AN ADDITIONAL GIBBS' STATE FOR THE CUBIC SCHRÖDINGER EQUATION ON THE CIRCLE

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Abstract. An invariant Gibbs' state for the nonlinear Schrödinger equation on the circle was constructed by Bourgain, [B], and McKean, [MC], out of the basic Hamiltonian using a trigonometric cut-off. The cubic nonlinear Schrödinger equation is a completely integrable system having an infinite number of additional integrals of motion. In this paper we construct the second invariant Gibbs' state from one of these additional integrals for the cubic NLS on the circle. This additional Gibbs' state is singular with respect to the Gibbs' state previously constructed from the basic Hamiltonian. Our approach employs the Ablowitz-Ladik system, a completely integrable discretization of the cubic Schrödinger equation.

1. Introduction. The nonlinear Schrödinger equation for the complex function $\psi(x,t),\ x\in\mathbb{S},\ t\in R^1$ is

$$i\psi^\bullet = -\psi'' + p|\psi|^{2p-2}\psi,$$

where p is an arbitrary integer. It can be written in Hamiltonian form. Let $\mathcal{M}=(\psi,\bar\psi)$ be a space of pairs of two complex functions $\psi=Q+iP$ and $\bar\psi=Q-iP$ on the circle $\mathbb{S}$ of perimeter 1. For any two functionals F and G on $\mathcal{M}$, define

$$\{F,G\}_\omega = i\int_{\mathbb{S}} \frac{\delta F}{\delta\psi}\frac{\delta G}{\delta\bar\psi} - \frac{\delta F}{\delta\bar\psi}\frac{\delta G}{\delta\psi}dx.$$

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The Hamiltonian $\mathcal{H}(\psi,\bar{\psi}) = \int_{\mathbb{S}} |\psi'|^2 + |\psi|^{2p}dx$ produces the nonlinear Schrö - dinger flow

$$\overset{\bullet}{\psi} = \{\psi, \mathcal{H}\}_\omega$$

The equation has two other generic integrals. They are $\mathcal{N} = \int |\psi|^2 dx$, the number of particles, and $\mathcal{P} = -\int i\psi'\bar{\psi}\, dx$, the momentum.

The Gibbs' measure associated with the basic Hamiltonian

$$e^{-\frac{1}{2}\mathcal{H}} d\,vol = e^{-\frac{1}{2}\mathcal{H}} \frac{1}{\infty!} \bigwedge_{\infty} \omega = e^{-\frac{1}{2}\mathcal{H}} \frac{1}{\infty!} \bigwedge_{x\in\mathbb{S}} i\delta\psi(x)\wedge\delta\bar{\psi}(x)$$

was constructed by Bourgain and McKean [B1-2, MC]; it is the product of two independent copies of circular Brownian motion for the components of the function $\psi = Q + iP$ coupled together by the nonlinear factor $e^{-\frac{1}{2}\int_{\mathbb{S}} |Q^2+P^2|^p dx}$. This measure is invariant under the flow; for any p the later exists almost everywhere with respect to the Gibbs' measure.

The cubic NLS corresponds to the case $p = 2$. It has infinite series of commuting Hamiltonians. The first five are listed below

$$\begin{aligned}
\mathcal{H}_1 &= \int |\psi|^2 dx,\\
\mathcal{H}_2 &= \int -i\psi'\bar{\psi} dx,\\
\mathcal{H}_3 &= \int |\psi'|^2 + |\psi|^4 dx,\\
\mathcal{H}_4 &= \int i\psi'''\bar{\psi} - i\bar{\psi}'\psi|\psi|^2 - 4|\psi|^2\bar{\psi}\psi' dx,\\
\mathcal{H}_5 &= \int |\psi''|^2 + 2|\psi|^6 + 8|\psi'|^2|\psi|^2 + \bar{\psi}'^2\psi^2 + \psi'^2\bar{\psi}^2 dx.
\end{aligned}$$

The classical generic integrals $\mathcal{N}$, $\mathcal{P}$ and $\mathcal{H}$ are the first three of this infinite series. All of them are integrals of the isobaric polynomials, counting ψ and d/dx as having degree 1; for example, $\mathcal{H}_5$ is an integral of an isobaric polynomial of degree 6.

We consider the phase space $\mathcal{M}$ of pairs $(\psi,\bar{\psi})$ from the Sobolev space H^1*. The original equation defines a global flow on $\mathcal{M}$, [B2]. In this paper we construct an invariant Gibbs' state from the higher Hamiltonian $\mathcal{H}_5$:

$$e^{-\frac{1}{2}\mathcal{H}_5} d\,vol = e^{-\frac{1}{2}\mathcal{H}_5} \frac{1}{\infty!} \bigwedge_{x\in\mathbb{S}} i\delta\psi(x)\wedge\delta\bar{\psi}(x).$$

*If $\psi(x) = \sum_k e^{2\pi i k x}\hat{\psi}(k)$, then the Sobolev norm of $\psi(x)$ is $|\psi|^2_{H^s} = \sum_k |\hat{\psi}(k)|^2(1+k^2)^s$.

This state is singular with respect to the Gibbs' state $e^{-\frac{1}{2}\mathcal{H}_3}dvol$. To construct the measure and prove its invariance we will use the Ablowitz-Ladik (AL) system.

The AL equation for the complex N-periodic function $\psi(n,t)$, $n \in Z^1$, $t \in R^1$, is*

$$i\overset{\bullet}{\psi}_n = -(\psi_{n+1} + \psi_{n-1} - 2\psi_n) + |\psi_n|^2(\psi_{n+1} + \psi_{n-1}).$$

It is easy to check that the quantity

$$D = \prod_{k=0}^{N-1} R_k = \prod_{k=0}^{N-1}(1 - |\psi_k|^2)$$

is an integral of motion. We assume that $|\psi_n| < 1$ for all n. This area of the phase space we call the box B; it is invariant under the flow. In the box all quantities R_n and D are *positive*.

In fact, AL has many integrals of motion. The first three interesting ones are $I_0,\ I_2,\ I_4$:

$$\begin{aligned}
NI_0 &\equiv -\frac{1}{2}\log D,\\
NI_2 &= \sum_{r=0}^{N-1} \psi_r \overline{\psi}_{r-1},\\
NI_4 &= \sum_{r=0}^{N-1} \psi_r \overline{\psi}_{r-2}\left(1 - |\psi_{r-1}|^2\right) - \frac{1}{2}(\psi_r\overline{\psi}_{r-1})^2.
\end{aligned}$$

From these integrals we form Hamiltonians

$$\begin{aligned}
H_1 &= N(I_0 + \bar{I}_0)\\
H_3 &= N(I_2 + \bar{I}_2 - 2I_0 - 2\bar{I}_0)\\
H_5 &= N(I_4 + \bar{I}_4 - 4I_2 - 4\bar{I}_2 + 6I_0 + 6\bar{I}_0).
\end{aligned}$$

Let M_N be a space of complex N-periodic sequences $\psi_n = Q_n + iP_n,\ n \in Z^1$ with $|\psi_n| < 1$. Introducing the bracket

$$\{f, g\}_{\omega_0} \equiv i\sum_{n=1}^{N} R_n \left(\frac{\delta f}{\delta\psi_n}\frac{\delta g}{\delta\bar{\psi}_n} - \frac{\delta f}{\delta\bar{\psi}_n}\frac{\delta g}{\delta\psi_n}\right),$$

for two functionals f and g on M_N we can write the original AL flow in Hamiltonian form

$$\overset{\bullet}{\psi}_n = \{\psi_n, H_3\}_{\omega_0}.$$

*• denotes a time derivative; $\psi_n = \psi(n,t)$; periodicity means $\psi_{n+N} = \psi_n$ for $n \in \mathbb{Z}$.

The symplectic form ω_0 producies the volume element

$$d\,\mathrm{vol} = \frac{1}{N!D_N}\bigwedge_n i\delta\bar{\psi}_n \wedge \delta\psi_n.$$

The volume of the box is infinite

$$\int\limits_B d\,\mathrm{vol} = \int\limits_B \frac{1}{N!D_N}\bigwedge_n i\delta\bar{\psi}_n \wedge \delta\psi_n = \infty$$

due to the singularity of the factor $\frac{1}{D_N}$. On the box we can define the Gibbs' state with the density

$$e^{-\frac{N^5}{2}H_5} d\,\mathrm{vol} = e^{-\frac{N^5}{2}H_5}\frac{1}{N!}\bigwedge_n \omega_0 = e^{-\frac{N^5}{2}H_5}\frac{1}{N!D_N}\bigwedge_n i\delta\psi_n \wedge \delta\bar{\psi}_n,$$

The singularity of the volume element near the boundary is rectified by the vanishing factor $e^{-\frac{N^5}{2}12NI_0} = D^{3N^5}$: the total mass

$$\int\limits_B e^{-\frac{N^5}{2}H_5} d\,\mathrm{vol} < \infty.$$

The area of the box close to the boundary has negligible probability. The measure of the area outside the box we take to be zero.

Let $\epsilon = \frac{1}{N}$. Under the scaling $\psi_n = \epsilon\psi(\frac{n}{N})$ and $\theta = \epsilon^{-2}t$, the AL vector field approaches the NLS vector field as $N \to \infty$. Indeed,

$$\psi_{n+1} = \psi_n + \psi_n'\epsilon + \frac{1}{2}\psi_n''\epsilon^2 \ldots, \qquad \psi_{n-1} = \psi_n - \psi_n'\epsilon + \frac{1}{2}\psi_n''\epsilon^2 \ldots.$$

So,

$$\psi_{n+1} + \psi_{n-1} - 2\psi_n = \psi_n''\epsilon^2 + \ldots.$$

and the flow in the time scale θ,

$$i\frac{\partial\psi_n}{\partial\theta} = -(\psi_{n+1} + \psi_{n-1} - 2\psi_n) + |\psi_n|^2(\psi_{n+1} + \psi_{n-1}),$$

scales as follows:

$$i\epsilon^3\frac{\partial\psi}{\partial t} = -\epsilon\psi''\epsilon^2 + \epsilon^3|\psi|^2 2\psi + \ldots.$$

This indicates that AL flow converges to the NLS flow, though statement can not be taken literaly because these two flows live in two different spaces M_N and $\mathcal{M}$.

The integrals H_1, H_3 and H_5 of the AL system converge to $\mathcal{H}_1$, $\mathcal{H}_3$ and $\mathcal{H}_5$:

$$\begin{aligned} N\, H_1(\psi_n,\bar\psi_n) &\to \mathcal{H}_1(\psi,\bar\psi),\\ -N^3 H_3(\psi_n,\bar\psi_n) &\to \mathcal{H}_3(\psi,\bar\psi),\\ N^5 H_5(\psi_n,\bar\psi_n) &\to \mathcal{H}_5(\psi,\bar\psi). \end{aligned}$$

Also, for any ψ, we have $D_N(\psi_n,\bar\psi_n)\to 1$, as $N\to\infty$. Therefore, one should expect to get from the Gibbs' state for AL, in the limit of $N\to\infty$, the desired invariant measure

$$e^{-\frac{1}{2}\mathcal{H}_5}d\,vol = e^{-\frac{1}{2}\mathcal{H}_5}\frac{1}{\infty!}\bigwedge_{x\in\mathbb{S}} i\delta\psi(x)\wedge\delta\bar\psi(x).$$

Again, this statement is correct but can not be taken literaly, because these measures do not live on the same space. To make the arguments rigorous we need to embed the AL flows and measures into the function space. For this, we use interpolating trigonometrical polynomials.

The main goal of the paper is the space-time random field $\psi(x,t)$, $x\in\mathbb{S}$, $t\in R^1$ such that:

(i) $\psi(x,t)$ is stationary respect x and t;

(ii) $\psi(\bullet,t)$ has Gibbs' distribution $e^{-\frac{1}{2}\mathcal{H}_5}d\,vol$;

(iii) the random variable $\psi(\bullet,t)$, $t\neq 0$ is measurable with respect to the σ-field generated by $\psi(\bullet,0)$; the measure is supported by the solutions of NLS;

(iv) The x-derivative of almost every realisation of the random field is Hölder continuous:

$$|\partial_x\psi(x_1,t_1)-\partial_x\psi(x_2,t_2)|\le K\left[|x_1-x_2|^{\frac{1}{2}-}+|t_1-t_2|^{\frac{1}{4}-}\right]$$

with a random constant K. The exponents $\frac{1}{2}$ and $\frac{1}{4}$ are optimal.

The paper is organised as follows. In the section 2 we introduce the commuting flows of the AL hierarchy and commuatator formulas for them. Section 3 is devoted to the study of the direct spectral problem for the auxiliary linear system. We explain the details of the spectral curve for various types of potentials. Invariant quantities are computed in section 4. The Floquet and dual Floquet solutions are studied in sections 5 and 6. Section 7 provides formulas for the symplectic structure and the Poisson bracket. This completes the first part of the paper about integrability properties and Hamiltonian formalism for the AL system.

The second part starts with section 8 where the strategy for constructing the measure is outlined; it is implemented in the subsequent sections 9-11.

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2. Ablowitz–Ladik hierarchy. The Ablowitz–Ladik equation for the complex N–periodic function $\psi(n,t)$ $n \in Z^1$, $t \in R^1$ is

$$i\overset{\bullet}{\psi}_n = -(\psi_{n+1} + \psi_{n-1} - 2\psi_n) + |\psi_n|^2(\psi_{n+1} + \psi_{n-1}). \tag{1}$$

Recall that

$$D = \prod_{k=0}^{N-1} R_k = \prod_{k=0}^{N-1} (1 - |\psi_k|^2)$$

is an integral of motion, whence the "box" $B = (\psi : \ |\psi_n| < 1$ for all $n)$ is invariant under the flow. In the box R_n and D are *positive.*

The flow (1) is one of infinitely many flows of the AL hierarchy. The rotation or the phase flow

$$i\overset{\bullet}{\psi}_n = -\psi_n \tag{2}$$

is the first flow of the hierarchy. Let

$$V_2(n,t,\lambda) = \frac{1}{\sqrt{R_n}} \begin{bmatrix} \lambda & \psi_n \\ \overline{\psi}_n & \lambda^{-1} \end{bmatrix}.$$

The phase flow is the compatibility condition for

$$[\partial_t - V_1, \Delta - V_2] = 0, \tag{3}$$

where $V_1 = i\sigma_3/2$ and $\Delta f_n = f_{n+1}$ is a shift operator. Here and below σ denotes the Pauli matricies

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The original AL equation (1) is the compatibility condition for*

$$[\partial_t - V_3, \Delta - V_2] = 0, \tag{4}$$

where

$$V_3(n,t,\lambda) = i \begin{bmatrix} \lambda^2 - 1 + \frac{3}{2}\psi_n\overline{\psi}_{n-1} - \frac{1}{2}\overline{\psi}_n\psi_{n-1} & \psi_n\lambda - \psi_{n-1}\lambda^{-1} \\ \overline{\psi}_{n-1}\lambda - \overline{\psi}_n\lambda^{-1} & 1 - \lambda^{-2} + \frac{1}{2}\overline{\psi}_n\psi_{n-1} + \frac{1}{2}\psi_n\overline{\psi}_{n-1} \end{bmatrix}.$$

The formula means that**

$$(\partial_t - V_3(n+1))(\Delta - V_2(n)) - (\Delta - V_2(n))(\partial_t - V_3(n)) = 0$$

i.e.

$$\overset{\bullet}{V}_2(n) = V_3(n+1)V_2(n) - V_2(n)V_3(n).$$

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*A similar form of the commutator formalism was considered in [AL, MEKL]. Our form has the small advantage, that it leads to a *unimodular* monodromy matrix.

**The operator $V = V_2$ acts like shift: $V_2(n)\mathbf{f}_n = \mathbf{f}_{n+1}$.

3. Monodromy matrix. The spectral curve. Since $\psi_{n+N} = \psi_n$, all matricies V_k, $k = 1, 2, \ldots$, satisfy the periodicity condition $V_k(n+N) = V_k(n)$. We introduce the transition matrix

$$T_{n,m}(t,\lambda) \equiv V(n-1,t,\lambda)V(n-2,t,\lambda)\ldots V(m,t,\lambda) \qquad (n > m).$$

It is easy to see that the spectrum of $T_{m+N,m}(t,\lambda)$ does not depend on m. Indeed,

$$T_{m+1+N,m+1}(t,\lambda) = V(m+N)\ldots V(m+1) = V(m+N)T_{m+N,m}V(m)^{-1}.$$

Also, the spectrum of $T_N \equiv T_{N,0}$ does not depend on time. This follows from the identity $\overset{\bullet}{T}_N = [V_3(0), T_N]$, which is proved as follows:

$$\begin{aligned}
&\partial_t[V\ldots V(k+1)V(k)\ldots V] \\
&\quad = \ldots + V\ldots \overset{\bullet}{V}(k+1)V(k)\ldots V + V\ldots V(k+1)\overset{\bullet}{V}(k)\ldots V + \ldots \\
&\quad = \ldots + V\ldots(V_3(k+2)V(k+1) - V(k+1)V_3(k+1))V(k)\ldots V \\
&\qquad\qquad + V\ldots V(k+1)(V_3(k+1)V(k) - V(k)V_3(k))\ldots V + \ldots \\
&\quad = V_3(N)T_N(\lambda) - T_N(\lambda)V_3(0).
\end{aligned}$$

Consider the special "Floquet" solution

$$\mathbf{g}_n = \begin{bmatrix} g_n^1(\lambda) \\ g_n^2(\lambda) \end{bmatrix}$$

of the eigenvalue problem $\mathbf{g}_{n+1}(\lambda) = V(n,\lambda)\mathbf{g}_n(\lambda)$ specified by the condition

$$\mathbf{g}_N(\lambda) = T_N\mathbf{g}_0(\lambda) = w\mathbf{g}_0(\lambda),$$

where w being the complex "multiplier" determined by

$$0 = \det\begin{bmatrix} T_N^{11} - w & T_N^{12} \\ T_N^{21} & T_N^{22} - w \end{bmatrix} = w^2 - 2w\Delta(\lambda) + 1,$$

where $\Delta(\lambda) = \operatorname{tr} T_N(\lambda)/2$. We know that $\Delta(\lambda)$ is an integral of motion. The multiplier is

$$w = \Delta(\lambda) + \sqrt{\Delta^2(\lambda) - 1};$$

It becomes single-valued on the spectral curve $\Gamma = \{Q = (\lambda, y) \in \mathbb{C}^2 : \ y^2 = \Delta^2(\lambda) - 1\}$.

The curve Γ inherits symmetries from $V_2(n,\lambda)$. It is easy to check that

$$\sigma_1 V\left(\frac{1}{\overline{\lambda}}\right)\sigma_1 = \overline{V(\lambda)}, \tag{1}$$

$$\sigma_3 V(-\lambda)\sigma_3 = -V(\lambda). \tag{2}$$

Obviously $T_N(\lambda)$ satisfies similar identities. Whence

$$\Delta\left(\frac{1}{\overline{\lambda}}\right) = \overline{\Delta(\lambda)}, \tag{3}$$

$$\Delta(-\lambda) = (-1)^N\Delta(\lambda). \tag{4}$$

The symmetries (3-4) produce an antiholomorhic involution τ_a and a holomorphic involution τ on the curve

$$\tau_a: \qquad (\lambda, y)\longrightarrow\left(\frac{1}{\overline{\lambda}}, \overline{y}\right), \tag{5}$$

$$\tau: \qquad (\lambda, y)\longrightarrow\left(-\lambda, (-1)^N y\right). \tag{6}$$

On Γ, there exists another holomorphic involution $\tau_\pm$ which permutes the sheets, viz. $\tau_\pm : (\lambda, y) \to (\lambda, -y)$. From the quadratic equation for w, we have $w(Q)w(\tau_\pm Q) = 1$.

Remark. Obviously, we have freedom in the choice of sign for the second coordinate of the involution, say

$$\tau_a: \qquad (\lambda, y)\longrightarrow\left(\frac{1}{\overline{\lambda}}, \pm\overline{y}\right).$$

The sign in (5–6) is chosen in such way that τ_a and τ preserve the infinities of the curve (see example 5 of this section).

Example 1: vanishing potential $\psi_n \equiv 0$. Then

$$T_N(\lambda) = \begin{pmatrix}\lambda^N & 0\\ 0 & \lambda^{-N}\end{pmatrix}, \qquad \Delta(\lambda) = \cosh N\log\lambda \quad \text{and} \quad w(Q) = e^{\pm N\log\lambda(Q)}.$$

The points $\lambda_k^{\pm} = e^{i\frac{2\pi}{2N}k}$, $k = 0,\cdots,2N-1$ satisfy the condition $\Delta^2(\lambda_k^{\pm}) = 1$ and at these points* $\Delta^\bullet(\lambda_k^{\pm}) = 0$. The points $\lambda_k^{\pm}$ are simple crossings of the curve Γ. They form periodic/antiperiodic spectrum; namely, $w = +1$ for k even and $w = -1$ for k odd. If $\lambda = e^{i\theta}$, then $\Delta(e^{i\theta}) = \left(e^{i\theta N} + e^{-i\theta N}\right)/2 = \cos\theta N$. The graph of $\Delta(\lambda)$ for $N = 2$ is shown in fig 1.

When ψ_n is not identicaly zero, then the double points $\lambda_k^{\pm}$ split into pairs λ_k^- and λ_k^+ of simple ramification. The symmetry (3) implies that $\Delta(\lambda)$ is always real for $|\lambda| = 1$. Note also that, due to the symmetry (4), the branch points form symmetric pairs λ_k^-, λ_k^+ and i $\lambda_{k'}^-$, $\lambda_{k'}^+$ such that $\lambda_k^{\pm} = -\lambda_{k'}^{\pm}$, when $k - k' \equiv 0 \pmod N$.

The branch points $\lambda_k^{\pm}$ lie only on the unit circle**. Indeed, the equation

*$\bullet$ now denotes derivative in λ variable.

**The statement is proved by the method of [AL].

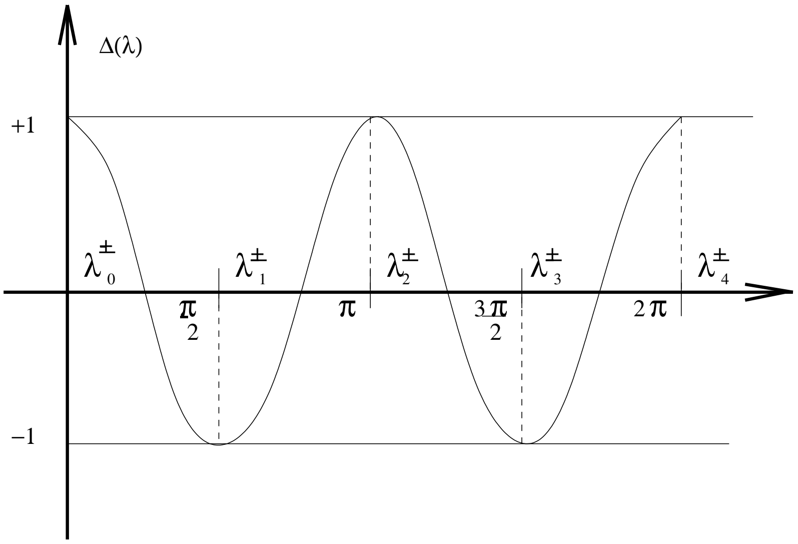

FIG. 1

$\mathbf{g}_{n+1}(\lambda) = V(n,\lambda)\mathbf{g}_n(\lambda)$ can be written in the form $\Lambda_n \mathbf{g}_n(\lambda) = \lambda \mathbf{g}_n(\lambda)$, where

$$\Lambda_n = \begin{bmatrix} \sqrt{R_n}\Delta & -\psi_n \\ \overline{\psi}_{n-1} & \sqrt{R_{n-1}}\Delta^{-1} \end{bmatrix} \qquad \text{and} \quad \Delta f_n = f_{n+1}.$$

It is easy to see that Λ is unitary in the space of $2N$–periodic vector functions with the complex inner product $< \mathbf{f}, \mathbf{g} >_{2\heartsuit} = \frac{1}{2N}\sum_{k=0}^{2N-1} f_k \overline{g}_k$. Indeed, introducing the formal inverse

$$\Lambda_n^{-1} = \begin{bmatrix} \sqrt{R_{n-1}}\Delta^{-1} & \psi_{n-1} \\ -\overline{\psi}_n & \sqrt{R_n}\Delta \end{bmatrix}$$

it is easy to derive the Cauchy identity

$$\begin{aligned} &< \Lambda \mathbf{f}, \mathbf{g} >_\heartsuit = < \mathbf{f}, \Lambda^{-1}\mathbf{g} >_\heartsuit + \\ &+ \frac{1}{N}\sqrt{R_{N-1}}\left(f_N^1 \overline{g}_{N-1}^1 - f_0^1 \overline{g}_{-1}^1\right) + \frac{1}{N}\sqrt{R_{N-1}}\left(f_{-1}^2 \overline{g}_0^2 - f_{N-1}^2 \overline{g}_N^2\right), \end{aligned} \tag{7}$$

where $< \mathbf{f}, \mathbf{g} >_\heartsuit = \frac{1}{N}\sum_{k=0}^{N-1} f_k \overline{g}_k$. Appling this formula twice, first to the interval $k = 0, \cdots, N-1$ and then to the interval $k = N, \cdots, 2N-1$ we obtain a similar identity for $< \bullet, \bullet >_{2\heartsuit}$ which implies the result.

Example 2: $N = 2$–periodic case. It is instructive to analyze this case complitely. The potential can be written in the form

$$\psi_n = \begin{cases} A + B, & n \text{ is even} \\ A - B, & n \text{ is odd.} \end{cases}$$

The matrix T_2 can be easily computed:

$$T_2 = \frac{1}{\sqrt{D}} \begin{bmatrix} \lambda^2 + (A-B)(\overline{A}+\overline{B}) & \lambda(A+B) + \lambda^{-1}(A-B) \\ \lambda(\overline{A}-\overline{B}) + \lambda^{-1}(\overline{A}+\overline{B}) & \lambda^{-2} + (A+B)(\overline{A}-\overline{B}) \end{bmatrix}.$$

Then $\Delta(\theta) = \frac{1}{\sqrt{D}}\left[\cos 2\theta + |A|^2 - |B|^2\right]$ for $\lambda = e^{i\theta}$, and $D = (1-|A+B|^2)\times(1-|A-B|^2)$. Consider the case $B \equiv 0$ and $A \neq 0$. Then $\Delta(\theta) = \frac{1}{\sqrt{D}}\left[\cos 2\theta + |A|^2\right]$, and $D = (1-|A|^2)^2$. The graph of $\Delta(\theta)$ is shown in fig. 2.

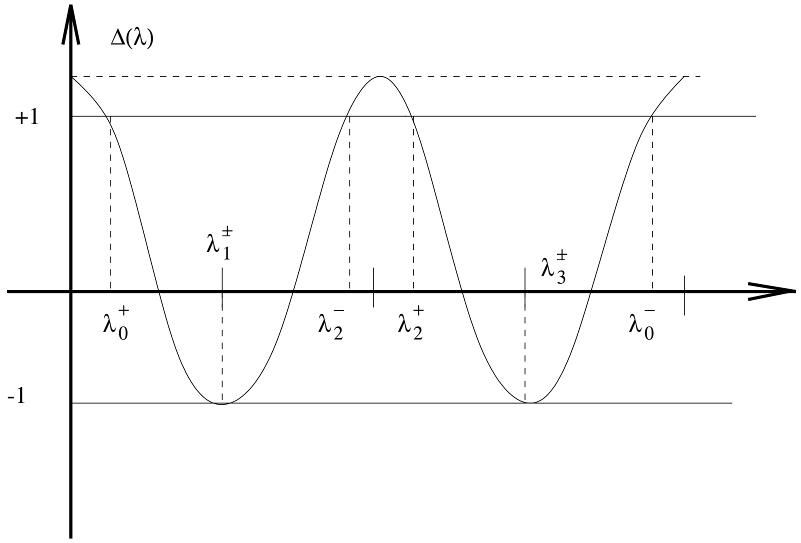

FIG. 2

The double points $\lambda_0^\pm$ and $\lambda_2^\pm$ split into pairs, while $\lambda_1^\pm$ and $\lambda_3^\pm$ remain double.

Now consider the case when $A = 0$ and $B \neq 0$. Then, $\Delta(\theta) = \frac{1}{\sqrt{D}}[\cos 2\theta - |B|^2]$, $D = (1-|B|^2)^2$. In this case, the double points $\lambda_0^\pm$ and $\lambda_2^\pm$ do not split, while $\lambda_1^\pm$ and $\lambda_3^\pm$ split into pairs of simple roots.

Finally, there is an open area in the space of parameters, say, $A \sim B$, where

$$\Delta(\theta) \approx \frac{1}{\sqrt{D}}\cos 2\theta, \qquad D \approx 1 - 4|A|^2 < 1.$$

In this case *all* double roots split into pairs of simple roots.

Example 3: constant potential $\psi_n = \psi_0$. Consider, first, the case $N = 2$. Then

$$T_2 = \frac{1}{\sqrt{D}} \begin{bmatrix} \lambda^2 + |\psi_0|^2 & \lambda\psi_0 + \lambda^{-1}\psi_0 \\ \lambda\overline{\psi}_0 + \lambda^{-1}\overline{\psi}_0 & \lambda^{-2} + |\psi_0|^2 \end{bmatrix},$$

$\Delta(\theta) = \frac{1}{\sqrt{D}}\left[\cos 2\theta + |\psi_0|^2\right]$, and $D = (1-|\psi_0|^2)^2$. The branch points $\lambda_k^\pm$ are given by the equation $\Delta^2(\lambda) = 1$. Obviously, the points $\lambda_1^\pm$ and $\lambda_3^\pm$ do not split. For $\lambda_0^\pm$ we have the equation

$$1 - |\psi_0|^2 = \cos 2\theta + |\psi_0|^2,$$

so for small $|\psi_0|$,

$$1 - 2|\psi_0|^2 = 1 - 2\theta^2 + O(\theta^4).$$

Finally, we arrive at $\theta = |\psi_0| + O(|\psi_0|^3)$ and

$$\lambda_0^\pm = e^{\pm i(|\psi_0| + O(|\psi_0|^3))} \qquad \lambda_2^\pm = e^{\pm i(|\psi_0| + O(|\psi_0|^3)) + i\pi}.$$

In words, the open gap is proportional to the absolute value of the potential $|\psi_0|$.

The case of general N can be treated easily. One has to compute spectrum of T_N. After simple algebra

$$\lambda_0^\pm = e^{\pm i(|\psi_0| + O(|\psi_0|^3))}, \qquad \lambda_N^\pm = e^{\pm i(|\psi_0| + O(|\psi_0|^3)) + i\pi}.$$

Example 4: two-gap trigonometric potential. Consider the N–periodic potential ψ^k:

$$\psi_n^k = e^{i\phi_k n}\psi_0, \qquad \phi_k = \frac{2\pi k}{N} \quad k = 0, \cdots, N-1.$$

We will show that this potential opens k–th and $k+N$–th gap as in

$$\lambda_k^\pm = e^{i\phi_k/2 \pm i(|\psi_0| + O(|\psi_0|^3))}, \tag{8}$$

and

$$\lambda_{k+N}^\pm = e^{i\phi_k/2 \pm i(|\psi_0| + O(|\psi_0|^3)) + i\pi}. \tag{9}$$

To emphasise the dependence on ψ^k we write $V(n, \lambda|\psi^k) = V^k(n)$, $T(\lambda|\psi^k) = T^k(\lambda)$ and $\Delta(\lambda|\psi^k) = \Delta^k(\lambda)$. Note, first, that

$$L^k(n) = \Phi^n V^k(0)\Phi^{-n}, \quad \text{where} \quad \Phi = e^{i\sigma_3\phi_k/2}.$$

Therefore,

$$\begin{aligned} T(\lambda|e^{i\phi_k n}\psi_0) &= V^k(N-1)\cdots V^k(0) = \Phi^N(\Phi^{-1}V^k(0))^N \\ &= (-1)^k(\Phi^{-1}V^k(0))^N = (-1)^k T(\lambda e^{-i\phi_k/2}|e^{-i\phi_k/2}\psi_0). \end{aligned}$$

Taking the trace we obtain

$$\Delta(\theta|e^{i\phi_k n}\psi_0) = (-1)^k \Delta(\theta - \frac{\phi_k}{2}|e^{-i\phi_k/2}\psi_0).$$

This and the result of Example 3 produce the formula (8). (9) follows from this and the symmetry (4).

Remark 1. The situation here is similar to the NLS equation, [MCV1], where the N–th gap opens in proportion to the N-th Fourier coefficient.

Remark 2. The $4N$ branch points $\lambda_0^\pm, \cdots, \lambda_{2N-1}^\pm$ determine the curve. The symmetry (4) leaves only $2N$ free parameters. In fact, there are only N free independent parameters due to periodicity conditions.

Example 5: generic periodic potential. The potential can be written as a sum of harmonics $\psi^k, \quad k = 0, \cdots, N-1$. All gaps are open. To construct the curve Γ, let us take two copies "+" and "-" of $\mathbb{C}P^1$ cut along circular arcs connecting λ_k^- and λ_k^+ (fig. 3)

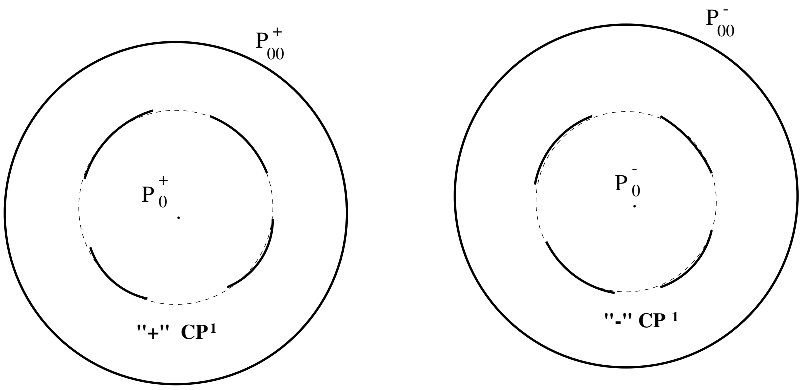

FIG. 3

Each copy $\mathbb{C}P^1$ has two marked points $P^+_{0/\infty}$ and $P^-_{0/\infty}$. The behavior of $\Delta(\lambda),\ \ y(Q) = \sqrt{\Delta(\lambda)^2 - 1}$, and $w(Q)$ near these points is this:

$$\Delta(\lambda) \sim \frac{1}{2}\lambda^N,\ y(Q) \sim \pm\frac{1}{2}\lambda^N,\ w(Q) \sim \lambda^{\pm N}, \qquad\qquad \lambda = \lambda(Q),\ Q \in (P_\infty^\pm);$$

and

$$\Delta(\lambda) \sim \frac{1}{2}\lambda^{-N},\ y(Q) \sim \pm\frac{1}{2}\lambda^{-N},\ w(Q) \sim \lambda^{\mp N}, \qquad\qquad \lambda = \lambda(Q),\ Q \in (P_0^\pm).$$

From this, we see that τ_a maps point $Q = (\lambda, y)$ in the vicinity of $P_\infty^{+/-}$ to the point $\tau_a Q = (\frac{1}{\lambda}, \overline{y})$ in the vicinity of $P_0^{+/-}$. The curve is obtained by gluing

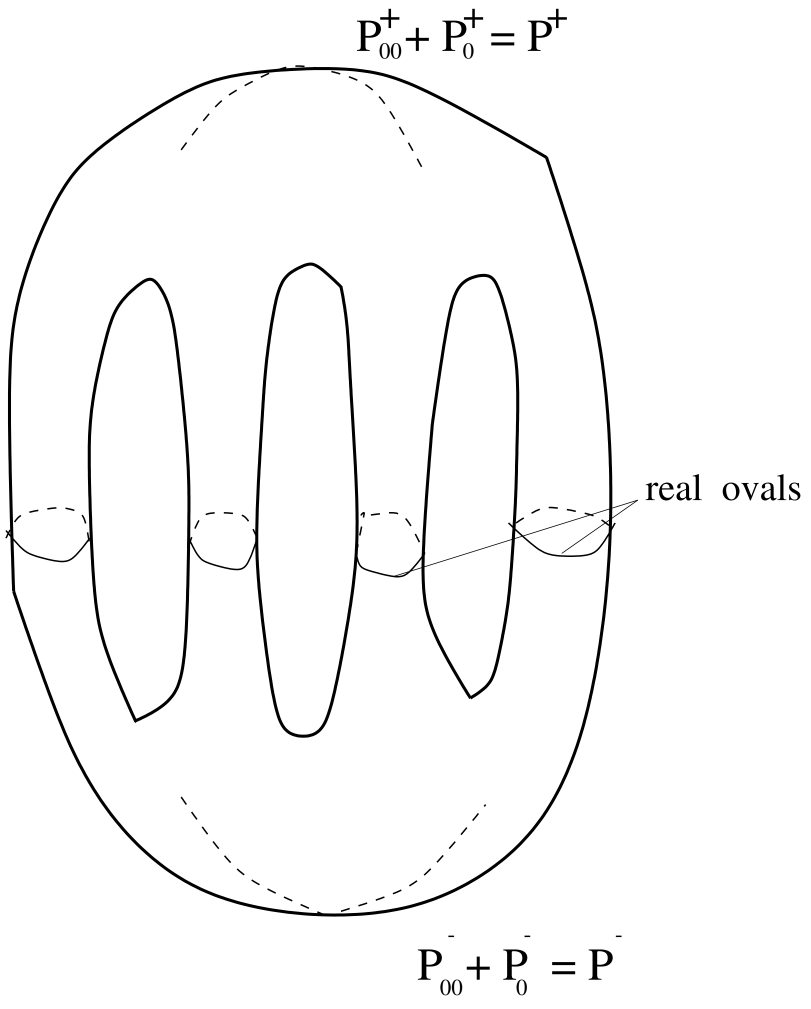

FIG. 4

together "±" copies of $\mathbb{C}P^1$ along the cuts and identifying the points $P_\infty^{+/-}$ with $P_0^{+/-}$ as specified

$$P^+ = P_\infty^+ + P_0^+, \qquad \text{and} \qquad P^- = P_\infty^- + P_0^-.$$

The cuts are the projections on the λ–plane of the real ovals of anti–holomorphic involution τ_a. The infinities P^+ and P^- are also fixed points of τ_a. The multiplier $w(Q)$ is single-valued on Γ and satisfies the identity $w(\tau_a Q) = \overline{w(Q)}$. This guaranties *continuity* of $w(Q)$ at the points $P^{+/-}$, where $w(Q) \longrightarrow \infty/0$ respectively. The involution τ is defined so that it preserves the infinities. The formula $w(Q) = \Delta(\lambda(Q)) + y(Q)$ implies $w(\tau Q) = (-1)^N w(Q)$.

Let us introduce a multivalued function $p(Q)$ on Γ by the formula $w(Q) = e^{p(Q)N}$. Obviously, $p(Q)$ is defined up to the integer multiple of $2\pi i/N$. The differential dp is of the third kind with poles at P^+ and P^-. The relation $w(Q)w(\tau_\pm Q) = 1$ implies that $p(Q) \equiv -p(\tau_\pm Q) \pmod{2\pi i/N}$. Therefore, in the vicinity of $P^\pm_\infty$,

$$p^+_\infty(\lambda) = +\log\lambda + I_0 + \frac{I_1}{\lambda} + \frac{I_2}{\lambda} + \cdots,$$
$$p^-_\infty(\lambda) = -\log\lambda - I_0 - \frac{I_1}{\lambda} - \frac{I_2}{\lambda} + \cdots,$$

where $\lambda = \lambda(Q)$. The fact that τ preserves $P^{+/-}_\infty$ and $w(\tau Q) = (-1)^N w(Q)$ implies $I_k = 0$ for k odd, so

$$p^+_\infty(\lambda) = +\log\lambda + I_0 + \frac{I_2}{\lambda^2} + \frac{I_4}{\lambda^4} + \cdots,$$
$$p^-_\infty(\lambda) = -\log\lambda - I_0 - \frac{I_2}{\lambda^2} - \frac{I_4}{\lambda^4} + \cdots.$$

The anti-involution τ_a maps $P^{+/-}_\infty$ to $P^{+/-}_0$. This and relation $w(\tau_a Q) = \overline{w(Q)}$ imply, in the vicinity of $P^\pm_0$,

$$p^+_0(\lambda) = -\log\lambda + \overline{I}_0 + \overline{I}_2\lambda^2 + \overline{I}_4\lambda^4 + \cdots,$$
$$p^-_0(\lambda) = +\log\lambda - \overline{I}_0 - \overline{I}_2\lambda^2 - \overline{I}_4\lambda^4 + \cdots.$$

The explicit form of the integrals $I_0, I_2, \cdots$ will be computed in the next section.

4. Integrals of motion. To simplify the calculations, we introduce $V_{\text{new}}(n) = \sqrt{R_n}V_{\text{old}}(n)$. The corresponding mulitipliers are $w_{\text{new}} = \sqrt{D}w_{\text{old}}$, and

$$p_{\text{old}} = p_{\text{new}} - \frac{1}{2N}\log D.$$

We write all formulas in this section for the *new* spectral problem.

We introduce $T_{n,m} = (1+W_n(\lambda))e^{Z_{n,m}(\lambda)}(1+W_m(\lambda))^{-1}$, where Z is a diagonal matrix and W is antidiagonal. Separating the diagonal and antidiagonal part in the equation $T_{n+1,m} = V(n)T_{n,m}$, we obtain

$$e^{Z_{n+1,m}-Z_{n,m}} = \Lambda + \Psi_n W_n, \tag{1}$$
$$W_{n+1}e^{Z_{n+1,m}-Z_{n,m}} = \Lambda W_n + \Psi_n, \tag{2}$$

where

$$\Lambda = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} \quad \text{and} \quad \Psi_n = \begin{bmatrix} 0 & \psi_n \\ \overline{\psi}_n & 0 \end{bmatrix}.$$

Let $W_n(\lambda) = \sum\limits_{k=1}^{\infty} W_n^k \lambda^{-k}$, where

$$W_n^k = \begin{bmatrix} 0 & w_n^{k+} \\ w_n^{k-} & 0 \end{bmatrix}.$$

From (1-2), we have $W_{n+1}(\Lambda + \Psi_n W_n) = \Lambda W_n + \Psi_n$. Introducing

$$\sigma_+ = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \sigma_- = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

we can write $\Lambda = \sigma_+\lambda + \sigma_-\lambda^{-1}$ and

$$\begin{aligned} \sum_{k\geq 1} W_{n+1}^k \lambda^{-k}(\lambda\sigma_+ + \lambda^{-1}\sigma_-) + \sum_{k,p\geq 1} W_{n+1}^r \Psi_n W_n^p \lambda^{-r-p} \\ = (\lambda\sigma_+ + \lambda^{-1}\sigma_-) \sum_{k\geq 1} W_n^k \lambda^{-k} + \Psi_n. \end{aligned} \tag{3}$$

Collecting terms with λ^0 in (3), we have $W_{n+1}^1 \sigma_+ = \sigma_+ W_n^1 + \Psi_n$ and

$$W_n^1 = \begin{bmatrix} 0 & -\psi_n \\ \overline{\psi}_{n-1} & 0 \end{bmatrix}.$$

Collecting terms with λ^{-1} in (3), we have $W_{n+1}^2 \sigma_+ = \sigma_+ W_n^2$. This implies $W_n^2 \equiv 0$.

Collecting terms in (3) of the order λ^{-k}, $k \geq 2$, we have the reccurence formula

$$W_{n+1}^{k+1}\sigma_+ - \sigma_+ W_n^{k+1} = \sigma_- W_n^{k-1} - W_{n+1}^{k-1}\sigma_- - \sum_{\substack{r+p=k, \\ r,p\geq 1}} W_{n+1}^r \Psi_n W_n^p; \tag{4}$$

whence, for $k = 2$,

$$W_n^3 = \begin{bmatrix} 0 & -\psi_{n+1} + \psi_{n+1}|\psi_n|^2 \\ \overline{\psi}_{n-2} - \overline{\psi}_{n-2}|\psi_{n-1}|^2 & 0 \end{bmatrix}.$$

Therefore, for $n = N$ and $m = 0$ we have

$$T_N(\lambda) = (1 + W_N(\lambda)) e^{Z_{N,0}(\lambda)} (1 + W_0(\lambda))^{-1}$$

and

$$e^{Z_{N,0}}(\lambda) = \prod_{r=0}^{N-1} (\Lambda + \Psi_r W_r).$$

Using the fact that $\Psi_r W_r$ is diagonal we infer that for $Q \in (P_\infty^+)$

$$w(Q) = e^{p_\infty^+(\lambda)N} = \prod_{r=0}^{N-1} \left(\lambda + \psi_r \sum_{k\geq 1} w_r^{k-} \lambda^{-k} \right).$$

Therefore,

$$\begin{aligned} p_\infty^+(\lambda)N = N \log \lambda + \\ + \frac{1}{\lambda^2} \sum_{r=0}^{N-1} \psi_r w_r^{1-} + \frac{1}{\lambda^4} \left(\sum_{r=0}^{N-1} \psi_r w_r^{3-} - \frac{1}{2} (\psi_r w_r^{1-})^2 \right) + \cdots . \end{aligned}$$

Using the explicit expression of W_n^1, W_n^3, we have*

$$\begin{aligned} NI_0 &\equiv -\frac{1}{2} \log D, \\ NI_2 &= \sum_{r=0}^{N-1} \psi_r \overline{\psi}_{r-1}, \\ NI_4 &= \sum_{r=0}^{N-1} \psi_r \overline{\psi}_{r-2} \left(1 - |\psi_{r-1}|^2\right) - \frac{1}{2} (\psi_r \overline{\psi}_{r-1})^2. \end{aligned}$$

5. Floquet solution. Baker–Akhiezer function. The Floquet solution was defined in section 3 as a special solution of the spectral problem $\mathbf{g}_{n+1} = V(n)\mathbf{g}_n$ with the property $\mathbf{g}_N = T_N \mathbf{g}_0 = w\mathbf{g}_0$. It is uniqlue specified by the boundary conditions $g_n^1 + g_n^2|_{n=0} = 1$.
Example: vanishing potential $\psi_n = 0$. Then,

$$V_2(n, \lambda) = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}, \qquad \text{and} \qquad T_N(\lambda) = \begin{bmatrix} \lambda^N & 0 \\ 0 & \lambda^{-N} \end{bmatrix}.$$

Therefore,** for $Q \in (P_\infty^+/P_0^+)$,

$$\mathbf{g}_n(Q) = e^{\pm n \log \lambda} \left[\mathbf{g}_0^+ / \hat{\mathbf{g}}_0^+ \right], \qquad w(Q) = e^{\pm N \log \lambda};$$

and for $Q \in (P_\infty^-/P_0^-)$

$$\mathbf{g}_n(Q) = e^{\mp n \log \lambda} \left[\mathbf{g}_0^- / \hat{\mathbf{g}}_0^- \right], \qquad w(Q) = e^{\mp N \log \lambda};$$

*We write the coefficients for the *old* spectral problem.

**For any $\mathbf{g} = \begin{bmatrix} a \\ c \end{bmatrix}$ we write $\hat{\mathbf{g}} = \begin{bmatrix} \bar{c} \\ \bar{a} \end{bmatrix}$.

vectors $\mathbf{g}_0^+$, $\mathbf{g}_0^-$ are

$$\mathbf{g}_0^+ = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \mathbf{g}_0^- = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

We introduce another Floquet solution $\tilde{\mathbf{g}}_n$ normalised by the condition $i\tilde{\mathbf{g}}_n^1 - i\tilde{\mathbf{g}}_n^2|_{n=0} = 1$. Let us define

$$D_0 = 1, \qquad D_n = \prod_{k=0}^{n-1} R_k, \quad n = 1, \ldots, N-1; \qquad D_N = D.$$

Lemma 1. *i. The Floquet solution* $\mathbf{g}_n(Q)$ *is:*

$$\mathbf{g}_n(Q) = C(Q) \begin{bmatrix} T_n^{11}(\lambda) \\ T_n^{21}(\lambda) \end{bmatrix} + (1 - C(Q)) \begin{bmatrix} T_n^{12}(\lambda) \\ T_n^{22}(\lambda) \end{bmatrix}, \qquad \lambda = \lambda(Q).$$

with

$$C(Q) = \frac{T^{12}}{T^{12} - T^{11} + w(Q)} = \frac{w(Q) - T^{22}}{T^{21} - T^{22} + w(Q)},$$

here, $T = T_N$.

ii. The Floquet solution $\tilde{\mathbf{g}}_n(Q)$ *is*

$$\tilde{\mathbf{g}}_n(Q) = R(Q) \begin{bmatrix} T_n^{11}(\lambda) \\ T_n^{21}(\lambda) \end{bmatrix} + (i + R(Q)) \begin{bmatrix} T_n^{12}(\lambda) \\ T_n^{22}(\lambda) \end{bmatrix},$$

with

$$R(Q) = \frac{iT^{12}}{w(Q) - T^{11} - T^{12}} = \frac{iT^{22} - iw(Q)}{w(Q) - T^{21} - T^{22}}.$$

iii. The following formulas hold:

$$\sigma_1 \mathbf{g}_n(\tau_a Q) = \bar{\mathbf{g}}_n(Q), \qquad \sigma_1 \tilde{\mathbf{g}}_n(Q)(\tau_a Q) = \overline{\tilde{\mathbf{g}}_n(Q)}.$$

and

$$\mathbf{g}_n(\tau Q) = (-1)^n i \sigma_3 \tilde{\mathbf{g}}_n(Q).$$

(iv). The Floquet solution $\mathbf{g}_n(Q)$ *has* $2N$ *poles at the points* γ*'s on the real ovals of the curve* Γ*. On each oval there is just one* γ*.*

(v). If $Q \in (P_\infty^+/P_0^+)$*, then* $\mathbf{g}_n(Q)$ *has the development*

$$\mathbf{g}_n(Q) = e^{\pm n \log \lambda} \frac{1}{\sqrt{D_n}} \left[\sum_{s=0}^{\infty} \mathbf{g}_s^+(n) \lambda^{-s} / \sum_{s=0}^{\infty} \hat{\mathbf{g}}_s^+(n) \lambda^s \right],$$

with

$$\mathbf{g}_0^+ = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \mathbf{g}_1^+(n) = \begin{bmatrix} -\bar{\psi}_{-1} \\ \bar{\psi}_{n-1} \end{bmatrix}.$$

Also for $Q \in (P_\infty^-/P_0^-)$,

$$\mathbf{g}_n(Q) = e^{\mp n\log\lambda}\sqrt{D_n}\left[\sum_{s=0}^{\infty}\mathbf{g}_s^-(n)\lambda^{-s}/\sum_{s=0}^{\infty}\hat{\mathbf{g}}_s^-(n)\lambda^{s}\right],$$

with

$$\mathbf{g}_0^- = \begin{bmatrix}0\\1\end{bmatrix}, \qquad \mathbf{g}_1^-(n) = \begin{bmatrix}-\psi_n\\ \psi_0\end{bmatrix}.$$

Proof. i. Obviously,

$$\mathbf{g}_n(Q) = C'(Q)\begin{bmatrix}T_n^{11}(\lambda)\\T_n^{11}(\lambda)i\end{bmatrix} + C''(Q)\begin{bmatrix}T_n^{12}(\lambda)\\T_n^{22}(\lambda)i\end{bmatrix}$$

with some $C'(Q)$ and $C''(Q)$. The normalisation $g_n^1 + g_n^2|_{n=0} = 1$ implies $C''(Q) = 1 - C'(Q)$. Another condition $T_n\mathbf{g}_0 = w\mathbf{g}_0$ produces the system

$$\begin{bmatrix}T_N^{11} & T_N^{12}\\ T_N^{21} & T_N^{22}\end{bmatrix}\begin{bmatrix}C\\1-C\end{bmatrix} = w\begin{bmatrix}C\\1-C\end{bmatrix}.$$

Solving the system, we obtain the stated formulas for $C(Q)$.

ii. The proof is identical to the proof of (i).

iii. The proof is based on explicit formulas of (i) and (ii). Using (1) of section (3), we have

$$T_{n,0}(\lambda(\tau_a Q)) = \sigma_1\overline{T_{n,0}(\lambda(Q))}\sigma_1 \qquad \text{and} \qquad w(\tau_a Q) = \overline{w(Q)}.$$

From this we have $1 - \overline{C(\tau_a Q)} = C(Q)$. The proof is finished by substituting $T_n(\lambda(\tau_a Q))$ and $C(\tau_a Q)$ into explicit formula of (i). Similarly $\overline{R(\tau_a Q)} = i + R(Q)$. This implies the stated formula for $\tilde{\mathbf{g}}_n$.

The last formula for $\mathbf{g}_n(\tau Q)$ is proved along the same lines using (2) and (6) of section 3.

(iv). It is easy to see, by perturbation arguments, that $\mu_k = \lambda(\gamma_k)$ are near the points $\lambda_k^{\pm} = e^{i\frac{2\pi}{2N}k}$, $k = 0, \ldots, 2N-1$ for a small potential. We will prove that $|\mu_k| = 1$ *always.*

Let $\mathbf{g}_n(Q)$ have a pole at some point γ. Then, from the formula of (i), we have

$$T^{11} - T^{12} = w(Q) \qquad \text{and} \qquad T^{22} - T^{11} = w(Q).$$

Consider the special solution $\mathbf{f}_n(\mu) = T_n^{(1)}(\mu) - T_n^{(2)}(\mu)$ of the eigenvalue problem $\Lambda_n\mathbf{f}_n = \mu\mathbf{f}_n$. We have

$$\mathbf{f}_0 = \begin{bmatrix}1\\-1\end{bmatrix} \qquad \text{and} \qquad \mathbf{f}_N = \begin{bmatrix}w(Q)\\-w(Q)\end{bmatrix}.$$

Now the Cauchy formula (7) of section 3 to obtain

$$< \Lambda \mathbf{f}, \mathbf{f} >_{\heartsuit} = < \mathbf{f}, \Lambda^{-1} \mathbf{f} >_{\heartsuit} + \text{boundary terms}.$$

Note, that $\Lambda \mathbf{f} = \mu \mathbf{f}$, $\Lambda^{-1}\mathbf{f} = \mu^{-1}\mathbf{f}$. It is easy to compute the boundary terms: to wit,

$$\frac{1}{N}(|w|^2 - 1)(\bar{\mu}^{-1} - \mu).$$

Substituting this into the Cauchy formula, we obtain

$$(1 - |\mu|^2)\left[\frac{1}{N}(|w|^2 - 1) - (\mathbf{f}, \mathbf{f})\right] = 0.$$

This implies $|\mu| = 1$.

(v). Consider P_∞^+/P_0^+. The relation between two asymptotic expansions follows from the formula $\sigma_1 \mathbf{g}_n(\tau_a Q) = \mathbf{g}_n(Q)$. The actual form of the coefficient is derived by substituting the asymptotic expansion

$$\mathbf{g}_n(Q) = e^{n \log \lambda} \frac{1}{\sqrt{D_n}} \sum_{s=0}^{\infty} \mathbf{g}_s^+(n) \lambda^{-s}$$

into the spectral problem $\mathbf{g}_{n+1} = V(n, \lambda)\mathbf{g}_n$, where $V(n, \lambda) = [\lambda \sigma_+ + \lambda^{-1}\sigma_- + \Psi_n]$. We arrive at the reccurence relation

$$\sigma_+ \mathbf{g}_{s+1}^+(n) + \sigma_- \mathbf{g}_{s-1}^+(n) + \Psi_n \mathbf{g}_s^+(n) = \mathbf{g}_{s+1}^+(n+1).$$

The boundary condition $g_n^1 + g_n^2|_{n=0} = 1$ implies

$$g_0^1(n) + g_0^2(n)|_{n=0} = 1; \qquad g_s^1(n) + g_s^2(n)|_{n=0} = 0 \quad \text{for} \quad s \geq 1.$$

Starting from $s = -1$, reccurently, one can compute $\mathbf{g}_0^+(n)$, $\mathbf{g}_1^+(n)$, *etc.* $\square$

On the curve Γ one can consider the Baker-Akhiezer (BA) function $\mathbf{g}(\tau, n, t, \lambda)$ with $2N$ poles on the real ovals. On each oval there is just one pole. The function $\mathbf{g}(\tau, n, t, \lambda)$ has the following assymptotics at the infinities:

$$\mathbf{g}(\tau, n, t, \lambda) = e^{\frac{i}{2}\tau + n \log \lambda + i(\lambda^2 - 1)t} \frac{1}{\sqrt{D_n}} \sum_{s=0}^{\infty} \mathbf{g}_s^+ \lambda^{-s}, \qquad Q \in (P_\infty^+),$$

$$\mathbf{g}(\tau, n, t, \lambda) = e^{-\frac{i}{2}\tau - n \log \lambda - i(\lambda^{-2} - 1)t} \sqrt{D_n} \sum_{s=0}^{\infty} \mathbf{g}_s^- \lambda^{-s}, \qquad Q \in (P_\infty^-),$$

$$\mathbf{g}(\tau, n, t, \lambda) = e^{-\frac{i}{2} - n \log \lambda - i(\lambda^{-2} - 1)t} \frac{1}{\sqrt{D_n}} \sum_{s=0}^{\infty} \hat{\mathbf{g}}_s^+ \lambda^{s}, \qquad Q \in (P_0^+),$$

$$\mathbf{g}(\tau, n, t, \lambda) = e^{\frac{i}{2} + n \log \lambda + i(\lambda^2 - 1)t} \sqrt{D_n} \sum_{s=0}^{\infty} \hat{\mathbf{g}}_s^- \lambda^{s}, \qquad Q \in (P_0^-).$$

The BA function with these properties exists and defined uniqly. The BA function $\mathbf{g}(\tau, n, t, Q)$ satisfies the identities

$$[\partial_\tau - V_1]\mathbf{g}(\tau, n, t, Q) = 0,$$
$$[\Delta - V_2]\mathbf{g}(\tau, n, t, Q) = 0,$$
$$[\partial_t - V_3]\mathbf{g}(\tau, n, t, Q) = 0.$$

The explicit form of the V's is given in section 2. The Floquet solution is a particular case of BA function with the variables τ, t fixed.

6. Dual Floquet solution. Variational identity. One can write the equation $V(n)\mathbf{g}_n = \mathbf{g}_{n+1}$ in the form*

$$[J\Delta - JV(n)]\mathbf{g}_n = 0.$$

Let us define the dual Floquet solution $\mathbf{g}^+ = \left[g^{1+}, g^{2+}\right]$ at the point Q by

$$\mathbf{g}_n^+(Q) = \mathbf{g}_n(\tau_\pm Q)^T.$$

Lemma 2. *The dual Floquet solution* $\mathbf{g}_n^+(Q)$ *satisfies* **

$$\mathbf{g}_n^+(Q)\,[J\Delta - JV(n-1, \lambda)] = 0.$$

Proof. The equation $V(n-1)\mathbf{g}_{n-1} = \mathbf{g}_n$ can be written as $\mathbf{g}_{n-1} = V^{-1}(n-1)\mathbf{g}_n$, where

$$V^{-1}(n-1) = \frac{1}{\sqrt{R_{n-1}}}\begin{bmatrix} \lambda^{-1} & -\psi_{n-1} \\ -\bar{\psi}_{n-1} & \lambda \end{bmatrix};$$

or, in coordinates,

$$g_{n-1}^1 = \frac{1}{\sqrt{R_{n-1}}}(\lambda^{-1}g_n^1 - \psi_{n-1}g_n^2),$$
$$g_{n-1}^2 = \frac{1}{\sqrt{R_{n-1}}}(-\bar{\psi}_{n-1}g_n^1 + \lambda g_n^2).$$

We can also rewrite it in the form

$$(g_n^1, g_n^2)\left(\begin{bmatrix} 0 & \Delta \\ -\Delta & 0 \end{bmatrix} - \frac{1}{\sqrt{R_{n-1}}}\begin{bmatrix} \bar{\psi}_{n-1} & \lambda^{-1} \\ -\lambda & -\psi_{n-1} \end{bmatrix}\right) = 0.$$

This is exactly the stated identity. $\square$

We need the standard formula [KP] for variations of the quasi-momentum.

---

*$J = i\sigma_2$.

**$\mathbf{f}_n\Delta = \mathbf{f}_{n-1}$.

Lemma 3. *i. The expression* $\mathbf{g}^+_n(Q)J\mathbf{g}_n(Q)$ *does not depend on* n *and*

$$\mathbf{g}^+_n(Q)J\mathbf{g}_n(Q) =< \mathbf{g}^+J\mathbf{g} >= \Psi(Q).$$

ii. The following identity holds

$$\delta p\Psi(Q) =< \mathbf{g}^+_n J\delta V(n-1)\mathbf{g}_{n-1} > .$$

Proof. i. Can be checked using difference equation for $\mathbf{g}_n(Q)$ and $\mathbf{g}^+_n(Q)$.

ii. Denote by $\tilde{V}(n,\lambda)$ and $\tilde{\mathbf{g}}_n(\lambda)$ deformed matrix $V(n,\lambda)$ and the Floquet solution $\mathbf{g}_n(\lambda)$; to wit, $\tilde{V} = V + \epsilon\delta V + o(\epsilon)$ and $\tilde{\mathbf{g}} = \mathbf{g} + \epsilon\delta\mathbf{g} + o(\epsilon)$. Then,

$$\mathbf{g}^+_n(Q)\left[(J\Delta - J\tilde{V}(n-1,\lambda))\tilde{\mathbf{g}}_{n-1}(Q)\right] = 0,$$
$$\left[\mathbf{g}^+_n(Q)(J\Delta - JV(n-1,\lambda))\right]\tilde{\mathbf{g}}_{n-1}(Q) = 0.$$

Subtracting, we obtain

$$\sum_{n=0}^{N-1}\mathbf{g}^+_n(J\Delta\tilde{\mathbf{g}}_{n-1}) - (\mathbf{g}^+_n J\Delta)\tilde{\mathbf{g}}_{n-1}$$
$$= \sum_{n=0}^{N-1}\mathbf{g}^+_n\left(J\tilde{V}(n-1)\tilde{\mathbf{g}}_{n-1}\right) - \left(\mathbf{g}^+_n JV(n-1)\right)\tilde{\mathbf{g}}_{n-1}.$$

It is easy to see, that

$$\text{RHS} = \epsilon\sum_{n=0}^{N-1}\mathbf{g}^+_n J\delta V(n-1)\mathbf{g}_{n-1} + o(\epsilon).$$

Using the formula

$$\sum_{n=0}^{N-1}(\mathbf{g}^+_n J\Delta\mathbf{g})\tilde{\mathbf{g}}_{n-1} = \sum_{n=0}^{N-1}\mathbf{g}^+_n(J\Delta\tilde{\mathbf{g}}_{n-1}) + \mathbf{g}^+_{-1}J\tilde{\mathbf{g}}_{-1} - \mathbf{g}^+_{N-1}J\tilde{\mathbf{g}}_{N-1},$$

we obtain

$$\text{LHS} = \mathbf{g}^+_{N-1}J\tilde{\mathbf{g}}_{N-1} - \mathbf{g}^+_{-1}J\tilde{\mathbf{g}}_{-1}.$$

From the definition of the Floquet solution

$$\mathbf{g}^+_{N-1} = e^{-Np}\mathbf{g}^+_{-1} \quad \text{and} \quad \tilde{\mathbf{g}}_{N-1} = e^{N\tilde{p}}\tilde{\mathbf{g}}_{-1},$$

we have

$$\text{LHS} = \left(e^{N(\tilde{p}-p)} - 1\right)\mathbf{g}^+_{-1}J\tilde{\mathbf{g}}_{-1} = \epsilon N\delta p\mathbf{g}^+_{-1}J\mathbf{g}_{-1} + o(\epsilon).$$

Collecting terems with ϵ, we obtain the stated identity $\square$

Let us introduce $\mathbf{g}^*_n(Q)$, the dual Floquet solution normalized by the condition $< \mathbf{g}^*J\mathbf{g} >= 1$. Obviously,

$$\mathbf{g}^*_n(Q) = \frac{\mathbf{g}^+_n(Q)}{\Psi(Q)};$$

$\mathbf{g}^*_n(Q)$ has poles at the branch points of the curve.

7. Hamiltonian formalism for the Ablowitz-Ladik system. As shown in [KP], the formula

$$\omega_0 = \frac{i}{2}\sum \operatorname{res} \frac{1}{\Psi(Q)} < \mathbf{g}^+(n) J \delta V(n-1) \wedge \delta \mathbf{g}(n-1) > \frac{d\lambda}{\lambda}.$$

defines a closed, nondegenerate 2 form on the space of operators $\Delta - V_2$ with periodic potential. Our goal to compute the residue *explicitly.*

Near P_0^+,

$$\frac{1}{\Psi(Q)} = \psi_0 + \psi_1\lambda^1 + \cdots, \qquad Q \in (P_0^+). \tag{1}$$

The identity

$$\begin{aligned}\Psi(\tau_\pm Q) = < \mathbf{g}^+(\tau_\pm Q) J \mathbf{g}(\tau_\pm Q) > &= < \mathbf{g}^T(Q) J \mathbf{g}^+(Q)^T > \\ &= - < \mathbf{g}^+(Q) J \mathbf{g}(Q) > = -\Psi(Q)\end{aligned}$$

implies

$$\frac{1}{\Psi(Q)} = -\psi_0 - \psi_1\lambda^1 - \cdots, \qquad Q \in (P_0^-). \tag{2}$$

The involution τ_a maps $P_\infty^{+/-}$ to $P_0^{+/-}$, and $\mathbf{g}_n(\tau_a Q) = \sigma_1 \bar{\mathbf{g}}_n(Q)$. Therefore, using $\tau_\pm\tau_a = \tau_a\tau_\pm$, we have

$$\begin{aligned}\Psi(\tau_a Q) &= \mathbf{g}^T(\tau_\pm\tau_a Q) J \mathbf{g}(\tau_a Q) = \mathbf{g}^T(\tau_a\tau_\pm Q) J \mathbf{g}(\tau_a Q) \\ &= \bar{\mathbf{g}}^T(\tau_\pm Q)\sigma_1^T J \sigma_1 \bar{\mathbf{g}}(Q) = -\overline{\Psi(Q)}\end{aligned}$$

so that, if $Q \in (P_\infty^+)$, then $\tau_a Q \in (P_0^+)$ and

$$\begin{aligned}\frac{1}{\Psi(Q)} &= -\overline{\frac{1}{\Psi(\tau_a Q)}} = -\overline{\left(\psi_0 + \frac{\psi_1}{\bar{\lambda}^1} + \cdots\right)} \\ &= -\bar{\psi}_0 - \frac{\bar{\psi}_1}{\lambda^1} - \cdots, \qquad Q \in (P_\infty^+).\end{aligned} \tag{3}$$

Again, applying $\tau_\pm$, we have

$$\frac{1}{\Psi(Q)} = \bar{\psi}_0 + \frac{\bar{\psi}_1}{\lambda^1} + \cdots, \qquad Q \in (P_\infty^-). \tag{4}$$

It is easy to compute

$$\psi_0 =< \hat{\mathbf{g}}_0^{-T} J \hat{\mathbf{g}}_0^{+} >= 1$$
$$\psi_1 =< \hat{\mathbf{g}}_0^{-T} J \hat{\mathbf{g}}_1^{+} > + < \hat{\mathbf{g}}_1^{-T} J \hat{\mathbf{g}}_0^{+} >= \bar{\psi}_0 - \psi_{-1}, \quad etc.$$

Similarly, introducing $S(Q) \equiv < \mathbf{g}_n^+(Q) J \delta V(n-1) \wedge \delta \mathbf{g}_{n-1}(Q) >$, we obtain

$$S(\tau_a Q) = -\overline{S(Q)}. \tag{5}$$

One can show that, in the vicinity of the infinities:

$$S(Q) = c_0 + c_1\lambda + \cdots, \qquad Q \in (P_0^+),$$
$$S(Q) = d_0 + \frac{d_{-1}}{\lambda} + \cdots, \qquad Q \in (P_\infty^+);$$

and

$$S(Q) = a_0 + a_1\lambda + \cdots \qquad Q \in (P_0^-),$$
$$S(Q) = b_0 + \frac{b_{-1}}{\lambda} + \cdots \qquad Q \in (P_\infty^-).$$

(5) implies $d_{-k} = -\bar{c}_k$ and $b_{-k} = -\bar{a}_k$. Computing residues, we have

$$\operatorname*{res}_{P_\infty^+} \frac{S}{\Psi}\frac{d\lambda}{\lambda} = \operatorname*{res}_{P_\infty^+} (d_0 + \cdots)\left(-\bar{\psi}_0 - \cdots\right)\frac{d\lambda}{\lambda} = d_0\bar{\psi}_0,$$
$$\operatorname*{res}_{P_\infty^-} \frac{S}{\Psi}\frac{d\lambda}{\lambda} = \operatorname*{res}_{P_\infty^-} (b_0 + \cdots)\left(\bar{\psi}_0 + \cdots\right)\frac{d\lambda}{\lambda} = -b_0\bar{\psi}_0,$$

and

$$\sum_{P_\infty^{+/-}} \operatorname{res} \frac{S}{\Psi}\frac{d\lambda}{\lambda} = d_0 - b_0.$$

Similarly,

$$\operatorname*{res}_{P_0^+} \frac{S}{\Psi}\frac{d\lambda}{\lambda} = \operatorname*{res}_{P_0^+} (c_0 + \cdots)(\psi_0 + \cdots)\frac{d\lambda}{\lambda} = c_0\psi_0,$$
$$\operatorname*{res}_{P_0^-} \frac{S}{\Psi}\frac{d\lambda}{\lambda} = \operatorname*{res}_{P_0^-} (a_0 + \cdots)(-\psi_0 - \cdots)\frac{d\lambda}{\lambda} = -a_0\psi_0,$$

and

$$\sum_{P_0^{+/-}} \operatorname{res} \frac{S}{\Psi}\frac{d\lambda}{\lambda} = c_0 - a_0 = -\overline{(d_0 - b_0)}.$$

Using the formulas for Floquet solutions from Lemma 1, we have

$$
\begin{aligned}
b_0 = \quad & < \frac{\mathbf{g}_0^{+T}}{\sqrt{D_n}} J\delta \frac{\sigma_-}{\sqrt{R_{n-1}}} \wedge \delta\sqrt{D_{n-1}}\mathbf{g}_0^- > \\
& + < \frac{\mathbf{g}_0^{+T}}{\sqrt{D_n}} J\delta \frac{\Psi_{n-1}}{\sqrt{R_{n-1}}} \wedge \delta\sqrt{D_{n-1}}\mathbf{g}_1^- > \\
& + < \frac{\mathbf{g}_1^{+T}}{\sqrt{D_n}} J\delta \frac{\sigma_+}{\sqrt{R_{n-1}}} \wedge \delta\sqrt{D_{n-1}}\mathbf{g}_1^- > \\
& + < \frac{\mathbf{g}_1^{+T}}{\sqrt{D_n}} J\delta \frac{\Psi_{n-1}}{\sqrt{R_{n-1}}} \wedge \delta\sqrt{D_{n-1}}\mathbf{g}_0^- > \\
= \quad & < \frac{1}{\sqrt{D_n}} \delta \frac{1}{\sqrt{R_{n-1}}} \wedge \delta\sqrt{D_{n-1}} > \\
& - < \frac{1}{\sqrt{D_n}} \delta \frac{\bar{\psi}_{n-1}}{\sqrt{R_{n-1}}} \wedge \delta\sqrt{D_{n-1}}\psi_{n-1} > \\
& + < \frac{\bar{\psi}_{n-1}}{\sqrt{D_n}} \delta \frac{1}{\sqrt{R_{n-1}}} \wedge \delta\sqrt{D_{n-1}}\psi_{n-1} > \\
& - < \frac{\bar{\psi}_{n-1}}{\sqrt{D_n}} \delta \frac{\Psi_{n-1}}{\sqrt{R_{n-1}}} \wedge \delta\sqrt{D_{n-1}} > .
\end{aligned}
$$

After simple algebra,

$$
\therefore= \frac{1}{4} < \frac{\delta R_{n-1}}{R_{n-1}} \wedge \frac{\delta D_{n-1}}{D_{n-1}} > - < \frac{1}{R_{n-1}} \delta\bar{\psi}_{n-1} \wedge \delta\psi_{n-1} > .
$$

Similarly,

$$
\begin{aligned}
d_0 =& < \sqrt{D_n}\mathbf{g}_0^{-T} J\delta \frac{\sigma_+}{\sqrt{R_{n-1}}} \wedge \delta \frac{\mathbf{g}_0^+}{\sqrt{D_{n-1}}} > \\
=& -\frac{1}{4} < \frac{\delta R_{n-1}}{R_{n-1}} \wedge \frac{\delta D_{n-1}}{D_{n-1}} > .
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\sum_{P_\infty^{+/-}} \operatorname{res} \frac{S}{\Psi}\frac{d\lambda}{\lambda} =& \quad < \frac{1}{R_{n-1}} \delta\bar{\psi}_{n-1} \wedge \delta\psi_{n-1} > -\frac{1}{2} < \frac{\delta R_{n-1}}{R_{n-1}} \wedge \frac{\delta D_{n-1}}{D_{n-1}} > \\
\sum_{P_0^{+/-}} \operatorname{res} \frac{S}{\Psi}\frac{d\lambda}{\lambda} =& - < \frac{1}{R_{n-1}} \delta\psi_{n-1} \wedge \delta\bar{\psi}_{n-1} > +\frac{1}{2} < \frac{\delta R_{n-1}}{R_{n-1}} \wedge \frac{\delta D_{n-1}}{D_{n-1}} > .
\end{aligned}
$$

Taking the sum, we obtain

$$
\omega_0 =< \frac{i}{R_{n-1}} \delta\bar{\psi}_{n-1} \wedge \delta\psi_{n-1} > .
$$

Lemma 5. *(i) The formula*

$$\xi_n(Q) = \lambda^n < \mathbf{g}^* J \delta V \wedge \delta \mathbf{g} > \frac{d\lambda}{\lambda}$$

$n = \dots, -1, 0, 1, \dots,$ *defines meromorphic in Q differential form on Γ with poles at $\gamma_1, \cdots, \gamma_{2N}$ and $P_\infty^{+/-}, P_0^{+/-}$.*

(ii) The symplectic 2-forms defined by the formula

$$\omega_n = \frac{i}{2} \sum_P res\, \xi_n(Q)$$

can be written as

$$\omega_n = -\frac{i}{2} \sum_{k=1}^{2N} \lambda^n(\gamma_k) \delta p(\gamma_k) \wedge \frac{\delta\lambda}{\lambda}(\gamma_k).$$

Proof. (i) The poles of $\mathbf{g}_n^*$ at the branch points $(\lambda^\pm, 0)$ are killed by the zeros of $d\lambda$. The rest are just $\gamma_1, \cdots, \gamma_{2N}$ and $P_\infty^{+/-}$ and $P_0^{+/-}$.

(ii) By Cauchy's theorem,

$$\sum_P \operatorname{res} \xi_n(Q) + \sum_{k=1}^{2N} \operatorname{res} \xi_n(Q) = 0.$$

Near γ_k,

$$\mathbf{g}_n = \frac{\operatorname{res} \mathbf{g}_n}{\lambda - \lambda(\gamma_k)} + O(1).$$

Therefore,

$$\delta \mathbf{g}_n(Q) = \frac{\operatorname{res} \mathbf{g}_n}{(\lambda - \lambda(\gamma_k))^2} \delta\lambda(\gamma_k) + O(1) = \frac{\mathbf{g}_n}{\lambda - \lambda(\gamma_k)} \delta\lambda(\gamma_k) + O(1).$$

Note that $\mathbf{g}_n^*(\gamma_k) = 0$ and, using Lemma 3,

$$\operatorname*{res}_{\gamma_k} \xi_n(Q) = \lambda^n < \mathbf{g}^* J \delta V \mathbf{g} > \wedge \frac{\delta\lambda}{\lambda}(\gamma_k) \operatorname*{res}_{\gamma_k} \frac{d\lambda}{\lambda - \lambda(\gamma_k)} = \lambda^n(\gamma_k) \delta p(\gamma_k) \wedge \frac{\delta\lambda}{\lambda}(\gamma_k).$$

We are done. $\square$

The bracket $\{\bullet, \bullet\}_{\omega_0}$ is constructed from the symplectic form ω_0

$$\{f, g\}_{\omega_0} \equiv i \sum_{n=1}^{N} R_n \left(\frac{\delta f}{\delta \psi_n} \frac{\delta g}{\delta \bar{\psi}_n} - \frac{\delta f}{\delta \bar{\psi}_n} \frac{\delta g}{\delta \psi_n} \right).$$

The original AL flow from section 2 can be written

$$\overset{\bullet}{\psi}_n = \{\psi_n, H\}_{\omega_0}, \qquad H = N\left(I_2 + \overline{I}_2 - 2I_0 - 2\overline{I}_0\right).$$

The phase flow is also Hamiltonian

$$\overset{\bullet}{\psi}_n = \{\psi_n, P\}_{\omega_0}, \qquad P = N(I_0 + \overline{I}_0).$$

8. Embedding of the AL system into the function space. First, we introduce interpolating trigonometrical polynomials.

Let $f(x)$ be a smooth 1-periodic complex function and associate to it the sequence of interpolating trigonometrical polynomials

$$f_N(x) = \sum_{|k|\leq m} e^{2\pi ik} \hat{f}_N(k), \qquad N = 2m+1, \;\; m = 1, 2 \dots.$$

These have the property that $f_N(x) = f(x)$ for $x \in \mathbb{S}_N = \{x \in \mathbb{S} : x = \frac{n}{N},\; n = 0, \dots, N-1\}$. Let $\mathcal{M}_N$ be a space of such trigonometric polynomials of degree m; it is in one-to-one correspondence with the space M_N of N-periodic complex sequences:

$$f_N(x) \longleftrightarrow f_n, \quad f_N(x) \in \mathcal{M}_N, f_n \in M_N$$

if we put $\epsilon f_N(\frac{n}{N}) = f_n, \;\; n = 0, \dots, N-1$.

Introduce $\mathcal{H}_1^N(\psi) \equiv H_1^N(\epsilon\psi)$. The region

$$\mathcal{B}^N = \{\psi_N \in \mathcal{M}_N : \; N\mathcal{H}_1^N(t\psi) < \infty, \;\; \text{for all} \;\; 0 \leq t \leq 1\}$$

is called the "box" of the space $\mathcal{M}_N$; the trigonometric polynomial ψ_N belongs $\mathcal{B}^N$ if and only if $|\psi_N(x)| < N, \; x \in \mathbb{S}_N$. The map $\psi_N(x) \leftrightarrow \psi_n$ allows us to define the flow $e^{t\mathcal{X}_3^N}$ on $\mathcal{B}^N$ by the formula

$$e^{t\mathcal{X}_3^N} \psi_N \equiv \epsilon^{-1} e^{\theta X_3^N} \epsilon\psi_N, \qquad \theta = \epsilon^{-2} t.$$

Evidently $\mathcal{B}^N$ is invariant under the flow ${}^{t\mathcal{X}_3^N}$. Due to the natural embedding $\mathcal{M}_N \subset \mathcal{M}$, the dynamics $e^{t\mathcal{X}_3^N}$ also can be defined by the formula $e^{t\mathcal{X}_3^N}\psi \equiv e^{t\mathcal{X}_3^N}\psi_N$ for any function $\psi \in \mathcal{M}$ which satisfies the inequality $|\psi(x)| < N, \; x \in \mathbb{S}_N$.

We define on $\mathcal{M}_N$ the volume form

$$d\,vol^N \equiv \frac{1}{N!\mathcal{D}_N(\psi)} \bigwedge_{x \in \mathbb{S}_N} i\delta\psi(x) \wedge \delta\bar{\psi}(x),$$

where $\mathcal{D}_N(\psi) \equiv D_N(\epsilon\psi)$ and the functional $\mathcal{H}_5^N(\psi)$ is by definition $H_5^N(\epsilon\psi)$. The flow $e^{\theta X_3}$ preserves both $H_3(\psi_n, \bar{\psi}_n)$ and the volume form

$$d\,\mathrm{vol} = \frac{1}{N!D_N} \bigwedge_n i\delta\psi_n \wedge \delta\bar{\psi}_n.$$

Therefore, the volume form $d\,vol^N = \epsilon^{-2N} d\,\mathrm{vol}$ and the functional $\mathcal{H}_5^N$ are invariant under the flow $e^{t\mathcal{X}_3^N}$. The finite measure $d\mu^N(\psi,\bar\psi)$ on $\mathcal{B}^N$ with density

$$e^{-\frac{N^5}{2}\mathcal{H}_5^N(\psi)} d\,vol^N = e^{-\frac{N^5}{2}\mathcal{H}_5^N(\psi)} \frac{1}{N!\mathcal{D}_N(\psi)} \bigwedge_{x\in\mathbb{S}_N} i\delta\psi(x)\wedge\delta\bar\psi(x)$$

is also invariant under the flow $e^{t\mathcal{X}_3^N}$.

To ensure proper analytic control* we need to introduce the function $h(x)$ (fig. 5)

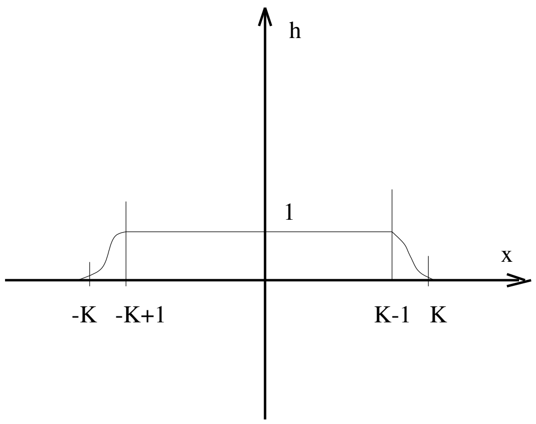

FIG. 5

and we define the invariant functional $\chi_K^N(\psi) \equiv h(N\mathcal{H}_1^N)h(-N^3\mathcal{H}_3^N)$. On the box $\mathcal{B}^N$ the probability measure $d\mu_K^N(\psi,\bar\psi)$

$$\frac{1}{\Xi_K^N}\chi_K^N(\psi)e^{-\frac{N^5}{2}\mathcal{H}_5^N(\psi)} d\,vol^N,$$

where Ξ_K^N is a normalisating factor**.

Due to the embedding $\mathcal{M}_N \subset \mathcal{M}$, the measure $d\mu_K^N(\psi,\bar\psi)$ can be transfered to the whole of $\mathcal{M}$. We define on $\mathcal{M}$ the probability measure $d\mu_K(\psi,\bar\psi)$ with the density

$$\frac{1}{\Xi_K}\chi_K(\psi)e^{-\frac{1}{2}\mathcal{H}_5} d\,vol = \frac{1}{\Xi_K}\chi_K(\psi)e^{-\frac{1}{2}\mathcal{H}_5}\frac{1}{\infty!}\bigwedge_{x\in\mathbb{S}} i\delta\psi(x)\wedge\delta\bar\psi(x),$$

where $\chi_K(\psi) = h(\mathcal{H}_1)h(\mathcal{H}_3)$ and Ξ_K is a normalisating factor.

*Similar cut-off was introduced in [MCV2].

**The precise definition of Ξ_K^N and Ξ_K will be given in Lemma 8.

In order to prove invariance of the measure $d\mu_K$, we use the method of weak solutions introduced by McKean, [MC]. The measure $d\mu_K^N$ on the initial data defines the measure $d\boldsymbol{M}_K^N$ in the space of paths $C\left([0,T]\to\mathcal{M}\right)$ for $T>0$. All information about the measure $d\mu_K^N$ and the flow $e^{t\mathcal{X}_3^N}$ is encoded now into the measure $d\boldsymbol{M}_K^N$. The proof of invariance takes three steps.
Step 1. The measures $d\mu_K^N$ converge to $d\mu_K$ weakly in H^s, $1\le s<\frac{3}{2}$, as $N\to\infty$.
Step 2. For any fixed K, the family $d\boldsymbol{M}_K^N$, $N=1,2\dots$ is tight and converges to measure $d\boldsymbol{M}_K$. The stationary measure $d\boldsymbol{M}_K$ for fixed t has marginal distribution $d\mu_K$.
Step 3. The measure $d\boldsymbol{M}_K$ is supported on the solutions of the NSL flow.

All three steps will be completed in subsequent sections. From the invariance of the measure $d\mu_K$, it is easy to infer the invariance of the the desired Gibbs' state with the density $\frac{1}{\Xi}e^{-\frac{1}{2}\mathcal{H}_5}d\,vol$.

9. Convergence of AL Gibbs' state to the NLS Gibbs' state. In the previous section, we introduced the family of probability measures $d\mu_K^N(\psi,\bar\psi)$ on $\mathcal{B}^N\subset\mathcal{M}$ with the densities*

$$\chi_K^N(\psi)e^{-\frac{N^5}{2}\mathcal{H}_5^N(\psi)}d\,vol^N=\chi_K^N(\psi)e^{-\frac{N^5}{2}\mathcal{H}_5^N(\psi)}\frac{1}{N!\,\mathcal{D}_N(\psi)}\bigwedge_{x\in\mathbb{S}_N}i\delta\psi(x)\wedge\delta\bar\psi(x).$$

We will show that $d\mu_K^N\to d\mu_K$, weakly in H^s, $1\le s<\frac{3}{2}$, as $N\to\infty$, where $d\mu_K(\psi,\bar\psi)$ has the density

$$\chi_K(\psi)e^{-\frac{1}{2}\mathcal{H}_5(\psi)}d\,vol=\chi_K(\psi)e^{-\frac{1}{2}\mathcal{H}_5(\psi)}\frac{1}{\infty!}\bigwedge_{x\in\mathbb{S}}i\delta\psi(x)\wedge\delta\bar\psi(x).$$

The cut-off in K can be removed easily and $d\mu_K$ converges to $d\mu$.

We split the integral H_5^N into a quadratic part $G^N(\psi)$ and a nonlinear part $R^N(\psi)$. Using the explicit expression for the integrals I's, we have

$$\begin{aligned}H_5^N(\psi_n)&=G^N(\psi_n)+R^N(\psi_n)\\&=\ \sum(\psi_{n+1}\bar\psi_{n-1}+\bar\psi_{n+1}\psi_{n-1})-4(\psi_n\bar\psi_{n+1}+\bar\psi_n\psi_{n+1})+6|\psi_n|^2\\&\quad-\sum(\psi_{n+1}\bar\psi_{n-1}+\bar\psi_{n+1}\psi_{n-1})|\psi_n|^2-\frac{1}{2}\sum(\psi_n\bar\psi_{n-1})^2+(\bar\psi_n\psi_{n-1})^2\\&\quad-\sum 6|\psi_n|^2+12NI_0.\end{aligned}$$

It is easy to check that

$$G^N(\psi_n)=\sum_{n=0}^{N-1}|\psi_{n+1}-2\psi_n+\psi_{n-1}|^2.$$

*To simplify notation, we omit the normalization factors Ξ for a moment.

Introduce

$$I^N(\psi_n) = \sum_{n=0}^{N-1} |\psi_n|^2$$

and also

$$\mathcal{G}^N(\psi) = G^N(\psi_n), \quad \mathcal{R}^N(\psi) = R^N(\psi_n), \quad \mathcal{I}^N(\psi) = I^N(\psi_n),$$

where $\psi_n = \epsilon\psi(\frac{n}{N})$. With the new notations,

$$\begin{aligned}
\chi_K^N(\psi)&e^{-\frac{N^5}{2}\mathcal{H}_5^N(\psi)}d\,vol^N \\
&=\chi_K^N(\psi)\frac{1}{\mathcal{D}_N(\psi)}e^{-\frac{1}{2}\left[-N\mathcal{I}^N(\psi)+N^5\mathcal{R}^N(\psi)\right]} \\
&\quad\times e^{-\frac{1}{2}\left[N^5\mathcal{G}^N(\psi)+N\mathcal{I}^N(\psi)\right]}\frac{1}{N!}\bigwedge_{x\in\mathbb{S}_N} i\delta\psi(x)\wedge\delta\bar{\psi}(x) \\
&\asymp\chi_K^N(\psi)\frac{1}{\mathcal{D}_N(\psi)}e^{-\frac{1}{2}\mathcal{W}^N(\psi)}\times d\gamma^N(\psi,\bar{\psi}).
\end{aligned}$$

Now we are ready to prove

Lemma 6. *The family of Gaussian probability measures $d\,\gamma^N(\psi,\bar{\psi})$ with the densities*

$$e^{-\frac{1}{2}\left[N^5\mathcal{G}^N(\psi)+N\mathcal{I}^N(\psi)\right]}\frac{1}{N!}\bigwedge_{x\in\mathbb{S}_N} i\delta\psi(x)\wedge\delta\bar{\psi}(x)$$

converges weakly in H^s, $s<\frac{3}{2}$, as $N\to\infty$, to the probability measure $d\gamma(\psi,\bar{\psi})$:

$$e^{-\frac{1}{2}\int|\psi''|^2+|\psi|^2}\frac{1}{\infty!}\bigwedge_{x\in\mathbb{S}} i\delta\psi(x)\wedge\delta\bar{\psi}(x).$$

Proof. For a smooth $\psi(x)=\sum_k e^{2\pi ixk}\hat{\psi}(k)$ and $N\to\infty$,

$$\begin{aligned}
&N^5\mathcal{G}^N(\psi)+N\mathcal{I}^N(\psi) \\
&\quad= N^5\epsilon^2\sum|\psi\left(n+1/N\right)-2\psi\left(n/N\right)+\psi\left(n-1/N\right)|^2+N\epsilon^2\sum|\psi\left(n/N\right)|^2 \\
&\quad\approx N^3\epsilon^4\sum|\psi''\left(n/N\right)|^2+\epsilon\sum|\psi\left(n/N\right)|^2\to\int_0^1|\psi''(x)|^2+|\psi(x)|^2dx.
\end{aligned}$$

This explains on a formal level, the convergence $d\,\gamma^N\to d\,\gamma$.

To obtain the full proof, we write all measures in terms of Fourier coefficients. First we note that

$$\int_0^1 |\psi''(x)|^2 + |\psi(x)|^2 dx = \sum_k (16\pi^4 k^4 + 1)|\hat{\psi}(k)|^2 = \sum_k \sigma^2(k)|\hat{\psi}(k)|^2.$$

Using the identity

$$\frac{1}{N}\sum_{p=0}^{N-1} |\psi(n/N)|^2 = \sum_{|k|\leq m} |\hat{\psi}_N(k)|^2,$$

for $\psi \in H^1$, we have

$$\begin{aligned} N^5 \mathcal{G}^N(\psi) + N\mathcal{I}^N(\psi) &= \sum_{|k|\leq m} \left[N^4 |e^{2\pi i k/N} - 2 + e^{-2\pi i k/N}|^2 + 1 \right] |\hat{\psi}_N(k)|^2 \\ &= \sum_{|k|\leq m} \sigma_N^2(k)|\hat{\psi}_N(k)|^2. \end{aligned}$$

Therefore, the measures $d\gamma^N(\psi,\bar{\psi})$ can be written as*

$$\asymp e^{-\frac{1}{2}\sum\limits_{|k|\leq m} \sigma_N^2(k)|\hat{\psi}(k)|^2} \bigwedge_{|k|\leq m} i\delta\hat{\psi}(k)\wedge\delta\bar{\hat{\psi}}(k).$$

Similarly, for $d\gamma(\psi,\bar{\psi})$

$$\asymp e^{-\frac{1}{2}\sum\limits_{k} \sigma^2(k)|\hat{\psi}(k)|^2} \bigwedge_{k} i\delta\hat{\psi}(k)\wedge\delta\bar{\hat{\psi}}(k).$$

For any fixed k,

$$\sigma_N^2(k) = N^4|e^{2\pi i k/N} - 2 + e^{-2\pi i k/N}|^2 + 1 \longrightarrow 16\pi^4 k^4 + 1 = \sigma^2(k),$$

as $N\to\infty$, whence the convergence of measures on finite-dimentional subspaces generated by the Fourier harmonics.

To prove tightness of the measures γ^N, we introduce a "brick" $B_a \equiv \{\psi : |\hat{\psi}(k)| \leq a_k,\ k\in\mathbb{Z}\}$, where $a_k \geq 0$. A brick is compact in H^s if and only if

$$\sum_k (1+k^2)^s a_k^2 < \infty.$$

*The sign $\asymp$ means up to unessential constant real factor.

Let us estimate

$$\gamma^N(B_a) = \prod_{|k|\le m} \frac{\sigma_N^2}{2\pi} \int\limits_{|\hat\psi(k)|^2\le a_k^2} e^{-\frac{\sigma_N^2}{2}|\hat\psi(k)|^2}\, \frac{i}{2}\, \delta\hat\psi(k)\wedge\delta\bar{\hat\psi}(k)$$
$$= \prod_{|k|\le m}\left[1 - e^{-\frac{\sigma_N^2(k)a_k^2}{2}}\right].$$

The inequalities $(1-b_1)\times\ldots\times(1-b_n) \ge 1 - b_1 - \ldots - b_n,\ b_k \ge 0$, and $e^{-x} \le p!x^{-p}$, $x\ge 0$, imply

$$\gamma^N(B_a) \ge 1 - \sum_{|k|\le m} e^{-\frac{\sigma_N^2(k)a_k^2}{2}}$$
$$\ge 1 - 2^p p!\sum_{|k|\le m}\frac{1}{\sigma_N^{2p}(k)a_k^{2p}}.$$

Now we obtain an estimate for $\sigma_N^2(k) = 4N^4(\cos\frac{2\pi k}{N}-1)^2+1,\ |k|\le m$. Obviously,

$$a'x^4 \le (\cos x - 1)^2 \le b'x^4 \qquad \text{for} \quad -\pi\le x\le\pi.$$

Therefore,

$$ak^4+1 \le \sigma_N^2(k) \le bk^4+1.$$

This, together with

$$a_k^2 = \frac{C}{(|k|+1)^\alpha}, \qquad C>0,$$

implies

$$\gamma^N(B_a) \ge 1 - \frac{2^p p!}{C^p}\sum_k \frac{(|k|+1)^{\alpha p}}{(ak^4+1)^p}.$$

Pick any $s<\frac{3}{2}$. Then $2s+1<4$. Pick any α such that $2s+1<\alpha<4$. Then the bricks B_a are compact in H^s. Now pick p such that $(4-\alpha)p>1$. Then the sum in the last estimate converges. Chose C so large as to make $\gamma^N(B_a)$ arbitrary close to 1. $\square$

Lemma 7. *(i). For any $\psi\in H^1$ the functionals*

$$\chi_K^N(\psi)\frac{1}{\mathcal{D}_N(\psi)}e^{-\frac{1}{2}\left[-N\mathcal{I}^N(\psi)+N^5\mathcal{R}^N(\psi)\right]} = \chi_K^N(\psi)\frac{1}{\mathcal{D}_N(\psi)}e^{-\frac{1}{2}\mathcal{W}^N(\psi)}$$

converge to

$$\chi_K(\psi)e^{-\frac{1}{2}\int -|\psi|^2+2|\psi|^6+6|\psi'|^2|\psi|^2+(\bar\psi'\psi+\psi'\bar\psi)^2} = \chi_K(\psi)e^{-\frac{1}{2}\mathcal{W}(\psi)} < \sqrt{e},$$

as $N \to \infty$.

(ii).

$$1 \geq \mathcal{D}_N(\psi) \geq e^{-\frac{K}{N}}$$

and

$$\mathcal{W}^N(\psi) \geq -c_4(K),$$

uniformly for all N*, provided* $N\mathcal{H}_1^N \leq K,\ -N^3\mathcal{H}_3^N \leq K$.

Proof. (i). The nonlinear term

$$\begin{aligned} R^N(\psi_n, \bar{\psi}_n) = - \quad & \sum (\psi_{n+1}\bar{\psi}_{n-1} + \bar{\psi}_{n+1}\psi_{n-1})|\psi_n|^2 \\ & - \frac{1}{2}\sum (\psi_n\bar{\psi}_{n-1})^2 + (\bar{\psi}_n\psi_{n-1})^2 \\ & - 6\sum |\psi_n|^2 + 12NI_0 \end{aligned}$$

can be expressed as

$R^N(\psi_n, \bar{\psi}_n) =$

$$\text{(A)} \qquad + \sum \left[(\psi_{n+1} - \psi_n)(\bar{\psi}_n - \bar{\psi}_{n-1}) + (\bar{\psi}_{n+1} - \bar{\psi}_n)(\psi_n - \psi_{n-1})\right] |\psi_n|^2$$

$$\text{(B)} \qquad - \sum \left[(\psi_{n+1} + \psi_{n-1} - 2\psi_n)\bar{\psi}_n + (\bar{\psi}_{n+1} + \bar{\psi}_{n-1} - 2\bar{\psi}_n)\psi_n\right] |\psi_n|^2$$

$$\text{(C)} \qquad -\frac{1}{2}\sum \psi_n^2(\bar{\psi}_{n-1} - \bar{\psi}_n)^2 + \bar{\psi}_n^2(\psi_{n-1} - \psi_n)^2$$

$$\text{(D)} \qquad - \sum \psi_n^2\bar{\psi}_n(\bar{\psi}_{n-1} - \bar{\psi}_n) + \bar{\psi}_n^2\psi_n(\psi_{n-1} - \psi_n)$$

$$\text{(E)} \qquad -3\sum |\psi_n|^4 - 6\sum |\psi_n|^2 + 12NI_0.$$

As in the continuous case, if one counts the difference of ψ's in two neighboring points $(\psi_{n+1} - \psi_n)$ and the function ψ_n itself of weight 1, then the terms A, B and C are isobaric polynomials of degree 6. The term D is isobaric polynomial of degree 5.

The term A can be reduced to the form

$$A(\psi_n, \bar{\psi}_n) = \sum |\psi_{n+1} - \psi_{n-1}|^2|\psi_n|^2 - \sum \{|\psi_{n+1} - \psi_n|^2 + |\psi_{n-1} - \psi_n|^2\}|\psi_n|^2.$$

We transform the terms B and D to a more convinient form also:

$$\begin{aligned} B(\psi_n, \bar{\psi}_n) = & \sum (\psi_{n+1} - \psi_n)^2\bar{\psi}_n^2 + (\bar{\psi}_{n+1} - \bar{\psi}_n)^2\psi_n^2 \\ & + \sum |\psi_{n+1} - \psi_n|^2 \left[\psi_{n+1}(\bar{\psi}_n + \bar{\psi}_{n+1}) + \bar{\psi}_{n+1}(\psi_n + \psi_{n+1})\right] \\ = & \sum \left[(\psi_{n+1} - \psi_n)\bar{\psi}_n + (\bar{\psi}_{n+1} - \bar{\psi}_n)\psi_n\right]^2 \\ & + 2\sum |\psi_{n+1} - \psi_n|^2|\psi_{n+1}|^2 \\ & + \sum |\psi_{n+1} - \psi_n|^2 \left[(\psi_{n+1} - \psi_n)\bar{\psi}_n + (\bar{\psi}_{n+1} - \bar{\psi}_n)\psi_n\right] \end{aligned}$$

and

$$\begin{aligned} D(\psi_n, \bar{\psi}_n) = & \frac{1}{2}\sum \psi_n^2(\bar{\psi}_n - \bar{\psi}_{n-1})^2 + \bar{\psi}_n^2(\psi_n - \psi_{n-1})^2 \\ & + 2\sum |\psi_{n+1} - \psi_n|^2|\psi_n|^2 \\ & + \frac{1}{2}\sum |\psi_{n+1} - \psi_n|^2 \left[\bar{\psi}_n(\psi_{n+1} - \psi_n) + \psi_n(\bar{\psi}_{n+1} - \bar{\psi}_n)\right] \\ & - \frac{1}{2}\sum |\psi_n|^2 \left[|\psi_{n+1} - \psi_n|^2 - |\psi_{n-1} - \psi_n|^2\right]. \end{aligned}$$

The first term in the formula is C with the opposite sign. Finally,

$$R(\psi_n, \bar{\psi}_n) =$$

$$-3\sum |\psi_n|^4 - 6\sum |\psi_n|^2 + 12NI_0 \tag{1}$$

$$+\sum |\psi_{n+1} - \psi_{n-1}|^2|\psi_n|^2 \tag{2}$$

$$+\frac{3}{2}\sum |\psi_{n-1} - \psi_n|^2|\psi_n|^2 \tag{3}$$

$$+\frac{1}{2}\sum |\psi_{n+1} - \psi_n|^2|\psi_n|^2 \tag{4}$$

$$+\sum \left[(\psi_{n+1} - \psi_n)\bar{\psi}_n + (\bar{\psi}_{n+1} - \bar{\psi}_n)\psi_n\right]^2 \tag{5}$$

$$+\frac{3}{2}\sum |\psi_{n+1} - \psi_n|^2 \left[\bar{\psi}_n(\psi_{n+1} - \psi_n) + \psi_n(\bar{\psi}_{n+1} - \bar{\psi}_n)\right] \tag{6}$$

The advantage of such a representation is that the terms (1)-(5) are nonnegative. The term (6) is the only term which is not sign-definite. This term vanishes when $N \to \infty$. Finally, as $N \to \infty$

$$\begin{aligned} & N^5\mathcal{R}^N(\psi) \longrightarrow \int_0^1 2|\psi|^6 + 6|\psi'|^2|\psi|^2 + \left(\bar{\psi}'\psi + \psi'\bar{\psi}\right)^2, \\ & N\mathcal{I}^N(\psi) \longrightarrow \int_0^1 -|\psi|^2 \\ & \mathcal{D}_N(\psi) \longrightarrow 1. \end{aligned}$$

(ii). From the definition $H_1 = N(I_0 + \bar{I}_0) = -\log D$ we have $\mathcal{H}_1^N(\psi) = -\log \mathcal{D}_N(\psi)$. The inequality $N\mathcal{H}_1^N \leq K$ implies, that $1 \geq \mathcal{D}_N \geq e^{-\frac{K}{N}}$.

To obtain the lower bound for $R^N(\psi_n, \bar{\psi}_n)$, we prove first that

$$-3\sum |\psi_n|^4 - 6\sum |\psi_n|^2 + 12NI_0 \geq 2\sum |\psi_n|^6. \tag{7}$$

Indeed, for $0 < x < 1$

$$-\log(1-x) \geq x + \frac{x^2}{2} + \frac{x^3}{3}.$$

Then,

$$-6\log(1-|\psi_n|^2) \geq 6|\psi_n|^2 + 3|\psi_n|^4 + 2|\psi_n|^6.$$

This and $12NI_0 = -6\log\prod(1-|\psi_n|^2)$ imply the inequality (7). Therefore, the term (1) is nonegative.

We have just one term (6) which is not sign-definite

$$\frac{3}{2}\sum|\psi_{n+1}-\psi_n|^2\left[\psi_n\bar{\psi}_{n+1} + \bar{\psi}_n\psi_{n+1} - 2|\psi_n|^2\right].$$

Using the identity

$$|\psi_{n+1}-\psi_n|^2 - |\psi_{n+1}|^2 - |\psi_n|^2 = \psi_{n+1}\bar{\psi}_n + \bar{\psi}_{n+1}\psi_n$$

and adding to it the two positive terms (3) and (4), we have

$$\begin{aligned}
&\frac{3}{2}\sum|\psi_{n+1}-\psi_n|^2\left(\psi_n\bar{\psi}_{n+1} + \bar{\psi}_n\psi_{n+1} - 2|\psi_n|^2\right)\\
+&\frac{3}{2}\sum|\psi_{n-1}-\psi_n|^2|\psi_n|^2\\
+&\frac{1}{2}\sum|\psi_{n+1}-\psi_n|^2|\psi_n|^2\\
&\geq -4\sum|\psi_{n+1}-\psi_n|^2|\psi_n|^2.
\end{aligned}$$

Therefore,

$$R(\psi_n,\bar{\psi}_n) \geq -4\sum|\psi_{n+1}-\psi_n|^2|\psi_n|^2.$$

and

$$N^5\mathcal{R}(\psi) \geq -4\sum N|\psi(n+1/N)-\psi(n/N)|^2|\psi(n/N)|^2.$$

Now we use the constraint $-N\mathcal{H}_3^N \leq K$ and the explicit formula for H_3:

$$-H_3^N(\psi_n,\bar{\psi}_n) = \sum|\psi_{n+1}-\psi_n|^2 - 2|\psi_n|^2 - 2\log(1-|\psi_n|^2).$$

The estimate $-\log(1-x) \geq x + \frac{x^2}{2}$ for $0 \leq x < 1$ implies

$$-2\log(1-|\psi_n|^2) \geq 2|\psi_n|^2 + |\psi_n|^4.$$

Therefore,

$$-H_3^N(\psi_n, \bar{\psi}_n) \geq \sum |\psi_{n+1} - \psi_n|^2 + |\psi_n|^4.$$

Finally,

$$K \geq -N^3 \mathcal{H}_3^N(\psi) \geq \sum N|\psi(n+1/N) - \psi(n/N)|^2 + N^{-1}|\psi(n/N)|^4.$$

This inequality also provides the estimate for $\max|\psi(n/N)|$*. Indeed,

$$\frac{1}{N} \sum |\psi(n/N)|^4 \leq K,$$

implies that there exists n'' such that $|\psi(n''/N)| \leq K^{1/4}$. Then, for any n',

$$\psi(n'/N) = \sum_{n' \leq n < n''} [\psi(n+1/N) - \psi(n/N)] + \psi(n''/N)$$

and, by Schwartz's inquality,

$$\begin{aligned}
|\psi(n'/N)| &\leq \sum |\psi(n+1/N) - \psi(n/N)| \frac{\sqrt{N}}{\sqrt{N}} + |\psi(n''/N)| \\
&\leq \sqrt{\sum N|\psi(n+1/N) - \psi(n/N)|^2} + |\psi(n''/N)| \\
&\leq c_2(K).
\end{aligned}$$

Therefore,

$$\begin{aligned}
N^5 \mathcal{R}^N(\psi) &\geq -4 \max|\psi(n/N)|^2 \sum N|\psi(n+1/N) - \psi(n/N)|^2 \\
&\geq -4c_2^2(K)K = -c_3(K).
\end{aligned}$$

Also,

$$N\mathcal{I}^N(\psi) = \sum N^{-1}|\psi(n/N)|^2 \leq c_2^2(K).$$

Finally,

$$\mathcal{W}^N(\psi) = -N\mathcal{I}^N(\psi) + N^5\mathcal{R}^N(\psi) \geq -c_2^2(K) - c_3(K) = -c_4(K).$$

□

*This is similar to the continuum case when the H^1-norm provides a bound in the sup-norm.

Lemma 8. *(i) The probability measures*

$$d\mu_K^N = \frac{1}{\Xi_K^N}\chi_K^N(\psi)\frac{1}{\mathcal{D}_N(\psi)}e^{-\frac{1}{2}\mathcal{W}^N(\psi)}d\gamma^N(\psi,\bar{\psi}),$$

with

$$\Xi_K^N \equiv \int\limits_{\mathcal{M}} \chi_K^N(\psi)\frac{1}{\mathcal{D}_N(\psi)}e^{-\frac{1}{2}\mathcal{W}^N(\psi)}d\gamma^N(\psi,\bar{\psi})$$

converge weakly in $H^s, 1 \le s < \frac{3}{2}$ as $N \to \infty$ to the probability measure

$$d\mu_K = \frac{1}{\Xi_K}\chi_K(\psi)e^{-\frac{1}{2}\mathcal{W}(\psi)}d\gamma(\psi,\bar{\psi}),$$

with

$$\Xi_K \equiv \int\limits_{\mathcal{M}} \chi_K(\psi)e^{-\frac{1}{2}\mathcal{W}(\psi)}d\gamma(\psi,\bar{\psi}).$$

(ii). The measure $d\mu_K$ converges in H^s, $1 \le s < \frac{3}{2}$ as $K \to \infty$ to the probability measure

$$d\mu = \frac{1}{\Xi}e^{-\frac{1}{2}\mathcal{W}(\psi)}d\gamma(\psi,\bar{\psi}),$$

with

$$\Xi \equiv \int\limits_{\mathcal{M}} e^{-\frac{1}{2}\mathcal{W}(\psi)}d\gamma(\psi,\bar{\psi}).$$

Proof. (i). Take $\mathcal{M} = H^s \times H^s$, $1 \le s \le \frac{3}{2}$ and f bounded and continuous. We will prove that

$$\int\limits_{\mathcal{M}} f\chi_K^N\frac{1}{\mathcal{D}_N(\psi)}\, e^{-\frac{1}{2}\mathcal{W}^N(\psi)}d\gamma^N(\psi,\bar{\psi}) \longrightarrow \int\limits_{\mathcal{M}} f\chi_K e^{-\frac{1}{2}\mathcal{W}}d\gamma(\psi,\bar{\psi}), \tag{8}$$

as $N \to \infty$. This implies that $\Xi_K^N \to \Xi_K$ as $N \to \infty$. Then, from (8), we have

$$\begin{aligned}\int f\, d\mu_K^N =& \frac{1}{\Xi_K^N}\int f\,\chi_K^N\frac{1}{\mathcal{D}_N(\psi)}e^{-\frac{1}{2}\mathcal{W}^N}d\gamma^N(\psi,\bar{\psi})\\ &\longrightarrow \frac{1}{\Xi_K}\int f\chi_K e^{-\frac{1}{2}\mathcal{W}}d\gamma(\psi,\bar{\psi}) \times \frac{\Xi_K}{\Xi_K^N} = \int f d\mu_K \times \frac{\Xi_K}{\Xi_K^N},\end{aligned}$$

as $N \to \infty$ and weak convergence is proved.

To begin with,

$$\begin{aligned}\int f\chi_K^N \frac{1}{\mathcal{D}_N(\psi)} e^{-\frac{1}{2}\mathcal{W}^N(\psi)} d\gamma^N(\psi,\bar\psi) \\ = \int f\left(\frac{1}{\mathcal{D}_N(\psi)} - 1\right)\chi_K^N e^{-\frac{1}{2}\mathcal{W}^N} d\gamma^N + \int f\chi_K^N e^{-\frac{1}{2}\mathcal{W}^N} d\gamma^N(\psi,\bar\psi).\end{aligned}$$

The first term can be estimated using the inequality of Lemma 8, item (ii):

$$\therefore \le \|f\|_\infty \, e^{\frac{1}{2}c_4(K)} \int_{\mathcal{M}} \left|\frac{1}{\mathcal{D}_N(\psi)} - 1\right| \chi_K^N d\gamma^N(\psi,\bar\psi) = o(1),$$

as $N \to \infty$. As to the second term,

$$\begin{aligned}\int f\chi_K^N e^{-\frac{1}{2}\mathcal{W}^N} d\gamma^N(\psi,\bar\psi) = & \int f\chi_K e^{-\frac{1}{2}\mathcal{W}} d\gamma^N(\psi,\bar\psi) \\ & + \int f\chi_K^N \left(e^{-\frac{1}{2}\mathcal{W}^N} - e^{-\frac{1}{2}\mathcal{W}}\right) d\gamma^N(\psi,\bar\psi) \qquad (9) \\ & + \int f\left(\chi_K^N - \chi_K\right) e^{-\frac{1}{2}\mathcal{W}} d\gamma^N(\psi,\bar\psi). \qquad (10)\end{aligned}$$

The first integral on the right converges to

$$\int f\chi_K e^{-\frac{1}{2}\mathcal{W}} d\gamma(\psi,\bar\psi).$$

The other two integrals vanish as $N \to \infty$. Indeed, (9) can be overestimated by

$$\frac{\|f\|_\infty}{2} e^{\frac{c_4(K)}{2}} \int \chi_K^N |\mathcal{W}^N - \mathcal{W}| \, d\,\gamma^N(\psi,\bar\psi). \qquad (11)$$

Using the explicit expressions for $\mathcal{W}^N$ and $\mathcal{W}$, we have

$$\mathcal{W}^N - \mathcal{W} =$$

$$-\sum N^{-1}|\psi(n/N)|^2 + \int |\psi|^2 dx \tag{12}$$

$$-3\sum N|\psi(n/N)|^4 - 6\sum N^3|\psi(n/N)|^2 + 12N^6\mathcal{I}_0(\psi) - 2\int |\psi|^6 dx \tag{13}$$

$$+\sum N|\psi(n+1/N) - \psi(n-1/N)|^2|\psi(n/N)|^2 - 4\int |\psi'|^2|\psi|^2 dx \tag{14}$$

$$+\frac{3}{2}\sum N|\psi(n-1/N) - \psi(n/N)|^2|\psi(n/N)|^2 - \frac{3}{2}\int |\psi'|^2|\psi|^2 dx \tag{15}$$

$$+\frac{1}{2}\sum N|\psi(n+1/N) - \psi(n/N)|^2|\psi(n/N)|^2 - \frac{1}{2}\int |\psi'|^2|\psi|^2 dx \tag{16}$$

$$\begin{aligned} &+\sum N[(\psi(n+1/N) - \psi(n/N))\bar{\psi}(n/N) + \\ &+ (\bar{\psi}(n+1/N) - \bar{\psi}(n/N))\psi(n/N)]^2 - \int [\psi'\bar{\psi} + \bar{\psi}\psi']^2 dx \end{aligned} \tag{17}$$

$$\begin{aligned} &+\frac{3}{2}\sum N|\psi(n+1/N) - \psi(n/N)|^2[\bar{\psi}(n/N)(\psi(n+1/N) - \psi(n/N)) + \\ &+ \psi(n/N)(\bar{\psi}(n+1/N) - \bar{\psi}(n/N))]. \end{aligned} \tag{18}$$

Here, each term contributes nothing when $N \to \infty$; for example, substituting (12) into (11):

$$\begin{aligned} &\le \int_{\mathcal{M}} \left| \sum N^{-1}|\psi(n/N)|^2 - \sum \int_{n/N}^{n+1/N} |\psi|^2 dx \right| d\gamma^N(\psi,\bar{\psi}) \\ &\le \sum \int_{\mathcal{M}} \left| N^{-1}|\psi(n/N)|^2 - \int_{n/N}^{n+1/N} |\psi|^2 dx \right| d\gamma^N(\psi,\bar{\psi}) \\ &\le N \int_{\mathcal{M}} \left| \int_0^{1/N} |\psi(x)|^2 - |\psi(0)|^2 dx \right| d\gamma^N(\psi,\bar{\psi}) \\ &\le N \int_0^{1/N} \int_{\mathcal{M}} \big| |\psi(x)|^2 - |\psi(0)|^2 \big| \, d\gamma^N(\psi,\bar{\psi}) dx. \end{aligned}$$

Now, by Schwartz, for $0 \le |x| \le 1/N$, we have

$$\begin{aligned}\left(\int_{\mathcal{M}} \left||\psi(x)|^2 - |\psi(0)|^2\right| d\gamma^N\right)^2 &\le \int_{\mathcal{M}} \left(|\psi(x)|^2 - |\psi(0)|^2\right)^2 d\gamma^N \\ &= \int_{\mathcal{M}} |\psi(x)|^4 - 2|\psi(x)|^2|\psi(0)|^2 + |\psi(0)|^4 d\gamma^N \\ &= 2\int_{\mathcal{M}} |\psi(0)|^4 - |\psi(x)|^2|\psi(0)|^2 d\gamma^N\end{aligned}$$

The stationary Gaussian measure $d\gamma^N$ has the spectral representation

$$\psi_N(x) = \sum_{|k|\le m} e^{2\pi i k x}\hat{\psi}(k)$$

where $\hat{\psi}$ are complex isotropic Gaussian variables $E^N|\hat{\psi}(k)|^2 = \sigma_N^{-2}(k)$. Now

$$|\psi(0)|^2|\psi(x)|^2 = \sum_{\substack{k_1,k_2\\k_3,k_4}} e^{2\pi i x(k_1-k_2)}\hat{\psi}(k_1)\overline{\hat{\psi}}(k_2)\hat{\psi}(k_3)\overline{\hat{\psi}}(k_4).$$

The terms in the last sum contributing into the expectation E^N are

$$\begin{array}{lll} k_1 = k_2 & k_1 = k_4 & k_1 = k_2 = k_3 = k_4. \\ k_3 = k_4 & k_3 = k_2 & \\ k_1 \ne k_3 & k_1 \ne k_3 & \end{array}$$

Now

$$\begin{aligned}E^N|\psi(0)|^2|\psi(x)|^2 = &\sum_{k_1\ne k_3} E^N|\hat{\psi}(k_1)|^2 E^N|\hat{\psi}(k_3)|^2 \\ &+ \sum_{k_1\ne k_2} e^{2\pi i x(k_1-k_2)} E^N|\hat{\psi}(k_1)|^2 E^N|\hat{\psi}(k_2)|^2 + \sum_{k_1} E^N|\hat{\psi}(k_1)|^4,\end{aligned}$$

So,

$$\begin{aligned}&E^N|\psi(0)|^4 - E^N|\psi(x)|^2|\psi(0)|^2 \\ &\qquad = \sum_{k_1\ne k_2}\left[1 - e^{2\pi i x(k_1-k_2)}\right] E^N|\hat{\psi}(k_1)|^2 E^N|\hat{\psi}(k_2)|^2 \\ &\qquad = \sum_{|k_i|\le m} |1 - e^{2\pi i x(k_1-k_2)}|\sigma_N^{-2}(k_1)\sigma_N^{-2}(k_2).\end{aligned}$$

Using the estimate $ak^4+1 \le \sigma_N^2(k)$ of Lemma 7, uniformly in $0 \le x \le 1/N$, the last sum can be overestimated as

$$\therefore \le \sum_{k_i \in \mathbb{Z}^1} \left|1 - e^{2\pi i x(k_1-k_2)}\right| \frac{1}{ak_1^4+1}\frac{1}{ak_2^4+1} = o(1),$$

as $N \to \infty$. This implies that the term (12) contributes nothing when $N \to \infty$.
To estimate the contribution of (13), expand the log in $\mathcal{I}_0$:

$$\begin{aligned} 12N^6\mathcal{I}_0(\psi) =& -6N^5 \log \prod (1 - N^{-2}|\psi(n/N)|^2) \\ =&6N^5[N^{-2}|\psi(n/N)|^2 + \frac{N^{-4}}{2}|\psi(n/N)|^4 + \frac{N^{-6}}{3}|\psi(n/N)|^6 \\ &+\frac{f^{(iv)}}{4!}N^{-8}|\psi(n/N)|^8] \end{aligned}$$

where $f(x) = -\log(1-x)$ and the fourth derivative $f^{(iv)} = 3!(1-x)^{-4}$ in the remainder term is bounded since $|\psi(n/N)| \le c_2(K)$ and $N^{-2}|\psi(n/N)|^2 \le \theta < 1$.
Now for (13), we have

$$2\sum N^{-1}|\psi(n/N)|^6 - 2\int_0^1 |\psi|^6 dx + \frac{6}{4!}N^{-3}\sum f^{(iv)}|\psi(n/N)|^6.$$

The part with the the fourth derivative can be overestimated by

$$\therefore \le \frac{6}{4!}N^{-3}\frac{3!}{(1-\theta)^4}c_2^6(K)N = o(1),$$

as $N \to \infty$. The contribution to (11) of first two terms can be overestimated by

$$\begin{aligned} \therefore \le& 2\left|\int_{\mathcal{M}} \sum N^{-1}|\psi(n/N)|^6 - \int_{n/N}^{n+1/N} |\psi|^6 dx\right| d\gamma^N \\ \le& 2N\int_0^{1/N}\int_{\mathcal{M}} \left||\psi(x)|^6 - |\psi(0)|^6\right| d\gamma^N dx. \end{aligned}$$

Again, by Schwarz,

$$\begin{aligned} &\left(\int_{\mathcal{M}} \left||\psi(x)|^6 - |\psi(0)|^6\right| d\gamma^N\right)^2 \\ \le& 2\int_{\mathcal{M}} |\psi(0)|^{12} - |\psi(0)|^6|\psi(x)|^6 d\gamma^N = o(1), \end{aligned}$$

as $N \to \infty$, uniformly in $0 \le x < 1/N$. This implies that the term (13) makes no contribution when $N \to \infty$. The estimation of (14-18) is similar.

(10) can be overestimated by

$$\|f\|_\infty \int |\chi_K^N - \chi_K| d\gamma^N. \tag{19}$$

Now, we have

$$\chi_K^N - \chi_K = \left[h(N\mathcal{H}_1^N) - h(\mathcal{H}_1)\right] h(-N^3\mathcal{H}_3^N) + \left[h(-N\mathcal{H}_3^N) - h(\mathcal{H}_3)\right] h(\mathcal{H}_1).$$

Therefore, (19) can be overestimated by

$$\therefore \le \|f\|_\infty \int |h(N\mathcal{H}_1^N) - h(\mathcal{H}_1)| h(-N^3\mathcal{H}_3^N) d\gamma^N \tag{20}$$

$$+\|f\|_\infty \int |h(-N^3\mathcal{H}_3^N) - h(\mathcal{H}_3)| h(\mathcal{H}_1) d\gamma^N. \tag{21}$$

To estimate (20) we note that $|\psi(n/N)| \le c_2(K)$ on the support of $h(-N^3\mathcal{H}_3^N)$ and that

$$\begin{aligned} N\mathcal{H}_1^N &= -N \log \prod (1 - N^{-2}|\psi(n/N)|^2) \\ &= N^{-1} \sum |\psi(n/N)|^2 + \frac{N^{-3}}{2!} \sum f^{(ii)} |\psi(n/N)|^4. \end{aligned}$$

Therefore,

$$\begin{aligned} (20) &\le \|f\|_\infty \|h'\|_\infty \int_{|\psi(n/N)| \le c_2(K)} |N\mathcal{H}_1^N - \mathcal{H}_1| \, d\gamma^N \\ &\le \|f\|_\infty \|h'\|_\infty \int_{|\psi(n/N)| \le c_2(K)} \left| N^{-1} \sum |\psi(n/N)|^2 - \int_0^1 |\psi|^2 \, dx \right| d\gamma^N \\ &\quad + \|f\|_\infty \|h'\|_\infty \int_{|\psi(n/N)| \le c_2(K)} \left| \sum \frac{f^{(ii)}}{2!} N^{-3} |\psi(n/N)|^4 \right| d\gamma^N. \end{aligned}$$

The first term vanishes as in the estimate for the term (20) and the second term vanishes as in the estimate for the term (21).

To estimate the term (21), note that for any $\epsilon > 0$,

$$\gamma^N \left(\|\psi\|_\infty \ge C \right) \ge 1 - \epsilon,$$

for all N provided C is sufficiently large. Then,

$$(21) \le 2\epsilon + \int_{\|\psi\|_\infty \le C} |h(-N^3\mathcal{H}_3^N) - h(\mathcal{H}_3)| h(\mathcal{H}_1) d\gamma^N,$$

in which the integral vanishes for $N \to \infty$ as for (20).

(iv). For any bounded continuous f

$$\int f\chi_K(\psi)e^{-\frac{1}{2}\mathcal{W}(\psi)}d\gamma(\psi,\bar{\psi}) \to \int fe^{-\frac{1}{2}\mathcal{W}(\psi)}d\gamma(\psi,\bar{\psi}),$$

as $K \to \infty$, by the bounded convergence theorem. This implies $\Xi_K \to \Xi$ as $K \to \infty$. Now the last statement is proved. □

10. Tightness of measures $d\boldsymbol{M}_K^N$. We assume that the initial data $\psi_N(x)$ has distribution $d\mu_N^K$. The solutions of initial value problem $\psi(x,t) = e^{t\mathcal{X}_3^N}\psi_N(x)$, $(x,t) \in \mathbb{S} \times [0,T]$, are realisations of the complex random field with distribution $d\boldsymbol{M}_K^N$. To prove tightness of measures we need

Lemma 9. *(Kolmogorov—Čentzov).* [Ku]. *Let the family of complex continuous random fields $\phi(x,t)$, $(x,t) \in \mathbb{S} \times [0,T]$, with distributions dm^N satisfy*

$$E^N|\phi(x,t)|^\gamma \le C \tag{1}$$

and

$$E^N|\phi(x_1,t_1) - \phi(x_2,t_2)|^\gamma \le C\left(|x_1 - x_2|^{\alpha_1} + |t_1 - t_2|^{\alpha_2}\right), \tag{2}$$

where $\alpha_1^{-1} + \alpha_2^{-1} < 1$ and $\gamma > 0$. Then the family of measures dm^N is tight with respect to the weak topology of $C\left(\phi : \mathbb{S} \times [0,T] \to \mathbb{C}^1\right)$

Since realisations of the field are continuously differentiable in the spatial variable x, we consider the random fields $\psi'(x,t)$. The main result of this section is

Lemma 10. *For any $n = 1,2,\ldots,$:*

$$E_K^N|\psi'(x,t)|^{2n} \le c(n,K) \tag{3}$$

$$E_K^N|\psi'(x_1,t_1) - \psi'(x_2,t_2)|^{2n} \le c(n,K)\left(|x_1 - x_2|^n + |t_1 - t_2|^{n/2}\right) \tag{4}$$

where E_K^N is expectation with respected to the measure $d\boldsymbol{M}_K^N$.

This, together with Lemma 9, implies tightness of the measures $d\boldsymbol{M'}_K^N$. Tightness of the family $d\boldsymbol{M}_K^N$ can be proved along the same lines. The distribution of the limiting stationary random field we denote by $d\boldsymbol{M}_K$. It follows from the previous section, that the measure $d\boldsymbol{M}_K$ has marginal distributions $d\mu_K$. In order to establish the Hölder continuity of the spatial derivatives of the random field we need another

Lemma 11. *(Kolmogorov–Čentzov).* [Ku]. *Let $\phi(x,t)$, $(x,t) \in \mathbb{S} \times [0,T]$, be a complex random field. Assume that there exist positive constants γ, C, α_1 and α_2 with $\alpha_1^{-1} + \alpha_2^{-1} < 1$ satisfying*

$$E|\phi(x_1,t_1) - \phi(x_2,t_2)|^\gamma \leq C\left(|x_1 - x_2|^{\alpha_1} + |t_1 - t_2|^{\alpha_2}\right).$$

Then the random field $\psi(x,t)$ has a continuous modification. Moreover $c_0 \equiv (1 - \alpha_1^{-1} - \alpha_2^{-1})/\gamma$, if β_1 and β_2 are positive numbers less then $\alpha_1 c_0$ or $\alpha_2 c_0$ respectively, then there exists a positive random constant C with $EC^\gamma < \infty$ such that

$$|\phi(x_1,t_1) - \phi(x_2,t_2)| < C\left(|x_1 - x_2|^{\beta_1} + |t_1 - t_2|^{\beta_2}\right).$$

By Fatou's lemma, passing to the limit $N \to \infty$ in the inequality (4), we have

$$E_K|\psi'(x_1,t_1) - \psi'(x_2,t_2)|^{2n} \leq C(n)\left(|x_1 - x_2|^n + |t_1 - t_2|^{n/2}\right)$$

Now Lemma 11 implies that the measure $d\boldsymbol{M}'_K$ is supported on paths satisfying

$$|\psi'(x_1,t_1) - \psi'(x_2,t_2)| \leq \left(|x_1 - x_2|^{1/2-} + |t_1 - t_2|^{1/4-}\right).$$

One can prove, using methods of [V], that the Hölder exponents $1/2-$ and $1/4-$ are optimal.

Proof of Lemma 10. To obtain (3), observe that by the invariance of $d\mu_K^N$

$$\begin{aligned} E_K^N|\psi'(x,t)|^{2n} &= E_K^N|\psi'(x,0)|^{2n} \\ &\leq \frac{e^{\frac{K}{N}} e^{\frac{c_4}{2}}}{\inf\limits_N \Xi_K^N} \int\limits_{\mathcal{M}} |\psi'(x,0)|^{2n} d\gamma^N \qquad \text{by Lemma 7} \end{aligned}$$

Due to Lemma 8, $\Xi_K^N \to \Xi_K > 0$, as $N \to \infty$. Using the Gaussian character of the measure $d\gamma^N$ the last integral can be overestimated by

$$\leq c_1(K,n)\left[\int\limits_{\mathcal{M}} |\psi'(0)|^2 d\gamma^N\right]^n = c_2(K,n)\left[\sum_{|k| \leq m} k^2 \sigma_N^{-2}(k)\right]^n$$

Now the estimate $ak^4 + 1 \leq \sigma_N^2(k)$ (see proof of Lemma 6) implies (3).

To prove (4), write the differential equation for the flow on $\mathcal{M}_N$ as in

$$\begin{aligned} i\frac{\partial\psi(x)}{\partial t} = -N^2(\psi(x+\frac{1}{N}) &+ \psi(x-\frac{1}{N}) - 2\psi(x)) \\ &+ |\psi(x)|^2(\psi(x-\frac{1}{N}) + \psi(x+\frac{1}{N})), \end{aligned}$$

where $\psi(x)=\psi_N(x,t)\in\mathcal{M}_N,\ \ x\in\mathbb{S}_N$; equivalently,

$$e^{t\mathcal{X}_3^N}\left[\psi(\bullet)\right](x)=e^{it\Delta_N}\left[\psi(\bullet)\right](x)$$
$$-i\int_0^t e^{i(t-s)\Delta_N}\left[|\psi(\bullet,s)|^2\left(\psi(\bullet+\frac{1}{N},s)+\psi(\bullet-\frac{1}{N},s)\right)\right]_N(x)ds, \tag{5}$$

where

$$e^{it\Delta_N}[\psi(\bullet)](x)\equiv\sum_{|k|\le m}e^{2\pi ikx}e^{it\Delta(k,N)}\hat{\psi}(k),\qquad\qquad \Delta(k,N)=N^2(\omega^k+\omega^{-k}-2)$$

and

$$\psi(x)=\sum_{|k|\le m}e^{2\pi ikx}\hat{\psi}(k).$$

Since (5) holds for all $x\in\mathbb{S}_N$ with left and right being polynomials, then (5) holds for all $x\in\mathbb{S}$.

First, we derive the estimate for spatial increments. Much as before,

$$\begin{aligned}&E_K^N|\psi'(x+h,t)-\psi'(x,t)|^{2n}\\&\quad=E_K^N|\psi'(x+h)-\psi'(x)|^{2n}\\&\quad\le c_2(K,n)\left[E_{\gamma^N}|\psi'(h)-\psi'(0)|^2\right]^n.\end{aligned}$$

Using the spectral representation of the Gaussian measure γ^N in the form

$$\psi(x)=\sum_{|k|\le m}e^{2\pi ikx}\hat{\psi}(k),\qquad\qquad E_{\gamma^N}|\hat{\psi}(k)|^2=\sigma_N^{-2}(k),$$

we obtain

$$E_{\gamma^N}|\psi'(h)-\psi'(0)|^2=\sum_{k\neq0}4\pi^2k^2|e^{2\pi ikh}-1|^2\sigma_N^{-2}(k).$$

The estimate $|e^{i2\pi x}-1|\le cx,\ \ 0\le|x|\le\frac{1}{2}$, implies

$$\begin{aligned}&\le c_3\sum_{0<|k|<h^{-1}/2}\frac{k^4h^2}{ak^4+1}+c_3\sum_{h^{-1}/2\le|k|}\frac{k^2}{ak^4+1}\\&\le c_4h^2\sum_{0<|k|<h^{-1}/2}1+c_4\sum_{h^{-1}/2\le|k|}\frac{1}{k^2}\le c_5h.\end{aligned}$$

Finally,

$$E_K^N|\psi'(x+h,t)-\psi'(x,t)|^{2n} \le c_6(K,n)h^n. \tag{6}$$

To obtain the estimate for temporal increments we use (5) and write*

$$E_K^N|\psi'(x,t+h)-\psi'(x,t)|^{2n} = E_K^N|\psi'(x,h)-\psi'(x,0)|^{2n}$$

$$\le c_7\, E_K^N \left|e^{ih\Delta_N}\psi'(x)-\psi'(x)\right|^{2n} \tag{7}$$

$$+c_7\, E_K^N \left| \int_0^h e^{i(h-s)\Delta_N} \left[|\psi(\bullet)|^2 \left(\psi(\bullet+\frac{1}{N},s)+\psi(\bullet-\frac{1}{N},s)\right)\right]_N' (x)ds\right|^{2n} \tag{8}$$

Here

$$(7) \quad \le c_8\, E_{\gamma^N}|e^{ih\Delta_N}\psi'(x)-\psi'(x)|^{2n} = c_9\left[E_{\gamma^N}|e^{ih\Delta_N}\psi'(x)-\psi'(x)|^2\right]^n.$$

Now, from the spectral respresentation,

$$E_{\gamma^N}|e^{ih\Delta_N}\psi'(0)-\psi'(0)|^2 = \sum_{|k|\le m} |e^{ihN^2(\omega^k+\omega^{-k}-2)}-1|^2 4\pi^2k^2\sigma_N^{-2}(k)$$

$$= \sum_{|k|\le m}\left(1-\cos hN^2(\omega^k+\omega^{-k}-2)\right)^2 4\pi^2k^2\sigma_N^{-2}(k) \tag{9}$$

$$+\sum_{|k|\le m}\sin^2 hN^2(\omega^k+\omega^{-k}-2)4\pi^2k^2\sigma_N^{-2}(k). \tag{10}$$

First, we note that for $|x|\le\pi$

$$c_{10}x^2 \le 1-\cos x \le c_{11}x^2 \tag{11}$$

Therefore, for $|k|\le m$, we have

$$c_{12}k^2 \le -N^2(\omega^k+\omega^{-k}-2) \le c_{13}k^2. \tag{12}$$

To estimate (8), we assume that $|k|\le h^{-1/2}\alpha$, where α is a suitably chosen constant. Then, by (12),

$$\left|hN^2(\omega^k+\omega^{-k}-2)\right| \le \pi.$$

---

*$e^{it\Delta_N}\psi(x) = e^{it\Delta_N}[\psi(\bullet)](x)$

Using (11) and (12), we find

$$(9) \le c_{14} \sum_{\substack{|k|\le m \\ |k|\le h^{-1/2}\alpha}} \frac{\left(hN^2(\omega^k+\omega^{-k}-2)\right)^4}{k^2+1} + c_{15} \sum_{\substack{|k|\le m \\ |k|> h^{-1/2}\alpha}} \frac{1}{k^2+1}$$
$$\le c_{16}h^4 \sum_{|k|\le h^{-1/2}\alpha} k^6 + c_{17} \sum_{|k|>h^{-1/2}\alpha} \frac{1}{k^2} \le c_{18}h^{1/2}.$$

The estimate of (10) is even simpler:

$$(10) \quad \le c_{19} \sum_{\substack{|k|\le m \\ |k|\le h^{-1/2}}} \frac{h^2k^4}{k^2+1} + c_{20} \sum_{\substack{|k|\le m \\ |k|> h^{-1/2}}} \frac{1}{k^2+1}$$
$$\le c_{21}h^2 \sum_{|k|\le h^{-1/2}} k^2 + c_{22} \sum_{|k|>h^{-1/2}} \frac{1}{k^2} \le c_{23}h^{1/2}.$$

Finally, for (7) we have the estimate

$$E_K^N|e^{it\Delta_N}\psi'(x)-\psi'(x)|^{2n} \le c_{24}(K,n)h^{n/2}. \tag{13}$$

To estimate (8), observe that

$$E_K^N\left|\int_0^h e^{i(h-s)\Delta_N}\left[|\psi(\bullet,s)|^2(\psi(\bullet+\frac{1}{N},s)+\psi(\bullet-\frac{1}{N},s))\right]_N'(x)ds\right|^{2n}$$
$$\le h^{2n-1}\int_0^h E_K^N\left|e^{i(h-s)\Delta_N}\left[|\psi(\bullet,s)|^2(\psi(\bullet+\frac{1}{N},s)+\psi(\bullet-\frac{1}{N},s))\right]_N'\right|^{2n} ds$$
$$\le h^{2n-1}\int_0^h E_K^N\left|e^{is\Delta_N}\left[|\psi(\bullet,s)|^2(\psi(\bullet+\frac{1}{N},s)+\psi(\bullet-\frac{1}{N},s))\right]_N'\right|^{2n} ds.$$

We will show that, for any $s\ge 0$,

$$E_K^N\left|e^{it\Delta_N}\left[|\psi(\bullet)|^2(\psi(\bullet+\frac{1}{N},s)+\psi(\bullet-\frac{1}{N},s))\right]_N'\right|^{2n} \le c_{25}(K,n). \tag{14}$$

Then,

$$(8) = E_K^N| \quad \ldots ditto \ldots \quad |^{2n} \le c_{25}h^{2n}. \tag{15}$$

The estimates (13) and (15) produce

$$E_K^N|\psi'(x,t+h)-\psi'(x,t)|^{2n}\le c_{26}(K,n)h^{n/2}.$$

This and (6) produce (4).

To prove (14) first we note, is $\psi(x)=\sum\hat\psi(k)e^{2\pi ikx}$, then

$$\psi_N(x)=\sum_{|k|\le m}\hat\psi_N(k)e^{2\pi ikx},\qquad\text{with}\qquad \hat\psi_N(k)=\sum_{n\equiv k,\ (\mathrm{mod}\,N)}\hat\psi(n).$$

To simplify notation we write $\psi=\psi_N$, then

$$|\psi(x)|^2\left(\psi(x+\frac{1}{N})+\psi(x-\frac{1}{N})\right)$$
$$=\sum_{|k_i|\le m}e^{2\pi i(k_1+k_3-k_2)x}2\cos\frac{2\pi k_3}{N}\hat\psi(k_1)\bar{\hat\psi}(k_2)\psi(k_3)$$

and

$$e^{it\Delta_N}\left[|\psi(\bullet)|^2\left(\psi(\bullet+\frac{1}{N})+\psi(\bullet-\frac{1}{N})\right)\right]_N'(x)$$
$$=\sum_{|k_i|\le m}e^{it\Delta(N,p)}e^{2\pi ip(k_1+k_3-k_2)x}2\cos\frac{2\pi k_3}{N}2\pi ip(k_1+k_3-k_2)\hat\psi(k_1)\bar{\hat\psi}(k_2)\hat\psi(k_3),$$

where $p(k)\equiv k\pmod N$ and $|p(k)|\le m$.

Now,

$$\left|\partial_x e^{it\Delta_N}\left[|\psi(\bullet)|^2\left(\psi(\bullet+\frac{1}{N})+\psi(\bullet-\frac{1}{N})\right)\right]_N'(x)\right|^2$$
$$=\sum_{|k_i|\le m}e^{it[\Delta(N,p)-\Delta(N,p')]}e^{2\pi ix(p(k_1+k_3-k_2)-p(k_4+k_6-k_5))}2\cos\frac{2\pi k_3}{N}2\cos\frac{2\pi k_6}{N}$$
$$\times 4\pi^2p(k_1+k_3-k_2)p'(k_1+k_3-k_5)\hat\psi(k_1)\bar{\hat\psi}(k_2)\hat\psi(k_3)\bar{\hat\psi}(k_4)\hat\psi(k_5)\bar{\hat\psi}(k_6).$$

Estimating

$$E_{\gamma^N}\big|\ldots\text{ditto}\ldots\big|^{2n}=\big|E_{\gamma^N}|\ldots\text{ditto}\ldots|^{2n}\big|$$

$$=|\sum_{j=1}^n\prod e^{it[\Delta(N,p^j)-\Delta(N,p'^j)]}e^{2\pi ix(p(k_1^j-k_3^j-k_2)-p(k_4^j+k_6^j-k_5^j))}2\cos\frac{2\pi k_3^j}{N}2\cos\frac{2\pi k_6^j}{N}$$

$$\times E_{\gamma^N} \prod 4\pi^2 p\,(k_1^j + k_3^j - k_2^j) p'(k_4^j + k_6^j - k_5^j)\hat{\psi}(k_1^j)\bar{\hat{\psi}}(k_2^j)\hat{\psi}(k_3^j)$$
$$\times \bar{\hat{\psi}}(k_4^j)\psi(k_5^j)\bar{\hat{\psi}}(k_6^j)|$$
$$\leq \sum_{j=1}^{n} E_{\gamma^N} \prod_{|k_i^j| \leq m} 4\pi^2 |p(k_1^j + k_3^j - k_2^j)||p(k_4^j + k_6^j - k_5^j)$$
$$\times \hat{\psi}(k_1^j)\bar{\hat{\psi}}(k_2^j)\hat{\psi}(k_3^j)\bar{\hat{\psi}}(k_4^j)\hat{\psi}(k_5^j)\bar{\hat{\psi}}(k_6^j).$$

Using the inequalities

$$|p(k_1 + k_3 - k_2)| \leq |k_1 + k_2 - k_3| \leq (1 + |k_1|)(1 + |k_2|)(1 + |k_3|),$$

and

$$(1 + |k|)^2 \leq 2(1 + |k|^2),$$

we overestimate the last sum by

$$\leq c_{27} \sum_{|p_i| \leq m} E_{\gamma^N}(1 + p_1^2)|\hat{\psi}(k_1)|^2 \ldots \qquad \ldots (1 + p_1^2)|\hat{\psi}(p_{3n})|^2.$$

The worst term

$$\sum E_{\gamma^N} p_1^2 |\hat{\psi}(p_1)|^2 \ldots \quad \ldots p_{3n}^2 |\hat{\psi}(3n)|^2.$$

can be overestimated by

$$\therefore \leq E_{\gamma^N} |\psi'(x)|^{2\times 3n} \leq C(n, N)$$

due to $ak^4 + 1 \leq \sigma_N^2(k)$. $\square$

11. Identification of measure $d\mathcal{M}_K$. This section shows that the measure $d\mathcal{M}_K$ is supported on the solutions of the NLS flow.

Multiplying both parts of the original equation

$$i\psi^{\bullet} = -\psi'' + 2|\psi|^2\psi$$

by the test function $f(x,t),\ (x,t) \in \mathbb{S} \times (0,T]$ and integrating produces

$$\int_0^1 \int_{-\infty}^{+\infty} dx\, dt \left[i\psi f^{\bullet} - \psi f'' + |\psi|^2 2\psi f \right] = 0.$$

Lemma 12. *In the statistical ensemble $d\mathcal{M}_K$*

$$E_{\mathcal{M}_K}\left|\int_0^1\int_{-\infty}^{+\infty} dxdt\,\left[i\psi f^{\bullet} - \psi f'' + |\psi|^2 2\psi f\right]\right|^2 = 0.$$

Proof. The equation for the AL flow on $\mathcal{M}_N$ is

$$i\psi^{\bullet}(x,t) = -\Delta_N\psi(x,t) + |\psi(x,t)|^2(\psi(x+\frac{1}{N},t)+\psi(x-\frac{1}{N},t)),$$

where $\Delta_N\psi(x) = N^2\left(\psi(x+\frac{1}{N})+\psi(x-\frac{1}{N})-2\psi(x)\right)$ and $x\in\mathbb{S}_N$. We multiply by the test function $f(x,t)$ and integrate over the variables t and x.

$$\int_0^1\int_{-\infty}^{+\infty} dx\,dt\,\delta_N(x)[i\psi(x,t)f^{\bullet} - \psi(x,t)\Delta_N f$$
$$+|\psi(x,t)|^2\left(\psi(x+\frac{1}{N},t)+\psi(x-\frac{1}{N},t)\right)f] = 0,$$

in which $\delta_N(x) = \frac{1}{N}\sum\limits_{x_0\in\mathbb{S}_N}\delta(x-x_0)$. Now

$$E_{\mathcal{M}_K}\left|\int\int dx\,dt\,\left[i\psi f^{\bullet} - \psi f'' + |\psi|^2 2\psi f\right]\right|^2$$
$$= \lim_{N\to\infty} E_{\mathcal{M}_K^N}|...\ ditto\ ...|^2$$
$$= \lim_{N\to\infty} E_{\mathcal{M}_K^N}|\int\int dx\,dt\; i\psi f^{\bullet}(1-\delta_N(x)) - \int\int dx\,dt\;\psi(\partial^2_{xx}-\delta_N\Delta_N)f$$
$$+\int\int dx\,dt\left[|\psi|^2 2\psi - |\psi|^2\left(\psi(x+\frac{1}{N})+\psi(x-\frac{1}{N})\right)\delta_N(x)\right]f|^2$$
$$\le \lim_{N\to\infty} cE_{\mathcal{M}_K^N}\left|\int\int dx\,dt\psi f^{\bullet}(1-\delta_N)\right|^2$$
$$+\lim_{N\to\infty} cE_{\mathcal{M}_K^N}\left|\int\int dxdt\psi(\delta^2_{xx}-\delta_N\Delta_N)f\right|^2$$

$$+\lim_{N\to\infty} cE_{\mathcal{M}_K^N}\left|\int\int dx\,dt\left[|\psi|^2 2\psi - |\psi|^2\left(\psi(x+\frac{1}{N})+\psi(x-\frac{1}{N})\right)\delta_N(x)\right]f\right|^2.$$

The first term overestimated by

$$
\begin{aligned}
\therefore \leq & |f^\bullet|_\infty E_{\mathcal{M}_K^N}\left(\int_0^1\int_0^T dx\,dt|\psi(x,t)-\psi(k/N,t)|dx\right)^2 \\
\leq & |f^\bullet|_\infty T\int_0^T E_{\mathcal{M}_K^N}\left(\int_0^1|\psi(x,t)-\psi(k/N,t)|dx\right)^2 \\
\leq & |f^\bullet|_\infty T^2\int_0^1 dx E_{\mathcal{M}_K^N}|\psi(x,t)-\psi(k/N,t)|^2 \\
= & |f^\bullet|_\infty T^2 N\int_0^{1/N} E_{\mathcal{M}_K^N}|\psi(x)-\psi(0)|^2dx \\
\leq & |f^\bullet|_\infty T^2\sup_{0\leq x\leq 1/N} E_{\mathcal{M}_K^N}|\psi(x)-\psi(0)|^2 = o(1),
\end{aligned}
$$

as $N\to\infty$ due to the stochastic continuity of the random field ψ. Estimates for the remaining terms can be obtained along the same lines. □

Due to the results of [MCV1, B2] the initial data $\psi(x,t)$ determines the flow $\psi(\bullet,t)$ for all t. In probabilistic language this means that $\psi(\bullet,t)$ is measurable with respect to the field generated by $\psi(\bullet,0)$.

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