Center conditions: Rigidity of logarithmic differential equations [1]

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Abstract

In this paper we prove that any degree d deformation of a generic logarithmic polynomial differential equation with a persistent center must be logarithmic again. This is a generalization of Ilyashenko's result on Hamiltonian differential equations. The main tools are Picard-Lefschetz theory of a polynomial with complex coefficients in two variables, specially the Gusein-Zade/A'Campo's theorem on calculating the Dynkin diagram of the polynomial, and the action of Gauss-Manin connection on the so called Brieskorn lattice/Petrov module of the polynomial. Some applications on the cyclicity of cycles and the Bautin ideals will be given.

0 Introduction

Let us be given the 1-form

$$\omega = P(x,y)dy - Q(x,y)dx \tag{1}$$

where P and Q are two relatively prime polynomials in $\mathbb{C}^2$. The degree of ω is the maximum of $deg(P)$ and $deg(Q)$. The space of degree d ω's up to multiplication by a constant, namely $\mathcal{F}(d)$, is a Zariski open subset of the projective space associated to the coefficient space of polynomial 1-forms (1) with $deg(P), deg(Q) \leq d$. We denote the elements of $\mathcal{F}(d)$ by $\mathcal{F}(\omega)$ or $\mathcal{F}$ if there is no confusion about the underlying 1-form ω in the text. Any $\mathcal{F}(\omega)$ induces a holomorphic foliation $\mathcal{F}$ on $\mathbb{C}^2$ i.e., the restrictions of ω to the leaves of $\mathcal{F}$ are identically zero. Therefore we name an element of $\mathcal{F}(d)$ a (holomorphic) foliation of degree d. But note that in the Introduction we do not need the geometric meaning of this word.

The points in $Sing(\mathcal{F}(\omega)) = \{P = 0, Q = 0\}$ are called the singularities of $\mathcal{F}$. A singularity $p \in \mathbb{C}^2$ of $\mathcal{F}(\omega)$ is called reduced if $D(P,Q)(p) \neq 0$. A reduced singularity p is called a center singularity or center for simplicity if there is a holomorphic coordinates system $(\tilde{x}, \tilde{y})$ around p with $\tilde{x}(p) = 0, \tilde{y}(p) = 0$ such that in this coordinate system $\omega \wedge d(\tilde{x}^2 + \tilde{y}^2) = 0$. The leaves of $\mathcal{F}$ around the center p are given by $\tilde{x}^2 + \tilde{y}^2 = const.$. The subset of $\mathcal{F}(d)$ containing $\mathcal{F}(\omega)$'s with at least one center, namely $\mathcal{M}(d)$, form an algebraic subset of $\mathcal{F}(d)$ (see Appendix A). Now the problem of identifying the irreducible components of $\mathcal{M}(d)$ arises. This problem is also known by the name "Center conditions" in the context of real polynomial differential equations.

H. Dulac in [**Du**] proves that $\mathcal{M}(2)$ has exactly 9 irreducible components (see also [**CL**] p. 601). In this case any foliation in $\mathcal{M}(2)$ has a Liouvillian first integral. Since this problem finds applications on the number of limit cycles in the context of real differential equations, this classification problem is very active. It is recommended to the reader to do a search with the title center/centre conditions in mathematical review to obtain many recent papers in this problem (We find also a similar problem for Abel equations $y' = p(x)y^2 + q(x)y^3$, p, q polynomials in x) In the context of holomorphic foliations we can

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refer to [**CL**],[**Mu**] and [**Mo**]. One of the main objectives in this paper is to introduce some new methods in this problem using an elementary algebraic geometry. We have borrowed many notions like Brieskorn lattice, Picard-Lefschetz theory and so on from the literature of singularities of holomorphic functions (see [**AGV**]) therefore a basic knowledge of this area of mathematics will be useful in understanding this work.

Let us be given the polynomials $f_i, deg(f_i) = d_i, 1 \leq i \leq s$ and non-zero complex numbers $\lambda_i \in \mathbb{C}^*, 1 \leq i \leq s$. The foliation

$$\mathcal{F} = \mathcal{F}(f_1 \cdots f_s \sum_{i=1}^{s} \lambda_i \frac{df_i}{f_i}) \tag{2}$$

is of degree $d = \sum_{i=1}^{s} d_i - 1$ and has the logarithmic first integral $f_1^{\lambda_1} \cdots f_s^{\lambda_s}$. Let $\mathcal{L}(d_1, \ldots, d_s)$ be the set of foliations (2). Since $\mathcal{L}(d_1, \ldots, d_s)$ is parameterized by λ_i's and the coefficients of f_i's, it is irreducible. The main theorem of this paper is:

Theorem 0.1. *$\overline{\mathcal{L}(d_1, \ldots, d_s)}$ is an irreducible component of $\mathcal{M}(d)$, where $d = \sum_{i=1}^{s} d_i - 1$ and $\overline{\mathcal{L}(d_1, \ldots, d_s)}$ is the closure of $\mathcal{L}(d_1, \ldots, d_s)$ in $\mathcal{F}(d)$.*

In the case $s = 1$ we can assume that $\lambda_1 = 1$ and so $\mathcal{L}(d+1)$ is the space of foliations of the type $\mathcal{F}(df)$, where f is a polynomial of degree $d+1$. This case is proved by Ilyashenko in [**Il**]. The similar result for foliations with a first integral of the type $\frac{F^p}{G^q}, \frac{deg(F)}{deg(G)} = \frac{q}{p}, g.c.d.(p,q) = 1$ is obtained in [**Mo**]. Some basic tools of this kind of generalizations for Lefschetz pencils on a manifold is worked in [**Mu**].

Let us reformulate our main theorem as follows: Let $\mathcal{F} \in \mathcal{L}(d_1, \ldots, d_s)$ be given by (2), p one of the center singularities of $\mathcal{F}$ and $\mathcal{F}_\epsilon$ a holomorphic deformation of $\mathcal{F}$ in $\mathcal{F}(d)$ such that its unique singularity p_ϵ near p is still a center.

Theorem 0.2. *In the above situation, there exists an open dense subset U of $\mathcal{L}(d_1, \ldots, d_s)$, such that for all $\mathcal{F} \in U$, $\mathcal{F}_\epsilon$ admits a logarithmic first integral. More precisely, there exist polynomials $f_{i\epsilon}, deg(f_{i\epsilon}) = d_i, i = 1, \ldots, s$ and non-zero complex numbers $\lambda_{i\epsilon}$ such that $\mathcal{F}_\epsilon$ is given by*

$$f_{1\epsilon} \cdots f_{s\epsilon} \sum_{i=1}^{s} \lambda_{i\epsilon} \frac{df_{i\epsilon}}{f_{i\epsilon}} = 0$$

$f_{i\epsilon}$ and $\lambda_{i\epsilon}$ are holomorphic in ϵ and $f_{i0} = f_i, \lambda_{i0} = \lambda_i, i = 1, \ldots, s$.

In the above theorem we can put U the complement of $\mathcal{L}(d_1, \ldots, d_s) \cap sing(\mathcal{M}(d))$ in $\mathcal{L}(d_1, \ldots, d_s)$. One may not be satisfied with this U and try to find explicit conditions like as: A foliation $\mathcal{F}(f_1 \cdots f_s \sum_{i=1}^{s} \lambda_i \frac{df_i}{f_i})$ in U satisfies 1. $\{f_i = 0\}$ intersects $\{f_j = 0\}$ transversally 2. $f_1^{\lambda_1} \cdots f_s^{\lambda_s}$ has nondegenerated critical points in $\mathbb{C}^2 - \cup_{i=1}^{s} \{f_i = 0\}$ and so on. In general one may be interested to identify the set $\mathcal{L}(d_1, \ldots, d_s) \cap sing(\mathcal{M}(d))$. In any case these questions are not in the focus of this paper.

The above theorem also says that the persistence of one center implies the persistence of all other centers and dicritical singularities $\{f_i = 0\} \cap \{f_j = 0\}, i, j = 1, \ldots, s$.

Since this paper is inspired by Ilyashenko's paper [**Il**] let us give a sketch of the proof in the case $\mathcal{L}(d+1)$:
Let f be a degree $d+1$ polynomial with the following condition

- f has d^2 non-degenerate critical points with distinct images.

This condition is generic, i.e. in the space of polynomials of degree $d+1$ there is an open dense subset such that for all f in this subset the condition is satisfied. Let $df+\epsilon\omega+h.o.t.$ be a deformation of df such that the singularity near a center singularity of df , namely p_1, persists in being center. Then $\int_\delta \omega=0$ for all vanishing cycles δ around p_1. The action of the monodromy on a single vanishing cycle $\delta\in H_1(f^{-1}(b),\mathbb{Z})$, where b is a regular value of f, generates the whole homology (the most significant part of the proof). Therefore our integral is zero in all cycles in the fibers of f and so it is relatively exact. Knowing the fact $deg(\omega)\leq d$ we conclude that $\omega=dP$, where P is a polynomial of degree less than $d+1$. Since $\mathcal{M}(d)$ is an algebraic set, the hypothesis on df is generic and the tangent vector ω of any deformation of df in $\mathcal{M}(d)$ is tangent also to $\mathcal{L}(d+1)$, we conclude that $\mathcal{L}(d+1)$ is an irreducible component of $\mathcal{F}(d)$.

Essentially the main steps of our proof is as the above proof. The main difference is that we have to work with a non-generic polynomial (Example 1.3) and in this case the associated foliation may lie in many irreducible components of $\mathcal{M}(d)$ (all $\mathcal{L}(d_1,\ldots,d_s)$). Therefore our strategy will be to identify as far as possible these irreducible components. Let us explain the structure of this paper:

In § 1 we introduce the Brieskorn lattice/Petrov module H associated to a polynomial f in $\mathbb{C}^2$ and the action of Gauss-Manin connection ∇ on it. The general objective is to obtain some information about H and ∇ when f has some bad critical fibers. Using a theorem of Mattei-Cerveau we prove Corollary 1.1 which is enough for our needs in this paper. This corollary classifies all $\omega\in H$ with $\nabla^n\omega=0$ for a given natural number n. In § 2 we analyze the action of the monodromy on a Lefschetz vanishing cycle in Example 1.3. Using the well-known Theorem 2.2 and Gusein-Zade/A'Campo's Theorem 2.1 we prove Theorem 2.3 and then Theorem 2.4. In § 3 we give the precise form of the tangent vector of a deformation of Example 1.3 with a persistent center. § 4 is devoted to the proof of Theorem 0.1. In § 5 we discuss our result in the context of real differential equations and its connection with concepts like cyclicity and Bautin Ideals. In § 6 we introduce some other problems concerning center conditions. In Appendix A we prove that $\mathcal{M}(d)$ is an algebraic set. The last Appendix is devoted to the proof of a lemma which does not fit in the paper. Following the arguments of this paper with Examples 1.1, 1.2 and specially 1.3 is recommended to the reader.

It is an interesting fact that we can restate center conditions for arbitrary algebraically closed field (see Appendix A). This arises the following question: Is Theorem 0.1 is true in an arbitrary algebraically closed field? Is there a completely algebraic proof for Theorem 0.1? The main obstacle in answering these questions is the Picard-Lefschetz theory which we use in this paper. One of the main source of this theory is the celebrated book of Lefschetz [**Le**], where Lefschetz's first, second and hard theorems is proved by topological methods. All of these theorems have latter completely algebraic proofs in algebraic geometry and this was the reason for me to put these questions in this paper.

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1 Brieskorn lattices/Petrov Modules

Let f be a non-composite polynomial of degree $d+1$ in $\mathbb{C}^2$, i.e. f cannot be composed as $p \circ g$, where p is a polynomial of degree greater than one in $\mathbb{C}$ and g is a polynomial in $\mathbb{C}^2$. It is easy to see that this condition is equivalent to the fact that for all $b \in \mathbb{C}$ except a finite number the fiber $f^{-1}(b)$ is irreducible. Let $\Omega^i, i = 0, 1, 2$ be the set of polynomial differential i-forms in $\mathbb{C}^2$ and $\mathbb{C}[t]$ be the ring of polynomials in t. Ω^i is a $\mathbb{C}[t]$-module in the following way

$$p(t).\omega = p(f)\omega, \ p \in \mathbb{C}[t], \omega \in \Omega^i$$

The Brieskorn lattice/Petrov module

$$H = \frac{\Omega^1}{d\Omega^0 + \Omega^0 df}$$

is a $\mathbb{C}[t]$-module. Also we define

$$V = \frac{\Omega^2}{df \wedge \Omega^1} \cong \frac{\mathbb{C}[x, y]}{< f_x, f_y >}$$

Multiplying by f defines a linear operator on V which is denoted by A.

Lemma 1.1. *If f has isolated singularities then the followings are true:*

1. *V is a $\mathbb{C}$-vector space of dimension μ, where μ is the sum of local Milnor numbers of f;*
2. *Eigenvalues of A are exactly the critical values of f.*

This lemma is a well-known fact, for instance see [**Ko**] but I did not find its proof in the literature and so I have proved it in Appendix B. In fact the canonical restriction $V \to \oplus_p \frac{\mathcal{O}_{\mathbb{C}^2,p}}{<f_x, f_y>}$, where $\mathcal{O}_{\mathbb{C}^2,p}$ is the ring of germs of holomorphic functions in a neighborhood of p in $\mathbb{C}^2$ and the sum runs through all critical points of f, is an isomorphism of vector spaces.

From now on we assume that f has isolated singularities and we denote the corresponding critical values by $c_1, c_2, \ldots, c_r$. We denote by $p(t)$ the minimal polynomial of A. The Gauss-Manin connection

$$\nabla : H \to H / < p >$$

where $H/ < p >$ is the localization of H by the multiplicative subgroup $< p >= \{p^i \mid i = 1, 2, \ldots\}$ of $\mathbb{C}[t]$, is defined as follows: For an $\omega \in H$ we have $p(f)d\omega = 0$ in V. Therefore there is a polynomial 1-form η in $\mathbb{C}^2$ such that

$$p(f)d\omega = df \wedge \eta \tag{3}$$

we define $\nabla\omega = \eta/p(t)$. Of course we must check that this operator is well-defined. If η_1 and η_2 are two polynomial 1-forms satisfying (3) then $(\eta_1 - \eta_2) \wedge df = 0$ and so by de Rham lemma $\eta_1 - \eta_2 = Pdf$ $(= 0$ in $H)$, for a P polynomial in $\mathbb{C}^2$. Also if $\omega = dP + Qdf$, P and Q two polynomials in $\mathbb{C}^2$, then $d\omega = dQ \wedge df$ and so $\nabla\omega = dQ$ $(= 0$ in $H)$.

Let $\tilde{H}$ be the localization of H by polynomials in t vanishing only on c_i's. We can extend ∇ as a function from $\tilde{H}$ to $\tilde{H}$ by the rule

$$\nabla(\omega/q) = (q\nabla\omega - q'\omega)/q^2$$

Let $f_1 = 0, f_2 = 0, \dots, f_k = 0$ be irreducible components of all critical fibers of f

$$D = \sum_{i=1}^{k} \{f_i = 0\}$$

and $\tilde{\Omega}^i$ be the set of meromorphic i-forms in $\mathbb{C}^2$ with poles in D. We define $\tilde{H}' = \frac{\tilde{\Omega}^1}{d\tilde{\Omega}^0 + \tilde{\Omega}^0 df}$ and in a similar way as for H a connection ∇' on $\tilde{H}'$.

Lemma 1.2. *$(\tilde{H}, \nabla)$ is isomorphic to $(\tilde{H}', \nabla')$.*

Proof. Every meromorphic 1-form in $\mathbb{C}^2$ with poles in D determines a unique element of $\tilde{H}$ as follows: if $\omega = \frac{\tilde{\omega}}{f_{i_1}^{r_1}}$ and $f - c_1 = f_{i_1}^{k_1} \cdots f_{i_l}^{k_l}$ is the decomposition of $f - c_1$ to irreducible factors then we multiply both $\tilde{\omega}$ and $f_{i_1}^{r_1}$ by $f_{i_1}^{k_1 m - r_1} f_2^{k_2 m} \cdots f_{i_l}^{k_l m}$, where m is an integer number satisfying $m - 1 < \frac{r_1}{k_1} \le m$, and we obtain $\omega = \frac{\tilde{\tilde{\omega}}}{(t-c_1)^m}$. Repeating this process by $\tilde{\tilde{\omega}}$ we obtain an element of $\tilde{H}$. If $\omega_1 = dP + Qdf$, where P and Q are two meromorphic functions on $\mathbb{C}^2$ with poles along D, then by applying the above method on P, Q we can see that ω is associated to zero in $\tilde{H}$. Therefore we obtain a map $\tilde{H}' \to \tilde{H}$ which is the inverse of the canonical map $\tilde{H} \to \tilde{H}'$ and so it is an isomorphism. Since ∇ and ∇' coincide on $H \subset \tilde{H}, \tilde{H}'$, the mentioned isomorphism sends ∇' to ∇. $\square$

Let b be a regular value of f and $\{\delta_t\}_{t \in (\mathbb{C}, b)}, \delta_t \in f^{-1}(t)$ be a continuous family of cycles in the fibers of f. For an $\omega \in \tilde{H}$ the integral $\int_\delta \omega$ is well-defined and

$$\frac{\partial}{\partial t} \int_{\delta_t} \omega = \int_{\delta_t} \nabla \omega \tag{4}$$

(see [**AGV**]). In fact this formula is a bridge between topology and algebra in this paper.

Our objective in this section is to analyze the action of ∇ on H. For this purpose let us state a classical theorem. Let ω be a meromorphic 1-form in $\mathbb{C}^2$ and $\cup_{i=1}^{k} \{f_i = 0\}$ be the pole locus of ω. Suppose that the multiplicity of ω along $\{f_i = 0\}$ is r_i.

Theorem 1.1. *([**CM**]) If ω is closed, i.e. $d\omega = 0$ then there are $\lambda_1, \dots, \lambda_k \in \mathbb{C}$ and a polynomial g such that*

1. *If $r_i = 1$ then $\lambda_i \neq 0$;*
2. *If $r_i > 1$ then f_i does not divide g;*
3. *ω can be written*

$$\omega = \left(\sum_{i=1}^{k} \lambda_i \frac{df_i}{f_i}\right) + d\left(\frac{g}{f_1^{r_1 - 1} \dots f_k^{r_k - 1}}\right)$$

Note that if ω has a pole of order r_∞ at the line at infinity then the degree of g is $\sum_{i=1}^{k} d_j (r_i - 1) + r_\infty - 1$ and $r_\infty + \sum_{i=1}^{k} \lambda_i d_i = 0$.

Let $\mathcal{L}$ be the subset of $\tilde{H}$ containing the 1-forms of the type $\sum_{i=1}^{k} \lambda_i \frac{df_i}{f_i}$, $\lambda_i \in \mathbb{C}$. We have the decomposition $\mathcal{L} = \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_r$, where $\mathcal{L}_i$ contains all the terms $\lambda_j \frac{df_j}{f_j}$, $f_j = 0$ being an irreducible component of the fiber $f^{-1}(c_i)$. Note that if $f^{-1}(c_i)$ is irreducible then $\mathcal{L}_i = 0$.

Corollary 1.1. *For the pair $(\tilde{H}, \nabla)$ and a positive integer number n*

1.

$$Kernel(\nabla^n) = \{\omega \in \tilde{H} \mid \omega = \sum_{j=0}^{n-1} \alpha_j t^j, \alpha_j \in \mathcal{L}\}$$

2.

$$Kernel(\nabla^n) \cap H = \{\omega \in H \mid \omega = \sum_{j=1}^{n-1} \sum_{i=1}^{r} \alpha_{ij}(t^j - c_i^j), \alpha_{ij} \in \mathcal{L}_i\}$$

where $\nabla^n = \nabla \circ \cdots \circ \nabla$ n-times.

Proof. Let us prove the first part by induction on n. We use the isomorphism in Lemma 1.2. If for $\omega \in \tilde{H}$, $\nabla\omega = 0$ then $d\omega = dP \wedge df$, where P is a meromorphic function in $\mathbb{C}^2$ with poles along D. Now $d(\omega - Pdf) = 0$ and by Theorem 1.1 we have $\omega \in \mathcal{L}$. This proves the first part $n = 1$ of the induction.
Now if $\nabla^{n+1}\omega = 0$ then by induction $\nabla\omega = \sum_{j=0}^{n-1} \alpha_j t^j = \nabla \sum_{j=0}^{n-1} \frac{\alpha_j}{j+1} t^{j+1}$ or equivalently $\nabla(\omega - \sum_{j=0}^{n-1} \frac{\alpha_j}{j+1} t^{j+1}) = 0$. Using the case $n = 1$ we finish the proof of the first part.
Now let us prove the second part. Let $\omega = \sum_{j=1}^{n-1} \alpha_j t^j + \alpha_0 \in H$ and $\alpha_j = \sum_{i=1}^{r} \alpha_{ij}$ be the decomposition of α_j. We write

$$\omega = \sum_{j=1}^{n-1} \sum_{i=1}^{r} \alpha_{ij} t^j + \alpha_0 =$$

$$\sum_{j=1}^{n-1} \sum_{i=1}^{r} \alpha_{ij}(t^j - c_i^j) + \sum_{j=1}^{n-1} \sum_{i=1}^{r} \alpha_{ij} c_i^j + \alpha_0$$

The first summand in the above belongs to H and hence the second one belongs to $\mathcal{L} \cap H$ and so the second summand must be zero. $\square$

Before we go to the next section let us give three simple but important examples. The last one has a very special role in this paper. Corollary 1.1 with $n = 2$ will be used in the next section. Therefore we explain it with these examples.

Example 1.1. $f = (x^2 + y^2 - 1)x$. Since $xy = 0$, $3x^2x^i = x^i$, $y^2y^i = y^i, i \geq 1$ in V, $1, x, y, x^2$ form a basis for the vector space V. f has four critical points $p_1 = (0, 1), p_2 = (0, -1), p_3 = (\sqrt{1/3}, 0), p_4 = (-\sqrt{1/3}, 0)$ with three critical values $c_1 = c_2 = 0, c_3 = -2/3\sqrt{1/3}, c_4 = 2/3\sqrt{1/3}$. We define $\omega_1 = (x^2 + y^2 - 1)dx$. We have

$$\nabla\omega_1 = -\frac{xd(x^2 + y^2 - 1)}{t} = \frac{\omega_1}{t}, \ \nabla^2\omega_1 = 0$$

Since $f^{-1}(0)$ is the only reducible fiber of f, by Corollary 1.1, 2 we know that any other 1-form in H with the property $\nabla^2\omega_1 = 0$ is some multiply of ω_1 by a constant.

Example 1.2. $f = xy(x + y - 1)$. There are four critical points $p_1 = (0, 0), p_2 = (1, 0), p_3 = (0, 1), p_4 = (1/3, 1/3)$ with two critical values $c_1 = c_2 = c_3 = 0, c_4 = -1/27$. Knowing Corollary 1.1, we can see that the 1-forms $\omega_1 = x(x+y-1)dy, \omega_2 = y(x+y-1)dx$ form a basis for the vector space $\{\omega \in H \mid \nabla^2\omega = 0\}$.

Example 1.3. The lines $l_p=(d-p)x+py-p(d-p)=0, p=0,1,\ldots,d$ are in a general position in $\mathbb{R}^2$ i.e., they are distinct and no three of them have a common intersection point (l_p is the line through $(p,0),(0,d-p)$). The polynomial $f=l_0l_1\cdots l_d$ satisfies the properties

1) All the critical points of f in $\mathbb{C}^2$ are real and non-degenerated;

2) The values of f at all saddle critical points equal zero;

By a small perturbation of the lines l_i we also get the property

3) The values of f at center critical points are distinct.

(See Appendix of [**Mo2**] for this kind of arguments). In a real coordinates system $(\tilde{x},\tilde{y})$ around a saddle (resp. center) critical point p the function f can be written as $f(p)+\tilde{x}^2-\tilde{y}^2$ (resp. $f(p)+\tilde{x}^2+\tilde{y}^2$). f has $a_2=\frac{d(d+1)}{2}$ saddle critical points, $a_1=\sum_{i=2}^{d}[\frac{i-1}{2}]$ center critical points with negative value and $a_3=\frac{d(d-1)}{2}-a_1$ center critical points with positive value, where $[q]$ is the integer number satisfying $[q]-1<q\leq[q]$. By Corollary 1.1 the set of $\omega\in H$ with $\nabla^2\omega=0$ is a vector space generated by

$$l_0l_1\cdots l_{p-1}l_{p+1}\cdots l_d dl_i, p=0,1,\ldots,d-1$$

Note that $\sum_{p=0}^{d} l_0l_1\cdots l_{p-1}l_{p+1}\cdots l_d dl_i=df=0$ in H.

In what follows when we refer to the polynomial f in Example 1.3 we mean the one with a small perturbation of the lines l_p and hence satisfying the property 3.

Remark 1.1. E. Brieskorn in [**Br**] introduced three $\mathcal{O}_{(\mathbb{C},0)}$-modules H,H',H'' in the context of singularity of holomorphic functions $f:(\mathbb{C}^n,0)\to(\mathbb{C},0)$ (The H used in this paper is the equivalent of Brieskorn's H'). After him these modules are called the Brieskorn lattices and recently the similar notions in a global context are introduced by many authors (see [**Sa**],[**DS**], [**BD**],[**Mo**]). In the context of differential equations H appears in the works of G.S. Petrov [**Pe**] on deformation of Hamiltonian equations of the type $d(y^2+p(x))$, where p is a polynomial in x and is named by L. Gavrilov in [**Ga2**] the Petrov module. For this reason I have used the both names Brieskorn lattice and Petrov module for H.

Restriction of an $\omega\in H$ to each fiber defines a global section of the cohomology fiber bundle of the function f and looking in this way ∇ is the usual Gauss-Manin connection in the literature. For this reason I have named ∇ again the Gauss-Manin connection. But of course we can name $\nabla\omega$ the Gelfand-Leray form of ω (see [**AGV**]).

2 Action of the monodromy

Suppose that f is a polynomial function in $\mathbb{R}^2$ with the properties 1,2,3 in Example 1.3. For a $c\in\mathbb{C}$ we define $L_c=f^{-1}(c)$ in $\mathbb{C}^2$. $\{\delta_t\}_{t\in(\mathbb{C},c)}$ with $\delta_t\in H_1(L_t,\mathbb{Z})$ denotes a continuous family of cycles.

Choose a value $b\in\mathbb{C}$ with $Im(b)>0$ and fix a system of paths joining b with the critical values of f, subject to the condition that these paths lie in their entirety in the upper half-plane $Im(z)>0$ except for the ends which coincide with the critical values. Now we can define a distinguished basis of vanishing cycles in $H_1(L_b,\mathbb{Z})$ (see [**AGV**] for the definition). Critical points of f are in one to one correspondence with the self intersection points of the real curve $f=0$, namely $p_j,\ a_1+1\leq j\leq a_1+a_2$, and relatively compact components of its complement, namely $U_i^0, 1\leq i\leq a_1, U_k^2, a_1+a_2+1\leq k\leq a=a_1+a_2+a_3$. U_i^0 contains a critical point of f with negative value and U_k^2 contains a critical point with positive value. We denote by

(5) $$\delta_i^0, \delta_j^1, \delta_k^2,\ 1 \leq i \leq a_1,\ a_1+1 \leq j \leq a_1+a_2,\ a_1+a_2+1 \leq k \leq a$$

the distinguished basis of vanishing cycles. δ_i^0, δ_k^2 are called the center vanishing cycles and δ_j^1 is called the saddle vanishing cycle.

Theorem 2.1. *(S. Gusein-Zade, N. A'Campo) After choosing a proper orientation for the cycles δ_i^σ we have*

- $< \delta_i^\sigma, \delta_j^\sigma >= 0$;
- $< \delta_i^0, \delta_j^1 >$ *equal to the number of vertices of the polygon U_i^0 coinciding with the point p_j;*
- $< \delta_k^2, \delta_j^1 >$ *equal to the number of vertices of the polygon U_k^2 coinciding with the point p_j;*
- $< \delta_k^2, \delta_i^0 >$ *equal to the number of common edges of U_k^2 and U_i^0.*

This theorem, in an apparently local context, is proved by S. Gusein-Zade [**Gu**],[**Gu1**] and N. A'Campo [**AC**],[**AC1**] independently. However the proof in our case is the same. The above theorem gives us the Dynkin diagram of f (see [**AGV**]).

Now let us state another theorem which we will use in this paper:

Theorem 2.2. *In the above situation, the vanishing cycles $\delta_i^0, \delta_j^1, \delta_k^2$ generate $H_1(L_b, \mathbb{Z})$ freely.*

The proof of the above theorems is classical and the reader can consult with [**DN**],[**Mo2**]. Also the main core of the proof can be found in [**La**].

Let us compactify $\mathbb{C}^2$ in $\mathbb{P}^2 = \{[x;y;z] \mid (x,y,z) \in \mathbb{C}^3 - 0\}$. Here $\mathbb{C}^2 = \{[x;y;1] \mid x, y \in \mathbb{C}\}$ and $\mathbb{L} = \{[x;y;0] \mid x, y \in \mathbb{C}\}$ is the projective line at infinity. Let $f = f_0 + f_1 + \cdots + f_{d+1}$ be the decomposition of f to homogeneous parts. We look at f as a meromorphic function on $\mathbb{P}^2$ by rewriting f as

$$f = f(x/z, y/z) = \frac{z^{d+1} f_0(x,y) + z^d f_1(x,y) + \cdots + f_{d+1}(x,y)}{z^{d+1}}$$

The indeterminacy set of f is given by

$$\mathcal{R} = \{[x;y;0] \mid f_{d+1}(x,y) = 0\}$$

Now suppose that $\mathcal{R}$ has $d+1$ distinct points. For instance the polynomial in Example 1.3 has this property. This implies that the fibers of f intersect the line at infinity transversally. Doing just one blow-up in each point of $\mathcal{R}$ and using Ehresmann's fibration theorem we conclude that the map f is a C^∞ fibration on $\mathbb{C} - C$, where $C = \{c_1, \ldots, c_r\}$ is the set of critical values of f. In general case we must do more blow-ups to obtain this conclusion and the set of critical points of f may be a proper subset of C. Therefore we have the action of the monodromy on the first (co)homology group of L_b:

$$h : \pi_1(\mathbb{C} - C, b) \times H_1(L_b, \mathbb{Z}) \to H_1(L_b, \mathbb{Z})$$

Recall the system of paths in the beginning of this section. When we say the "Monodromy around a critical value " we mean the monodromy associated to the path which gets out of b, goes along γ (the path connecting b to the critical value in this system of paths) , turns around the critical value counterclockwise and then comes back to b along γ. By Picard-Lefschetz formula (see [**La**]) the monodromy around zero is given by

$$\delta \to \delta - \sum_{j=a_1+1}^{a_2} < \delta, \delta_j^1 > \delta_j^1 \tag{6}$$

and the monodromy around a center critical value c_i is given by

$$\delta \to \delta - < \delta, \delta_i > \delta_i \tag{7}$$

For the regular value $b \in \mathbb{C} - C$, $\overline{L_b} = L_b \cup \mathcal{R}$ is a compact Riemann surface. Let I be the subgroup of $H_1(L_b, \mathbb{Z})$ generated by the cycles around the points of $\mathcal{R}$ in $\overline{L_b}$. We have

$$I = \{\delta \in H_1(L_b, \mathbb{Z}) \mid < \delta, \delta' >= 0, \ \forall \delta' \in H_1(L_b, \mathbb{Z})\} \tag{8}$$

Elements of I are fixed under the action of the monodromy.

Now we are in a position such that we can look at our objects in an abstract way: We have a union of curves $\mathcal{C} = f^{-1}(0)$ in $\mathbb{R}^2$. To $\mathcal{C}$ we associate an Abelian group G ($= H_1(L_b, \mathbb{Z})$) freely generated by the symbols (5). These symbols are in one to one correspondence with the self intersection points of $\mathcal{C}$ and the relatively compact components of the complement of $\mathcal{C}$ in $\mathbb{R}^2$. We have an antisymmetric pairing $< . >$ given by the items of Theorem 2.1. Also the non-Abelian freely generated group $\pi_1(\mathbb{C} - C, b)$ acts from left on G with the rules (6), (7). We have also the subgroup I of G defined by (8). These are all we are going to need. From now on we can think about (vanishing) cycles in this abstract context.

We will apply the above arguments for the Example 1.3. For the line $l_p = 0, p = 0, 1, \dots, d$ we can associate the saddle critical points of f on $l_p = 0$ and the corresponding vanishing cycles. We rename these vanishing cycles by $\delta_j^{l_p}, j = 1, 2, \dots, d$ and suppose that the ordering by the index j is the same as the ordering of corresponding saddle points in the line $l_p = 0$. We define

$$\delta^{l_p} = \sum_{j=1}^{d} (-1)^j \delta_j^{l_p} \in G, p = 0, 1, \dots, d$$

Now reindex all relatively compact polygons by $U_i, 1 \le i \le a_1 + a_3$. In our example the polygons are triangles or cubics. For any polygon U_i we denote by δ^i ($\in G$) the sum of vanishing cycles in the vertices of U_i. Also we denote by δ_i the vanishing cycle associated to U_i

Lemma 2.1. *The cycles δ^{l_p}, $1 \le p \le d$ and $\delta^i, 1 \le i \le a_1 + a_3$ generate all saddle vanishing cycles in $G \otimes \mathbb{Q}$ and so they are linearly independent in $G \otimes \mathbb{Q}$.*

Note that the number of the cycles considered in the lemma is equal to the number of saddle vanishing cycles.

Proof. This is a nice high school problem. It would be more difficult if we assume only that the lines l_p are in a general position in $\mathbb{R}^2$ i.e., they are distinct and no three of them have a common intersection point. For our example we give the following hint: 1. First draw the lines for a small value of d 2. Let $\delta_{p,p+1}$ denote the vanishing cycle associated to the intersection of l_p and l_{p+1}, $p = 0, 1, \ldots, d-1$. Try to write $d.\delta_{p,p+1}$ as an integral sum of δ^{l_p} and $\delta^{l_{p+1}}$ and δ^i, where i runs through the index of all polygons between (the angle less than $90°$) the lines l_p and l_{p+1}. 3. Now it is easy to conclude that every vanishing cycle associated to the intersection points multiplied by d can be written as an integral sum of δ^{l_p}, $0 \leq p \leq d$ and $\delta^i, 1 \leq i \leq a_1 + a_3$. 4. After choosing a proper sign for δ^{l_p}, $0 \leq p \leq d$ prove that $\sum_{i=0}^{d} \delta^{l_p} = 0$. $\square$

Now let us state the geometric meaning of δ^i and δ^{l_p}.

Lemma 2.2. *We have*

1. $\delta^i = \delta_i - h_0(\delta_i)$, *where h_0 is the monodromy around* 0;
2. $I \otimes \mathbb{Q}$ *is generated by the cycles* $\delta^{l_p}, 1 \leq p \leq d$.

Proof. The first part is a direct consequence of Picard-Lefschetz formula and Theorem 2.1.
By Theorems 2.1 and 2.2 we have $< \delta^{l_p}, \delta' >= 0$, $\forall \delta' \in H_1(L_b, \mathbb{Z})$ and so δ^{l_p} is in I. By Lemma 2.1 $\delta^{l_p}, k = 1, 2, \ldots, d$ are linearly independent in G and we know that I is freely generated of rank d. Therefore $\delta^{l_p}, k = 1, 2, \ldots, d$ generate $I \otimes \mathbb{Q}$ freely. $\square$

Because of the symmetry in Example 1.3, one may conjecture that δ^{l_p} is the cycle around $l_p \cap \mathbb{L}$ in $\overline{L_b}$ (multiplied by a rational number). Since we don't need this statement we don't try to prove it.

Theorem 2.3. *In the Example 1.3 the action of the monodromy on a Lefschetz vanishing cycle generates the homology* $H_1(\overline{L_b}, \mathbb{Q})$.

Proof. First consider the case where δ is a center vanishing cycle. By Theorem 2.1 and Picard-Lefschetz formula the action of the monodromy generates δ_i and then δ_i's. By Lemma 2.1 and Theorem 2.2 these cycles generate $H_1(\overline{L_b}, \mathbb{Q})$. Now suppose that δ is a saddle vanishing cycle. There is a center vanishing cycle δ' such that $< \delta', \delta > \neq 0$. Performing a monodromy of δ around the critical value associated to δ' and subtracting the obtained cycle by δ we obtain δ' and so we fall in the first case. $\square$

In the beginning of this section we defined the degree of a polynomial 1-form $\omega = Pdy - Qdx$ to be the maximum of $deg(F)$ and $deg(Q)$. This definition is no more useful when we look at ω as a meromorphic 1-form in $\mathbb{P}^2$ or when we consider the foliation induced by ω in $\mathbb{P}^2$. Let us introduce a new definition of degree. For a polynomial 1-form ω we define $deg_1(\omega)$ to be the order of ω along the line at infinity mines two. We can see easily that if $deg_1(\omega) \leq d$ then $\omega = Pdy - Qdx + G(xdy - ydx)$, where P, Q are two polynomials of degree at most d and G is zero or a homogeneous polynomial of degree d. Therefore $deg_1(\omega) - deg(\omega) = 0, 1$. Naturally for a $\omega \in H$ we define $deg_1(\omega)$ to be the minimum of the deg_1's of the elements of ω. Let q be an indeterminacy point of f at the line at infinity $\mathbb{L}$. Recall that the fibers of f intersect $\mathbb{L}$ transversally. Now we can choose a continuous family of cycles $\{\delta_t\}_{t \in \mathbb{C}}$ such that δ_t is a cycle in L_t around q. Latter we will need the following lemma.

Lemma 2.3. *For $\omega \in H$ the integral $\int_{\delta_t} \omega$ is a polynomial of degree at most $[\frac{n}{d+1}]$, where $n - 2 = deg_1(\omega)$ and $d + 1 = deg(f)$. $\nabla^i \omega, i > [\frac{n}{d+1}]$ restricted to each fiber $\overline{f^{-1}(t)}$ has not residues in $\mathcal{R}$ and hence is a 1-form of the second type.*

Proof. We have $p(t) := \int_{\delta_t} \omega = t^{\frac{n}{d+1}} \int_{\delta_t} \frac{\omega}{f^{\frac{n}{d+1}}}$. Since the 1-form $\frac{\omega}{f^{\frac{n}{d+1}}}$ has not pole along the line at infinity, $\frac{p(t)}{t^{\frac{n}{m}}}$ has finite growth at $t=\infty$. Since $p(t)$ is is holomorphic in $\mathbb{C}$, we conclude that $p(t)$ is a polynomial of degree at most $[\frac{n}{d+1}]$. The second part is a direct consequence of the first one and the formula (4). □

Let f be a polynomial and ω be a 1-form in $\mathbb{C}^2$. ω is called relatively exact modulo f, or simply relatively exact if the underlying f is known, if the restriction of ω to each fiber L_b is exact. 1-forms of the type $dP + Qdf$, where P,Q are polynomials in $\mathbb{C}^2$, are relatively exact. Latter we will need the following lemma:

Lemma 2.4. *If L_t is connected for all $t \in \mathbb{C}$ then every relatively exact 1-form ω with $deg_1(\omega) = d$ is of the form $dP+Qdf$, where P,Q are two polynomials in $\mathbb{C}^2$ with $deg(P) = deg_1(\omega) + 2$ and $\deg(Q) = deg_1(\omega) + 2 - deg(f)$.*

The proof can be found in [**BD**], [**Bo**]. (For the assertion about the degrees see [**Mo**]). The main objective of this section is the following theorem.

Theorem 2.4. *In the Example 1.3 let δ_t be a continuous family of vanishing cycles and ω be a degree d 1-form in $\mathbb{C}^2$ such that $\int_{\delta_t} \omega = 0, t \in (\mathbb{C}, b)$. Then ω is of the form*

$$\omega = l_0 \cdots l_d \alpha + d(P),\ \ \alpha = \sum_{i=0}^{d} \lambda_i \frac{dl_i}{l_i}, \lambda_i \in \mathbb{C}$$

where P is a polynomial of degree not greater than $d+1$.

Proof. By the hypothesis and (4) we have $\int_{\delta_t} \nabla^2\omega = 0, t \in (\mathbb{C}, b)$. Lemma 2.3 implies that $\nabla^2\omega$ is a 1-form without residue in each fiber and Theorem 2.3 implies that $\nabla^2\omega$ is a relatively exact 1-form and hence by Lemma 2.4 it is zero in $\tilde{H}$. Since L_0 is the only reducible fiber of f, by Corollary 1.1 ω must be of the form $f\sum_{i=0}^{d} \lambda_i \frac{dl_i}{l_i} + dP + Qdf$, where P and Q are two polynomials in $\mathbb{C}^2$ and λ_i is a complex number. Recall that ω has degree less than d. By Lemma 2.4 we can suppose that P (resp. Q) has degree less than $d+2$ (resp. 1). We have $dP_{d+2} + Q_1 df_{d+1} = 0$, where P_{d+2} denote the homogeneous part of P of degree $d+2$ and so on. It is easy to check that if $Q_1 \not\equiv 0$ then we have $f_{d+1} = (ax+by)^{d+1}$ which is not our case. Therefore Q must be constant and P of degree less than $d+1$. We substitute $P + Qf$ by P and so we can assume that Q is zero. We obtain the desired equality. □

3 Deformation and Melnikov functions

As a first attempt to prove Theorem 0.1 one may fix a generic $\mathcal{F} \in \mathcal{L}(d_1, \ldots, d_s)$ and perform a deformation such that one of the center singularities of $\mathcal{F}$ persists. Then one may try to find some tools for finding a logarithmic first integral for the deformed foliation. These tools in the Ilyashenko's case $\mathcal{L}(d+1)$ were Picard-Lefschetz theory of a polynomial and the classification of relatively exact 1-forms modulo the polynomial. Developing these tools for a generic point of other irreducible components of $\mathcal{M}(d)$, for instance $\mathcal{L}(d_1, \ldots, d_s)$, seems to be difficult and wasting the time.

In this section we want to explain this idea that it is not necessary to take a generic point of $\mathcal{L}(d_1, \ldots, d_s)$. For instance suppose that the variety $X_1 = \mathcal{L}(d_1, \ldots, d_s)$ has a point $\mathcal{F}(df)$ which is Hamiltonian. This point may lie in other irreducible components of

$\mathcal{M}(d)$. Let $X_1, X_2, \cdots, X_q$ be all irreducible components of $\mathcal{M}(d)$ containing $\mathcal{F}$. Now the idea is to deform $\mathcal{F}$ in such a way that one of its centers persists and since we can develop our tools for the Hamiltonian system $\mathcal{F}(df)$, we try to prove that the deformed foliation lies in one of X_i's. Note that in the same time we prove that $X_1, X_2, \ldots, X_q$ are irreducible components of $\mathcal{M}(d)$. Now we are going to realize this idea for the Hamiltonian foliation with the polynomial in Example 1.3. This will lead to the proof of our main theorem. In this section $\mathcal{L}$ denotes the set of meromorphic 1-forms of the type $\sum_{i=0}^{d} \lambda_i \frac{dl_i}{l_i}$ and $\mathcal{P}_n$ (resp. $\mathcal{P}_*$) denotes the set of polynomials of degree not greater than n (resp. arbitrary degree) in $\mathbb{C}^2$. $I = \{0, 1, \ldots, d\}$ and we denote by J an equivalence relation in I. If $J_1, J_2, \ldots, J_{s_J}$ are all equivalence classes of J then we define $f_i^J = \Pi_{j \in J_i} l_j, i = 1, 2, \ldots, s_J$.

Let f be the polynomial considered in Example 1.3 and

$$\mathcal{F}_\epsilon : \omega_\epsilon = df + \epsilon^k \omega_k + \epsilon^{k+1} \omega_{k+1} + \cdots + \epsilon^{2k} \omega_{2k} + h.o.t. = 0, \ \omega_k \neq 0 \tag{9}$$

where $\omega_i, i = k, k+1, \ldots$ are polynomial 1-forms in $\mathbb{C}^2$ and k is a natural number, be a one parameter degree d deformation of $\mathcal{F}(df)$.

Definition 3.1. *Let p be a center singularity of $\mathcal{F} = \mathcal{F}(df)$. p is called a persistent center if there exists a sequence $\epsilon_i, i = 1, 2, 3, \ldots,\ \epsilon_i \to 0$ such that the singularity of $\mathcal{F}_{\epsilon_i}$ near $p_0 = p$, namely p_{ϵ_i}, is center for all i.*

Latter we will see that if p is persistent then p_ϵ is center for all ϵ. Our main theorem in this section is the following:

Theorem 3.1. *If p is a persistent center in the degree d deformation (9) then ω_k is of the form*

$$\omega_k = f \sum_J \sum_{i=1}^{s_J} (\lambda_i^J \frac{df_i^J}{f_i^J} + \frac{A_i^J}{f_i^J})$$

where in the first sum J runs through all equivalence relations in I, for each J the complex numbers $\lambda_i^J, i = 1, 2, \ldots, s_J$ are distinct and $A_i^J \in \mathcal{P}_{deg(f_i^J)}$.

Let $\delta_t, t \in (\mathbb{C}, b)$ be a continuous family of vanishing cycles around p and Σ be a transverse section to $\mathcal{F}$ at some point of δ_b. We write the Taylor expansion of the deformed holonomy $h_\epsilon(t)$

$$h_\epsilon(t) - t = M_1(t)\epsilon + M_2(t)\epsilon^2 + \cdots + M_i(t)\epsilon^i + h.o.t.$$

$M_i(t)$ is called the i-th Melnikov function of the deformation. If the center p is persistent under the deformation then $M_i = 0$ for all i. But we don't need to use all these equalities. For instance in Ilyashenko's case $\mathcal{L}(d+1), k = 1$ we need only $M_1 = 0$. To prove Theorem 3.1 and our main theorem we will need to use $M_k = M_{k+1} = \cdots = M_{2k} \equiv 0$.

Lemma 3.1. *In the above situation 1) $M_1 = M_2 = \cdots = M_{k-1} \equiv 0$ and 2) if Σ is parameterized by the image of f then*

$$M_k(t) = -\int_{\delta_t} \omega_k$$

3) If $M_k = M_{k+1} = \cdots = M_{2k-1} \equiv 0$ then

$$\omega_i = f\alpha_i + P_i, \ P_i \in \mathcal{P}_{d+1}, \ \alpha_i \in \mathcal{L}, \ k \leq i \leq 2k-1$$

4) Moreover if the transverse section is parameterized by the image of a branch of lnf then

$$M_{2k}(t) = -\int_{\delta_t} \frac{\omega_{2k}}{f} - \frac{P_k}{f}.\alpha_k$$

Proof. Let $\delta_{t,h_\epsilon(t)}$ be the path connecting t and $h_\epsilon(t)$ along δ_b in the leaf of $\mathcal{F}_\epsilon$ through t. We take the integral $\int_{\delta_{t,h_\epsilon(t)}}$ of (9). Now the equalities associated to the coefficients of $\epsilon^i, 1 \leq i \leq k-1$ prove the first part. The equality associated to the coefficient of ϵ^k proves the second part (for more detail see [**Ro**][**Mo**]).
We prove the third and fourth part by induction on i. First $i = k$. $M_k \equiv 0$ implies that $\int_{\delta_t} \omega_k = 0,\ t \in (\mathbb{C}, b)$ and so by Theorem 2.4 ω_k is of the form

$$\omega_k = f\alpha_k + P_k,\ \alpha_k \in \mathcal{L}, P_k \in \mathcal{P}_{d+1} \tag{10}$$

Now let us suppose that

$$\omega_j = f\alpha_j + P_j,\ \alpha_j = \sum_{p=0}^{d} \lambda_{j,p} \frac{dl_p}{l_p},\ k \leq j \leq i$$

Let $\tilde{\omega}_\epsilon = \omega_\epsilon / f$, $\tilde{\omega} = \omega / f$ and so on. With this new notation we have

$$\tilde{\omega}_j = \alpha_j + d\tilde{P}_j = d(\tilde{P}_j + lnl_1^{\lambda_{j,1}} \cdots l_{d+1}^{\lambda_{j,d+1}}) + \tilde{P}_j\tilde{df} = dg_j + \tilde{P}_j\tilde{df}$$

From now on suppose that Σ is parameterized by the image of lnf. We have

$$(1 - \tilde{P}_k\epsilon^k) \cdots (1 - \tilde{P}_i\epsilon^i)\tilde{\omega}_\epsilon$$

$$\begin{aligned} (1 - \tilde{P}_k\epsilon^k) \cdots (1 - \tilde{P}_i\epsilon^i)(\tilde{df} + \epsilon^k\tilde{\omega}_k + \epsilon^{k+1}\tilde{\omega}_{k+1} + \cdots + \epsilon^{2k}\tilde{\omega}_{2k} + h.o.t.) \\ = \tilde{df} + \epsilon^k dg_k + \epsilon^{k+1} dg_{k+1} + \cdots + \epsilon^i dg_i \\ +\epsilon^{i+1}\tilde{\omega}_{i+1} + \cdots + \epsilon^{2k-1}\tilde{\omega}_{2k-1} + \epsilon^{2k}(\tilde{\omega}_{2k} - \tilde{P}_k\tilde{\omega}_k) + h.o.t. \end{aligned} \tag{11}$$

We take the integral $\int_{\delta_{t,h_\epsilon(t)}}$ of (11). Now the equality associated to the coefficients of ϵ^{i+1} is $M_{i+1}(t) + \int_{\delta_t} \omega_{i+1} = 0$. $M_{i+1} \equiv 0$ and Theorem 2.4 imply that ω_{i+1} is of the desired form. In the last step $i = 2k-1$ the fourth part of the lemma is proved. Note that $\int_{\delta_t} \tilde{\omega}_{2k} - \tilde{P}_k\tilde{\omega}_k = \int_{\delta_t} \tilde{\omega}_{2k} - \tilde{P}_k\alpha_k$. □

Now $M_{2k} = 0$ implies that $\int_{\delta_t} f\omega_{2k} - P_k f\alpha_k = 0$. $\omega_{2k} - P_k f\alpha_k$ has a pole of order at most $2d+3$ at the line at infinity. Therefore by Lemma 2.3 $\nabla^3(\omega_{2k} - P_k f\alpha_k)$ restricted to fibers $\overline{f^{-1}(b)}$ has not residues. By Theorem 2.3, Corollary 1.1 we conclude that

$$f\omega_{2k} - P_k.f\alpha_k = \alpha_1.f + \alpha_2 f^2 + dg + pdf, g, p \in \mathcal{P}_*, \alpha_i \in \mathcal{L}, i = 1, 2$$

The restriction of the above equality to the $L_0 = f^{-1}(0)$ implies that g is constant on L_0. Since L_0 is connected in $\mathbb{C}^2$, we conclude that $dg = d(fg'), g' \in \mathcal{P}_*$. From now on we write $\lambda_{k,i} = \lambda_i$. The above equality modulo multiplications by l_i gives us

$$l_i \mid \lambda_i P_k + g',\quad i = 0, 1, \ldots, d \tag{12}$$

Let $I = \{0, 1, \ldots, d\}$. Define $i \cong j$ if $\lambda_i = \lambda_j$. $\cong$ is an equivalence relation. Let $J_1, J_2, \ldots, J_s$ be the equivalence classes of $\cong$. We define $f_i = \Pi_{j \in J_i} l_j$. Note that we have $f = f_1 f_2 \cdots f_s$. The following lemma finishes the proof of Theorem 3.1.

Lemma 3.2. *In the above situation, P_k must be of the form*

$$P_k = f\sum_{i=1}^{s}\frac{A_i}{f_i},\ A_i\in\mathcal{P}_{\#J_i} \tag{13}$$

Proof. Let $d_i = deg(f_i)$. (12) implies that P_k is zero in $\{f_i = 0\}\cap\{f_j = 0\}$. The space of degree $d+1$ polynomials in $\mathbb{C}^2$ vanishing in $\{f_i = 0\}\cap\{f_j = 0\}$ for all $1\le i<j\le s$, namely G, is of dimension $\frac{(d+2)(d+3)}{2} - \sum_{1\le i<j\le s} d_i d_j$. But the space of polynomials in (13) is a subset of G and has dimension $-1+\sum_{i=1}^{s}\frac{(a_i+1)(a_i+2)}{2}$. Since $d+1 = \sum_{i=1}^{s} d_i$, these two numbers are equal and so P_k must be of the form (13). □

4 Proof of Theorem 0.1

Let $(X,0)$ be a germ of an analytic variety in $(\mathbb{C}^n,0)$. The tangent cone TC_0X of X at 0 is defined as follows: Let $\gamma:(\mathbb{C},0)\to(\mathbb{C}^n,0)$ be an analytic map such that its image lies in X and has the Taylor series $\gamma = \omega\epsilon^l + \omega'\epsilon^{l+1}+\cdots,\omega,\omega',\ldots\in\mathbb{C}^n$. T_l is the set of all ω and $TC_0X = \cup_{i=1}^{\infty}T_l$.

We have $\mathbb{C}.TC_0X = TC_0X$ therefore we can projectivize TC_0X and obtain a subset, namely Y, of $\mathbb{P}^{n-1}$. Suppose that X is irreducible. Let $\pi: M\to(\mathbb{C}^n,0)$ be a blow-up at 0 with the divisor $\pi^{-1}(0)\cong\mathbb{P}^{n-1}$. The closure $\tilde{X}$ of $\pi^{-1}(X-\{0\})$ in M is an irreducible analytic set and we can see easily that Y is isomorphic to the intersection of $\tilde{X}$ and $\mathbb{P}^{n-1}\subset M$, and so it is algebraic compact subset of $\mathbb{P}^{n-1}$. Moreover since $dim(Y)\ge dim(\tilde{X}) + dim\mathbb{P}^{n-1} - n$ (see [**Ke**] Theorem 3.6.1) and Y cannot be the whole $\mathbb{P}^{n-1}$, Y is of pure dimension $dimX-1$, i.e. each irreducible component of Y is of dimension $dimX-1$. We conclude that TC_0X is an algebraic subset of $\mathbb{C}^n$ of pure dimension $dim(X)$. If X is smooth then TC_0X is the usual tangent space of X at 0 and hence it is a vector space. For more information about the tangent cone of a singularity and its definition by the leading terms of the polynomial defining the singularity see [**Mu**] and [**Ke**] Section 6.2.

The variety $\mathcal{L}(d_1,\cdots,d_s)$ is parameterized by

$$\tau:\mathbb{C}^s\times\mathcal{P}_{d_1}\times\cdots\times\mathcal{P}_{d_s}\to\mathcal{F}(d)$$

$$\tau(\lambda_1,\ldots,\lambda_s,f_1,\ldots,f_s) = f_1\cdots f_s\sum_{i=1}^{s}\frac{df_i}{f_i} \tag{14}$$

and so it is irreducible. Let J be an equivalence relation in $I=\{0,1,\ldots,d\}$ with s equivalence classes, namely $J_1,\ldots,J_s$. Let also f be the polynomial in Example 1.3 and $\mathcal{F}_0 = \mathcal{F}(df)$. In a neighborhood of $\mathcal{F}_0$ in $\mathcal{F}(d)$, $\mathcal{L}(d_1,\cdots,d_s)$ has many irreducible components corresponding to the J's as follows: The above parameterization near $(1,\ldots,1,\Pi_{i\in J_1}l_i,\ldots,\Pi_{i\in J_s}l_i)$ determines an irreducible component, namely $\mathcal{L}(d_1,\cdots,d_s)_J$, of $(\mathcal{L}(d_1,\cdots,d_s)\ ,\mathcal{F}_0)$ corresponding to J. Fix one of these branches and name it X. Now to prove our main theorem it is enough to prove that X is an irreducible component of $(\mathcal{M}(d),\mathcal{F}_0)$.

What we have proved in Theorem 3.1 is:

$$TC_{\mathcal{F}_0}\mathcal{M}(d) = \cup TC_{\mathcal{F}_0}\mathcal{L}(d_1,\cdots,d_s)_J$$

where the union is taken over all $1\le d_i\le d+1,\ 1\le i\le s\le d+1,\ \sum_{i=1}^{s}d_i = d+1$ and all equivalence relations J. Now let $X\subset X'$, where X' is an irreducible component of $(\mathcal{M}(d), \mathcal{F}_0)$. Since the above union is the decomposition of $TC_{\mathcal{F}_0}\mathcal{M}(d)$ to irreducible components, the irreducible component of $TC_{\mathcal{F}_0}X'$ containing $TC_{\mathcal{F}_0}X$ must be a subset of $TC_{\mathcal{F}_0}X$ and so is equal $TC_{\mathcal{F}_0}X$. An irreducible component of $TC_{\mathcal{F}_0}X'$ is of dimension $dim(X)$ and so X' is of dimension $dim(X)$. Since $X \subset X'$ and X, X' are irreducible, we conclude that $X = X'$.

5 Limit cycles and Bautin Ideals

Let $\mathcal{F}_0 \in \mathcal{M}(d)$, p_0 be a center singularity of $\mathcal{F}_0$, $\delta_t, t \in (\mathbb{C}, 0)$ be a continuous family of cycles invariant by $\mathcal{F}_0$ around p_0 and Σ be a transverse section to $\mathcal{F}_0$ at some point of δ_0. Let also $(\mathbb{C}^\mu, \psi)$ be an affine chart of $\mathcal{F}(d)$ with $\psi(\mathcal{F}_0) = 0$. We use also ψ for the points in $\mathbb{C}^\mu$. For instance we denote by $\mathcal{F}_\psi$ the foliation associated to $\psi \in \mathbb{C}^\mu$ by this affine chart.

The holonomy of $\mathcal{F}_0$ along δ_0 in Σ is identity. Now considering a ψ near 0, we have the holonomy h_ψ of $\mathcal{F}_\psi$ along δ_0 in Σ which is called the deformed holonomy. We write the Taylor expansion

$$h_\psi(t) - t = \sum_{i=0}^{\infty} a_i(\psi) t^i$$

The ideal generated by $a_i(\psi), 0 \leq i$ is called the Bautin ideal of δ_0 in the deformation space $\mathcal{F}(d)$. If $\psi \in Zero(I)$ then the holonomy of $\mathcal{F}_\psi$ along δ_0 is identity. Using the below lemma we conclude that $zero(I) \subset \mathcal{M}(d)$.

Lemma 5.1. *In the above situation, the singularity p_ψ near p_0 is center and so $\mathcal{F}_\psi \in \mathcal{M}(d)$.*

Proof. Let $(D_a \times D_a, (x, y))$ be a coordinate system around p_0 such that $p_0 = (0, 0)$ and the foliations is given by $f_0 = xy = const.$, where $D_a = \{z \in \mathbb{C} \mid\mid z \mid\leq a\}$. Let $K = \overline{(D_{a/2} - D_{a/3}) \times D_{a/2} \cup D_{a/2} \times (D_{a/2} - D_{a/3})}$ and Σ be a transverse section to $\mathcal{F}_0$ at a point in $(D_{a/2} - D_{a/3}) \times \{0\}$. Since K is compact and the holonomy of $\mathcal{F}_\psi$ along a closed path in $(D_{a/2} - D_{a/3}) \times \{0\}$ is identity, there exists $0 < \epsilon$ such that for all $\mid \psi \mid < \epsilon$ we can associate to each point (x, y) of K the intersection point, namely $A(x, y)$, of the leaf passing through p and Σ. Now $f_\psi = f_0(A(x, y))$ is a holomorphic map in K and so by Hartogs's theorem it extends to a neighborhood of p_0. Since $f_\psi \mid_{\psi=0} = xy$ and f_ψ is constant in the leaves of $\mathcal{F}_\psi$, we conclude that f_ψ has a unique non-degenerate critical point near p_0 which is p_ψ. □

The center p_0 of $\mathcal{F}_0$ is called stable if for any deformation $\mathcal{F}_\tau, \tau \in (\mathbb{C}^k, 0)$ of $\mathcal{F}_0$ inside $\mathcal{M}(d)$, the deformed singularity p_τ is again a center.

Let $\mathcal{F}_0 \in \mathcal{M}(d)$ and $\mathcal{F}_\tau$ be a deformation of $\mathcal{F}_0$ inside $\mathcal{M}(d)$. Since each $\mathcal{F}_\tau$ has at least on center, there is a sequence p_{τ_i} of centers converging to a singularity of $\mathcal{F}_0$. We conclude that the deformed holonomy along the vanishing cycles around p_0 is identity and so by the above lemma we conclude that p_τ is center for all τ. From this argument we conclude that every $\mathcal{F}_0$ with $(\mathcal{M}(d), \mathcal{F}_0)$ irreducible has at least one stable center. In particular generic points of irreducible components have stable centers. It is an interesting problem to show that a generic point of $\mathcal{L}(d_1, \ldots, d_s)$ has $d^2 - \sum_{i<j} d_i d_j$ stable centers. For the stable center p_0 we have $zero(I) = (\mathcal{M}(d), \mathcal{F}_0)$, where I is the Bautin ideal associated to a vanishing cycle around p_0 and the deformation space $\mathcal{F}(d)$.

Now let X be an irreducible component of $\mathcal{M}(d)$, $\mathcal{F} \in X - sing(\mathcal{M}(d))$ be a real foliation, i.e. its equation has real coefficients, p be a real center singularity and $\delta_t, t \in (\mathbb{R}, 0)$ be a family of real vanishing cycles around p.

Theorem 5.1. *The cyclicity of δ_0 in a deformation of $\mathcal{F}$ inside $\mathcal{F}(d)$ is greater than $codim_{\mathcal{F}(d)}(X)-1$*

Roughly speaking, the cyclicity of δ_0 is the maximum number of limit cycles appearing near δ_0 after a deformation of $\mathcal{F}$ in $\mathcal{F}(d)$. The proof of this Theorem and the exact definition of cyclicity can be found in [**Ro**]. $codim_{\mathcal{F}(d)}(\mathcal{L}(d+1))-1=\frac{(d+2)(d-1)}{2}$ and this is the number obtained by Yu. Ilyashenko in [**Il**].

Now Let $X=\mathcal{L}(d_1,\ldots,d_s)$ and $\mathcal{F}=\mathcal{F}(f\sum_{i=1}^{s}\lambda_i\frac{df_i}{f_i})\in\mathcal{L}(d_1,\ldots,d_s)-sing(\mathcal{M}(d))$. Suppose that λ_i's and the coefficients of f_i's are real numbers and $\mathcal{F}$ has a (real) center singularity at $0\in\mathbb{R}^2$. As a direct consequence of Theorem 0.1 and Theorem 5.1.

Corollary 5.1. *Suppose that $s>1$. The cyclycity of δ_0 in a deformation of $\mathcal{F}$ in $\mathcal{F}(d)$ is not less than*

$$(d+1)(d+2)-\sum_{i=1}^{s}(\frac{(d_i+1)(d_i+2)}{2})-1$$

The above lower bound reaches to its maximum when $d_1=d_2=\ldots=d_s=1, s=d+1$. In this case the cyclicity of δ_0 is not less than d^2-2.

Until the time of writing this paper, the best upper bound for the cyclicity of a vanishing cycle of a Hamiltonian equation is the P. Mardesic's result $\frac{d^4+d^2-2}{2}$ in [**Ma**]. Nothing is known about the exact calculation of these numbers.

6 Challenge

In this section I would like to give some highlights into the difficulty and bigness of the center conditions problem.

All the varieties $\mathcal{L}(d_1,\ldots,d_s)$ pass through $\mathcal{F}=\mathcal{F}(df)$, where f is the polynomial in Example 1.3.

Problem 6.1. Are $\mathcal{L}(d_1,\ldots,d_s)$'s all irreducible components of $\mathcal{M}(d)$ through $\mathcal{F}$?

Note that the argument used in § 4 does not give an answer to this problem. There may be an irreducible component of $\mathcal{M}(d)$ through $\mathcal{F}$ such that its tangent cone at $\mathcal{F}$ is a subset of $\cup TC_{\mathcal{F}_0}\mathcal{L}(d_1,\cdots,d_s)_J$. In this case the definition of other notions of tangent cone based on higher order 1-forms in the deformation (9) seems to be necessary.

Problem 6.2. For $i=0,\ldots,d$ let l_i be a line in the real plane and m_i be an integer number. Put $f=l_0^{m_0}\cdots l_d^{m_d}$. Find all irreducible components of $\mathcal{M}(d)$ through $\mathcal{F}(df)$.

In this problem the line l_i has multiplicity m_i and it would be interesting to see how the classification of irreducible components through $\mathcal{F}(df)$ depends on the different arrangements of the lines l_i in the real plane and the associated multiplicities. In particular we may allow several lines to pass through a point or to be parallel. When there are lines with negative multiplicities then we have a third kind of singularities $\{l_i=0\}\cap\{l_j=0\}$ called dicritical singularities, where l_i (resp. l_j) has positive (resp. negative) multiplicity. They are indeterminacy points of f and are characterized by this property that there are infinitely many leaves of the foliation passing through the singularity. Also in this case there are saddle critical points of f which are not due to the intersection points of the lines with positive (resp. negative) multiplicity. The reader may analyze the situation by the example $f=\frac{l_0l_1}{l_2l_3}$. As we saw in this paper, our problem breaks into two main parts

1. Study the action of Gauss-Manin connection on the Brieskorn lattice/Petrov module of f 2. Study the action of monodromy on the first homology group of a regular fiber of f. These two studies enable us to calculate Melnikov functions, as many as necessary, and obtain an explicit form for the 1-forms appearing in a deformation of $\mathcal{F}(df)$ with a persistent center. The reader may consult with [**OrTe**] about the arrangements of lines. May be one of the interesting parts of Problem 6.2 is:

Problem 6.3. Generalize A'Campo/Gusein zade theorem on calculating the Dynkin diagram of f in the Problem 6.2 with some m_i's negative.

Another way to look for more irreducible components of $\mathcal{M}(d)$ is as follows. Take a polynomial of degree d in $\mathbb{C}^2$ with the generic conditions considered by Ilyashenko (see Introduction). Now $\mathcal{F}(df)$ has degree $d-1$. Let us suppose that we have compactified $\mathcal{F}(d)$ in a projective space by considering the low degree foliations.

Problem 6.4. Classify all irreducible components of $\mathcal{M}(d)$ through $\mathcal{F}(df)$.

All $\mathcal{L}(d_1,\dots,d_s)$'s pass through $\mathcal{F}(df)$. There are other candidates as follows

1. $A_i = \{\mathcal{F}(\frac{dp}{p} + d(\frac{q}{p^i})) \mid deg(p) = 1, deg(q) = d\}$ $i = 0, 1, 2, \dots, d$;
2. $B_1 = \{\mathcal{F}(\frac{dq}{q} + d(p)) \mid deg(p) = 1, deg(q) = d\}$;

An element of A_i (resp. B_1) has a first integral of the type pe^{q/p^i} (resp. qe^p). These candidates are supported by Dulac's classification (see [**Du**] and [**CL**] p.601) in the case $d = 2$.

In the beginning of this paper we defined the degree of a polynomial 1-form ω to be the maximum of degrees of underlying polynomials. But in § 2 we came to know deg_1 which is more geometric and is due to to compactification of $\mathbb{C}^2$ in $\mathbb{P}^2$. In fact it is the number of tangency points of the foliation induced by ω in $\mathbb{P}^2$ with a generic line. Now we redefine $\mathcal{F}(d)$ to be the space of holomorphic foliations $\mathcal{F}$ in $\mathbb{P}^2$ with $deg_1(\mathcal{F}) = d$.

Problem 6.5. Prove a theorem similar to Theorem 0.1 in the above space of parameters.

For this problem a solution to Problem 6.3 is necessary. For more information about this point of view see [**Mo**].

A Center conditions are algebraic

Let k be an arbitrary algebraically closed field. We can develop our notations in Introduction for k except the center singularity. Now let us see how we can define the notion of a center singularity when we are working with k.

Suppose that the origin $0 = (0, 0)$ is a reduced singularity of $\mathcal{F}(\omega) \in \mathcal{F}(d)$. 0 is called the center singularity of ω if there is a formal power series

$$f = xy + f_3 + f_4 + \cdots + f_n + \cdots \tag{15}$$

where f_n is a homogeneous polynomial of degree n and with coefficients in k, such that

$$df \wedge \omega = 0$$

A singularity p of ω is called center if the origin is a center singularity of $i^*\omega$, where i is the linear transformation $a \to a + p$ in k^2. Our definition is complete.

Now let $k = \mathbb{C}$ and the origin is a center singularity of a 1-form ω. By theorem A in [**MaMo**] the existence of a formal series (15) implies the existence of a convergent one, namely g. Using the complex Morse theorem we find a coordinates system $(\tilde{x}, \tilde{y})$ around the origin such that $g = \tilde{x}^2 + \tilde{y}^2$. So our definition of a center singularity coincides with the definition in Introduction.

Theorem A.1. *$\mathcal{M}(d)$ is an algebraic subset of $\mathcal{F}(d)$.*

Proof. Let $\mathcal{M}_0(d)$ be the set of 1-forms ω in $\mathcal{F}(d)$ with a center at the origin. Since $\mathcal{M}(d)$ is obtained by the action of the group $\{a \to a + p \mid p \in k^2\}$ on $\mathcal{M}_0(d)$, it is enough to prove that $\mathcal{M}_0(d)$ is algebraic. Now we reserve the p for the characteristic number of the field k.

Let $\omega \in \mathcal{F}(d)$ and $\omega = \omega_1 + \omega_2 + \omega_3 + \ldots + \omega_d$ be the homogeneous decomposition of ω, then

$$\omega \wedge df = 0 \Rightarrow (\omega_1 + \omega_2 + \omega_3 + \cdots + \omega_{d+1}) \wedge (d(xy) + df_3 + df_4 + \cdots) = 0$$

Putting the homogeneous parts of the above equation equal to zero, we obtain

$$\tag{16} \begin{cases} \omega_1 \wedge d(xy) = 0 \Rightarrow \omega_1 = c.d(xy), c \in k \\ \omega_1 \wedge df_3 = -\omega_2 \wedge d(xy) \\ \ldots \\ \omega_1 \wedge df_n = -\omega_2 \wedge df_{n-1} - \cdots - \omega_{n-1} \wedge d(xy) \\ \ldots \end{cases}$$

Since 0 is a reduced singularity, we have $c \neq 0$. Dividing the 1-form ω by c, we can assume that $c = 1$. Let $\mathcal{P}_n$ denote the set of homogeneous polynomials of degree n. Define the operator :

$$S_n : \mathcal{P}_n \to (\mathcal{P}_n dx \wedge dy)$$
$$S_n(g) = \omega_1 \wedge d(g)$$

We have

$$\begin{aligned} S_{i+j}(x^i y^j) = d(xy) \wedge d(x^i y^j) &= (xdy + ydx) \wedge (x^{i-1} y^{j-1}(jxdy + iydx)) \\ &= (j - i) x^i y^j dx \wedge dy \end{aligned}$$

This implies that

$$Im(S_n) = A_n dx \wedge dy$$

where A_n is the subspace generated by the monomials $x^i y^j$, $i \neq j \mod p$. The existence of f_n implies that the coefficient of $x^i y^j$ $i = j \mod p$ in

$$-\omega_2 \wedge df_{n-1} - \cdots - \omega_{n-1} \wedge d(xy)$$

which is a polynomial, say P_{ij}, with variables

$$\text{coefficients of } \omega_2 \ldots \omega_{n-1}, f_2, \ldots, f_{n-1}$$

is zero. The coefficients of $f_i, i \leq n - 1$ is recursively given as polynomials in coefficients of $\omega_i, i \leq n - 1$ and so the algebraic set

$$X : \cap_{i=j \text{ mod } p} \{P_{ij} = 0\}$$

consists of all $\mathcal{F}(\omega)$ in $\mathcal{F}(d)$ with a center singularity at the origin. Note that by Hilbert zeroes theorem, a finite number of P_{ij}'s defines $\mathcal{M}_0(d)$. □

B Proof of Lemma 1.1

Let us compactify $\mathbb{C}^2$ in $\mathbb{P}^2 = \mathbb{C}^2 \cup \mathbb{L}$, where $\mathbb{L}$ is the line at infinity and look f as a meromorphic function in $\mathbb{P}^2$ with indeterminacy points at $\mathbb{L}$. Let $\mathcal{F}$ be the foliation in $\mathbb{P}^2$ given by $df = 0$ and $\tilde{\mathcal{F}}$ be the sheaf of holomorphic 1-forms in $\mathbb{P}^2$ inducing the foliation $\mathcal{F}$. The foliation $\mathcal{F}$ have two kind of singularities 1. the critical points of f 2. the indeterminacy points of f at the line at infinity. We denote by $sing(\mathcal{F})$ the singular points of $\mathcal{F}$.

It is easy to see that $\tilde{\mathcal{F}}$ is a freely generated sheaf and hence it is coherent. Let $\tilde{\Omega}^i, i = 1, 2$ denote the sheaf of i-forms in $\mathbb{P}^2$. Since the wedge product map $\wedge : \tilde{\Omega}^1 \times \tilde{\mathcal{F}} \to \tilde{\Omega}^2$ is an analytic map between two coherent sheaves, its image, namely $\tilde{\mathcal{F}} \wedge \tilde{\Omega}^1$, is coherent (see [**GR**]). The sheaf

$$\mathcal{L} = \frac{\tilde{\Omega}^2}{\tilde{\mathcal{F}} \wedge \tilde{\Omega}^1}$$

is a coherent discrete sheaf whose support is a subset of $sing(\mathcal{F})$. If the foliation around a singular point p is given by $Pdx + Qdy = 0, g.c.d.(P, Q) = 1$ then the stalk of $\mathcal{L}$ at p is $\frac{\mathcal{O}_{\mathbb{C}^2,p}}{<P,Q>}$. For an arbitrary analytic sheaf $\mathcal{S}$ on $\mathbb{P}^2$ we define $\mathcal{S}(*\mathbb{L})$ to be the sheaf of meromorphic sections of $\mathcal{S}$ with poles of arbitrary order along $\mathbb{L}$. The sheaf $\mathcal{L}(*\mathbb{L})$ is obtained from $\mathcal{L}$ by putting zero the nonzero stalks of $\mathcal{L}$ associated to the singularities of $\mathcal{F}$ at $\mathbb{L}$. Therefore $H^0(\mathbb{P}^2, \mathcal{L}(*\mathbb{L}))$ is a vector space of dimension μ, where μ is the sum of local Milnor numbers of f.

Now $\tilde{\mathcal{F}} \wedge \tilde{\Omega}^1$ is coherent and $\mathbb{L}$ is a positive divisor in $\mathbb{P}^2$, therefore by Kodaira vanishing theorem $H^1(\mathbb{P}^2, (\tilde{\mathcal{F}} \wedge \tilde{\Omega}^1(*\mathbb{L})) = 0$ (see [**Nr**]). Writing the long exact sequence associated to

$$0 \to (\tilde{\mathcal{F}} \wedge \tilde{\Omega}^1)(*\mathbb{L}) \to \tilde{\Omega}^2(*\mathbb{L}) \to \mathcal{L}(*\mathbb{L}) \to 0$$

we conclude that $\frac{H^0(\mathbb{P}^2, \Omega^2(*\mathbb{L}))}{H^0(\mathbb{P}^2, (\tilde{\mathcal{F}} \wedge \Omega^1)(*\mathbb{L}))}$ is a vector space of dimension μ. But $H^0(\mathbb{P}^2, (\tilde{\mathcal{F}} \wedge \Omega^1)(*\mathbb{L})) = df \wedge \Omega^1$ and $H^0(\mathbb{P}^2, \Omega^2(*\mathbb{L})) = \Omega^2$. So we have proved the first part.

For a critical value $p_i, i = 1, \ldots, s$ of f let V_i be the stalk of $\mathcal{L}$ in p_i. In fact in the above argument we have proved that the natural canonical map $V \to \oplus_{i=1}^s V_i$ is an isomorphism of vector spaces. V_i's are invariant by the linear operator $A : V \to V$ associated to the multiplication by f. Also $A - c_i I$ restricted to V_i is a nilpotent matrix (see [**Br**]), where $c_i = f(p_i)$. This finishes the proof of the second part.

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