On the graded ring of modular forms of the Siegel paramodular group of level 2

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In this paper, we shall describe the concrete ring structure of the graded rings of modular forms belonging to the Siegel paramodular group $\Gamma^{para}(2)$ of degree two with polarization $\mathrm{diag}(1,2)$. We also show that the Satake compactification of the quotient variety by this group is rational. Here, for each prime p, we define the group $\Gamma^{para}(p)$ by

$$\Gamma^{para}(p) = \{g \in M_4(\mathbb{Z});\ {}^t g J_2(p) g = J_2(p)\},$$

where for any number d, we put

$$J_2(d) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & d \\ -1 & 0 & 0 & 0 \\ 0 & -d & 0 & 0 \end{pmatrix}.$$

The main results will be given in section 1.

Historically, Freitag [1] has obtained the ring structure for a certain group which contains our group $\Gamma^{para}(2)$ with index 2. He used some geometrical method. Since the dimension formula for $\Gamma^{para}(p)$ has been known by Ibukiyama [9], we can use more direct method here, and his result is also obtained as a corollary of our result. Various generators of the ring have been considered by various approach. (cf. Gritsenko [2], [3], Gritsenko and Nikulin [4], [5], or Runge [15]). But the ring structure was not known as far as the authors know.

Actually we treat the discrete subgroup $K(p)$ of $Sp(2,\mathbb{R})$ which is $GL_4(\mathbb{Q})$-conjugate to $\Gamma^{para}(p)$ and defined by

$$K(p) = Sp(2,\mathbb{Q}) \cap \begin{pmatrix} \mathbb{Z} & \mathbb{Z} & p^{-1}\mathbb{Z} & \mathbb{Z} \\ p\mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ p\mathbb{Z} & p\mathbb{Z} & \mathbb{Z} & p\mathbb{Z} \\ p\mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{pmatrix}.$$

(The fact that $K(p)$ is conjugate to $\Gamma^{para}(p)$ is well known and was remarked also in the introduction of Ibukiyama [9] without proof. As for the written proof, see e.g. Hulek, Kahn and Weintraub [7] or Gritsenko [2].) This group $K(p)$ has been treated in Ibukiyama [8], [9] as one of standard parahoric subgroups in some different context and several results there are applicable here. For example, the dimension formula for Siegel cusp forms of weight $k \geq 5$ belonging to $K(p)$ has been known in [9] for each prime p, and some forms of small weights belonging to $K(2)$ have been given explicitly with their L functions in [8]. If we take the Iwahori subgroup $B(2)$ of level 2 defined e.g. in [8], then $K(2)$ contains $B(2)$ (cf. [6]) and the ring structure of modular forms belonging to $B(2)$ has been known in Ibukiyama [10]. We shall use these facts. By the way, the structure of $A(\Gamma(2))$ for the principal congruence subgroup of degree 2 is well known by Igusa [12] and our $K(2)$ contains $\Gamma(2)$. But $\Gamma(2)$ is not a normal subgroup of $K(2)$, and we need some work to get $A(K(2))$. In section 1, we shall state the main result, and the proof will be given in section 2.

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1 Main results

1.1 Preliminary definitions

For any ring S, we denote by $Sp(n, S)$ the usual symplectic group of size $2n$ defined by

$$Sp(n, S) = \{g \in M_{2n}(S);\ {}^t gJg = J\},$$

where

$$J = \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}.$$

We denote by H_n the Siegel upper half space of degree n defined by

$$H_n = \{Z \in M_n(\mathbb{C});\ {}^t Z = Z,\ \mathrm{Im}(Z) > 0\}.$$

Let Γ be a discrete subgroup of $Sp(n, \mathbb{R})$ with covolume finite. We denote by $A_k(\Gamma)$ or $S_k(\Gamma)$ the space of modular forms, or cusp forms, of weight k belonging to Γ, respectively. We define two graded rings as follows.

$$A(\Gamma) = \oplus_{k=0}^{\infty} A_k(\Gamma) \qquad \text{and} \qquad A_{even}(\Gamma) = \oplus_{k=0}^{\infty} A_{2k}(\Gamma).$$

For any $F(Z) \in A_k(\Gamma)$, we write

$$F|[g] = F|_k[g] = F(gZ)\det(CZ + D)^{-k}, \qquad \text{for } g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{R}).$$

Next, we define several discrete subgroups of $Sp(2,\mathbb{R})$. For each prime p, we define "Iwahori subgroup" $B(p)$ by

$$B(p) = Sp(2,\mathbb{Z}) \cap \begin{pmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ p\mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ p\mathbb{Z} & p\mathbb{Z} & \mathbb{Z} & p\mathbb{Z} \\ p\mathbb{Z} & p\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{pmatrix}.$$

We need several groups which contain $B(2)$. Put

$$s_0 = \begin{pmatrix} 0 & 0 & -p^{-1} & 0 \\ 0 & 1 & 0 & 0 \\ p & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad s_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

We also put $\Gamma_0^{'}(p) = B(p) \cup B(p)s_2B(p)$, and $\Gamma_0^{''}(p) = B(p) \cup B(p)s_0B(p)$. These are groups. As for more explicit description, cf.[8] p.601. By the general theory of Bruhat-Tits, we get $K(p) = B(p) \cup B(p)s_0B(p) \cup B(p)s_2B(p) \cup B(p)s_0s_2B(p)$ and the group $K(p)$ is generated by $\Gamma_0^{'}(p)$ and $\Gamma_0^{''}(p)$. Now, put

$$\rho = \frac{1}{\sqrt{p}} \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & p & 0 & 0 \\ p & 0 & 0 & 0 \end{pmatrix}.$$

Then we get $\Gamma_0^{''}(p) = \rho\Gamma_0^{'}(p)\rho^{-1}$ and $\rho K(p)\rho^{-1} = K(p)$. We denote by $K^*(p)$ the group generated by $K(p)$ and ρ. We have $[K^*(p) : K(p)] = 2$.

1.2 Generaters of Modular forms

For any $m = (m^{'}, m'') \in \mathbb{Z}^{2n}$ ($m^{'}, m'' \in \mathbb{Z}^n$), we define a theta constant $\theta_{m',m''} = \theta_{m',m''}(Z)$ by the following function of $Z \in H_n$.

$$\theta_{m',m''}(Z) = \sum_{p \in Z^n} \exp(2\pi i({}^t(p + m'/2)Z(p + m'/2)/2 + {}^t(p + m'/2)m''/2)).$$

We also put

$$\begin{aligned} X &= (\theta_{0000}^4 + \theta_{0001}^4 + \theta_{0010}^4 + \theta_{0011}^4)/4, \\ Y &= (\theta_{0000}\theta_{0001}\theta_{0010}\theta_{0011})^2, \\ Z &= (\theta_{0100}^4 - \theta_{0110}^4)^2/16384, \\ K &= (\theta_{0100}\theta_{0110}\theta_{1000}\theta_{1001}\theta_{1100}\theta_{1111})^2/4096, \\ T &= (\theta_{0100}\theta_{0110})^4/256, \end{aligned}$$

where the theta constants are for $n=2$. Each of the above is a modular form which belongs to $B(2)$ of weight 2, 4, 4, 6, or 4, respectively. We define the following functions.

$$
\begin{aligned}
F_4 &= X^2+3Y+3072Z+960T,\\
F_6 &= X^3-9XY-9216XZ+27648K+4032TX,\\
F_8 &= 16YZ-16XK+64T^2-TX^2+1024TZ+TY,\\
F_{12} &= 32X^3K+64X^2YZ-96XYK-98304XZK+5X^4T-14X^2YT\\
&\quad -14336X^2ZT-6144XKT+9Y^2T+18432YZT+9437184Z^2T\\
&\quad -896X^2T^2+1152YT^2+1179648ZT^2+36864T^3\\
G_{10} &= 4X^2K-16XYZ+12YK+12288ZK+X^3T-XYT-1024XZT\\
&\quad +768KT-64XT^2\\
G_{12} &= 3014656TX^2Z-2944TX^2Y+12582912KXZ-12288KXY\\
&\quad +184320T^2Y-188743680T^2Z-1152TY^2+1207959552TZ^2-1024X^4Z\\
&\quad +2097152X^2Z^2+3145728YZ^2-1073741824Z^3+X^4Y-2X^2Y^2-3072Y^2Z+Y^3\\
G_{11} &= \theta_{0000}\theta_{0001}\theta_{0010}\theta_{0011}\theta_{0100}\theta_{0110}\theta_{1000}\theta_{1001}\theta_{1100}\theta_{1111}\\
&\quad \times(\theta_{1000}^{12}-\theta_{1001}^{12}-\theta_{1100}^{12}+\theta_{1111}^{12})/1536(\quad =\chi_{11} \text{ in [10].})
\end{aligned}
$$

Proposition 1 *The function F_4, F_6, F_8, F_{12}, G_{10}, G_{12}, or G_{11} defined above is a modular form which belongs to $K(2)$ and of weight 4, 6, 8, 12, 10, 12 or 11, respectively. The first 5 forms belong also to $K^*(2)$, and we get $G_{11}|[\rho]=-G_{11}$ and $G_{12}|[\rho]=-G_{12}$. Besides, F_8, F_{12}, G_{10} and G_{11} are cusp forms.*

The proof of this Proposition will be given in section 2.

1.3 Main results

We denote by B the following subring of $A(K(2))$

$$B=\mathbb{C}[F_4,F_6,F_8,F_{12}].$$

Theorem 1 *The modular forms F_4, F_6, F_8, F_{12} are algebraically independent and B is a weighted polynomial ring. The graded ring $A_{even}(K(2))$ is given by*

$$A_{even}(K(2))=B\oplus(G_{12})B\oplus(G_{10})B\oplus(G_{10}G_{12})B,$$

and we get

$$A(K(2))=A_{even}(K(2))\oplus(G_{11})A_{even}(K(2)),$$

where $\oplus$ means the direct sum as modules. The ideal of cusp forms of $A(K(2))$ is spanned by F_8, F_{12}, G_{10}, and G_{11}.

The fundamental relations of the generators of the above graded ring are given as follows.

$$\begin{aligned}
G_{10}^2 &= F_8F_{12}/4,\\
729G_{12}^2 &= 26873856F_{12}^2 - 10368F_4^3F_{12} - 71663616F_4F_8F_{12} - 10368F_6^2F_{12}\\
&\quad +F_4^6 - 6912F_4^4F_8 - 2F_4^3F_6^2 + 15925248F_4^2F_8^2\\
&\quad -13824F_4F_6^2F_8 + F_6^4 - 12230590464F_8^3\\
&\quad +G_{10}(82944F_4^2F_6 + 63700992F_6F_8),\\
G_{11}^2 &= 3^{-3}\cdot 2^6(-F_6F_8^2 + 3F_4F_8G_{10} - F_{12}G_{10}).
\end{aligned}$$

Next Corollary is first proved by Freitag [1] for even weights.

Corollary 1 *We get*

$$A(K^*(2)) = B \oplus (G_{10})B \oplus (G_{11}G_{12})B \oplus (G_{11}G_{12}G_{10})B.$$

Now, we will give a result of the structure of the variety. For the sake of simplicity, we put

$$\begin{aligned}
\alpha &= G_{10}/F_4F_6\\
\beta &= F_6^2/F_4^3\\
\gamma &= F_8/F_4^2
\end{aligned}$$

Further, we define the automorphic functions A, B, C belonging to $K(2)$ by

$$\begin{aligned}
A &= 27\gamma(20736\alpha^2 - \gamma)(G_{12}/F_4^3)\\
B &= \beta(20736\alpha^2 - \gamma)^2 - 20736\alpha^2\gamma - 143327232\alpha^2\gamma^2 - \gamma^2\\
&\quad -6912\gamma^3 + 41472\alpha\gamma^2 + 31850496\alpha\gamma^3\\
C &= \gamma(768\gamma^2 + \gamma - 13824\alpha\gamma + 15925248\alpha^2\gamma - 2\alpha + 20736\alpha^2)
\end{aligned}$$

As an application of the above theorem, we get

Corollary 2 *The Satake compactification $\mathcal{S}(K(2)\backslash H_2) = Proj(A(K(2)))$ is a rational variety. The function field is given by*

$$\mathbb{C}(\frac{G_{10}}{F_4F_6}, \frac{A}{C}, \frac{B}{C}).$$

2 Proofs

2.1

We first review on dimension formulas.

Proposition 2 (cf. [9]) *We get*

$$\sum_{k=0}^{\infty} \dim A_k(K(2))t^k = \frac{(1+t^{10})(1+t^{12})(1+t^{11})}{(1-t^4)(1-t^6)(1-t^8)(1-t^{12})},$$

$$\sum_{k=0}^{\infty} \dim S_k(K(2))t^k = \frac{(t^8+t^{10}+t^{12}-t^{20})(1+t^{12})}{(1-t^4)(1-t^6)(1-t^8)(1-t^{12})},$$

$$\sum_{k=0}^{\infty} \dim A_k(\Gamma_0''(2))t^k = \frac{(1+t^{11})(1+t^6+t^8+t^{10}+t^{12}+t^{18})}{(1-t^4)^2(1-t^6)(1-t^{12})}.$$

Proof. As for the formula for $\Gamma_0'(2)$, this is an easy corollary to Igusa [12]. When the weight $k \leq 4$, the results for $K(2)$ is obtained from explicit structure of $A_k(B(2))$ very easily. For general k with $k \geq 5$, as for $\dim S_k(K(2))$, the above formula is the special case of [9] Theorem 4. As for $\dim A_k(K(2))$, it is easily obtained by the surjectivity of Φ operator by Satake [16] and the explicit description of cusps of $K(p)$ given e.g. in [11] for each prime p. We omit the details. q.e.d.

It has been written in [8] how to obtain forms in $A_k(K(2))$. We review this shortly for readers convenience. For automorphic form $F \in A_k(B(2))$, write the Fourier expansion as

$$F(Z) = \sum_T a(T)\exp(2\pi i \mathrm{tr}(TZ)),$$

where T runs over positive semi-definite half integral matrices. The subspace $A_k(\Gamma_0''(2))$ is characterized by those forms $F \in A_k(B(2))$ such that $a(T) = 0$ for all T with odd (1,1) component. (As for the proof, see [8].) Since we know dim $A_k(\Gamma_0''(2))$ and the generators of $A(B(2))$ (cf. [10]), we can determine $A_k(\Gamma_0''(2))$ if k is given explicitly and enough Fourier coefficients are known. Now, we have $A_k(K(2)) = A_k(\Gamma_0''(2)) \cap A_k(\Gamma_0'(2))$ and $A_k(\Gamma_0'(2)) = A_k(\Gamma_0''(2))|\rho$. So we can get $A_k(K(2))$ for given small k. We know that X, K, T are invariant by the action of ρ and $Y|_4[\rho] = 1024Z$, $Z|_4[\rho] = Y/1024$. Hence, we can also show that F_4, F_6, F_8, F_{12}, $G_{10} \in A_k(K^*(2))$. We can show $G_{12}|[\rho] = -G_{12}$ easily from the above. If we put $G = G_{11}|[\rho_1]$ for

$$\rho_1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

then obviously $(G_{11}|[\rho])(Z) = 2^{11}G(2Z)$. By the theta transformation formula (cf. [13]), we get

$$G(Z) = -(\theta_{0001}^{12} - \theta_{1001}^{12} - \theta_{0011}^{12} + \theta_{1111}^{12})\prod_m \theta_m/1536,$$

where m runs over ten even characteristics mod 2. Since $\dim A_{11}(K(2)) = 1$, obviously $G_{11}|[\rho]$ is G_{11} or $-G_{11}$, hence comparing one non vanishing Fourier coefficient, we can show it is $-G_{11}$. If you prefer more theoretical proof, you can prove this by using the following relations (cf. [13] p.232)

$$\begin{aligned}
\theta_{0000}(2Z)\theta_{0100}(2Z) &= (\theta_{0100}(Z)^2 + \theta_{0110}(Z)^2)/4, \\
\theta_{1000}(2Z)\theta_{1100}(2Z) &= (\theta_{0100}(Z)^2 - \theta_{0110}(Z)^2)/4, \\
\theta_{0011}(2Z)\theta_{1111}(2Z) &= (\theta_{1100}(Z)\theta_{1111}(Z))/2, \\
\theta_{0010}(2Z)\theta_{0110}(2Z) &= (\theta_{0100}(Z)\theta_{0110}(Z))/2, \\
\theta_{0001}(2Z)\theta_{1001}(2Z) &= (\theta_{1000}(Z)\theta_{1001}(Z))/2, \\
\theta_{0001}(2Z)^2 &= (\theta_{0000}(Z)\theta_{0001}(Z) + \theta_{0010}(Z)\theta_{0011}(Z))/2, \\
\theta_{1001}(2Z)^2 &= (\theta_{0000}(Z)\theta_{0001}(Z) - \theta_{0010}(Z)\theta_{0011}(Z))/2, \\
\theta_{0011}(2Z)^2 &= (\theta_{0000}(Z)\theta_{0011}(Z) + \theta_{0010}(Z)\theta_{0001}(Z))/2, \\
\theta_{1111}(2Z)^2 &= (\theta_{0000}(Z)\theta_{0011}(Z) - \theta_{0010}(Z)\theta_{0001}(Z))/2,
\end{aligned}$$

and Riemann's theta formula (cf. [12].) Since K, YZ, and $T(X^2 - Y - 1024Z - 64T)$ are cusp forms (cf. [10]), it is easy to see F_8, F_{12} and G_{10} are cusp forms. Hence, Proposition 1 is proved.

Now we introduce the Witt operator. For any function $F(Z)$ on H_2, we put

$$(WF)(z_1, z_2) = F\begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix} \qquad (z_1,\, z_2 \in H_1).$$

We denote by $E_k(z)$ $(z \in H_1)$ the Eisenstein series of weight k belonging to $SL_2(\mathbb{Z})$ having 1 as the constant term. For short we write $E_k = E_k(z_2)$ and $E_k^{'} = E_k(2z_1)$ for mutually independent variables $z_1,\, z_2 \in H_1$. The image of theta constants by W is again easily expressed by theta constants. Also, it is easy to see the following relations.

$$\begin{aligned}
E_4 &= \theta_{01}^8(z_2) + \theta_{01}^4(z_2)\theta_{10}^4(z_2) + \theta_{10}^8(z_2), \\
E_4^{'} &= (16\theta_{01}^8(z_1) + 16\theta_{01}^4(z_1)\theta_{10}^4(z_1) + \theta_{10}^8(z_1))/16, \\
E_6 &= (2\theta_{01}^4(z_2) + \theta_{10}^4(z_2))(\theta_{01}^8(z_2) + \theta_{01}^4(z_2)\theta_{10}^4(z_2) - 2\theta_{10}^8(z_2))/2, \\
E_6^{'} &= (2\theta_{01}^4(z_1) + \theta_{10}^4(z_1))(32\theta_{01}^8(z_1) + 32\theta_{01}^4(z_1)\theta_{10}^4(z_1) - \theta_{10}^8(z_1))/64,
\end{aligned}$$

Hence, it is easy to show that $W(F_8) = W(G_{10}) = 0$ and

$$\begin{aligned}
W(F_4) &= 4E_4^{'}E_4, \\
W(F_6) &= -8E_6^{'}E_6, \\
W(F_{12}) &= (E_4^{'3} - E_6^{'2})(E_4^3 - E_6^2)/81, \\
W(G_{12}) &= 3^{-3}\cdot 2^6(E_4^{'3}E_6^2 - E_4^3E_6^{'2}).
\end{aligned}$$

Lemma 1 *Let P and Q be polynomials of three variables which satisfy the following relation*

$$P(W(F_4), W(F_6), W(F_{12})) + W(G_{12})Q(W(F_4), W(F_6), W(F_{12})) = 0.$$

Then we get $P = Q = 0$.

Proof. It is clear that E_4, $E_4^{'}$, E_6, $E_6^{'}$ are algebraically independent. The forms $W(F_4)$, $W(F_6)$ and $W(F_{12})$ are invariant under the exchange between E_k and $E_k^{'}$ ($k = 4$, 6) and G_{12} becomes $-G_{12}$ by this exchange. So, we get $P(W(F_4), W(F_6), W(F_{12})) = 0$. But obviously, $W(F_4)$, $W(F_6)$ and $W(F_{12})$ are algebraically independent. Hence $P = Q = 0$ as polynomials.

Now we prove Theorem 1. First, it is easy to show the formula for G_k^2 for $k = 10$, 11, 12 in Theorem, since we know the relation

$$64K^2 = -16XTK - T(-16YZ + X^2T - YT - 1024ZT - 64T^2),$$

and the formula to express G_{11}^2 by X, Y, Z, K and T in Appendix of [10]. All we must do is to express both sides of the formulas as polynomials of X, Y, Z, K, T of degree one with respect to K to find out they are the same. In particular we get

$$G_{10}^2 = F_8 F_{12}/4.$$

Proposition 3 *Let $P_i(x_1, x_2, x_3, x_4)$ ($1 \leq i \leq 4$) be polynomials of four variables which satisfiy the following relation*

$$\begin{aligned} & P_1(F_4, F_6, F_8, F_{12}) + G_{12}P_2(F_4, F_6, F_8, F_{12}) \\ + \quad & G_{10}P_3(F_4, F_6, F_8, F_{12}) + G_{10}G_{12}P_4(F_4, F_6, F_8, F_{12}) = 0. \end{aligned}$$

Then $P_i = 0$ ($i = 1, \ldots, 4$) as polynomials. In particular, F_4, F_6, F_8, F_{12} are algebraically independent.

Proof. We take the image under the Witt operator of the both sides of the above relation. Since $W(G_{10}) = 0$, by the above lemma we get

$$P_1(x_1, x_2, 0, x_4) = P_2(x_1, x_2, 0, x_4) = 0.$$

That is, for $i = 1$, 2, we have $P_i = x_3 Q_i$ for some polynomials Q_i. Now, multiplying G_{10} to both sides, we get

$$\begin{aligned} & F_8 G_{10} Q_1(F_4, F_6, F_8, F_{12}) + F_8 G_{10} G_{12} Q_2(F_4, F_6, F_8, F_{12}) \\ + \quad & 4^{-1} F_8 F_{12} P_3(F_4, F_6, F_8, F_{12}) + 4^{-1} F_8 F_{12} G_{12} P_4(F_4, F_6, F_8, F_{12}) = 0. \end{aligned}$$

Now dividing both sides by F_8, then applying W on both sides, and dividing by $W(F_{12}) \neq 0$, we can see as before that $P_i = x_3 Q_i$ for $i = 3, 4$, for some polynomials Q_i. Repeating this process, we can conclude that $P_i = 0$ for all $i = 1,\ldots,4$. q.e.d.

By the above proposition, we can calculate the dimensions of $(B + G_{10}B + G_{12}B + G_{10}G_{12}B) \cap A_k(K(2))$ for each k to find that it is equal to $\dim A_k(K(2))$. Hence our main theorem is proved. Corollary 1 is clear from Theorem. We now prove Corollary 2. We denote by $\mathcal{K}$ the function field of $Proj(A(K(2)))$. We define elements A, B, C, α, β, $\gamma \in \mathcal{K}$ as in section 1. By the formula for G_{10}^2, we get $F_{12}/F_4^3 = 4\alpha^2\beta/\gamma$. By Theorem 1, it is easy to see that $\mathcal{K}$ is generated by α, β, γ, and G_{12}/F_4^3. Now, we define the field $\mathcal{K}'$ by $\mathcal{K}' = \mathbb{C}(\alpha, A/C, B/C)$. By modifying the formula for G_{12}^2, we can show that $A^2 = B^2 - \gamma C^2$. Hence $\gamma = (B/C)^2 - (A/C)^2 \in \mathcal{K}'$. Since C is a polynomial of α and γ, we get $C \in \mathcal{K}'$. Hence A, $B \in \mathcal{K}'$ and also G_{12}/F_4^3, $\beta \in \mathcal{K}'$. Thus we get $\mathcal{K}' = \mathcal{K}$ and Corollary 2.

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