A P-ADIC PROPERTY OF HODGE CLASSES ON ABELIAN VARIETIES

Don Blasius

University of California
Dept. of Mathematics
Los Angeles, CA 90024

USA

Max-Planck-Institut für Mathematik
Gottfried-Claren-Straße 26
D-5300 Bonn 3

Germany

MPI/92-59

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DON BLASIUS

UCLA

Introduction.

(0.1). The Hodge conjecture asserts that every Hodge class is the image under the cycle map of an algebraic cycle. If true, then the Hodge classes possess arithmetic properties which will sometimes admit definitions independent of the conjecture. Thus, the problem arises of proving such properties unconditionally.

In a basic paper ([D1]), Deligne took an important step along these lines by defining an absolute Hodge class and proving that on an abelian variety every Hodge class is an absolute Hodge class. This result has powerful consequences which stem mostly from the strengthening of the Shimura-Taniyama reciprocity law, in a motivic setting, that it makes possible ([DM], [D3]).

(0.2). In the present paper, we prove another such arithmetic property of Hodge classes on abelian varieties. Let p be a rational prime and let $\sigma_p : \overline{\mathbf{Q}} \to \overline{\mathbf{Q}_p}$ be an embedding. Let X be a proper smooth variety defined over $\overline{\mathbf{Q}}$ and let $\gamma_B \in H_B^{2j}(X)(j)$ be an absolute Hodge class (see 1.3 below). Let $\gamma_p = I_p(\gamma_B) \in H_p^{2j}(X)(j)$ and $\gamma_{DR} = I_\infty(\gamma_B) \in H_{DR}^{2j}(X)(j)$ be the images of γ_B in p-adic étale and algebraic De Rham cohomology, respectively, under the comparison maps. Recall that Faltings ([F]) has shown that there exists an isomorphism

$$I_{DR} : H_p^j(\sigma_p X)(k) \otimes B_{DR} \to H_{DR}^j(\sigma_p X)(k) \otimes_{\sigma_p(\overline{\mathbf{Q}})} B_{DR}$$

where B_{DR} is the ring introduced by Fontaine ([Fo]). Consequently, it is natural to make the following definition: an absolute Hodge class γ_B is **De Rham** if, for all primes p and all embeddings $\sigma_p : \overline{\mathbf{Q}} \to \overline{\mathbf{Q}_p}$, we have

$$I_{DR}(\sigma_p \gamma_p) = \sigma_p \gamma_{DR}.$$

Of course, every algebraic class is a De Rham class.

Partially supported by NSF

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

(0.3) Theorem. Let X be an abelian variety defined over $\overline{\mathbf{Q}}$. Then every Hodge class on X is De Rham.

The purpose of this paper is to prove this result while at the same time sketching a variant proof of Deligne's theorem. The proof of the new result here employs the methods of [D1], replacing the use of the usual De Rham isomorphism I_∞ in [D1] by Faltings' I_{DR}. However, at a crucial point in [D1], the argument (proving Priniciple B) employs the Gauss-Manin connection. As it appears that the behavior of I_{DR} relative to Gauss-Manin is not well-understood, we find it necessary to change this part of the argument in the p-adic case. Indeed, we replace the use of Gauss-Manin by an appeal to general facts about the structure of the cohomology of a family of varieties. Apart from this simplification, the proof of the main result here follows very closely the argument of [D1]. However, for clarity, we have taken throughout a motivic, i.e. Tannakian viewpoint, which provides a slight change of perspective on Principle A of [D1]. For progress concerning Gauss-Manin itself in the p-adic setting, see [W].

(0.4). As an application of (0.3), we deduce formally in the last section that Hodge classes are *crystalline* in the case of primes of good reduction. This fact has several applications in the case of abelian varieties of CM type ([Co], [O], [W2]). We have neglected here to include applications of (0.3) itself. Nevertheless, one should note that it is an easy exercise to define B_{DR}-valued p-adic periods of CM motives and to develop a formalism of p-adic period relations parametrized by identities of tensor products (monomial relations) which occur between such motives, extending the archimedian formalism given in [D2], [Sch] and [Sh]. In fact, one may easily proceed further to obtain a theory of p-adic periods with values in $\mathbf{C}_p$, at least once one has chosen an identification of $\mathbf{C}$ with $\mathbf{C}_p$, and obtained thereby a trivialization of the p-adic Tate module. This follows because the functoriality of I_{DR} enables one to define periods in the graded ring B_{HT} associated to the filtration on B_{DR} in such a way that an eigenperiod for the complex multiplication is supported on a single graded component, necessarily isomorphic to $\mathbf{C}_p(n)$. After invoking the trivialization, this component is identified with $\mathbf{C}_p$.

That such a formalism should exist was proven using Dwork theory in the ordinary case by Gillard ([G]) by a method which imitates Shimura's proof of his monomial relations theorem ([Sh]).

More systematically, one may define, as in [DM], a category of motives starting from abelian varieties of CM type where the morphisms are given by absolute Hodge cycles which are also De Rham. Then (0.3) says that this category is the same as the category defined using just absolute Hodge cycles as morphisms. In particular, the Taniyama group is also the motivic Galois group of this category ([D3], [L]).

(0.5) Acknowledgement. I thank A. Ogus for conversations which led to substantial simplification and restructuring of the proof, and G. Faltings for a conversation concerning the compatibility of the De Rham and crystalline comparison maps. I also thank the Max Planck Institut in Bonn for its hospitality during the preparation of the paper.

1. Cohomologies and comparison maps.

1.1. Let K be a subfield of $\mathbf{C}$ and let X be a smooth projective variety defined over K. On X we have several cohomology functors.

First, we have

$$H_B^*(X) = \bigoplus_{j=0}^{2dimX} H_B^j(X),$$

the topological cohomology of $X(\mathbf{C})$ with rational coefficients. Each $H_B^j(X) \otimes \mathbf{C}$ is equipped with a Hodge decomposition

$$H_B^j(X) \otimes \mathbf{C} = \bigoplus_{p+q=j} H^{p,q}(X).$$

Next we have

$$H_{DR}^*(X) = \bigoplus_{j=0}^{2dimX} H_{DR}^j(X),$$

the algebraic De Rham cohomology of X. Each $H_{DR}^j(X)$ is a K vector space equipped with the decreasing (Hodge) filtration $F^* H_{DR}^j(X)$.

Last, for each rational prime p, we have

$$H_p^*(X) = \bigoplus_{j=0}^{2dimX} H_p^j(X),$$

the p-étale cohomology of $X \times_K \overline{K}$. Each $H_p^j(X)$ is a $\mathbf{Q}_p$ vector space equipped with a continuous action of $Gal(\overline{K}/K)$.

Between these cohomology theories we have the graded comparison isomorphisms:

$$I_\infty : H_B^*(X) \otimes \mathbf{C} \to H_{DR}^* \otimes_K \mathbf{C}$$

and,for p a prime,

$$I_p : H_B^* \otimes \mathbf{Q}_p \to H_p^*(X).$$

Of course, the map I_∞satisfies:

$$I_\infty(\bigoplus_{\substack{p+q=j \\ p \geq p_0}} H_B^{p,q}) = F_j^{p_0} H_{DR}^j \otimes_K \mathbf{C}$$

If $K \subseteq \overline{\mathbf{Q}} \subseteq \mathbf{C}$ is a number field, we obtain further structures. Let $\mathbf{C}_p$ be a completion of an algebraic closure of $\mathbf{Q}_p$, and let σ_p be an embedding of $\overline{\mathbf{Q}}$ in $\mathbf{C}_p$. Let $\widehat{\sigma_p(K)}$ be the topological closure of $\sigma_p(K)$. Let B_{DR} be Fontaine's ([Fo]) $\mathbf{Z}$-filtered $\overline{\mathbf{Q}_p}$ algebra. If D_p denotes the group of continuous automorphisms of $\overline{\mathbf{Q}_p}$, then B_{DR} is a D_p module for which the action is semilinear: $\tau(\alpha b) = \tau(\alpha)\tau(b)$, if $\alpha \in \overline{\mathbf{Q}_p}$, $b \in B_{DR}$, and $\tau \in D_p$. Each element of D_{σ_p} extends uniquely to a continuous automorphism of $\mathbf{C}_p$. Let D_{σ_p} be the subgroup of D_p consisting of the elements which fix $\widehat{\sigma_p(K)}$ pointwise and let V_p be a finite dimensional $\mathbf{Q}_p$ vector space on which D_{σ_p} acts continuously. Then $V_p \otimes_{\mathbf{Q}_p} B_{DR}$ acquires a filtration from that on B_{DR} and is naturally a D_{σ_p}-module: one puts $\tau(v \otimes b) = \tau(v) \otimes \tau(b)$.

For a variety X defined over a field $K \subseteq \overline{\mathbf{Q}}$, let $\sigma_p X$ be the conjugate of X by σ_p. Basic to our work is the following result of Faltings, already cited in the Introduction:

Theorem. *For each prime p, there is a functorial D_{σ_p} equivariant filtered isomorphism*

$$I_{DR} : H^*_p(\sigma_p X) \otimes_{\mathbf{Q}_p} B_{DR} \to H^j_{DR}(\sigma_p X) \otimes_{\sigma_p(K)} B_{DR}$$

where on the right hand side D_{σ_p} acts via the right factor and the filtration is that defined by the tensor product of the filtrations on B_{DR} and $H^j_{DR}(\sigma_p X)$. The isomorphism I_{DR} is compatible with cycle maps and with extension of the ground field.

1.2. Tate twists. Let

$$\mathbf{Q}_p(1) = \varprojlim \mu_{p^n} \qquad \text{(for each prime p)}$$

$$\mathbf{Q}_B(1) = 2\pi i \mathbf{Q} \subset \mathbf{C}$$

$$\mathbf{Q}_{DR}(1) = \mathbf{Q}$$

Let

$$\chi_p : Gal(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathbf{Z}_p^* = Aut(\mathbf{Q}_p(1))$$

be the p-adic cyclotomic character; it gives the natural action of $Gal(\overline{\mathbf{Q}}/\mathbf{Q})$ on $\mathbf{Q}_p(1)$ which we always regard as a Galois module. Let $\mathbf{Q}_B(1)$ have the unique Hodge structure which is purely of type (-1,-1), and let $\mathbf{Q}_{DR}(1)$ have the Hodge filtration $F^j_{DR}\mathbf{Q}_{DR}(1) = \mathbf{Q}_{DR}(1)$ for $j \leq -1$ and $F^j_{DR}\mathbf{Q}_{DR}(1) = 0$ if $j \geq 0$. Let

$$I_\infty : \mathbf{Q}_B(1) \otimes \mathbf{C} \to \mathbf{Q}_{DR}(1) \otimes \mathbf{C} = \mathbf{C}$$

be defined by

$$I_\infty(\alpha \otimes z) = \alpha z$$

for $\alpha \in (2\pi i)\mathbf{Q}$ and $z \in \mathbf{C}$. Let

$$I_p : \mathbf{Q}_B(1) \otimes \mathbf{Q}_p \to \mathbf{Q}_p(1)$$

be the map defined via the inverse limit of the isomorphisms

$$exp : (p^{-n} 2\pi i \mathbf{Z})/(2\pi i \mathbf{Z}) \to \mu_{p^n}.$$

We have, for any n, and any finite extension L of $\mathbf{Q}_p$

$$(B_{DR} \otimes (\mathbf{Q}_p(1)^{\otimes n}))^{D_{\sigma_p}} = \widehat{\sigma_p(K)}$$

for all K and σ_p, where for $n \leq 0$, $\mathbf{Q}_p(1)^{\otimes n} = (\mathbf{Q}_p(1)^\vee)^{\otimes |n|}$. Let

$$I_{DR} : \mathbf{Q}_p(1) \otimes B_{DR} \to B_{DR}$$

be the B_{DR}-linear extension of this identity. For each subscript $! = B, DR, p$, let $\mathbf{Q}_!(n) = \mathbf{Q}_!(1)^{\otimes n}$, with the same convention as above if $n \leq 0$. In this case, let I_∞, I_p, and I_{DR} denote the maps between these objects naturally defined via those just introduced and having the same symbol.

1.3 Hodge Classes. Let

$$H_B^{2n}(X)(n) = H_B^{2n}(X) \otimes \mathbf{Q}_B(n).$$

Then $H_B^{2n}(X)(n)$ carries a Hodge structure of weight 0 defined by

$$H_B(X)^{p,q} = H_B(X)(n)^{p-n,q-n}.$$

A **Hodge class** $\gamma_B \in H_B^{2n}(X)(n)$ is an element of type (0,0). Let

$$\gamma_{DR} = I_\infty(\gamma_B)$$

and

$$\gamma_p = I_p(\gamma_B).$$

Then γ_B is **Absolutely Hodge** if, for any automorphism τ of $\mathbf{C}$, there exists a Hodge class

$$\gamma_B(\tau) \in H_B^{2n}(\tau X)(n)$$

such that

$$I_\infty(\gamma_B(\tau)) = \tau\gamma_{DR}$$

and

$$I_p(\gamma_B(\tau)) = \tau\gamma_p$$

for each p. In this case, we define $\tau\gamma_B = \gamma_B(\tau)$. Note that if γ_B is absolutely Hodge then $\gamma_{DR} \in H_{DR}^{2n}(X)(n) \otimes_K L$ for a finite extension L of K. Indeed, $Aut(\mathbf{C}/K)$ acts on the finite dimensional rational vector subspace of $H_p^{2n}(X)(n)$ generated be the absolute Hodge cycles. If the image were infinite, it would necessarily be uncountable, which is impossible. Since $\tau\gamma_{DR} = \gamma_{DR}$ if and only if $\tau\gamma_p = \gamma_p$, γ_{DR} is defined over a finite extension. The smallest such extension is called the field of definition of γ_B; it is the field defined, via Galois theory, by the stabilizer of γ_B in $Gal(\overline{K}/K)$.

Suppose now that K is a number field and γ_B is an absolute Hodge cycle. As in the Introduction, we say that γ_B is **De Rham** if, for all p and for all embeddings $\sigma_p : \overline{\mathbf{Q}} \to \mathbf{C}_p$,

$$I_{DR}(\sigma_p\gamma_p) = \sigma_p\gamma_{DR}.$$

Let X and Y be smooth connected projective varieties defined over $K \subseteq \mathbf{C}$. Suppose that X has dimension n. Define $Mor_H(X,Y)$ to be the space of all Hodge classes in $H_B^{2n}(X \times Y)(n)$ and let $Mor_{AH}(X,Y)$ be the space of all absolute Hodge classes in $H_B^{2n}(X \times Y)(n)$. If K is again a number field, let $Mor_{DR}(X,Y)$ be the space of all De Rham classes in $H_B^{2n}(X \times Y)(n)$. Finally, for X and Y not necessarily connected, define $Mor_?(X,Y) = \bigoplus_{i,j} Mor_?(X_i, Y_j)$ where the collections $\{X_i\}$ and $\{Y_j\}$ are the connected components of X and Y and $? = H, AH, or\, DR$.

2. Motives.

(2.1) Effective Motives. We now briefly sketch the construction of some categories of motives. See [DM] for more details on the formalism. Let $\mathcal{C}$ be a sub-category of the category of smooth projective varieties defined over a given field $L \subseteq \mathbf{C}$. Assume that $\mathcal{C}$ is closed under disjoint unions and products. Defined $\otimes_?^*\mathcal{C}$ ($? = H, AH, DR$) to be the category with objects the symbols $h(X)$ for X in $\mathcal{C}$, and with morphisms given by $Hom(h(X), h(Y)) = Mor_?(X,Y)$. Then $\otimes_?^*\mathcal{C}$ is a $\mathbf{Q}$-linear category for which we put $h(X) \oplus h(Y) = h(X \coprod Y)$ and $h(X) \otimes h(Y) = h(X \times Y)$. Let $\otimes_?^+\mathcal{C}$ be the category whose objects are pairs (M,p) with $M \in \otimes_?^*\mathcal{C}$ and p an idempotent element in $End(M)$. The morphisms are

$$Hom_?((M_1,p_1),(M_2,p_2)) = \{f : M_1 \to M_2 | f \circ p_1 = p_2 \circ f\}/\sim$$

where

$$\sim = \{f : M_1 \to M_2 | f \circ p_1 = p_2 \circ f = 0\}$$

We put

$$(M_1,p_1) \otimes (M_2,p_2) = (M_1 \otimes M_2, p_1 \otimes p_2)$$

The definition is so arranged that any $\mathbf{Q}$-linear functor

$$\omega : \otimes_?^*\mathcal{C} \to \{\mathbf{Q} - vectorspaces\}$$

extends to $\otimes_?^+\mathcal{C}$ and we have

$$\omega((M,p)) = Im(\omega(p))$$

Especially, the rational Hodge structure valued functor H_B^* on $\mathcal{C}$ extends to a functor ω_B on $\otimes_?^+\mathcal{C}$, and we put, for $M = (h(X),p)$,

$$M_B = p(H_B^*(X))$$

(2.2) Remark on tensor structure. Note that the canonical decomposition

$$H_B^*(X) = \bigoplus_{j=0}^{2dim X} H_B^j(X)$$

provides a family of idempotents $\{p_j\}$ so that $h(X) = \bigoplus_{j=0}^{2dim X} h^j$ with $h^j = (h(X), p_j)$ and $h_B^j = H_B^j(X)$. The functor sending M to M_B is faithful and $\mathbf{Q}$-linear. However, it is not a tensor functor (c.f. [DM]) since the commutativity isomorphism $c^* : h(X) \otimes h(Y) \to h(Y) \otimes h(X)$ given by the natural permutation isomorphism $X \times Y \to Y \times X$ sends $\gamma = \gamma_X \otimes \gamma_Y \in H_B^j(X) \otimes H_B^k(Y)$ to $c_B^*(\gamma) = (-1)^{jk}\gamma_Y \otimes \gamma_X \in H_B^j(Y) \otimes H_B^k(X)$. Hence, one replaces $c^* = \sum_{j,k} c_{j,k}^*$ by $c = \sum_{j,k} c_{j,k}$ where $c_{j,k} = (-1)^{jk} c_{j,k}^*$. With this change of the commutativity isomorphisms on $\otimes_?^+\mathcal{C}$, the category becomes a tensor category and H_B^* becomes a tensor functor.

(2.3) Motives. Suppose that $\mathcal{C}$ contains a curve Γ so that $\otimes_?^+\mathcal{C}$ contains $L = h^2(\Gamma)$. Then $L_B \cong \mathbf{Q}_B(-1)$, and, for any $n \geq 0$, $Hom(M,N)$ is canonically isomorphic to $Hom(M \otimes L^n, N \otimes L^n)$ via $\phi \to \phi \otimes 1$ for $\phi \in Hom(M,N)$. Let $\otimes_?\mathcal{C}$ denote the category obtained by inverting L: an object of $\otimes_?\mathcal{C}$ is a pair (M,n) with $M \in \otimes_?^+\mathcal{C}$ and $n \geq 0$, and the morphisms are given by

$$Hom((M,m),(N,n)) = Hom(M \otimes L^{N-m}, N \otimes L^{N-n})$$

for any $N \geq m,n$. For $n \leq 0$, put $(M,n) = M \otimes L^{|n|}$. It is conventional to define $M(n) = (M,n)$. Then the rule $M(n)_B = M_B(n)$ extends the functor $M \to M_B$ to $\otimes_?\mathcal{C}$. The categories $\otimes_H\mathcal{C}$, $\otimes_{AH}\mathcal{C}$, and $\otimes_{DR}\mathcal{C}$ are called the categories of **motives for Hodge, Absolute Hodge,and De Rham classes** , generated by $\mathcal{C}$, respectively. Note that we have obvious inclusions:

$$\otimes_{DR}\mathcal{C} \subseteq \otimes_{AH}\mathcal{C} \subseteq \otimes_H\mathcal{C}.$$

If $\mathcal{C}$ is the category generated by a single variety X, we write $\otimes_H X$, $\otimes_{AH} X$, and $\otimes_{DR} X$. Finally, we sometimes write ω_B for the functor which to a motive M attaches its topological cohomology M_B, viewed just as a rational vector space.

(2.4) Proposition. The categories $\otimes_H\mathcal{C}$, $\otimes_{AH}\mathcal{C}$, and $\otimes_{DR}\mathcal{C}$ are semisimple, Tannakian categories for which ω_B is a fiber functor.

Proof. This is proved in [DM, Section 6] for the AH case. The other cases are identical.

(2.5) Other realizations. Let $M = (h(X), e)$ be a motive in one of the categories of motives just constructed. Via the comparison isomorphisms I_∞ and I_p, the idempotent class $e = e_B$ defines $e_{DR} = I_\infty \circ e_B \circ I_\infty^{-1}$ in $End_{\mathbf{C}}(H^*_{DR}(X) \otimes_K \mathbf{C})$ and $e_p = I_p \circ e_B \circ I_p^{-1}$ in $End_{\mathbf{Q}_p}(H^*_p(X))$. Put $M_{DR,\mathbf{C}} = Im(e_{DR})$, $M_p = Im(e_p)$, and extend these functors to all of $\otimes_?\mathcal{C}$ in the evident way, i.e. via the rule $M(n)_B = M_B(n)$, etc.

If $M \in \otimes_{AH}\mathcal{C}$, and e_B is defined over K, then M_p is a $Gal(\overline{K}/K)$-module, and $e_{DR} \in End_K(H^*_{DR}(X)$ so that $Im(e_{DR}) = M_{DR}$ is a K-vector space such that $M_{DR} \otimes_K \mathbf{C} = M_{DR,\mathbf{C}}$. Then M_{DR} carries a K-rational filtration $F^\cdot M_{DR}$ such that

$$I_\infty(\bigoplus_{p \geq p_0} M_B^{p,q}) = F^{p_0} M_{DR} \otimes_K \mathbf{C}$$

If $M \in \otimes_{DR}\mathcal{C}$, then we have also, for each prime p and each $\sigma_p : \overline{\mathbf{Q}} \to \mathbf{C}_p$, the comparison isomorphism

$$I_{DR} : (\sigma_p M_p) \otimes_{\mathbf{Q}_p} B_{DR} \to (\sigma_p M_{DR}) \otimes_{\overline{\sigma_p(K)}} B_{DR}$$

(2.6) Dual groups. Let

$$\mathcal{G}_? = Aut^{\otimes}(\omega_B, \otimes_?\mathcal{C})$$

be the group of automorphisms of ω_B which respect the tensor structures (see [DM]). Then $\mathcal{G}_?$ is a a connected reductive pro-algebraic group defined over $\mathbf{Q}$; it is algebraic if and only if the ring of isomorphism classes of objects of $\otimes_?\mathcal{C}$ is finitely generated. Note that

$$\mathcal{G}_H \subseteq \mathcal{G}_{AH} \subseteq \mathcal{G}_{DR}$$

Each $\mathcal{G}_?$ acts on each M_B ($M \in \otimes_?\mathcal{C}$) via a representation ρ_M and the correspondence $M \to \rho_M$, extended to morphisms, defines an equivalence of categories for which

$$\omega_B(Hom_?(M,N)) = Hom_{\mathcal{G}_?}(M_B,N_B)$$

We have evident notions of Hodge (resp. absolute Hodge, resp. De Rham) classes on a motive M in the category $\otimes_H\mathcal{C}$ (resp. $\otimes_{AH}\mathcal{C}$, resp. $\otimes_{DR}\mathcal{C}$). Especially, if $\tau \in Aut(\mathbf{C})$, and $M \in \otimes_?\mathcal{C}$, $(? = AH or DR)$, and $M = (X,e)(n)$, then $\tau M = (\tau X, \tau e)(n)$ is defined.

(2.7) Proposition. Let $M \in \otimes_?\mathcal{C}$. Then the subspace $M_B^{\mathcal{G}_?}$ of $\mathcal{G}_?$- invariants in M_B is the subspace of all ?-classes, for $? = H, AH, or DR$.

(2.8) Proof. We give the proof for $\mathcal{G}_{DR}$. The other cases are the same. Note first that the space of De Rham classes on M is

$$\{\phi_B(1) | \phi \in Hom(\mathbf{Q}(0), M)\}$$

where $1 \in \mathbf{Q} = \mathbf{Q}(0)_B$. Thus every De Rham class is $\mathcal{G}_{DR}$-invariant. If $\gamma_B \in M_B$ is fixed by $\mathcal{G}_{DR}$, then the map $\phi_{\gamma,B} : \mathbf{Q}_B(0) \to M_B$ such that $\phi_{\gamma,B}(1) = \gamma_B$ is $\mathcal{G}_{DR}$-invariant. Hence it belongs to $Hom(\mathbf{Q}(0), M)$ and so γ_B is De Rham.

(2.9) Proposition. Let $\mathcal{A}$ and $\mathcal{B}$ be $\mathbf{Q}$-linear Tannakian categories with $\mathcal{A} \subseteq \mathcal{B}$. Let $\omega : \mathcal{B} \to \{\mathbf{Q} - vectorspaces\}$ be a fiber functor, and denote its restriction to $\mathcal{A}$ by $\omega_{\mathcal{A}}$. Let $\mathcal{G}_{\mathcal{B}} = Aut^{\otimes}(\omega, \mathcal{B})$ and $\mathcal{G}_{\mathcal{A}} = Aut^{\otimes}(\omega_A, \mathcal{A})$, so that $\mathcal{G}_{\mathcal{B}} \subseteq \mathcal{G}_{\mathcal{A}}$. Suppose that

$$\mathcal{G}_{\mathcal{B}} = \mathcal{G}_{\mathcal{A}}$$

Then the inclusion of $\mathcal{A}$ into $\mathcal{B}$ is an equivalence of categories.

(2.10) Proof. This is evident: $\mathcal{A}$ and $\mathcal{B}$ are both equivalent to the category or representations of the same group, hence they are equivalent. That the inclusion of $\mathcal{A}$ into $\mathcal{B}$ defines an equivalence is also clear since under $\omega_{\mathcal{A}}$ the inclusion of the representations of $\mathcal{G}_{\mathcal{A}}$ obtained from $\mathcal{A}$ into those obtained from $\mathcal{B}$ is an equivalence of categories.

(2.11) Proposition(Principle A). Let X be a smooth projective variety defined over the complex numbers. The following are equivalent:

(1) $\otimes_{AH}X = \otimes_H X$
(2) $\mathcal{G}_H = \mathcal{G}_{AH}$
(3) Every Hodge class in $\otimes_H X$ is absolutely Hodge.

Suppose that X is defined over a number field. Then the following are equivalent:

(4) $\otimes_{AH}X = \otimes_{DR}X$
(5) $\mathcal{G}_{DR} = \mathcal{G}_{AH}$
(6) Every absolute Hodge class in $\otimes_{AH}X$ is De Rham.

(2.12) Proof. By Prop.(2.7), it is clear that [1] is equivalent to [3] and [4] is equivalent to [6]. Furthermore, [3] implies [2] and [6] implies [5]. Hence we need only show that [2] implies [1] and [5] implies [4]. Let $M \in \otimes_{AH}X$ with associated representation ρ_M of $\mathcal{G}_{AH}$. Then M is indecomposable in $\otimes_{AH}X$ if and only if ρ_M is irreducible, since the category is semisimple. Suppose that, as an element of $\otimes_H X$, $M = M_1 \oplus M_2$. Then the restriction of ρ_M to $\mathcal{G}_H$ is a non-trivial direct sum $\rho_1 \oplus \rho_2$. But since $\mathcal{G}_H = \mathcal{G}_{AH}$, this cannot happen. Thus each M which is irreducible in $\otimes_{AH} X$ remains irreducible in $\otimes_H X$. On the other hand, every irreducible object of $\otimes_H X$ is a constituent of a $\otimes_{AH} X$ irreducible object, by definition of the categories. Hence, $\otimes_{AH} X$ and $\otimes_H X$ have the same objects. Since

$$\omega_B(Hom_H(M,N)) = Hom_{\mathcal{G}_{\mathcal{H}}}(M_B, N_B) = Hom_{\mathcal{G}_{\mathcal{AH}}}(M_B, N_B) = \omega_B(Hom_{AH}(M,N))$$

we conclude that

$$Hom_H(M,N) = Hom_{AH}(M,N)$$

as well. Thus , $\otimes_{AH} X = \otimes_H X$, as was to be shown.

The proof for the second case is exactly parallel, with "AH" replacing "H" and "DR" replacing "AH". (See [D1] for another approach which does not employ Tannakian duality.)

3. Principle B.

(3.1) Theorem. Let S be a smooth, geometrically connected variety defined over the subfield K of $\mathbf{C}$. Let $\pi : X \to S$ be a smooth proper morphism defined over K. Let $\gamma_B \in H^0(S, \mathbf{R}^{2n}\pi_*\mathbf{Q})(n)$. For $s \in S(L)$, let $\gamma_B(s) \in H_B^{2n}(X_s)(n)$ be the restriction of γ_B to the fiber $X_s = \pi^{-1}(s)$. Let $s_0 \in S(K)$. Then:

(1) Suppose $K = \mathbf{C}$. If $\gamma_B(s_0)$ is a Hodge class, then $\gamma_B(s)$ is a Hodge class for all $s \in S(\mathbf{C})$.
(2) Suppose $K = \mathbf{C}$. If $\gamma_B(s_0)$ is an absolute Hodge class, then $\gamma_B(s)$ is an absolute Hodge class for all $s \in S(\mathbf{C})$.
(3) Suppose $K \subseteq \overline{\mathbf{Q}}$. If $\gamma_B(s_0)$ is De Rham, then $\gamma_B(s)$ is De Rham for all $s \in S(\overline{\mathbf{Q}})$.

(3.2) Proof. The Leray spectral sequence degenerates at E_2 and provides a surjection $\alpha : H_B^{2n}(X)(n) \to H^0(S, \mathbf{R}^{2n}\pi_*\mathbf{Q})(n)$ whose kernel we denote K_B. For $s \in S(K)$, the restriction $\beta_s : H^0(S, \mathbf{R}^{2n}\pi_*\mathbf{Q})(n) \to H_B^{2n}(X_s)(n)$ is injective. Let $\widehat{\gamma_B} \in H_B^{2n}(X)(n)$ satisfy $\alpha(\widehat{\gamma_B}) = \gamma_B$. Then $\gamma_B(s) = \beta_s \circ \alpha(\widehat{\gamma_B})$, and the kernel of $\beta_s \circ \alpha : H_B^{2n}(X)(n) \to H_B^{2n}(X_s)(n)$ equals K_B and is independent of $s \in S(K)$. Since $\beta_s \circ \alpha$ is a morphism of mixed Hodge structures, $\beta_s \circ \alpha$ identifies $H_B^{2n}(X)(n)/K_B$ with a pure sub-Hodge structure of $H_B^{2n}(X_s)(n)$ which is independent of s. Let $\overline{\gamma_B}$ be the image of $\widehat{\gamma_B}$ in $H_B^{2n}(X)(n)/K_B$.Since $\beta_s \circ \alpha(\widehat{\gamma_B})$ is a Hodge class, so is $\overline{\gamma_B}$. Hence $\beta_s \circ \alpha(\overline{\gamma_B}) = \gamma_B(s)$ is a Hodge class for all $s \in S(\mathbf{C})$. This proves the first claim.

We now prove the second claim. Let $\widehat{\gamma_{DR}} = I_\infty(\widehat{\gamma_B})$, $\widehat{\gamma_p} = I_p(\widehat{\gamma_B})$, $K_{DR} = I_\infty(K_B \otimes \mathbf{C})$ and $K_p = I_p(K_B \otimes \mathbf{Q}_p)$. Restricting to X_{s_0}, we have $\widehat{\gamma_{DR}}(s_0) = I_\infty(\gamma_B(s_0))$ and $\widehat{\gamma_p}(s_0) = I_p(\gamma_B(s_0))$. Let $\sigma \in Aut(\mathbf{C})$. Then $\sigma(K_p)$ is the kernel of restriction $H_p^{2n}(\sigma X)(n) \to H_p^{2n}(X_{\sigma(s)})(n)$ and σK_{DR} is the kernel of restriction $H_{DR}^{2n}(\sigma X)(n) \to H_{DR}^{2n}(X_{\sigma(s)})(n)$ for all $s \in S(\mathbf{C})$. Since $\sigma(\widehat{\gamma_p})$ restricts to $\sigma(I_p(\gamma_B(s_0)))$, $\gamma^* = I_p^{-1}(\sigma(\widehat{\gamma_p})) \in H_B^{2n}(\sigma X)(n) \otimes \mathbf{Q}_p$ restricts to $I_p^{-1}(\sigma\gamma_p(s_0)) = \sigma\gamma_B(s_0)$. But if $\phi : V \to W$ is a linear map of rational vector spaces, and there exists $v \in V \otimes \mathbf{Q}_p$ such that $\phi(v) = w \in W$, then there exists $v' \in V$ such that $\phi(v') = w$. Hence there exists $\widehat{\gamma_B}(\sigma) \in H_B^{2n}(\sigma X)(n)$ whose restriction to $H_B^{2n}(X_{\sigma(s_0)})(n)$ is $\sigma\gamma_B(s_0)$. Note that i) $I_p(\widehat{\gamma_B}(\sigma)) - \sigma(\widehat{\gamma_p}) \in \sigma(K_p)$ and ii) $I_\infty(\widehat{\gamma_B}(\sigma)) - \sigma(\widehat{\gamma_{DR}}) \in \sigma(K_{DR})$. Let $K_B(\sigma)$ denote the kernel of restriction $H_B^{2n}(\sigma X)(n) \to$

$H_B^{2n}(X_{\sigma(s_0)})(n)$. Then $I_p(K_B(\sigma)) \otimes \mathbf{Q}_p = \sigma(K_p)$ and hence $K_B(\sigma)$ is also the kernel of restriction $H_B^{2n}(\sigma X)(n) \to H_B^{2n}(X_{\sigma(s)})(n)$ for all $s \in S(\mathbf{C})$. We now argue as before. Let $\overline{\gamma_B}(\sigma)$ be the image of $\widehat{\gamma_B}(\sigma))$ in $H_B^{2n}(\sigma X)(n)/K_B(\sigma)$. Then $\overline{\gamma_B}(\sigma)$ is a Hodge class , and hence its image $\overline{\gamma_B}(\sigma, s) \in H_B^{2n}(X_{\sigma(s)})(n)$ is a Hodge class for every $s \in S(\mathbf{C})$. Finally, that $I_\infty(\overline{\gamma_B}(\sigma, s)) = \sigma(\gamma_{DR}(s))$ and $I_p(\overline{\gamma_B}(\sigma, s)) = \sigma(\gamma_p(s))$ follows at once from i) and ii) above. This proves the second part.

To prove the third part, let $\sigma_p : \overline{\mathbf{Q}} \to \mathbf{C}_p$ be an embedding and define the maps I_{DR} relative to this σ_p. Let $\overline{X}$ be a smooth compactification of X, defined over K, with $\overline{X} - X$ a union of smooth divisors having normal crossings. By [D4, Thm. 4.1.1], the natural map

$$\tilde{\alpha} : H_B^{2n}(\overline{X})(n) \to H^0(S, \mathbf{R}^{2n}\pi_*\mathbf{Q})(n)$$

is surjective. As before, this means that the kernel $\widetilde{K_B}$ of the restriction $\beta_s o\tilde{\alpha} : H_B^{2n}(\overline{X})(n) \to H_B^{2n}(X_s)(n)$ is independent of $s \in S(L)$. Let $\widetilde{K_{DR}} \subseteq H_{DR}^{2n}(\overline{X})(n)$ be the kernel of restriction to $H_{DR}^{2n}(X_{s_0})(n)$. Since $\widetilde{K_{DR}} = I_\infty(K_B \otimes \mathbf{C}$, it is also the kernel of restriction for all $s \in S(L)$.

Let $\overline{\gamma_B} \in H_B^{2n}(\overline{X})(n)$ restrict to γ_B. Then $\overline{\gamma_B}(s_0) = \gamma_B(s_0)$ is a De Rham class by hypothesis. Replacing K by a finite extension, if necessary, we see that the restriction to X_{s_0} of $I_\infty(\overline{\gamma_B})$ equals $I_\infty(\overline{\gamma_B}(s_0)) \in H_{DR}^{2n}(\overline{X}_{s_0})(n)$. Hence, there exists $\overline{\gamma_{DR}} \in H_{DR}^{2n}(\overline{X})(n)$ such that $\overline{\gamma_{DR}}(s_0) = I_\infty(\overline{\gamma_B}(s_0))$. Similarly, there exists $\overline{\gamma_p} \in H_p^{2n}(\overline{X})(n)$ such that $\overline{\gamma_p}(s_0) = I_p(\overline{\gamma_B}(s_0))$. By assumption, we have $I_{DR}(\sigma_p\overline{\gamma_p}(s_0)) = \sigma_p\overline{\gamma_{DR}}(s_0)$. Hence $I_{DR}(\sigma_p\overline{\gamma_p}) - \overline{\gamma_{DR}}$ belongs to $\sigma_p\widetilde{K_{DR}} \otimes B_{DR}$. Let $s \in S(L)$. Then

$$I_{DR}(\sigma_p(\overline{\gamma_p}(s))) = I_{DR}(\sigma_p\overline{\gamma_p})(s) = \sigma_p(\overline{\gamma_{DR}}(s))$$

since I_{DR} commutes with restriction and $\sigma_p\widetilde{K_{DR}}$ restricts to 0. Since p, σ_p and $s \in S(L)$ are arbitrary, we are done.

4. Completion of the proof. In this section, we review the objects and steps of Deligne's proof of his absolute Hodge cycles theorem, giving also the extension to the p-adic maps I_{DR}. However, our exposition is not fully self-contained and the reader will need to consult [D1] to fill in the details.

Let A be an abelian variety. We have attached to A the 3 Tannakian categories $\otimes_H\mathcal{C}$, $\otimes_{AH}\mathcal{C}$, and $\otimes_{DR}\mathcal{C}$, and hence three groups

$$\mathcal{G}_H \subseteq \mathcal{G}_{AH} \subseteq \mathcal{G}_{DR}.$$

By Proposition (2.11), it is enough for us to prove

$$\mathcal{G}_H = \mathcal{G}_{AH} = \mathcal{G}_{DR}.$$

(4.1) The CM case: a reduction. Let K be a CM field (i.e. a totally imaginary quadratic extension of a totally real number field) which is Galois over $\mathbf{Q}$. A *CM type* of K is a set Φ of complex embeddings of K such that $\Phi \cup \Phi\rho$ is all embeddings and $\Phi \cap \Phi\rho$ is empty. Embedding K into $\mathbf{C}^\Phi$ by sending $k \in K$ to $z(k) \in \mathbf{C}^\Phi$ defined by $z(k(\sigma)) = \sigma(k)$, the ring of integers $\mathcal{O}_K$ of K becomes a lattice in $\mathbf{C}^\Phi$ and $A_\Phi = \mathbf{C}^\Phi/\mathcal{O}_K$ is an abelian variety. Let S be the set of all CM types. By definition , an abelian variety of CM type is an abelian variety which is isogenous to a product of such A_Φ, where K is allowed to vary. All such abelian varieties admit projective models defined over $\overline{\mathbf{Q}}$

Note that, if $A \subseteq B$ and if the above equality of groups holds for B, then it holds for A, by Prop.(2.11). Furthermore, the equality i) holds for A if and only if it holds for A^n with any positive integer n, and ii) holds for A if and only if it holds for abelian variety isogenous to A. Note finally that if A has an action of K, then A^n carries an action of any field extension of K of degree n. Hence, starting from any A of CM type we can find a Galois CM extension L of $\mathbf{Q}$ such that A is isogenous to a quotient of a power of

$$B = \prod_{\Phi\in S} A_\Phi$$

where S denotes the set of all CM types of L. By the above remarks, to prove our claim for A, it is enough to prove it for B. (c.f. [D1,p.65] for more detail about this reduction.)

(4.2) Special De Rham classes. Deligne finds three special types of absolute Hodge classes:

[1] Classes of the graphs of endomorphisms given by the evident embedding of L into $End(A_\Phi)$ for each Φ.

[2] Let $\sigma \in Gal(L/\mathbf{Q})$. Via $\sigma : L \to L$, each A_Φ becomes also of type $A_{\Phi\sigma}$, and we have a natural isomorphism of A_Φ with $A_{\Phi\sigma}$.

[3] Let $T \subseteq S$ have d elements. Let

$$B_T = \prod_{\Phi\in T} A_\Phi$$

Suppose that for the action of L on $H^{10}(B_T)$ each embedding of L occurs with equal multiplicity, necessarily equal to $d/2$. Then

$$\bigwedge_L^d H_B^1(B_T)(d/2) \subseteq H_B^d(B_T)(d/2)$$

consists of De Rham classes.

The classes of types [1] and [2] are De Rham because they are algebraic and a principal point of the argument of [D1] is to show that the classes of type [3] are absolutely Hodge using Principle B. To do this, one first constructs a universal family of abelian varieties $\pi : \mathcal{A} \to X$ parametrized by an arithmetic quotient X of the symmetric space attached to a certain unitary group in d variables associated to the quadratic extension defined by L relative to its maximal totally real subfield. This family contains B_T, carries an action of $\mathcal{O}_L$, and has a fiber of the form $A_0^{[L:\mathbf{Q}]}$ where the L action is that defined by an embedding of L into

$$M_{[L:\mathbf{Q}]}(\mathbf{Q}) \subseteq End(A_0^{[L:\mathbf{Q}]})$$

The association to $x \in X(\mathbf{C})$ of $\wedge_L^d H_B^1(A_x)(d/2)$ is a constant local sub-system on X whose global sections lie in $H^0(X, R^d\pi_*\mathbf{Q})$. Further, at the point x_0 with fiber $A_0^{[L:\mathbf{Q}]}$ all the elements of $\bigwedge_L^d H_B^1(A_{x_\bullet})(d/2)$ are algebraic, hence absolutely Hodge and De Rham. In fact, this space is generated over L by the class of the cycle $A_0^{[L:\mathbf{Q}]-1} \times \{0\} \subset A_0^{[L:\mathbf{Q}]}$. Thus, Principle B applies: the classes of type [3] are absolutely Hodge. To see that they are De Rham we need only note also that i) $\pi : \mathcal{A} \to X$ is defined over a number field, since it is a universal family attached to a moduli problem of PEL type, ii) the point of X corresponding to B_T is algebraic, since B_T is of CM type, and iii) the A_0 of Deligne's construction is, up to isogeny, any abelian variety of dimension $d/2$, and hence both A_0 and x_0 can be taken to be defined also over a number field. Thus, Principle B applies and the classes in $\bigwedge_L^d H_B^1(B_T)(d/2)$ are De Rham.

(4.3) Completion of the CM case. Note first that the classes of type [1] force

$$\mathcal{G}_H \subseteq \mathcal{G}_{AH} \subseteq \mathcal{G}_{DR} \subseteq (\prod_{\Phi} \mathrm{L}^*) \times G_m \overset{def}{=} G$$

Next, observe that $\mathcal{G}_H$ can be explicitly described: it is the $\mathbf{Q}$-Zariski closure in $G(\mathbf{C})$ of the cocharacter $\mu : G_m/\mathbf{C} \to G/\mathbf{C}$ which acts on the (1,0) classes in $H_B^1(A_\Phi)$ by sending z to z^{-1} , acts on the (0,1) classes trivially, and projects to the identity on the G_m factor. Since G acts on $H_B^1(B)$ and $\mathbf{Q}(1)_B$ via projection on the first and second factors, respectively, it acts on all tensor expressions

$$(H_B^1(B)^{\otimes n} \otimes H_B^1(B)^{\vee \otimes m})(k) = W$$

and the subspaces of W_H, W_{AH}, and W_{DR} of Hodge, absolutely Hodge, and De Rham classes are stable for this action. If $\gamma \in W_H \otimes \overline{\mathbf{Q}}$ (resp. $W_{AH} \otimes \overline{\mathbf{Q}}$, resp. $W_{DR} \otimes \overline{\mathbf{Q}}$) transforms according to the character χ of $G/\overline{\mathbf{Q}}$, then χ has trivial restriction to $\mathcal{G}_H$ (resp. $\mathcal{G}_{AH}$, resp. $\mathcal{G}_{DR}$). The classes of types[2] and [3] are De Rham and therefore provide an explicit submodule $\mathcal{X}$ of the character group $\mathcal{X}_G$ of G whose elements restrict trivially to $\mathcal{G}_{DR}$. On the other hand, using the explicit description of $\mathcal{G}_H$ via μ, Deligne shows by linear algebra that any element of $\mathcal{X}_G$ which restricts trivially to $\mathcal{G}_H$ belongs to $\mathcal{X}$. Thus, every character of G which restricts trivially to $\mathcal{G}_H$ also restricts trivially to $\mathcal{G}_{DR}$. Hence $\mathcal{G}_{DR} \subseteq \mathcal{G}_H$ and so $\mathcal{G}_{DR} = \mathcal{G}_H$ and we are done.

(4.4) Completion of the proof. Let A be an abelian variety, not of CM type. The data $(\mathcal{G}_H, \mu)$, with $\mu : G_m/\mathbf{C} \to \mathcal{G}_H/\mathbf{C}$ defined as above for A, define (choosing additional structure, in particular a sufficiently small open compact subgroup U of $\mathcal{G}_H(\mathbf{A}_f)$) a Shimura variety Sh_U which carries a natural family of abelian varieties $\pi : \mathcal{A} \to Sh_U$, such that there exists $s_0 \in Sh_U(\mathbf{C})$ for which $\pi^{-1}(s_0)$ is isogenous to A. Identify $\pi^{-1}(s_0)$ with A. If

$$\gamma_{s_0} \in (H_B^1(A)^{\otimes n} \otimes H_B^1(A)^{\vee \otimes m})(\frac{n-m}{2}) = W_{s_0}$$

is a Hodge class, then this family has the property that γ_{s_0} extends to a global section γ_B of $H^0(Sh_U, R^{m+n}\pi_*\mathbf{Q})$, where we have used the identification $H_B^1(C)^\vee = H_B^1(C)(1)$ for any abelian variety C. Further , since A is not of CM type, $dim(Sh_U) > 0$ and, by a general principle (c.f.[D1]), there exists $s_1 \in Sh_U(\mathbf{C})$ such that $\pi^{-1}(s_1)$ is of CM type. Hence, by Principle B, $\gamma_B(s_0)$ is an absolute Hodge class since this is true of $\gamma_B(s_1)$.

To obtain the De Rham result, we must only check that $\pi : \mathcal{A} \to Sh_U$ and s_1 are defined over $\overline{\mathbf{Q}}$. This is clear for Sh_U itself since it is a Shimura variety, and to see it for the family, one may remark either: 1) that $\pi : \mathcal{A} \to Sh_U$ is the universal family attached to a moduli problem defined by absolute Hodge cycles, polarization and level structure, or 2) that there is a natural $\overline{\mathbf{Q}}$-embedding of Sh into $\mathcal{A}_{\delta,n}$, the moduli space of abelian varieties of dimension $n = \frac{d}{2}[L : \mathbf{Q}]$ of polarization degree δ (determined by the degree of the polarization chosen on A), and that $\pi : \mathcal{A} \to Sh_U$ is just the pullback of the universal family over $\mathcal{A}_{\delta,n}$. Finally, it is clear that s_1 is defined over $\overline{\mathbf{Q}}$: from the first viewpoint, it is the modulus point on Sh_U associated to $\pi^{-1}(s_1)$ and its additional structure, and since $\pi^{-1}(s_1)$ is of CM type, this data can have only finitely many distinct isomorphism classes of conjugates under the elements of $Aut(\mathbf{C})$; from the second viewpoint, the image of s_1 in $\mathcal{A}_{\delta,n}$ is algebraic, by the same argument, and hence so is s_1. Hence, we can apply Principle B to conclude that γ_{s_0} is De Rham.

5. A crystalline consequence.

5.1. Let X be a proper smooth variety defined over the number field K. Suppose that $\sigma_p X \times_{\sigma_p(K)} \widehat{\sigma_p(K)}$ has good reduction. Then:

(1) the crystalline cohomology groups $H^j_{cris}(\sigma_p X)$ are defined for all $j \geq 0$. They are vector spaces over the maximal subextension $W(\sigma_p)$ of $\widehat{\sigma_p(K)}$ which is unramified over $\mathbf{Q}_p$. Each carries a $\mathbf{Q}_p$ linear automorphism Φ which is ϕ semi-linear, where ϕ denotes the Frobenius automorphism of $W(\sigma_p)$: $\Phi(\alpha v) = \phi(\alpha)\Phi(v)$ for $v \in H^j_{cris}(\sigma_p X)$ and $\alpha \in W(\sigma_p)$.

(2) There is a canonical identification
$$H^j_{cris}(\sigma_p X) \otimes_{W(\sigma_p)} \widehat{\sigma_p(K)} = H^j_{DR}(\sigma_p X)$$

(3) Let B_{cris} denote the algebra introduced by Fontaine in [Fo]. It contains the maximal unramified extension of $\mathbf{Q}_p$ inside $\mathbf{C}_p$ and it carries a D_{σ_p} action extending the natural action on the maximal unramified extension. Further, it carries an automorphism Φ_{cris} which extends the action of Frobenius on the maximal unramified extension and commutes with the action of D_{σ_p}. Then
there is B_{cris} linear isomorphism
$$I_{cris} : H^j_p(\sigma_p X) \otimes_{\mathbf{Q}_p} B_{cris} \to H^j_{cris}(\sigma_p X) \otimes_{W(\sigma_p)} B_{cris}$$
which is D_{σ_p} equivariant. Here we use the same definitions for the D_{σ_p} action on each side as in the De Rham case. The isomorphism is compatible with products and with cycle maps.

(4) The isomorphism I_{cris} satisfies
$$I_{cris} \circ (1 \otimes \Phi_{cris}) = (\Phi \otimes \Phi_{cris}) \circ I_{cris}$$

(5) We have
$$I_{cris} \otimes 1 = I_{DR}$$

Remarks. The first two properties are basic to the theory of crystalline cohomology. The third and fourth are fundamental results of [F] and the fifth, while not explicitly claimed in [F], follows easily ([F2]) from the compatiblity of the constructions of I_{cris} and I_{DR}.

We extend these assertions to the Tate-twisted case by setting

$$H^j_{cris}(\sigma_p X)(k) = H^j_{cris}(\sigma_p X),$$

and by putting

$$\Phi_{H^j_{cris}(\sigma_p X)(k)} = p^{-k}\Phi_{H^j_{cris}(\sigma_p X)}.$$

5.2. Let $\gamma_B \in H^{2j}_B(X)(j)$ be a De Rham class which is defined over the number field K. Let $\sigma_p : K \to \mathbf{C}_p$ be an embedding. We say that γ_B is **crystalline** at σ_p if

(1) X has good reduction at σ_p.
(2) γ_{DR} belongs to the crystalline subspace $H^{2j}_{cris}(\sigma_p X)(j)$ of $H^{2j}_{DR}(\sigma_p X)(j)$
(3) $\Phi(\gamma_{DR}) = \gamma_{DR}$

5.3 Theorem. *Let A be an abelian variety defined over K with good reduction at σ_p. Let $\gamma_B \in H^{2j}_B(A)(j)$ be a Hodge class defined over K. Then γ_B is crystalline at σ_p.*

Proof. We have

$$\sigma_p\gamma_{DR} = I_{DR}(\sigma_p\gamma_p) = I_{cris}(\sigma_p\gamma_p) \in H^{2j}_{cris}(\sigma_p A)(j),$$

thus proving the first claim. Since

$$\Phi(\sigma_p\gamma_{DR}) = \Phi(I_{cris}(\sigma_p\gamma_p)) = I_{cris}((1\otimes\Phi_{cris})(\sigma_p\gamma_p\otimes 1)) = I_{cris}(\sigma_p\gamma_p) = \sigma_p\gamma_{DR},$$

the second claim follows as well.

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Los Angeles, CA 90024