =$ Div $Rp_*(F \otimes L)$ ([KM], Prop. 9 (b)), we may choose a very ample line bundle $\mathcal{O}_X(1)$ on X and replace $K(E)$ by $K(E) \otimes O_X(m)$ with m large so that $H^i(X, (\wedge^j E)(m)) = 0$ $(i > 0, j \geq 0)$. Then R_E equals Div of the complex (see Cayley-Koszul complex, [GKZ])

$$0 \longrightarrow H^0(\mathcal{O}_X(m)) \otimes \mathcal{O} \xrightarrow{\wedge s} H^0(E(m)) \otimes \mathcal{O}(1) \xrightarrow{\wedge s} \dots \xrightarrow{\wedge s} H^0((\wedge^{n+1} E)(m)) \otimes \mathcal{O}(n+1) \longrightarrow 0$$

on the projective space $\mathbb{P}H^0(X,E)$.

§5 Hessians.

In the first part of this section, which is independent of the previous three sections, we analyse the existence and uniqueness of Hessians via a standard argument. The result is stated in (5.11). After this, we set up the determinantal description of Hessians, which is the main result of this article.

(5.1) Let $f : X \to S$ be a smooth and proper morphism of relative dimension one, where the varieties X and S are assumed to be smooth and irreducible. Let L be a line bundle on X such that $f_*(L)$ is locally free of rank $n+1$. Consider the natural homomorphism

$$\tau : f^* f_*(L) \to \mathcal{P}^n_{X/S}(L)$$

of locally free modules of rank $n+1$ on X and denote by $w_{f,L} = \det \tau$ the relative Wronskian and by $F_{f,L}$ the divisor of zeros of $w_{f,L}$.

(5.2) For the linear equivalence class of $F_{f,L}$ we have

$$\begin{aligned}[F_{f,L}] &= c_1 \mathcal{P}^n_{X/S}(L) - c_1 f^* f_*(L) \\ &= (n+1)\ c_1 L + \binom{n+1}{2}\ c_1 \Omega^1_{X/S} - c_1 f^* f_*(L)\end{aligned}$$

where the second equality is proved using (2.8).

(5.3) Suppose that our family of curves is embeded in projective space, so that we have a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{\ i\ } & S \times \mathbb{P}^r = Y & \xrightarrow{\ q\ } & \mathbb{P}^r \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle p} & & \\ S & = = & S & & \end{array}$$

where i is a closed embeding and p and q are the canonical projections. We shall use the notation $\mathcal{O}_Y(d) = q^* \mathcal{O}_{\mathbb{P}^r}(d)$ and $\mathcal{O}_X(d) = (q \circ i)^* \mathcal{O}_{\mathbb{P}^r}(d)$ for $d \in \mathbb{Z}$. The line bundle L will be chosen to be $L = \mathcal{O}_X(m)$ for a certain $m \in \mathbb{N}$.

(5.4) If $\mathcal{I}$ is the ideal sheaf of X in Y, from the exact sequence

$$0 \to \mathcal{I}/\mathcal{I}^2 \to q^* \Omega^1_{\mathbb{P}^r}|_X = \Omega^1_{Y/S}|_X \to \Omega^1_{X/S} \to 0$$

we obtain

$$c_1 \Omega^1_{X/S} = (-r-1)\ c_1 \mathcal{O}_X(1) - c_1 \mathcal{I}/\mathcal{I}^2$$

(5.5) Let us also assume that our family is the family of smooth complete intersection curves of multi-degree $(d_1, \ldots, d_{r-1})$ in $\mathbb{P}^r$, so that S is an open subvariety of the product $P = \prod_{1 \le i \le r-1} P_i$ of projective spaces $P_i = \mathbb{P}H^0(\mathbb{P}^r, \mathcal{O}(d_i))$, and X is given as

$$X = \{(F_1, \ldots, F_{r-1}, x)/F_i(x) = 0 \ \forall i\} \subset Y$$

We denote by $\mathcal{O}_i(1)$ the pull-back of $\mathcal{O}_{P_i}(1)$ by the canonical projection $S \to P_i$. Also, we write $L_i = p^*\mathcal{O}_i(1) \otimes \mathcal{O}_Y(d_i)$, so that X is the scheme of zeros of a regular section of the vector bundle $E = \bigoplus_{i=1}^{r-1} L_i$ on Y. It follows that $\mathcal{I}/\mathcal{I}^2 = E^*|_X$ and hence $c_1\mathcal{I}/\mathcal{I}^2 = -c_1 E = -\sum_i p^* c_1\mathcal{O}_i(1) + d_i c_1 \mathcal{O}_Y(1)$. We obtain from (5.4)

$$c_1\Omega^1_{X/S} = (-r - 1 + \sum_i d_i)\ c_1\mathcal{O}_Y(1) + \sum_i p^* c_1\mathcal{O}_i(1)$$

(5.6) To determine $c_1 f_*(L)$, we consider the Koszul resolution of $\mathcal{O}_X$ tensored by $\mathcal{O}_Y(m)$

$$0 \to \mathcal{O}_Y(m) \otimes \wedge^{r-1}E^* \to \cdots \to \mathcal{O}_Y(m) \otimes E^* \to \mathcal{O}_Y(m) \to L \to 0$$

We start by calculating

$$\wedge^\alpha E^* = \wedge^\alpha \bigoplus_i L_i^* = \bigoplus_{|J|=\alpha} \bigotimes_{j \in J} L_j^* = \bigoplus_{|J|=\alpha} \mathcal{O}_Y(-d_J) \otimes p^* \bigotimes_{j \in J} \mathcal{O}_j(-1)$$

where J denotes a subset of $\{1, \ldots, r-1\}$ and $d_J := \sum_{j \in J} d_j$. It follows from the projection formula that

$$p_*(\mathcal{O}_Y(m) \otimes \wedge^\alpha E^*) = \bigoplus_{|J|=\alpha} H^0(\mathbb{P}^r, \mathcal{O}(m - d_J)) \otimes \bigotimes_{j \in J} \mathcal{O}_j(-1)$$

and that $R^i p_*(\mathcal{O}_Y(m) \otimes \wedge^\alpha E^*) = 0$ for $1 \le i \le r-1$. This last vanishing implies that p_* of the resolution of L above is exact, as is easily seen by breaking the resolution into 3-term exact sequences. Therefore,

$$\begin{aligned}
c_1 f_*(L) &= \sum_{\alpha=0}^{r-1} (-1)^\alpha \ c_1 p_*(\mathcal{O}_Y(m) \otimes \wedge^\alpha E^*) \\
&= \sum_{\alpha=0}^{r-1} (-1)^\alpha \sum_{|J|=\alpha} h^0(\mathbb{P}^r, \mathcal{O}(m - d_J)) \sum_{j \in J} c_1\mathcal{O}_j(-1) \\
&= \sum_{\{(J,j)/j \in J\}} (-1)^{|J|}\ h^0(\mathbb{P}^r, \mathcal{O}(m - d_J))\ c_1\mathcal{O}_j(-1) \\
&= \sum_{j=1}^{r-1} \left(\sum_{J \ni j} (-1)^{|J|} \binom{m - d_J + r}{r} \right) \ c_1\mathcal{O}_j(-1)
\end{aligned}$$

where we use the convention $\binom{a}{b} = 0$ if $a < b$.

(5.7) It also follows from (5.6) that

$$n + 1 = \text{rank } f_*(L) = \sum_{J \subset \{1,\dots,r-1\}} (-1)^{|J|} \binom{m - d_J + r}{r}$$

(5.8) Combining the results above, we obtain

$$[F_{f,L}] = a\ c_1\mathcal{O}_X(1) + \sum_{j=1}^{r-1} b_j\ c_1\mathcal{O}_j(1)$$

with $a = (n+1)m + \binom{n+1}{2}(-r - 1 + \sum_i d_i)$ as predicted by the Plücker formula [ACGH] and

$$b_j = \binom{n+1}{2} + \sum_{J \ni j} (-1)^{|J|} \binom{m - d_J + r}{r}$$

(5.9) PROPOSITION: Let $\bar{F}_{f,L}$ and $\bar{X}$ denote the closures of $F_{f,L}$ and X in $P \times \mathbb{P}^r$, with P as in (5.5). Then the equality

$$[\bar{F}_{f,L}] = a\ c_1\mathcal{O}_{\bar{X}}(1) + \sum_{j=1}^{r-1} b_j\ c_1\mathcal{O}_j(1)$$

holds in $\text{Pic}(\bar{X})$.

PROOF: Let us recall that S in (5.5) is the open subvariety of all $F = (F_1, \dots, F_{r-1}) \in P$ such that $F_1 = \cdots = F_{r-1} = 0$ is a smooth curve, and the point of the present proof is that now we have to deal with bad F's that violate this condition. More precisely, let $\bar{f} : \bar{X} \to P$ be the restriction of the projection $p : \mathbb{P}^r \times P \to P$ as in (5.3), and denote by $\Sigma \subset \bar{X}$ the subvariety where $\bar{f}$ is not smooth (Σ has codimension at least two in $\bar{X}$). Let $w_{\bar{f},L}$ denote the corresponding Wronskian, defined on $\bar{X} - \Sigma$, and $F_{\bar{f},L}$ its scheme of zeros. The class of $F_{\bar{f},L}$ in $\text{Pic}(\bar{X} - \Sigma) \cong \text{Pic}(\bar{X})$ is given by the same formula (5.8), since the previous argument goes through (see [C]). What we need to check is that $F_{\bar{f},L} - F_{f,L}$ does not have divisorial components, so that the closures in $\bar{X}$ of $F_{\bar{f},L}$ and $F_{f,L}$ coincide. For this, we only need to consider phenomena that occur in codimension one in the smooth variety $\bar{X}$. First let us look at the subvariety $J \subset P$ where the dimension of the fibers of $\bar{f} : \bar{X} \to P$ jumps: this is under control since $\bar{f}^{-1}(J) \subset \Sigma$ has no divisorial components. Next, let $\Delta \subset P$ denote the divisor parametrizing F's such that $F = 0$ is a singular curve, with one node as its only singularities. The divisor Δ is irreducible, and for $F \in \Delta$ the corresponding curve $F = 0$ is irreducible and it has points that are not L-flexes (i.e. its Wronskian, defined away from the singular point, is not identically zero). Therefore, the irreducible divisor $f^{-1}(\Delta)$ is not contained in $\bar{F}_{f,L}$. Finally, we observe that $\bar{f}^{-1}(\bar{\Delta}-\Delta-J)$ has codimension two, and hence the proposition is proved.

(5.10) Now we analyse the existence and uniqueness of m-Hessians. We use the notation $\bar{Y}=\mathbb{P}^r\times P$, $\bar{F}=\bar{F}_{f,L}$ and $\mathcal{F}=\mathcal{O}_{\mathbb{P}^r}(a)\otimes\bigotimes_{j=1}^{r-1}\mathcal{O}_j(b_j)$ with a, b_j as in (5.8). Consider the natural exact sequence of ideal sheaves on $\bar{Y}$, twisted by $\mathcal{F}$:

$$0\longrightarrow\mathcal{F}\otimes\mathcal{I}_{\bar{X}/\bar{Y}}\longrightarrow\mathcal{F}\otimes\mathcal{I}_{\bar{F}/\bar{Y}}\xrightarrow{\rho}\mathcal{F}\otimes\mathcal{I}_{\bar{F}/\bar{X}}\cong\mathcal{O}_{\bar{X}}\longrightarrow 0$$

The general linear group $G=GL_{r+1}(k)$ acts on $\bar{Y}$, preserving $\bar{X}$ and $\bar{F}$, so that the sequence above is of G-Modules. It follows from (5.6) and Kunneth that $H^1(\bar{X},\mathcal{F}\otimes\mathcal{I}_{\bar{X}/\bar{Y}})=0$, and taking cohomology we obtain an exact sequence of finite dimensional G-vector spaces

$$0\longrightarrow\left\{\begin{array}{l}\text{forms vanishing}\\ \text{on } \bar{X}\end{array}\right\}\longrightarrow\left\{\begin{array}{l}\text{forms vanishing on}\\ \text{m-flexes of curves}\end{array}\right\}\xrightarrow{H^0(\rho)}H^0(\bar{X},\mathcal{F}\otimes\mathcal{I}_{\bar{F}/\bar{X}})=k.w_{f,L}\longrightarrow 0$$

where by forms we mean elements in

$$\begin{aligned}H^0(\mathbb{P}^r\times P,\mathcal{F})&\cong H^0(\mathbb{P}^r,\mathcal{O}(a))\otimes\bigotimes_{j=1}^{r-1}\mathrm{Sym}^{b_j}H^0(\mathbb{P}^r,\mathcal{O}(d_j))^*\\ &\cong\mathrm{Sym}^a(k^{r+1})\otimes\bigotimes_{j=1}^{r-1}\mathrm{Sym}^{b_j}\mathrm{Sym}^{d_j}(k^{r+1})^*\end{aligned}$$

An m-Hessian is a G semi-invariant element in $H^0(\bar{X},\mathcal{F}\otimes\mathcal{I}_{\bar{F}/\bar{Y}})$, and hence corresponds to a G-spliting of the sequence above. Since G is reductive, such a spliting exists. It also follows that if H_m denotes an m-Hessian then any other m-Hessian may be represented, up to multiplicative constant, as

$$H_m+\sum_{j=1}^{r-1}G_j\ F_j$$

where F_j is the general polynomial of degree d_j and $G_j=G_j(x,F_1,\ldots,F_{r-1})$ is zero or a G semi-invariant of the appropriate multi-degree, when they exist.

To summarize, we may state:

(5.11) PROPOSITION: For any $r, m, d_1,\ldots,d_{r-1}$ an m-Hessian H_m exists and is esentially unique. It may be represented as a form $H_m=H_m(x,F_1,\ldots,F_{r-1})$ of multi-degree $(a,b_1,\ldots,b_{r-1})$, where a, b_j are as in (5.8).

(5.12) It is verified that the classical formulas in the Introduction have the degree stated in (5.11). Notice that in all those cases one has $m < d_j$ for all j, and hence $b_j = \binom{n+1}{2}$.

(5.13) Now we turn to the description of the complex of graded free modules representing an m-Hessian. Let $X \subset \mathbb{P}^r_k = P$ be a smooth complete intersection curve defined by the sequence of polynomials $F_1, \dots, F_{r-1} \in A = k[x_0, \dots, x_r]$ of degrees $d_1, \dots, d_{r-1}$. Denote $E = \bigoplus_{i=1}^{r-1} \mathcal{O}_P(d_i)$, so that $F = (F_1, \dots, F_{r-1}) \in H^0(P, E)$ and $X = (F = 0)$. The Koszul complex K_F

$$0 \to \mathcal{O}_P(m) \otimes \wedge^{r-1} E^* \to \cdots \to \mathcal{O}_P(m) \otimes E^* \to \mathcal{O}_P(m)$$

provides a resolution of $\mathcal{O}_X(m)$ by graded free $\mathcal{O}_P$-Modules.

(5.14) Let us consider the commutative diagram of complexes of $\mathcal{O}_X$-Modules

$$\begin{array}{ccccc} \mathcal{H} := H^0(P, K_F) \otimes \mathcal{O}_X & \xrightarrow{\rho} & H^0(X, \mathcal{O}_X(m)) \otimes \mathcal{O}_X & \longrightarrow & 0 \\ \bar{\tau} \downarrow & & \downarrow \tau & & \\ \mathcal{P} := \mathcal{P}^n(K_F) & \xrightarrow{\rho} & \mathcal{P}^n_{X/k}(\mathcal{O}_X(m)) & \longrightarrow & 0 \end{array}$$

with $\mathcal{P}^n(K_F)$ as in (3.10), using K_F for (M, μ), $\bar{\tau}$ as in (3.11) and n as in (5.7).

(5.15) The Euler sequences (2.25) combine to give an exact sequence of complexes of $\mathcal{O}_X$-Modules

$$0 \to \mathcal{P}^n(K_F) \xrightarrow{j} \mathrm{P}^n := \iota^*(\mathrm{P}^n_{A/k}(F) \otimes A(m))^\sim \xrightarrow{\epsilon} \mathrm{P}^{n-1} := \iota^*(\mathrm{P}^{n-1}_{A/k}(F) \otimes A(m))^\sim \to 0$$

where $(\mathrm{P}^n_{A/k}(F) \otimes A(m))^\sim$ is the complex of (3.6) or (3.9) with each module in the complex twisted by m and then sheafified, ϵ is defined by duals of Euler operators $.(E - s)^*$ as in (2.27), for various constants s, and $\iota : X \to P$ is the inclusion map.

(5.16) REMARK: The complexes $(\mathrm{P}^n_{A/k}(F) \otimes A(m))^\sim$ are defined on P, and the maps ϵ are defined on P, but they commute with the differentials of the complex only after applying ι^* (i.e. modulo multiples of the F_i's).

(5.17) It is easy to check that the upper row of (5.14) is exact. The lower row is also exact, as it follows from (3.7) and the long exact sequence of homology for (5.15). In other words, the maps ρ are quasi-isomorphisms, and hence $\mathrm{Div}(\rho) = 0$. Now [KM] (Thm. 3 (i), p. 48) implies that

$$\mathrm{Div}\ (\tau) = \mathrm{Div}\ (\bar{\tau})$$

which achieves the first step in the direction of lifting the Wronskian $\det(\tau)$ to $\mathbb{P}^r$.

(5.18) Next we would like to replace $\bar{\tau} : \mathcal{H} \to \mathcal{P}$ by a homomorphism of complexes of graded free modules. This will be done by a cone construction. Consider the diagram

$$\begin{array}{ccccccccc} & & \mathcal{H} & & & & & & \\ & & \bar{\tau}\downarrow & & & & & & \\ 0 & \longrightarrow & \mathcal{P} & \xrightarrow{j} & \mathrm{P}^n & \xrightarrow{\epsilon} & \mathrm{P}^{n-1} & \longrightarrow & 0 \\ & & \alpha\downarrow & & & & & & \\ & & \mathcal{C} & & & & & & \end{array}$$

where $\mathcal{C} = \text{cone}(\epsilon)(1) = \mathrm{P}^n \oplus \mathrm{P}^{n-1}(1)$ is the cone complex of ϵ shifted by one. It follows from [Bo,2] (Ex. 9, p. 174) that the natural map $\alpha(x) = (j(x), 0)$ is a quasi-isomorphism, and therefore $\text{Div}(\alpha) = 0$. Defining $\tilde{\tau} = \alpha \circ \bar{\tau}$ we have the equality

$$\text{Div}(\bar{\tau}) = \text{Div}(\tilde{\tau})$$

We state the result in the following way

(5.19) THEOREM: If $\tilde{\tau}$ denotes the natural homomorphism of complexes of $\mathcal{O}_X$-Modules

$$H^0(P, K_F) \otimes \mathcal{O}_X \xrightarrow{\tilde{\tau}} \iota^*(\mathrm{P}^n_{A/k}(F) \otimes A(m))^{\sim} \oplus \iota^*((\mathrm{P}^{n-1}_{A/k}(F) \otimes A(m))^{\sim})(1)$$

then $\text{Div}(\tilde{\tau})$ is a rational m-Hessian for X.

We need to explain what we mean by rational m-Hessian and why they appear in our statement. As mentioned in (5.16), the differentials of the complexes in question, and the map $\tilde{\tau}$, are all given by matrices of homogenous polynomials, that is, they are defined over P. The difficulty is that our cone $\mathcal{C}$ is a complex only on X, that is, the condition $d^2 = 0$ is satisfied on P only modulo the F_i's. To calculate $\text{Div}(\tilde{\tau})$ we carry out the process of (4.5) with these matrices of polynomials. To do this we need to make a choice of admissible minors. We don't need the condition $d^2 = 0$ to calculate the alternating product of determinants of the chosen minors, but different choices will give different answers, all equal on X (by (5.18), compare (5.11)). These various "Div's" are quotients A/B of homogeneous polynomials on P. We obtain a finite number of rational m-Hessians and our computations seem to indicate that none of these is a polynomial Hessian. It would be desirable to find a further manipulation of these complexes that leads to a polynomial Hessian.

(5.20) For plane curves the diagrams above take a simpler form that perhaps it is worth making explicit. Let $X = (F = 0) \subset P = \mathbb{P}^2$ with F of degree d and fix $m \in \mathbb{N}$. The basic diagram is

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & H^0(P,\mathcal{O}(m-d))\otimes\mathcal{O}_X & \xrightarrow{.F} & H^0(P,\mathcal{O}(m))\otimes\mathcal{O}_X & \longrightarrow & H^0(X,\mathcal{O}(m))\otimes\mathcal{O}_X & \longrightarrow & 0 \\
 & & \tau\downarrow & & \tau\downarrow & & \tau\downarrow & & \\
0 & \longrightarrow & \iota^*\mathcal{P}^{n-1}_{P/k}\mathcal{O}(m-d) & \longrightarrow & \iota^*\mathcal{P}^{n}_{P/k}\mathcal{O}(m) & \longrightarrow & \mathcal{P}^{n}_{X/k}\mathcal{O}(m) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & \iota^*\mathrm{P}^{n-1}_{A/k}(m-d)^\sim & \xrightarrow{.\Delta F} & \iota^*\mathrm{P}^{n}_{A/k}(m)^\sim & & & & \\
 & & \epsilon\downarrow & & \epsilon\downarrow & & & & \\
 & & \iota^*\mathrm{P}^{n-2}_{A/k}(m-d)^\sim & \xrightarrow{.\Delta F} & \iota^*\mathrm{P}^{n-1}_{A/k}(m)^\sim & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

where $.\Delta F$ is multiplication by $\Delta F = \sum_\alpha \partial^\alpha(F)\ (dx)^\alpha$ in the symmetric algebra of $\Omega^1_{A/k}$ (see (2.34) or (3.9)). According to our previous analysis, the divisor of m-flexes is the Div of the double complex

$$
\begin{array}{ccc}
H^0(P,\mathcal{O}(m-d))\otimes\mathcal{O}_X & \xrightarrow{.F} & H^0(P,\mathcal{O}(m))\otimes\mathcal{O}_X \\
\tau\downarrow & & \tau\downarrow \\
\iota^*\mathrm{P}^{n-1}_{A/k}(m-d)^\sim & \xrightarrow{.\Delta F} & \iota^*\mathrm{P}^{n}_{A/k}(m)^\sim \\
\epsilon\downarrow & & \epsilon\downarrow \\
\iota^*\mathrm{P}^{n-2}_{A/k}(m-d)^\sim & \xrightarrow{.\Delta F} & \iota^*\mathrm{P}^{n-1}_{A/k}(m)^\sim
\end{array}
$$

that is, the Div of the associated total complex

$$
H^0(P,\mathcal{O}(m-d))\otimes\mathcal{O}_X \xrightarrow{\delta_2} H^0(P,\mathcal{O}(m))\otimes\mathcal{O}_X \ \oplus\ \iota^*\mathrm{P}^{n-1}_{A/k}(m-d)^\sim \xrightarrow{\delta_1}
$$
$$
\iota^*\mathrm{P}^{n}_{A/k}(m)^\sim \ \oplus\ \iota^*\mathrm{P}^{n-2}_{A/k}(m-d)^\sim \xrightarrow{\delta_0} \iota^*\mathrm{P}^{n-1}_{A/k}(m)^\sim
$$

where $\delta_2(a) = (-F.a, -\tau(a))$, $\delta_1(b,c) = (c.\Delta F - \tau(b), -\epsilon(c))$ and $\delta_0(d,e) = e.\Delta F - \epsilon(d)$.

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