The Siegel modular forms of genus 2 with the simplest divisor

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Abstract

We prove that there exist exactly eight Siegel modular forms with respect to the congruence subgroups of Hecke type of the paramodular groups of genus two vanishing precisely along the diagonal of the Siegel upper half-plane. This is a solution of a problem formulated during the conference "*Black holes, Black Rings and Modular Forms*" (ENS, Paris, August 2007). These modular forms generalize the classical Igusa form and the forms constructed by Gritsenko and Nikulin in 1998.

Introduction: dd-modular forms

The first cusp form for the Siegel modular group $\mathrm{Sp}_2(\mathbb{Z})$ is the Igusa form Ψ_{10}. In fact $\Psi_{10} = \Delta_5^2$ where Δ_5 is the product of the ten even theta-constants (see [F]). This modular form has a lot of remarkable properties. One of the main features of Δ_5 is that it vanishes (with order one) precisely along the diagonal

$$\mathcal{H}_1 = \Big\{ \begin{pmatrix} \tau & 0 \\ 0 & \omega \end{pmatrix}, \ \tau, \omega \in \mathbb{H}_1 \Big\} \subset \mathbb{H}_2$$

of the Siegel upper half-plane $\mathbb{H}_2 = \{Z = \left(\begin{smallmatrix} \tau & z \\ z & \omega \end{smallmatrix}\right) \in M_2(\mathbb{C}), \, \mathrm{Im}(Z) > 0\}$. It is known that Δ_5 determines a Lorentzian Kac–Moody super Lie algebra of Borcherds type. See [GN1]–[GN2] where two lifting constructions of Δ_5 were proposed

$$\Delta_5(Z) = \mathrm{Lift}(\eta^9(\tau)\vartheta(\tau, z)) = \mathrm{B}(\phi_{0,1})(Z)$$

where η is the Dedekind eta-function and ϑ is the Jacobi theta-series (see (7)). This relation gives the two parts of the denominator identity for the Borcherds algebra determined by Δ_5. There exists a geometric interpretation of this identity in terms of the arithmetic mirror symmetry (see [GN4]). Moreover $2\phi_{0,1}$ is the elliptic genus of a K3 surface and Ψ_{10}^{-2} is related to the so-called second quantized elliptic genus of K3 surfaces (see [DMVV], [G4]). These facts explain the importance of Δ_5 in the theory of strings (see [DVV], [Ka]). During the conference "*Black holes, Black Rings and*

Modular Forms" (ENS, Paris, August 2007) there was formulated a problem on the existence of Siegel modular forms similar to Δ_5 with respect to the congruence subgroup of Hecke type

$$\Gamma_0^{(2)}(N) = \{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_2(\mathbb{Z}) \mid C \equiv 0 \mod N\}.$$

Such Siegel modular forms can characterize the black holes entropy and the degeneracy of dyons for some class of CHL string compactification (see [DG], [DN]). Mathematically we can reformulate this question as follows: *finding all Siegel modular forms F with respect to $\Gamma_0^{(2)}(N)$ (with a character or a multiplier system) such that F vanishes exactly along the $\Gamma_0^{(2)}(N)$-translates of the diagonal $\mathcal{H}_1 \subset \mathbb{H}_2$ and with vanishing order one.*

We call such functions **dd-modular forms**: modular forms with the diagonal divisor. A dd-modular form is a natural generalization of Δ_5. In this paper we give the complete answer to this problem.

Theorem 0.1 *For the congruence subgroups $\Gamma_0^{(2)}(N)$ with $N > 1$ there are exactly three dd-modular forms: ∇_3 of weight 3 for $\Gamma_0^{(2)}(2)$ with a character of order 2, ∇_2 of weight 2 for $\Gamma_0^{(2)}(3)$ with a character of order 2 and $\nabla_{3/2}$ of weight 3/2 for $\Gamma_0^{(2)}(4)$ with a multiplier system of order 4.*

In fact we get a result which is stronger than the theorem above. We give the full classification of the dd-modular forms for the Hecke subgroups $\Gamma_t(N)$ (see (1)) of the symplectic paramodular groups Γ_t. Theorem 1.4 claims that there are exactly *eight* such dd-modular forms. Four of them, Δ_5 and the modular forms Δ_2, Δ_1 and $\Delta_{1/2}$ of weights 2, 1 and 1/2 with respect to the paramodular groups Γ_2, Γ_3 and Γ_4 were constructed in [GN2]. The other four modular forms are the three functions of Theorem 0.1 and the dd-modular form Q_1 of weight 1 and character of order 4 with respect to the congruence subgroup $\Gamma_2(2)$ of the paramodular group Γ_2.

These eight remarkable modular forms can be considered as the best possible three dimensional analogues of the Dedekind η-function. We expect a number of interesting applications of these new functions in the string theory, in the theory of Lorentzian Kac–Moody algebras and in algebraic geometry.

The paper contains three sections. In §1 we prove that there might exist only nine dd-modular forms with respect to $\Gamma_t(N)$. In §2 using the lifting of Theorem 2.2 we construct seven dd-forms and the square of $\nabla_{3/2}$. Moreover using the particular form of Q_1 we prove that the ninth dd-form does not exist. In §3 using Theorem 3.1 about the Borcherds automorphic products for congruence subgroup $\Gamma_t(N)$ we construct the last dd-modular form $\nabla_{3/2}$ of weight 3/2.

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1 Classification of the dd-modular forms

One of the main idea of our approach (see [G1]–[G3]) is that in order to understand better the properties of Siegel modular forms of genus two one has to consider not only the modular group $\Gamma_1 = \mathrm{Sp}_2(\mathbb{Z})$ and its congruence subgroups, but the integral symplectic groups Γ_t, the paramodular groups, for all $t \geq 1$. In this section we give the complete classification of the dd-modular forms for the most natural congruence subgroups of the paramodular groups. Let t and N be positive integers. We put

$$\Gamma_t(N) = \left\{ \begin{pmatrix} * & *t & * & * \\ * & * & * & *t^{-1} \\ *N & *Nt & * & * \\ *Nt & *Nt & *t & * \end{pmatrix} \in \mathrm{Sp}_2(\mathbb{Q}), \quad \text{all } * \in \mathbb{Z} \right\}. \tag{1}$$

The group $\Gamma_t = \Gamma_t(1)$ is conjugated to the integral symplectic group of the integral skew-symmetric form with elementary divisors $(1, t)$ (see [G2], [GH2]). The quotient $\Gamma_t \setminus \mathbb{H}_2$ is the moduli space of the $(1, t)$-polarized Abelian surfaces. If $t = 1$ then $\Gamma_1 = \mathrm{Sp}_2(\mathbb{Z})$ and $\Gamma_1(N) = \Gamma_0^{(2)}(N)$ is the Hecke subgroup from the introduction.

Let $\Gamma_t(N)^+ = \Gamma_t(N) \cup \Gamma_t(N)V_t$ be a normal double extension of $\Gamma_t(N)$ in $\mathrm{Sp}_2(\mathbb{R})$ where

$$V_t = \frac{1}{\sqrt{t}} \begin{pmatrix} 0 & t & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & t & 0 \end{pmatrix} \in \mathrm{Sp}_2(\mathbb{R}). \tag{2}$$

Repeating the proof of Lemma 2.2 of [G2] we obtain

Lemma 1.1 *The group $\Gamma_t(N)^+$ is generated by V_t and by its parabolic subgroup*

$$\Gamma_t^{\infty}(N) = \left\{ \pm \begin{pmatrix} * & 0 & * & * \\ * & 1 & * & */t \\ N* & 0 & * & * \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \Gamma_t(N), \ \text{all} \quad * \in \mathbb{Z} \right\}. \tag{3}$$

Let $\Gamma < \mathrm{Sp}_2(\mathbb{R})$ be an arithmetic subgroup. In this paper Γ will be one of the groups $\Gamma_t(N)$ or $\Gamma_t(N)^+$. *A modular form* of weight k (k is integral or half-integral) for the subgroup Γ with a character (or a multiplier system) $\chi : \Gamma \to \mathbb{C}^\times$ is a holomorphic function on $\mathbb{H}_2$ which satisfies the functional equation

$$(F|_k\gamma)(Z) = \chi(\gamma)F(Z) \qquad \text{for any} \quad \gamma \in \Gamma.$$

We denote by $|_k$ the standard slash operator on the space of functions on $\mathbb{H}_2$:

$$(F|_k\gamma)(Z) := \det(CZ + D)^{-k} F(M\langle Z\rangle)$$

where $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}_2(\mathbb{R})$ and $M\langle Z\rangle = (AZ + B)(CZ + D)^{-1}$. For a half-integral k we choose one of the holomorphic square roots by the condition $\sqrt{\det(Z/i)} > 0$ for $Z = iY \in \mathbb{H}_2$. We denote by $M_k(\Gamma, \chi)$ the finite dimensional space of all modular forms of this type.

A dd-modular form is a particular case of reflective modular forms whose divisor is defined by reflections in the corresponding integral orthogonal group. In [GN2]–[GN3] we classified the reflective modular forms for the paramodular groups Γ_t. In particular we found all Siegel modular forms for the paramodular groups Γ_t with the diagonal divisor. To classify all possible dd-modular forms for the congruence subgroups $\Gamma_t(N)$ we use the method of multiplicative symmetrization (see [GN2, §3.1]). The next proposition is a generalization of Proposition 1.1 of [GH1] in which we studied the case $N = 1$.

Proposition 1.2 *If F_k is a dd-modular form of integral (or half-integral) weight k with a character (or a multiplier system) with respect to $\Gamma_t(N)$ then the triplet $(t, N; k)$ can take one of the nine values*

$$(1, 1; 5),\ (2, 1; 2),\ (3, 1; 1),\ (4, 1; \frac{1}{2}),\ (1, 2; 3),\ (1, 3; 2),\ (1, 4; \frac{3}{2}),$$

$$(2, 2; 1),\ (2, 4; \frac{1}{2}).$$

The corresponding dd-modular forms are, if they exist, unique up to a scalar.

Proof. Uniqueness of a dd-modular form for a fixed group follows from the Koecher principle (see [F]). Let F be a non-zero modular form of weight k with respect to $\Gamma_t(N)$. We use the following operator of multiplicative symmetrization

$$[F]_1 = \prod_{\gamma \in \Gamma_t^{(int)}(N)\backslash \mathrm{Sp}_2(\mathbb{Z})} F|_k\gamma \qquad \text{where} \quad \Gamma_t^{(int)}(N) = \Gamma_t(N) \cap \mathrm{Sp}_2(\mathbb{Z}).$$

It is clear that $[F]_1$ is a non-zero modular form with respect to $\Gamma_1 = \mathrm{Sp}_2(\mathbb{Z})$.

Lemma 1.3 *For any integral $t \geq 1$ and $N \geq 1$ we have*

$$[\Gamma_1 : \Gamma_t^{(int)}(N)] = \left((Nt)^3 \prod_{p\,|(tN)} (1 + p^{-1})(1 + p^{-2})\right) \cdot \prod_{p\,|(t,N)} (1 + p^{-1}).$$

Proof. The diagram of the subgroups shows that

$$[\Gamma_1 : \Gamma_t^{(int)}(N)] = \frac{[\Gamma_1 : \Gamma_{tN}^{(int)}] \cdot [\Gamma_{tN}^{(int)} : \Gamma_{tN}^{(int)} \cap \Gamma_t(N)]}{[\Gamma_t^{(int)}(N) : \Gamma_{tN}^{(int)} \cap \Gamma_t(N)]}$$

where $\Gamma_d^{(int)} = \Gamma_d \cap \Gamma_1$ is a subgroup of the paramodular group. It is known (see [GH1, §1]) that

$$[\Gamma_1 : \Gamma_{tN}^{(int)}] = (tN)^3 \prod_{p\,|(tN)} (1 + \frac{1}{p})(1 + \frac{1}{p^2}).$$

Analyzing the form of the elements in the subgroups we obtain

$$[\Gamma_{tN}^{(int)} : \Gamma_{tN}^{(int)} \cap \Gamma_t(N)] = [\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)] = N \prod_{p\,|N} (1 + \frac{1}{p}),$$

and

$$[\Gamma_t^{(int)}(N) : \Gamma_t^{(int)} \cap \Gamma_{tN}] = [\Gamma_0(t) : \Gamma_0(tN)].$$

This gives us the formula of the lemma. □

Let $\pi_{t,N} : \mathbb{H}_2 \to \mathcal{A}_t(N) = \Gamma_t(N)\backslash\mathbb{H}_2$ be the quotient map. Note that $\mathcal{A}_t = \mathcal{A}_t(1)$ is the moduli space of $(1,t)$-polarized Abelian surfaces. For $N = 1$ the image $\pi_{t,1}(\mathcal{H}_1)$ in $\mathcal{A}_t$ parameterizes split polarized Abelian surfaces. For $t = N = 1$ this is the Humbert surface H_1 of discriminant 1 in $\mathcal{A}_1$ and one can consider the divisor $\pi_{1,1}(\mathcal{H}_1)$ as the discriminant of the moduli space of curves of genus 2.

Let us assume that F has a diagonal divisor of multiplicity $m \geq 1$, i.e., $\mathrm{div}_{\mathcal{A}_t(N)} F = m \cdot \pi_{t,N}(\mathcal{H}_1)$. We note that H_1 is irreducible in $\mathcal{A}_1$ (for the theory of Humbert surfaces see [vdG] and [GH2]). It follows that $\mathrm{div}_{\mathcal{A}_1}([F]_1)$ is the Humbert surface H_1 with some multiplicity d. Therefore according to the Koecher principle

$$[F]_1(Z) = C \cdot \Delta_5(Z)^d$$

where C is a non-zero constant. In order to calculate the multiplicity d we note that the stabilizer of $\mathcal{H}_1$ in $\mathrm{Sp}_2(\mathbb{R})$ is the group generated by the direct product of two copies of $\mathrm{SL}_2(\mathbb{R})$ in $\mathrm{Sp}_2(\mathbb{R})$ and the involution J:

$$\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R}) \cong \left\{ \begin{pmatrix} a & 0 & b & 0 \\ 0 & a_1 & 0 & b_1 \\ c & 0 & d & 0 \\ 0 & c_1 & 0 & d_1 \end{pmatrix} \in \mathrm{Sp}(\mathbb{R}) \right\}, \quad J = \begin{pmatrix} 0&1&0&0 \\ 1&0&0&0 \\ 0&0&0&1 \\ 0&0&1&0 \end{pmatrix}. \qquad (4)$$

The order of zero of $[F]_1$ along H_1 is equal to the number of left cosets $\Gamma_t^{(int)}(N)M$ in $\mathrm{Sp}_2(\mathbb{Z})$ containing an element from the stabilizer of $\mathcal{H}_1$. Therefore we have to find the number of the distinct cosets $\Gamma_t^{(int)}(N)M$ with $M \in \mathrm{St}_{\mathrm{Sp}_2(\mathbb{Z})}(\mathcal{H}_1)$. The involution J permuting the diagonal elements

τ and ω of $Z \in \mathbb{H}_2$ belongs to $\Gamma_t^{(int)}(N)$ if and only if $t = 1$. If $t > 1$ then $\Gamma_t^{(int)}(N)M_1 \neq \Gamma_t^{(int)}(N)JM_2$ for any M_1, M_2 in $\mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})$. It gives us the factor 2 if $t > 1$. Therefore $[F]_1$ vanishes along H_1 with order

$$d = 2^{\delta(t)}\, m[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)] \cdot [\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(tN)].$$

where $\delta(t) = 0$ if $t = 1$ and $\delta(t) = 1$ if $t > 1$.

The weight of the symmetrization$[F]_1$ equals the weight of F multiplied by the index of the subgroup calculated in the above lemma. The relation between $[F]_1$ and Δ_5 gives us the following identity between the weights of these modular forms

$$\Bigl(kN \prod_{\substack{p\,|N\\ p\nmid t}} \frac{p^2+1}{p(p+1)} \Bigr) \cdot t^2 \prod_{p\,|t} \frac{p^2+1}{p^2} = 2^{\delta(t)}\, 5m$$

where $k \in \mathbb{Z}/2$ is the weight of F. For any fixed m simple arguments of divisibility show that there exist only a few possibilities for $(t, N; k)$. If F is a dd-modular form of weight k (i.e., if $m = 1$), then there are only four triplets $(t, N; k)$ with $t = 1$ and five more $(t, N; k)$ for $t > 1$. This proves the proposition. $\square$

In what follows we construct eight dd-modular forms and we prove that a dd-modular form of type $(2, 4; \frac{1}{2})$ does not exist.

Theorem 1.4 *For the Hecke congruence subgroups $\Gamma_t(N) < \Gamma_t$ there are exactly eight dd-modular forms. They belong to the spaces*

$$M_5(\Gamma_1, \chi_2); \quad M_2(\Gamma_2, \chi_4),\ M_3(\Gamma_0^{(2)}(2), \chi_2); \quad M_1(\Gamma_3, \chi_6),\ M_2(\Gamma_0^{(2)}(3), \chi_2);$$

$$M_{\frac{1}{2}}(\Gamma_4, \chi_8),\ M_{\frac{3}{2}}(\Gamma_0^{(2)}(4), \chi_4); \qquad M_1(\Gamma_2(2), \chi_4)$$

where χ_d is a character (or a multiplier system) of order d of the corresponding modular group.

To prove this theorem we describe two lifting constructions for the congruence subgroups of Hecke type of the paramodular groups of genus 2.

2 Additive construction of dd-modular forms

The four dd-modular forms for $N = 1$ are the modular forms Δ_5 for $\mathrm{Sp}_2(\mathbb{Z})$ with a character of order 2, Δ_2 for Γ_2 with a character of order 4, Δ_1 for Γ_3 with a character of order 6 and $\Delta_{1/2}$ for Γ_4 with a multiplier system of order 8 (see [GN2]). In this section we construct the dd-modular forms for

$N > 1$. For this aim we use special Jacobi modular forms of index 1/2 with respect to the Jacobi group of level N

$$\Gamma^J(N) = (\Gamma_t^\infty(N) \cap \mathrm{Sp}_2(\mathbb{Z}))/\{\pm 1_4\} \cong \Gamma_0(N) \ltimes H(\mathbb{Z})$$

(see (3)). The Jacobi group is the semi-direct product of the Heisenberg group

$$H(\mathbb{Z}) = \left\{ [\lambda, \mu; \kappa] = \begin{pmatrix} 1 & 0 & 0 & \mu \\ \lambda & 1 & \mu & \kappa \\ 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \lambda, \mu, \kappa \in \mathbb{Z} \right\}$$

and

$$\Gamma_0(N) = \Big\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}), \quad c \equiv 0 \mod N \Big\}.$$

We embed $\gamma \in \Gamma_0(N)$ in $\Gamma_t(N)$ using the first copy of SL_2 in (4) (the second term is the unit matrix). We denote this matrix by $\widetilde{\gamma}$ and we identify $\Gamma_0(N)$ with this subgroup of $\Gamma_t(N)$.

Let t and k be integral or half-integral positive numbers. A holomorphic function ϕ on $\mathbb{H}_1 \times \mathbb{C}$ is called *holomorphic Jacobi form for* $\Gamma_0(N)$ *of weight* k *and index* t with a character (or a multiplier system) $v : \Gamma^J(N) \to \mathbb{C}^*$ if the function $\widetilde{\phi}(Z) := \phi(\tau, z) e^{2i\pi t\omega}$ of $Z \in \mathbb{H}_2$ is a $\Gamma^J(N)$-modular form with character (or multiplier system) v, i.e., if it satisfies

$$(\widetilde{\phi}|_k \gamma)(Z) = v(\gamma) \widetilde{\phi}(Z) \quad \text{for any} \quad \gamma \in \Gamma^J(N) \tag{5}$$

and for each $M \in \mathrm{SL}(2, \mathbb{Z})$ it has a Fourier expansion of the following type

$$\big(\widetilde{\phi}|_k \widetilde{M}\big)(Z) = \sum_{\substack{n,\, l \\ 4nt - l^2 \geqslant 0}} c_M(n, l)\, q^n r^l s^t \tag{6}$$

where n, l are in $\mathbb{Q}$, $q = e^{2i\pi\tau}$, $r = e^{2i\pi z}$ and $s = e^{2i\pi\omega}$. The last condition means that ϕ is holomorphic at the cusp determined by M (see [EZ]). The form ϕ is called *cusp form* if $c_M(n, l) \neq 0$ only for $4nt - l^2 > 0$ for all M. We call the form ϕ a *weak* Jacobi form if in its Fourier expansions $c_M(n, l) \neq 0$ only for $n \geqslant 0$. The Jacobi form ϕ is called *nearly holomorphic* if there exists $n \in \mathbb{N}$ such that $\Delta^n \phi$ is a weak Jacobi form where Δ is the Ramanujan Δ–function.

We denote by $J_{k,t}(\Gamma_0(N), v)$ the space of all Jacobi forms with a character (or a multiplier system) for $\Gamma^J(N) = \Gamma_0(N) \ltimes H(\mathbb{Z})$. We denote the space of corresponding weak (resp. nearly holomorphic) Jacobi forms by $J^w_{k,t}(\Gamma_0(N))$ (resp. $J^{nh}_{k,t}(\Gamma_0(N))$). In §4, we use nearly holomorphic Jacobi forms of weight 0 for $\Gamma_0(N)$ in order to construct Borcherds products. In this section we work with holomorphic Jacobi forms.

The main example of Jacobi forms of half-integral index is the Jacobi theta-function of level 2 (see (28)):

$$\vartheta(\tau,z) = -i\vartheta^{(2)}_{1,1}(\tau,z) = \sum_{m\in\mathbb{Z}} \left(\frac{-4}{m}\right) q^{m^2/8}\, r^{m/2}$$
$$= -q^{1/8} r^{-1/2} \prod_{n\geq 1} (1-q^{n-1}r)(1-q^n r^{-1})(1-q^n) \quad (7)$$

is an element of $J_{\frac{1}{2},\frac{1}{2}}(SL(2,\mathbb{Z}), v_\eta^3\times v_H)$ where v_η^3 is the multiplier system of the cube of the Dedekind η-function and

$$v_H([\lambda,\mu;\kappa]) = (-1)^{\lambda+\mu+\lambda\mu+\kappa}.$$

is a character of the Heisenberg group. (See [GN2] for more details on the Jacobi forms of half-integral index.)

We denote by $\chi\times v_H^{\varepsilon}$ the character of $\Gamma^J(N)$ induced by the character (or multiplier system of finite order) $\chi : \Gamma_0(N) \to \mathbb{C}^*$ and by a power $v_H^{\varepsilon} : H(\mathbb{Z}) \to \{\pm 1\}$. It is easy to see from the definition that the non trivial binary character v_H can appear only if the index t is half-integral.

Let

$$\phi \in J_{k,t}(\Gamma_0(N), \chi\times v_H^{2t}) \quad (8)$$

where $k\in\mathbb{N}$, $t\in\mathbb{N}/2$, $\chi : \Gamma_0(N)\to\mathbb{C}^*$ is a character of finite order. We suppose that

$$\mathrm{Ker}(\chi) \supset \Gamma_1(Nq,q) \quad (9)$$

for some q where the last group is defined as follows $\Gamma_1(Nq,q) =$

$$\Big\{ \begin{pmatrix} a & b\\ c & d\end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}),\ c\equiv 0 \bmod Nq,\ b\equiv 0\bmod q,\ a\equiv d\equiv 1 \bmod Nq \Big\}.$$

For this group we introduce the Hecke operator (see [Sh, Ch.3])

$$T^{(N)}(m) = \sum_{\substack{ad=m\\ (a,Nq)=1\\ b \bmod d}} \Gamma_1(Nq,q)\,\sigma_a \cdot \begin{pmatrix} a & qb\\ 0 & d\end{pmatrix}$$

where $a>0$ and $\sigma_a\in\mathrm{SL}_2(\mathbb{Z})$ such that $\sigma_a \equiv \left(\begin{smallmatrix} a^{-1} & 0\\ 0 & a\end{smallmatrix}\right) \bmod Nq$. This element induces the Hecke operator on Jacobi form $\widetilde{\phi}(Z) = \phi(\tau,z)\exp(2\pi i\omega)$:

$$\widetilde{\phi}|_k T_-^{(N)}(m)(Z) = m^{k-1} \sum_{\substack{ad=m\\ (a,Nq)=1\\ b \bmod d}} d^{-k}\chi(\sigma_a)\phi(\frac{a\tau+bq}{d},\, az)\, e^{2\pi i m t\omega} \quad (10)$$

(compare with [EZ] and [GN2, (1.11)–(1.12)]).

Lemma 2.1 *Let ϕ be as in (8) and (9). We suppose that m is coprime to q and that m is odd if t is half-integral. Then*

$$\phi|_k\, T_-^{(N)}(m) \in J_{k,mt}(\Gamma_0(N), \chi_m \times v_H^{2t})$$

where χ_m is a character of $\mathrm{SL}_2(\mathbb{Z})$ defined by

$$\chi_m(\alpha) := \chi(\alpha_m).$$

For any $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$ the matrix $\alpha_m \in \Gamma_0(N)$ is defined by the condition

$$\alpha_m \equiv \begin{pmatrix} a \mod Nq & m^{-1}b \mod q \\ mc \mod Nq & d \mod Nq \end{pmatrix}.$$

The proof of the lemma is similar to the proof of [GN2, Lemma 1.7]. One has to use that $\Gamma_1(Nq, q)$ is a normal subgroup of $\Gamma_0(N)$. We note that we do not assume that m is coprime to N.

In this paper we consider Jacobi forms with special characters such that $\mathrm{Ker}(\chi) \supset \Gamma_1(Nq, q)$. If $q = 1$, then χ is induced by a Dirichlet character χ_N modulo N:

$$\chi\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right) = \chi_N(d). \tag{11}$$

To construct all dd-modular forms we have to use characters which appear in the theory of η-products. For example, we shall use the following Jacobi forms

$$\eta(\tau)\eta(2\tau)^4\vartheta(\tau, z) \quad \text{or} \quad \frac{\eta(2\tau)^2\eta(4\tau)^4}{\eta(\tau)^2}\,\vartheta(\tau, z)^2$$

(see the proof of Theorem 1.4 below). The corresponding characters can be calculated using the conjugation of the multiplier system v_η of order 24 of the Dedekind eta-function. This explains the role of the number 24 in the lifting construction of Theorem 2.2. This theorem generalizes to the congruence subgroups the lifting constructions of [G1] and [GN2].

Theorem 2.2 *Let $\phi \in J_{k,t}(\Gamma_0(N), \chi \times v_H^{2t})$ be a holomorphic Jacobi form where $k \in \mathbb{N}$, $t \in \mathbb{N}/2$ and $\chi : \Gamma_0(N) \to \mathbb{C}^*$ is a character of finite order such that $\mathrm{Ker}(\chi) \supset \Gamma_1(Nq, q)$. We assume that q is a divisor of 24, $qt \in \mathbb{N}$ and $\chi\left(\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\right) = e^{\frac{2\pi i}{q}}$.*

1. *Let $q > 1$ or $q = 1$ and $c(0,0) = 0$ where $c(0,0)$ is the constant coefficient in the Fourier expansion of ϕ at ∞. We fix $\mu \in (\mathbb{Z}/q\mathbb{Z})^\times$. Then the function*

$$F_\phi(Z) = \mathrm{Lift}_\mu(\phi)(Z) = \sum_{\substack{m \equiv \mu \bmod q \\ m>0}} \widetilde{\phi}\,|_k\, T_-^{(N)}(m)(Z)$$

is a modular form for $\Gamma_{qt}(N)^+$ with a character $\chi_{t,\mu}$. The lifting is a cusp form if ϕ is a cusp form. If $\mu = 1$, then $\mathrm{Lift}(\phi) = \mathrm{Lift}_1(\phi) \not\equiv 0$ for $\phi \not\equiv 0$. If

$\mathrm{Lift}_\mu(\phi) \not\equiv 0$, *then the character* $\chi_{t,\mu}$ *is induced by the character* $\chi_\mu \times v_H^{2t}$ *of the Jacobi group, where* χ_μ *is a character of* $\mathrm{SL}_2(\mathbb{Z})$ μ*-conjugated to* χ *(see Lemma 2.1), and by the relations*

$$\chi_{t,\mu}(V_{qt}) = (-1)^k, \quad \chi_{t,\mu}([0,0;\frac{\kappa}{qt}]) = \exp{(2\pi i\,\frac{\mu\kappa}{q})} \quad (\kappa\in\mathbb{Z}).$$

2. *Let* $q=1$ *and* $c(0,0)\neq 0$. *We assume that the character* χ *of* $\Gamma_0(N)$ *is induced by a primitive Dirichlet character* χ_N *modulo* N *(see (11)). Then*

$$F_\phi(Z) = \mathrm{Lift}(\phi)(Z) = c(0,0)E_k(\tau,\chi_N) + \sum_{m\ge 1} \widetilde{\phi}\,|_k\, T_-^{(N)}(m)(Z)$$

where

$$E_k(\tau,\chi_N) = 2^{-1}L(1-k,\chi_N) + \sum_{n\ge 1}\sum_{a|n} \chi_N(a)\,a^{k-1}\exp{(2\pi i n\tau)}$$

is the Eisenstein series of weight k *for* $\Gamma_0(N)$ *with character* χ_N.

Remark. There is a variant of this theorem if qt is half-integral. One has to add a conjugation with respect to an element of the symplectic group over $\mathbb{Q}$ in order to obtain a modular form for a congruence subgroup of the paramodular group Γ_{4qt}. (See [GN2, Theorem 1.12]).

Proof. First we prove the convergence of the series defining $\mathrm{Lift}_\mu(\phi)$. We put

$$Z = X+iY = \begin{pmatrix} \tau & z\\ z & \omega\end{pmatrix} = \begin{pmatrix} u & x\\ x & u_1\end{pmatrix} + i\begin{pmatrix} v & y\\ y & v_1\end{pmatrix} \in \mathbb{H}_2.$$

Then $\det Y = vv_1 - y^2 = v(v_1 - \frac{y^2}{v}) = v\cdot\tilde{v}$ where $\tilde{v}$ is invariant under the action of the Jacobi group. If ϕ is a holomorphic function with Fourier expansion of type (6), then

$$|\phi(\tau,z)e^{2\pi i t\omega}|e^{2\pi t\tilde{v}} = |\phi(\tau,z)|e^{-2\pi t y^2/v}$$

does not depend on u_1 and $\tilde{v}$ and it is bounded in the domain $v>\varepsilon$ (see [Kl]). We introduce

$$\widetilde{\psi}(Z) = \sum_{M_i\in\Gamma_0(N)\backslash\mathrm{SL}_2(\mathbb{Z})} |\widetilde{\phi}|_k\widetilde{M}_i(Z)|.$$

Then the function $\widetilde{\psi}$ is $|_k$-invariant with respect to the full Jacobi group $\Gamma^J = \mathrm{SL}_2(\mathbb{Z})\ltimes H(\mathbb{Z})$ and $\widetilde{\psi}(Z)e^{2\pi t\tilde{v}}$ (depending only on τ and z) is bounded for $v>\varepsilon$. If $0<v<\varepsilon$ then there exists $M = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in\mathrm{SL}_2(\mathbb{Z})$ such that $\mathrm{Im}(M\langle\tau\rangle)>\varepsilon$ and

$$\widetilde{\psi}(Z) = |c\tau+d|^{-k}\widetilde{\psi}(\widetilde{M}\langle Z\rangle).$$

Therefore $\widetilde{\psi}(Z)e^{2\pi t\tilde{v}} = \mathrm{O}(v^{-k})$ if $v \to 0$. Using, if necessary, this estimation for all terms $\psi(\frac{a\tau+b}{d}, az)$ we see that the series which defines F_ϕ has a majorant of type

$$C\sum_{m\ge 1} m^{k+1}e^{-2\pi t m v_1}$$

in any compact subset of $\mathbb{H}_2$. If $c(0,0)\neq 0$ then we add an Eisenstein series $E_k(\tau,\chi_N)$ with respect to $\Gamma_0(N)$ in the lifting construction. This series is well defined for any weight $k\ge 1$ according to Hecke (see [Hi, Proposition 5.1.2]).

Now we prove that the lifting is a modular form. According to the conditions of the theorem ϕ has the Fourier expansion of the following type

$$\phi(\tau,\, z) = \sum_{\substack{l\equiv 2t \bmod 2\\ n\ge 0,\ n\equiv 1 \bmod q\\ \frac{4nt}{q}\ge \frac{l^2}{4}}} c(n,l)\,\exp\big(2\pi i(\frac{n}{q}\tau + \frac{l}{2}z)\big).$$

By the definition (10) of the Hecke operators we have

$$\big(\widetilde{\phi}|_k\, T_-^{(N)}(m)\big)(Z) = \tag{12}$$

$$m^{k-1}\sum_{\substack{ad=m\\ (a,Nq)=1\\ b \bmod d}} d^{-k}\chi(\sigma_a)\sum_{\substack{l\equiv 2t \bmod 2\\ n\ge 0,\ n\equiv 1 \bmod q}} c(n,l)\,\exp\big(2\pi i(\frac{n(a\tau+bq)}{dq}+\frac{al}{2}z + mt\omega)\big) =$$

$$\sum_{\substack{ad=m\\ (a,Nq)=1}} a^{k-1}\chi(\sigma_a)\sum_{\substack{l\equiv 2t \bmod 2,\ n_1\ge 0\\ dn_1\equiv 1 \bmod q}} c(dn_1,l)\,\exp\big(2\pi i(\frac{an_1}{q}\tau+\frac{al}{2}z+adt\omega)\big).$$

The Jacobi form ϕ has a nontrivial character v_H of the Heisenberg subgroup of the Jacobi group if and only if $2t\equiv 1 \mod 2$. If t is half-integral, then q is pair because $tq\in\mathbb{N}$. Therefore in this case for any m coprime to q the character of the Heisenberg group of the Jacobi form $\widetilde{\phi}|_k T_-^{(N)}(m)$ is equal to v_H. The $\Gamma_0(N)$-part of the character of $\widetilde{\phi}|_k T_-^{(N)}(m)$ depends only on m modulo q according to Lemma 2.1 because ϕ satisfies (9). In the definition of F_ϕ we have $m\equiv\mu \mod q$. Therefore, if $m\equiv \mu \mod q$, then $\widetilde{\phi}|_k T_-^{(N)}(m)$ is a Jacobi form with character $\chi_\mu\times v_H^{2t}$.

The number q is a divisor of 24 and μ is coprime to q. For any $x\in(\mathbb{Z}/24\mathbb{Z})^\times$ we have $x^2\equiv 1 \mod 24$. (24 is the maximal number with this property. The same is true for any divisor of 24.) Therefore in the formula for the Fourier expansion of $\widetilde{\phi}|_k T_-^{(N)}(m)$ we have that the coefficient at τ under the exponent satisfies the relations $an_1 = \frac{an}{d}\equiv mn\equiv \mu \mod q$.

Now we assume that $c(0,0)=0$. Taking the summation over all positive $m\equiv\mu \mod q$ we get

$$F_\phi(Z) =$$

$$\sum_{\substack{a>0,\,d>0\\ ad\equiv\mu \bmod q\\ (a,N)=1}} a^{k-1}\chi(\sigma_a) \sum_{\substack{l\equiv 2t \bmod 2\\ n_1>0,\ dn_1\equiv 1 \bmod q}} c(dn_1,l)\,\exp\big(2\pi i(\frac{an_1}{q}\tau+\frac{al}{2}z+adt\omega)\big)=$$

$$\sum_{\substack{n,\,m>0\\ n,m\equiv\mu \bmod q\\ l\equiv 2t \bmod 2\\ \frac{4nmt}{q}\ge\frac{l^2}{4}}} \sum_{\substack{a|(n,l,m)\\ (a,N)=1\\ a>0}} a^{k-1}\,\chi(\sigma_a)\,c(\frac{nm}{a^2},\frac{l}{a})\,\exp\big(2\pi i(\frac{n}{q}\tau+\frac{l}{2}z+mt\omega)\big).$$

The Jacobi forms $\widetilde{\phi}|_k\,T_-^{(N)}(m)$ are modular forms with respect the Jacobi group $\Gamma^J(N)$. The parabolic subgroup $\Gamma_{qt}^{\infty}(N)$ (see (3)) differs from the Jacobi group $\Gamma^J(N)$ by its center. For $m\equiv\mu \mod q$ the action of the center is given by

$$\big(\widetilde{\phi}|_k\,T_-^{(N)}(m)\big)\big|_k\,[0,0;\frac{\kappa}{qt}]=\exp\,(2\pi i\frac{\kappa\mu}{q})\,\big(\widetilde{\phi}|_k\,T_-^{(N)}(m)\big).$$

Therefore the lifting $F_\phi(Z)$ is a $\Gamma_{qt}^{\infty}(N)$-modular form of weight k with character $\chi_\mu\times v_H^{2t}\times\exp\,(2\pi i\frac{*\mu}{q})$. The Fourier expansion of F_ϕ is also invariant under the transformation $\{\tau\to qt\omega,\ \omega\to(qt)^{-1}\tau\}$. It is induced by V_{qt} (see (2)). Therefore

$$(F_\phi|_k\,V_{qt})(Z)=(-1)^kF_\phi(Z).$$

The subgroup $\Gamma_{qt}^{\infty}(N)$ and V_{qt} generate the group $\Gamma_{qt}^{+}(N)$ (see Lemma 1.1) and the lifting is a $\Gamma_{qt}^{+}(N)$-modular form if $c(0,0)\neq 0$.

If ϕ is a cusp form then $\phi|_kT_-^{(N)}(m)$ is also a Jacobi cusp form. In order to prove this we note that $T^{(N)}(m)$ is a part of the full Hecke operator for the congruence subgroup $\Gamma_0(N)$ (see (23)). Therefore for any $M\in\mathrm{SL}_2(\mathbb{Z})$ we have $(\widetilde{\phi}|_kT_-^{(N)}(m))|_k\widetilde{M}=\sum_i(\widetilde{\phi}|_k\widetilde{M_i})|_kP_i$ for some $M_i\in\mathrm{SL}_2(\mathbb{Z})$ and integral upper triangular matrices P_i with $\det P_i=m$. All indices of the Fourier coefficients of $\widetilde{\phi}|_k\widetilde{M_i}$ have positive hyperbolic norm $4nt-l^2>0$ like in (6). The action by upper triangular P_i does not change this property. It follows that $\widetilde{\phi}|_kT_-^{(N)}(m)$ is a cusp form. We have proved that the index (n,l) of arbitrary non-zero Fourier coefficient of the lifting is non degenerate (i.e., $4nt-l^2>0$) for all 0-dimensional cusps of the 1-dimensional cusp determined by $\Gamma^J(N)$. $\Gamma_t(N)$ and the full Jacobi group $SL_2(\mathbb{Z})\ltimes H(\mathbb{Z})$ generate the paramodular group Γ_t. In order to obtain all cusps of $\Gamma_t(N)$ we can use the parabolic subgroup $\Gamma_\infty=\{P=\left(\begin{smallmatrix}A&B\\0&D\end{smallmatrix}\right)\in\mathrm{Sp}_2(\mathbb{Z})\}$ because $\langle\Gamma_t,\Gamma_\infty\rangle=\mathrm{Sp}_2(\mathbb{Z})$ (see [G2]). We have considered above the action of $\mathrm{SL}_2(\mathbb{Z})$ on the lifting. After the action by any upper triangular matrix P Fourier coefficients with degenerate index do not appear. Therefore the lifting of a Jacobi cusp form is a cusp form.

Let us consider the case $c(0,0)\neq 0$. Since $\chi\big(\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)\big)=e^{\frac{2\pi i}{q}}$ we have that $q=1$, $t\in\mathbb{N}$, the character χ is induced by a Dirichlet character χ_N modulo

N (see (11)) and $(-1)^k = \chi_N(-1)$. In the sum $\sum_{m\ge 1} \widetilde{\phi}\,|_k\, T_-^{(N)}(m)(Z)$ we have an additional term

$$c(0,0)\sum_{m\ge 1}\sum_{a|m}\chi_N(a)\,a^{k-1}\exp{(2\pi i m t\omega)}.$$

To make the lifting invariant with respect to V_t ($\omega\mapsto\tau/t$, $\tau\mapsto t\omega$) we have to add a similar term with respect to τ. For that we use the Eisenstein series

$$E_k(\tau,\chi_N) = 2^{-1}L(1-k,\chi_N)+\sum_{n\ge 1}\sum_{a|n}\chi_N(a)\,a^{k-1}e^{2\pi i n\tau}\in M_k(\Gamma_0(N),\chi_N)$$

(see [Hi, Proposition 5.1.2]). The Eisenstein series is a Jacobi form of weight k and index 0. The theorem is proved. $\square$

Proof of Theorem 1.4. We consider the nine possibilities for dd-modular forms given in Proposition 1.2.

1. $N=1$. The dd-modular forms for the full paramodular group Γ_t with $t=1$, 2, 3, 4 were constructed in [GN1]–[GN2]:

$$\begin{aligned}\Delta_5(Z) &= \mathrm{Lift}(\eta(\tau)^9\vartheta(\tau,z))\in M_5(\Gamma_1, v_\eta^{12}\times v_H),\\ \Delta_2(Z) &= \mathrm{Lift}(\eta(\tau)^3\vartheta(\tau,z))\in M_2(\Gamma_2, v_\eta^{6}\times v_H),\\ \Delta_1(Z) &= \mathrm{Lift}(\eta(\tau)\vartheta(\tau,z))\in M_1(\Gamma_3, v_\eta^{4}\times v_H).\end{aligned}\tag{13}$$

They are cusp forms with character of order 2, 4 and 6 respectively. Moreover

$$\Delta_{1/2}(Z) = \mathrm{Trivial-Lift}(\vartheta(\tau,z))\in M_{1/2}(\Gamma_4, v_\eta^{3}\times v_H)\tag{14}$$

is the most odd Siegel even theta-function $\theta_{1111}(Z)$ of level 2 which is a modular form of weight 1/2 and a multiplier system of degree 8 with respect to Γ_4. We construct below the four new Siegel dd-modular forms ∇_3, ∇_2, $\nabla_{3/2}$ and Q_1 for the congruence subgroups. The index denotes the weight of the corresponding modular form.

2. Let $N=2$. Two groups of level $N=2$ appear in Proposition 1.2. We consider two Jacobi forms of index $\frac12$ with respect to the Hecke congruence subgroup $\Gamma_0(2)$:

$$\eta(\tau)\eta(2\tau)^4\,\vartheta(\tau,z)\in J^{cusp}_{3,\frac12}(\Gamma_0(2),\chi_2^{(2)}\times v_H),$$

$$\frac{\eta(2\tau)^2}{\eta(\tau)}\,\vartheta(\tau,z)\in J_{1,\frac12}(\Gamma_0(2),\chi_4^{(2)}\times v_H).$$

Every cusp p of $\Gamma_0(N)$ has a representative of the form $p=a/c$ where c is a positive divisor of N and a is taken $\mod (c,\frac{N}{c})$. For any divisor n of N the order of $\eta(n\tau)$ at p is equal to $\frac{(c,n)^2}{24n}$. Using this we check that the

$\Gamma_0(2)$-modular form $\frac{\eta(2\tau)^2}{\eta(\tau)}$ has a zero of order $\frac{1}{8}$ at $p=\infty$ and is equal to $1/2$ at the second cusp.

The Jacobi theta-series $\vartheta(\tau,z)$ has the multiplier system $v_\eta^3\times v_H$ of order 8. $\eta(2\tau)^8\eta(\tau)^8$ is a well known example of the modular forms with respect to $\Gamma_0(2)$. The powers $\eta(2\tau)^4\eta(\tau)^4$ and $\eta(2\tau)^2\eta(\tau)^2$ are cusp forms for $\Gamma_0(2)$ with characters $\chi_2^{(2)}$ and $\chi_4^{(2)}$ of $\Gamma_0(2)$ of order 2 and 4 respectively. Using the exact formula for v_η^2 (see, for example, [GN2, Lemma 1.2]) we obtain that

$$\chi_2^{(2)}\left(\left(\begin{smallmatrix} a & b\\ 2c & d\end{smallmatrix}\right)\right)=(-1)^{b-c},\qquad \chi_4^{(2)}\left(\left(\begin{smallmatrix} a & b\\ 2c & d\end{smallmatrix}\right)\right)=e^{\frac{2\pi i}{4}d(b-c)}$$

for any matrix in $\Gamma_0(2)$. In particular

$$\ker\chi_2^{(2)}\supset\Gamma_1(4,2),\quad \ker\chi_4^{(2)}\supset\Gamma_1(8,4),$$

$$\chi_2^{(2)}\left(\left(\begin{smallmatrix} 1 & 1\\ 0 & 1\end{smallmatrix}\right)\right)=e^{\frac{2\pi i}{2}},\quad \chi_4^{(2)}\left(\left(\begin{smallmatrix} 1 & 1\\ 0 & 1\end{smallmatrix}\right)\right)=e^{\frac{2\pi i}{4}}.$$

The lifting construction gives us

$$\nabla_3:=\mathrm{Lift}(\eta(\tau)\eta(2\tau)^4\,\vartheta(\tau,z))\in S_3(\Gamma_0^{(2)}(2),\chi_2^{(2)}\times v_H),\tag{15}$$

$$Q_1:=\mathrm{Lift}(\frac{\eta(2\tau)^2}{\eta(\tau)}\,\vartheta(\tau,z))\in M_1(\Gamma_2(2),\chi_4^{(2)}\times v_H).\tag{16}$$

3. Let $N=3$. It is known that $\eta(3\tau)^6\eta(\tau)^6\in S_6(\Gamma_0(3))$. We consider

$$\eta(3\tau)^3\vartheta(\tau,z)\in J_{2,\frac{1}{2}}^{cusp}(\Gamma_0(3),\chi_2^{(3)}\times v_H)$$

where $\chi_2^{(3)}$ is a character of order 2. Similar to the case $N=2$ one can check that

$$\chi_2^{(3)}(M)=(-1)^{a+d+1}\left(\frac{d}{3}\right)\qquad M=\begin{pmatrix} a & b\\ 3c & d\end{pmatrix}\in\Gamma_0(3)\quad\text{if}\ \ c\equiv 1\mod 2$$

and

$$\chi_2^{(3)}(M)=(-1)^{b}\left(\frac{d}{3}\right)\qquad\text{if}\ \ c\equiv 0\mod 2.$$

Therefore

$$\ker\chi_2^{(3)}\supset\Gamma_1(6,2),\quad \chi_2^{(3)}\left(\left(\begin{smallmatrix} 1 & 1\\ 0 & 1\end{smallmatrix}\right)\right)=e^{\pi i}.$$

Applying Theorem 2.2 we obtain

$$\nabla_2:=\mathrm{Lift}(\eta(3\tau)^3\vartheta(\tau,z))\in S_2(\Gamma_0^{(2)}(3),\chi_2^{(3)}\times v_H).\tag{17}$$

4. Let $N=4$. We define

$$h_{\frac{3}{2}}(\tau,z)=\frac{\eta(2\tau)\eta(4\tau)^2}{\eta(\tau)}\,\vartheta(\tau,z)\in J_{\frac{3}{2},\frac{1}{2}}(\Gamma_0(4),\chi_4^{(4)}\times v_H)\tag{18}$$

where $\chi_4^{(4)}$ is a multiplier system of order 4. The $\Gamma_0(4)$-modular form $\eta(2\tau)\eta(4\tau)^2/\eta(\tau)$ vanishes at two cusps ∞ and $\frac{1}{2}$ and takes non-zero value at 1. Let us consider

$$h_{\frac{3}{2}}(\tau,z)^2=\frac{\eta(2\tau)^2\eta(4\tau)^4}{\eta(\tau)^2}\,\vartheta(\tau,z)^2\in J_{3,1}(\Gamma_0(4),\chi_2^{(4)}).$$

The $\Gamma_0(4)$-modular form $\eta(4\tau)^4\eta(2\tau)^2\eta(\tau)^4$ has the non-trivial quadratic character modulo 4

$$\chi_2^{(4)}(\left(\begin{smallmatrix} a & b\\ 4c & d\end{smallmatrix}\right))=(-1)^{\frac{(d-1)}{2}}\qquad\text{where}\qquad \left(\begin{smallmatrix} a & b\\ 4c & d\end{smallmatrix}\right)\in\Gamma_0(4).$$

We have

$$F_3:=\operatorname{Lift}(h_{3/2}^2)\in M_3(\Gamma_0^{(2)}(4),\chi_2^{(4)}).\tag{19}$$

The Jacobi form $h_{3/2}(\tau,z)^2$ has zero of order 2 for $z=0$. The Hecke operators of the lifting keep this divisor. Therefore F_3 vanishes with order 2 along $\mathcal{H}_1=\{z=0\}$. The proof of Proposition 1.2 shows that

$$\operatorname{div}_{\mathbb{H}_2}(F_3)=2\big(\bigcup_{\gamma\in\Gamma_0^{(2)}(4)}\gamma\langle\mathcal{H}_1\rangle\big).$$

In the next section we construct a modular form $\nabla_{3/2}$ such that $F_3=\nabla_{3/2}^2$ using the Borcherds automorphic products.

5. The last case of Proposition 1.2 is a possible dd-modular form of type $(N,t;k)=(4,2;\frac{1}{2})$. Using the exact construction of the dd-modular form Q_1 for $\Gamma_2(2)$ we prove that a dd-modular form of weight $\frac{1}{2}$ with respect to $\Gamma_2(4)$ does not exist.

Let assume that $D\in M_{\frac{1}{2}}(\Gamma_2(4),\chi)$ is a dd-modular form. Q_1 can be considered as a modular form with respect to $\Gamma_2(4)<\Gamma_2(2)$. Q_1 vanishes along the diagonal $\mathcal{H}_1$ but its divisor modulo $\Gamma_2(4)$ contains several irreducible components. Therefore $F=Q_1/D$ is a holomorphic function on $\mathbb{H}_2$ and it is a $\Gamma_2(4)$-modular form of weight $\frac{1}{2}$ according to the Koecher principle. Then it has the following Fourier-Jacobi expansion

$$F(Z)=f_0(\tau)+\sum_{m\geq\frac{1}{2}}f_{\frac{1}{2},m}(\tau,z)\exp(2\pi im\omega)$$

where the constant term f_0 is a modular form of weight $\frac{1}{2}$ with respect to $\Gamma_0(4)$. The zeroth Fourier-Jacobi coefficient of D is identically equal to zero

$$d_0(\tau)=\lim_{v_1\to\infty}D(\begin{pmatrix}\tau & z\\ z & iv_1\end{pmatrix})\equiv 0$$

because D is zero for $z=0$. Considering the Fourier-Jacobi expansions of the both part of the identity $Q_1=D\cdot F$ we obtain that

$$\frac{\eta(2\tau)^2}{\eta(\tau)}\,\vartheta(\tau,z)=f_0(\tau)\cdot d_{\frac{1}{2},\frac{1}{2}}(\tau,z).$$

Therefore the first non-trivial Fourier-Jacobi coefficient $d_{\frac12,\frac12}$ of D is equal to $g\vartheta$ where g is an automorphic form of weight 0 with respect to $\Gamma_0(4)$. The Jacobi theta-series is a modular form of singular weight 1/2. For every Fourier coefficient $c(n,l)$ of ϑ we have $2n^2-l^2=0$ (see (7)). The automorphic form g has a pole at some cusp. Therefore the Jacobi form $g\vartheta$ cannot be holomorphic at this cusp. It follows that D is not holomorphic. We finish the proof of Theorem 1.4 modulo existence of the dd-modular form $\nabla_{3/2}$.

We note that two new dd-modular forms ∇_2 and Q_1 have elementary formulae for the Fourier coefficients. You can compare them with cusp forms $\Delta_2 \in S_2(\Gamma_2,\chi_4)$ and $\Delta_1 \in S_1(\Gamma_3,\chi_6)$ (see [GN1] and [GN2, Example 1.14]). According to Euler and Jacobi

$$\eta(\tau)^3=\sum_{n>0}\left(\frac{-4}{n}\right) n q^{n^2/8}.$$

Then we obtain

$$\nabla_2(Z)=\sum_{N>0}\ \sum_{\substack{m,\,n\in 2\mathbb{N}+1\\ 3N^2=4mn-l^2}} N\left(\frac{-4}{Nl}\right)\sum_{\substack{a\mid(l,m,n)\\ a>0}} a\left(\frac{a}{3}\right) q^{\frac{n}{2}} r^{\frac{l}{2}} s^{\frac{m}{2}} \tag{20}$$

because in the lifting formula $\chi(\sigma_a)=\left(\frac{a}{3}\right)$. To calculate Q_1 we note that $\eta(2\tau)^2/\eta(\tau)=\frac12\vartheta_{1,0}^{(2)}(\tau,0)$ where

$$\vartheta_{1,0}^{(2)}(\tau,0)=q^{\frac18}\prod_{n\ge1}(1-q^n)(1+q^{n-1})(1+q^n)=2\sum_{n\in\mathbb{N}} q^{\frac{(2n+1)^2}{8}}.$$

Using the last formula we obtain that

$$Q_1(Z)=\sum_{N>0}\ \sum_{\substack{n,\,m\in 4\mathbb{N}+1\\ l\in 2\mathbb{Z}+1\\ (2N+1)^2=2mn-l^2}} \left(\frac{-4}{l}\right)\sigma_0((n,l,m))\, q^{\frac{n}{4}} r^{\frac{l}{2}} s^{\frac{m}{2}} \tag{21}$$

where $\sigma_0((n,l,m))$ is the number of divisors of the greatest common divisor of n, l, m.

From the proof of theorem given above we obtain also a description of the squares of dd-forms as liftings.

Corollary 2.3 *The following identities are true*

$$\nabla_3(Z)^2=\mathrm{Lift}\big(\eta(\tau)^2\eta(2\tau)^8\,\vartheta(\tau,z)^2\big)\in S_6(\Gamma_0(2)),$$
$$\nabla_2(Z)^2=\mathrm{Lift}\big(\eta(3\tau)^6\vartheta(\tau,z)^2\big)\in S_4(\Gamma_0(3)),$$
$$Q_1(Z)^2=\mathrm{Lift}\big(\frac{\eta(2\tau)^4}{\eta(\tau)^2}\,\vartheta(\tau,z)^2\big)\in M_2(\Gamma_2(2),\chi_2),$$
$$Q_1(Z)^4=\mathrm{Lift}\big(\frac{\eta(2\tau)^8}{\eta(\tau)^4}\,\vartheta(\tau,z)^4\big)\in M_4(\Gamma_2(2)).$$

Proof. All identities are similar. We prove the last one. First $f_1^4 \in J_{4,2}(\Gamma_0(2))$. According to Theorem 2.2 we get $\mathrm{Lift}(f_1^4) \in M_4(\Gamma_2(2))$. The Jacobi form f_1^4 has zero of order 4 along $z = 0$. The Hecke operators in the lifting construction preserve this divisor. Therefore the quotient $\mathrm{Lift}(f_1^4)/Q_1^4$ is a constant according to the Koecher principle. This constant is one. To see this we compare the first Fourier-Jacobi coefficients. □

We make two remarks on Theorem 2.2.

The modular forms ∇_3^2 and ∇_2^2 coincide with generators of the graded rings of Siegel modular forms for $\Gamma_0^{(2)}(2)$ and $\Gamma_0^{(2)}(3)$ (see [Ib]). The lifting construction gives us an universal approach to the generators. Moreover we obtain more fundamental functions like ∇_2 or $\nabla_{3/2}$ which are roots from generators of the corresponding graded rings. We give the relations between dd-modular forms and the generators proposed in the papers of Ibukiyama. First, we have

$$\nabla_3(Z)^2 = \mathrm{Lift}\big(\eta(\tau)^2\eta(2\tau)^8\,\vartheta(\tau,z)^2\big) = K(Z) \in S_6(\Gamma_0(2))$$

where

$$K(Z) = \frac{1}{4096}\big(\theta_{0100}(Z)\theta_{0110}(Z)\theta_{1000}(Z)\theta_{1001}(Z)\theta_{1100}(Z)\theta_{1111}(Z)\big)^2$$

and

$$\nabla_2(Z)^2 = \mathrm{Lift}\big(\eta(3\tau)^6\vartheta(\tau,z)^2\big) = \frac{1}{24}\,\Theta_4(Z) \in S_4(\Gamma_0(3)).$$

θ_{abcd} denotes the Siegel theta-series with characteristic $(abcd)$ of level 2 and Θ_4 is a theta-series with a spherical function. Using the dd-function Q_1 we can construct the generators of the graded ring of the modular forms with respect to the congruence subgroup $\Gamma_2(2)$. The details will be published in a separate paper.

The second remark is related to differential equations. In [CYY] it was proved that the monodromy group of Picard–Fuchs equations associated with one parameter families of Calabi–Yau threefolds is a subgroup of certain congruence subgroup $\Gamma(d_1, d_2)$ in $\mathrm{Sp}_2(\mathbb{Z})$ where d_2 is a divisor of d_1. Therefore one can put a question on Siegel modular forms with respect to this group. In order to construct such modular forms we can use Theorem 2.2 because this subgroup is the integral part of the intersection of two modular groups considered in this theorem

$$\Gamma(d_1, d_2) = \Gamma_{t_1}(q_1) \cap \Gamma_{t_2}(q_2) \cap \mathrm{Sp}_2(\mathbb{Z})$$

where we have the following relations for the least common multiples $d_1 = [t_1, t_2]$ and $d_2 = [q_1, q_2]$. If t_1 and t_2 are coprime we do not need to make the intersection with $\mathrm{Sp}_2(\mathbb{Z})$. In particular, $\Gamma(d_1, d_2) = \Gamma_0^{(2)}(d_2) \cap \Gamma_{d_1}$. According to Theorem 2.2 for any $f_i \in J_{k_i,t_i}(\Gamma_0(q_i))$ $(i = 1, 2)$ the product

$\mathrm{Lift}(f_1)\cdot\mathrm{Lift}(f_2)$ is a modular form of weight k_1+k_2 with respect to $\Gamma(d_1,d_2)$. (Under some conditions on t_i and q_i one can consider Jacobi forms with some characters.)

To finish the proof of Theorem 1.4 we have to construct a square root from the $\Gamma_0^{(2)}(4)$-modular form F_3 (see (19)). For this aim we consider the Borcherds automorphic products.

3 Borcherds products for $\Gamma_t(N)$

In this section we consider Borcherds automorphic products related to the Jacobi forms of weight 0 with respect to the congruence subgroup $\Gamma_0(N)$. This construction gives us the dd-modular forms of §2 as automorphic products. In particular, we construct the last dd-modular form $\nabla_{3/2}$ with respect to $\Gamma_0^{(2)}(4)$. In [B1] the language of the orthogonal groups and the vector valued automorphic forms was used. The Jacobi forms are very useful in the framework of Siegel modular forms because we have many methods to construct Jacobi forms of weight 0. The case of the symplectic paramodular group Γ_t was considered in [GN1]–[GN2]. A similar result one can obtain for the congruence subgroups. Some examples of Borcherds automorphic products for $\Gamma_0^{(2)}(N) < \mathrm{Sp}_2(\mathbb{Z})$ in terms of Jacobi forms were constructed in [AI] but they could not prove that the construction works for arbitrary Jacobi forms (see Lemma 3.2 below and the remark before it). In this section we construct the automorphic products for the subgroups $\Gamma_t(N)$ of the paramodular groups Γ_t for any t and N.

First we recall some well known facts about the Hecke congruence subgroup $\Gamma_0(N)$ (see [Sh], [Mi]). The number of non-equivalent cusps of $\Gamma_0(N)$ is equal to $\displaystyle\sum_{e|N,\,e>0}\varphi((e,\frac{N}{e}))$ where φ is the Euler's function and (a,b) is the greatest common divisor of a and b. We denote by $\mathcal{P}$ the set of cusps

$$\mathcal{P}=\left\{\frac{f}{e},\ e|N,\ e\ge 1,\ f \mod (e,\frac{N}{e}),\ (e,f)=1\right\}.$$

To each cusp $f/e\in\mathcal{P}$ of $\Gamma_0(N)$, we associate a matrix

$$\frac{f}{e}\mapsto M_{f/e}=\begin{pmatrix} f & * \\ e & * \end{pmatrix}\in \mathrm{SL}(2,\mathbb{Z}),\quad M_{f/e}\langle\infty\rangle=f/e.$$

Let $h_e=N/(e^2,N)$ be the width of the cusp $f/e\in\mathcal{P}$. The sum of the widths is $N\cdot\prod_{p\,|N}(1+p^{-1})$ (p is prime) which is the index of $\Gamma_0(N)$ in $\mathrm{SL}(2,\mathbb{Z})$. We also put $N_e=\frac{N}{e}$. In order to construct the dd-modular forms we need two particular cases when $N=p$ or p^2.

Example. $\Gamma_0(p)$ and $\Gamma_0(p^2)$.

i) If $N = p$, p prime, then there are two cusps: $\frac{1}{p}$ which is $\Gamma_0(p)$-equivalent to ∞ and 0 of width 1 and p respectively.
ii) If $N = p^2$, p prime , there are $(p+1)$ cusps: $\frac{1}{p^2}$ which is $\Gamma_0(p^2)$-equivalent to ∞, 0 and $\left\{ \frac{f}{p},\ 1 \le f \le p-1 \right\}$ of width 1, p^2 and 1 respectively.

As we mentioned above our datum for the automorphic Borcherds product for the congruence subgroup $\Gamma_t(N) < \Gamma_t$ is a nearly holomorphic Jacobi form of weight 0 and index t with respect to $\Gamma_0(N)$ (see §2). The character of this form is trivial. In the Borcherds automorphic products [B1] vector valued modular forms were used. In the case of a Jacobi modular form with respect to the congruence subgroup $\Gamma_0(N)$ one has to use its Fourier coefficients at all cusps of $\Gamma_0(N)$ (see [B1, Examples 2.2 and 2.3]). In order to realize this one can use the complete Hecke operator $T_N(m)$ for $\Gamma_0(N)$ which contains more classes than the operator $T_-^{(N)}(m)$ defined in (10) if $(m, N) \neq 1$. The operator $T_N(m)$ was introduced in [He] and it was used in [AI]. For $m \in \mathbb{N}^*$, we set

$$M_N(m) = \left\{ M = \begin{pmatrix} a & b \\ cN & d \end{pmatrix} \in \mathcal{M}_2(\mathbb{Z}) \mid \det(M) = m \right\}.$$

Similar to (10) we can consider the Hecke operator with respect to the parabolic subgroup of $\Gamma_t(N)$ acting on the modular forms $\tilde{\phi}(Z) = \phi(\tau, z)e^{2\pi i t\omega}$. This gives us for any $\phi \in J^{nh}_{0,t}(\Gamma_0(N))$ the Hecke operator

$$\phi\big|_{0,t} T_N(m)(\tau, z) = m^{-1} \sum_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)\backslash M_N(m)} e^{-2i\pi m t \frac{cz^2}{c\tau+d}}\, \phi\left(\frac{a\tau+b}{c\tau+d}, \frac{mz}{c\tau+d}\right). \tag{22}$$

Then $\phi\big|_{0,t} T_N(m) \in J^{nh}_{0,mt}(\Gamma_0(N))$. This operator transfers the weak (holomorphic) Jacobi forms into weak (holomorphic) Jacobi forms.

We can write the Fourier expansion of $\phi \in J^{nh}_{0,t}(\Gamma_0(N))$ (see (6)) at the corresponding cusp f/e using $M_{f/e}$

$$(\phi\big|_{0,t} M_{f/e})(\tau, z) = \sum_{n \in \mathbb{Z}/h_e} \sum_{l \in \mathbb{Z}} c_{f/e}(n, l) q^n r^l.$$

We note that $c_{1/N}(n, l)$ is the Fourier coefficient of ϕ at infinity. For a weak Jacobi form we have $n \ge 0$ if $c_{f/e}(n, l) \neq 0$.

Theorem 3.1 *Let $\phi \in J^{nh}_{0,t}(\Gamma_0(N))$. Assume that for all cusps of $\Gamma_0(N)$ we have that $\frac{h_e}{N_e} c_{f/e}(n, l) \in \mathbb{Z}$ if $4nmt - l^2 \le 0$. Then the product*

$$B_\phi(Z) = q^A r^B s^C \prod_{f/e \in \mathcal{P}} \prod_{\substack{n,l,m \in \mathbb{Z} \\ (n,l,m) > 0}} \left(1 - (q^n r^l s^{tm})^{N_e}\right)^{\frac{h_e}{N_e} c_{f/e}(nm, l)},$$

where $(n,l,m)>0$ means that if $m>0$, then $n\in\mathbb{Z}$ and $l\in\mathbb{Z}$, if $m=0$ and $n>0$, then $l\in\mathbb{Z}$, if $m=n=0$, then $l<0$, and

$$A=\frac{1}{24}\sum_{\substack{f/e\in\mathcal{P}\\ l\in\mathbb{Z}}} h_e c_{f/e}(0,l),\quad B=\frac{1}{2}\sum_{\substack{f/e\in\mathcal{P}\\ l\in\mathbb{Z},\, l>0}} l h_e c_{f/e}(0,l),\quad C=\frac{1}{4}\sum_{\substack{f/e\in\mathcal{P}\\ l\in\mathbb{Z}}} l^2 h_e c_{f/e}(0,l),$$

defines a meromorphic modular form of weight

$$k=\frac{1}{2}\sum_{f/e\in\mathcal{P}}\frac{h_e}{N_e}\,c_{f/e}(0,0)$$

with respect to $\Gamma_t(N)^+$ with a character (or a multiplier system) χ. In particular

$$\frac{B_\phi(V_t\langle Z\rangle)}{B_\phi(Z)}=(-1)^{D_0}\quad \textit{with}\quad D_0=\sum_{f/e\in\mathcal{P}}\sum_{l\in\mathbb{Z},\ n<0}\frac{h_e}{N_e}\sigma_0(-n)c_{f/e}(n,0)$$

where $V_t\langle Z\rangle=V_t\langle\left(\begin{smallmatrix}\tau & z\\ z & \omega\end{smallmatrix}\right)\rangle=\left(\begin{smallmatrix}t\omega & z\\ z & \tau/t\end{smallmatrix}\right)$ and $\sigma_j(m)=\sum_{d|m}d^j$. The poles and zeros of B_ϕ lie on the rational quadratic divisors defined by the Fourier coefficients $c_{f/e}(n,l)$ with $4nmt-l^2<0$. In particular B_ϕ is holomorphic if all such coefficients are positive. The character χ is induces by the following relations

$$\chi(\widetilde{M})=\prod_{f/e\in\mathcal{P}}\left(v_\eta^{(N_e)}(M)\right)^{\frac{h_e}{N_e}\sum_{l\in\mathbb{Z}}c_{f/e}(0,l)}$$

for $M\in\Gamma_0(N_e)$ where $v_\eta^{(N_e)}(M)=v_\eta(\alpha_e M\alpha_e^{-1})$ with $\alpha_e=\left(\begin{smallmatrix}N_e & 0\\ 0 & 1\end{smallmatrix}\right)$ and

$$\chi([\lambda,\mu;0])=\prod_{f/e\in\mathcal{P},\ l>0} v_{H,N_e}^{\frac{h_e}{N_e}lc_{f/e}(0,l)}([\lambda,\mu;0])$$

where $v_{H,N_e}([\lambda,\mu;0])=(-1)^{\lambda+N_e\mu+N_e\lambda\mu}$ for $\lambda,\ \mu\in\mathbb{Z}$ and for all $\kappa\in\mathbb{Z}$ $\chi([0,0;\frac{\kappa}{t}])=e^{2i\pi C\kappa/t}$.

Proof. The paramodular group Γ_t can be realized as the stable orthogonal group of the lattice $2U\oplus\langle -2t\rangle$ of signature $(2,3)$ where $U=\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ is the hyperbolic plane (see [G1], [GH2]). Using the similar arguments we can realize $\Gamma_t(N)$ as a subgroup of the orthogonal group of the lattice $U\oplus U(N)\oplus\langle -2t\rangle$ where $U(N)=\left(\begin{smallmatrix}0&N\\N&0\end{smallmatrix}\right)$. The product of the theorem is a specialization of the Borcherds automorphic product considered in [B1, Theorem 13.3]. It converges if $Y=\operatorname{Im}Z$ lies in a Weyl chamber determined by the action of $\Gamma_0(N)$ on $Y>0$ with $\det(Y)>C$ for a sufficiently large C. The product can be extended to a meromorphic function on $\mathbb{H}_2$ whose poles and zeros lie on rational quadratic divisor of $\mathbb{H}_2$. We define below the invariants (the modular group, the weight, the character, the first and the second Fourier-Jacobi coefficients) of this modular form in terms of the Fourier coefficients of the lifted Jacobi form of weight 0 using a representation similar to [GN1]–[GN2].

We have the following decomposition (see [He])

$$\Gamma_0(N)\backslash M_N(m) = \tag{23}$$
$$\bigsqcup_{f/e\in\mathcal{P}} \left\{ M_{f/e} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mid ad = m, \quad ae \equiv 0 \bmod N, \quad b \bmod h_e d \right\}.$$

For $\phi \in J_{0,t}^{nh}(\Gamma_0(N))$ and $Z = \left(\begin{smallmatrix} \tau & z \\ z & \omega \end{smallmatrix}\right) \in \mathbb{H}_2$ we set

$$L_\phi(Z) = \sum_{m=1}^{\infty} \phi|_{0,t} T_N(m)(\tau, z) e^{2i\pi t m\omega}. \tag{24}$$

Using the decomposition of $\Gamma_0(N)\backslash M_N(m)$ and the formula for the action of $T_N(m)$ (see the proof of Theorem 2.2) we have (whenever the product converges):

$$\mathrm{Exp}(-L_\phi(Z)) = \prod_{f/e\in\mathcal{P}} \prod_{\substack{m\geqslant 1 \\ n,\, l\in\mathbb{Z}}} \left(1 - (q^n r^l s^{tm})^{N_e}\right)^{\frac{h_e}{N_e} c_{f/e}(nm,l)}.$$

This product is invariant with respect to the action of the Jacobi group. We introduce one more factor(the "zeroth" Hecke operator or the Hodge correction in the geometric terms of [G4])

$$T_\phi^{(0)}(Z) = \prod_{f/e\in\mathcal{P}} \eta(N_e\tau)^{\frac{h_e}{N_e} c_{f/e}(0,0)} \prod_{l>0} \left(\frac{\vartheta(N_e\tau, N_e l z)}{\eta(N_e\tau)}\, e^{i\pi N_e l^2 \omega} \right)^{\frac{h_e}{N_e} c_{f/e}(0,l)}. \tag{25}$$

So as in [GN2, (2.7)] we obtain that

$$B_\phi(Z) = T_\phi^{(0)}(Z) \cdot \mathrm{Exp}(-L_\phi(Z)). \tag{26}$$

The additional term $T_\phi^{(0)}(Z)$ is a nearly holomorphic Jacobi form of weight k indicated in the theorem and of index $C \in \mathbb{N}/2$ with respect to $\Gamma_0(N)$. This is *the first Fourier-Jacobi coefficient* of the automorphic product B_ϕ. (It might be that this is a Jacobi form of index zero, i.e., an automorphic form in τ.) The Jacobi form is a modular form with respect to the parabolic subgroup $\Gamma_t^\infty(N)$. Like in the proof of Theorem 2.2 we use that $\Gamma_t(N)^+ = \langle \Gamma_t^\infty(N), V_t \rangle$. We have to analyze the behavior of B_ϕ under V_t-action. Like in [GN2] a straightforward calculation shows that

$$\frac{B_\phi(V_t\langle Z\rangle)}{B_\phi(Z)} = (-1)^{D_0} (q^{1/t} s^{-1})^{tD_1 + C - tA}$$

where D_0 is given in the theorem and

$$D_1 = \sum_{f/e\in\mathcal{P}} \sum_{l\in\mathbb{Z},\ n<0} h_e \sigma_1(-n) c_{f/e}(n,l).$$

We note that in [AI] the approach of [GN1]–[GN2] was also used. But in [AI] it was not proved that $tD_1 + C - tA = 0$. To show that the automorphic product is V_t-invariant we prove Lemma 3.2 (see below) similar to [GN2, Lemma 2.2]. We note also that the automorphic product of the theorem is defined at the "standard" 0-dimensional cusp ∞ of $\Gamma_t(N)$. If $N = 1$ then the Γ_t-orbit of any rational quadratic divisor (a Humbert modular surface) has a representative containing ∞ (see [GH2] and [GN2]). If $N > 1$ then there are more orbits. Not all of them have a non-trivial intersection with infinity. Therefore the arguments in the construction of some examples of the automorphic products in [AI] are not complete. One has to use [B1, Theorem 13.3] in the proof. We add that as we mentioned in the beginning of this proof we do not agree with [AI, page 262] that "$\Gamma_0^{(2)}(N)$ is not an automorphism group of a lattice."

Lemma 3.2 *For any $\phi \in J^{nh}_{0,t}(\Gamma_0(N))$ we have $tD_1 + C - tA = 0$.*

Proof. We give a proof based on the method of the automorphic correction proposed in [G4] which is more simple than the proof of [GN2, Lemma 2.2]. For any $\phi \in J^{nh}_{k,t}(\mathrm{SL}_2(\mathbb{Z}))$, we consider the following automorphic correction of ϕ:

$$\Psi(\tau,z) = e^{-8\pi^2 t G_2(\tau)}\phi(\tau,z) \quad \text{where} \quad G_2(\tau) = -\frac{1}{24} + \sum_{n\geqslant 1}\sigma_1(n)q^n$$

is the quasi-modular Eisenstein series of weight 2. The corrected form satisfies the functional equation

$$\Psi(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}) = (c\tau+d)^k\Psi(\tau,z), \qquad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

We consider the Taylor expansion of Ψ around $z = 0$

$$\Psi(\tau,z) = \sum_{\nu\geqslant 0} f_\nu(\tau) z^\nu.$$

The Taylor coefficient $f_\nu \in M^{(mer)}_{k+\nu}(\mathrm{SL}_2(\mathbb{Z}))$ are modular forms with a possible pole of finite order at the cusp. If $\phi(\tau,z) = \sum_{n\in\mathbb{Z}}\sum_{l\in\mathbb{Z}} c(n,l)q^n r^l$ is in $J^{nh}_{0,t}(\mathrm{SL}_2(\mathbb{Z}))$, then

$$f_2(\tau) = \frac{\partial^2\phi}{\partial z^2}(\tau,0) - 16\pi^2 t G_2(\tau)\phi(\tau,0) \in M^{(mer)}_2(SL_2(\mathbb{Z})).$$

But the constant term of any nearly holomorphic modular form of weight two is zero (see [B2, Lemma 9.2]). Therefore

$$t\sum_{l\in\mathbb{Z}} c(0,l) - 24t\sum_{\substack{n<0\\ l\in\mathbb{Z}}} \sigma_1(-n)c(n,l) - 6\sum_{l\in\mathbb{Z}} l^2c(0,l) = 0. \tag{27}$$

For a Jacobi form with respect to a congruence subgroup we use the trace operator. Let $\phi\in J_{0,t}^{nh}(\Gamma_0(N))$. We set

$$\psi = \mathrm{Tr}_{\mathrm{SL}_2(\mathbb{Z})}(\phi) = \sum_{\gamma\in\Gamma_0(N)\setminus\mathrm{SL}_2(\mathbb{Z})} \phi\big|_{0,t}\gamma \in J_{0,t}^{nh}(\mathrm{SL}_2(\mathbb{Z})).$$

This Jacobi form has the following Fourier expansion

$$\psi(\tau,z) = \sum_{n\in\mathbb{Z}}\sum_{l\in\mathbb{Z}} c(n,l)q^nr^l \quad \text{where} \quad c(n,l) = \sum_{f/e\in\mathcal{P}} h_e c_{f/e}(n,l).$$

The last expression is obtained by noticing that

$$\mathrm{SL}_2(\mathbb{Z}) = \bigsqcup_{f/e\in\mathcal{P},\ 0\leqslant a\leqslant h_e-1} \Gamma_0(N)M_{f/e}\begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix}.$$

The claim of the lemma follows from (27). □

The formula for the character of $\Gamma_t(N)^+ = \langle\Gamma_t(N), V_t\rangle$ follows directly form the calculation with η- and ϑ-factors in $T_\phi^{(0)}$ (see (25)) which is the first Fourier-Jacobi coefficient of B_ϕ. More exactly the SL_2-part of the character (or the multiplier system) of this Jacobi form is equal to the character of the η-product

$$\prod_{f/e\in\mathcal{P}} \eta(N_e\tau)^{\frac{h_e}{N_e}\sum_l c_{f/e}(0,l)}$$

which is a $\Gamma_0(N)$-modular form because N_e is a divisor of N. Its character is the character $\chi(\widetilde{M})$ of the theorem. The Heisenberg part of the character of the Jacobi form $\vartheta(M\tau, Mz)$ of index $M/2$ is equal to

$$v_{H,M}([\lambda,\mu;0]) = v_H([\lambda, M\mu; 0]) = (-1)^{\lambda+M\mu+M\lambda\mu}.$$

It gives us the Heisenberg part of the character. We note that the second Fourier-Jacobi coefficient is equal to $T_\phi^{(0)}\cdot\widetilde{\phi}$. We note that if a Siegel modular form F is a Borcherds automorphic product B_ϕ we can find ϕ taking the quotient of the first two non-zero Fourier–Jacobi coefficients of F. □

In order to obtain the Borcherds products for the dd-modular forms we propose a method of construction of weak Jacobi forms of weight 0 for $\Gamma_0(N)$ using the Jacobi theta-series with characteristics (see [Mu]). Let $N\in\mathbb{N}$ and

$(a,b)\in\mathbb{Z}^2$. We call the theta-series of level N with characteristic (a,b), the series

$$\vartheta_{a,b}^{(N)}(\tau,z)=\sum_{n\in\mathbb{Z}} e^{i\pi(n+\frac{a}{N})^2\tau+2i\pi(n+\frac{a}{N})(z+\frac{b}{N})}. \tag{28}$$

This is a holomorphic function on $\mathbb{H}_1\times\mathbb{C}$. Among these series, there is a special one for $(a,b)=(0,0)$

$$\vartheta_{00}(\tau,z)=\sum_{n\in\mathbb{Z}} e^{i\pi n^2\tau+2i\pi nz}=\prod_{n\geqslant 1}(1-q^n)(1+q^{\frac{2n-1}{2}}r)(1+q^{\frac{2n-1}{2}}r^{-1}).$$

All the theta-series with characteristics can be expressed by the mean of ϑ_{00}

$$\vartheta_{a,b}^{(N)}(\tau,z)=e^{2i\pi\frac{ab}{N^2}}q^{\frac{a^2}{2N^2}}r^{\frac{a}{N}}\vartheta_{00}(\tau,z+\frac{a}{N}\tau+\frac{b}{N}).$$

We also have for any $(a',b')\in\mathbb{Z}^2$

$$\vartheta_{a+a'N,b+b'N}^{(N)}(\tau,z)=e^{2i\pi\frac{ab'}{N}}\vartheta_{a,b}^{(N)}(\tau,z).$$

The last formula allows us to take the characteristics (a,b) modulo N.

To construct Jacobi forms of weight 0 we consider quotients of theta-series. We put

$$\xi_{a,b}^{(N)}(\tau,z)=\frac{\vartheta_{a,b}^{(N)}(\tau,z)}{\vartheta_{a,b}^{(N)}(\tau,0)}.$$

This function is holomorphic on $\mathbb{H}_1\times\mathbb{C}$ for any (a,b) if N is odd. For N even, as $\vartheta_{00}(\tau,\frac{\tau}{2}+\frac{1}{2})=0$, we cannot make the quotient $\xi_{\frac{N}{2},\frac{N}{2}}^{(N)}$. When we write $\xi_{a,b}^{(N)}$ for even N then we assume that $(a,b)\neq(\frac{N}{2},\frac{N}{2})$. In fact $\xi_{a,b}^{(N)}$ is a weak Jacobi forms of weight 0 and index 1/2 with respect to the principal congruence subgroup of level N (see (5)):

$$\xi_{a,b}^{(N)}|_{0,\frac{1}{2}}M=\xi_{a,b}^{(N)},\qquad M\in\Gamma(N). \tag{29}$$

More exactly we have the following functional equations with respect to the generators of the full Jacobi group (see §2)

$$\xi_{a,b}^{(N)}|_{0,\frac{1}{2}}[\lambda,\mu;0]=e^{2i\pi\frac{a}{N}\mu}e^{-2i\pi\frac{b}{N}\lambda}\xi_{a,b}^{(N)},\qquad (\lambda,\mu)\in\mathbb{Z}^2,$$

$$\xi_{a,b}^{(N)}|_{0,\frac{1}{2}}S=\xi_{b,\overline{-a}}^{(N)},\qquad S=\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$$

where $\overline{-a}$ is the unique representant of $-a$ modulo N such that $\overline{-a}\in\{0,...,N-1\}$,

$$\xi_{a,b}^{(2N)}|_{0,\frac{1}{2}}T=\xi_{a,\overline{a+b+N}}^{(2N)},\qquad T=\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$$

where $\overline{a+b+N}$ is the unique representant of $a+b+N$ modulo $2N$ such that $\overline{a+b+N} \in \{0,...,2N-1\}$ and

$$\xi^{(2N'+1)}_{a,b}\Big|_{0,\frac{1}{2}} T = \xi^{(4N'+2)}_{2a,\overline{2(a+b+N')+1}}$$

where $\overline{2(a+b+N')+1}$ is the unique representant of $2(a+b+N')+1$ modulo $4N'+2$ such that $\overline{2(a+b+N')+1} \in \{0,...,4N'+1\}$. These formulae lead us to construct weak Jacobi forms for $\Gamma_0(N)$ in the following way:
(i) we consider the quotient group $G = \Gamma(N)\backslash\Gamma_0(N)$ if N is even or $G = \Gamma(2N)\backslash\Gamma_0(N)$ if N is odd since according to the T-transformation formula we have to double the level;
(ii) we compute the orbits of $\xi^{(N)}_{a,b}$ under G;
(iii) in a fixed orbit of $\xi^{(N)}_{a,b}$, we take some powers of elements or products of them in order to obtain the trivial character of the Jacobi group.

In this paper we only construct the Jacobi forms of weight 0 which generate dd-modular forms. We are planing to obtain results similar to [G4] about the structure of the graded rings of weak Jacobi forms with respect to $\Gamma_0(N)$ for small N in a separate publication.

Examples 3.3 $\mathbf{N=2}$. We have $G = \{I_2, T\}$ (the group of order two) and the orbit $O_G(\xi^{(2)}_{1,0})$ contains the only element $\xi^{(2)}_{1,0}$. The formula for the $[\mu,\nu;0]$-action implies that $\xi^{(2)}_{1,0}$ has a character of order two. Therefore

$$(\xi^{(2)}_{1,0})^2 \in J^w_{0,1}(\Gamma_0(2)).$$

$\mathbf{N=3}$. In this case G is non abelian group of order 36. It contains the set $\Sigma = \{\pm T^k, \pm ST^3ST^k, 0 \leqslant k \leqslant 5\}$. Therefore $O_G(\xi^{(6)}_{3,1}) \supseteq O_\Sigma(\xi^{(6)}_{3,1}) = \{\xi^{(6)}_{3,1}, \xi^{(6)}_{3,5}\}$ and using the standard generators of $\Gamma_0(3)$ we have equality. Therefore

$$\xi^{(6)}_{3,1} \cdot \xi^{(6)}_{3,5} \in J^w_{0,1}(\Gamma_0(3)).$$

$\mathbf{N=4}$. We have that $G = \{I_2, T, T^2, T^3, -I_2, TST^4S, TST^4ST, TST^4ST^2\}$ is the group of order 8 isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$. We see that $O_G(\xi^{(4)}_{0,1}) = \{\xi^{(4)}_{0,1}, \xi^{(4)}_{0,3}\}$ and $O_G(\xi^{(4)}_{2,1}) = \{\xi^{(4)}_{2,1}, \xi^{(4)}_{2,3}\}$. Therefore

$$\xi^{(4)}_{0,1} \cdot \xi^{(4)}_{0,3}, \quad \text{and} \quad \xi^{(4)}_{2,1} \cdot \xi^{(4)}_{2,3} \in J^w_{0,1}(\Gamma_0(4)).$$

The dd-modular forms as Borcherds products. Now we can finish the proof of Theorem 1.4 and to construct the last dd-modular form $\nabla_{3/2}$ of weight 3/2 with respect to $\Gamma^{(2)}_0(4)$. Moreover we give the Borcherds automorphic product for all new Siegel dd-modular forms with respect to the congruence subgroups. (The Borcherds products of the dd-modular forms for the full paramodular group were found in [GN1]–[GN2].)

We start with $N = 2$. Let

$$\phi_2(\tau, z) = 4(\xi^{(2)}_{1,0})^2(\tau, z) \in J^w_{0,1}(\Gamma_0(2)).$$

There are two cusps and ϕ_2 has the two Fourier expansions with integral Fourier coefficients

$$\phi_2(\tau, z) = (r^{-1} + 2 + r) + 2(r^{-2} - 2 + r^2)q + \cdots = \sum_{n\in\mathbb{N},\, l\in\mathbb{Z}} c(n, l)q^n r^l,$$

$$(\phi_2|_{0,1}S)(\tau, z) = 4 - 8(r^{-1} - 2 + r)q^{\frac{1}{2}} + \cdots = \sum_{n\in\frac{1}{2}\mathbb{N},\, l\in\mathbb{Z}} c_S(n, l)q^n r^l.$$

The only orbit of the Fourier coefficients with negative hyperbolic norm $4nt - l^2$ of its index is $c(0, 1) = 1$. Then applying Theorem 3.1 to ϕ_2, we obtain

$$B_{\phi_2}(Z) = q^{\frac{1}{2}} r^{\frac{1}{2}} s^{\frac{1}{2}} \prod_{(n,l,m)>0} (1 - q^n r^l s^m)^{c(nm,l)}(1 - q^{2n} r^{2l} s^{2m})^{c_S(nm,l)}$$

$$= \eta(\tau)\eta(2\tau)^4\vartheta(\tau, z)e^{i\pi\omega} \cdot \mathrm{Exp}(-L_{\phi_2})(Z).$$

This is a holomorphic Siegel modular form of weight 3 with respect to $\Gamma_0(2)$. According to the Koecher principle a Siegel dd-modular form is defined up to a constant. Comparing the first Fourier coefficients we obtain

$$\nabla_3(Z) = \mathrm{Lift}(\eta(\tau)\eta(2\tau)^4\vartheta(\tau, z)) = B_{\phi_2}(Z).$$

For $N = 3$ we take

$$\phi_3(\tau, z) = 3(\xi^{(6)}_{3,1}\xi^{(6)}_{3,5})(\tau, z) \in J^w_{0,1}(\Gamma_0(3)).$$

We again have two Fourier expansions containing only integral Fourier coefficients

$$\phi_3(\tau, z) = (r^{-1} + 1 + r) + (r^{-2} - r^{-1} - r + r^2)q + \cdots = \sum_{n\in\mathbb{N},\, l\in\mathbb{Z}} c(n, l)q^n r^l,$$

$$(\phi_3|_{0,1}S)(\tau, z) = 3 - 3(r^{-1} - 2 + r)q^{\frac{1}{3}} + \cdots = \sum_{n\in\frac{1}{3}\mathbb{N},\, l\in\mathbb{Z}} c_S(n, l)q^n r^l.$$

The both Fourier expansions contain only one type of coefficients with negative norm of its index. This is $c(0, 1) = 1$. According to Theorem 3.1 we obtain

$$B_{\phi_3}(Z) = q^{\frac{1}{2}} r^{\frac{1}{2}} s^{\frac{1}{2}} \prod_{(n,l,m)>0} (1 - q^n r^l s^m)^{c(nm,l)}(1 - q^{3n} r^{3l} s^{3m})^{c_S(nm,l)}$$

and

$$\nabla_2(Z) = \mathrm{Lift}(\eta(3\tau)^3\vartheta(\tau,z)) = B_{\phi_3}(Z).$$

The dd-modular form $\nabla_{3/2}$. The case of $N=4$ is a little bit more difficult because there are three different cusps. Let

$$\phi_4(\tau,z) = 2(\xi^{(4)}_{2,1}\xi^{(4)}_{2,3})(\tau,z) \in J^w_{0,1}(\Gamma_0(4)).$$

We have the following Fourier expansions

$$\phi_4(\tau,z) = (r^{-1}+r) + (r^{-3}-r^{-1}-r^1+r^3)q^2 + \cdots = \sum_{n\in\mathbb{N},\, l\in\mathbb{Z}} c(n,l)q^nr^l,$$

$$(\phi_4|_{0,1}S)(\tau,z) = 2 - 2(r^{-1}-2+r)q^{\frac{1}{4}} + \cdots = \sum_{n\in\frac{1}{4}\mathbb{N},\, l\in\mathbb{Z}} c_S(n,l)q^nr^l,$$

$$(\phi_4|_{0,1}M)(\tau,z) = 2 + 2(r^{-2}-2+r^2)q + \cdots = \sum_{n\in\mathbb{N},\, l\in\mathbb{Z}} c_M(n,l)q^nr^l$$

where $M = \left(\begin{smallmatrix} 1 & -1 \\ 2 & -1\end{smallmatrix}\right)$. All Fourier coefficients are integral and there exists the only type of coefficients with negative index norm $c(0,1)=1$. We obtain the Siegel modular form $\nabla_{3/2} = B_{\phi_4}$ of weight $3/2$ for $\Gamma_0^{(2)}(4)$ given by

$$\nabla_{3/2}(Z) = B_{\phi_4}(Z) = \frac{\eta(2\tau)\eta(4\tau)^2}{\eta(\tau)}\,\vartheta(\tau,z)e^{i\pi\omega}\,\mathrm{Exp}(-L_{\phi_4})(Z) =$$

$$q^{\frac{1}{2}}r^{\frac{1}{2}}s^{\frac{1}{2}}\prod_{(n,l,m)>0}(1-q^nr^ls^m)^{c(nm,l)}(1-q^{2n}r^{2l}s^{2m})^{\frac{1}{2}c_M(nm,l)}(1-q^{4n}r^{4l}s^{4m})^{c_S(nm,l)}.$$

The modular form $\nabla_{3/2}$ is the last Siegel dd-modular form which we need in order *to finish the proof of Theorem 1.4*. Using the Koecher principle we obtain that

$$\nabla_{3/2}(Z)^2 = B^2_{\phi_4}(Z) = \mathrm{Lift}\big(\frac{\eta(2\tau)^2\eta(4\tau)^4}{\eta(\tau)^2}\,\vartheta(\tau,z)^2\big).$$

The last example is the automorphic product of the dd-modular form Q_1 with $N=2$, $t=2$ and $k=1$. Let

$$\psi(\tau,z) = 2\xi^{(2)}_{1,0}(\tau,2z) \in J^w_{0,2}(\Gamma_0(2)).$$

We have the following Fourier expansions

$$\psi(\tau,z) = (r^{-1}+r) + (r^{-3}-r^{-1}-r+r^3)q + \cdots = \sum_{n\in\mathbb{N},\, l\in\mathbb{Z}} c(n,l)q^nr^l,$$

$$(\psi\big|_{0,2}S)(\tau,z) = 2-2(r^{-2}-2+r^2)q^{\frac{1}{2}}-4(r^{-2}-2+r^2)q^1-8(r^{-2}-2+r^2)q^{\frac{3}{2}}+\ldots$$

$$= \sum_{n\in\frac{1}{2}\mathbb{N},\, l\in\mathbb{Z}} c_S(n,l)q^n r^l.$$

Then applying Theorem 3.1, we obtain

$$B_\psi(Z) = q^{\frac{1}{4}} r^{\frac{1}{2}} s^{\frac{1}{2}} \prod_{(n,l,m)>0} (1-q^n r^l s^{2m})^{c(nm,l)}(1-q^{2n}r^{2l}s^{4m})^{c_S(nm,l)}$$

$$= Q_1(Z) = \operatorname{Lift}\big(\frac{\eta(2\tau)^2}{\eta(\tau)}\vartheta(\tau,z)\big).$$

A traced form of Borcherds product and reflective modular forms. For each dd-modular form we have the identity between the known (due to the Jacobi lifting) Fourier expansion and the Borcherds products. We note that such examples are rather rare. Below we give more examples of this type analyzing new reflective modular forms, i.e., the modular forms with divisor determined by some reflections in the corresponding modular group (see [GN2]–[GN3]). Every dd-modular form is reflective. We construct new examples as the quotient of dd-modular forms. To represent the quotient of two dd-modular functions in a better form we give a new representation for the automorphic product in Theorem 3.1. For that we rewrite the full Hecke operator $T_N(m)$ using the summation with respect to the classes from the same subgroup $\Gamma_0(N_a)$ where $N_a = N/(a,N)$:

$$T_N(m) = \sum_{a|m} \sum_{M\in\Gamma_0(N)\backslash\Gamma_0(N_a)} \sum_{b \bmod m/a} \Gamma_0(N)M\begin{pmatrix} a & b\\ 0 & m/a\end{pmatrix}.$$

Let us reorganize the formal Hecke sum $L_T = \sum_{m=1}^{\infty} m^{-1}T_N(m)$ using the last representation. Formally we have

$$L_T = \sum_{e|N} \sum_{\substack{a'\ge 1\\ (a',N_e)=1\\ (a=ea')}} \sum_{M\in\Gamma_0(N)\backslash\Gamma_0(N_e)} \sum_{\substack{n\ge 1\\ (m=an)}} (an)^{-1} \sum_{b \bmod n} \Gamma_0(N)M\begin{pmatrix} a & 0 & b & 0\\ 0 & an & 0 & 0\\ 0 & 0 & n & 0\\ 0 & 0 & 0 & 1\end{pmatrix}.$$

We can rewrite the last class as

$$\Gamma_0(N)M\begin{pmatrix} \frac{a}{e} & 0 & b & 0\\ 0 & \frac{an}{e} & 0 & 0\\ 0 & 0 & n & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\cdot\begin{pmatrix} e & 0 & 0 & 0\\ 0 & e & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}.$$

Therefore we have a new representation for (24)

$$L_\phi(Z) = \sum_{e|N}\sum_{m\ge 1} e^{-1}\big(\widetilde{\psi}_{N_e}|_0\, T_-^{(N_e)}(m)\big)(eZ) = \sum_{e|N} e^{-1} L_{\psi_{N_e}}(eZ) \qquad (30)$$

where

$$\widetilde{\psi}_{N_e}(Z) = \psi_{N_e}(\tau, z)e^{2\pi i t\omega} = \mathrm{Tr}_{\Gamma_0(N_e)}\widetilde{\phi}(Z) = \sum_{M\in\Gamma_0(N)\backslash\Gamma_0(N_e)} \big(\widetilde{\phi}\,|_0\,\widetilde{M}\big)(Z)$$

is a Jacobi form of weight 0 and index t with respect to $\Gamma_0(N_e)$ and $T_-^{(N_e)}(m)$ is the Hecke operator which we used in the additive lifting in §2.

$$T^{(N_e)}(m) = \sum_{\substack{ad=m,\ (a,N_e)=1\\ b \bmod d}} \Gamma_0(N_e)\begin{pmatrix} a & b\\ 0 & d\end{pmatrix}.$$

We consider the Fourier expansion of the traced Jacobi form $\mathrm{Tr}_{\Gamma_0(N_e)}\widetilde{\phi}$ at infinity

$$\psi_{N_e}(\tau, z) = \sum_{n,\, l\in\mathbb{Z}} f_{N_e}(n, l)q^n r^l.$$

We note that for $e = N$ we have $\psi_1 = \phi$ and $f_1(n, l)$ is the Fourier coefficient of ϕ at infinity denoted by $c_{1/N}(n, l)$ in Theorem 3.1. As in the proof of Theorem 3.1 we have

$$e^{-1}L_{\psi_{N_e}}(eZ) = \sum_{\substack{m\ge1\\ n,l\in\mathbb{Z}}}\ \sum_{\substack{a\ge1\\ (a,N_e)=1}} \frac{1}{ae} f_{N_e}(mn, l)\left(q^n r^l s^{tm}\right)^{ea}$$

and

$$\sum_{\substack{m\ge1\\ (m,N)=1}} \frac{x^m}{m} = -\sum_{b|N} \frac{\mu(b)}{b}\mathrm{Log}(1 - x^b)$$

where μ stands for the Moebius function. Therefore

$$L_\phi(Z) = -\sum_{e|N}\ \sum_{b|N_e}\ \sum_{\substack{m\ge1\\ n,\, l\in\mathbb{Z}}} \mathrm{Log}\left(1 - \left(q^n r^l s^{tm}\right)^{be}\right)^{\mu(b)\frac{f_{N_e}(mn,l)}{be}}.$$

The advantage of this new representation of the Borcherds product is evident. We use in it only the Fourier expansion of the traced Jacobi forms ϕ_{N_e} at infinity. For the group $\Gamma_0^{(2)}(p)$ this expression contains only two functions and one of them is well known.

$$\mathrm{Tr}_{\mathrm{SL}_2(\mathbb{Z})} : J_{0,1}^w(\Gamma_0(p)) \to J_{0,1}^w(\mathrm{SL}_2(\mathbb{Z})) = \mathbb{C}\phi_{0,1}$$

where

$$\phi_{0,1}(\tau, z) = -\frac{3}{\pi^2}\frac{\wp(\tau, z)\vartheta(\tau, z)^2}{\eta(\tau)^6} = \sum_{n\ge0,\, l\in\mathbb{Z}} a(n, l)q^n r^l = (r + 10 + r^{-1}) + \dots$$

is one of the main generators of the graded ring of weak Jacobi forms (see [EZ], [G4]). We note that $\phi_{0,1}$ is the elliptic genus of Enriques surfaces and $2\phi_{0,1}$ is the elliptic genus of K3 surfaces. For any $\phi_p \in J_{0,t}^{nh}(\Gamma_0(p))$ we have

$$\operatorname{Exp}(-L_{\phi_p}(Z)) = \prod_{\substack{m\ge 1\\ n,\,l\in\mathbb{Z}}} \bigl(1-q^n r^l s^{tm}\bigr)^{c_{\phi_p}(nm,l)} \bigl(1-q^{pn} r^{pl} s^{pmt}\bigr)^{\frac{1}{p}(f(nm,l)-c_{\phi_p}(nm,l))}$$

where $c_{\phi_p}(n,l)$ and $f(n,l)$ are the Fourier coefficients of ϕ_p and $\operatorname{Tr}_{\mathrm{SL}_2(\mathbb{Z})}(\phi_p)$ at infinity.

Let us consider ∇_3 $(N=2)$ and ∇_2 $(N=3)$. By comparing the Fourier expansions we conclude that

$$\phi_{0,1} = \operatorname{Tr}_{\mathrm{SL}_2(\mathbb{Z})}\phi_2 = 4(\xi_{1,0}^{(2)})^2 + 4(\xi_{0,1}^{(2)})^2 + 4(\xi_{0,0}^{(2)})^2,$$

$$\phi_{0,1} = \operatorname{Tr}_{\mathrm{SL}_2(\mathbb{Z})}\phi_3 = 3(\xi_{3,1}^{(6)}\xi_{3,5}^{(6)}) + 3(\xi_{1,3}^{(6)}\xi_{5,3}^{(6)}) + 3(\xi_{1,1}^{(6)}\xi_{5,5}^{(6)}) + 3(\xi_{1,5}^{(6)}\xi_{5,1}^{(6)}).$$

Moreover

$$J_{0,1}^{w}(\Gamma_0(2)) = \langle \phi_{0,1},\, \phi_2\rangle_{\mathbb{C}} \quad\text{and}\quad J_{0,1}^{w}(\Gamma_0(3)) = \langle \phi_{0,1},\, \phi_3\rangle_{\mathbb{C}}.$$

Therefore for $p=2$ or 3

$$\operatorname{Exp}(-L_{\phi_p}(Z)) = \prod_{\substack{m\ge 1\\ n,\,l\in\mathbb{Z}}} \bigl(1-q^n r^l s^{pm}\bigr)^{c_{\phi_p}(nm,l)} \bigl(1-q^{pn} r^{pl} s^{pm}\bigr)^{\frac{1}{p}(a(nm,l)-c_{\phi_p}(nm,l))}$$

where $a(n,l)$ is the Fourier coefficient of $\phi_{0,1}$.

Using this approach we can easy calculate the product formulae for new reflective modular forms of weight 2 for $\Gamma_0(2)$, weight 3 for $\Gamma_0(3)$, weight 3/2 and 7/2 for $\Gamma_0(4)$ and weight 1 for $\Gamma_2(2)$:

$$\frac{\Delta_5(2Z)}{\nabla_3(Z)},\quad \frac{\Delta_2(2Z)}{Q_1(Z)},\quad \frac{\nabla_3(2Z)}{\nabla_{3/2}(Z)},\quad \frac{\Delta_5(2Z)}{\nabla_{3/2}(Z)}$$

and

$$\frac{\Delta_5(Z)}{\nabla_3(Z)},\quad \frac{\Delta_5(Z)}{\nabla_2(Z)},\quad \frac{\Delta_2(Z)}{Q_1(Z)},\quad \frac{\nabla_3(Z)}{\nabla_{3/2}(Z)},\quad \frac{\Delta_5(Z)}{\nabla_{3/2}(Z)}.$$

The dd-modular forms and all these reflective modular forms are related to Lorentzian Kac–Moody super Lie algebras of Borcherds type. This object will be similar to the algebras constructed in [GN1]–[GN4]. We are planning to consider them in a separate publication.

Using the formula $\Delta_5 = B_{\phi_{0,1}}$ (see [GN2, (2.16)]) and the trace formula for ϕ_2, we deduce an infinite product expansion

$$\frac{\Delta_5(Z)}{\nabla_3(Z)} = \frac{\eta(\tau)^8}{\eta(2\tau)^4} \prod_{\substack{m\ge 1\\ n,\,l\in\mathbb{Z}}} \left(\frac{1-q^n r^l s^m}{1+q^n r^l s^m}\right)^{\frac{1}{2}((a(nm,l)-c_{\phi_2}(nm,l))}$$

where $a(n,l)$ and $c_{\phi_2}(n,l)$ are respectively the Fourier coefficients of $\varphi_{0,1}$ and ϕ_2 at ∞. For $N=3$ we obtain

$$\frac{\Delta_5(Z)}{\nabla_2(Z)}=\frac{\eta(\tau)^9}{\eta(3\tau)^3}\prod_{\substack{m\ge 1\\ n,\,l\in\mathbb{Z}}}(1-q^nr^ls^m)^{b(nm,l)}(1-q^{3n}r^{3l}s^{3m})^{-\frac{1}{3}b(nm,l)}$$

where $b(n,l)=a(nm,l)-c_{\phi_3}(nm,l)$. The both modular forms are holomorphic because the divisor of $\Delta_5(Z)$ is larger than the divisor of $\nabla_3(Z)$ or $\nabla_2(Z)$. They are non-cusp forms because the zeroth Fourier-Jacobi coefficient is non zero.

Analyzing the examples of the reflective modular forms constructed above and in [GN2]–[GN3] we see that the first non-zero coefficient of the Taylor expansion of a reflective form F at $z=0$ is an η-product or an η-quotient of the type considered by J. McKay and Y. Martin (see [Ma]). We can assume that *every η-quotients of this type is the first coefficient of a Taylor expansion of some power of a reflective modular form.*

The reflective modular forms in the first line above are more regular. Then we have

$$\frac{\Delta_5(2Z)}{\nabla_3(Z)}=\widetilde{\phi}_{2,\frac{1}{2}}(Z)\prod_{\substack{m\ge 1\\ n,\,l\in\mathbb{Z}}}(1-q^{2n}r^{2l}s^{2m})^{\frac{1}{2}(a(nm,l)+c_{\phi_2}(nm,l))}(1-q^nr^ls^m)^{-c_{\phi_2}(nm,l)}$$

where

$$\phi_{2,\frac{1}{2}}(\tau,z)=\frac{\eta(2\tau)^5}{\eta(\tau)}\frac{\vartheta(2\tau,2z)}{\vartheta(\tau,z)}\in J_{2,\frac{1}{2}}(\Gamma_0(2),\chi_2)$$

is a Jacobi cusp form of weight 2 with a character of order 2. More exactly, $\chi_2(\left(\begin{smallmatrix}a&b\\2c&d\end{smallmatrix}\right))=(-1)^b$ and $\chi_2([\lambda,\mu;0])=(-1)^\lambda$. This reflective form and its square are the lifting of the first Fourier–Jacobi coefficient

$$\frac{\Delta_5(2Z)}{\nabla_3(Z)}=\mathrm{Lift}(\phi_{2,\frac{1}{2}})\in M_2(\Gamma_0^{(2)}(2),\chi_2),$$

$$\frac{\Delta_5(2Z)^2}{\nabla_3(Z)^2}=\mathrm{Lift}(\phi_{2,\frac{1}{2}}^2)\in M_4(\Gamma_0^{(2)}(2)).\tag{31}$$

We have a similar formula for $N=3$

$$\frac{\Delta_5(3Z)}{\nabla_2(Z)}=\mathrm{Lift}(\phi_{3,1})\in M_3(\Gamma_0^{(2)}(3),\left(\frac{\det D}{3}\right))\tag{32}$$

where

$$\phi_{3,1}(\tau,z)=\eta(3\tau)^6\,\frac{\vartheta(3\tau,3z)}{\vartheta(\tau,z)}\in J_{3,1}(\Gamma_0(3),\left(\frac{d}{3}\right)),$$

$$\frac{\Delta_5(3Z)}{\nabla_2(Z)}=\widetilde{\phi}_{3,1}(Z)\prod_{\substack{m\ge 1\\ n,\,l\in\mathbb{Z}}}(1-q^{3n}r^{3l}s^{3m})^{\frac{1}{3}(2a(nm,l)+c_{\phi_3}(nm,l))}(1-q^nr^ls^m)^{-c_{\phi_3}(nm,l)}.$$

For $N=4$ we get two new traced functions defined by $\phi_4 = 2\xi^{(4)}_{2,1}\xi^{(4)}_{2,3}$. They are

$$\phi_{0,1} = \mathrm{Tr}_{\mathrm{SL}_2(\mathbb{Z})}\phi_4 \quad \text{and} \quad \phi_2 = \mathrm{Tr}_{\Gamma_0(2)}\phi_4 = \sum_{M\in\Gamma_0(4)\backslash\Gamma_0(2)} \phi_4|_0 M.$$

To get the second identity we take into account that $\dim J^w_{0,1}(\Gamma_0(2)) = 2$. So we are able to write the infinite product expansions for the four reflective modular forms of type $\Delta_5/\nabla_{3/2}$ and $\nabla_3/\nabla_{3/2}$ from our list using the Fourier coefficients of $\phi_{0,1}$, ϕ_2 and ϕ_4 at infinity.

We finish with the case $N=2$ and $t=2$. In order to construct Q_1 we used $\psi = 2\xi^{(2)}_{1,0}(\tau, 2z)$. As before, we get only one new traced function

$$\phi_{0,2} = \mathrm{Tr}_{\mathrm{SL}_2(\mathbb{Z})}\psi = \psi + 2\xi^{(2)}_{0,1}(\tau, 2z) + 2\xi^{(2)}_{0,0}(\tau, 2z)$$

where $\phi_{0,2} \in J^w_{0,2}(\mathrm{SL}_2(\mathbb{Z}))$ is the second generator of the graded ring of the weak Jacobi forms of weight 0 with integral Fourier coefficients (see [GN2, (2.18)] and [G4]). Then we get a reflective holomorphic modular form of weight 1 with respect to $\Gamma_2(2) < \Gamma_2$

$$\frac{\Delta_2(Z)}{Q_1(Z)} = \frac{\eta(\tau)^4}{\eta(2\tau)^2} \prod_{\substack{m\geq 1\\ n,\, l\in\mathbb{Z}}} \left(\frac{1-q^n r^l s^{2m}}{1+q^n r^l s^{2m}}\right)^{\frac{1}{2}(a_2(nm,l)-c_\psi(nm,l))}$$

where $a_2(n,l)$ is the Fourier coefficient of $\phi_{0,2}$. In the same way we obtain that

$$\frac{\Delta_2(2Z)}{Q_1(Z)} = \widetilde{\phi}_{1,\frac{1}{2}}(Z) \prod_{\substack{m\geq 1\\ n,\, l\in\mathbb{Z}}} (1-q^{2n}r^{2l}s^{4m})^{\frac{1}{2}(a_2(nm,l)+c_\psi(nm,l))}(1-q^n r^l s^{2m})^{-c_\psi(nm,l)}$$

where

$$\phi_{1,\frac{1}{2}}(\tau, z) = \eta(2\tau)\eta(\tau)\frac{\vartheta(2\tau, 2z)}{\vartheta(\tau, z)} \in J_{1,\frac{1}{2}}(\Gamma_0(2), \chi_4).$$

This reflective modular form of weight one has elementary Fourier coefficients like Q_1. The character of $\phi_{1,\frac{1}{2}}$ is given by the following formula

$$\chi_4(M) = e^{\frac{2i\pi}{4}(bd+d-1)}$$

for $M = \left(\begin{smallmatrix} a & b\\ 2c & d\end{smallmatrix}\right) \in \Gamma_0(2)$. Then we have $\Gamma_1(8,4) \subset \mathrm{Ker}(\chi_4)$ so $q=4$. We also have

$$\left(\phi_{1,\frac{1}{2}}|_{\frac{1}{2}}[\lambda, \mu; 0]\right)(\tau, z) = (-1)^\mu \phi_{1,\frac{1}{2}}(\tau, z).$$

Then we obtain that $\frac{\Delta_2(2Z)}{Q_1(Z)} = \mathrm{Lift}(\phi_{1,\frac{1}{2}})$. This is not a cusp form because $\phi_{1,\frac{1}{2}}(\tau, z) = \frac{1}{2}\vartheta^{(2)}_{1,0}(\tau, z)\vartheta^{(2)}_{1,0}(\tau, 0)$. For $(a, 8) = 1$, we have $\chi_4(\sigma_a) = \left(\frac{-4}{a}\right)$

then we deduce as for Q_1 that

$$\frac{\Delta_2(2Z)}{Q_1(Z)} = \mathrm{Lift}(\phi_{1,\frac{1}{2}})(Z) = \frac{1}{2}\sum_{N\geq 1}\sum_{\substack{n,m\in 4\mathbb{N}+1\\ l\in 2\mathbb{Z}+1\\ 2nm-l^2=N^2}}\sum_{\substack{a|(n,l,m)\\ a>0}}\left(\frac{-4}{a}\right) q^{\frac{n}{4}} r^{\frac{l}{2}} s^{\frac{m}{2}}.$$

References

[AI] H. Aoki, T. Ibukiyama, *Simple graded rings of Siegel modular forms, differential operators and Borcherds products,* Int. J. of Math. **16** (2005), 249–279.

[B1] R.E. Borcherds, *Automorphic forms with singularities on Grassmannians.* Invent. Math. **132** (1998), 491–562.

[B2] R.E. Borcherds, *Automorphic forms on $O_{s+2,2}(R)$ and infinite products.* Invent. Math. **120** (1995), 161–213.

[CYY] Y.-H. Cheng, Y. Yang, N. Yui, *Monodromy of Picard–Fuchs differential equations for Calabi–Yau threefolds.* J. Reine Angew. Math. **616** (2008), 167–203.

[DG] A. Dabholkar, D. Gaiotto, *Spectrum of CHL Dyons from Genus-Two Partition Function.* J. High Energy Phys. **12**, 087 (2007), 23 pp.

[DN] A. Dabholkar, S. Nampuri, *Spectrum of Dyons and Black Holes in CHL orbifolds using Borcherds Lift.* J. High Energy Phys. **11**, 077 (2007), 20 pp.

[DVV] R. Dijkgraaf, E. Verlinde, H. Verlinde, *Counting dyons in $N = 4$ string theory* Nucl. Phys. **B484** (1997), 543–561.

[DMVV] R. Dijkgraaf, G. Moore, E. Verlinde, H. Verlinde, *Elliptic genera of symmetric products and second quantized strings.* Commun. Math. Phys. **185** (1997), 197–209.

[EZ] M. Eichler, D. Zagier, *The theory of Jacobi forms.* Progress in Mathematics **55**. Birkhäuser, Boston, Mass., 1985.

[F] E. Freitag, Siegelsche Modulfunktionen. Grunglehren Math. Wiss., vol. 254. Springer Verlag, 1983.

[vdG] G. van der Geer, *Hilbert modular surfaces.* Erg. Math. Grenzgeb., vol. 16, Springer Verlag, 1988.

[G1] V. Gritsenko, *Modular forms and moduli spaces of abelian and* K3 *surfaces.* Algebra i Analiz **6** (1994), 65–102; English translation in St. Petersburg Math. J. **6** (1995), 1179–1208.

[G2] V. Gritsenko, *Irrationality of the moduli spaces of polarized Abelian surfaces.* Int. Math. Research Notices **6** (1994), 235–243.

[G3] V. Gritsenko, *Arithmetical lifting and its applications.* Number Theory. Proceedings of Paris Seminar 1992–93 (eds. S. David), Cambridge Univ. Press, 1995, 103–126.

[G4] V. Gritsenko, *Elliptic genus of Calabi-Yau manifolds and Jacobi and Siegel modular forms.* St. Petersburg Math. J. **11** (1999), 100–125.

[GH1] V. Gritsenko, K. Hulek, *The modular form of the Barth–Nieto quintic.* Intern. Math. Res. Notices **17** (1999), 915–938.

[GH2] V. Gritsenko, K. Hulek, *Minimal Siegel modular threefolds.* Math. Proc. Cambridge Philos. Soc. **123** (1998), 461–485.

[GN1] V. Gritsenko, V. Nikulin, Siegel automorphic form correction of some Lorentzian Kac–Moody Lie algebras. Amer. J. Math. **119** (1997), 181–224.

[GN2] V. Gritsenko, V. Nikulin, *Automorphic forms and Lorentzian Kac-Moody algebras. I, II.* International J. Math. **9** (2) (1998), 153–275.

[GN3] V. Gritsenko, V. Nikulin, On classification of Lorentzian Kac-Moody algebras. Russ. Math. Survey **57**, (2002), 921–979

[GN4] V. Gritsenko, V. Nikulin, *The arithmetic mirror symmetry and Calabi–Yau manifolds.* Comm. Math. Phys. **200** (2000), 1–11.

[He] H. O. Herøy, *Hecke-Operatoren auf* $\Gamma_0(N)$. Bonner Mathem. Schriften. Nr. 292 (1996), 84pp. Bonn.

[Hi] H. Hida, *Elementary theory of L-functions and Eisenstein series.* London Math. Soc. Student Texts **26**. Cambridge Univ. Press, 1993.

[Ib] T. Ibukiyama, *On Siegel modular variety of level* 3*,* Int. J. of Math. **2** (1991), 17–35.

[Ka] T. Kawai, *String duality and modular forms.* Phys. Lett. **B397** (1997), 51–62.

[Kl] H. Klingen, *Über Kernfunktionen für Jacobiformen und Siegelsche Modulformen.* Math. Ann. **285** (1989), 405–416.

[Ma] Y. Martin, *Multiplicative η-quotients.* Transactions of AMS **348** (1996), 4825–4856.

[Mi] T. Miyake, *Modular forms.* Springer, 1976.

[Mu] D. Mumford, *Tata lectures on theta I*, Progress in Mathem. **28**, Birkhäuser, Boston, Mass., 1983.

[Sh] G. Shimura, *Introduction to the arithmetic theory of automorphic functions.* Princeton Univ. Press, 1971.

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