RING THEORETIC ASPECTS OF WEAK HOPF ACTIONS

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Abstract. Following Linchenko and Montgomery's arguments we show that the smash product of a semiprime module algebra, satisfying a polynomial identity and an involutive weak Hopf algebra is semiprime. We get new insight into the existence of non-trivial central invariant elements in non-trivial H-stable ideals of subdirect products of certain H-prime module algebras satisfying a polynomial identity by considering an adapted version of Kaplansky's theorem and by introducing a Brown-McCoy radical for module algebras. We extend Puczylowski and Smoktunowicz's description of the Brown-McCoy radical of a polynomial ring to module algebras and apply our result to left bialgebroid measurings, gradings and involutions. The paper finishes with an extension of results by Bergen et al. and Cohen at al. on irreducible module algebras to weak Hopf actions.

1. Introduction

Group actions, Lie algebras acting as derivations and finite group gradings are typical examples of Hopf algebra actions which have been studied for many years. Several generalizations of Hopf algebras have emerged in recent years, like weak Hopf algebras (or quantum groupoids) introduced by Böhm et al. [6]. The action of such objects on algebras, as given by quantum groupoids acting on C^*-algebras, [29] or weak Hopf algebras arising from Jones towers [14] are particularly interesting. New examples of weak Hopf algebras arose from double groupoids [3], which were also used to find new weak Hopf actions (see [29]).

A long-standing open problem in the theory of Hopf action is to show that the smash product $A\#H$ of a semiprime module algebra A and a semisimple Hopf algebra H is again semiprime (see [10]). The most recent partial answer to this problem has been given by Linchenko and Montgomery in [17] where they prove the semiprimness of $A\#H$ under the condition of A satisfying a polynomial identity. We will see that their result carries over to actions of weak Hopf algebras. We reach more generality by considering actions of linear operators that satisfy certain intertwining relations with the regular multiplications on the algebra.

In the fourth section we will consider weak Hopf action H on algebras A satisfying a polynomial identity and prove that any non-zero H-stable ideal contains a non-zero central H-invariant element if A is a subdirect product of H-compressible module algebras. For that we consider a kind of Brown-McCoy radical for algebras relative to the action of an algebra of linear operators. We will show an analogous description of the Brown-McCoy radical of a polynomial ring as given by Puczylowski and Smoktunowicz. We also adept a version of Martindale's central closure relative to an operator action and consider strongly prime algebras. In the fifth section we

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apply the previously obtained results to measurings of left bialgebroids, gradings and algebras with involutions.

In the final section we consider weak Hopf actions on irreducible algebras extending some results of Cohen et al. to this setting.

2. Linear operators acting on algebras satisfying a polynomial identity.

The aim of this section is to extend a recent solution to the semiprimness problem for smash products to weak Hopf actions.

2.1. Let k be a field and let A be an associative unital k-algebra. For any $a \in A$ define two linear operators L_a and R_a in $\mathrm{End}_k(A)$ given by $\langle L(a), x\rangle = ax$ and $\langle R(a), x\rangle = xa$ for all $x \in A$. We identify A with the subalgebra $L(A)$ of $\mathrm{End}_k(A)$ generated by all left multiplications $L(a)$ and denote the subalgebra generated by all operators $L(a)$ and $R(a)$ by $M(A)$, which is also sometimes referred to as the *multiplication algebra* of A. As a left $L(A)$-module, A is isomorphic to $L(A)$ since we assume A to be unital.

Let $L(A) \subseteq B \subseteq \mathrm{End}_k(A)$ be any intermediate algebra. Then A becomes a cyclic left B-module by the evaluation pairing $\langle\ ,\ \rangle : \mathrm{End}_k(A) \times A \to A$ with $\psi \cdot a := \langle \psi, a\rangle = \psi(a)$. A subset I of A is called B-stable if $B \cdot I \subseteq I$. The B-stable left ideals are precisely the (left) B-submodules of A.

2.2. The first technical Lemma generalizes a corresponding result of Linchenko [16, Theorem 3.1] for Hopf actions and Nikshych [23, Theorem 6.1.3] for weak Hopf actions. Recall that an ideal whose elements are nilpotent is called a nil ideal.

Lemma. *Let $L(A) \subseteq B \subseteq \mathrm{End}_k(A)$ and suppose that for all $\psi \in B$ there exist $m \geq 1$ and elements $\psi_1^1, \ldots, \psi_m^1, \psi_1^2, \ldots, \psi_m^2 \in B$ such that*

$$L(\langle\psi, a\rangle) = \sum_{i=1}^{m} \psi_i^1 \circ L(a) \circ \psi_i^2 \quad \text{and} \quad \sum_{i=1}^{n} \psi_i^2 \circ \psi_i^1 \in L(A) \tag{1}$$

for any $a \in A$. If A is finite dimensional over a field of characteristic 0 and if I is a nil ideal, then $B \cdot I$ is nil. In particular the Jacobson radical of A is B-stable.

Proof. Denote the trace of a k-linear endomorphism f of A by $\mathrm{Tr}(f)$. Let $\psi \in B$, $a \in A$. Using $\mathrm{Tr}(fgh) = \mathrm{Tr}(hfg)$ and the hypotheses we get:

$$\mathrm{Tr}(L(\langle\psi, a\rangle)) = \mathrm{Tr}\left(\left(\sum_{i=1}^{n} \psi_i^2 \circ \psi_i^1\right) \circ L(a)\right) = \mathrm{Tr}(L(y) \circ L(a)) = \mathrm{Tr}(L(ya))$$

for some $y \in A$. Suppose that $a \in I$ with I a nil ideal, then $ya \in I$ is nilpotent, hence $\mathrm{Tr}(L(\langle\psi, a\rangle)) = \mathrm{Tr}(L(ya)) = 0$. For any $k > 0$ set $z^k := \langle\psi, a\rangle^k$. Then

$$z^k = \langle L(\langle\psi, a\rangle), z^{k-1}\rangle = \sum_{i=1}^{n} \langle\psi_i^1, a_i\rangle$$

for $a_i = a\langle\psi_i^2, z^{k-1}\rangle$. Since I is an ideal, $a_i \in I$. Hence

$$\mathrm{Tr}\big(L(z)^k\big) = \mathrm{Tr}\big(L(z^k)\big) = \sum_{i=1}^{n} \mathrm{Tr}\big(L(\langle\psi_i^1, a_i\rangle)\big) = 0$$

Since A is finite dimensional, $char(k) = 0$ and the trace of all powers of $L(z)$ is zero, $L(z)$ is a nilpotent operator, i.e. $z = \langle\psi, a\rangle$ is nilpotent. Thus $B \cdot I$ is a nil ideal. Since the Jacobson radical of an Artinian ring is the largest nilpotent ideal, we have $B \cdot \mathrm{Jac}(A) = \mathrm{Jac}(A)$. $\square$

2.3. The last Lemma, which had been proven first by Linchenko for Hopf actions and then by Nikshych for weak Hopf actions allows us to show the stability of the Jacobson radical of an algebra A which satisfies a polynomial identity and on which act some operator algebra B which is finitely generated over A. The hypotheses of the following Theorem allow the reduction to finite dimensional factors.

Theorem. *Let $L(A) \subseteq B \subseteq \mathrm{End}_k(A)$ over some field k of characteristic 0 with B being finitely generated as right A-module. Suppose that for all $\psi \in B$ there exist $n \geq 1$ and elements $\psi_1^1, \ldots, \psi_m^1, \psi_1^2, \ldots, \psi_m^2 \in B$ satisfying*

$$L(\langle \psi, a\rangle) = \sum_{i=1}^m \psi_i^1 \circ L(a) \circ \psi_i^2 \quad \text{and} \quad \sum_{i=1}^m \psi_i^2 \circ \psi_i^1 \in L(A) \tag{2}$$

for any $a \in A$. If A satisfies a polynomial identity or if k is an uncountable algebraically closed field, A countably generated and all left primitive factor rings of A are Artinian, then $B \cdot I \subseteq \mathrm{Jac}(A)$ for all nil ideals I of A.

Proof. Let I be a nil ideal. It is enough to show that $(B \cdot I)V = 0$ for all simple left A-modules V, then $B \cdot I \subseteq \mathrm{Jac}(A)$. Let V be a simple left A-module and $P = \mathrm{Ann}_A(V)$ be its annihilator. If k is an uncountable algebraically closed field and A is countably generated, then it satisfies the Nullstellensatz, hence $\mathrm{End}_k(V) = k$ (see [20, 9.1.8]). If primitive factors of A are Artinian, then by the Weddeburn-Artin Theorem $A/P \simeq M_n(k)$ for some n, hence V is a finite dimensional simple left A-module. On the other hand, if A satisfies a polynomial identity, then $A/P \simeq M_n(D)$ where D is a finite dimensional division algebra over $F = Z(A/P)$ by Kaplansky's Theorem [20, 13.3.8]. Tensoring A by F yields an F-algebra $\widetilde{A} := A \otimes F$ with F-action on the right. Then

$$L(\widetilde{A}) = L(A) \otimes F \subseteq B \otimes F \subseteq \mathrm{End}_k(A) \otimes F \subseteq \mathrm{End}_F(A \otimes F) = \mathrm{End}_F(\widetilde{A}).$$

Moreover V is a finite dimensional simple left $\widetilde{A}$-module since $\mathrm{Ann}_{\widetilde{A}}(V) = P \otimes F =: \widetilde{P}$ and $\widetilde{A}/\widetilde{P} \simeq M_n(D)$ is finite dimensional over F. Note also that the nil ideal I extends to a nil ideal $\widetilde{I} := I \otimes F$ since by [15, Theorem 5] I is a locally nilpotent algebra and hence any element $\sum_{i=1}^m a_i \otimes f_i$ lies in a nilpotent finitely generated subalgebra generated by the a_i's and F.

To summarize, our hypothesis on A allows us to consider V to be a finite dimensional simple left A-module, where A and B are algebras over some field k of characteristic 0. Denote by $W = B \otimes_A V$ the induced left B-module. Since B_A is finitely generated and V is finite dimensional, W is finite dimensional. Note that the left B-action on W is given by $\psi \cdot (\phi \otimes v) := (\psi \circ \phi) \otimes v$. Let $Q = \mathrm{Ann}_A(W)$. Then Q is B-stable, because if $a \in Q$ and $\psi \in B$, then by hypothesis there exist elements $\psi_1^1, \ldots, \psi_m^1, \psi_1^2, \ldots, \psi_m^2 \in B$ satisfying equation (2). Thus for any $w = \phi \otimes v \in W$ we have

$$\langle \psi, a\rangle \cdot (\phi \otimes v) = (L(\langle \psi, a\rangle) \circ \phi) \otimes v = \sum_{i=1}^m \psi_i^1 \circ L(a) \circ \psi_i^2 \circ \phi \otimes v = \sum_{i=1}^m \psi_i^1 \cdot (a \cdot (\psi_i^2 \cdot w)) = 0,$$

since $\psi_i^2 \cdot w \in W$ and $a \cdot W = 0$. Hence $B \cdot Q \subseteq Q$. Let $Q_B = \mathrm{Ann}_B(A/Q)$. Then

$$A/Q \simeq L(A/Q) \subseteq B/Q_B \subseteq \mathrm{End}_k(A/Q).$$

Since W is finite dimensional, A/Q is finite dimensional. Note that V is a simple left A/Q-module. Any nil ideal I of A yields a nil ideal $(I+Q)/Q$ of A/Q. Moreover every element $\psi + Q_B \in B/Q_B$ satisfies (2). By Lemma 2.2, $((B \cdot I) + Q)/Q = B/Q_B \cdot (I+Q)/Q$ is included in $\mathrm{Jac}(A/Q)$. Thus

$$(B \cdot I)V = (B/Q_B \cdot (I+Q)/Q) \cdot V \subseteq \mathrm{Jac}(A/Q) \cdot V = 0.$$

Hence $B \cdot I \subseteq \mathrm{Jac}(A)$ for any nil ideal I of A. □

3. Weak Hopf actions on algebras satisfying a polynomial identity

Before we apply the results from the previous section, we first recall the definition of weak Hopf algebras (or quantum groupoids) as introduced by G. Böhm et al. in [6].

Definition. *An associative k-algebra H with multiplication m and unit 1 which is also a coassociative coalgebra with comultiplication Δ and counit ϵ is called a weak Hopf algebra if it satisfies the following properties:*

(1) *the comultiplication is multiplicative, i.e. for all $g,h \in H$:*
$$\Delta(gh) = \Delta(g)\Delta(h). \tag{3}$$
(2) *the unit and counit satisfy:*
$$(\Delta \otimes id)\Delta(1) = (\Delta(1) \otimes 1)(1 \otimes \Delta(1) = (1 \otimes \Delta(1))(\Delta(1) \otimes 1) \tag{4}$$
$$\epsilon(fgh) = \epsilon(fg_1)\epsilon(g_2h) = \epsilon(fg_2)\epsilon(g_1h) \tag{5}$$
(3) *there exists a linear map $S : A \to A$, called antipode, such that*
$$h_1S(h_2) = (\epsilon \otimes id)(\Delta(1)(h \otimes 1)) =: \epsilon_t(h) \tag{6}$$
$$S(h_1)h_2 = (id \otimes \epsilon)((1 \otimes h)\Delta(1)) =: \epsilon_s(h) \tag{7}$$
$$S(h) = S(h_1)h_2S(h_3) \tag{8}$$
Note that we will use Sweedler's notation for the comultiplication with suppressed summation symbol.

3.1. The image of ϵ_t and ϵ_s are subalgebras H_t and H_s of H which are separable over k ([24, 2.3.4]) and their images commute with each other. Those subalgebras are also characterized by $H_t = \{h \in H : \Delta(h) = 1_1h \otimes 1_2\}$ respectively $H_s = \{h \in H : \Delta(h) = 1_1 \otimes 1_2h\}$.

3.2. A left H-module algebra A over a weak Hopf algebra H is an associative unital algebra A such that A is a left H-module and for all $a, b \in A, h \in H$:
$$h \cdot (ab) = (h_1 \cdot a)(h_2 \cdot b) \text{ and } h \cdot 1_A = \epsilon_t(h) \cdot 1_A \tag{9}$$

3.3. Let A be a left H-module algebra over a weak Hopf algebra H and let λ be the ring homomorphism from H to $\mathrm{End}_k(A)$ that defines the left module structure on A, i.e. $\langle\lambda(h), a\rangle := h \cdot a$ for all $h \in H, a \in A$. Property (9) of the definition above can be interpreted as an intertwining relation $\lambda(h) \circ L(a) = L(h_1 \cdot a) \circ \lambda(h_2)$ of left multiplications $L(a)$ and left H-actions $\lambda(h)$.

The following properties are now deduced from the axioms:

Lemma. *Let A be a left H-module algebra over a weak Hopf algebra H. Then*

(1) $\forall z \in H_t : \lambda(z) = L(z \cdot 1_A)$ *and* $\forall z \in H_s : \lambda(z) = R(z \cdot 1_A)$*;*
(2) $\forall h \in H, a \in A : L(h \cdot a) = \lambda(h_1) \circ L(a) \circ \lambda(S(h_2))$
(3) *If $S^2 = id$, then $\lambda(S(h_2)) \circ \lambda(h_1) \in L(A)$ for all $h \in H$.*

Proof. (1) Let $z \in H_t$. Since $\Delta(z) = 1_1z \otimes 1_2$, we have for all $a \in A$:
$$z \cdot a = z \cdot (1_Aa) = (1_1z \cdot 1_A)(1_2 \cdot a) = 1_H \cdot ((z \cdot 1_A)a) = (z \cdot 1_A)a$$
The proof of the second statement is analogous.

(2) For $h \in H, a, x \in A$ we have
$$\begin{aligned}(h \cdot a)x &= (h_1 \cdot a)(\epsilon_t(h_2) \cdot 1_A)x \\ &= (h_1 \cdot a)(\epsilon_t(h_2) \cdot x) = (h_1 \cdot a)(h_2S(h_3) \cdot x) = h_1 \cdot (a(S(h_3) \cdot x))\end{aligned}$$

(3) Suppose $S^2 = id$, then $S(\epsilon_s(h)) = S(h_2)h_1$ and as $S(H_s) \subseteq H_t$, we have using (1):
$$\lambda(S(h_2)) \circ \lambda(h_1) = \lambda(S(\epsilon_s(h))) = L(S(\epsilon_s(h)) \cdot 1_A) \in L(A).$$
□

3.4. We say that a weak Hopf algebra H is involutive if its antipode is an involution. Say that H acts finitely on a left H-module algebra A if the image of $\lambda : H \to \mathrm{End}_k(A)$ is finite dimensional. The following statement follows from the last Lemma and Theorem 2.3

Theorem. *Let H be an involutive weak Hopf algebra over a field k of characteristic zero acting finitely on a left H-module algebra A. If A satisfies a polynomial identity or if k is an uncountable algebraically closed field, A is countably generated and all left primitive factor rings of A are Artinian, then the Jacobson radical of A is H-stable.*

Proof. Let $\lambda : H \to \mathrm{End}_k(A)$ be the ring homomorphism inducing the left H-module structure on A. Denote by B the subalgebra of $\mathrm{End}_k(A)$ generated by $L(A)$ and $\lambda(H)$. Let $h_1, \ldots, h_m$ be elements of H such that $\{\lambda(h_1), \ldots, \lambda(h_m)\}$ forms a basis of $\lambda(H)$. We claim that any element of B is of the form $\sum_{i=1}^m \lambda(h_i) \circ L(a_i)$ for some $a_i \in A$. It is enough to show $L(A)\lambda(H) \subseteq \lambda(H)L(A)$. So take elements $h \in H$ and $a, b \in A$. Then using Lemma 3.3(2), $S^{-1} = S$ and $S(H_t) = H_s$ we have

$$\begin{aligned} h_2 \cdot ((S(h_1) \cdot a)b) &= (h_2 S(h_1) \cdot a)(h_3 \cdot b) \\ &= (S(\epsilon_t(h_1)) \cdot a)(h_2 \cdot b) = a(\epsilon_t(h_1) \cdot 1_A)(h_2 \cdot b) = a(h \cdot b). \end{aligned} \tag{10}$$

This shows the intertwining relation $L(a) \circ \lambda(h) = \lambda(h_2) \circ L(S(h_1) \cdot a)$ in B which yields that B is finitely generated as a right A-module. By the definition of module algebras, we also have that $\lambda(h) \circ L(a) = L(h_1 \cdot a) \circ \lambda(h_2)$. Hence $\lambda(H)L(A) = L(A)\lambda(H)$. For any $a \in A$ and $\psi = \sum_{i=1}^m L(a_i) \circ \lambda(h_i) \in B$ we have by Lemma 3.3(3) and by equation (10):

$$\begin{aligned} L(\langle \psi, a \rangle) &= \sum L(a_i) \circ L(\langle h_i, a \rangle) \\ &= \sum L(a_i) \circ \lambda(h_{i1}) \circ L(a) \circ \lambda(S(h_{i2})) \\ &= \sum \lambda(h_{i2}) \circ L((S(h_{i1}) \cdot a_i)) \circ L(a) \circ \lambda(S(h_{i3})) = \sum_j \psi_j^1 \circ L(a) \circ \psi_j^2. \end{aligned}$$

for $\psi_j^1 = \lambda(h_{i2}) \circ L(S(h_{i1}) \cdot a_i)$, $\psi_j^2 = \lambda(S(h_{i3})$ and some appropriate choice of indices j. Moreover

$$\sum \psi_j^2 \circ \psi_j^1 = \sum \lambda(S(h_{i3})) \circ \lambda(h_{i2}) \circ L(S(h_{i1}) \cdot a_i) = \sum L(y_{h_{i2}}) \circ L(S(h_{i1}) \cdot a_i) \in L(A)$$

for some elements $y_{h_{i2}} \in A$ that exist by Lemma 3.3(4). Therefore the hypotheses of Theorem 2.3 are fulfilled and the statement follows.

□

3.5. Smash products of weak Hopf actions. Recall that the smash product $A\#H$ of a left H-module algebra A and a weak Hopf algebra H is defined on the tensor product $A \otimes_{H_t} H$ where A is considered a right H_t-module by $a \cdot z = a(z \cdot 1_A)$ for $a \in A, z \in H_t$. The (k-linear) dual H^* of H becomes also a weak Hopf algebra and acts on $A\#H$ by $\phi \cdot (a\#h) := a\#\phi(\rightharpoonup h)$, where $\phi \rightharpoonup h = h_1\langle \phi, h_2 \rangle$. Using the Montgomery-Blattner duality theorem for weak Hopf algebras proven by Nikshych we have:

Lemma. *Let H be a finite dimensional weak Hopf algebra and A a left H-module algebra. Then $A\#H$ is a finitely generated projective right A-module and $A\#H\#H^* \simeq eM_n(A)e$ for some idempotent $e \in M_n(A)$ where $M_n(A)$ denotes the ring of $n \times n$-matrixes for some number $n > 0$.*

Proof. By [21, Theorem 3.3] $A\#H\#H^* \simeq \mathrm{End}(A\#H_A)$. Since H_t is a separable k-algebra, it is semisimple Artinian. Hence H is a (finitely generated) projective right H_t-module and H is a direct summand of H_t^n for some $n > 0$. Moreover it follows from the proof of Lemma 3.3 that $A\#H = (1\#H)(A\#1)$. Thus $H \otimes_{H_t} A \simeq A\#H$

as right A-modules by $h \otimes a \mapsto (1\#h)(a\#1)$. On the other hand $H \otimes_{H_t} A$ is a direct summand of $H_t^n \otimes_{H_t} A \simeq A^n$ as right A-module. Hence $A\#H$ is a projective right A-module of rank n and $\mathrm{End}(A\#H)_A \simeq eM_n(A)e$ for some idempotent $e \in M_n(A)$. □

3.6. Semiprime smash products for weak Hopf actions. We can now transfer Linchenko and Montgomery's result [17, Theorem 3.4] on the semiprimness of smash products to weak Hopf actions.

Theorem. *Let A be a left H-module algebra over a finite dimensional involutive weak Hopf algebra H over a field of characteristic zero. If A is semiprime and satisfies a polynomial identity, then $A\#H$ is semiprime.*

Proof. Set $B = A\#H\#H^*$. Note that H^* is also involutive since its antipode is defined by $\langle S^*(\phi), h\rangle := \langle \phi, S(h)\rangle$ for all $\phi \in H^*, h \in H$. By [22, Corollary 6.5] H^* is semisimple and by [6, 3.13] there exists a normalized left integral $\Lambda \in H^*$. This implies that $A\#H$ is a projective left B-module as the left B-linear map $A\#H \to B$ with $a\#h \mapsto a\#h\#\Lambda$ splits the projection $B \to A\#H$ given by $a\#h\#\phi \mapsto a\#h(\phi \rightharpoonup 1)$.

First suppose that $\mathrm{Jac}(A) = 0$. By Lemma 3.5, $\mathrm{Jac}(B) \simeq eM_n(\mathrm{Jac}(A))e = 0$ for some idempotent e. This implies also that $\mathrm{Rad}_B(A\#H) = 0$ as well, since $A\#H$ is supposed to be a projective left B-module. Recall that the radical $\mathrm{Rad}(M)$ of a module M is the intersection of all maximal submodules of M or equivalently the sum of all small submodules, i.e. of those submodules N of M such that $N + L \neq M$ for all $L \neq M$.

Since $A\#H$ is a finite extension of A, also $A\#H$ satisfies a polynomial identity and since H^* is finite dimensional it acts finitely on $A\#H$. Thus 3.4 applies and for any nil ideal I of $A\#H$ we have $B \cdot I \subseteq \mathrm{Jac}(A\#H)$. On the other hand any B-submodule N of $\mathrm{Jac}(A\#H)$ is contained in $\mathrm{Rad}_B(A\#H)$, which is zero. Hence $I = 0$ and $A\#H$ is semiprime.

In general, if A is semiprime, we can extend the H-action of A to the polynomial ring $A[x]$ by identifying $A[x]$ with $A \otimes_{H_t} H_t[x]$, which is a left H-module algebra, where H acts on x by $h \cdot x = (\epsilon_t(h) \cdot 1_A)x$. Since A is semiprime, satisfying a polynomial identity, $\mathrm{Jac}(A[x]) = 0$ by [2]. Moreover $A[x]$ also satisfies a polynomial identity and by the argument above $A[x]\#H$ is semiprime. As any ideal I of $A\#H$ can be extended to an ideal $I[x]$ of $A\#H[x] = A[x]\#H$, also $A\#H$ is semiprime. □

4. Kaplansky's theorem and the Brown-McCoy radical

As seen in the last section, the Jacobson radical of the polynomial ring $A[x]$ has been shown very useful to prove the desired result. The Brown-McCoy radical of $A[x]$, which includes the Jacobson radical, had been described by E. Puczylowski and A. Smoktunowicz in [25]. In this section we will give a weak Hopf theoretical analog of their description by considering the intersection of maximal H-stable ideals. This also leads to consider compressible and strongly prime module algebras. Eventually in analogy of Rowen's theorem on semiprime PI-algebras we show the existence of non-zero central H-invariant elements in non-trivial H-stable ideals for certain H-module algebras that satisfy a polynomial identity. Throughout this section we will consider intermediate algebras $M(A) \subseteq B \subseteq \mathrm{End}_k(A)$. Recall that $A^B = \{(1)f \mid f \in \mathrm{End}_B(A)\} \simeq \mathrm{End}_B(A)$. We have $A^B = \{a \in A \mid \forall \psi \in B : \langle \psi, a\rangle = \langle \psi, 1\rangle a\}$. Since $M(A) \subseteq B$, $A^B \subseteq Z(A)$. The B-stable ideals of A are precisely the (left) B-submodules of A. We say that A^B is large if it intersects all non-zero B-stable ideals non-trivially. If I is a B-stable ideal, then B also acts on A/I. In particular $(A/I)^B \subseteq Z(A/I)$. Be aware that in general $(A^B + I)/I \subseteq (A/I)^B$ might be a proper inclusion.

4.1. Kaplansky's theorem. Kaplansky's theorem says that any primitive algebra satisfying a polynomial identity is a matrix ring over a division ring which is finite dimensional over its center.

Proposition. *Let $M(A) \subseteq B \subseteq \mathrm{End}_k(A)$ such that B is finitely generated as a right module over $M(A)$. Suppose that A satisfies a polynomial identity. Then there exists a number d such that for all maximal B-stable ideals P of A, there exists an embedding $A/P \hookrightarrow M_n((A/P)^B)$ where $n \leq d/2$.*

Proof. Since $A \otimes A^{op}$ satisfies a polynomial identity, also its factor $M(A)$ satisfies a polynomial identity. As a finite extension, also B satisfies a polynomial identity of some minimal degree d. Let P be a maximal B-stable ideal of A and $B' := B/\mathrm{Ann}_B(A/P)$. Then A/P is a simple faithful left B'-module with $\mathrm{End}_{B'}(A/P) \simeq (A/P)^B$. By Kaplansky's Theorem [20, 13.3.8] $B' \simeq M_n((A/P)^B)$ with $n \leq d/2$. Since $A/P \subseteq B'$ as rings the statement follows. □

In particular if A is B-simple and B is finitely generated over $M(A)$, then A is finite dimensional over A^B if (and only if) A satisfies a polynomial identity.

4.2. Subdirect product of B-simple algebras. As a consequence we will get that if the intersection of maximal B-stable ideals in A is zero, then A^B is large. First we will recall some terminology. A polynomial $f(x_1, \ldots, x_m)$ in non-commuting indeterminates $x_1, \ldots, x_m$ and integer coefficients is called a *central polynomial* for an algebra A if all evaluations are central, i.e. for all $a_1, \ldots, a_m \in A : f(a_1, \ldots, a_m) \in Z(A)$ and furthermore f is non-trivial, i.e. $f(x_1, \ldots, x_m) \neq z + g(x_1, \ldots, x_m)$ with $z \in \mathbb{Z}$ and $g(x_1, \ldots, x_m)$ a polynomial identity for A. The *Capelli polynomials* are the polynomials

$$c_{2t-1}(x_1, \ldots, x_t, y_1, \ldots, y_{t-1}) = \sum_{\sigma \in S_t} (\mathrm{sgn}(\sigma)) x_{\sigma(1)} y_1 x_{\sigma(2)} y_2 \cdots x_{\sigma(t-1)} y_{t-1} x_{\sigma(t)}$$

and

$$c_{2t} = (c_{2t-1}(x_1, \ldots, x_t, y_1, \ldots y_{t-1})) y_t$$

for $t \geq 2$ (see [20, 13.5.5]). There exists an operator ∇ on non-commutative polynomials, such that for any commutative algebra A, ∇c_{2n^2} is a central polynomial for $M_n(A)$ (see [20, 13.5.10]). The discovery of such central polynomials is usually attributed to Formanek and Razmyslov. Since for lower sizes $m \leq n$, $M_m(A)$ can be considered a subring of $M_n(A)$ with equal center, we also have that ∇c_{2n^2} is a central polynomial for $M_m(A)$ for all $1 \leq m \leq n$.

4.3. Following Rowen's argument in [27] and using Proposition 4.1 we will show that subdirect products A of B-simple algebras have large ring of invariants A^B.

Theorem. *Let $M(A) \subseteq B \subseteq \mathrm{End}_k(A)$ with B being finitely generated as left $M(A)$-module. If A satisfies a polynomial identity and the intersection of maximal B-stable ideals is zero, then any non-zero B-stable ideal intersects A^B non-trivially.*

Proof. The proof follows Rowen's argument in [27]. Let $\{P_i\}_{i \in I}$ be a collection of maximal B-stable ideals of A whose intersection is equal to zero. Denote by $\pi_i : A \to A/P_i$ the canonical projections. For any non-zero B-stable ideal K of A, denote by I' the set of indices with $\pi_i(K) \neq 0$. By Lemma 4.1, all factors A/P_i can be embedded into a matrix ring $M_{n_i}(C_i)$ whose center is $C_i = (A/P_i)^B$. Choose an index $j \in I'$ such that $n = n_j$ is maximal (but less than $d/2$ from the Lemma). As mentioned in the last paragraph, there exists a central polynomial $f(x_1, \ldots, x_m)$ for $M_{n_j}(C_j)$ and also for all matrix algebras $M_{n_i}(C_i)$, with $i \in I'$. Since f is not-trivial, there exist elements $a_1, \ldots, a_m \in A/P_j$ such that $0 \neq f(a_1, \ldots, a_m) \in C_j = (A/P_j)^B$. Since $\pi_j(K) = A/P_j$ there exist elements $k_1, \ldots, k_m \in K$ such that

$\pi_j(k_r) = a_r$ for all $1 \leq r \leq m$. Since f is a central polynomial for each A/P_i, we also have $\pi_i(f(k_1,\ldots,k_m)) \in (A/P_i)^B$ for all i. For all $\psi \in B$ and $i \in I$ we have

$$\pi_i\left(\langle \psi - L(\psi\cdot 1), f(k_1,\ldots,k_m)\rangle\right) = \langle \psi - L(\psi\cdot 1), \pi_i(f(k_1,\ldots,k_m))\rangle = 0,$$

as π_i is B-linear. So $\langle \psi - L(\psi\cdot 1), f(k_1,\ldots,k_m)\rangle$ belongs to the intersection of all kernels of the π_i's which is zero. This show $f(k_1,\ldots,k_m) \in K \cap A^B$. □

4.4. A Brown-McCoy radical. Denote by $\mathrm{BMc}(A)$ the intersection of all maximal B-stable ideals. We saw that if A is a PI-algebra with $\mathrm{BMc}(A) = 0$, then A^B is large. A Theorem by Rowen says that a PI-algebra whose prime radical is zero has large center and we will attempt to generalize his result. Note that since the B-submodules of A are precisely the B-stable two-sided ideals, we have $\mathrm{BMc}(A) = \mathrm{Rad}({}_BA)$, where $\mathrm{Rad}({}_BA)$ is the (module theoretic) radical of A as left B-module. When considering semiprime PI algebras A, one typically reduces to study the polynomial ring $A[x]$ which is semiprimitive since $\mathrm{Jac}(A[x]) = N[x]$ for some nil ideal N of A which will be zero in this case. Puczylowski and Smoktunowicz showed that the Brown-McCoy radical $\mathrm{BM}(A[x])$ equals $S(A)[x]$ where $S(A)$ is the intersection of all prime factors with large center. Following their ideas we will give a similar characterization for the $\mathrm{BMc}(A[x])$.

4.5. Compressible algebras. The algebra A is called *B-prime* if $IJ \neq 0$ for all B-stable non-zero ideals I, J of A. Note that if A is B-prime, then all non-zero elements of A^B are central non-zero divisors of A, because if $0 \neq x \in A^B$ and $xy = 0$ for some $y \in A$, then $\langle\psi, x\rangle\langle\phi, y\rangle = \langle\psi, 1\rangle\langle\phi, xy\rangle = 0$ for all $\psi, \phi \in B$. Note that for all $y \in A, x \in A^B$ and $\psi \in B$ we have $\langle\psi, yx\rangle = \langle\psi\circ L(y), x\rangle = \langle\psi\circ L(y), 1\rangle x = \langle\psi, y\rangle x$. Analogously we have $\langle\psi, xy\rangle = x\langle\psi, y\rangle$. Thus $(B\cdot x)(B\cdot y) = 0$ and hence $y = 0$ as A was B-prime. In particular A^B is an integral domain. A is called *B-compressible* if A can be embedded as left B-module into each non-zero B-stable ideal I of A.

Lemma. *A is B-compressible if and only if A is B-prime and A^B is large.*

Proof. If A is B-compressible and $IJ = 0$ for some B-stable ideals I and J of A. If $J \neq 0$, then there exists a B-linear embedding $f : A \to J$. Hence $(I)f = I(1)f \subseteq IJ = 0$ and as f was injective, $I = 0$. On the other hand $0 \neq (1)f \in J \cap A^B$. Conversely if A is B-prime, then A^B is an integral domain. For any B-stable non-zero ideal I of A and $0 \neq x \in I \cap A^B$, if $yx = 0$ for $y \in A$, then $\langle\psi, y\rangle\langle\psi', x\rangle = \langle\psi, yx\rangle\langle\psi', 1\rangle = 0$, i.e. $(B\cdot y)(B\cdot x) = 0$. Hence $y = 0$ and the map $a \mapsto ax$ is an injective B-linear map. □

4.6. We define $S(A) = \bigcap\{P \subseteq A \mid P \text{ is } B\text{-stable and } A/P \text{ is } B\text{-compressible}\}$. Considering $A[x]$ as a free left A-module of countable rank, the action of B extends to $A[x]$ by $\langle\psi, ax^n\rangle := \langle\psi, a\rangle x^n$ for all monomial ax^n and $\psi \in B$. Our aim is to show that $\mathrm{BMc}(A[x]) = S(A)[x]$ as in [25].

Lemma. $\mathrm{BMc}(A) \cap A^B \subseteq \mathrm{Jac}(A^B)$.

Proof. Note that $\mathrm{BMc}(A) = \mathrm{Rad}({}_BA)$ since the B-stable ideals are precisely the B-submodules of A. If $x \in \mathrm{Rad}({}_BA) \cap A^B$, then $B\cdot(1-x) = A$ since $B\cdot x$ is a small B-submodule of A. Hence there exists $\psi \in B$ with $1 = \langle\psi, 1-x\rangle = \langle\psi, 1\rangle(1-x)$. Set $y = \langle\psi, 1\rangle$. For all $\phi \in B$ we have

$$\langle\phi, y\rangle = \langle\phi, y\rangle(1-x)y = \langle\phi, y(1-x)\rangle y = \langle\phi, 1\rangle y.$$

Thus $y \in A^B$ is a quasi-inverse of x. The same argument holds for all elements $1 - rx$ with $r \in A^B$ and shows that $x \in \mathrm{Jac}(A^B)$. □

4.7. We extended the B-action to $A[x]$, but the B-submodules are in general not ideals of $A[x]$. Let $\widehat{B}$ be the subalgebra of $\mathrm{End}_k(A[x])$ generated by $M(A[x])$ and B, then the B-stable ideals of $A[x]$ are precisely the $\widehat{B}$-submodules of $A[x]$. Furthermore $\mathrm{BMc}(A[x]) = \mathrm{Rad}({}_{\widehat{B}}A[x])$. Note also that $A[x]^{\widehat{B}} = A^B[x]$. Hence by Lemma 4.6 we have

$$\mathrm{BMc}(A[x]) \cap A^B[x] = \mathrm{Rad}({}_{\widehat{B}}A[x]) \cap A[x]^{\widehat{B}} \subseteq \mathrm{Jac}(A[x]^{\widehat{B}}) = \mathrm{Jac}(A^B[x]) = N[x] \quad (11)$$

for some nil ideal N of A^B.

Lemma. *If A is B-compressible, then* $\mathrm{BMc}(A[x]) = 0$.

Proof. Suppose that A is B-compressible, then A^B is an integral domain and hence has no non-zero nil ideals. Thus $\mathrm{BMc}(A[x]) \cap A^B[x] = 0$ by equation (11). If $\mathrm{BMc}(A[x]) \neq 0$ we may choose a non-zero polynomial $0 \neq f(x) = \sum_{i=1}^n a_i x^i \in \mathrm{BMc}(A[x])$ of minimal degree. Let I be the ideal in A generated by all leading coefficients of polynomials in $\mathrm{BMc}(A[x])$ of degree n. Since $\mathrm{BMc}(A[x])$ is B-stable and since B-acts homogeneously, I is also B-stable. Since A is B-compressible, $I \cap A^B$ is non-zero and we may assume that the leading coefficient of $f(x)$ is in $I \cap A^B$. Since for all $\psi \in B : \langle \psi, f(x) \rangle - \langle \psi, 1 \rangle f(x) \in \mathrm{BMc}(A[x])$ has degree less than $f(x)$ and since $f(x)$ had minimal degree in $\mathrm{BMc}(A[x])$ we have $\langle \psi, f(x) \rangle = \langle \psi, 1 \rangle f(x)$. Hence $f(x) \in A^B[x] \cap \mathrm{BMc}(A[x]) = 0$ a contradiction. $\square$

As a consequence we have $\mathrm{BMc}(A[x]) \subseteq S(A)[x]$, since we showed that any B-stable ideal P with A/P being B-compressible is the intersection of maximal B-stable ideals of $A[x]$.

4.8. In order to show the reversed inclusion we closely follow Puczylowski and Smoktunowicz's proof. Let $f(x) = \sum_{i=m}^n a_i x^i \in A[x]$ with $a_m \neq 0 \neq a_n$. Set $deg(f) = n$ for the degree of f and $min(f) = m$. The coefficient of $x^{min(f)}$ is denoted by $\mathrm{ls}(f) = a_m$. The length of $f(x)$ is defined as $len(f) = deg(f) - min(f) + 1$. The following operation had been introduced in [25] for $w(x) \in A[x]$ with $min(w(x)) \geq 1$ and any $f(x) \in A[x]$:

$$f^*(x) := f(x) - (1 - w(x))\mathrm{ls}(f)x^{min(f)}.$$

Then the following statements hold:

(1) $min(f^*) \geq min(f) + 1$;
(2) $deg(f^*) \leq \max\{deg(f), min(f) + deg(w)\}$;
(3) $len(f^*) \leq \max\{len(f) - 1, deg(w)\}$

In particular there exists a number d such that applying the $*$-operation d times yields a polynomial $p(x) \in A[x]$ with

$$len(p) \leq deg(w) \leq min(p).$$

Moreover if $f(x) \in I[x]$, then also $f^*(x) \in I[x]$ and if M is an ideal of $A[x]$ and $1 - w(x) \in M$, then $1 - f(x) \in M$ implies $1 - f^*(x) \in M$. Thus we can conclude

Lemma. *Let M be an ideal of $A[x]$ and $w(x) \in A[x]$ with $min(x) \geq 1$ and $1 - w(x) \in M$. If I is a proper ideal of A such that $xI[x] + M = A[x]$, then there exists a polynomial $p(x) \in xI[x]$ such that $1 - p(x) \in M$ and $len(p) \leq deg(w) \leq min(p)$.*

4.9. We need some condition on the B-stableness of powers of B-stable ideals that in our application will be satisfied.

Lemma. *Suppose that for any B-stable ideal I of A all powers I^m are B-stable. Let M be a maximal B-stable ideal of $A[x]$ and $P = A \cap M$. Then P is B-stable and A/P is B-compressible.*

Proof. We may assume $P = 0$, i.e. $A \cap M = 0$. Our aim is to show that A is B-compressible. Certainly A is B-prime since for all non-trivial B-stable ideals I, J of A, $I[x] + M = A[x]$ and $J[x] = JI[x] + M$ hold. Hence $JI \neq 0$ - otherwise $J \subseteq M \cap A = 0$. Let J be a non-zero proper B-stable ideal of A. We need to show the existence of a non-trivial B-invariant element in J. Note that $xJ[x]+M = A[x]$ since otherwise $xJ[x] \subseteq M$ and $J[x] + M = A[M]$ implies $J = A$. So let $w(x) = \sum_{i=1}^n a_i x^i \in xJ[x]$ be a non-zero polynomial of minimal length with $1 - w(x) \in M$. We claim that $w(x) \in A^B[x]$. If not, then let k be the maximal number such that $a_k \notin A^B$. Hence there exist $\psi^* \in B$ with $c := \langle \psi^*, a_k\rangle - \langle \psi^*, 1\rangle a_k \neq 0$. Let $I = B \cdot c$ be the B-stable ideal in A generated by c. Note that for any $m \geq 1$, I^m is B-stable by hypothesis and $xI^m[x] + M = A[x]$ since otherwise I^m is improper. By Lemma 4.8 there exist polynomials $p_m(x) \in xI^m[x]$ such that $1 - p_m(x) \in M$ and $1 - p(x) \in M$ and $len(p_m) \leq deg(w) \leq min(p_m)$ for all m. For each m we suppose to have chosen a polynomial $p_m(x)$ of minimal length. Since $w(x)$ had minimal length in $xJ[x]$ among all those with $1 - w(x) \in M$ and since $I^m \subseteq J$, we have also $len(w) \leq len(p_m)$. By showing that actually

$$len(w) \leq len(p_1) < len(p_2) < \cdots < len(p_{m-1}) < len(p_m) < \cdots \leq deg(w) \qquad (12)$$

for all m, we will reach a contradiction since $len(p_m)$ is bound by $deg(w)$. So let $m \geq 2$ and let a be the leading coefficient of $p_m(x)$ which belongs to I^m. Since $I^m = I^{m-1}(B \cdot c)$ there exist elements $u_j \in I^{m-1}$ and $\psi_j \in B$ such that $a = \sum_{j=1}^s u_j \langle \psi_j, c\rangle$. Denote by $g(x)$ the polynomial $g(x) = \langle \psi^*, w(x)\rangle - \langle \psi^*, 1\rangle w(x)$ whose leading coefficient is c and whose degree is $deg(g) = k$. Note that since $1 - w(x) \in M$ we have

$$\begin{aligned} g(x) &= \langle \psi^*, w(x)\rangle - \langle \psi^*, 1\rangle + \langle \psi^*, 1\rangle - \langle \psi^*, 1\rangle w(x) \\ &= \langle \psi^*, 1\rangle(1 - w(x)) - \langle \psi^*, 1 - w(x)\rangle \in M. \end{aligned}$$

Now define $h(x) = \sum_{j=1}^s u_j \langle \psi_j, g(x)\rangle$ which has degree k and leading coefficient a. Thus the polynomial $q(x) = p_m(x) - h(x)x^{deg(p_m)-k}$ has degree less than $deg(p_m)$ and since $len(p_m) \geq len(w) \geq len(g) \geq len(h)$, we have $len(p_m) > len(q)$. On the other hand $q \in xI^{m-1}[x]$ and $1 - q(x) \in M$. Thus by the minimality of the length of p_{m-1} we get $len(p_{m-1}) \leq len(q)$, which proves the chain (12). □

4.10. Summarizing the Lemmas 4.7 and 4.9 we can now state an extension of Puczylowski and Smoktunowicz's description of the Brown-McCoy radical of $A[x]$ under some operator action.

Theorem. *Let $M(A) \subseteq B \subseteq \mathrm{End}_k(A)$ such that powers of B-stable ideals of A are B-stable and denote by $S(A)$ the intersection of B-stable ideals I whose factors A/I are B-compressible. Then* $\mathrm{BMc}(A[x]) = S(A)[x]$.

The description of the Brown-McCoy radical and Theorem 4.3 allows us to strengthen 4.3.

Corollary. *Let $M(A) \subseteq B \subseteq \mathrm{End}_k(A)$ such that B is finitely generated over $M(A)$. Suppose that A satisfies a polynomial identity and that A is a subdirect product of B-compressible algebras. Then A^B is large in A.*

4.11. The central closure of a module algebra. Before applying the obtained results we like to look at another related concept. A general central closure construction for an algebra A and some intermediate algebra $M(A) \subseteq B \subseteq \mathrm{End}_k(A)$ such that A does not contain any non-zero nilpotent B-stable ideal, was given in [18], namely by defining a ring structure on the self-injective hull $\widehat{A}$ of A as left B-module. To be more precise, there exists a left B-linear map

$$\psi : A \otimes_{A^B} \mathrm{End}_B(\widehat{A}) \to \widehat{A} \ \text{ with } \ (a \otimes t)\psi := (a)t$$

with $A^B = \{(1)f \mid f \in \mathrm{End}_B(A)\}$, whose kernel is an ideal in the ordinary tensor product of A and $\mathrm{End}_B(\widehat{A})$ (see [18, Theorem 3.2]). The multiplication on $\widehat{A}$ is then defined as

$$(a)s \cdot (b)t = (ab)st \ \forall a, b \in A, s, t \in \mathrm{End}_B(\widehat{A}).$$

There exists also an injective ring homomorphism $\mathrm{End}_B(\widehat{A}) \to \widehat{A}\ t \mapsto (1)t$ whose image is denoted by $\widehat{A}^B$. Define $\widehat{B} = \langle M(\widehat{A}), B\rangle$ to be the subalgebra of $\mathrm{End}_k(\widehat{A})$ generated by $M(\widehat{A})$ and the B-action. Then $\mathrm{End}_{\widehat{B}}(\widehat{A}) \simeq \widehat{A}^B$.

4.12. Strongly B-prime algebras. Those algebras whose central closure is simple are called strongly prime.To have an analog for module algebras we first recall that for any left R-module M over some ring R, the smallest Grothendieck category in R-Mod containing M is called the *Wisbauer category* $\sigma[M]$, whose objects are all submodules of quotients of direct sums of copies of M. A module M is called *strongly prime* if for every submodule $N \subseteq M$, $\sigma[M] = \sigma[N]$. Strongly prime left R-modules were defined in [13] and form the elements of Rosenberg's "left spectrum" of a ring (see [26]). Let $M(A) \subseteq B \subseteq \mathrm{End}_k(A)$ be as above. Then we call A strongly B-prime if A is a strongly prime left B-module. For $B = M(A)$ this notion reduces to the ordinary notion of a strongly prime algebra. Since the self-injective hull of a strongly prime module has no fully invariant submodules ([30, 13.3]) and since every ideal of $\widehat{A}$ is fully invariant as $\mathrm{End}_B(\widehat{A}) \simeq \widehat{A}^B \subseteq Z(\widehat{A})$, we get that the central closure $\widehat{A}$ of a strongly B-prime algebra A is $\widehat{B}$-simple and $\widehat{A}^B$ is a field. With this and with Lemma 4.1 we get:

Corollary. *Let $M(A) \subseteq B \subseteq \mathrm{End}_k(A)$ with B being finitely generated as $M(A)$-module. If A is a strongly B-prime algebra satisfying a polynomial identity, then its central closure $\widehat{A}$ is a finite dimensional over $\widehat{A}^B$. Moreover A embeds into a a matrix ring $M_n(\widehat{A}^B)$.*

5. Applications to measurings, gradings and involutions

In this section we will apply the obtained results to (weak) Hopf actions or more generally to measurings of left bialgebroids.

5.1. Bialgebroids. Hopf algebroids generalize the notion of (weak) Hopf algebras, but instead have two bialgebroid structure. Recall from [5] or [8] that a k-vector space C is an R-coring (for a k-algebra R) if there exists some R-bilinear maps $\Delta : C \to C \otimes_R C$ and $\epsilon : C \to R$ satisfying the coassociativity and counit condition. An associative unital k-algebra A is an R-ring, with R being an associative unital k-algebra if there exists a homomorphism of k-algebras $\eta : R \to A$. A left character of an R-ring A is a k-linear map $\chi : A \to R$ such that for all $a, a' \in A, r \in R$:

$$\chi(1_A) = 1_R, \qquad \chi(aa') = \chi(a\eta(\chi(a'))), \qquad \chi(\eta(r)a) = r\chi(a).$$

Right character being defined analogously. An $R \otimes R^{op}$-ring A can be described by a pair of k-algebra maps $s : R \to A$ and $t : R^{op} \to A$ with commuting range.

Definition. *Let L be a k-algebra. A left bialgebroid H over L is a k-algebra with an $L \otimes L^{op}$-ring structure (H, s, t) and an L-coring structure (H, Δ, ϵ) subject to the following compatibility axioms.*

(i) *The bimodule structure in the L-coring (H, Δ, ϵ) is related to the s and t via*
$$l \cdot h \cdot l' := s(l)t(l')h, \quad \text{for all } l, l' \in L,\ h \in H. \tag{13}$$
(ii) *Considering H as an L-bimodule as in (13), the coproduct Δ restricts to a k-algebra map from H to*

$$H \times_L H := \left\{ \sum_i h_i \otimes_L h'_i \ : \ \sum_i h_i t(l) \otimes_L h'_i = \sum_i h_i \otimes_L h'_i s(l) \quad \forall l \in L \right\},$$

where $H \times_L H$ is an algebra via factorwise multiplication.

(iii) *The counit ϵ is a left character on the L-ring (H,s).*

A *bialgebra* H over k is of course an example of left bialgebroid over k. A weak Hopf algebra H has a left bialgebroid structure over $L = H_t$ with $\epsilon = \epsilon_t$.

5.2. Following [7] we say that a bialgebroid H over L *measures* a left L-ring A with unit map $\iota : L \to A$ if there exits a k-linear map, called a *measuring*, $H \otimes_k A \to A, h \otimes a \mapsto h \cdot a$ such that for all $h \in H, l \in L, a, b \in A$,

(i) $h \cdot 1_A = \iota(\epsilon(h))$;

(ii) $(t(l)h) \cdot a = (h \cdot a)\iota(l)$ and $(s(l)h) \cdot a = \iota(l)(h \cdot a)$;

(iii) $h \cdot (ab) = (h_1 \cdot a)(h_2 \cdot b)$;

Weak Hopf actions are examples of left bialgebroid measurings. Note that an H-module algebra A over a weak Hopf algebra H is an H_t-ring, with $\iota(z) = z \cdot 1_A$ for $z \in H_t$.

5.3. Given a measuring of a left bialgebroid H on A we consider the subalgebra B of $\mathrm{End}_k(A)$ generated by $M(A)$ and $\lambda(H)$ where $\lambda(h)$ denotes the action of h on A. The invariants of such an H-action, are defined as

$$A^H = \{a \in A \mid h \cdot a = \iota(\epsilon(h))a \ \ \forall h \in H\}.$$

Then $Z(A)^H := Z(A) \cap A^H = A^B$ and $Z(A)^H$ being large means that any non-zero H-stable ideal contains a non-zero central H-invariant element. We say that H acts finitely if $\lambda(H)$ is finite dimensional over k. Note that by property (iii) we also have some intertwining relations between $L(a)$ resp. $R(b)$ and $\lambda(h)$ namely:

$$\lambda(h) \circ L(a) = L(h_1 \cdot a) \circ \lambda(h_2) \ \text{ and } \ \lambda(h) \circ R(b) = R(h_2 \cdot b) \circ \lambda(h_1).$$

Hence if H acts finitely on A, then B is finitely generated as right $M(A)$-module. Instead of B-simple, B-prime, B-compressible, etc. we will write H-simple, H-prime, H-compressible. We call an H-stable ideal I of A H-prime (strongly H-prime resp. H-compressible) if A/I is H-prime (strongly H-prime, resp. H-compressible). Summarizing we have the following results on Hopf algebroid actions on A:

Theorem. *Let H be a left bialgebroid measuring A. Then*

(1) *A is H-compressible if and only if A is H-prime and $Z(A)^H$ is large in A.*

(2) *The intersection of maximal H-stable ideals of $A[x]$ is given by $S(A)[x]$ where $S(A)$ is the intersection of H-compressible ideals.*

(3) *Suppose that A satisfies a polynomial identity and that H acts finitely on A, then*

 (a) *If A is H-simple, then it embeds into a matrix ring $M_n(Z(A)^H)$.*

 (b) *If A is strongly H-prime, then $\widehat{A}$ is finite dimensional over $Z(\widehat{A})^H$.*

 (c) *If A is a subdirect product of H-compressible module algebras, then $Z(A)^H$ is large in A.*

5.4. Gradings. If M is a monoid and A is an M-graded k-algebra, then $H = k[M]$ is a bialgebra and there are projections $\pi_g : A \to A_g$ onto the g-components for all $g \in M$. Those projections define linear operators. Let B be the subalgebra of $\mathrm{End}_k(A)$ generated by $M(A)$ and all projections π_g with $g \in M$. The B-stable ideals are precisely the graded ideals of A. The polynomial ring $A[x]$ is naturally graded with g-components being $A_g[x]$. Note that we also have some intertwining of $L(a)$ and π_g, namely if a_h is a homogeneous element of A_h, $h \in M$, then $L(a_h) \circ \pi_g = \pi_{hg} \circ L(a_h)$ and $R(a_h) \circ \pi_g = \pi_{gh} \circ R(a_h)$. Hence if M is finite, then B is finitely generated as $M(A)$-module. The subring of invariants A^B is the center of the e-component $Z(A_e)$, where e is the neutral element of M. Applying the results of the last section we get similar statements as in the last theorem. For instance that the intersection of maximal graeded ideals of $A[x]$ is given by $S(A)[x]$ where $S(A)$ is the intersection of those graded ideals P of A with A/P being graded prime and having large subring $Z(A/P_e)$. And if A satisfies a polynomial identity, M is finite and A is strongly graded prime, then its graded central closure $\widehat{A}$ is finite dimensional over $Z(\widehat{A}_e)$.

5.5. Involutions. Our approach also yields corresponding results for algebras A with involutions $*$, by considering the subalgebra B generated by $L(A)$ and $*$, that automatically will contain also all right multiplications $R(a) = * \circ L(a^*) \circ *$. The $*$-ideals are the left ideals I of A with $I^* \subseteq I$. The subring of invariants A^B is the subring of central symmetric elements, i.e $a \in Z(A)$ with $a^* = a$. We can rephrase the above results in the $*$-setting. For instance, that the intersection of maximal $*$-ideals of $A[x]$ is given by the intersection of $*$-ideals P of A with A/P being $*$-prime and having a large subring of central symmetric elements.

6. One-sided irreducible module algebra

The left uniform dimension of A is denoted by $\mathrm{udim}(A)$ and is the supremum of the cardinalities of the index sets of direct sums of left ideals contained in A.

6.1. A result [4, Theorem 2.2] by Bergen, Cohen and Fishman says that if a Hopf algebra H acts finitely on a module algebra A with finite uniform dimension, such that A is a simple $A\#H$-module, then A has finite dimension over A^H.

Lemma. *Suppose $L(A) \subseteq B \subseteq \mathrm{End}_k(A)$ such that A is a simple left B-module. If ${}_AB$ is generated by n elements then $[A : A^B] \leq n \cdot \mathrm{udim}(A)$.*

Proof. Since A is a simple left B-module, $\mathrm{End}_B(A)$ and hence A^B is a division ring by Schur's Lemma. Let $x_1, \ldots, x_k$ be linearly independent elements of A_{A^B}. Then by the Jacobson Density theorem there exist elements $\psi_1, \ldots, \psi_k$ such that $\psi_j \cdot x_i = \delta_{ij} 1$. Since B is n-generated over A, there exist an epimorphism of left A-modules $\varphi : A^n \to B$ therefore elements $t_i \in A^n$ such that $(t_i)\varphi = \psi_i$ for all i. If $\sum_{i=1}^k a_i t_i = 0$ for some $a_i \in A$, then

$$0 = \left(\sum_{i=1}^n a_i t_i \right) \varphi \cdot x_j = \sum_{i=1}^n a_i (\psi_i \cdot x_j) = L(a_j)$$

for all j. Thus $\bigoplus_{i=1}^k A t_i$ is a direct sum in A^n and $k \leq \mathrm{udim}(L(A)^n) = n \cdot \mathrm{udim}(A)$. Hence $[A : A^B] \leq n \cdot \mathrm{udim}(A)$. □

6.2. Before we apply this result to weak Hopf actions, we first generalize another result from the theory of Hopf actions to a more general setting. Let A be an R-ring with k-algebra homomorphism $\eta : R \to A$. The existence of a left character $\chi : A \to R$ is equivalent to R being a left A-module, namely through the following action $a \cdot r := \chi(a\eta(r))$ for all $a \in A, r \in R$. It also allows to define invariants of left A-modules M as

$$M_\chi = \{ m \in M \mid am = \chi(a)m \, \forall a \in A \} \simeq \mathrm{Hom}_A(R, M).$$

In particular $\mathrm{End}_A(R) \simeq R_\chi$ can be considered a subalgebra of R. A is called a *Galois* R-ring if the homomorphism $A \to \mathrm{End}(R_{R_\chi}), a \mapsto \chi(a\eta(-))$ that defines the A-module structure on R is an isomorphism.

6.3. The smash product $A\#H$ of a module algebra A and a weak Hopf algebra H is an A-ring with left character χ given by $a\#h \mapsto a\epsilon_t(h) \cdot 1$. So the next result will automatically apply to this case.

Proposition. *Let A be an R-ring with left character χ such that A is finitely generated as left and right R-module . Suppose R has finite left uniform dimension, R is simple as left A-module. The following statements are equivalent:*

(a) *A is a simple ring;*
(b) *A is a Galois R-ring and R_{R_χ} is finitely generated projective;*
(c) *R is a faithful A-module and $A_\chi \neq 0$;*

If A_R is free of rank n, then $(a-c)$ are also equivalent to:

(d) *$dim(R_{R_\chi}) = n$ and $A_\chi \neq 0$;*

Proof. To put us in the situation where we can apply Lemma 6.1 we consider the subalgebra B of $\mathrm{End}_k(R)$ generated by all operators $\lambda(a) := [r \mapsto r \cdot a = \chi(a\eta(r))]$. The left B-module structure of R is the same as the left A-module structure. Hence R is a simple (faithful) left B-module. Note that $R^B \simeq \mathrm{End}_B(R) = R_\chi$ is a division algebra. Furthermore, B is finitely generated as left $L(R)$-module since A is finitely generated as left R-module and $R \simeq L(R) \subseteq B$. By Lemma 6.1 $dim(R_{R_\chi})$ is finite. Since A_R is finitely generated, say by R^k, we have $k \cdot dim(R_{R_\chi}) \geq dim(A_{R_\chi})$. Hence A is finite dimensional over R_χ.

$(a) \Rightarrow (b)$ Suppose A is simple, then it is a matrix ring over a division ring, hence χ splits as left A-module by some map $\phi : R \to A$. Since A is simple, the trace ideal $(R)\phi A$ equals A, i.e. R is a generator in A-Mod. Hence by a standard module theoretic argument $A \simeq \mathrm{End}(R_{R_\chi})$ and R_{R_χ} is finitely generated projective (see [30, 5.5]).

$(b) \Rightarrow (c)$ is clear since R is a generator. Recall that $A_\chi \simeq \mathrm{Hom}_A(R, A)$.

$(c) \Rightarrow (a)$ $A_\chi \neq 0$ implies that R is isomorphic to a left ideal of A (since R is a simple A-module). Hence also B contains a copy of R as a left ideal and is therefore a left primitive ring having a faithful minimal left ideal, i.e. $B \simeq M_m(R_\chi)$ with $m = dim(R_{R_\chi})$. If R is a faithful left A-module, then $A \simeq B$ is simple.

If A_R is free of rank n, then $n \cdot dim(R_{R_\chi}) = dim(A_{R_\chi})$ holds. $(c) \Leftrightarrow (d)$ As seen in the last step $A_\chi \neq 0$ implies $B \simeq M_m(R_\chi)$ with $m = dim(R_{R_\chi})$. On the other hand $n \cdot m = dim(A_{R_\chi}) \geq dim(B_{R_\chi}) = m^2$ holds. Hence $n = m$ if and only if $A \simeq B$ if and only if R is a faithful left A-module.

□

6.4. Considering $A\#H$ as an A-ring with left character and by considering the subalgebra of $\mathrm{End}_k(A)$ generated by the H-action and $L(A)$ we can apply 6.1 and 6.3 to obtain the weak Hopf analogs of results by Bergen et al. [4, Theorem 2.2] and Cohen et al. [11, Theorem 3.3].

Corollary. *Let A be a left H-module algebra over a finite dimensional weak Hopf algebra H. If A has finite left uniform dimension and is a simple left $A\#H$-module, then it is finite dimensional over A^H and satisfies $[A : A^H] \leq \dim(H) \cdot \mathrm{udim}(A)$. Moreover the following statements are equivalent:*

(a) *$A\#H$ is simple;*
(b) *$A^H \subseteq A$ is an H^*-Galois extension;*
(c) *A is a faithful left $A\#H$-module;*

If $A\#H_A$ is free of rank $dim(H)$, then $(a-c)$ are also equivalent to:

(d) *$dim(A_{A^H}) = dim(H)$.*

6.5. Concluding remarks. The associativity of A is not needed to prove Lemma 6.1. Non-associative examples of module algebras are given by module algebras over quasi-Hopf algebras. Let H be a quasi-Hopf algebra, that is H is an associative algebra which is a not necessarily coassociative coalgebra satisfying some compatibility conditions (see [12]). A left H-module algebra A is a unital algebra in the category of left H-modules (see [9]). In particular its multiplication satisfies $(ab)c = \sum (x^1 \cdot a)[(x^2 \cdot b)(x^3 \cdot c)]$ where $\phi^{-1} = x^1 \otimes x^2 \otimes x^3$ is the inverse of the Drinfeld reassociator of H. By the proof of 6.1, substituting A by $L(A)$ we get the following

Corollary. *Let A be a left H-module algebra over a quasi-Hopf algebra H which acts finitely on it. If A has finite left uniform dimension and is a simple left $A\#H$-module, then it is finite dimensional over A^H.*

This applies in particular to finite quasi-Hopf action on non-associative division rings, which are now seen to be finite extensions of their (associative) subring of invariants. Quasi-Hopf actions on non-associative division rings were considered for example by Albuquerque and Majid in [1].

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