Koecher-Maass series for positive definite Fourier coefficients of real analytic Siegel-Eisenstein series of degree two

Yoshinori Mizuno

Abstract

We give a meromorphic continuation and a functional equation for the Koecher-Maass series for positive definite Fourier coefficients of real analytic Siegel-Eisenstein series of degree two.

1 Introduction

The purpose of this paper is to give a meromorphic continuation and a functional equation of the Koecher-Maass series for positive definite Fourier coefficients of real analytic Siegel-Eisenstein series of degree two. It is well known that Koecher-Maass series associated with any holomorphic Siegel modular form has a meromorphic continuation and a functional equation [14], [12]. As for non holomorphic Siegel modular froms, the Koecher-Maass series associated with real analytic Siegel-Eisenstein series of degree $n \geq 3$ for any signature was introduced by Arakawa [1]. Their meromorphic continuation and a vector type functional equation were shown. See also [7] in which explicit forms of the Koecher-Maass series were given and Arakawa's functional equation was simplified employing results in [16]. The case degree is two has a special difficulty to study associated Koecher-Maass series. In this paper we will consider easier half. In particular a meromorphic continuation and a functional equation for the Koecher-Maass series for positive definite Fourier coefficients are given.

Let k be an even integer and σ a complex number such that $2\Re\sigma + k > 3$. A real analytic Siegel-Eisenstein series of degree two and weight k is defined by

$$E_{2,k}(Z,\sigma) = \sum_{\{C,D\}} \det(CZ+D)^{-k} |\det(CZ+D)|^{-2\sigma}, \qquad Z \in H_2,$$

where the sum is taken over all pairs $\{C, D\}$ which occur as the second matrix row of representatives of $\Gamma_\infty^{(2)} \setminus Sp_2(\mathbf{Z})$ with the standard notations and $H_2 = \{Z = {}^t Z \in M_2(\mathbf{C}); \Im Z > O\}$ is the Siegel upper half-space of degree 2. It has a Fourier expansion

$$E_{2,k}(Z,\sigma) = \sum_T C(T,\sigma,Y) e(\mathrm{tr}(TX)), \qquad Z = X + iY,$$

where the summation extends over all half-integral symmetric matrices of size two and $e(x) = e^{2\pi i x}$ as usual. By [19], [20], [14], [13], if $\det T \neq 0$ then the Fourier coefficients can be written as a product of the Siegel series $b(T, k+2\sigma)$ and a certain function $\xi(Y, T, \sigma+k, \sigma)$ (essentially the confluent hypergeometric function of degree two):

$$C(T,\sigma,Y) = b(T,k+2\sigma)\xi(Y,T,\sigma+k,\sigma),$$

$$b(T,\sigma) = \sum_{R \in S_2(\mathbf{Q})/S_2(\mathbf{Z})} \nu(R)^{-\sigma} e(\mathrm{tr}(TR)),$$

$$\xi(Y,T,\alpha,\beta) = \int_{S_2(\mathbf{R})} e(-\mathrm{tr}(TX)) \det(X+iY)^{-\alpha} \det(X-iY)^{-\beta} dX,$$

where $S_2(\mathbf{K})$ is the set of all symmetric matrices of size two whose components are in $\mathbf{K}$ and $\nu(R) = |\det C|$ when we write $R = C^{-1}D$, $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in Sp_2(\mathbf{Z})$.

Let $a_{2,k}(T,\sigma)$ be an arithmetic part of $C(T,\sigma,Y)$ defined by

$$a_{2,k}(T,\sigma) = \gamma_2(k+2\sigma)|\det 2T|^{k+2\sigma-3/2} 2^2 b(T, k+2\sigma),$$

where $\gamma_2(\sigma) = e^{\pi i \sigma} \pi^{2\sigma-1/2} \Gamma(\sigma)^{-1} \Gamma(\sigma-1/2)^{-1}$. Note that $b(T, k+2\sigma)$ has a meromorphic continuation to all σ. Then following Ibukiyama and Katsurada [7], the Koecher-Maass series for positive definite Fourier coefficients is defined by

$$L_{2,k}^{(2)}(s,\sigma) = \sum_{T \in L_2^+/SL_2(\mathbf{Z})} \frac{a_{2,k}(T,\sigma)}{\epsilon(T)(\det T)^s},$$

where L_2^+ is the set of all half-integral positive definite symmetric matrices of size two, the summation extends over all $T \in L_2^+$ modulo the usual action $T \to T[U] = {}^tUTU$ of the group $SL_2(\mathbf{Z})$ and $\epsilon(T) = \sharp\{U \in SL_2(\mathbf{Z}); T[U] = T\}$ is the order of the unit group of T. Put

$$L_{2,k}^*(s,\sigma) = (2\pi)^{-2s} \Gamma(s) \Gamma(s-2\sigma-k+3/2) L_{2,k}^{(2)}(s,\sigma).$$

The main result is the following.

Theorem 1. *Suppose that $\sigma \notin 1/4 + \mathbf{Z}/2$. Then the Koecher-Maass series $L_{2,k}^*(s,\sigma)$ can be meromorphically continued to the whole s-plane. It satisfies a functional equation*

$$\begin{aligned} & L_{2,k}^*(k+2\sigma-s,\sigma) = L_{2,k}^*(s,\sigma) \\ + \quad & 2\pi^{-k-2\sigma+1/2} \frac{\gamma_2(k+2\sigma)\zeta(k+2\sigma-1)}{\zeta(k+2\sigma)\zeta(2k+4\sigma-2)} \\ \times \quad & \frac{\sin\pi\sigma \sin\pi(s-\sigma)}{\cos\pi s \sin\pi(s-2\sigma)} \frac{\Gamma(s)\Gamma(s-2\sigma-k+3/2)}{\Gamma(s-1/2)\Gamma(s-2\sigma-k+1)} \zeta^*(2s-1)\zeta^*(2s-4\sigma-2k+2), \end{aligned}$$

where $\zeta^*(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ is the completed Riemann zeta function.

As a special case we have analytic properties of a Dirichlet series defined by

$$\xi(s) = \pi^{-2s}\Gamma(s)\Gamma(s-1/2)\zeta(2s-1) \sum_{d=1}^{\infty} \frac{H(d)^2}{d^s},$$

where $H(d)$ is the weighted class number given by

$$H(d) = \sum_{T \in L_2^+/SL_2(\mathbf{Z}),\ \det 2T = d} \epsilon(T)^{-1}.$$

Then as a corollary to Theorem 1 we have the following result. In fact $\xi(s)$ is the Koecher-Maass series for the non-holomorphic Siegel-Eisenstein series of degree two and weight two ($k=2, \sigma=0$). It should be compaired with [21, p.42, Theorem 2 (i)], [23, p.229, Theorem 3], [17], [10].

Corollary 1. *The Dirichlet series $\xi(s)$ can be meromorphically continued to the whole s-plane. It satisfies a functional equation*

$$\xi(2-s) = \xi(s) + 2^{-3}\pi^{-3/2}\frac{\Gamma(s)}{\cos\pi s\Gamma(s-1)}\zeta^*(2s-1)\zeta^*(2s-2).$$

We want to define a Koecher-Maass series not only for positive definite Fourier coefficients, but also for indefinite Fourier coefficients. We can expect to replace $\epsilon(T)^{-1}$ by a certain volume $\mu(T)$ introduced by Siegel. See [21], [7], [8, p.1100] for its definition. However it is known that if $-\det T$ is a square of a rational number then $\mu(T)$ is not finite. The same difficulty comes up when we treat the prehomogeneous zeta function associated with the space of two by two symmetric matrices. This case was solved by Shintani [21]. There are different approaches due to Sato [17] and Ibukiyama and Saito [10]. Ibukiyama and Saito [10] proved all of the Shintani's results by using certain real analytic Eisenstein series of half-integral weight. See also [23]. Our treatment of the Koecher-Maass series is a "convolution version" of their method. In the future we will use results in this paper to get a reasonable definition of Koecher-Maass series for the indefinite case by following the approach developed in [9], [10].

2 Koecher-Maass series and certain convolution product

In this section an explicit formula for the Koecher-Maass series will be given as a convolution product. Then we summarize analytic properties of this convolution product. Throughout this paper, the branch of z^α is taken so that $-\pi < \arg z \le \pi$.

Proposition 1. *One has*

$$L_{2,k}^{(2)}(s,\sigma) = 2^{2s+2}\gamma_2(k+2\sigma) \times\frac{\zeta(2s-k-2\sigma+1)}{\zeta(k+2\sigma)\zeta(2k+4\sigma-2)}\sum_{d>0}H(d)L_{-d}(k+2\sigma-1)d^{k+2\sigma-3/2-s}.$$

Here

$$L_D(s) = \begin{cases} \zeta(2s-1), & D=0 \\ L(s,\chi_K)\sum_{a|f}\mu(a)\chi_K(a)a^{-s}\sigma_{1-2s}(f/a), & D\neq 0,\ D\equiv 0,1 \pmod 4 \\ 0, & D\equiv 2,3 \pmod 4), \end{cases}$$

where the natural number f is defined by $D = d_K f^2$ with the discriminant d_K of $K = \mathbf{Q}(\sqrt{D})$, χ_K is the Kronecker symbol, μ is the Möbius function and $\sigma_s(n) = \sum_{d|n} d^s$.

Proof. For non-degenerate T, the explicit formula due to Kaufhold [13] implies

$$b(T,\sigma) = \frac{1}{\zeta(\sigma)\zeta(2\sigma-2)}\sum_{d|e(T)} d^{2-\sigma}L_{\frac{-\det 2T}{d^2}}(\sigma-1),$$

where $e(T) = (n,r,m)$ is the greatest common divisor of n,r,m for $T = \begin{pmatrix} n & r/2 \\ r/2 & m\end{pmatrix}$. Hence by Böcherer [2, p.20, Satz 3 (d)], we get

$$\sum_{T\in L_2^+/SL_2(\mathbf{Z})} \frac{b(T,\sigma)}{\epsilon(T)\det T^s} = 2^{2s}\frac{\zeta(2s+\sigma-2)}{\zeta(\sigma)\zeta(2\sigma-2)}\sum_{d>0} H(d)L_{-d}(\sigma-1)d^{-s}$$

and thus complete the proof.

For a complex number σ, odd k and an integer d, put

$$c(d,\sigma,k) = 2^{k+3/2-2\sigma}e^{(-1)^{(k+1)/2}\frac{\pi i}{4}}\frac{L_{(-1)^{(k+1)/2}d}(\sigma-\frac{k+1}{2})}{\zeta(2\sigma-k-1)}.$$

This is an arithmetic part of the d-th Fourier coefficient of a real analytic Eisenstein series of half-integral weight $-k/2$ with a parameter σ on $\Gamma_0(4)$ called Cohen's Eisenstein series ([10], [16]). We fix even k and a complex number σ. Then for complex numbers η and s such that $\Re s$ sufficiently large, a kind of convolution product $R_\infty(s,\eta)$ is defined by

$$R_\infty(s,\eta) = \sum_{d\neq 0} c(d,\eta,-3)\overline{c(d,2\sigma+2,-2k+5)}I_d(s,\eta,2\sigma+2,-3,-2k+5),$$

where $I_d(s,\sigma_1,\sigma_2,k_1,k_2)$ is the Mellin transform of a product of two Whittaker functions $W_{\alpha,\beta}(y)$,

$$W_{\alpha,\beta}(y) = y^\alpha e^{-y/2}\omega(y;1/2+\alpha+\beta,1/2-\alpha+\beta),$$

$$\omega(y;\alpha,\beta) = y^\beta\Gamma(\beta)^{-1}\int_0^\infty (1+u)^{\alpha-1}u^{\beta-1}e^{-yu}du.$$

As for detail expositions on these special functions, see [18] and [15, section 7.2]. By definition I_d has the form

$$\begin{aligned} I_d(s,\sigma_1,\sigma_2,k_1,k_2) &= |d|^{\frac{\sigma_1+\overline{\sigma_2}}{2}-\frac{k_1+k_2}{2}-1-s}e(\frac{k_1-k_2}{8})(2\pi)^{\sigma_1+\overline{\sigma_2}-\frac{k_1+k_2}{2}}(4\pi)^{-\frac{\sigma_1+\overline{\sigma_2}}{2}-s+1} \\ &\times \int_0^\infty y^{\frac{k_1+k_2}{4}-2+s}W_{\frac{-sgn(d)k_1}{4},\frac{\sigma_1}{2}-\frac{k_1}{4}-\frac{1}{2}}(y)W_{\frac{-sgn(d)k_2}{4},\frac{\overline{\sigma_2}}{2}-\frac{k_2}{4}-\frac{1}{2}}(y)dy \end{aligned}$$

$$\times\begin{cases} \Gamma(\frac{\sigma_1-k_1}{2})^{-1}\Gamma(\frac{\overline{\sigma_2}-k_2}{2})^{-1}, & \text{for } d>0 \\ \Gamma(\frac{\sigma_1}{2})^{-1}\Gamma(\frac{\overline{\sigma_2}}{2})^{-1}, & \text{for } d<0. \end{cases}$$

By [16, Theorem 2], the function $\Omega_{k,\sigma}(s,\eta)$ defined by

$$\Omega_{k,\sigma}(s,\eta) = 2^{2s}\pi^{-s}\Gamma\left(s-3/2\right)\zeta(2s-k+1)R_\infty(s,\eta)$$

can be meromorphically continued to the whole s-plane and satisfies a functional equation

$$\Omega_{k,\sigma}(s,\eta) = \Omega_{k,\sigma}(k-s,\eta).$$

For any sign $\epsilon=\pm$, let $R_\infty^\epsilon(s,\eta)$ be the subseries of $R_\infty(s,\eta)$ indexed by positive or negative integers d:

$$R_\infty^\epsilon(s,\eta) = \sum_{\epsilon d>0} c(d,\eta,-3)\overline{c(d,2\sigma+2,-2k+5)}I_d(s,\eta,2\sigma+2,-3,-2k+5), \tag{1}$$

$$\Omega_{k,\sigma}^\epsilon(s,\eta) = 2^{2s}\pi^{-s}\Gamma\left(s-3/2\right)\zeta(2s-k+1)R_\infty^\epsilon(s,\eta). \tag{2}$$

Since it is easy to see that $\Omega_{k,\sigma}(s,\eta)$ is holomorphic at $\eta=0$, if we put

$$\Omega_{k,\sigma}(s)=\Omega_{k,\sigma}(s,0), \qquad \Omega^{\epsilon}_{k,\sigma}(s)=\Omega^{\epsilon}_{k,\sigma}(s,0), \tag{3}$$

then the functional equation of $\Omega_{k,\sigma}(s,\eta)$ implies

$$\Omega_{k,\sigma}(s)=\Omega_{k,\sigma}(k-s). \tag{4}$$

Let us relate $\Omega^{+}_{k,\sigma}(s)$ with the Koecher-Maass series $L^{(2)}_{2,k}(s,\sigma)$. For $d>0$, the class number formula $L_{-d}(1)=\frac{2\pi}{d^{1/2}}H(d)$ implies

$$c(d,0,-3)\overline{c(d,2\sigma+2,-2k+5)}=\frac{2^{-2k-4\overline{\sigma}+2}\pi}{\zeta(2)\zeta(4\overline{\sigma}+2k-2)}H(d)L_{-d}(2\overline{\sigma}+k-1)d^{-1/2}.$$

On the other hand, the formula [6, p.816, 7.621.11] combined with $W_{\mu+1/2,\mu}(y)=e^{-y/2}y^{\mu+1/2}$ (the formula in [5, p.432] line 3 from the bottom) yields

$$I_d(s,0,2\sigma+2,-3,-2k+5)=d^{\overline{\sigma}+k-1-s}\frac{e(k/4)(2\pi)^{2\overline{\sigma}+k+1}\Gamma(s+\overline{\sigma})\Gamma(s-\overline{\sigma}-k+\frac{3}{2})}{(4\pi)^{\overline{\sigma}+s}\Gamma(\frac{3}{2})\Gamma(\overline{\sigma}+k-\frac{3}{2})\Gamma(s-k+\frac{5}{2})}.$$

These two equations combined with (1), (2) and Proposition 1 tell us that

$$\overline{\Omega^{+}_{k,\sigma}(\overline{s}-\overline{\sigma})}=\frac{e(-k/4)2^{-k-4\sigma+1}\pi^{3\sigma+k+2}\zeta(2\sigma+k)}{\Gamma(\frac{3}{2})\Gamma(k+\sigma-3/2)\zeta(2)\gamma_2(k+2\sigma)}\frac{\Gamma(s-\sigma-3/2)}{\Gamma(s-k-\sigma+5/2)}L^{*}_{2,k}(s,\sigma). \tag{5}$$

3 A meromorphic continuation of $\Omega^{-}_{k,\sigma}(s)$

In order to obtain a meromorphic continuation and a functional equation of $\Omega^{+}_{k,\sigma}(s)$, we first study the function $\Omega^{-}_{k,\sigma}(s)$. Because of zero of $\Gamma(\eta/2)^{-1}$ and the holomorphy of $L(\eta+1,\chi_K)$ for $d_K\neq 1$, one has

$$\begin{aligned}\Omega^{-}_{k,\sigma}(s)&=2^{2s}\pi^{-s}\Gamma\left(s-3/2\right)\zeta(2s-k+1)\\ &\times\lim_{\eta\to 0}\sum_{f\geq 1}c(-f^2,\eta,-3)\overline{c(-f^2,2\sigma+2,-2k+5)}I_{-f^2}(s,\eta,2\sigma+2,-3,-2k+5),\end{aligned}$$

where

$$\begin{aligned}I_{-f^2}(s,\eta,2\sigma+2,-3,-2k+5)&=f^{\eta+2\overline{\sigma}+2k-2-2s}\frac{e(k/4)(2\pi)^{\eta+2\overline{\sigma}+k+1}}{\Gamma(\eta/2)\Gamma(\overline{\sigma}+1)(4\pi)^{\eta/2+\overline{\sigma}+s}}\\ &\times\int_0^{\infty}y^{(s-\frac{k+1}{2})-1}W_{-\frac{3}{4},\frac{\eta}{2}+\frac{1}{4}}(y)W_{-\frac{k}{2}+\frac{5}{4},\overline{\sigma}+\frac{k}{2}-\frac{3}{4}}(y)dy.\end{aligned} \tag{6}$$

Lemma 1. *One has*

$$\begin{aligned}&\lim_{\eta\to 0}\sum_{f\geq 1}\Gamma(\eta/2)^{-1}c(-f^2,\eta,-3)\overline{c(-f^2,2\sigma+2,-2k+5)}f^{\eta+2\overline{\sigma}+2k-2-2s}\\ =&\ \frac{e(1/4)\zeta(2\overline{\sigma}+k-1)}{2P(0,-3)P(2\overline{\sigma}+2,-2k+5)}\frac{\zeta(2s-2\overline{\sigma}-2k+2)\zeta(2s+2\overline{\sigma}-1)}{\zeta(2s-k+1)},\end{aligned}$$

where $P(\sigma,k)=e^{\frac{\pi i}{4}}2^{-k+2\sigma-3/2}\zeta(2\sigma-k-1)$ *for* $k\equiv 1 \pmod 4$.

Proof. A simple calculation implies

$$\sum_{f\geq 1}\left(\sum_{d|f}\mu(d)d^{-\alpha}\sigma_{1-2\alpha}(f/d)\right)\left(\sum_{d|f}\mu(d)d^{-\beta}\sigma_{1-2\beta}(f/d)\right)f^{\gamma}$$
$$= \frac{\zeta(-\gamma)\zeta(2\beta-\gamma-1)\zeta(2\alpha-\gamma-1)\zeta(2\alpha+2\beta-\gamma-2)}{\zeta(\alpha+\beta-\gamma-1)}$$
$$\times\prod_{prime\ p}\{(1+p^{1-\alpha-\beta+\gamma})(1+p^{-\alpha-\beta+\gamma})-(p^{\gamma-\alpha}+p^{\gamma-\beta})(1+p^{1-\alpha-\beta})\}.$$

This yields Lemma 1.

To study the gamma like factor of above series, we define three functions

$$K(s)=\int_0^{\infty}y^{(s-\frac{k+1}{2})-1}W_{-\frac{3}{4},\frac{1}{4}}(y)W_{-\frac{k}{2}+\frac{5}{4},\overline{\sigma}+\frac{k}{2}-\frac{3}{4}}(y)dy, \tag{7}$$

$$K_1(s) = \frac{\Gamma(s+\overline{\sigma})\Gamma(s+\overline{\sigma}-\frac{1}{2})\Gamma(-2\overline{\sigma}-k+\frac{3}{2})}{\Gamma(-\overline{\sigma})\Gamma(s+\overline{\sigma}+1)} \times {}_3F_2\left[\begin{array}{ccc} s+\overline{\sigma}, & s+\overline{\sigma}-\frac{1}{2}, & k+\overline{\sigma}-\frac{3}{2} \\ 2\overline{\sigma}+k-\frac{1}{2}, & s+\overline{\sigma}+1 & \end{array}\right], \tag{8}$$

$$K_2(s)=\frac{\Gamma(s-\overline{\sigma}-k+\frac{3}{2})\Gamma(s-\overline{\sigma}-k+1)\Gamma(2\overline{\sigma}+k-\frac{3}{2})}{\Gamma(k+\overline{\sigma}-\frac{3}{2})\Gamma(s-\overline{\sigma}-k+\frac{5}{2})} \times {}_3F_2\left[\begin{array}{ccc} s-\overline{\sigma}-k+\frac{3}{2}, & s-\overline{\sigma}-k+1, & -\overline{\sigma} \\ -2\overline{\sigma}-k+\frac{5}{2}, & s-\overline{\sigma}-k+\frac{5}{2} & \end{array}\right]. \tag{9}$$

Here ${}_3F_2$ is generalized hypergeometric series at unit argument given by

$${}_3F_2\left[\begin{array}{ccc} a, & b, & c \\ e, & f & \end{array}\right]=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n(c)_n}{(e)_n(f)_n n!}, \qquad (x)_n=\Gamma(x+n)/\Gamma(x),\ (x)_0=1,$$

where $e,f\notin\{0,-1,-2,....\}$ and $\Re(e+f-a-b-c)>0$.

The integral defining $K(s)$ converges for $\Re s>\Re\sigma+k-1$ if $\Re\sigma+\frac{k}{2}\geq\frac{3}{4}$ and for $\Re s>\frac{1}{2}-\Re\sigma$ if $\Re\sigma+\frac{k}{2}<\frac{3}{4}$. The series defining $K_j(s)$ converges for $\Re s<\frac{5}{2}$. The following lemma gives meromorphic continuations of these functions to the whole s-plane.

Lemma 2. *Let k be an even integer and $\sigma\notin 1/4+\mathbf{Z}/2$ a complex number. Suppose $k<5$. Then $K_j(s)$ can be meromorphically continued to all s. Suppose $k<5$, $\Re\sigma+k<\frac{7}{2}$ if $\Re\sigma+\frac{k}{2}\geq\frac{3}{4}$ and suppose $k<5$, $-2<\Re\sigma$ if $\Re\sigma+\frac{k}{2}<\frac{3}{4}$. Then $K(s)$ can be meromorphically continued to the whole s-plane by the relation*

$$K(s)=K_1(s)+K_2(s). \tag{10}$$

Proof. If we put

$$A(s)={}_3F_2\left[\begin{array}{ccc} s+\overline{\sigma}, & s+\overline{\sigma}-\frac{1}{2}, & k+\overline{\sigma}-\frac{3}{2} \\ 2\overline{\sigma}+k-\frac{1}{2}, & s+\overline{\sigma}+1 & \end{array}\right], \tag{11}$$

$$B(s) = {}_3F_2\left[\begin{array}{lll} s-\overline{\sigma}-k+\frac{3}{2}, & s-\overline{\sigma}-k+1, & -\overline{\sigma} \\ -2\overline{\sigma}-k+\frac{5}{2}, & s-\overline{\sigma}-k+\frac{5}{2} & \end{array}\right], \tag{12}$$

then

$$K_1(s) = \frac{\Gamma(s+\overline{\sigma})\Gamma(s+\overline{\sigma}-\frac{1}{2})\Gamma(-2\overline{\sigma}-k+\frac{3}{2})}{\Gamma(-\overline{\sigma})\Gamma(s+\overline{\sigma}+1)}A(s), \tag{13}$$

$$K_2(s) = \frac{\Gamma(s-\overline{\sigma}-k+\frac{3}{2})\Gamma(s-\overline{\sigma}-k+1)\Gamma(2\overline{\sigma}+k-\frac{3}{2})}{\Gamma(k+\overline{\sigma}-\frac{3}{2})\Gamma(s-\overline{\sigma}-k+\frac{5}{2})}B(s). \tag{14}$$

It follows from the formula [22, (4.3.1.3)] (see also [11, p.312, (3.4.2)]) that

$$A(s) = \frac{\Gamma(\frac{5}{2}-s)\Gamma(s+\overline{\sigma}+1)}{\Gamma(s-k+\frac{5}{2})\Gamma(k-s+\overline{\sigma}+1)}A(k-s),$$

$$B(s) = \frac{\Gamma(\frac{5}{2}-s)\Gamma(s-k-\overline{\sigma}+\frac{5}{2})}{\Gamma(s-k+\frac{5}{2})\Gamma(\frac{5}{2}-s-\overline{\sigma})}B(k-s).$$

These equations hold for $k-\frac{5}{2} < \Re s < \frac{5}{2}$ and give a meromorphic continuation of $A(s)$ and $B(s)$ to the whole s-plane. Hence $K_1(s)$ and $K_2(s)$ can be meromorphically continued to the whole s-plane if we define their values on the domain $\Re s > k-\frac{5}{2}$ by

$$K_1(s) = \frac{\Gamma(-2\overline{\sigma}-k+\frac{3}{2})}{\Gamma(-\overline{\sigma})}\frac{\Gamma(s+\overline{\sigma})\Gamma(s+\overline{\sigma}-\frac{1}{2})\Gamma(\frac{5}{2}-s)}{\Gamma(s-k+\frac{5}{2})\Gamma(k-s+\overline{\sigma}+1)}A(k-s), \tag{15}$$

$$K_2(s) = \frac{\Gamma(2\overline{\sigma}+k-\frac{3}{2})}{\Gamma(k+\overline{\sigma}-\frac{3}{2})}\frac{\Gamma(s-k-\overline{\sigma}+\frac{3}{2})\Gamma(\frac{5}{2}-s)\Gamma(s-k-\overline{\sigma}+1)}{\Gamma(s-k+\frac{5}{2})\Gamma(\frac{5}{2}-s-\overline{\sigma})}B(k-s). \tag{16}$$

The equation (10) follows from the formula [6, p.814, 7.611.7] (or [5, p.410, (42)]) for $\Re\sigma+k-1 < \Re s < \frac{5}{2}$ if $\Re\sigma+\frac{k}{2} \geq \frac{3}{4}$ and for $\frac{1}{2}-\Re\sigma < \Re s < \frac{5}{2}$ if $\Re\sigma+\frac{k}{2} < \frac{3}{4}$. By the meromorphic continuation of $K_1(s)$ and $K_2(s)$ by means of (8), (15), (9), (16), we get that of $K(s)$ by the relation (10). (Note that there is a misprint in the formula [6, p.814, 7.611.7]. In the first ${}_3F_2$, the third parameter $\frac{1}{2}-\lambda-\nu$ should be $\frac{1}{2}-\lambda+\nu$.)

It follows from (6), (7) and Lemma 2 that

$$\Omega^-_{k,\sigma}(s) = \frac{\pi^{-k+1/2}C(\sigma,k)\Gamma(s-\frac{3}{2})K(s)}{\Gamma(s+\overline{\sigma}-\frac{1}{2})\Gamma(s-\overline{\sigma}-k+1)}\zeta^*(2s+2\overline{\sigma}-1)\zeta^*(2s-2\overline{\sigma}-2k+2), \tag{17}$$

where

$$C(\sigma,k) = \frac{e((k+1)/4)(2\pi)^{2\overline{\sigma}+k+1}\zeta(2\overline{\sigma}+k-1)}{2\Gamma(\overline{\sigma}+1)(4\pi)^{\overline{\sigma}}P(0,-3)P(2\overline{\sigma}+2,-2k+5)}.$$

This equation combined with the meromorphic continuations of $\zeta^*(s)$ and $K(s)$ obtained in Lemma 2 implies a meromorphic continuation of $\Omega^-_{k,\sigma}(s)$ to the whole s-plane under the assumptions in Lemma 2.

4 Functional equation of $\Omega^+_{k,\sigma}(s)$: proof of Theorem 1

Theorem 1 follows from the next proposition.

Proposition 2. *Let $k < 5$ be an even integer and $\sigma \notin 1/4 + \mathbf{Z}/2$ a complex number. Suppose $\Re\sigma + k < \frac{7}{2}$ if $\Re\sigma + \frac{k}{2} \geq \frac{3}{4}$ and suppose $-2 < \Re\sigma$ if $\Re\sigma + \frac{k}{2} < \frac{3}{4}$. Then $\Omega^+_{k,\sigma}(s)$ defined by (2) can be meromorphically continued to the whole s-plane. It satisfies a functional equation*

$$\begin{aligned} &\Omega^+_{k,\sigma}(k-s) = \Omega^+_{k,\sigma}(s) \\ +\ & \frac{e(k/4)2^{-k+2-4\overline{\sigma}}\pi^{\overline{\sigma}+5/2}\zeta(2\overline{\sigma}+k-1)}{\Gamma(\frac{3}{2})\Gamma(k+\overline{\sigma}-3/2)\zeta(2)\zeta(4\overline{\sigma}+2k-2)}\zeta^*(2s+2\overline{\sigma}-1)\zeta^*(2s-2\overline{\sigma}-2k+2) \\ & \times \frac{\sin\pi\overline{\sigma}\sin\pi s}{\cos\pi(s+\overline{\sigma})\sin\pi(s-\overline{\sigma})}\frac{\Gamma(s-\frac{3}{2})\Gamma(s+\overline{\sigma})\Gamma(s-\overline{\sigma}-k+\frac{3}{2})}{\Gamma(s-k+\frac{5}{2})\Gamma(s-\overline{\sigma}-k+1)\Gamma(s+\overline{\sigma}-\frac{1}{2})}. \end{aligned}$$

Proof. A meromorphic continuation of $\Omega^+_{k,\sigma}(s) = \Omega_{k,\sigma}(s) - \Omega^-_{k,\sigma}(s)$ to the whole s-plane follows from section 2 and 3. By the functional equation (4), we have

$$\Omega^+_{k,\sigma}(s) + \Omega^-_{k,\sigma}(s) = \Omega^+_{k,\sigma}(k-s) + \Omega^-_{k,\sigma}(k-s). \tag{18}$$

We want to simplify $\Omega^-_{k,\sigma}(s) - \Omega^-_{k,\sigma}(k-s)$. Applying the functional equation of $\zeta^*(s)$ to (17) implies

$$\begin{aligned} &\Omega^-_{k,\sigma}(s) - \Omega^-_{k,\sigma}(k-s) = \pi^{-k+1/2}C(\sigma,k)\zeta^*(2s+2\overline{\sigma}-1)\zeta^*(2s-2\overline{\sigma}-2k+2) \\ \times & \left\{\frac{\Gamma(s-\frac{3}{2})K(s)}{\Gamma(s+\overline{\sigma}-\frac{1}{2})\Gamma(s-\overline{\sigma}-k+1)} - \frac{\Gamma(k-s-\frac{3}{2})K(k-s)}{\Gamma(k-s+\overline{\sigma}-\frac{1}{2})\Gamma(1-s-\overline{\sigma})}\right\}. \end{aligned} \tag{19}$$

Suppose $\Re s < \frac{5}{2}$. By (15) and (16), we get

$$K_1(k-s) = \frac{\Gamma(-2\overline{\sigma}-k+\frac{3}{2})}{\Gamma(-\overline{\sigma})}\frac{\Gamma(k-s+\overline{\sigma})\Gamma(k-s+\overline{\sigma}-\frac{1}{2})\Gamma(s-k+\frac{5}{2})}{\Gamma(-s+\frac{5}{2})\Gamma(s+\overline{\sigma}+1)}A(s),$$

$$K_2(k-s) = \frac{\Gamma(2\overline{\sigma}+k-\frac{3}{2})}{\Gamma(k+\overline{\sigma}-\frac{3}{2})}\frac{\Gamma(-s-\overline{\sigma}+\frac{3}{2})\Gamma(-k+s+\frac{5}{2})\Gamma(-s-\overline{\sigma}+1)}{\Gamma(-s+\frac{5}{2})\Gamma(s-\overline{\sigma}-k+\frac{5}{2})}B(s).$$

These equations combined with (10) imply that (19) can be written as

$$\begin{aligned} &\Omega^-_{k,\sigma}(s) - \Omega^-_{k,\sigma}(k-s) = \pi^{-k+1/2}C(\sigma,k)\zeta^*(2s+2\overline{\sigma}-1)\zeta^*(2s-2\overline{\sigma}-2k+2) \\ \times & \left\{\frac{\Gamma(-2\overline{\sigma}-k+\frac{3}{2})}{\Gamma(-\overline{\sigma})}\left(\frac{\Gamma(s-\frac{3}{2})\Gamma(s+\overline{\sigma})\Gamma(s+\overline{\sigma}-\frac{1}{2})}{\Gamma(s+\overline{\sigma}-\frac{1}{2})\Gamma(s-\overline{\sigma}-k+1)\Gamma(s+\overline{\sigma}+1)}\right.\right. \\ & \left. - \frac{\Gamma(k-s-\frac{3}{2})\Gamma(k-s+\overline{\sigma})\Gamma(s-k+\frac{5}{2})}{\Gamma(1-s-\overline{\sigma})\Gamma(-s+\frac{5}{2})\Gamma(s+\overline{\sigma}+1)}\right)A(s) \\ & + \frac{\Gamma(2\overline{\sigma}+k-\frac{3}{2})}{\Gamma(k+\overline{\sigma}-\frac{3}{2})}\left(\frac{\Gamma(s-\frac{3}{2})\Gamma(s-\overline{\sigma}-k+\frac{3}{2})\Gamma(s-\overline{\sigma}-k+1)}{\Gamma(s+\overline{\sigma}-\frac{1}{2})\Gamma(s-\overline{\sigma}-k+1)\Gamma(s-\overline{\sigma}-k+\frac{5}{2})}\right. \\ & \left.\left. - \frac{\Gamma(k-s-\frac{3}{2})\Gamma(-s-\overline{\sigma}+\frac{3}{2})\Gamma(s-k+\frac{5}{2})}{\Gamma(k-s+\overline{\sigma}-\frac{1}{2})\Gamma(-s+\frac{5}{2})\Gamma(s-\overline{\sigma}-k+\frac{5}{2})}\right)B(s)\right\}. \end{aligned} \tag{20}$$

It follows from the formula [3, p.15, (2)] (or [22, p.115, (4.3.4)]) that

$$
\begin{aligned}
A(s) &= \frac{\Gamma(\frac{3}{2}-s-\overline{\sigma})\Gamma(2\overline{\sigma}+k-\frac{1}{2})\Gamma(s+\overline{\sigma}+1)\Gamma(k-s-\frac{3}{2})}{\Gamma(\overline{\sigma}+k-\frac{1}{2}-s)\Gamma(\frac{3}{2})\Gamma(k+\overline{\sigma}-\frac{3}{2})} \\
&+ \frac{\Gamma(\frac{3}{2}-s-\overline{\sigma})\Gamma(2\overline{\sigma}+k-\frac{1}{2})\Gamma(s+\overline{\sigma}+1)\Gamma(s-k+\frac{3}{2})}{\Gamma(\overline{\sigma}+1)\Gamma(s-k+\frac{5}{2})\Gamma(k-s)\Gamma(s+\overline{\sigma})} \\
&\times {}_3F_2\left[\begin{matrix} k+\overline{\sigma}-\frac{3}{2}, & -\overline{\sigma}, & k-s-\frac{3}{2} \\ k-s-\frac{1}{2}, & k-s & \end{matrix}\right],
\end{aligned}
$$

$$
\begin{aligned}
B(s) &= \frac{\Gamma(-s+\overline{\sigma}+k)\Gamma(-2\overline{\sigma}-k+\frac{5}{2})\Gamma(s-\overline{\sigma}-k+\frac{5}{2})\Gamma(-s+k-\frac{3}{2})}{\Gamma(-s-\overline{\sigma}+1)\Gamma(\frac{3}{2})\Gamma(-\overline{\sigma})} \\
&+ \frac{\Gamma(-s+\overline{\sigma}+k)\Gamma(-2\overline{\sigma}-k+\frac{5}{2})\Gamma(s-\overline{\sigma}-k+\frac{5}{2})\Gamma(s-k+\frac{3}{2})}{\Gamma(-\overline{\sigma}-k+\frac{5}{2})\Gamma(s-k+\frac{5}{2})\Gamma(k-s)\Gamma(s-\overline{\sigma}-k+\frac{3}{2})} \\
&\times {}_3F_2\left[\begin{matrix} -\overline{\sigma}, & \overline{\sigma}+k-\frac{3}{2}, & k-s-\frac{3}{2} \\ k-s-\frac{1}{2}, & k-s & \end{matrix}\right].
\end{aligned}
$$

Substituting these equations into (20), we get

$$
\begin{aligned}
&\Omega^{-}_{k,\sigma}(s)-\Omega^{-}_{k,\sigma}(k-s) = \zeta^*(2s+2\overline{\sigma}-1)\zeta^*(2s-2\overline{\sigma}-2k+2) \\
&\times \pi^{-k+1/2}C(\sigma,k)\left\{X(s)+Y(s)\,{}_3F_2\left[\begin{matrix} k+\overline{\sigma}-\frac{3}{2}, & -\overline{\sigma}, & k-s-\frac{3}{2} \\ k-s-\frac{1}{2}, & k-s & \end{matrix}\right]\right\}, \qquad (21)
\end{aligned}
$$

where $X(s)$ and $Y(s)$ are defined by

$$
\begin{aligned}
Y(s) &= \frac{\Gamma(-2\overline{\sigma}-k+\frac{3}{2})\Gamma(2\overline{\sigma}+k-\frac{1}{2})}{\Gamma(-\overline{\sigma})\Gamma(\overline{\sigma}+1)}\frac{\Gamma(\frac{3}{2}-s-\overline{\sigma})\Gamma(s-k+\frac{3}{2})}{\Gamma(k-s)} \\
&\times\left(\frac{\Gamma(s-\frac{3}{2})}{\Gamma(s-k+\frac{5}{2})\Gamma(s-\overline{\sigma}-k+1)} - \frac{\Gamma(k-s-\frac{3}{2})\Gamma(k-s+\overline{\sigma})}{\Gamma(s+\overline{\sigma})\Gamma(1-s-\overline{\sigma})\Gamma(-s+\frac{5}{2})}\right) \\
&+ \frac{\Gamma(2\overline{\sigma}+k-\frac{3}{2})\Gamma(-2\overline{\sigma}-k+\frac{5}{2})}{\Gamma(k+\overline{\sigma}-\frac{3}{2})\Gamma(-\overline{\sigma}-k+\frac{5}{2})}\frac{\Gamma(-s+k+\overline{\sigma})\Gamma(s-k+\frac{3}{2})}{\Gamma(k-s)} \\
&\times\left(\frac{\Gamma(s-\frac{3}{2})}{\Gamma(s-k+\frac{5}{2})\Gamma(s+\overline{\sigma}-\frac{1}{2})} - \frac{\Gamma(k-s-\frac{3}{2})\Gamma(-s-\overline{\sigma}+\frac{3}{2})}{\Gamma(s-\overline{\sigma}-k+\frac{3}{2})\Gamma(k-s+\overline{\sigma}-\frac{1}{2})\Gamma(-s+\frac{5}{2})}\right),
\end{aligned}
$$

$$
\begin{aligned}
X(s) &= \frac{\Gamma(-2\overline{\sigma}-k+\frac{3}{2})\Gamma(2\overline{\sigma}+k-\frac{1}{2})}{\Gamma(-\overline{\sigma})\Gamma(\frac{3}{2})\Gamma(k+\overline{\sigma}-\frac{3}{2})}\frac{\Gamma(\frac{3}{2}-s-\overline{\sigma})\Gamma(k-s-\frac{3}{2})}{\Gamma(\overline{\sigma}+k-s-\frac{1}{2})} \\
&\times\left(\frac{\Gamma(s-\frac{3}{2})\Gamma(s+\overline{\sigma})}{\Gamma(s-\overline{\sigma}-k+1)} - \frac{\Gamma(k-s-\frac{3}{2})\Gamma(k-s+\overline{\sigma})\Gamma(-k+s+\frac{5}{2})}{\Gamma(1-s-\overline{\sigma})\Gamma(-s+\frac{5}{2})}\right) \\
&+ \frac{\Gamma(2\overline{\sigma}+k-\frac{3}{2})\Gamma(-2\overline{\sigma}-k+\frac{5}{2})}{\Gamma(-\overline{\sigma})\Gamma(\frac{3}{2})\Gamma(k+\overline{\sigma}-\frac{3}{2})}\frac{\Gamma(-s+k+\overline{\sigma})\Gamma(-s+k-\frac{3}{2})}{\Gamma(-\overline{\sigma}-s+1)} \\
&\times\left(\frac{\Gamma(s-\frac{3}{2})\Gamma(s-\overline{\sigma}-k+\frac{3}{2})}{\Gamma(s+\overline{\sigma}-\frac{1}{2})} - \frac{\Gamma(k-s-\frac{3}{2})\Gamma(-s-\overline{\sigma}+\frac{3}{2})\Gamma(-k+s+\frac{5}{2})}{\Gamma(k-s+\overline{\sigma}-\frac{1}{2})\Gamma(-s+\frac{5}{2})}\right).
\end{aligned}
$$

The reflection formula of the gamma function implies

$$Y(s) = \frac{-\pi^2}{\cos 2\pi\overline{\sigma}} \frac{\Gamma(s-k+\frac{3}{2})\{\cos\pi(s+\overline{\sigma})\cos\pi s\sin\pi(s-\overline{\sigma})\}^{-1}}{\Gamma(k-s)\Gamma(s-k+\frac{5}{2})\Gamma(-s+\frac{5}{2})\Gamma(s-\overline{\sigma}-k+1)\Gamma(s+\overline{\sigma}-\frac{1}{2})}$$
$$\times(\cos\pi\overline{\sigma}\cos\pi(s+\overline{\sigma})+\sin\pi\overline{\sigma}\sin\pi(s+\overline{\sigma})-\cos\pi\overline{\sigma}\cos\pi(s-\overline{\sigma})+\sin\pi\overline{\sigma}\sin\pi(s-\overline{\sigma}))$$

and the additive formulas of trigonometric functions imply $Y(s)=0$. By a similar way we have

$$X(s) = \frac{-\pi^2}{\cos 2\pi\overline{\sigma}\Gamma(-\overline{\sigma})\Gamma(\frac{3}{2})\Gamma(k+\overline{\sigma}-\frac{3}{2})}$$
$$\times \quad \frac{\Gamma(s-\frac{3}{2})\Gamma(s+\overline{\sigma})\Gamma(s-\overline{\sigma}-k+\frac{3}{2})}{\Gamma(s-k+\frac{5}{2})\Gamma(s-\overline{\sigma}-k+1)\Gamma(s+\overline{\sigma}-\frac{1}{2})} \frac{\cos\pi(s-\overline{\sigma})\sin\pi(s+\overline{\sigma})}{\cos\pi s}$$
$$\times \quad \left(\frac{1}{\sin\pi(s+\overline{\sigma})\cos\pi(s+\overline{\sigma})}+\frac{1}{\sin\pi(s-\overline{\sigma})\cos\pi(s-\overline{\sigma})}\right).$$

A simplification combined with (18) and (21) implies Proposition 2.

Let us prove Theorem 1. For an even integer k and a complex number σ, we take a natural number l so that the assumptions in Proposition 2 are satisfied for $k' = k-4l$ and $\sigma' = \sigma+2l$. Then Proposition 2 combined with (5) implies Theorem 1 for k', σ'. Using $k'+2\sigma' = k+2\sigma$, we complete the proof of Theorem 1.

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Yoshinori Mizuno
Max-Planck-Institut für Mathematik,
Vivatsgasse 7, D-53111 Bonn, Germany
e-mail: mizuno@mpim-bonn.mpg.de