Differential calculus in noncommutative algebraic geometry I. D-calculus on noncommutative rings

Valery Lunts, Alexander Rosenberg

Department of Mathematics
Indiana University
Bloomington, Indiana 47405

USA

Max-Planck-Institut
für Mathematik
Gottfried-Claren-Str. 26
53225 Bonn

Germany

MPI 96-53

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Contents:

Introduction.

Systems of q-differential equations have been studied for at least a century by ruther classical methods involving lots of computations and producing cumbersome formulas. With the appearance of quantized enveloping algebras, q-differential equations and operators became very popular, since they were regarded, at the beginning, as principal characters of a future quantized D-module theory. The general expectation was that, for any simple Lie algebra, there exist quantized versions of the flag variety and of the principal homogenious space, otherwise called the 'base affine space' (cf. [GK], [BB1]), and an analog of the canonical homomorphism of the quantized enveloping algebra into the *algebra of differential operators on the base affine space.* This homomorphism should imply the so-called 'localization construction' [BB1], [Be] which realizes representations of a quantized enveloping algebra as modules over (twisted) differential operators on the flag variety. And the algebras of (twisted) differential operators on the flag varieties were expected to be 'locally' isomorphic to the algebras of q-differential operators.

Quantized analogs of the flag variety [TT], [LaR], [So] and a quantized version of the base affine space [Jo] were proposed.

But, with the exception of the (quantized) sl_2-case investigated by T. Hodges [H], q-differential operators do not provide the local picture for any version of Bernstein-Beilinson 'localization construction'. As well as other, quite interesting, analogs of Weyl algebras (see [Ha], [Ma]).

Thus two questions arise:

(a) What are differential operators on 'quantized spaces'?

(b) What is a natural quantized version of the Beilinson-Bernstein localization construction?

Answering to these questions was the motivation to undertake this work.

In this paper, we explain what are differential operators on general associative rings. Since the known examples of 'quantized spaces' are (identified with) algebras of certain type, or 'projective spaces' associated with them, this seems to give an answer to the question (a) above, at least for affine spaces. However, this is not exactly the case.

The point is that quantized 'spaces', in particular, quantized enveloping algebras, are algebras in certain, naturally related to them, monoidal categories. Thus, a quantized enveloping algebra $U_q(\mathfrak{g})$ is an algebra in the monoidal category of $\mathbb{Z}^r$-graded modules, where r is the rank of the Lie algebra $\mathfrak{g}$. A choice of a quasi-symmetry β (i.e. a solution of the Yang-Baxter equation) in any monoidal category $\mathcal{C}^{\tilde{}}$ determines a differential calculus in this monoidal category. And after a quasi-symmetry β is fixed, any (not necessarily) finite set X of invertible objects of $\mathcal{C}^{\tilde{}}$ determines a Hopf algebra $U_{\beta,X}$ in $\mathcal{C}^{\tilde{}}$. The quantized algebras by Drinfeld and Jimbo are particular cases of this construction. Using this setting, one can define the 'base affine space', the flag variety, and the Beilinson-Bernstein construction for quantized and 'classical' enveloping algebras simultaneously. Note that everything is going on inside of the monoidal category $\mathcal{C}^{\tilde{}}$ (of graded modules) which, therefore, can be regarded as a natural universum of the theory. The same way as the monoidal category of super (i.e. $\mathbb{Z}/2\mathbb{Z}$-graded) vector spaces can be viewed as a natural universum of super-mathematics.

Below follows a more detailed account on the contents of this paper.

Section 1 is a continuation of Introduction. We remind shortly the conventional, Grothendieck's, differential calculus [Gr]. Then we explain how the approach should be modified to obtain the notion of a differential operator on a noncommutative ring. Finally, we give, for a reader's convenience, a short outline of some of the basic notions and results of this paper and its continuation with a more algebraic (and less geometric) ring-theoretic flavor than in the main body of the work.

The conventional D-calculus deals with quasi-coherent sheaves of differential algebras on a scheme. We switch to a noncommutative picture by identifying a scheme $\mathbf{X} = (X, \mathcal{O})$ with the category $\mathcal{A} = \mathfrak{Qcoh}_{\mathbf{X}}$ of quasi-coherent sheaves on $\mathbf{X}$, and replacing quasi-coherent (sheaves of) algebras $\mathcal{R}$ with the monads $\mathcal{R}\otimes_{\mathcal{O}}$ in $\mathcal{A}$. Thus, abelian categories in this work are thought as categories of quasi-coherent sheaves on a scheme. Following this dictionary, (closed) subschemes of a 'scheme' $\mathcal{A}$ are identified with a certain class of topologizing subcategories of $\mathcal{A}$. We outline this philosophy (together with preliminaries on topologizing subcategories) in Section 2.

In Section 3, we introduce some of the first notions of formal differential geometry, like formal neighborhood and the conormal bundle of a 'subscheme'.

In Section 4, differential endofunctors and monads appear.

In Section 5, we apply the general approach to the category $R-mod$ of left modules over an associative ring R. In particular, we define differential operators from one R-module to another.

Section 6 is concerned with properties of differential bimodules and algebras with respect to localizations. It is known that differential bimodules, in particular differential algebras (in the sense of [BB]) over a commutative ring R behave well with respect to localizations at finitely generated multiplicative subsets in R. This means that such a localization of a differential bimodule has a natural structure of a differential bimodule over the localized ring. We establish much stronger properties in the general, noncommutative, case. However, in order to obtain natural localization properties in noncommutative setting we have to switch to derived categories already in the affine case.

In 'Complementary facts' we discuss the localization of D-modules at points of the spectrum and a suggestive analog of the D-affinity of endofunctors.

We were trying to introduce in this paper the main ideas and some facts used in the next part of the work. In particular we make more stress on categorical and geometrical point of view than is strictly necessary to obtain the main results of the paper.

In order to show the direction of the futher development, we will sketch here the contents of the next two parts of this work which shall appear in the subsequent paper.

In Part II, we extend and apply facts of the previous sections to sketch D-calculus in monoidal categories.

After introducing general constructions and some of basic facts of module theory in monoidal categories, we describe the algebra of differential operators on a skew polynomial algebra. The latter is regarded as a 'commutative' algebra in the monoidal category of $\mathbb{Z}^r$-graded modules over a commutative ring K with a fixed symmetry. We call it sometimes, by abuse of language, *an affine space.*

After that we define an 'affine algebra' when instead of a symmetry in the monoidal category of $\mathbb{Z}^r$-graded modules we have a quasi-symmetry, β. In this case, the 'affine algebra' appeares together with the β- *Weyl algebra* of differential operators on it.

The third part of the work is dedicated to the construction of Hopf algebras naturally associated with a quasi-symmetry. We define Hopf algebras in a quasi-symmetric category and discuss some of relevant examples.

Fix a monoidal category and its symmetry σ. Given a quasi-symmetry β, we can construct Weyl algebras (in the sense of Part II) which are β-Hopf algebras. Our goal is to 'extend' them naturally to σ-Hopf algebras. This is possible to do under certain natural requirements which are satisfied in all known cases of interest.

We study Hopf actions and crossed products in monoidal categories. A special case of this construction is the 'affine base space' for any (quantized) enveloping algebra of a reductive or Kac-Moody Lie algebra.

We construct, in an arbitrary monoidal category, *a Weyl algebra* associated with a bilinear form and a quasi-symmetry.

Then we study relations between quasi-symmetries, the *Translation group* (– the group of isomorphy classes of invertible objects), and the *fundamental group* of our monoidal category.

As a result of this study, we construct Hopf algebras associated with a quasi-symmetry. This way we obtain a family of Hopf algebras with a quasi-symmetry playing the role of a parameter. The quantized enveloping algebras by Drinfeld and Jimbo are particular cases of this construction.

We conclude with a short presentation of a quantized version of the Beilinson-Bernstein localization construction.

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1. Grothendieck's differential calculus and its noncommutative version.

1.0. Differential bimodules and differential rings. First we recall shortly the differential calculus on commutative rings and schemes following [BB].

Fix a commutative ring k. If not specified otherwise, $\otimes$ means $\otimes_k$.

Let R be a commutative k-algebra; and let M be an R-bimodule such that, for any $z \in M$ and any $\lambda \in k$, $\lambda \cdot z = z \cdot \lambda$. The bimodule M can be regarded as an $R \otimes R$-module.

For any $r \in R$, define the endomorphism ad_r of the bimodule M (- the adjoint action of r) by $ad_r(z) = r \cdot z - z \cdot r$ for all $z \in M$.

An increasing filtration $\{M_i \mid i \geq -1\}$ on M is called a *D-filtration* if $M_{-1} = 0$ and $ad_r(M_i) \subseteq M_{i-1}$ for all $r \in R$ and $i \geq 0$.

There is the largest (with respect to the inclusion) D-filtration $M_.^{\sim}$ on M defined by $M_i^{\sim} := \{z \in M \mid ad_r(z) \in M_{i-1}^{\sim}$ for all $r \in R\}$, $i \geq 0$. The subbimodule $M^{\sim} := \bigcup_{i \geq 0} M_i^{\sim}$ is called *the differential part of* M. And M is called *a differential bimodule* if $M^{\sim}$ coincides with M.

Let A be an associative ring equipped with an algebra morphism $\iota : R \longrightarrow$A. An increasing ring filtration $A_. = \{A_i \mid i \geq -1\}$ is called a *D-ring filtration* if it is a D-filtration of the R-bimodule A such that $\iota(R) \subseteq A_0$ and $\iota(R)$ lies in the center of the associated graded algebra. One can observe that the largest D-filtration $A_.^{\sim}$ on A is a D-ring filtration. And $\iota : R \longrightarrow A$ is called *an R-differential algebra* if $A = A^{\sim}$, i.e. when A is a differential R-bimodule.

Note that, if we regard a bimodule M as an $R \otimes R$-module, $M_i^{\sim} := \{z \in M \mid \mathcal{I}^{i+1} z = 0\}$, where $\mathcal{I}$ is the kernel of the multiplication $m : R \otimes R \longrightarrow$R. It follows from this description that the canonical D-filtration is compatible with localizations: for any $t \in R$, there are natural isomorphisms $(M_i^{\sim})_t \simeq (M_t)_i^{\sim} \simeq R_t \otimes M_i^{\sim} \simeq M_i^{\sim} \otimes R_t$. Here M_t denotes the localization of M at t; i.e. $M_t := R_t \otimes M \otimes R_t$.

The compatibility with localizations allows to globalize the notion of a differential bimodule. *A differential bimodule on a scheme* X is a quasi-coherent sheaf M on X having the following properties:

(i) for any open $U \subseteq X$, $M(U)$ is a differential $\mathcal{O}_X(U)$-bimodule;

(ii) if U is affine and $t \in \mathcal{O}_X(U)$, then $M(U_t) \simeq M(U)_t$.

If the scheme X is locally noetherian, the direct image functor of the diagonal $\Delta : X \longrightarrow X \times X$ provides an equivalence between the category Diff(X) of differential bimodules on X and the full subcategory of the category $\mathrm{Qcoh}_{X \times X}$ of quasi-coherent sheaves on $X \times X$ generated by sheaves supported on the diagonal.

A differential $\mathcal{O}_X$*-algebra* (or *D-algebra on* X) is a sheaf of associative algebras on X equipped with a morphism of algebras $\iota : \mathcal{O}_X \longrightarrow A$ which makes A a differential $\mathcal{O}_X$-bimodule.

For a D-algebra A on X, an A-module is a sheaf of A-modules which is quasi-coherent as an $\mathcal{O}_X$-module.

1.0.1. Differential operators. Let $\mathbf{X} = (X, \mathcal{O}_X)$ be a scheme over a commutative ring k; and let L and N be quasi-coherent $\mathcal{O}_X$-modules. A morphism $f : L \longrightarrow N$ of sheaves of abelian groups is called *a differential operator* if, for any open affine $U \subseteq X$, the morphism $f_U : L(U) \longrightarrow N(U)$ lies in the differential part of the $\mathcal{O}(U)$-bimodule $\mathrm{Hom}_k(L(U), N(U))$.

1.0.2. The same constructions don't work in the noncommutative case. Suppose now that R is a noncommutative ring. We still have the adjoint action of R on any R-bimodule M, $r \mapsto ad_r$, $r \in R$.

Note by passing that, for any $z \in M$, the map $ad : R \longrightarrow M$, $r \mapsto ad_r(z)$ is a derivation of the ring R in M; i.e. the Leibniz's rule holds:

$$ad_{rs}(z) := rs \cdot z - z \cdot rs = r \cdot ad_s(z) + ad_s(z) \cdot \mathrm{r}.$$

The derivations $r \mapsto ad_r(z)$ are called *inner derivations* in the bimodule M. One can check that the map $ad : R \longrightarrow End_k(M)$, $r \mapsto ad_r$, is a Lie algebra morphism (with respect to commutators in R and $End_k(M)$).

Having the adjoint action of R, one can repeat the constructions above. Namely, define the canonical filtration of a bimodule M by:

$$M_{-1}^{\check{}} = 0, M_i^{\check{}} := \{z \in M \mid \ \mathrm{ad}_r(z) \in M_{i-1}^{\check{}} \text{ for all } r \in R\}.$$

Again, identifying R-bimodules with $R \otimes R^o$-modules (where R^o denotes the ring opposite to R), one have:

$$M_i^{\check{}} := \{z \in M \mid \mathcal{I}^{i+1} z = 0\} \text{ for all } i \geq 0, \qquad (1)$$

where $\mathcal{J}$ is the kernel of the multiplication $R \otimes R^o \longrightarrow$R.

Note that, in general, $M_i^{\check{}}$ and $M^{\check{}} := \bigcup_{i \geq 0} M_i^{\check{}}$ are not R-bimodules: they have a structure of bimodules only over the center of R.. This indicates that the copying the commutative constructions does not lead to adequate notions of a differential bimodule and (therefore) a differential operator.

It shall become clear from what follows that the failure of a direct imitation of the commutative setting is due to the fact that, for a noncommutative ring (this time we mean $R \otimes R^o$), left ideals (in particular $\mathcal{I}$) do not define a subscheme and its formal neighborhood in a way they do in the commutative case.

So that to find a 'right' notion of a differential operator one needs first to have an adequate notion of a noncommutative subscheme. Or at least to understand what is the diagonal.

1.0.3. The diagonal. For two left ideals, m and n, of an associative unital ring A, we write $m \leq n$ if there exists a finite set x of elements of A such that the left ideal $(m : x) := \{b \in A \mid bx \subset m\}$ is contained in n. The equality $((m : x) : y) = (m : yx)$ implies that $\leq$ is a preorder on the set $I_l A$ of left ideals in A (cf. [R], Lemma I.1.1).

Note that if m is a two-sided ideal, $m \subseteq (m : x)$ for any subset $x \subset A$ which means that $m \leq n$ iff $m \subseteq$n. In particular, the preorder $\leq$ coincides with $\subseteq$ when the ring A is commutative.

Fix an associative (always unital) k-algebra R, and take $A = R \otimes R^o$, where R^o is the k-algebra opposite to R. Denote by K_μ the kernel of the multiplication $\mu : R \otimes R^o \longrightarrow$R. One can check that K_μ is a left ideal in $R \otimes R^o$ which is two-sided if and only if the k-algebra R (hence $R \otimes R^o$) is commutative (Lemma 5.1 and Note 5.1.2).

As we have mentioned in Introduction, a way to switch from the commutative setting to the noncommutative one is to replace schemes by the categories of quasi-coherent sheaves on these schemes.

Naturally, the category of quasi-coherent sheaves on the 'affine scheme' $\mathbf{X}$ corresponding to a k-algebra R should be $R - mod$.

The category of quasi-coherent sheaves on the affine scheme $\mathbf{X} \times \mathbf{X}$ is (equivalent to) the category $R - bi$ of R-bimodules which we identify, whenever it is convenient, with $R \otimes R^o - mod$ of left $R \otimes R^o$-modules.

More generally, if $\mathbf{Y}$ is an affine scheme isomorphic to $\mathbf{Spec}S$ for another k-algebra, S, then the category of quasi-coherent sheaves on the product $\mathbf{X} \times \mathbf{Y}$ is (equivalent to) $R \otimes S^o - mod$.

Finally, we define the (category of quasi-coherent sheaves on the) diagonal of $\mathbf{X} \times \mathbf{X}$ as the full subcategory, Δ_R, of the category $R \otimes R^o - mod$ generated by all $R \otimes R^o$-modules M such that, for any $z \in M$, $K_\mu \leq$Ann(z).

Fix an $R \otimes R^o$-module M. We call an increasing filtration $\{M_i \mid i \geq -1\}$, where all M_i are $R \otimes R^o$-submodules of M, a *D-filtration* if $M_{-1} = 0$ and $M_i/M_{i-1} \in Ob\Delta_R$ for all $i \geq 0$. And we call an $R \otimes R^o$-module M *differential* if it has a D-filtration $\{M_i\}$ such that $\bigcup_{i \geq 0} M_i$=M.

Every $R \otimes R^o$-module contains the biggest differential submodule which is called *the differential part of* M. For any two left R-modules, L and N, the k-module $\mathrm{Hom}_k(L, N)$ is naturally a $R \otimes R^o$-module. In particular, it contains the biggest differential $R \otimes R^o$-submodule $Diff_k(L, N)$. Morphisms from $Diff_k(L, N)$ are called *differential operators from L to* N.

For readers' convenience, we give in the next section an 'elementary' definition of differential operators and outline some of their main properties.

1.1. A definition and an outline of main properties of differential operators. Fix an associative unital algebra R over a commutative ring k. If not specified otherwise, $\otimes$ means $\otimes_k$.

1.1.1. Definition. An $R \otimes R^o$-module M is *differential* iff, for any finitely generated submodule M' of M, there is an increasing finite filtration $\{M_i \mid -1 \leq i \leq n\}$ such that $M_{-1} = 0$, $M_n = M'$, and the $R \otimes R^o$-module M_{i+1}/M_i is a subquotient of a finite direct sum of copies of R.

A differential R-algebra is an algebra morphism $R \longrightarrow S$ which makes S a differential $R \otimes R^o$-module.

1.1.2. Proposition. *Any $R \otimes R^o$-module M contains the biggest differential submodule, M_{diff}, called the differential part of M. The correspondence $M \mapsto M_{diff}$ is functorial: for any $R \otimes R^o$-module morphism $\varphi : M \longrightarrow N$, $\varphi(M_{diff}) \subseteq N_{diff}$.*

In particular, for any pair of R-modules, L and N, the $R \otimes R^o$-module $\mathrm{Hom}_k(L, N)$ contains the biggest differential submodule $Diff_k(L, N)$. Morphisms from $Diff_k(L, N)$ are called *(k-linear) differential operators from L to* N.

1.1.3. Proposition. *(a) If M and M' are differential $R \otimes R^o$-modules, then their tensor product over R, the $R \otimes R^o$-module $M \otimes_R M'$, is differential too.*

(b) For any ring morphism $R \longrightarrow B$, the differential part of the $R \otimes R^o$-module B is a subring of B; i.e. B_{diff} is a differential R-algebra.

In particular, for any left R-module L, the $R \otimes R^o$-module $Diff_k(L, L)$ of differential operators from L to L is a differential R-algebra.

If $L = R$, we shall write $D_k(R)$, or simply $D(R)$, instead of $Diff_k(R, R)$ and call $D_k(R)$ *the algebra of* (k-linear) *differential operators on* R.

Probably, the most significant property of differential $R \otimes R^o$-modules and algebras (in particular, algebras of differential operators) is their compatibility with localizations which we establish in Section 6. One of the consequences (and special cases) of this compatibility is the following assertion (which is a particular case of Proposition 6.5.1):

1.1.4. Proposition. *Let $R \longrightarrow R'$ be an algebra morphism such that the functor $Q = R' \otimes_R$ is an exact localization and R' is flat as a left R-module too. (for instance, R' is the localization of R at a right and left Ore set). Then*

(a) For any differential $R \otimes R^o$-module M , the functor $M \otimes_R$ is compatible with the localization $Q = R' \otimes_R$. And $Q \circ (M \otimes_R) \simeq M' \otimes_{R'}$, where $M' = R' \otimes_R M \otimes_R R'$. The canonical (R', R)-bimodule morphism $R' \otimes_R M \longrightarrow R' \otimes_R M \otimes_R R'$ is an isomorphism.

(b) If $M \in Ob\Delta_R^{(n)}$, i.e. if M is a differential $R \otimes R^o$-module of n-th order, then the $R' \otimes R^o$-module M' has the same order: $M' \in Ob\Delta_{R'}^{(n)}$.

(c) Let $\varphi : R \longrightarrow A$ be a differential algebra (i.e. φ is a k-algebra morphism turning A into a differential $R \otimes R^o$-module. Then $R' \otimes_R A$ has a unique k-algebra structure such that the canonical maps $A \longrightarrow R' \otimes_R A \longleftarrow R'$ are k-algebra morphisms. And $R' \otimes_R A$ is a differential $R' \otimes R^o$-module.

The following Proposition (6.5.2 in the main body of the text) shows that, under the conditions of Proposition 1.1.4, any differential bimodule over the localized ring is the localization of a differential bimodule.

1.1.5. Proposition. *Let $R \longrightarrow R'$ be an algebra morphism such that the functor*

$$Q = R' \otimes_R : R - mod \longrightarrow R' - mod$$

is an exact localization and the R' is flat as a left R-module. Let M' be a differential $R' \otimes R'^o$-module. And let $M := Q^{\hat{}}(M')_{diff}$ (i.e. M is the differential part of the $R \otimes R^o$-module M'). Then the canonical morphism $\varphi : R' \otimes_R M \longrightarrow M'$ is an isomorphism of $R \otimes R^o$-modules. Moreover, the isomorphism φ induces, for any $n \geq 0$, an isomorphism $R' \otimes_R M_n \longrightarrow M'_n$, where M'_n (resp. M_n) denotes the $\Delta_{R'}^{(n+1)}-$ (resp.$\Delta_R^{(n+1)}-$) torsion of M' (resp. of M).

The restrictions on a localization can be considerably weakened if differential bimodules and operators are replaced by *strongly differential* ones.

1.1.6. Strongly differential bimodules and operators. We call an $R \otimes R^o$-module M is *strongly differential* iff, for any finitely generated submodule M' of M, there is an increasing finite filtration $\{M_i \mid -1 \leq i \leq n\}$ such that $M_{-1} = 0$, $M_n = M'$, and the $R \otimes R^o$-module M_{i+1}/M_i is a quotient of a finite direct sum of copies of R.

Clearly any strongly differential bimodule is differential. If the base ring R is commutative, then the converse statement is true: these two classes coincide. In the general case, one can characterize differential bimodules in terms of strongly differential ones:

1.1.6.1. Proposition. *An $R \otimes R^o$-module is differential iff it is a submodule of a strongly differential $R \otimes R^o$-module.*

(This is a corollary of Proposition 5.11.4.3.)

The exact analogs of Propositions 1.1.2 and 1.1.3 hold for strongly differential bimodules. We denote strongly differential operators on a k-algebra R by $D^s_k(R)$ or simply by $D^s(R)$

Strongly differential $R \otimes R^o$-modules which are flat as left R-modules are compatible with any exact localizations. In particular, we have the following assertion (Proposition 6.5.4.1):

1.1.6.2. Proposition. *Let $R \longrightarrow R'$ be an algebra morphism such that the functor*

$$R' \otimes_R : R - mod \longrightarrow R' - mod$$

is an exact localization (say the ring R' is the localization of R at a left Ore set). Then

(a) The action of $D^s(R)$ on R extends naturally to an action on R' giving a canonical ring homomorphism $D^s(R) \longrightarrow D^s(R')$ which induces a left R'-module isomorphism

$$R' \otimes_R D^s(R) \longrightarrow D^s(R').$$

(b) For any $D^s(R)$-module M, the R'-module $R' \otimes_R M$ has a natural, in particular compatible with $D^s(R) \longrightarrow D^s(R')$, structure of a $D^s(R')$-module.

(c) If the ring R' is such that the functor $\otimes_R R' : mod-R \longrightarrow mod-R'$ is a localization (e.g. R' is the localization of R at a left and right Ore set), then we also get an induced right R'-module isomorphism $D^s(R) \otimes_R R' \longrightarrow D^s(R')$.

Let us point out an important connection with enveloping algebras. Let k be a commutative ring. Let $U_k(\mathfrak{g}) = U(\mathfrak{g})$ be the enveloping k-algebra of a Lie algebra $\mathfrak{g}$. Recall that $U(\mathfrak{g})$ is a Hopf algebra. There is a following assertion (Proposition 5.10.1):

1.1.6.3. Proposition. *Let $\varphi : U(\mathfrak{g}) \longrightarrow End_k(R)$ be a Hopf action on the algebra R (i.e. the multiplication $R \otimes_k R \longrightarrow R$ is a $U(\mathfrak{g})$-module morphism). Then $U(\mathfrak{g})$ acts by strongly differential operators.*

1.1.6.4. Corollary. *Let $\mathfrak{g}$ be a Lie algebra over a field k of zero characteristic. And let $\varphi : U(\mathfrak{g}) \longrightarrow End_k(R)$ be a Hopf action on the algebra R. Let R' be an Ore localization of* R. *Then the action of $U(\mathfrak{g})$ extends uniquely to a Hopf action on R'.*

To give a flavor of the next part of this work (which will appear in a subsequent paper), we shall continue this sketch a little bit futher.

1.2. Graded differential operators. Let k be a commutative ring. Fix an abelian group Γ and a group homomorphism (bicharacter) $\beta : \Gamma \times \Gamma \longrightarrow k^*$. In our main examples, the group Γ can be assumed to be free of finite rank, i.e. $\Gamma \simeq \mathbb{Z}^r$. Note that once an isomorphism $\Gamma \simeq \mathbb{Z}^r$ is fixed, β is canonically determined by its values on the basis of $\mathbb{Z}^r$; i.e. by an $r \times r$ matrix with entrees in k^*.

Let R be a Γ-graded associative k-algebra. Consider the category $\mathfrak{gr}_\Gamma R - mod$ of graded R-modules. We want to define differential operators in this graded setting. To do this, we need a natural substitute for $R \otimes R^o$-module R.

Note that the bicharacter β determines an action of Γ on any graded module M. In particular, β defines an action of Γ on R by ring automorphisms. The corresponding crossed product $R\#k[\Gamma]$ (where $k[\Gamma]$ is the group algebra of Γ with the natural Γ-grading) regarded as a graded $R \otimes R^o$-module, is our substitute for the bimodule R.

1.2.1. Definitions. A graded $R \otimes R^o$-module M is *differential* (resp. *strongly differential*) iff, for any finitely generated graded submodule M' of M, there is an increasing filtration $\{M_i \mid i \geq -1\}$ such that $M_{-1} = 0$, $M_n = M'$ for some n, and the $R \otimes R^o$-module M_{i+1}/M_i is a subquotient (resp. a quotient) of a direct sum of copies of $R\#\Gamma$.

A differential (resp. a strongly differential) R-algebra is a graded ring morphism $R \longrightarrow S$ such that S is a differential (resp. strongly differential) R-module.

Propositions 1.1.2 – 1.1.5 and their analogs for strongly differential bimodules hold word by word if we replace modules by graded modules and morphisms by graded morphisms. For example, when we define differential operators from a graded R-module L to a graded R-module M, we should take the graded part, $\mathfrak{gr}\mathrm{Hom}_k(L, M)$, of $\mathrm{Hom}_k(L, M)$. In particular, we obtain the ring of Γ-graded differential operators, $D^{\#}_\beta(R)$, and the ring $D^{\#s}_\beta(R)$ of Γ-graded strongly differential operators on R. Note that $D^{\#}_\beta(R)$ is a graded subbimodule of $\mathfrak{gr}\mathrm{Hom}_k(L, M)$.

Note that when the group Γ is trivial, we get our previous notions of differential and strongly differential bimodules and operators.

The correspondence $R \mapsto D_\beta(R)$ depends naturally on the grading group Γ. Namely, we have the following

1.2.2. Proposition. *Suppose that Γ' is a subgroup of the group Γ, and β' is the restriction to $\Gamma' \times \Gamma'$ of the bicharacter $\beta : \Gamma \times \Gamma \to k^*$. Then the subring of $D^{\#}_{\beta'}(R)$ generated by Γ-graded endomorphisms is a subring of $D^{\#}_\beta(R)$.*

We have the following graded analog of Proposition 1.1.6.2:

1.2.3. Proposition. *Let $R \longrightarrow R'$ be a graded ring morphism such that the functor*

$$R' \otimes_R : \mathfrak{gr}_\Gamma R - mod \longrightarrow \mathfrak{gr}_\Gamma R' - mod$$

is an exact localization (for instance, the ring R' is the localization of R at a left Ore set consisting of homogeneous elements). Then

(a) The action of $D^{\#s}_\beta(R)$ on R extends naturally to an action on R' giving a canonical ring homomorphism

$$D^{\#s}_\beta(R) \longrightarrow D^{\#s}_\beta(R')$$

which induces an isomorphism of left R'-modules

$$R' \otimes_R D_\beta^{\#_s}(R) \longrightarrow D_\beta^{\#_s}(R').$$

(b) For any $D_\beta^{\#_s}(R)$-module M, the R'-module $R' \otimes_R M$ has a natural, in particular compatible with $D_\beta^{\#_s}(R) \longrightarrow D_\beta^{\#_s}(R')$, structure of a $D_\beta^{\#_s}(R')$-module.

(c) If the ring R' is such that the functor $\otimes_R R' : \mathfrak{gr}_\Gamma mod - R \longrightarrow \mathfrak{gr}_\Gamma mod - R'$ is a localization (e.g. R' is the localization of R at a left and right homogeneous Ore set), then we also get an induced right R'-module isomorphism

$$D_\beta^{\#_s}(R) \otimes_R R' \longrightarrow D_\beta^{\#_s}(R').$$

The above definition of differential operators is quite satisfactory in the skew commutative situation as is shown in the following example.

1.2.4. Example: differential operators on a skew affine space. Let $\mathbf{q} = (q_{ij})$ be an $r \times r$ matrix with $q_{ij} \in k^*$ such that $q_{ji}q_{ij} = 1$ for all i, j. Let R be the corresponding skew polynomial k-algebra; i.e. R is generated by $x_1, \dots, x_r$ subject to the following relations:

$$x_i x_j = q_{ij} x_j x_i \quad \text{for all} \quad i,\ j. \tag{1}$$

The algebra R is regarded as the algebra of regular functions on the *skew* (more specifically, $\mathbf{q}-$)*affine space.*

Take $\Gamma = \mathbb{Z}^r$, and β the bicharacter $\Gamma \times \Gamma \longrightarrow k^*$ determined by the matrix $\mathbf{q}$. Assume that k is a field of characteristic zero. One can show (Proposition 8.4) that the ring of graded differential operators, $D_\beta(R) = D_\mathbf{q}(R)$ is generated by left multiplication by elements of R and by corresponding β-derivations. This implies that $D_\mathbf{q}(R)$ coincides with the ring $D_\mathbf{q}^{\#_s}(R)$ of graded strongly differential operators.

1.3. Taking into account a canonical action of Γ. It was one of our first conclusions that, in the graded situation, all 'schemes' should be considered over the group algebra $k[\Gamma]$. Therefore a more natural definition of differential operators on R should include the action $k[\Gamma]$.

1.3.1. Definition of β-differential operators. Set for convenience $R_\Gamma := R\#k[\Gamma]$. Note that any graded R-module L is automatically a graded R_Γ-module and any graded $R \otimes R^o$-module is a graded $R_\Gamma \otimes R_\Gamma^o$-module.

We define differential and strongly differential R_Γ-bimodule as in 1.1.1 with 'modules' replaced by 'graded modules' and R replaced by R_Γ.

For any graded R-modules L and N, $Diff_\beta(L, N)$ is the (graded) differential part of the graded $R_\Gamma \otimes R_\Gamma^o$-module $\mathfrak{gr}\mathrm{Hom}_k(L, N)$. In particular, we obtain the (graded) ring of *β-differential operators*, $D_\beta(R)$, acting on the graded ring R.

1.3.2. Proposition. *The algebra $D_\beta(R)$ is the subalgebra of $End_k(R)$ generated by $D_\beta^{\#}(R)$ and $k[\Gamma]$.*

Similarly, the algebra $D_\beta^s(R)$ of strongly differential operators on R is the subalgebra of $End_k(R)$ generated by $D_\beta^{\#_s}(R)$ and $k[\Gamma]$.

In particular, we have a direct analog (and consequence) of Proposition 1.2.2:

1.3.2.1. Proposition. *Suppose that Γ' is a subgroup of the group Γ, and β' is the restriction to $\Gamma' \times \Gamma'$ of the bicharacter $\beta : \Gamma \times \Gamma \to k^*$. Then the subring of $D_{\beta'}(R)$ generated by Γ-graded endomorphisms is a subring of $D_\beta(R)$.*

1.3.3. Localization. Proposition 1.2.3 above holds literally with $D_\beta^{\#^{\mathfrak{s}}}$ replaced by $D_\beta^{\mathfrak{s}}$.

1.3.4. Main example. Let $(a_{ij})_{1\leq i,j\leq r}$ be a Cartan matrix of finite type. Let $\mathfrak{g}$ be the corresponding semisimple Lie algebra, P the weight lattice. Let $Q \subseteq P$ be the root lattice, and $\alpha_1, \ldots, \alpha_r$ a basis of simple roots. Let q be an indeterminate, k a field containing $\mathbb{Q}(q)$ and all roots of q. Let the set $\{d_i\}$, $d_i \in \{1,2,3\}$ for all i, be such that the matrix $(a_{ij}d_i)$ is symmetric. Let $\langle , \rangle : P \times P \longrightarrow \mathbb{Q}$ be a nondegenerate symmetric pairing determined by $\langle \alpha_i, \alpha_j \rangle = a_{ij}d_i$. Take $\Gamma = P$, and define a bicharacter $\beta : \Gamma \times \Gamma \longrightarrow k^*$ as $\beta(\gamma,\sigma) = q^{\langle\gamma,\sigma\rangle}$.

Let $U_q = U_q(\mathfrak{g})$ be the quantized enveloping algebra corresponding to the Cartan matrix (a_{ij}). Then U_q admits a triangular decomposition $U_q = U^- \otimes_k U^o \otimes_k U^+$, where $U^o \simeq k[Q]$ – the group algebra of Q. The algebra U_q is naturally Q-graded, hence it is Γ-graded. The action of $k[Q]$ induced by the grading and the bicharacter β is nothing but the adjoint action of U^o on U_q.

Denote by U_q' the subalgebra of U_q generated by U^- and U^+.

For any Γ-graded k-algebra R, we put $D_q^{\#}(R) := D_\beta^{\#}(R)$, $D_q^{\#^{\mathfrak{s}}}(R) := D_\beta^{\#^{\mathfrak{s}}}(R)$ and $D_q(R) := D_\beta(R)$, $D_q^{\mathfrak{s}}(R) := D_\beta^{\mathfrak{s}}(R)$, and call these algebras respectively *the ring of graded quantum differential and strongly differential operators on* R and *the ring of quantum differential and strongly differential operators on* R.

1.3.4.1. Note. In practice, we usually meet graded algebras R which are defined over $\mathbb{Q}(q)$ and such that the Γ-action on R is defined over $\mathbb{Q}(q)$. In this case one can take $k = \mathbb{Q}(q)$. ∎

One of important facts of this paper is the following quantum analog of Proposition 1.1.5 above:

1.3.4.2. Proposition. *Let R be* a *Γ-graded k-algebra with a Hopf action of U_q such that the canonical action of $k[\Gamma]$ on R when restricted to $k[Q]$ coincides with the $U^o(\simeq k[Q])$-action. Then*

(a) The algebra U_q acts on R by quantum strongly differential operators; i.e. the action is given by a homomorphism $U_q \longrightarrow D_q^{\mathfrak{s}}(R)$.

(b) The subalgebra U_q' acts on R by graded quantum strongly differential operators; i.e. the action is given by a homomorphism $U_q' \longrightarrow D_q^{\#^{\mathfrak{s}}}(R)$.

1.3.4.3. Corollary. *Let R be as in Proposition 1.3.4.2; and let $R \longrightarrow R'$ be a graded Ore localization. Then the action of U_q on R extends uniquely to a Hopf action on R'.*

1.3.4.4. Remarks. 1) Proposition 1.3.4.2 and Corollary 1.3.4.3 are essential for our localization construction for quantized enveloping algebras.

2) It is worth to mention that the proofs of these (and a number of other) statements do not belong to the world of rings and ideals. And even a proper formulation of the localization assertions requires a richer envirement of abelian categories and monads. ∎

1.3.5. Example: algebras of differential operators on the quantum line. To illustrate the difference between $D^\#_\beta$ and D_β, consider the simplest possible example of a 'noncommutative space' – the 'quantum line'.

Let k be a field of characteristic zero. The algebra of functions on a quantum line over k is the algebra $R = k[x]$ of polynomials in one variable regarded as an algebra in the category $\mathfrak{gr}_{\mathbb{Z}} Vec_k$ of $\mathbb{Z}$-graded k-vector spaces – the parity of x is 1). We define the bicharacter β by (the necessary conditions) $\beta(1,0) = 1 = \beta(0,1)$, and $\beta(1,1) = q$ for some $q \in k^*$. We assume that q is not a root of one. Then the algebra $D^\#_\beta = D^\#_q$ is generated by (multiplications by) R and the β-derivation $\partial = \partial_q$ (see Example 1.2.4). In particular it coincides with the algebra $D^{\#s}_q$ of strongly differential graded operators.

Note that the β-derivation $\partial = \partial_q$ happens to be the so called *q-derivation* – an operator acting on polinomials by the formula:

$$\partial = \partial_q : f(x) \mapsto (f(qx) - f(x))/x(q-1) \tag{1}$$

– known for at least a hundred years.

Thus $D^\#_q(R)$ is a k-algebra generated by x and ∂ subject to the relation:

$$\partial x - qx\partial = 1. \tag{2}$$

When $q = 1, D^\#_q(R)$ is the first Weyl algebra, $D^\#_1 = A_1$; i.e. it is isomorphic to the algebra of differential operators on the one-dimensional affine space. A remarkable property of the Weyl algebras is the Bernstein's Theorem (cf. [B]) which in the case of A_1 claims that any nonzero A_1-module is of infinite dimension over k. This property does not hold for $D^\#_q(R)$ if $q \neq 1$.

Indeed, one can check that the left ideal μ of the algebra $D^\#_q(R)$ generated by the element $\eta = \partial x - 1/(1-q)$ is two-sided, and the corresponding quotient algebra, $D^\#_q(R)/\mu$, is (isomorphic to) the commutative algebra of functions on the hyperbola given by the equation $\partial x = 1/(1-q)$. In other words, $D^\#_q(R)/\mu$ is isomorphic to the algebra of Laurent polynomials $k[x, x^{-1}]$ in one variable. In particular, the algebra $D^\#_q(R)$ has a parametrized by k^* family of one-dimensional representations. Note however that if M is a finite dimensional $D^\#_q(R)$-module, then it is annihilated by the ideal μ (see [R], II.4).

Consider now the algebra $D_q(R)$ defined in 1.3.1. It is generated by $D^\#_q(R)$ and the automorphism h sending x into qx and the inverse to h. We claim that the Bernstein's property – any nonzero $D_\beta(R)$-module is infinite dimensional – holds.

In fact, one can check that

$$x\partial(x^n) = x^n(q^n - 1)/(q-1), \quad \text{and} \quad h(x^n) = q^n x^n;$$

for all n; i.e. $x\partial = (h-1)/(q-1)$, or

$$h = (q-1)x\partial + 1 \in D^\#_\beta(R). \tag{3}$$

But, of course, $h^{-1} \notin D^\#_\beta(R)$. Let M be a $D_\beta(R)$-module of finite k-dimension. Then, since M is finite dimensional $D^\#_\beta(R)$-module, it is annihilated by the element $\eta =$

$\partial x - 1/(1-q)$ (cf. [R], II.4), hence by the left ideal generated by η. But, it follows from (3) that the left ideal generated by η coincides with the whole algebra $D^{\#}_{\beta}(R)$. So that the module M is zero.

Note that the algebra $D_\beta(R)$ is isomorphic to the (Ore) localization of the algebra $D^{\#}_{\beta}(R)$ at the multiplicative set (η) generated by the normal element η. One can deduce from this fact that the algebra $D_\beta(R)$ enjoys the same nice properties as the first Weyl algebra A_1: its Krull, homological, and Gelfand-Kirillov dimensions coincide and are equal to 1.

We shall show in one of the subsequent papers that the coincidence of the three dimensions, and the Bernstein's property hold for the algebra of β-differential operators $D_\beta(R)$, where R is a skew polynomial algebra of any dimension.

2. Preliminaries: topologizing subcategories as subschemes.

2.1. From schemes to categories. The first natural step on the way of finding non-commutative analogs of constructions and notions of commutative algebraic geometry is to identify schemes with the categories of quasi-coherent sheaves on these schemes. If Y is a closed subscheme of a scheme X, than the category $\mathbb{Y}$ of quasi-coherent sheaves on Y is (identified with) a certain subcategory of the category $\mathbb{X}$ of quasi-coherent sheaves on X. The subcategory $\mathbb{Y}$ has the following properties:

(a) It is full and closed with respect to finite direct sums (taken in $\mathbb{X}$);

(b) With any object, it contains all its subquotients (taken in $\mathbb{X}$);

(c) The subcategory $\mathbb{Y}$ is *coreflective* which means that the inclusion functor $\mathbb{Y} \longrightarrow \mathbb{X}$ has a right adjoint.

(d) The subcategory $\mathbb{Y}$ is *reflective*, i.e. the inclusion functor $\mathbb{Y} \longrightarrow \mathbb{X}$ has a left adjoint.

The right adjoint functor of (c) assigns to any object of $\mathbb{X}$ its biggest subobject from $\mathbb{Y}$. The left adjoint functor of (d) is the tensoring over $\mathcal{O}_X$ by $\mathcal{O}_X/\mathcal{I}$, where $\mathcal{I}$ is the defining ideal of the subscheme Y.

A subcategory $\mathbb{Y}$ of an abelian category $\mathbb{X}$ satisfying the conditions (a) and (b) is called *topologizing*.

If X is an affine scheme, i.e. X=SpecR for a commutative ring R, then $\mathbb{X}$ is the category $R-mod$ of left R-modules. And $\mathbb{Y}$ is generated by all R-modules annihilated by some ('defining') ideal $\mathcal{I} \subset R$. The left adjoint functor of (d) is the tensoring by $R/\mathcal{I}$ over R. In other words, it sends every R-module M into $M/\mathcal{I}M$.

There is the following fact ([R], Section III.6).

2.2. Proposition. *For an arbitrary associative ring R, there is one-to-one correspondence between reflective topologizing subcategories of $R-mod$ and two-sided ideals of the ring R: to any two-sided ideal α , there corresponds the full subcategory $[R/\alpha]$ generated by all modules annihilated by α. In particular, any reflective topologizing subcategory of $R-mod$ is coreflective.*

2.3. Subschemes. Fix an abelian category $\mathcal{A}$. We shall call topologizing coreflective subcategories of $\mathcal{A}$ *subschemes of $\mathcal{A}$*.

A subscheme shall be called *Zariski closed* (or simply *closed*), if it is a reflective subcategory of $\mathcal{A}$.

Proposition 2.2 shows the category $R - mod$ could have very few Zariski closed subschemes. For instance, if $\mathcal{A} = R - mod$ for a simple ring R (say, a Weyl algebra), then there are only two trivial Zariski closed subcategories: $\mathbf{0}$ and $R - mod$.

On the other hand, there are, usually, lots of non-closed subschemes as the following subsection shows.

2.4. Subschemes of the category of modules. Let $\mathcal{A}$ be the category $R - mod$ of left modules over an associative ring R. And let $\mathbb{T}$ be any topologizing subcategory of $\mathcal{A}$. Denote by $F_{\mathbb{T}}$ the set of all left ideals m in R such that $R/m \in Ob\mathbb{T}$.

Conversely, for any set F of left ideals in R, denote by $\mathbb{T}_F$ the full subcategory of $R - mod$ generated by all modules M such that, for any $z \in M$, $Ann(z) \in$F.

2.4.1. Lemma. *1) For any topologizing subcategory $\mathbb{T}$ of $R - mod$, the set $F = F_{\mathbb{T}}$ has the following properties:*

(a) $m, n \in F$ implies that $m \cap n \in F$;

(b) if $m \in F$, then any left ideal n containing m belongs to F;

(c) for any $m \in F$ and any finite subset x of elements of R, $(m : x) \in$F.

2) If F is a subset of the set I_lR of left ideals of R having the properties (a), (b), (c), then the subcategory $\mathbb{T}_F$ is topologizing and coreflective.

Proof. 1) *(a)* is a consequence of the fact that the quotient module $R/m \cap n$ is a submodule of the direct sum $R/m \oplus R/n$.

(b) The module R/n is a quotient of R/m; hence $R/n \in Ob\mathbb{T}$ together with R/m.

(c) Let u denote the image of the identity element in R/m. The left ideal $(m : x)$ is the annihilator of the element $\oplus_{r \in x} ru$ of the direct sum of $\mid x \mid$ copies of R/m; hence $R/(m : x)$, being a submodule of a module from $\mathbb{T}$, belongs to $\mathbb{T}$.

2) For any module M the set $M_F := \{z \in M \mid Ann(z) \in F\}$ is a submodule.

In fact, for any $z, z' \in M$ and any $r \in R$, we have:

$$Ann(z + z') \supseteq Ann(z) \cap Ann(z'), \text{ and } Ann(rz) = (Ann(z) : r).$$

Clearly M_F is the largest submodule of M which belongs to $\mathbb{T}_F$. This means that the subcategory $\mathbb{T}_F$ is coreflective.

If $M, M' \in Ob\mathbb{T}_F$, then $M \oplus M' \in Ob\mathbb{T}_F$, since for any two elements $z \in M$ and $z' \in M'$, the annihilator of $z \oplus z'$ equals to the intersection of $Ann(z)$ and $Ann(z')$. Clearly any subobject of an object of $\mathbb{T}_F$ belongs to $\mathbb{T}_F$. Finally, a quotient of any object of $\mathbb{T}_F$ belongs to $\mathbb{T}_F$. So that the subcategory $\mathbb{T}_F$ is topologizing. ∎

The sets F of left ideals satisfying the conditions of Lemma 2.4.1 are called *topologizing filters.*

2.4.2. Note. For any topologizing subcategory $\mathbb{T}$, the subcategory $\mathbb{T}_F$, where $F = F_{\mathbb{T}}$ is the set $\{m \in I_lR \mid R/m \in Ob\mathbb{T}\}$ is the intersection of all coreflective topologizing subcategories of $R - mod$ containing $\mathbb{T}$. ∎

2.4.3. Example. Fix an associative ring R. Let $\leq$ denote a preorder in the set I_lR of left ideals in R defined as follows: $m \leq n$ if there exists a finite subset x of elements of R such that $(m : x) := \{r \in R \mid rx \subset m\} \subseteq$n.

Let m be any left ideal in R. Denote by $[R/m]$ the full subcategory of $\mathcal{A}$ generated by all modules M such that, for any $z \in M$, $m \leq Ann(z)$. One can check that the subcategory $[R/m]$ is topologizing and coreflective. Moreover, $[R/m]$ is the smallest coreflective topologizing subcategory of $\mathcal{A}$ containing the module R/m. One can see that $[R/m] = \mathbb{T}_{[m]}$, where $[m] := \{n \in I_l R \mid m \leq n\}$.

The topologizing subcategories $\mathbb{T}_{[m]}$ are minimal in the following sense: for any topologizing filter F of left ideals in R, $\mathbb{T}_F = \bigcup_{m \in F} \mathbb{T}_{[m]}$. ∎

2.5. Subschemes of an abelian category. We shall call coreflective topologizing subcategories of the category $\mathcal{A}$ *subschemes of* $\mathcal{A}$. The subschemes which are also reflective subcategories shall be called *Zariski closed* or simply *closed* if this does not create any ambiguity.

Note that if the ring R is simple (like algebras of differential operators with polynomial coefficients), there are only trivial Zariski closed subschemes of $R - mod$. While we have lots of subschemes.

From now on we shall assume that the abelian categories under consideration have the property

(sup) For any ascending chain Ω of subobjects of an object M, the supremum of Ω exists; and, for any subobject L of M, the natural morphism

$$\sup\{X \cap L \mid X \in \Omega\} \longrightarrow (\sup \Omega) \cap L$$

is an isomorphism.

Examples of the categories with the property (sup):

1) The category $R - mod$ of left modules over an associative ring R.
2) The category of sheaves of R-modules on an arbitrary topological space.
3) The category of quasi-coherent sheaves on an arbitrary scheme.
4) Any noetherian abelian category.

2.5.1. Note. Recall that the property (sup) is a part of the definition of Grothendieck categories examples of which are categories $R - mod$ for all rings and categories of abelian sheaves on topological spaces (Examples 1) and 2) above).

Apparently, it is not known if the category $\mathfrak{Qcoh}_X$ of quasi-coherent sheaves on an arbitrary scheme X is a Grothendieck category (cf. the Appendix B in [ThT]). One can easily check, however, that $\mathfrak{Qcoh}_X$ has the property (sup). ∎

Example 2.4.3 is extended to any abelian category $\mathcal{A}$ with the property (sup) as follows.

For two objects, X and Y of $\mathcal{A}$, we write $X \succ Y$ if Y is a subquotient of a finite direct sum of copies of X. One can check that the relation $\succ$ is a preorder (cf. [R], III.1). For any object V of $\mathcal{A}$, denote by $V_\succ$ the full subcategory of $\mathcal{A}$ generated by all objects X such that $V \succ X$.

Note that $V_\succ$ is topologizing, since it is closed under finite direct sums; and if $X \in ObV_\succ$ and $X \succ Y$, then $Y \in ObV_\succ$. But, in general, the subcategory $V_\succ$ is not coreflective. The full subcategory $[V]$ of $\mathcal{A}$ generated by all $X \in Ob\mathcal{A}$ which are supremums of their subobjects from $V_\succ$ is both topologizing and coreflective.

Note that any coreflective topologizing subcategory $\mathbb{T}$ of $\mathcal{A}$ can be represented as $\bigcup_{V \in \mathfrak{X}}[V]$, where $\mathfrak{X}$ is a class of objects of $\mathbb{T}$ having the property: for any $Y \in Ob\mathbb{T}$, there exists $X \in \mathfrak{X}$ such that $X \succ Y$.

Clearly $V \succ W$ if and only if $W_\succ \subseteq V_\succ$. In particular, the subcategories $V_\succ$ and $[V]$ depend only on the equivalence class $\langle V \rangle$ of the object V with respect to $\succ$.

The following remark is independent on the main body of the text.

2.5.2. Remark: relations with the spectrum. For the notion of the spectrum of an abelian category and related notions used below, the reader is referred to [R], Chapter III, or to [R1]. One can see that $\mathbf{Spec}[V] = \mathbf{Spec}V_\succ = Supp(V)$. So if $V \in Spec\mathcal{A}$, then $\mathbf{Spec}V_\succ$ is the set of all specializations of $\langle V \rangle$.

Assume that the category $\mathcal{A}$ has no nonzero objects with empty support. In this case, if $V \in Spec\mathcal{A}$ and is a closed point, then all nonzero objects of $V_\succ$ are equivalent to V.

A nonzero object V of an abelian category $\mathcal{A}$ is called *quasifinal* if $X \succ V$ for any nonzero object X of $\mathcal{A}$. A category $\mathcal{A}$ having a quasifinal object is called *local*. Recall that all simple objects (if any) of a local category are isomorphic one to another and are quasifinal (cf. [R], Lemma III.3.1.2). In particular, any nonzero object of $\mathcal{A}$ has simple subquotients.

If $\mathcal{A}$ is a local category and V is a quasifinal object, then $[V]$ is called the residue category of $\mathcal{A}$. If V is a simple object, the residue category is equivalent to the category of vector spaces over residue skew field of $K(\mathcal{A}) = End(V)$ (cf. [R], III.5.4). ∎

If $\mathcal{A} = R - mod$ and $V = R/m$ for some left ideal m in R, the subcategory $[V]$ coincides with the subcategory $[R/m]$ of Example 2.4.3.

2.6. Serre subcategories. Recall that a full subcategory $\mathbb{T}$ of an abelian category $\mathcal{A}$ is called *thick* if, for any exact sequence $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ in $\mathcal{A}$, the object M belongs to $\mathbb{T}$ iff M' and M'' belong to $\mathbb{T}$. In other words, $\mathbb{T}$ is thick iff it is topologizing and closed under extensions.

For any subcategory $\mathbb{S}$ of $\mathcal{A}$, denote by $\mathbb{S}^-$ the full subcategory of $\mathcal{A}$ generated by all objects M such that any nonzero subquotient of M has a nonzero subobject from $\mathbb{S}$.

2.6.1. Lemma. *For any subcategory* $\mathbb{S}$ *of an abelian category* $\mathcal{A}$,

(a) the subcategory $\mathbb{S}^-$ *is thick;*

(b) $(\mathbb{S}^-)^- = \mathbb{S}^-$*;*

(c) $\mathbb{S} \subseteq \mathbb{S}^-$ *if* $\mathbb{S}$ *is topologizing.*

Proof. See Lemma III.2.3.2.1 in [R]. ∎

The subcategory $\mathbb{S}$ of $\mathcal{A}$ is called *a Serre subcategory* if $\mathbb{S} = \mathbb{S}^-$.

The proof of the following observation is left as an exercise for a reader:

2.6.2. Lemma. *Suppose that* $\mathcal{A}$ *has supremums of subobjects (which is the case if, for example,* $\mathcal{A}$ *has small direct sums), then a thick subcategory* $\mathbb{T}$ *is a Serre subcategory iff it is a subscheme.*

2.7. Operations on subschemes. Fix an abelian category $\mathcal{A}$. We shall assume whenever it is required (in particular through this subsection) that $\mathcal{A}$ has the property (sup) (cf. Subsection 2.5).

2.7.1. Lemma. *(a) The intersection of any set of subschemes of $\mathcal{A}$ is a subscheme.*

(b) The intersection of any set of Zariski closed subschemes of $\mathcal{A}$ is a Zariski closed subscheme.

Proof. *(a)* Clearly the intersection of any set of topologizing subcategories is a topologizing subcategory. Similarly, the intersection of any family $\mathfrak{X}$ of coreflective subcategories is a coreflective subcategory.

In fact, let Ω be a family of subobjects of an object Y which belong to the intersection $\bigcap_{\mathbb{S}\in\mathfrak{X}}\mathbb{S}$. Since each of the subcategories $\mathbb{S}\in\mathfrak{X}$ is coreflective, $\sup\Omega$ belongs to this intersection too. This implies the coreflectivity of $\bigcap_{\mathbb{S}\in\mathfrak{X}}\mathbb{S}$.

(b) Let now $\mathfrak{F}$ be a family of Zariski closed subschemes. And let, for any $\mathbb{T}$ in $\mathfrak{F}$, $\hat{}J_{\mathbb{T}}$ denote a left adjoint to the inclusion $J_{\mathbb{T}}:\mathbb{T}\longrightarrow\mathcal{A}$, and $\eta_{\mathbb{T}}$ the adjunction arrow $Id_{\mathcal{A}}\longrightarrow J_{\mathbb{T}}\circ\hat{}J_{\mathbb{T}}$. Let $K_{\mathbb{T}}$ denote the kernel of $\eta_{\mathbb{T}}$. Note that $\eta_{\mathbb{T}}$ is an epimorphism; so that $J_{\mathbb{T}}\circ\hat{}J_{\mathbb{T}}\simeq Cok(\eta_{\mathbb{T}})$. Set $K\mathfrak{F}:=\sup\{K_{\mathbb{T}}\mid\mathbb{T}\in\mathfrak{F}\}$. For any $M\in Ob\mathcal{A}, M/K\mathfrak{F}(\mathrm{M})$ is a quotient of $M/K_{\mathbb{T}}(M)$ for any $\mathbb{T}\in\mathfrak{F}$; hence it is an object of $\bigcap_{\mathbb{T}\in\mathfrak{F}}\mathbb{T}$. Conversely, if Y is an object of $\bigcap_{\mathbb{T}\in\mathfrak{F}}\mathbb{T}$, then an arbitrary morphism $f:M\longrightarrow Y$ factors by $M\longrightarrow M/K_{\mathbb{T}}(M)$. So that Kerf 'contains' $K\mathfrak{F}(\mathrm{M})$. All together shows that the map $M\mapsto M/K\mathfrak{F}(\mathrm{M})$ extends to a left adjoint to the inclusion functor $\bigcap_{\mathbb{T}\in\mathfrak{F}}\mathbb{T}\longrightarrow\mathcal{A}$; i.e. $\bigcap_{\mathbb{T}\in\mathfrak{F}}\mathbb{T}$ is a reflective subcategory of $\mathcal{A}$. ■

2.7.2. The supremum of subschemes. The *supremum*, $\sup\mathfrak{F}$, of a family $\mathfrak{F}=\{\mathbb{S}_i\mid i\in J\}$ of subschemes is the smallest subscheme of $\mathcal{A}$ containing all the subschemes of the family $\mathfrak{F}$.

Let $\{\mathbb{S}_i\mid i\in J\}$ be any family of topologizing subcategories of $\mathcal{A}$. Then the smallest topologizing subcategory containing all the subcategories $\mathbb{S}_i$ equals to the union of the subcategories $X_{\succ}$, where X runs through $\oplus_{i\in J}X_i$ in which $X_i\in Ob\mathbb{S}_i$ for all $i\in J$, and only finite number of X_i are nonzero. If all the subcategories $\mathbb{S}_i$ are coreflective and arbitrary direct sums $\oplus_{i\in J}X_i$, $X_i\in Ob\mathbb{S}_i$, exist, then we have an analogous description of the smallest subscheme $\mathbb{S}$ containing all $\mathbb{S}_i$: the subcategory $\mathbb{S}$ is the union of the subcategories $[X]$, where X runs through all sums $\oplus_{i\in J}X_i$ with $X_i\in Ob\mathbb{S}_i$.

Note that 'all sums' in this description can be replaced by the requirement $X_i\in\Xi_i$, where Ξ_i is a set of objects of $\mathbb{S}_i$ such that $\mathbb{S}_i=\bigcup_{Y\in\Xi_i}[Y]$.

For instance, if $\mathbb{S}_i=[X_i]$ for some $X_i\in Ob\mathcal{A}$, $i\in J$, then $\sup\{\mathbb{S}_i\mid i\in J\}=[\oplus_{i\in J}X_i]$.

2.7.2.1. Lemma. *The supremum of a finite number of Zariski closed subschemes is a Zariski closed subscheme.*

Proof. We shall use the notations of the argument of Lemma 2.7.1.

Let $\mathfrak{J}$ be a finite family of Zariski closed subschemes of $\mathcal{A}$. Denote by $K_{\mathfrak{J}}$ the functor which assigns to any $M\in Ob\mathcal{A}$ the intersection $\bigcap_{\mathbb{T}\in\mathfrak{J}}K_{\mathbb{T}}(M)$. Since $\mathfrak{J}$ is finite, $M/K_{\mathfrak{J}}(M)$ is a subobject of $\oplus_{\mathbb{T}\in\mathfrak{J}}M/K_{\mathbb{T}}(M)=\oplus_{\mathbb{T}\in\mathfrak{J}}\hat{}J_{\mathbb{T}}(M)$. Denote by $\Psi_{\mathfrak{J}}$ the (uniquely defined) extension of the map $M\mapsto M/K_{\mathfrak{J}}(M)$ to a functor from $\mathcal{A}$ to $\mathcal{A}$. Since the direct sum $\oplus_{\mathbb{T}\in\mathfrak{J}}M/K_{\mathbb{T}}(M)$ is an object of $\sup\mathfrak{J}$, the functor $\Psi_{\mathfrak{J}}$ takes values in the subcategory $\sup\mathfrak{J}$. On the other hand, if $M\in Ob(sup\mathfrak{J})$, then $K_{\mathfrak{J}}(M)=0$; i.e. the natural epimorphism $M\longrightarrow\Psi_{\mathfrak{J}}(M)$ is an isomorphism. This shows that $\Psi_{\mathfrak{J}}$ is left adjoint to the inclusion $\sup\mathfrak{J}\longrightarrow\mathcal{A}$. ■

2.7.3. Gabriel multiplication. For any two subcategories $\mathbb{S}, \mathbb{T}$ of an abelian category $\mathcal{A}$, define their product $\mathbb{S} \bullet \mathbb{T}$ as the full subcategory of $\mathcal{A}$ generated by all objects M of $\mathcal{A}$ such that there exists an exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

with $M' \in Ob\mathbb{T}$ and $M'' \in Ob\mathbb{S}$.

2.7.3.1. Lemma. *(a) If* $\mathbb{S}$ *and* $\mathbb{T}$ *are topologizing subcategories, then such is* $\mathbb{S} \bullet \mathbb{T}$.
(b) For any topologizing subcategories $\mathbb{S}$, $\mathbb{T}$, $\mathbb{U}$ *of* $\mathcal{A}$,

$$\mathbb{S} \bullet (\mathbb{T} \bullet \mathbb{U}) = (\mathbb{S} \bullet \mathbb{T}) \cdot \mathbb{U} \quad \textit{and} \quad \mathbf{0} \bullet \mathbb{S} = \mathbb{S} = \mathbb{S} \bullet \mathbf{0}.$$

Proof is left to a reader. ■

It follows from the definition of $\bullet$ that a topologizing category $\mathbb{T}$ is thick iff $\mathbb{T}$ is idempotent: $\mathbb{T} = \mathbb{T} \bullet \mathbb{T}$.

2.7.3.2. Lemma. *For any topologizing subcategories* $\mathbb{S}$, $\mathbb{T}$, *and* $\mathbb{X}$,

$$(\mathbb{S} \bullet \mathbb{T}) \cap (\mathbb{S} \bullet \mathbb{X}) = \mathbb{S} \cdot (\mathbb{T} \cap \mathbb{X}) \quad \textit{and} \quad (\mathbb{T} \bullet \mathbb{S}) \cap (\mathbb{X} \bullet \mathbb{S}) = (\mathbb{T} \cap \mathbb{X}) \bullet \mathbb{S}.$$

Proof. Clearly $(\mathbb{S} \bullet \mathbb{T}) \cap (\mathbb{S} \cdot \mathbb{X}) \supseteq \mathbb{S} \bullet (\mathbb{T} \cap \mathbb{X})$. To show the inverse inclusion, pick any object M of $(\mathbb{S} \bullet \mathbb{T}) \cap (\mathbb{S} \bullet \mathbb{X})$. By definition of $\bullet$, there exist exact sequences

$$0 \longrightarrow T \longrightarrow M \longrightarrow S \longrightarrow 0 \text{ and } 0 \longrightarrow X \longrightarrow M \longrightarrow S' \longrightarrow 0,$$

where $T \in Ob\mathbb{T}$, $X \in Ob\mathbb{X}$, $S \in Ob\mathbb{S} \ni S'$. Since the sequence

$$0 \longrightarrow T \cap X \longrightarrow M \longrightarrow S \oplus S'$$

is exact, and $S \oplus S' \in \mathbb{S}$, the quotient object $M/(T \cap X)$, being a subobject of $S \oplus S'$, belongs to $\mathbb{S}$. While $T \cap X$, being a subobject of T and X, belongs to $\mathbb{T} \cap \mathbb{X}$.
The second assertion coincides with the first one in the dual category. ■

Thus, topologizing subcategories form a semiring with a commutative operation $\cap$ and a noncommutative operation $\bullet$; i.e. $\cap$ might be thought as an addition, and $\bullet$ as a multiplication.

The following assertion shows that the Gabriel multiplication is compatible with our notions of a subscheme and a Zariski closed subscheme.

2.7.3.3. Lemma. *If topologizing subcategories* $\mathbb{S}$ *and* $\mathbb{T}$ *of* $\mathcal{A}$ *are reflective (resp. coreflective), then such is* $\mathbb{S} \bullet \mathbb{T}$.

Proof is that of Lemma III.6.2.1 in [R]. ■

2.7.3.1. Example. Let $\mathcal{A} = R - mod$ for some ring R. Let $\mathbb{S}$ and $\mathbb{T}$ be subschemes of $\mathcal{A}$ defined by left ideals resp. $\mathcal{J}$ and $\mathcal{I}$: $\mathbb{S} = [R/\mathcal{J}], \mathbb{T} = [R/\mathcal{I}]$. If the subschemes $\mathbb{S}$ and

$\mathbb{T}$ are closed, i.e. the ideals $\mathcal{J}$ and $\mathcal{I}$ are two-sided (cf. Proposition 2.2), then $\mathbb{S} \bullet \mathbb{T}$ is a closed subscheme with the defining ideal $\mathcal{J}\mathcal{I} : \mathbb{S} \cdot \mathbb{T} = [R/\mathcal{J}\mathcal{I}] \simeq R/\mathcal{J}\mathcal{I} - mod$.

If $\mathbb{S}$ and|or $\mathbb{T}$ are not closed, $\mathbb{S} \cdot \mathbb{T}$ is not, in general, of the form $[R/m]$ for some left ideal m. ∎

2.7.4. The n-th neighborhood of a topologizing subcategory. Given a topologizing subcategory $\mathbb{T}$ of $\mathcal{A}$, define the n-th neighborhood of $\mathbb{T}$ as the n-th power of $\mathbb{T}$:

$$\mathbb{T}^{(n)} := \mathbb{T} \bullet \ldots \bullet \mathbb{T} \ (n \text{ times}).$$

One can check that $\mathbb{T}^{(\infty)} := \bigcup_{n\geq 1} \mathbb{T}^{(n)}$ is a thick subcategory of $\mathcal{A}$ which coincides with the intersection of all thick subcategories containing $\mathbb{T}$.

2.7.4.1. Remark. If $\mathbb{T}$ is a subscheme (resp. a closed subscheme), then, by Lemma 2.7.3.3, all the subcategories $\mathbb{T}^{(n)}$ are (resp. closed) subschemes. However, the thick subcategory $\mathbb{T}^{(\infty)}$ is not, in general, a subscheme.

On the other hand, the minimal subscheme $\mathbb{T}^{\infty} = [\mathbb{T}^{(\infty)}]$ containing $\mathbb{T}^{(\infty)}$ is not, in general, thick any more. We shall see that $\mathbb{T}^{\infty}$ is thick (hence Serre) if the category $\mathcal{A}$ is locally noetherian (Proposition 3.1.2.1). ∎

3. Relative differential calculus in abelian categories.

Fix an abelian category $\mathcal{A}$ with the property (sup) and its subscheme $\mathbb{T}$.

3.1. $\mathbb{T}$-filtrations and $\mathbb{T}$-objects. Fix an object M of $\mathcal{A}$. An increasing filtration $M_. = \{M_i \mid i \geq -1\}$ on M is a $\mathbb{T}$-*filtration* if $M_{-1} = 0$, and $M_i/M_{i-1} \in Ob\mathbb{T}$ for any $i \geq 0$.

Thanks to the coreflectiveness of $\mathbb{T}$, there is a canonical $\mathbb{T}$-filtration defined by: for any $i \geq 0$, $M_i := \mathbb{T}^{(i+1)}M := \mathbb{T}^{(i+1)}$-torsion of M. We are using here the fact that if $\mathbb{T}$ and $\mathbb{S}$ are coreflective topologizing categories than such is $\mathbb{T} \bullet \mathbb{S}$. Clearly the canonical $\mathbb{T}$-filtration is the biggest one in an obviuos sense.

For any $M \in Ob\mathcal{A}$, we call the subobject $\mathbb{T}^{\infty}(M) := \sup\{\mathbb{T}^{(i)}M\}$ *the* $\mathbb{T}$-*part of* M.

We call M a $\mathbb{T}$-*object* if $M = \mathbb{T}^{\infty}$M. The full subcategory of the category $\mathcal{A}$ generated by $\mathbb{T}$-objects shall be denoted by $\mathbb{T}^{\infty}$.

3.1.1. Lemma. *The subcategory* $\mathbb{T}^{\infty}$ *is a subscheme.*

Proof. For any object M of $\mathcal{A}$, $\mathbb{T}^{\infty}M$ is the largest $\mathbb{T}$-subobject of M. Since any morphism $f : M \longrightarrow M'$ induces morphisms $f^{(n)} : \mathbb{T}^{(n)}M \longrightarrow \mathbb{T}^{(n)}M'$ for all n, the map $M \mapsto \mathbb{T}^{\infty}M$ extends to a functor $\mathcal{A} \longrightarrow \mathbb{T}^{\infty}$ which is left adjoint to the inclusion functor $\mathbb{T}^{\infty} \longrightarrow \mathcal{A}$; i.e. $\mathbb{T}^{\infty}$ is coreflective.

It follows from the property (sup) and the fact that subcategories $\mathbb{T}^{(n)}$ are topologizing that any subquotient of a $\mathbb{T}$-object is a $\mathbb{T}$-object. And direct sum of any number of $\mathbb{T}$-objects is a $\mathbb{T}$-object. In particular, the subcategory $\mathbb{T}^{\infty}$ is topologizing. ∎

3.1.2. Digression: $\mathbb{T}$-objects and $\mathbb{T}^-$-objects. For any $M \in Ob\mathcal{A}$, the $\mathbb{T}$-part of M is a (proper in general) subobject of the $\mathbb{T}^-$-torsion of M. To see the difference between being a $\mathbb{T}$-object and a $\mathbb{T}^-$-torsion ($= \mathbb{T}^-$-object), we consider, for any object M of $\mathcal{A}$, the following increasing filtration:

$M_0 = 0$;

M_i is the preimage of the $\mathbb{T}$-torsion of M/M_{i-1}, if i is a limit ordinal;

$M_i = \sup\{M_\nu \mid \nu < i\}$, if i is a limit ordinal.

Set $M_\omega := \sup\{M_i\}$. One can see that M_ω belongs to $\mathbb{T}^-$, and M/M_ω is $\mathbb{T}$-torsion free. The latter implies that M/M_ω is $\mathbb{T}^-$-torsion free; i.e. M_ω is the $\mathbb{T}^-$-torsion of M.

Recall that 'M is locally noetherian' means that M is the supremum of its noetherian subobjects. For example, any left module over a left noetherian ring is a locally noetherian object of $R-mod$. And any quasi-coherent sheaf on a noetherian scheme $\mathbf{X}$ is a locally noetherian object of the category $\mathfrak{Qcoh}_{\mathbf{X}}$ of quasi-coherent sheaves on $\mathbf{X}$.

3.1.2.1. Proposition. *Suppose that M is a locally noetherian object of $\mathcal{A}$, then $M \in Ob\mathbb{T}^-$ iff M is a $\mathbb{T}$-object.*

Proof. a) Suppose first that M is noetherian. Then $M \in Ob\mathbb{T}^-$ iff $M \in Ob\mathbb{T}^{(n)}$ for some n.

b) Suppose that M is the supremum of a set $\{M_\alpha\}$ of noetherian subobjects. If M belongs to $\mathbb{T}^-$, then, according to a), all M_α are $\mathbb{T}$-objects. It remains to observe that, since all subcategories $\mathbb{T}^{(n)}$ are coreflective, supremum of any family of $\mathbb{T}$-subobject is a $\mathbb{T}$-subobject. ∎

3.2. Conormal bundle and $\mathbb{T}$-filtrations. For any topologizing subcategory $\mathbb{S}$, denote by $\mathcal{I}_{\mathbb{S}}$ the subfunctor of $Id_{\mathcal{A}}$ assigning to any object M of $\mathcal{A}$ the intersection of $Ker(f)$, where f runs over all arrows $f: M \longrightarrow X$ with $X \in Ob\mathbb{S}$. We call $\mathcal{I}_{\mathbb{S}}$ *the defining ideal of the subcategory* $\mathbb{S}$.

Clearly $\mathcal{I}_{\mathbb{S}}$ is a subfunctor of $\mathcal{I}_{\mathbb{T}}$ if $\mathbb{T} \subseteq \mathbb{S}$. In particular, $\mathcal{I}_{\mathbb{T}\bullet\mathbb{T}}$ is a subfunctor of $\mathcal{I}_{\mathbb{T}}$. For any topologizing subcategory $\mathbb{T}$ of $\mathcal{A}$, we define the *conormal bundle* $\Omega_{\mathbb{T}}$ of $\mathbb{T}$ as $\mathcal{I}_{\mathbb{T}}/\mathcal{I}_{\mathbb{T}\bullet\mathbb{T}}$.

Let $M_. = \{M_i \mid i \geq -1\}$ be any $\mathbb{T}$-filtration of an object M. One can see that, for any i, $\mathcal{I}_{\mathbb{T}}(M_i) \subseteq M_{i-1}$ and $\mathcal{I}_{\mathbb{T}\bullet\mathbb{T}}(M_i) \subseteq M_{i-2}$ (we set $M_i = 0$ for all negative i). This implies the existence of canonical morphisms

$$\delta_i : \Omega_{\mathbb{T}}(gr_i M) \longrightarrow gr_{i-1}M \tag{1}$$

3.3. The case of the category of modules. Suppose that $\mathcal{A}$ is the category $R-mod$ of left modules over a ring R. And let F be any functor from $\mathcal{A}$ to $\mathcal{A}$. Then, for any R-bimodule M, $F(M)$ has a natural R-bimodule structure which is the composition of the bimodule structure $R^0 \longrightarrow End_R(M,M)$ and the provided by the functor F mapping $End_R(M,M) \longrightarrow End_R(F(M),F(M))$. In particular, the R-module $F(R)$ has a natural R-bimodule structure.

There is a functor morphism $\varphi : F(R)\otimes_R \longrightarrow F$, where, for any $X \in Ob\mathcal{A}$, the morphism $\varphi(X)$ is defined as the image of id_X under the composition

$$\mathcal{A}(X,X) \simeq \mathcal{A}(X, \mathrm{Hom}_R(R,X)) \longrightarrow \mathcal{A}(X, \mathrm{Hom}_R(F(R),F(X))) \simeq \mathcal{A}(F(R)\otimes_R X, F(X)).$$

The morphism φ is an isomorphism iff the functor F has a right adjoint (cf. [Bass], Proposition I.2.2). The functor $h : M \mapsto M\otimes_R$ from the category R-bi of R-bimodules to the category $End\mathcal{A}$ of endofunctors of the category $\mathcal{A} = R-mod$ is fully faithful and is a left adjoint to the functor assigning to any endofunctor F of $\mathcal{A}$ into the R-bimodule $F(R)$.

Consider now the setting of Example 3.1: $\mathcal{A} = R-mod$, J is a left ideal in R; $\mathbb{T} = [J]$ is the minimal subscheme of $\mathcal{A}$ containing the module R/J. One can check that $\mathcal{I}_{\mathbf{T}}(R)$ coincides with the biggest two-sided ideal $J_t := (J : R) = \{r \in R \mid Jr \subseteq J\}$ contained in J. And $\mathcal{I}_{\mathbf{T}\bullet\mathbb{T}}(R) = J_t^2$. So that $\Omega_{\mathbf{T}}(R) \simeq J_t/J_t^2$.

It follows from Proposition 6.4.1 in [R1] that the canonical functor morphism

$$\mathcal{I}_{\mathbf{T}}(R)\otimes_R = J_t\otimes_R \longrightarrow \mathcal{I}_{\mathbf{T}}$$

is an isomorphism iff J is a two-sided ideal; i.e. if $J = J_t$. In this case $\mathcal{I}_{\mathbf{T}\bullet\mathbb{T}}(R) = J^2$ and, by the same proposition, $\mathcal{I}_{\mathbf{T}\bullet\mathbf{T}} \simeq \mathcal{I}_{\mathbf{T}\bullet\mathbf{T}}(R)\otimes_R$.

In particular, if J is a two-sided ideal, the canonical functor morphism $\Omega_{\mathbf{T}}(R)\otimes_R \longrightarrow \Omega_{\mathbf{T}}$ is an isomorphism.

4. Differential endofunctors and monads.

Let $\mathcal{A}$ be an abelian category. Let $\mathcal{B}$ denote the category $End(\mathcal{A})$ of functors from $\mathcal{A}$ to $\mathcal{A}$ (we assume that $\mathcal{A}$ is equivalent to a 'small' category). Clearly $\mathcal{B}$ is an abelian category which inherits many of properties of $\mathcal{A}$. For example, $\mathcal{B}$ has the property (sup) if $\mathcal{A}$ has it. And $\mathcal{B}$ has the same kind of limits as $\mathcal{A}$ has.

We take as $\mathbb{T}$ the diagonal Δ which is the minimal subscheme of $\mathcal{B}$ containing $Id_{\mathcal{A}}$. We shall call the Δ-part of any object M of $\mathcal{B}$ *the differential part of* M, and Δ-objects *differential objects* (functors).

And we have the corresponding conormal bundle $\Omega_\Delta : \mathcal{B} \longrightarrow \mathcal{B}$.

4.1. Lemma. *For any* $M, N \in Ob\mathcal{B}$ *and* $n, m \geq 0$, *the natural morphism*

$$\Delta^{(m)}M \circ \Delta^{(n)}N \longrightarrow M \circ N$$

factors through $\Delta^{(mn)}(M \circ N)$.

Proof. (a) If $X \in Ob\Delta$ and $Y \in Ob\Delta^{(n)}$, then $X \circ Y \in Ob\Delta^{(n)}$.

In fact, the assertion is trivially true for $X = Id$. And the class of Ξ_Y of all $X \in Ob\mathcal{B}$ such that $X \circ Y \in Ob\Delta^{(n)}$ is closed with respect to taking direct sums and subquotients, since $\Delta^{(n)}$ is topologizing and coreflective. Since Δ is the minimal full subcategory of $\mathcal{B}$ containing Id and closed with respect to these operations, the class Ξ_Y contains $Ob\Delta$.

(b) Suppose that $X \in Ob\Delta^{(m)}$ for some $m \geq 2$, and $Y \in Ob\Delta^{(n)}$. Then there exists an exact sequence $0 \longrightarrow L \longrightarrow X \longrightarrow M \longrightarrow 0$ such that $L \in Ob\Delta$, $M \in Ob\Delta^{(m-1)}$. Clearly

$$0 \longrightarrow L \circ Y \longrightarrow X \circ Y \longrightarrow M \circ Y \longrightarrow 0 \tag{1}$$

is an exact sequence in which, according to (a), $L \circ Y \in Ob\Delta^{(n)}$. And, by the induction hypothesis, $M \circ Y \in Ob\Delta^{(nm-n)}$. It follows from (1) that $X \circ Y \in Ob\Delta^{(mn)}$. ∎

4.2. Corollary. *If M and N are differential objects of $\mathcal{B}$, then $M \circ N$ is differential too.*

4.3. Differential monads. Recall that a monad in a category $\mathcal{A}$ is a pair (F, μ), where F is a functor from $\mathcal{A}$ to $\mathcal{A}$ and μa morphism $F \circ F \longrightarrow F$ such that

(a) $\mu \circ F\mu = \mu \circ \mu F$;

(b) there exists a morphism $\eta : Id_{\mathcal{A}} \longrightarrow F$ such that $\mu \circ F\eta = id_F = \mu \circ \eta F$.

Note that the latter equality determines η uniquely.

In fact, if $\eta' : Id_{\mathcal{A}} \longrightarrow F$ is another morphism with the same properties, then we have: $\eta' = \mu \circ \eta F \circ \eta' = \mu \circ F\eta' \circ \eta = \eta$.

4.3.1. Examples of monads. **(a)** A standart example of a monad is associated with a ring morphism $R \longrightarrow$A. The corresponding category $\mathcal{A}$ is $R-mod$; and the monad is $(A\otimes_R, \mu)$, where μ is determined by the multiplication $A \otimes A \longrightarrow A$ (or, ruther, by the morphism $A \otimes_R A \longrightarrow A$ induced from the multiplication) in A.

(b) Similarly, a quasi-coherent sheaf of algebras A on a scheme X equipped with a morphism $\iota : \mathcal{O}_X \longrightarrow A$ of sheaves of rings defines a monad $(A\otimes_{\mathcal{O}_X}, \mu)$ on the category $\mathcal{A} = \mathfrak{Qcoh}_X$.

(c) Let $G : \mathcal{A} \longrightarrow \mathcal{B}$ and $G^{\wedge} : \mathcal{B} \longrightarrow \mathcal{A}$ be adjoint functors; and let

$$\epsilon : G \circ G^{\wedge} \longrightarrow Id_{\mathcal{B}}, \quad \eta : Id_{\mathcal{A}} \longrightarrow G^{\wedge} \circ G$$

be adjunction arrows. Then $(G^{\wedge} \circ G, G^{\wedge}\epsilon G)$ is a monad in $\mathcal{A}$. ∎

Fix a monad $\mathbb{F} = (F, \mu)$. An $\mathbb{F}$-module is a pair (M, m), where $M \in Ob\mathcal{A}$ and m is a morphism $F(M) \longrightarrow M$ with the following properties: $m \circ \eta(M) = id_M$, $m \circ \mu(M) = m \circ Fm$.

A module morphism from $\mathcal{M} = (M, m)$ to $\mathcal{M}' = (M', m')$ is any triple $(\mathcal{M}, f, \mathcal{M}')$, where f is a morphism from M to M' compatible with actions: $m' \circ Ff = f$om. The composition is defined in an obvious fashion. Thus we have the category $\mathbb{F} - mod$ and the forgetting functor $\mathcal{F} : \mathbb{F} - mod \longrightarrow \mathcal{A}$. The functor $\mathcal{F}$ is right adjoint to the functor assigning to any $M \in Ob\mathcal{A}$ the $\mathbb{F}$-module $^{\wedge}\mathcal{F}(M) := (F(M), \mu(M))$ and to any arrow $f : X \longrightarrow X'$ the morphism $Ff : {^{\wedge}\mathcal{F}}(X) \longrightarrow {^{\wedge}\mathcal{F}}(X')$ of $\mathbb{F}$-modules. One can see that the corresponding to the pair of adjoint functors $(^{\wedge}\mathcal{F}, \mathcal{F})$ monad coincides with $\mathbb{F}$.

So the general nonsense example (c) provides a universal way of constructing monads.

4.3.2. Proposition. *Let* $\mathbb{F} = (F, \mu)$ *be a monad in* $\mathcal{A}$. *Then* $\Delta^{(\infty)}F$ *is a submonad of* $\mathbb{F}$.

Proof. By Lemma 4.1, for any $n, m \geq 0$, the canonical morphism

$$\Delta^{(m)}F \circ \Delta^{(n)}F \longrightarrow F \circ F \tag{2}$$

factors through $\Delta^{(m)}F \circ \Delta^{(n)}F \longrightarrow \Delta^{(mn)}F \circ F$. This implies (because the subcategory $\Delta^{(mn)}$ is topologizing) that the image of the composition of (2) with the multiplication μ is a subobject of $\Delta^{(mn)}$F. ∎

We call a monad $\mathbb{F} = (F, \mu)$ *differential*, or a *D-monad*, if F is differential.

The full subcategory of differential objects of the category $\mathcal{B}$ will be denoted by $D-\mathcal{A}$.

4.4. Remark. Instead of taking $\mathcal{B}$ equal to the entire category $End\mathcal{A}$, we might be interested in choosing $\mathcal{B}$ to be a certain full subcategory of $End\mathcal{A}$ closed with respect to composition of functors and colimits (taken in $End\mathcal{A}$), and containing $Id_{\mathcal{A}}$. Two important for this work choices are $\mathcal{B} = End'\mathcal{A}$ – the full subcategory of $End\mathcal{A}$ generated by all right exact functors - and $\mathcal{B} = \mathfrak{End}\mathcal{A}$ – the full subcategory of $End\mathcal{A}$ generated by functors having a right adjoint. ∎

5. Differential calculus over noncommutative rings.

5.1. Differential bimodules and rings. For any two rings A, B, we shall identify (A, B)-bimodules with the corresponding $A \otimes B^o$-modules. Here B^o is the ring opposite to B. In particular, R-bimodules will be identified with left $R \otimes R^o$-modules.

To make the contents of this section more conveniently applicable, we shall consider algebras and bimodules over a commutative ring k. This means that, instead of the category (A, B)-bi of all (A, B)-modules, we single out the full subcategory $(A, B) - bi/k$ of (A, B)-bi generated by bimodules M having the property: for any $y \in M$ and $\lambda \in k$, $\lambda y = y\lambda$. The canonical equivalence $(A, B) - bi \longrightarrow A \otimes B^o - mod$ induces an equivalence $(A, B) - bi/k \longrightarrow A \otimes_k B^o - mod$. In particular $R - bi/k$ is identified with the category $R \otimes_k R^o - mod$.

Fix a k-algebra $R = (R, \mu)$.

5.1.1. Lemma. *The kernel K_μ of the multiplication μ is a left ideal in $R \otimes_k R^o$.*

Proof. By definition of the multiplication in $R \otimes_k R^o$, $(a \otimes b) \sum s_i \otimes t_i = \sum as_i \otimes t_i b$. Therefore, if $\sum s_i \otimes t_i \in K_\mu$, then $\mu((a \otimes b) \sum s_i \otimes t_i) = a(\sum s_i t_i)b = 0$. ∎

5.1.2. Note. The kernel K_μ of the multiplication μ is a right (hence two-sided) ideal if and only if the algebra R is commutative.

In fact, if R is commutative, then K_μ is two-sided. On the other hand, for any $r \in R$, the element $r \otimes 1 - 1 \otimes r$ belongs to K_μ. The element $(r \otimes 1 - 1 \otimes r)(a \otimes 1) = ra \otimes 1 - a \otimes r$ belongs to K_μ iff $ra = ar$. ∎

5.2. Lemma. *The full subcategory Δ_R of $R \otimes_k R^o - mod$ generated by all $R \otimes_k R^o$-modules M such that, for any $z \in M$, $K_\mu \leq Ann(z)$ is a subscheme.*

The subscheme Δ_R is closed (i.e. the subcategory Δ_R is reflective) if and only if the algebra R is commutative.

Proof. This is a special case of Example 3.1: for any ring A and any left ideal ν in A, the full subcategory $[R/\nu]$ generated by all A-modules M such that $\nu \leq Ann(z)$ for all $z \in M$, is topologizing and coreflective.

If R is commutative, then such is $R \otimes_k R^o = R \otimes_k R$; and the subcategory Δ_R (which coincides with $[R \otimes_k R^o / K_\mu]$ in the notation above) is generated by all $R \otimes_k R$-modules M which are annihilated by K_μ. It is reflective: a functor right adjoint to the inclusion $\Delta_R \longrightarrow R \otimes_k R - mod$ is tensoring by $R \otimes_k R/K_\mu$ over $R \otimes_k R$.

It follows from Proposition III.6.4.1 in [R], that the topologizing subcategory Δ_R is reflective if and only if K_μ is a two-sided ideal. And, according to Note 5.1.2, the ideal K_μ is two-sided iff the algebra R is commutative. ∎

For any R-bimodule M, the subbimodule $\Delta_R^\infty M := \sup\{\Delta_R^{(i)} M \mid i \geq 1\}$ of M will be called *the differential part of* M. We shall call an $R \otimes_k R^o$-modules M *differential* if $M = \Delta_R^\infty M$.

Clearly the differential part of M is contained (usually properly) in the Δ_R^--torsion $\Delta_R^-(M)$ of M.

5.3. D-filtrations. We call an increasing filtration $\{M_i \mid i \geq -1\}$ of an R-bimodule M a *D-filtration* if $M_{-1} = 0$ and $M_i/M_{i-1} \in \Delta_R$ for any $i \geq 0$.

One can produce the canonical D-filtration of any bimodule M by taking as M_i the $\Delta_R^{(i+1)}$-torsion of M : $M_i := \Delta_R^{(i+1)}M$. In other words, for any $i \geq 0$, M_i is the preimage in M of $\{z \in M/M_{i-1} \mid K_\mu \leq Ann(z)\}$.

This D-filtration is the biggest one with respect to the natural preordering of filtrations.

5.4. Remark. Let $M_. = \{M_i \mid i \geq -1\}$ be a D-filtration of an $R \otimes_k R^o$-module. If R is commutative, the action of K_μ on M sends M_i into M_{i-1}. Therefore this action induces actions ($R \otimes_k R^o$-module morphisms) $K_\mu \otimes_k gr_iM \longrightarrow gr_{i-1}$M. Note that the ideal K_μ^2 acts trivially. Which means that the latter actions induce bimodule morphisms

$$\Omega^1 R \otimes_{R \otimes_k R} gr_iM \longrightarrow gr_{i-1}M \tag{1}$$

where $\Omega^1 R := K_\mu/(K_\mu)^2$ is the bimodule of Kähler differentials.

We denote by

$$\delta_i : gr_iM \longrightarrow \mathrm{Hom}_R(\Omega^1 R, gr_{i-1}M). \tag{2}$$

the dual morphisms. One can check that the D-filtration $M_.$ is maximal iff M_ω coincides with the Δ_R^--torsion of M, and δ_i are monomorphims for all $i \geq 1$.

If R is noncommutative, this construction does not work. And the reason is that the topologizing subcategory Δ_R is not reflective (in other words, it is not Zariski closed). But there is a natural replacement for $\Omega^1 R$ – the conormal bundle of the subscheme Δ_R which is a functor from $R \otimes_k R^o - mod$ to $R \otimes_k R^o - mod$. ∎

5.5. A reformulation. Now we shall consider the equivalence of the category R-bi/k of R-bimodules over k (which is identified whenever it is convenient with the category $R \otimes_k R^o - mod$) and the category of k-linear functors $R - mod \longrightarrow R - mod$ having a right adjoint. Following H. Bass [Ba], we shall call such functors *continuous*. Recall that this equivalence sends any R-bimodule M into the functor $M \otimes_R$.

5.6. Lemma. *The equivalence $\mathfrak{F}_R$ between $R \otimes_k R^o - mod$ and the category $\mathfrak{End}_k(R - mod)$ of continuous k-linear endofunctors $R - mod \longrightarrow R - mod$ sends the subcategory Δ_R into the minimal subscheme $\Delta_\mathfrak{c}$ of $\mathfrak{End}_k(R - mod)$ containing the identical functor. The induced functor from Δ_R to $\Delta_\mathfrak{c}$ is an equivalence of categories.*

Proof. Note that $R \simeq R \otimes_k R^o/K_\mu$ as left $R \otimes_k R^o$-modules. The isomorphism is the composition of the map

$$R \longrightarrow R \otimes_k R^o, \quad r \mapsto r \otimes 1, r \in R,$$

and the projection $R \otimes_k R^o \longrightarrow R \otimes_k R^o/K_\mu$ (we are using the fact that $r \otimes s - rs \otimes 1 \in K_\mu$ for all $r, s \in R$). This implies that Δ_R is the minimal topologizing coreflective subcategory of $R \otimes_k R^o - mod$ containing the $R \otimes_k R^o$-module R.

Note now that the equivalence between the category $R \otimes_k R^o - mod$ and the category of continuous k-linear functors sends R into a functor isomorphic to the identical functor. This implies the assertion. ∎

5.7. Lemma. *If M and N are differential bimodules, then $M \otimes_R N$ is a differential bimodule.*

Proof. The assertion follows from the following statement:

For any $n, m \geq 0$, the natural morphism $\Delta_R^{(m)} M \otimes_R \Delta_R^{(n)} N \longrightarrow M \otimes_R N$ factors through the subbimodule $\Delta_R^{(mn)}(M \otimes_R N)$.

This can be proved the same way as Lemma 4.1, first using the fact that the equivalence between the category $R \otimes_k R^o - mod$ and $\mathfrak{End}_k(R-mod)$ sends $M \otimes_R N$ into the composition of the functors corresponding to M and N (cf. Remark 4.4). ∎

5.8. Proposition. *Let $R \longrightarrow A$ be a ring morphism. Then $\Delta_R^\infty A$ is a subring of A.*

Proof. The fact follows from Lemma 5.7 (as Proposition 4.3.2 follows from Lemma 4.1). ∎

We call $R \longrightarrow A$ *a differential R-algebra* (or simply a D-algebra) if $A = \Delta_R^\infty$A.

5.9. Differential operators. For any two R-modules V and W, $\mathrm{Hom}_k(V, W)$ has a natural structure of an $R \otimes_k R^o$-module. We denote by $Diff_k(V, W)$ the differential part of $\mathrm{Hom}_k(V, W)$. We shall call elements of $Diff_k(V, W)$ *k-differential operators* (or simply *differential operators*) from V to W.

Clearly $\mathrm{Hom}_R(V, W)$ is contained in $Diff_k^{(0)}(V, W) := \Delta(End_k(V, W))$; i.e. R-module morphisms are differential operators of the zero order.

We shall write $D_k(R)$ instead of $Diff_k(R, R)$. Note that there is a natural ring monomorphism from R to $End_k(R)$ assigning to each element $r \in R$ the left multiplication by r. The bimodule structure on $End_k(R)$ defined above coincides with the one induced from the morphism $\iota : R \longrightarrow End_k(R)$. Thus $R \longrightarrow D_k(R)$ is a D-algebra.

Note that the right multiplications by elements of R (=endomorphisms of the left R-module R) also belong to $D_k^{(0)}(R)$. Therefore the image of the canonical map

$$R \otimes_k R^o \longrightarrow End_k(R)$$

belongs to the subbimodule $D_k^{(0)}(R)$ of differential operators of zero order.

Any derivation $d : R \longrightarrow R$ belongs to $\Delta^{(2)} End_k(R) := D^{(1)}(R)$.

In fact, for any $r \in R$, $d \circ r \cdot - r \cdot \circ d = d(r) \cdot$. Here $r\cdot$ denotes the operator of the left multiplication by r. Therefore, for any element $t \in \mathrm{Ker}(\mu)$, $td \in \iota(R) = D_k^{(0)}(R)$.

Thus $D(R)$ contains all operators of left and right multiplication by elements of R and all derivations. Recall that if R is commutative and regular, $D(R)$ is generated by $D_0(R)$ and $Der(R)$.

5.10. Enveloping algebras and differential operators. Fix a commutative ring k. Let $\mathcal{H} = (\delta, H, \mu)$ be a Hopf k-algebra; and let $\mathcal{R} = (R, m)$ be any associative k-algebra. Recall that a $\mathcal{H}$-module structure $\tau : H \otimes_k R \longrightarrow R$ is called a Hopf action if the multiplication $m : R \otimes_k R \longrightarrow R$ is an $\mathcal{H}$-module morphism. The latter means that the diagram

$$\begin{array}{ccccc} H \otimes_k R \otimes_k R & \xrightarrow{id_H \otimes_k m} & H \otimes_k R & \xrightarrow{\tau} & R \\ \delta \otimes id \downarrow & & & & \uparrow m \\ H \otimes_k H \otimes_k R \otimes_k R & \xrightarrow{id_H \otimes \sigma \otimes id_R} & H \otimes_k R \otimes_k H \otimes_k R & \xrightarrow{\tau \otimes \tau} & R \otimes_k R \end{array}$$

commutes. Here σ denotes the standart isomorphism $H \otimes_k R \longrightarrow R \otimes_k H$, $h \otimes r \mapsto r \otimes h$.

Let $U_k(\mathfrak{g}) = U(\mathfrak{g})$ be the enveloping k-algebra of a Lie algebra $\mathfrak{g}$. Recall that $U(\mathfrak{g})$ is a Hopf algebra.

5.10.1. Proposition. *Let $\varphi : U(\mathfrak{g}) \otimes_k R \longrightarrow R$ be a Hopf action of $U(\mathfrak{g})$ on any k-algebra* R. *Then $U(\mathfrak{g})$ acts by differential operators.*

Proof. The coproduct $\delta : U(\mathfrak{g}) \longrightarrow U(\mathfrak{g}) \otimes_k U(\mathfrak{g})$ is uniquely determined by the its values on $\mathfrak{g}$, and $\delta(x) = x \otimes 1 + 1 \otimes x$ for all $x \in \mathfrak{g}$. The action of $U(\mathfrak{g})$ on R being Hopf and the above formula for δ mean that $\mathfrak{g}$ acts on R by derivations which are differential operators of the first order (cf. Section 5.9). Since $U(\mathfrak{g})$ is generated by $\mathfrak{g}$, the assertion follows from the fact that $D_k(R)$ is a subalgebra of $End_k(R)$ (cf. Proposition 5.8). ∎

5.11. Upper central filtration and strongly differential bimodules. Recall that the *center* of an R-bimodule M is the set $\mathfrak{Z}$(M):$= \{z \in M \mid rz = zr$ for all $r \in R\}$. An R-bimodule L is called *artinian* if it is generated as a left (or right) R-module by its center: $L = R\mathfrak{Z}$(L).

We denote the full subcategory of $R - bi/k = R \otimes_k R^o - mod$ generated by artinian R-bimodules by Art_R. Clearly Art_R is a subcategory of the 'diagonal' Δ_R. And if the ring R is commutative, they coincide: $Art_R = \Delta_R$.

Note that the inclusion functor $Art_R \longrightarrow R \otimes_k R^o - mod$ has a right adjoint which assigns to any R-bimodule M the artinian bimodule $R\mathfrak{Z}$(M). In other words, Art_R is a full coreflective subcategory of $R \otimes_k R^o - mod$.

Define *the upper central series* $\{\mathfrak{z}_n M \mid n \geq -1\}$ of an R-bimodule M by

$\mathfrak{z}_{-1}M = 0$; and for any $n \geq 0$, $\mathfrak{z}_n M = R\mathfrak{Z}_n(M)$, where $\mathfrak{Z}_n(M) := \{z \in M \mid ad_r(z) \in \mathfrak{z}_{n-1}(M)$ for all $r \in R\}$. Here $ad_r(z) = rz - zr$.

In particular, $\mathfrak{Z}_0(M) = \mathfrak{Z}(M)$ is the center of M; hence $\mathfrak{z}_0(M) = R\mathfrak{Z}(M)$ is the maximal artinian subbimodule of M.

Clearly thus defined the upper central series of M is an increasing filtration. We denote the union $\bigcup_{n \geq -1} \mathfrak{z}_n M$ by $\mathfrak{z}_\infty$M.

5.11.1. Note. The upper central filtration of a bimodule M is the $\mathbb{T}$-filtration of M in the sense of Subsection 3.1, where $\mathbb{T} = Art_R$: $\mathfrak{z}_n M$ is the $Art_R^{(n)}$-torsion of M. Only this time $\mathbb{T}$ is not a topologizing subcategory. ∎

Since artinian bimodules belong to the diagonal Δ_R, it follows from the definition of the upper central series that $\mathfrak{z}_n M \subseteq \Delta_R^{(n)} M$ for all n. Therefore $\mathfrak{z}_\infty M$ is contained in the differential part of M: $\mathfrak{z}_\infty M \subseteq \Delta_R^\infty M$.

If R is commutative, $\mathfrak{z}_n M = \Delta_R^{(n)} M$ for all n; hence $\mathfrak{z}_\infty M = \Delta_R^\infty M$. In the noncommutative case, $\mathfrak{z}_\infty M$ is, in general, a proper subbimodule of $\Delta_R^\infty M$.

5.11.2. Example. Let V, W be left R-modules; and let M be the $R \otimes_k R^o$-module $\mathrm{Hom}_k(V, W)$. Then $\mathfrak{Z}$(M)= $\mathrm{Hom}_R(V, W)$, hence $\mathfrak{z}_0 M = R \cdot \mathrm{Hom}_R(V, W)$. In other words, $\mathfrak{z}_0 M$ consists of all left linear combinations with coefficients in R of R-module morphisms from V to W. If $V = W = R$, where R is regarded as a left R-module, then $\mathrm{Hom}_R(V, W)$ is naturally isomorphic to (the right R-module) R: the isomorphism assigns to each element $r \in R$ the right multiplication by r. So that $\mathfrak{z}_0 M = \mathfrak{z}_0 End_k(R)$ is the image of the natural morphism $R \otimes_k R^o \longrightarrow End_k(R)$.

By definition, $\mathfrak{Z}_1(End_k(R))$ consists of all endomorphisms $d : R \longrightarrow R$ such that, for any $r \in R, dr - rd \in \mathfrak{z}_0 End_k(R)$, where r means the operator of left multiplication by the element r; i.e. $dr - rd$ is a sum of compositions of operators of left and right multiplications (=the image of $R \otimes_k R^o$ in $End_k(R)$). Note that derivations of R are exactly k-endomorphisms ∂ of R such that, for any $r \in R$, the commutator $\partial r - r\partial$ is an operator of the left multiplication by an element of R. In particular, any derivation of R belongs to $\mathfrak{Z}_1(End_k(R))$. ■

5.11.3. Strongly differential bimodules and operators. We shall call objects of the subcategory Art_R^∞ *strongly differential R-bimodules* (or $R \otimes_k R^o$*-modules*). More explicitly, an $R \otimes_k R^o$-module M is *strongly differential of order n* if $\mathfrak{z}_{n+1} M = M$ and $M \neq \mathfrak{z}_m M$ if $m \leq n$.

For any R-modules L and N, we call the elements of $\mathfrak{z}_\infty \mathrm{Hom}_k(L, N)$ *strongly differential operators from L to* N.

Note that if the ring R is commutative, *strongly* can be droped, since in this case $Art_R^{(n)} = \Delta_R^{(n)}$.

5.11.4. The difference between the category of differential and that of strongly differential bimodules. We begin with a couple of general assertions.

5.11.4.1. Proposition. *(a) Let $\mathcal{S}$ be a full subcategory of an abelian category $\mathcal{A}$ closed with respect to finite direct sums (taken in $\mathcal{A}$) and containing all quotients of each of its objects. Then the full subcategory $\mathbb{S}$ of $\mathcal{A}$ generated by all subobjects of objects of $\mathcal{S}$ is the smallest topologizing subcategory of $\mathcal{A}$ containing $\mathcal{S}$.*

(b) Suppose the abelian category $\mathcal{A}$ has the property (sup) and direct sums of (small) sets of objects. Let $\mathcal{S}$ be a full coreflective subcategory of $\mathcal{A}$ containing all quotients (in $\mathcal{A}$) of each of its objects. Then the full subcategory $\mathbb{S}$ of $\mathcal{A}$ generated by all subobjects of objects of $\mathcal{S}$ is topologizing and coreflective. Therefore $\mathbb{S}$ is the minimal subscheme of $\mathcal{A}$ containing $\mathcal{S}$.

Proof. (a) The subcategory $\mathbb{S}$ is, evidently, closed with respect to taking subobjects (in $\mathcal{A}$) of any of its objects. It is closed with respect to taking any quotients too.

In fact, let $f : X \longrightarrow Y$ be an epimorphism with $X \in Ob\mathbb{S}$. By definition of $\mathbb{S}$, there exists a monoarrow $\iota : X \longrightarrow W$, where $W \in Ob\mathcal{S}$. Then in the universal (push-forward) square

$$\begin{array}{ccc} X & \xrightarrow{\iota} & W \\ f \downarrow & & \downarrow f' \\ Y & \xrightarrow{\iota'} & W' \end{array}$$

the arrows ι' and f' are resp. a monomorphism and an epimorphism. Since the subcategory $\mathcal{S}$ is closed with respect to taking quotients in $\mathcal{A}$, the object W' belongs to $\mathcal{S}$. Therefore Y, being a subobject of W', belongs to $\mathbb{S}$.

(b) Since $\mathcal{S}$ is a full coreflective subcategory of $\mathcal{A}$, for any diagram $D \longrightarrow \mathcal{S}$, the existence of resp. $limJ \circ D$ and $colimJ \circ D$ garantees the existence of resp. $limD$ and $colimD$, where J is the inclusion functor $\mathcal{S} \longrightarrow \mathcal{A}$. And

$$limD = J^{\wedge}(limJ \circ D), \text{ (resp. } colimD = J^{\wedge}(colimJ \circ D))$$

whenever $limJ \circ D$ (resp. $colimJ \circ D$) exist (cf. [GZ], Proposition I.1.4). Since J has a right adjoint, $colimJ \circ D \simeq J(colimD)$.

Under conditions, since $\mathcal{A}$ is closed with respect to small direct sums, hence with respect to any colimits, $\mathcal{S}$ is closed with respect to colimits, and the last formula means that the colimits in $\mathcal{S}$ are those taken in $\mathcal{A}$.

Since $\mathcal{A}$ has the property (sup), the direct sum of any set of monomorphisms is a monomorphism. Therefore, since $\mathcal{S}$ is closed with respect to direct sums, any direct sum of a set of objects of $\mathbb{S}$ is an object of $\mathbb{S}$. This proves that $\mathbb{S}$ is a coreflective topologizing subcategory. ■

5.11.4.2. Proposition. *Let $\mathcal{A}$ be an abelian category with the property (sup). And let $\mathcal{S}$ be a full coreflective subcategory of $\mathcal{A}$ (in particular, $\mathcal{S}$ is closed with respect to colimits taken in $\mathcal{A}$) and containing all quotients of each of its objects. Then the subcategory $\mathcal{S}^{(n)}$ for any positive n and the subcategory $\mathcal{S}^{\infty}$ have the same properties.*

Proof. (a) It is convenient to split the proof of this assertion into a couple of useful lemmas.

5.11.4.2.1. Lemma. *Let $\mathcal{S}$ and $\mathcal{T}$ be full subcategories of $\mathcal{A}$ containing with any* of *its objects all its quotients in $\mathcal{A}$. Then $\mathcal{S} \bullet \mathcal{T}$ enjoys the same property.*

Proof. In fact, let $f : X \longrightarrow Y$ be any epimorphism with $X \in Ob\mathcal{S} \bullet \mathcal{T}$. The latter means that there exists an exact sequence

$$0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0 \qquad (1)$$

with $X' \in Ob\mathcal{T}$ and $X'' \in Ob\mathcal{S}$. We can include the sequence (1) and the morphism f into a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X' & \xrightarrow{i} & X & \xrightarrow{e} & X'' & \longrightarrow & 0 \\ & & f' \downarrow & & f \downarrow & & \downarrow f'' & & \\ 0 & \longrightarrow & Y' & \xrightarrow{i'} & Y & \xrightarrow{e'} & Y'' & \longrightarrow & 0 \end{array} \qquad (2)$$

where Y' is the image of $f \circ i$, and Y'' is a cokernel of the monomorphism i'. Thus f' is an epimorphism by construction. Since e' and f are epimorphisms, $f'' \circ e = e' \circ f$ is an epimorphism. Therefore f'' is an epimorphism. By assumptions on the subcategories $\mathcal{S}$ and $\mathcal{T}, Y' \in Ob\mathcal{T}$ and $Y'' \in Ob\mathcal{S}$; hence $Y \in Ob\mathcal{S} \bullet \mathcal{T}$. ∎

5.11.4.2.2. Lemma. *If $\mathcal{S}$ and $\mathcal{T}$ are full coreflective subcategories of an abelian category $\mathcal{A}$, then $\mathcal{S} \bullet \mathcal{T}$ is coreflective.*

Proof. For any $X \in Ob\mathcal{A}$, denote by $X_\mathcal{S}$ (resp. $X_\mathcal{T}$) the $\mathcal{S}$-torsion of X which is the maximal subobject of X belonging to $\mathcal{S}$ (resp. to $\mathcal{T}$). Fix an object X of $\mathcal{A}$. We have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X_\mathcal{T} & \longrightarrow & X & \xrightarrow{\mathfrak{e}} & X/X_\mathcal{T} & \longrightarrow & 0 \\ & & id \uparrow & & \iota' \uparrow & & \uparrow \iota & & \\ 0 & \longrightarrow & X_\mathcal{T} & \longrightarrow & X_{\mathcal{S}\bullet\mathcal{T}} & \xrightarrow{\mathfrak{e}'} & (X/X_\mathcal{T})_\mathcal{S} & \longrightarrow & 0 \end{array} \qquad (1)$$

where $X_{\mathcal{S}\bullet\mathcal{T}}$ is the pull-back of the arrows $\mathfrak{e}$ and ι. Clearly $X_{\mathcal{S}\bullet\mathcal{T}}$ is an object of $\mathcal{S} \bullet \mathcal{T}$. We claim that any monomorphism $Y \longrightarrow X$ with $Y \in Ob\mathcal{S} \bullet \mathcal{T}$ factors through $X_{\mathcal{S}\bullet\mathcal{T}} \longrightarrow X$.

This follows from the fact that $Y_\mathcal{T} = Y \cap X_\mathcal{T}$ which implies that $Y/Y_\mathcal{T}$ is a subobject of $X/X_\mathcal{T}$. Therefore, being an object of $\mathcal{S}$, $Y/Y_\mathcal{T}$ factors through $(X/X_\mathcal{T})_\mathcal{S}$. The latter means that Y is a subobject of $X_{\mathcal{S}\bullet\mathcal{T}}$. ∎

5.11.4.2.3. Lemma. *Let $\{\mathcal{S}_n \mid n \geq 1\}$ be an increasing (with respect to $\subseteq$) sequence of full coreflective subcategories of an abelian category $\mathcal{A}$. Suppose that $\mathcal{A}$ has the property (sup). Then the full coreflective subcategory $colim\{\mathcal{S}_n\}$ generated by all objects X which are supremums of families of objects from $\{\mathcal{S}_n\}$ is coreflective.*

If, for any $n \in \mathbb{N}$, the subcategory $\mathcal{S}_n$ contains all quotients of any of its objects, then $colim\{\mathcal{S}_n\}$ has the same property.

Proof. (a) Let, for any $X \in Ob\mathcal{A}, X_\mathcal{S}$ denote the $\mathcal{S}$-torsion of X. Since $\mathcal{A}$ has the property (sup), $\sup\{X_{\mathcal{S}_n} \mid n \geq 1\}$ is the $\lim\{\mathcal{S}_n\}$-torsion of X.

(b) Let X is an object of $\lim\{\mathcal{S}_n\}$; i.e. $\sup\{X_{\mathcal{S}_n} \mid n \geq 1\} \simeq X$. And let $f : X \longrightarrow Y$ be an epimorphism. Denote by Y_n the image of the composition $X_{\mathcal{S}_n} \longrightarrow X$ and f. It follows from the property (sup) that the canonical morphism $\sup\{Y_n\} \longrightarrow Y$ is an isomorphism. Since by assumption $Y_n \in Ob\mathcal{S}_n$ for any n, this implies that $Y \in colim\{\mathcal{S}_n\}$. ∎

(b) *The proof of Proposition 5.11.4.2.* By Lemmas 5.11.4.1 and 5.11.4.2, the Gabriel product $\mathcal{S} \bullet \mathcal{T}$ of coreflective full subcategories of $\mathcal{A}$ containing with each object all its quotients (in $\mathcal{A}$) enjoys the same properties. Therefore $\mathcal{S}^{(n)}$ enjoys these properties for any positive integer n. Now it follows from Lemma 5.11.4.3 that $\mathcal{S}^\infty := colim\{\mathcal{S}^{(n)}\}$ is also coreflective and contains all quotients of any of its objects. ∎

Now we go back to R-bimodules.

5.11.4.3. Proposition. *The diagonal Δ_R is the full subcategory of $R \otimes_k R^o - mod$ generated by subbimodules of artinian bimodules. Moreover, for any $n \in \mathbb{N}$, $\Delta_R^{(n)}$ is the full subcategory of $R \otimes_k R^o - mod$ generated by all subbimodules of bimodules of $Art_R^{(n)}$. And*

Δ_R^∞ *is the full subcategory of* $R \otimes_k R^o$ *– mod generated by all subbimodules of bimodules of* Art_R^∞.

Proof. In fact, the subcategory Art_R of artinian bimodules is coreflective and any quotient module of an artinian bimodule is artinian too. So the assertion follows from Propositions 5.11.4.2 and 5.11.4.1. ∎

One of the big advantages of 'artinian formal neighborhoods' $Art_R^{(n)}$ is the following fact.

5.11.4.4. Proposition. *The functor* $F_R : R-bi \longrightarrow End(R-mod)$, $M \mapsto M\otimes_R$, *sends, for any* $n \in \mathbb{N}$, *the subcategory* $Art_R^{(n)}$ *into the* n*-th neighborhood* $\Delta^{(n)}$ *of the diagonal* Δ *in* $End(R-mod)$*; and it sends* Art_R^∞ *into* Δ^∞.

Proof. (a) The functor F_R is right exact. And, for any artinian bimodule M, there exists a bimodule epimorphism $L \longrightarrow M$, where L is a free artinian bimodule; that is $L = (J)R$ – the direct sum of a set J of copies of R. Since $L\otimes_R$ is isomorphic to the direct sum of J copies of the identical functor, $M\otimes_R$ belongs to the diagonal Δ.

(b) Suppose now that F_R sends $Art_R^{(n)}$ into $\Delta^{(n)}$. And let $M \in Art_R^{(n+1)}$; i.e. there exists an exact bimodule sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0 \tag{1}$$

with M' artinian and $M'' \in ObArt_R^{(n)}$. Since the functor F_R is right exact, to the sequence (1) there corresponds an exact sequence of endofunctors:

$$M'\otimes_R \longrightarrow M\otimes_R \longrightarrow M''\otimes_R \longrightarrow 0. \tag{2}$$

By the induction hypothesis, $M''\otimes_R$ belongs to $\Delta^{(n)}$, and we have checked that $M'\otimes_R$ belongs to Δ. Therefore $M\otimes_R$ is an object of $\Delta^{(n+1)}$.

(c) It remains the case $n = \infty$. But the functor F_R is compatible with any colimits. So that if $M \in Art_R^\infty$, i.e. $M = \sup\{M_n \mid n \geq 0\}$, where $\{M_n \mid n \geq 0\}$ is an increasing filtration with $M_n \in ObArt_R^{(n)}$ for every $n \in \mathbb{N}$, then $M\otimes_R = \sup\{M_n\otimes_R \mid n \geq 0\}$. Therefore, since $M_n\otimes_R$ belongs to $\Delta^{(n)}$ for each $n, M\otimes_R$ belongs to Δ^∞. ∎

5.11.5. Strongly differential operators and the Weyl algebra of a noncommutative algebra. The following proposition is analogous to Proposition 5.7.

5.11.5.1. Proposition. *For any pair of bimodules,* $M \in ObArt_R^{(n)}$ *and* $L \in ObArt_R^{(m)}$, *their tensor product,* $M \otimes_R N$, *belongs to* $Art_R^{(nm)}$.

Proof. 1) The category Art_R of artinian R-bimodules is closed with respect to $\otimes_R$, since the center, $\mathfrak{Z}_0(M \otimes_R N)$, of the tensor product of bimodules M and N contains the image of the product, $\mathfrak{Z}_0(M)\otimes_k\mathfrak{Z}_0(N)$, of centers of M and N under canonical epimorphism $M \otimes_k N \longrightarrow M\otimes_R$N. And one can see that, for artinian M and N, the map

$$R\mathfrak{Z}_0(M) \otimes_k \mathfrak{Z}_0(N) = M \otimes_k \mathfrak{Z}_0(N) \longrightarrow M \otimes_R N$$

is epimorphic.

2) Suppose that $M \in ObArt_R^{(n)}, n \geq 2$, and $L \in ObArt_R$. Then $L \otimes_R M \subseteq ObArt_R^{(n)}$. In fact, there is an exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0 \qquad (1)$$

where $M' \in ObArt_R$ and $M'' \in ObArt_R^{(n-1)}$. The sequence

$$L \otimes_R M' \longrightarrow L \otimes_R M \longrightarrow L \otimes_R M'' \longrightarrow 0 \qquad (2)$$

is exact. The bimodule $L \otimes_R M'$ is artinian and, by the induction hypothesis, $L \otimes_R M''$ belongs to $Art_R^{(n-1)}$. This together with the fact that the image of an artinian bimodule is shows that $L \otimes_R M \in ObArt_R^{(n)}$.

3) Take $M \in ObArt_R^{(n)}, n \geq 2$, so that there exists a exact sequence (1) with M' artinian and $M'' \in ObArt_R^{(n-1)}$. And let $L \in ObArt_R^{(m)}$ for some $m \geq 2$. Consider the exact sequence

$$0 \longrightarrow M' \otimes_R L \longrightarrow M \otimes_R L \longrightarrow M'' \otimes_R L \longrightarrow 0 \qquad (3)$$

According to 1), $M'' \otimes_R L \in ObArt_R^{(m)}$. By the induction hypothesis, $M' \otimes_R L$ belongs to $Art_R^{(mn-m)}$. It follows from Proposition 5.11.4.2 and the fact that Art_R contains all quotients of each of its objects, that the image of $M' \otimes_R L$ in $M \otimes_R L$ is an object of $Art_R^{(mn-m)}$. Therefore $M \otimes_R L \in ObArt_R^{(mn)}$. ∎

This implies that the full subcategory Art_R^{∞} generated by all bimodules M such that $M = \mathfrak{z}_\infty M$ is closed with respect to $\otimes_R$ too. One of the consequences of these facts is the following

5.11.5.2. Proposition. *Let A be a k-algebra with an R-bimodule structure determined by a k-algebra morphism $R \longrightarrow A$. Then $\mathfrak{z}_\infty A$ is a subalgebra of A.*

In particular, we have the k-algebra $D_k^{\mathfrak{s}}(R) := \mathfrak{z}_\infty End_k(R)$ of strongly differential operators on R.

5.11.5.3. The Weyl algebra of an algebra. Denote by $A_k(R)$ the subalgebra of $End_k(R)$ generated by the image of $R \otimes_k R^o$ in $End_k R$ (which is the subalgebra in $End_k(R)$ generated by left and right multiplications by elements of R) and by the k-module $Der_k(R)$ of k-derivations of R.

We call $A_k(R)$ *the Weyl algebra of* R.

Since all derivations of a ring R belong to $\mathfrak{z}_1 End_k(R)$ (cf. Example 5.11.2), and the image of $R \otimes_k R^o$ in $End_k R$ belongs to $\mathfrak{z}_0 End_k(R)$, the Weyl algebra $A_k(R)$ is a subalgebra of the algebra $\mathfrak{z}_\infty End_k(R) = D_k^{\mathfrak{s}}(R)$ of strongly differential operators on R.

5.11.5.3.1. Proposition. *Let $\varphi : U(\mathfrak{g}) \otimes_k R \longrightarrow R$ be a Hopf action of $U(\mathfrak{g})$ on any k-algebra R. Then the image of the algebra homomorphism $U(\mathfrak{g}) \longrightarrow End_k(R)$ is contained in the Weyl algebra of R. In particular $U(\mathfrak{g})$ acts by strongly differential operators.*

Proof. This follows from the fact that the Lie algebra $\mathfrak{g}$ acts by derivations (cf. the argument of Proposition 5.10.1 and Example 5.11.2). ∎

5.11.6. Upper central filtration and differential continuous endofunctors. It is useful to extend the facts of Subsections 5.11.1–5.11.4 to a more general setting which includes in particular sheaves of quasi-coherent bimodules on schemes.

Fix an abelian category $\mathcal{A}$ with the property (sup). And fix a Zariski closed subscheme (i.e. reflective and coreflective topologizing subcategory) $\mathbb{T}$ of the category $\mathfrak{End}\mathcal{A}$ of continuous (i.e. having a right adjoint) functors from $\mathcal{A}$ to $\mathcal{A}$. We assume also that $\mathbb{T}$ is closed with respect to compositions of functors and contains the identical functor $Id_{\mathcal{A}}$.

We call a continuous functor $F \in Ob\mathbb{T}$ *artinian* if for any nonzero morphism $g : F \longrightarrow G$, there exists a morphism $f : Id_{\mathcal{A}} \longrightarrow F$ such that $f \circ g \neq 0$.

We denote the full subcategory of $\mathbb{T}$ generated by artinian functors by $Art\mathbb{T}$.

5.11.6.1. Example. Let R be a k-algebra and $\mathcal{A} = R - mod$. Take as $\mathbb{T}$ the full subcategory $\mathfrak{End}_k\mathcal{A}$ of $\mathfrak{End}\mathcal{A}$ generated by k-linear continuous functors. The canonical equivalence $R - bi \longrightarrow \mathfrak{End}\mathcal{A}$, $M \mapsto M\otimes_R$, establishes an equivalence between $\mathbb{T}$ and the Zariski closed subscheme $R \otimes_k R^o - mod$ of $R - bi = R \otimes R^o - mod$. We have already observed (cf. the part (a) of the proof of Proposition 5.11.4.4) that the same equivalence establishes an equivalence between the category $Art\mathbb{T}$ of artinian k-linear endofunctors and the category Art_R of artinian R-bimodules. ∎

5.11.6.2. Proposition. *The subcategory $Art\mathbb{T}$ of $\mathbb{T}$ is coreflective and contains all quotients of any of its objects. It is also a monoidal subcategory of $\mathbb{T}$: i.e. it contains $Id_{\mathcal{A}}$ and closed with respect to the composition of functors.*

Proof. (a) Let $F \in ObArt\mathbb{T}$, $G \in Ob\mathbb{T}$, and let $e : F \longrightarrow G$ be an epimorphism. If g is a nonzero morphism $G \longrightarrow G'$, then $g \circ e : F \longrightarrow G'$ is nonzero. Therefore, since F is artinian, there exists $f : Id_{\mathcal{A}} \longrightarrow F$ such that $(g \circ e) \circ f = g \circ (e \circ f) \neq 0$ which shows that G is artinian.

(b) To prove that $Art\mathbb{T}$ is coreflective, we need to show that every $F \in Ob\mathbb{T}$ has the biggest artinian subobject. Let Arts(F) denote the class of all artinian subobjects of F.

Note that Arts(F) is filtered. In fact, for any two monoarrows $G \longrightarrow F \longleftarrow G'$ in $\mathbb{T}$ with G and G' artinian, the supremum of the subobjects G and G' is the image of the corresponding morphism $G \oplus G' \longrightarrow$F. Clearly the direct sum of artinian object is an artinian object. Therefore $\sup(G, G')$, being an epimorphic image of an artinian object $G \oplus G'$, is artinian too (cf. (a)).

Let now $\{G_i \mid i \in J\}$ be an increasing family of artinian subobjects of F. We claim that $G := \sup\{G_i \mid i \in J\}$ is artinian. Note first that the category $\mathbb{T}$ inherits from $\mathfrak{End}\mathcal{A}$ the property (sup) which implies that the canonical morphism

$$colim\{G_i \mid i \in J\} \longrightarrow \sup\{G_i \mid i \in J\}$$

is an isomorphism. In particular, if g is a nonzero morphism $\sup\{G_i\} \longrightarrow G'$, there exists $j \in J$ such that the composition, g_j, of $\pi_j : G_j \longrightarrow \sup\{G_i\}$ and g is nonzero. Since G_j is artinian, there exists a morphism $f : Id_{\mathcal{A}} \longrightarrow G_j$ such that $g_j \circ f = g \circ (\pi_j \circ f)$ is nonzero.
∎

Denote by $\Delta_{\mathbb{T}}$ the diagonal in $\mathbb{T}$ – the minimal subscheme of $\mathbb{T}$ containing $Id_{\mathcal{A}}$. Applying Propositions 5.11.4.1 and 5.11.4.2, we a obtain direct analog of Proposition 5.11.4.3:

5.11.6.3. Proposition. *The diagonal $\Delta_{\mathbb{T}}$ is the full subcategory of $\mathbb{T}$ generated by all subobjects of artinian objects. Moreover, for any $n \in \mathbb{N}$, $\Delta_{\mathbb{T}}^{(n)}$ is the full subcategory of $\mathbb{T}$ generated by all subobjects of objects of $Art\mathbb{T}^{(n)}$.* And *$\Delta_{\mathbb{T}}^{\infty}$ is the full subcategory of $\mathbb{T}$ generated by all subobjects of objects of $Art\mathbb{T}^{\infty}$.*

Proof. By Proposition 5.11.6.2, the subcategory $Art\mathbb{T}$ of artinian objects is coreflective and contains all quotients of any of its objects. So the assertion follows from Propositions 5.11.4.2 and 5.11.4.1. ■

We have also an analog of Proposition 5.11.4.4:

5.11.6.4. Proposition. *The inclusion functor $\mathbb{T} \longrightarrow End\mathcal{A}$ sends, for any $n \in \mathbb{N}$, the subcategory $Art\mathbb{T}^{(n)}$ into the n-th neighborhood Δ^{∞} of the diagonal Δ in $End\mathcal{A}$. And it sends $Art\mathbb{T}^{(n)}$ into the Δ^{∞}.*

Proof is a simplified version of the argument proving Proposition 5.11.4.4. Details are left to the reader. ■

Mimiking 5.11.4.5, we shall call objects of $Art\mathbb{T}^{(n)}$ *strongly differential objects of $\mathbb{T}$ of order $\leq n$.*

5.12. D$^-$-bimodules and D$^-$-rings. We call an R-bimodule M a D^--bimodule if it belongs to the minimal Serre subcategory Δ_R^- containing Δ_R. Clearly any D-bimodule is a D^--bimodule. The opposite is not true in general. However, we have the following assertion:

5.12.1. Proposition. *Let R be a left noetherian ring. Then any $R \otimes_k R^o$-module which is a D^--torsion is differential.*

Proof. This is a corollary of Proposition 3.1.2.1. ■

5.12.2. Lemma. *The minimal Serre subcategory Δ^- of the category $\mathfrak{End}_k(R-mod)$ containing the identical functor Id_{R-mod} is closed with respect to the composition of functors. Moreover, $\Delta^- \circ \mathbb{T} \subseteq \mathbb{T}$ for any Serre subcategory $\mathbb{T}$ of $\mathfrak{End}_k(R-mod)$.*

Proof. Note first that any topologizing subcategory $\mathbb{T}$ of $\mathfrak{End}_k(R-mod)$ is closed with respect to composition from the left with functors from some family Ξ, then it is closed with respect to composition from the left with functors of the minimal topologizing subcategory $[\Xi]$ containing Ξ.

This follows from the fact that $[\Xi]$ is obtained from Ξ by taking subquotients and direct sums.

Similarly, if a family of functors Ξ stabilizes a thick (resp. Serre) subcategory $\mathbb{T}$ of $\mathfrak{End}_k(R-mod)$, then so does the minimal thick (resp. Serre) subcategory containing Ξ.

Now take $\Xi = \{Id\}$. Clearly, any subcategory $\mathbb{T}$ of $\mathfrak{End}_k(R-mod)$ is stable with respect to the composition with Id. Therefore, if $\mathbb{T}$ is a Serre subcategory, it is stable with respect to the composition with any functor from Δ^-. ■

5.12.3. Corollary. *Let $R \longrightarrow A$ be an algebra morphism. Then Δ^--torsion of* A *is a subring of A.*

We call an algebra morphism $R \longrightarrow A$ (or, by abuse of language, the algebra A itself) a D^--algebra if the R-bimodule A belongs to Δ^-.

6. Differential operators and localizations.

6.1. Differential functors and formal neighborhoods of subschemes. Fix an abelian category $\mathcal{A}$ with the property (sup) (cf. 2.7).

6.1.1. Lemma. *(a) Let $F : \mathcal{A} \longrightarrow \mathcal{A}$ belong to $\Delta^{(n)}$, for a positive integer n (resp. $F \in Ob\Delta^{\infty}$). Then, for any subscheme $\mathbb{T}$ of $\mathcal{A}$, $F(\mathbb{T}) \subseteq \mathbb{T}^{(n)}$ (resp. $F(\mathbb{T}) \subseteq \mathbb{T}^{\infty}$).*

(b) If $F : \mathcal{A} \longrightarrow \mathcal{A}$ belongs to Δ^-, then every Serre subcategory, $\mathbb{S}$, of $\mathcal{A}$ is stable with respect to F; i.e. $F(\mathbb{S}) \subseteq \mathbb{S}$.

Proof. (a) Fix a subscheme $\mathbb{T}$ of $\mathcal{A}$.

1) Note that if $F \in Ob\Delta$, then $F(\mathbb{T}) \subseteq \mathbb{T}$.

In fact, let $\Xi_{\mathbb{T}}$ denote the full subcategory of $End\mathcal{A}$ generated by all endofunctors F such that $F(\mathbb{T}) \subseteq \mathbb{T}$. If $F \in Ob\Xi_{\mathbb{T}}$, than every subquotient F' of F belongs to $\Xi_{\mathbb{T}}$, since, for any $X \in Ob\mathbb{T}$, the object $F'(X)$, being a subquotient of $F(X)$, is contained in $Ob\mathbb{T}$. Similarly, $\Xi_{\mathbb{T}}$ is closed with respect to direct sums and the taking supremums of subobjects. In other words, $\Xi_{\mathbb{T}}$ is a subscheme of $End\mathcal{A}$. Clearly $\Xi_{\mathbb{T}}$ contains $Id_{\mathcal{A}}$. Therefore, it contains the minimal subscheme, Δ, generated by $Id_{\mathcal{A}}$.

2) Let n be a positive integer. Suppose it is established that $G(\mathbb{T}) \subseteq \mathbb{T}^{(n)}$ for any $G \in Ob\Delta^{(n)}$. And let $F \in Ob\Delta^{(n+1)}$. The latter means that there exists an exact sequence

$$0 \longrightarrow G \longrightarrow F \longrightarrow G' \longrightarrow 0 \qquad (1)$$

with $G \in Ob\Delta^{(n)}$ and $G' \in Ob\Delta$. The exactness of (1) means exactly that, for any $X \in Ob\mathcal{A}$, the sequence

$$0 \longrightarrow G(X) \longrightarrow F(X) \longrightarrow G'(X) \longrightarrow 0$$

is exact. If $X \in Ob\mathbb{T}$, then $G(X) \in Ob\Delta^{(n)}$ by assumption, and $G'(X) \in Ob\Delta$ by 1). Therefore $F(X) \in Ob\Delta^{(n+1)}$.

3) Suppose now that $F \in Ob\Delta^{\infty}$. This means that F is the supremum of an increasing family $\{F_n \mid n \geq 1\}$ of its subfunctors such that, for any $n, F_n \in Ob\Delta^{(n)}$. Therefore, for any $X \in Ob\mathbb{T}, F(X) = \sup\{F_n(X) \mid n \geq 1\}$; and, for any n, $F_n(X) \in Ob\mathbb{T}^{(n)}$ (cf. 2)); i.e. $F(X) \in Ob\mathbb{T}^{\infty}$.

(b) Suppose now that $\mathbf{S}$ is a thick subcategory of $\mathcal{A}$. Then $\Xi_{\mathbf{S}}$ is also closed with respect to extensions, hence thick (cf. the argument in 1)). Thus, if $\mathbb{S}$ is a Serre subcategory of $\mathcal{A}$, then $\Xi_{\mathbf{S}}$ is a thick subscheme of $End\mathcal{A}$. By Lemma 2.6.2, $\Xi_{\mathbf{S}}$ is a Serre subcategory of $End\mathcal{A}$. Therefore, since $Id_{\mathcal{A}} \in \Xi_{\mathbf{S}}$, $\Delta^- \subseteq \Xi_{\mathbf{S}}$. ∎

6.2. Differential endofunctors and localizations. Here we shall show that exact differential endofunctors and monads are compatible with localizations at Serre subcategories.

Fix a thick category $\mathbb{T}$ of an abelian category $\mathcal{A}$ and a localization $Q : \mathcal{A} \longrightarrow \mathcal{A}/\mathbb{T}$ at $\mathbb{T}$. Denote by $\Sigma_{\mathbf{T}}$ the class of all morphisms s of $\mathcal{A}$ such that $Ker(s)$ and $Cok(s)$ are objects of $\mathbb{T}$. Or, equivalently, $\Sigma_{\mathbb{T}} = \{s \in \mathrm{Hom}\mathcal{A} \mid Qs \text{ is invertible}\}$.

6.2.1. Lemma. *Let F be a functor $\mathcal{A} \longrightarrow \mathcal{A}$.*

(a) The following conditions on F are equivalent:

(i) $F(\Sigma_{\mathbf{T}}) \subseteq \Sigma_{\mathbf{T}}$

(ii) There exists a unique functor $F_{\mathbb{T}}$ such that $Q \circ F = F_{\mathbf{T}} \circ Q$.

(b) If $F(\Sigma_{\mathbf{T}}) \subseteq \Sigma_{\mathbf{T}}$, then $F(\mathbb{T}) \subseteq \mathbb{T}$.

(c) Suppose that $F(\mathbb{T}) \subseteq \mathbb{T}$ and F is exact. Then $F(\Sigma_{\mathbb{T}}) \subseteq \Sigma_{\mathbb{T}}$, and the functor $F_{\mathbf{T}}$ defined by the equality $Q \circ F = F_{\mathbf{T}} \circ Q$ is exact too.

Proof. *(a)* The inclusion $F(\Sigma_{\mathbf{T}}) \subseteq \Sigma_{\mathbf{T}}$ means exactly that $Q \circ F(\Sigma_{\mathbf{T}})$ consists of only invertible morphisms. The latter implies, by the universal property of localizations, that there exists a unique functor $F_{\mathbb{T}} : \mathcal{A}/\mathbb{T} \longrightarrow \mathcal{A}/\mathbb{T}$ such that $Q \circ F = F_{\mathbb{T}} \circ Q$.

Conversely, the equality $Q \circ F = F_{\mathbf{T}} \circ Q$ implies that the morphism $Q \circ F(s)$ is invertible for any $s \in \Sigma_{\mathbf{T}}$.

(b) The subcategory $\mathbb{T}$ is the kernel of the localization Q, i.e. $Ob\mathbb{T} = \{X \in Ob\mathcal{A} \mid Q(X) = 0\}$. It follows from the equality $Q \circ F = F_{\mathbb{T}} \circ Q$ that $Q \circ F(X) = 0$ for any $X \in Ob\mathbb{T}$; i.e. $F(X) \in Ob\mathbb{T}$.

(c) Suppose that the conditions of *(c)* hold; i.e. F is exact and $F(\mathbb{T}) \subseteq \mathbb{T}$. Let $s : L \longrightarrow M$ be any morphism of $\mathcal{A}$ such that Qs is invertible; i.e. $Ker(s)$ and $Cok(s)$ are objects of $\mathbb{T}$. Since the functor F is exact, it sends the exact sequence

$$0 \longrightarrow Ker(s) \longrightarrow L \xrightarrow{s} M \longrightarrow Cok(s) \longrightarrow 0$$

into the exact sequence

$$0 \longrightarrow F(Ker(s)) \longrightarrow F(L) \xrightarrow{Fs} F(M) \longrightarrow F(Cok(s)) \longrightarrow 0 \qquad (2)$$

By assumption, $F(Ker(s)) \simeq Ker(Fs)$ and $F(Cok(s)) \simeq Cok(Fs)$ are objects of the subcategory $\mathbb{T}$ which means that $Fs \in \Sigma_{\mathbf{T}}$.

According to *(a)*, $Q \circ F = F_{\mathbf{T}} \circ Q$ for a uniquely determined $F_{\mathbb{T}}$. The functor $F_{\mathbf{T}}$ is exact by Proposition I.3.4 in [GZ]. ∎

We shall say that a functor $F : \mathcal{A} \longrightarrow \mathcal{A}$ is compatible with localizations at $\mathbb{T}$ if F satisfies the equivalent conditions *(i)*, *(ii)* of Lemma 6.2.1.

6.2.2. Proposition. *Let $\mathbb{T}$ be any thick subcategory of $\mathcal{A}$ and $Q : \mathcal{A} \longrightarrow \mathcal{A}/\mathbb{T}$ a localization at $\mathbb{T}$. Let F be a functor from $\mathcal{A}$ to $\mathcal{A}$ compatible with Q; i.e. $Q \circ F = F_{\mathbf{T}} \circ Q$ for a (unique) functor $F_{\mathbb{T}} : \mathcal{A}/\mathbb{T} \longrightarrow \mathcal{A}/\mathbb{T}$.*

(a) If the localization Q and the functor F have right adjoints, then $F_{\mathbf{T}}$ has a right adjoint.

(b) Let $\mu : F \circ F \longrightarrow F$ be a structure of a monad in $\mathcal{A}$. Then $\mathbb{F} = (F, \mu)$ defines uniquely a monad $\mathbb{F}_{\mathbb{T}} = (F_{\mathbf{T}}, \mu_{\mathbf{T}})$ in $\mathcal{A}/\mathbb{T}$ *such that the localization $Q : \mathcal{A} \longrightarrow \mathcal{A}/\mathbb{T}$ induces an exact and faithful functor*

$$\Psi : \mathbb{F} - mod/\mathfrak{F}^{-1}(\mathbb{T}) \longrightarrow \mathbb{F}_{\mathbb{T}} - mod.$$

Here $\mathfrak{F}$ is the forgetting functor $\mathbb{F} - mod \longrightarrow \mathcal{A}$.

(b1) If the localization Q has a right adjoint, then the functor Ψ is an equivalence of categories.

Proof. *(a)* Let $Q^{\wedge}$ and $F^{\wedge}$ be right adjoints of resp. Q and F. Denote by G the composition $Q \circ F^{\wedge} \circ Q^{\wedge} : \mathcal{A}/\mathbb{T} \longrightarrow \mathcal{A}/\mathbb{T}$. We have canonical morphisms:

$$F_{\mathbb{T}} \circ G = F_{\mathbb{T}} \circ Q \circ F^{\wedge} \circ Q = Q \circ F \circ F^{\wedge} \circ Q^{\wedge} \longrightarrow Q \circ Q^{\wedge} \longrightarrow Id_{\mathcal{A}/\mathbf{T}} \qquad (1)$$

the right arrow being the adjunction isomorphism. Denote the composition of these morphisms (1) by ϵ.

On the other hand, there are arrows

$$Q \longrightarrow Q \circ F^{\wedge} \circ F \longrightarrow Q \circ F^{\wedge} \circ Q^{\wedge} \circ Q \circ F = (Q \circ F^{\wedge} \circ Q^{\wedge} \circ F_{\mathbf{T}}) \circ Q = G \circ F_{\mathbf{T}} \circ Q$$

By the universal property of localizations, the composition of arrows (2) is equal to ηQ for a uniquely defined morphism $\eta : Id_{\mathcal{A}/\mathbf{T}} \longrightarrow G \circ F_{\mathbf{T}}$. It follows from the definition of ϵ and η that these morphisms are adjunction arrows (details of the checking are left to a reader); hence G is a left adjoint to $F_{\mathbb{T}}$.

(b) Suppose now that $\mathbb{F} = (F, \mu)$ is a monad with an exact functor F such that $F(\mathbb{T}) \subseteq \mathbb{T}$. Then the multiplication μ determines a multiplication $\mu' : F_{\mathbf{T}} \circ F_{\mathbf{T}} \longrightarrow F_{\mathbf{T}}$.

Thanks to the equality (1), we can define a functor, $\Phi : \mathbb{F} - mod \longrightarrow F_{\mathbb{T}} - mod$, by $\Phi(M, m) = (Q(M), Qm)$, $\Phi f = Qf$ for any $\mathbb{F}$-module (M, m) and for any $\mathbb{F}$-module morphism f.

Since the localization Q is exact, the functor Φ is exact, and $\mathrm{Ker}\Phi = \mathfrak{F}^{-1}(\mathbb{T})$. Therefore $\Phi = \Psi \circ Q'$, where Q' is a localization $\mathbb{F} - mod \longrightarrow \mathbb{F} - mod/\mathfrak{F}^{-1}(\mathbb{T})$ at $\mathfrak{F}^{-1}(\mathbb{T})$, Ψ a uniquely defined exact and faithful functor from $\mathbb{F} - mod/\mathfrak{F}^{-1}(\mathbb{T})$ to $\mathbb{F}_{\mathbb{T}} - mod$.

(b1) Suppose now that the localization Q has a right adjoint, $Q^{\wedge}$. Let η, ϵ denote the adjunction morphisms respectively $Id_{\mathcal{A}} \longrightarrow Q^{\wedge} \circ Q$ and $Q \circ Q^{\wedge} \longrightarrow Id_{\mathcal{A}/\mathbf{T}}$. For any $\mathbb{F}_{\mathbf{T}}$-module (M, m), set $\Phi^{\wedge}(M, m) := (Q^{\wedge}(M), m')$, where m' is the composition of

$$\eta F Q^{\wedge}(M) : F Q^{\wedge}(M) \longrightarrow Q^{\wedge} Q F Q^{\wedge}(M),$$

$$Q^{\wedge} F_{\mathbf{T}} \epsilon(M) : Q^{\wedge} Q F Q^{\wedge}(M) = Q^{\wedge} F_{\mathbf{T}} Q Q^{\wedge}(M) \longrightarrow Q^{\wedge} F_{\mathbf{T}}(M),$$

and $Q^{\wedge} m : Q^{\wedge} F_{\mathbb{T}}(M) \longrightarrow Q^{\wedge}(M)$.

One can check that the map $(M, m) \mapsto (Q^{\wedge} M, m')$, $f \mapsto Q^{\wedge} f$ for any $\mathbb{F}_{\mathbf{T}}$-module (M, m) and any $\mathbb{F}_{\mathbf{T}}$-module morphism f defines a functor $\Phi^{\wedge}$ from the category $\mathbb{F}_{\mathbb{T}} - mod$ to $\mathbb{F} - mod$ which is right adjoint to the functor Φ defined above. The adjunction arrows $\eta' : Id \longrightarrow \Phi^{\wedge} \circ \Phi$ and $\epsilon' : \Phi \circ \Phi^{\wedge} \longrightarrow Id$ are induced from the adjunction arrows respectively η and ϵ. In particular, ϵ' is an isomorphism which means that the functor $\Phi^{\wedge}$ is fully faithful. Therefore Φ is a localization (cf. [GZ], Proposition I.1.3). Since Φ is exact, the induced functor

$$\Psi : \mathbb{F} - mod/\mathfrak{F}^{-1}(\mathbb{T}) \longrightarrow \mathbb{F}_{\mathbf{T}} - mod$$

is an equivalence of categories. ■

Let $Q : \mathcal{A} \longrightarrow \mathcal{A}/\mathbb{T}$ be an exact localization having a right adjoint. For any functor $F : \mathcal{A} \longrightarrow \mathcal{A}$, one has an endofunctor $F' = Q \circ F \circ Q\hat{} : \mathcal{A}/\mathbb{T} \longrightarrow \mathcal{A}/\mathbb{T}$. So that we have a canonical functor morphism:

$$\eta' := Q \circ F\eta : Q \circ F \longrightarrow F' \circ Q = Q \circ F \circ Q\hat{} \circ Q, \tag{1}$$

where $\eta : Id_{\mathcal{A}} \longrightarrow Q\hat{} \circ Q$ is an adjunction arrow.

6.2.3. Lemma. *The following conditions are equivalent:*

(i) The morphism $\eta' := Q \circ F\eta : Q \circ F \longrightarrow F' \circ Q = Q \circ F \circ Q\hat{} \circ Q$ *is an isomorphism.*

(ii) There is a functor $G : \mathcal{A}/\mathbb{T} \longrightarrow \mathcal{A}/\mathbb{T}$ *such that* $Q \circ F \simeq G \circ Q$.

(iii) There exists a functor $F_{\mathbf{T}} : \mathcal{A}/\mathbb{T} \longrightarrow \mathcal{A}/\mathbb{T}$ *uniquely determined by the equality* $F_{\mathbf{T}} \circ Q = Q \circ F$.

Proof. *(i)*$\Leftrightarrow$ *(ii)* is trivial.

(ii)$\Leftrightarrow$ *(iii)*. The isomorphism $Q \circ F \simeq G \circ Q$ shows that the functor $Q \circ F$ makes invertible all arrows the localization Q makes invertible. Therefore, by the universal property of localizations, there exists a unique functor $F_{\mathbf{T}} : \mathcal{A}/\mathbb{T} \longrightarrow \mathcal{A}/\mathbb{T}$ such that $Q \circ F = F_{\mathbf{T}} \circ Q$.

(iii)$\Leftrightarrow$ *(i)*. The equality $Q \circ F = F_{\mathbf{T}} \circ Q$ implies that $\eta' := Q \circ F\eta = F_{\mathbf{T}} \circ Q\eta$. But $Q\eta$ is an isomorphism (because $Q\hat{}$ is fully faithful). Therefore η' is an isomorphism. ∎

6.2.3.1. Note. Let $\mathcal{Q} : \mathcal{A} \longrightarrow \mathcal{B}$ be a functor having a fully faithful right adjoint, $\mathcal{Q}\hat{}$. Then, for any functor $G : \mathcal{B} \longrightarrow \mathcal{B}$, the functor $G'' := \mathcal{Q}\hat{} \circ G \circ \mathcal{Q} : \mathcal{A} \longrightarrow \mathcal{A}$ has the desired property: $\mathcal{Q} \circ G'' = \mathcal{Q} \circ \mathcal{Q}\hat{} \circ G \circ \mathcal{Q} \simeq G \circ \mathcal{Q}$. This follows from the fact that the adjunction morphism $\epsilon : \mathcal{Q} \circ \mathcal{Q}\hat{} \longrightarrow Id_{\mathcal{B}}$ is, thanks to the full faithfulness of $\mathcal{Q}\hat{}$, an isomorphism (cf. [GZ], Proposition I.1.3). ∎

6.2.3.2. Remark. Let $\mathcal{Q} : \mathcal{A} \longrightarrow \mathcal{B}$ be a functor having a fully faithful right adjoint, $\mathcal{Q}\hat{}$. And let $F : \mathcal{A} \longrightarrow \mathcal{A}$ be any endofunctor. Consider the full subcategory $\mathcal{C} = \mathcal{C}_F$ of $\mathcal{A}$ generated by all objects X of $\mathcal{A}$ such that the canonical morphism

$$\mathcal{Q} \circ F\eta(X) : \mathcal{Q} \circ F(X) \longrightarrow \mathcal{Q} \circ F \circ \mathcal{Q}\hat{} \circ \mathcal{Q}(X)$$

is an isomorphism. Note that, thanks to the full faithfulness of $\mathcal{Q}\hat{}$, the functor $\mathcal{Q}\hat{}$ takes values in $\mathcal{C}$. In particular, the restriction, $\mathcal{Q}_F$, of the functor $\mathcal{Q}$ to $\mathcal{C}$ has a fully faithful right adjoint, $\mathcal{Q}_F\hat{} = \mathcal{Q}\hat{} |^{\mathcal{C}}$ (– the corestriction of $\mathcal{Q}\hat{}$ to $\mathcal{C}$); i.e. $\mathcal{Q}_F$ is a localization of $\mathcal{C}$ with the same quotient category $\mathcal{B}$.

The restriction of the functor F to $\mathcal{C}$ is compatible with the functor (localization) $\mathcal{Q}$. Which means that there exists a functor $F' : \mathcal{B} \longrightarrow \mathcal{B}$ ($F' = \mathcal{Q} \circ F \circ \mathcal{Q}\hat{}$) such that $F' \circ \mathcal{Q} |_{\mathcal{C}} \simeq \mathcal{Q} \circ F |_{\mathcal{C}}$. ∎

6.2.4. Proposition. *(a) Let* $F : \mathcal{A} \longrightarrow \mathcal{A}$ *be an exact* Δ^-*-functor. Then, for any localization* $Q : \mathcal{A} \longrightarrow \mathcal{A}/\mathbb{T}$ *at a Serre subcategory* $\mathbb{T}$ *of* $\mathcal{A}$*, there exists a unique exact functor* $F_{\mathbf{T}} : \mathcal{A}/\mathbb{T} \longrightarrow \mathcal{A}/\mathbb{T}$ *such that* $Q \circ F = F_{\mathbf{T}} \circ Q$.

(b) Suppose that the subcategory $\mathbb{T}$ *is 'localizable'; i.e. the localization* $Q : \mathcal{A} \longrightarrow \mathcal{A}/\mathbb{T}$ *has a right adjoint. Then* $F_{\mathbf{T}} \in Ob\Delta^-$*. Moreover, if* $F \in Ob\Delta^{(n)}$*, where* n *is any positive integer, then* $F_{\mathbf{T}} \in Ob\Delta^{(n)}$*. If* $F \in Ob\Delta^{\infty}$*, then* $F_{\mathbf{T}} \in Ob\Delta^{\infty}$

(c) Let $\mathbb{F} = (F, \mu)$ be a D^--monad such that the functor F is exact. Then, for any Serre subcategory $\mathbb{T}$, the monad $\mathbb{F}$ induces a monad $\mathbb{F}_{\mathbb{T}}$ in the quotient category $\mathcal{A}/\mathbb{T}$ and a canonical exact and faithful functor

$$\Psi_{\mathbb{T}} : \mathbb{F} - mod/\mathfrak{F}^{-1}(\mathbb{T}) \longrightarrow \mathbb{F}_{\mathbb{T}} - \mathrm{mod}.$$

Suppose that the subcategory $\mathbb{T}$ is localizable. Then the functor $\Psi_{\mathbb{T}}$ is an equivalence of categories. And $\mathbb{F}_{\mathbb{T}}$ is a D^--monad.

If $\mathbb{F}$ is a differential monad, then $\mathbb{F}_{\mathbb{T}}$ is differential.

Proof. (a) Suppose that $F \in Ob\Delta^{(n)}$. Then, by Lemma 6.1.1, any Serre subcategory $\mathbb{T}$ of $\mathcal{A}$ is F-stable. It follows now from Lemma 6.2.1 and Propositions 6.2.2 that $Q \circ F = F_{\mathbb{T}} \circ Q$ for a unique functor $F_{\mathbb{T}}$. According to Proposition I.3.4 in [GZ], the functor $F_{\mathbb{T}}$ is exact.

(b) Suppose now that $\mathbb{T}$ is localizable, and fix a right adjoint $Q^\wedge$ to Q. Note that, thanks to the exactness of Q, the functor $\mathcal{L} : F \mapsto Q \circ F \circ Q^\wedge$ from $End\mathcal{A}$ to $End\mathcal{A}/\mathbb{T}$ is exact. Moreover, since Q respects colimits, the functor $\mathcal{L}$ enjoys the same property.

For any family $\Omega \subset End\mathcal{A}$, denote by $\mathcal{S}(\Omega)$ the minimal Serre subcategory of $End\mathcal{A}$ contained Ω. Note that $Ob\mathcal{S}(\Omega)$ is obtained from Ω by taking subquotients, direct sums, and supremums. Since the functor $\mathcal{L}$ is compatible with these operations, the image $\mathcal{L}(\mathcal{S}(\Omega))$ of $\mathcal{S}(\Omega)$ is contained in $\mathcal{S}(\mathcal{L}(\Omega))$ and the embedding $\mathcal{L}(\mathcal{S}(\Omega)) \subseteq \mathcal{S}(\mathcal{L}(\Omega))$ is an equivalence. Applying this observation to $\Omega = \{Id_{\mathcal{A}}\}$, we see that $\mathcal{L}$ assigns to any functor from Δ^- a functor from Δ^-.

This shows also that, for all $n \in \mathbb{Z}_+$, $\mathcal{L}$ sends $\Delta^{(n)}$ into $\Delta^{(n)}$. Therefore $\mathcal{L}$ sends Δ^∞ into Δ^∞.

It follows from Lemma 6.2.3 that $\mathcal{L}(F)$ is canonically isomorphic to $F_{\mathbb{T}}$ whenever $F_{\mathbb{T}}$ exists. This implies the assertions of *(b)*.

(c) The functor $\Psi_{\mathbb{T}}$ being an equivalence of categories follows from Proposition 6.2.2. The rest follows from *(b)*. ∎

6.2.5. Remarks and observations. Proposition 6.2.3 is a fundamental fact having one annoying condition – that of the differential functor F being exact. The following two lemmas provide a partial remedy.

6.2.5.1. Lemma. *Let F be a functor from $\mathcal{A}$ to $\mathcal{A}$; and let $\mathbb{T}$ be a thick subcategory of $\mathcal{A}$ stable with resp. to the functor* F. *Any of the following conditions garantees the compatibility of F with the localization at $\mathbb{T}$:*

(a) For any monomorphism (resp. epimorphism) $f : X \longrightarrow Y$, the object $Ker(Ff)$ (resp. $Cok(Ff)$) belongs to the subcategory $\mathbb{T}$.

(b) The functor F is right exact and $S_1F(\mathbb{T}) \subseteq \mathbb{T}$. Here S_1F is the left satellite of F.

(c) The functor F is left exact and, for any object X of $\mathbb{T}$, there exists an injective embedding $X \longrightarrow M$ in $\mathcal{A}$ such that $M \in Ob\mathbb{T}$ (for instance, an injective hull of any object of $\mathbb{T}$ belongs to $\mathbb{T}$).

(d) Both S_1F and S^1F exist, and the subcategory $\mathbb{T}$ is stable with respect to S_1F and S^1F.

Proof. (a) The condition *(a)* implies that the functor $Q \circ F$ sends epimorphisms into epimorphisms and monomorphisms into monomorphisms. Thus $Q \circ F$ is an exact functor annihilating the subcategory $\mathbb{T}$ which implies that $Q \circ F(s)$ is invertible for any $s \in \Sigma_{\mathbf{T}} := \{s \in \mathrm{Hom}\mathcal{A} \mid\ Ker(s),\ Cok(s) \in Ob\mathbb{T}\}$. Therefore, by the universal property of localizations, $Q \circ F = F_{\mathbf{T}} \circ Q$ for a unique functor $F_{\mathbf{T}}$.

(b) Let $s : X \longrightarrow Y$ be an epimorphism in $\mathcal{A}$ such that $K = Ker(s)$ is an object of $\mathbb{T}$. Since the functor F is right exact, the sequence

$$F(K) \longrightarrow F(X) \longrightarrow F(Y) \longrightarrow 0 \qquad (1)$$

is exact. Applying to (1) the localization $Q = Q_{\mathbb{T}}$, and using that Q is exact and, by assumption, $F(K) \in Ob\mathbb{T}$, hence $Q \circ F(K) = 0$, we get an exact sequence

$$0 \longrightarrow QF(X) \xrightarrow{QFs} QF(Y) \longrightarrow 0$$

i.e. QFs is an isomorphism. Note that S_1F is not used in this argument.

Suppose now that $t : X \longrightarrow V$ is a monomorphism such that $W := Cok(t) \in Ob\mathbb{T}$. Then we have an exact sequence

$$S_1F(W) \longrightarrow F(X) \xrightarrow{Ft} F(V) \longrightarrow F(W). \qquad (2)$$

Applying to (2) the functor Q and using the fact that $S_1F(W)$ and $F(W)$ belong to $\mathbb{T}$, hence $QS_1F(W) = 0 = QF(W)$, we obtain that $QF(t)$ is an isomorphism.

It remains to notice that any morphism u of $\mathcal{A}$ such that Ker(u) and Cok(u) are objects of $\mathbb{T}$, has a decomposition $u = t \circ s$, where s is an epimorphism and t is a monomorphism such that $Ker(s)$ and Cok(t) belong to $\mathbb{T}$.

(c) The condition of *(c)* implies that $S^1F(\mathbb{T}) \subseteq \mathbb{T}$. The assertion follows from *(b)* by switching to the opposite category.

(d) The argument is similar to that of *(b)*. We leave detailes to a reader. ∎

6.2.5.2. Lemma. *Let Ω_Q be the class of all functors $F : \mathcal{A} \longrightarrow \mathcal{A}$ satisfying the equivalent conditions of Lemma 6.2.3. Suppose that $F \longrightarrow G \longrightarrow H \longrightarrow K \longrightarrow L$ is an exact sequence of functors $\mathcal{A} \longrightarrow \mathcal{A}$ such that $F, G, K, L \in \Omega_Q$. Then $H \in \Omega_Q$.*

Proof. Consider the commutative diagram with the canonical vertical arrows:

$$\begin{array}{ccccccccc} Q\circ F & \longrightarrow & Q\circ G & \longrightarrow & Q\circ H & \longrightarrow & Q\circ K & \longrightarrow & Q\circ L \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ F'\circ Q & \longrightarrow & F'\circ Q & \longrightarrow & F'\circ Q & \longrightarrow & F'\circ Q & \longrightarrow & L'\circ Q \end{array} \qquad (1)$$

where $F' = Q \circ F \circ Q^{\wedge}$, etc. (cf. Lemma 6.2.1.2.1). By assumption, all vertical arrows except possibly the central one are isomorphisms. By the five-lemma, the central vertical arrow is an isomorphism too. ∎

6.3. Localization of differential bimodules. Fix a commutative ring k. Let R be an associative k-algebra. Set $R^{\mathfrak{e}} := R \otimes_k R^o$ (– *enveloping k-algebra of R*). Denote by $\mathcal{A}$ the category $R - mod$ of left R-modules. We shall regard $R^{\mathfrak{e}} - mod$ as the category of quasi-coherent sheaves on the square of $SpecR$ over $Speck$. The diagonal, Δ_R, is the minimal subscheme of $R^{\mathfrak{e}} - mod$ containing the $R^{\mathfrak{e}}$-module R.

6.3.1. Proposition. *(a) Let M be a strongly differential $R^{\mathfrak{e}}$-module. If M is flat as a right R-module, then, for any Serre subcategory $\mathbb{T}$ of $R-mod$, the functor $M\otimes_R$ induces a functor $M_{\mathbb{T}} : R-mod/\mathbb{T} \longrightarrow R-mod/\mathbb{T}$.*

(b) Let $R \longrightarrow A$ be an algebra morphism such that A is a strongly differential $R^{\mathfrak{e}}$-module flat as a right R-module. Then, for any Serre subcategory $\mathbb{T}$ of the category $R-mod$, the algebra A induces a monad, $A_{\mathbf{T}}$, on $R-mod/\mathbb{T}$.

Proof. The fact follows from Proposition 6.2.2 and Proposition 5.11.4.3. ■

6.3.2. Proposition. *Let $R \longrightarrow R'$ be an algebra morphism such that the functor $Q = R'\otimes_R$ is an exact localization. Then*

(a) Any strongly differential $R^{\mathfrak{e}}$-module M which is flat as a right R-module determines a strongly differential $R'^{\mathfrak{e}}$-module $M' = R'\otimes_R M\otimes_R R'$. And M' is isomorphic to $R'\otimes_R M$ as (R', R)-bimodules.

(b) If $M \in ObArt_R^{(n)}$, i.e. if M is a strongly differential $R^{\mathfrak{e}}$-module of n-th order, then the $R'^{\mathfrak{e}}$-module M' has the same order: $M' \in ObArt_{R'}^{(n)}$.

(c) Let $R \longrightarrow A$ be an algebra morphism such that A is a strongly differential $R^{\mathfrak{e}}$-module flat as a right R-module. Then $R'\otimes_R A$ has a unique k-algebra structure such that the canonical maps $A \longrightarrow R'\otimes_R A \longleftarrow R'$ are k-algebra morphisms. And $R'\otimes_R A$ is a strongly differential $R'^{\mathfrak{e}}$-module.

Proof. 1) Let M be an $R^{\mathfrak{e}}$-module. By Lemma 6.2.3, the functor $M\otimes_R$ is compatible with the localization $Q : R-mod \longrightarrow R-mod/\mathbb{S}$ iff the canonical morphism

$$Q\circ(M\otimes_R) \longrightarrow Q\circ(M\otimes_R)\circ Q^{\wedge}\circ Q \qquad (1)$$

is an isomorphism. In the case when $R-mod/\mathbb{S} = R'-mod$ for some k-algebra R', hence Q can be taken equal to $R'\otimes_R$, the isomorphness of (1) means that the canonical $R'\otimes_k R^o$-module morphism

$$R'\otimes_R M \longrightarrow R'\otimes_R M\otimes_R R' \qquad (2)$$

is an isomorphism.

(a) Let M be a strongly differential $R^{\mathfrak{e}}$-module. By Proposition 6.3.1, the functor $M\otimes_R$ induces a functor $M_{\mathbb{T}}$, where $\mathbb{T}$ is the kernel of the localization Q. Since $Q = R'\otimes_R$ for some k-algebra morphism $R \longrightarrow R'$, the canonical morphism (2) is an isomorphism. This proves the assertion *(a)*.

(b) The assertion *(b)* follows from the fact that the functor $R'\otimes_R$, being a localization, is exact and, for any R'-module L, the natural R'-module morphism $R'\otimes_R L \longrightarrow L$ is an isomorphism. In particular, we have: $R'\otimes_R R' \simeq R' \simeq R'\otimes_R$R. Therefore if $M \in ObArt_R$, i.e. M is a quotient of a direct sum of a set of copies of R, then $R'\otimes_R M$ is a quotient of a direct sum of a set of copies of R'. The rest of the proof is a standart induction argument which goes through thanks to the exactness of of the localization $R'\otimes_R$. Details are left to a reader.

(c) The fact follows from *(a)* and the assertion *(b)* of Proposition 6.3.1. ■

6.3.3. Enveloping algebras and differential operators. Let $U(\mathfrak{g})$ be the enveloping algebra of a Lie algebra $\mathfrak{g}$ over a field k of zero characteristic.

6.3.3.1. Proposition. *Let $\tau : U(\mathfrak{g}) \otimes_k R \longrightarrow R$ be a Hopf action on* a *k-algebra* R. *Let $R \longrightarrow R'$ be* a *k-algebra morphism such that $R' \otimes_R$ is a localization functor (e.g. R' is a localization of R at a left Ore set). Then the action of $U(\mathfrak{g})$ extends uniquely to a Hopf action on R'.*

Proof. By Proposition 5.10.1, if $\varphi : U(\mathfrak{g}) \otimes_k R \longrightarrow R$ is a Hopf action of $U(\mathfrak{g})$ on a k-algebra R, then $U(\mathfrak{g})$ acts by strongly differential operators. Therefore the crossed product $R\#U(\mathfrak{g})$ acts on R by strongly differential operators. Note that $R\#U(\mathfrak{g})$ is a flat right (and left) R-module. By Proposition 6.3.2, $R' \otimes_R R\#U(\mathfrak{g})$ has a uniquely defined structure of a k-algebra such that $R' \longrightarrow R' \otimes_R R\#U(\mathfrak{g})$ is an algebra morphism turning $R' \otimes_R R\#U(\mathfrak{g})$ into a strongly differential $R'^{\mathfrak{e}}$-bimodule.

For any $R\#U(\mathfrak{g})$-module M, the algebra $R' \otimes_R R\#U(\mathfrak{g})$ acts on $R' \otimes_R$M. In particular, we have a uniquely defined action τ' of $U(\mathfrak{g})$ on R' by strongly differential operators. We claim that τ' is a Hopf action.

Denote for convenience $U(\mathfrak{g})$ by U and $R\#U(\mathfrak{g})$ by $\mathfrak{U}$.

1) Let $(M, \rho : U \otimes M \longrightarrow M)$ be an U-module. Then the composition

$$U \otimes (R' \otimes M) \xrightarrow{\delta} U \otimes U \otimes (R' \otimes M) \xrightarrow{\sigma} (U \otimes R') \otimes (U \otimes M) \xrightarrow{\tau' \otimes \rho} R' \otimes M \qquad (1)$$

defines an action of $R' \otimes_R \mathfrak{U}$ on $R' \otimes M$ (- the 'tensor action') which we denote by $\tau' \odot \rho$. Clearly the diagram

$$\begin{array}{ccccc} R' \otimes_R \mathfrak{U} \otimes (R' \otimes M) & \xrightarrow{\tau' \otimes \rho} & R' \otimes M & \xrightarrow{m'} & R' \otimes_R M \\ \uparrow & & \uparrow & & \uparrow \\ \mathfrak{U} \otimes (R \otimes M) & \xrightarrow{\tau \otimes \rho} & R \otimes M & \xrightarrow{m} & M \end{array} \qquad (2)$$

commutes. Applying the localization functor $R' \otimes_R : R-mod \longrightarrow R-mod$ to the diagram (2), we obtain the diagram

$$\begin{array}{ccccc} R' \otimes_R \mathfrak{U} \otimes (R' \otimes M) & \xrightarrow{\tau' \odot \rho} & R' \otimes M & \xrightarrow{m'} & R' \otimes_R M \\ \uparrow & & \uparrow & & \uparrow \\ R' \otimes_R (\mathfrak{U} \otimes (R \otimes M) & \xrightarrow{\tau \odot \rho} & R \otimes M & \xrightarrow{m} & M) \end{array} \qquad (3)$$

with all vertical arrows isomorphisms. Here we are using the fact that the functor $R' \otimes_R$ is idempotent: $R' \otimes_R R' \simeq R'$. This shows that the localization of the natural ('tensor') action of $\mathfrak{U} := R\#U$ is the natural action of $R' \otimes_R \mathfrak{U}$. Now the commutativity of the diagram

$$\begin{array}{ccc} \mathfrak{U} \otimes (R \otimes M) & \xrightarrow{\tau \odot \rho} & R \otimes M \\ {\scriptstyle \mathfrak{U} \otimes m} \downarrow & & \downarrow {\scriptstyle m} \\ \mathfrak{U} \otimes M & \xrightarrow{\rho} & M \end{array} \qquad (4)$$

implies the commutativity of the corresponding localized diagram

$$\begin{array}{ccc} R'\otimes_R \mathfrak{U}\otimes (R\otimes M) & \xrightarrow{\tau'\odot\rho} & R'\otimes M \\ R'\otimes_R \mathfrak{U}\otimes m' \Big\downarrow & & \Big\downarrow m \\ R'\otimes_R \mathfrak{U}\otimes (R'\otimes_R M) & \xrightarrow{\rho'} & R'\otimes_R M \end{array} \qquad (5)$$

This shows in particular (for $M = R'$ and ρ induced by τ) that τ' is a Hopf action. ∎

6.3.4. The commutative case. Propositions 6.3.1 and 6.3.2 provide the following assertion.

6.3.4.1. Proposition. *Let R be a commutative k-algebra. And let M be a differential $R^{\mathfrak{e}}$-module which is flat as a right R-module. Then*

(a) For any Serre subcategory $\mathbb{T}$ of $\mathcal{A}$, the functor $M\otimes_R$ induces a (unique up to isomorphism) differential functor $M_{\mathbb{T}} : \mathcal{A}/\mathbb{T} \longrightarrow \mathcal{A}/\mathbb{T}$.

If $M \in Ob\Delta_R^{(n)}$, then $M_{\mathbb{T}} \in Ob\Delta^{(n)}$.

(b) If the quotient category $\mathcal{A}/\mathbb{T}$ is equivalent to $R' - mod$ for some R-algebra R', then $M_{\mathbb{T}}$ is isomorphic to the functor $R' \otimes_R M\otimes_R$.

Proof. The fact follows from Propositions 6.3.1 and 6.3.2 and the coincedence, for a commutative ring R, of the subcategory Art_R of artinian $R^{\mathfrak{e}}$-modules and the diagonal Δ_R. ∎

6.3.4.2. Remark. One can show (using Lemma 6.2.5.1) that if $\mathcal{A} \longrightarrow \mathcal{A}/\mathbb{T}$ is a localization at a principal open set (i.e. a localization at any finitely generated multiplicative set), then one can drop the right flatness condition in Proposition 6.3.4.1. Since the principal open sets form a base of the Zariski topology of schemes, they provide satisfactory 'local properties' of differential bimodules and, in particular, of differential operators. Thus, in the commutative case, the necessity to switch to derived categories is first manifested in the non-affine situation. In the noncommutative case the principal open sets do not make much sense in general. Therefore we have to work in the derived categories already in the affine case. Considering that the diagonal Δ_R is not Zariski closed if the ring R is noncommutative, this is not very surprizing. ∎

6.4. Localization of differential actions in derived categories of categories of modules. Let k be a commutative ring, R a k-algebra. Let $\mathcal{A} = R - mod$ – the category of left R-modules; and let $\mathcal{B} = R^{\mathfrak{e}} - mod$, where $R^{\mathfrak{e}} := R\otimes_k R^o$. The natural action

$$\mathcal{B}\times\mathcal{A} \longrightarrow \mathcal{A}, \quad (M,N) \mapsto M\otimes_R N$$

(we identify $\mathcal{B}$ with the corresponding full subcategory of the category R-bi of R-bimodules) is an action of the monoidal category $R^{\mathfrak{e}}$-modules, $\mathcal{B}^{\tilde{}} = (\mathcal{B}, \otimes_R, R)$, on $\mathcal{A}$. This action induces an action

$$\Phi : \mathfrak{D}^-(\mathcal{B}) \times \mathfrak{D}^-(\mathcal{A}) \longrightarrow \mathfrak{D}^-(\mathcal{A})$$

of the monoidal derived category $\mathfrak{D}^-(\mathcal{B})^{\tilde{}}$ of the bounded from above complexes over $\mathcal{B}$ on $\mathfrak{D}^-(\mathcal{A})$.

Fix a Serre subcategory $\mathbb{S}$ of $\mathcal{A}$. And let $\mathcal{B}_{\mathbb{S}}$ denote the full subcategory of $\mathcal{B}$ generated by all $R^{\mathfrak{e}}$-modules M such that the functor $M \otimes_R$ preserves $\mathbb{S}$. Denote by $\mathfrak{D}^-\mathbb{S}$ the full subcategory of $\mathfrak{D}^-(\mathcal{A})$ generated by all complexes X of R-modules such that $H^n(X) \in Ob\mathbb{S}$ for all n. Recall that $\mathfrak{D}^-\mathbb{S}$ is a thick subcategory of $\mathfrak{D}^-(\mathcal{A})$. By a standart argument (using spectral sequence) one can show that the action of the subcategory $\mathfrak{D}^-\mathcal{B}_{\mathbb{S}}$ of $\mathfrak{D}^-(\mathcal{B})$ preserves $\mathfrak{D}^-\mathbb{S}$; i.e. the restriction to $\mathfrak{D}^-\mathcal{B}_{\mathbb{S}} \times \mathfrak{D}^-\mathbb{S}$ of the functor $\otimes^L_R : \mathfrak{D}^-(\mathcal{B}) \times \mathfrak{D}^-(\mathcal{A}) \longrightarrow \mathfrak{D}^-(\mathcal{A})$ takes values in $\mathfrak{D}^-\mathbb{S}$.

One of consequences of this fact is the following

6.4.1. Proposition. *For any Serre subcategory $\mathbb{S}$ of $\mathcal{A} = R - mod$, the action of $\mathfrak{D}^-\mathcal{B}_{\mathbb{S}}$ on $\mathfrak{D}^-(\mathcal{A})$ induces an action of $\mathfrak{D}^-\mathcal{B}_{\mathbb{S}}$ on the quotient triangulated category*

$$\mathfrak{D}^-\mathcal{B}_{\mathbb{S}} \times \mathfrak{D}^-(\mathcal{A})/\mathfrak{D}^-\mathbb{S} \longrightarrow \mathfrak{D}^-(\mathcal{A})/\mathfrak{D}^-\mathbb{S}.$$

In particular, it induces an action of the category Art^{∞}_R of strongly differential $R^{\mathfrak{e}}$-modules on $\mathfrak{D}^-(\mathcal{A})/\mathfrak{D}^-\mathbb{S}$.

6.4.2. Proposition. *Let $\mathbb{F} = (F, \mu)$ be an algebra in $\mathfrak{D}^-\mathcal{B}_{\mathbb{S}}$ (i.e. $\mathbb{F}$ is an algebra in the monoidal category $\mathfrak{D}^-(\mathcal{B})^{\tilde{}}$ such that $F \in Ob\mathfrak{D}^-\mathcal{B}_{\mathbb{S}}$). Then $\mathbb{F}$ determines a monad $\mathbb{F}_{\mathbb{S}} = (F_{\mathbb{S}}, \mu_{\mathbb{S}})$ on $\mathfrak{D}^-(\mathcal{A})/\mathfrak{D}^-\mathbb{S}$.*

A localization $Q : \mathfrak{D}^-(\mathcal{A}) \longrightarrow \mathfrak{D}^-(\mathcal{A})/\mathfrak{D}^-\mathbb{S}$ induces an equivalence of triangulated categories

$$\Psi : \mathbb{F} - mod/\mathfrak{F}^{-1}(\mathfrak{D}^-\mathbb{T}) \longrightarrow \mathbb{F}_{\mathbb{S}} - mod,$$

where $\mathfrak{F}$ is a forgetting functor $\mathbb{F} - mod \longrightarrow \mathfrak{D}^-(\mathcal{A})$.

Proof. The assertion can be proved by the argument used for a similar statement in Proposition 6.2.2. ∎

Denote by ${}^{\mathfrak{e}}\mathfrak{D}^-(\mathcal{B})$ the full subcategory of $\mathfrak{D}^-(\mathcal{B})$ generated by all complexes X of $R^{\mathfrak{e}}$-modules such that $H^n(X)$ is a strongly differential bimodule for all n.

6.4.3. Corollary. *Let $\mathbb{F} = (F, \mu)$ be an algebra in $\mathcal{B} = R^{\mathfrak{e}} - mod$ such that F is a strongly differential $R^{\mathfrak{e}}$-module. Then, for any Serre subcategory $\mathbb{T}$ of $\mathcal{A} = R - mod$, $\mathbb{F}$ induces a unique up to isomorphism monad $\mathbb{F}_{\mathbb{T}} = (F_{\mathbb{T}}, \mu_{\mathbb{T}})$ on the triangulated category $\mathfrak{D}^-(\mathcal{A})/\mathfrak{D}^-\mathbb{T}$.*

A localization $Q : \mathfrak{D}^-(\mathcal{A}) \longrightarrow \mathfrak{D}^-(\mathcal{A})/\mathfrak{D}^-\mathbb{T}$ induces an equivalence of triangulated categories

$$\Psi : \mathbb{F} - mod/\mathfrak{F}^{-1}(\mathbb{T}) \longrightarrow \mathbb{F}_{\mathbb{T}} - mod,$$

where $\mathfrak{F}$ is a forgetting functor $\mathbb{F} - mod \longrightarrow \mathfrak{D}^-(\mathcal{A})$.

Note by passing that the triangulated category $\mathfrak{D}^-(\mathcal{A})/\mathfrak{D}^-\mathbb{T}$ is naturally equivalent to the derived category $\mathfrak{D}^-(\mathcal{A}/\mathbb{T})$. And this is true in the general case, when $\mathcal{A}$ is an arbitrary abelian category, $\mathbb{S}$ is a thick subcategory in $\mathcal{A}$, and $\mathbb{T}$ is the corresponding thick subcategory in $\mathfrak{D}^-(\mathcal{A})$ (cf. [BO]).

6.5. Localization of differential operators. The (not necessarily strongly) differential bimodules are compatible with localizations given by $R' \otimes_R$ for an algebra morphism $R \longrightarrow R'$ such that R' is a flat left R-module as well. For instance, R' is the localization of R at a left and right Ore set.

6.5.1. Proposition. *Let $R \longrightarrow R'$ be an algebra morphism such that the functor $Q = R'\otimes_R$ is an exact localization and R' is flat as a left R-module too. Then*

(a) For any $R^{\mathfrak{e}}$-module M which belongs to Δ_R^-, the functor $M\otimes_R$ is compatible with the localization $Q = R'\otimes_R$. And $Q \circ (M\otimes_R) \simeq M'\otimes_{R'}$, where $M' = R' \otimes_R M \otimes_R R'$. The canonical (R', R)-bimodule morphism $R' \otimes_R M \longrightarrow R' \otimes_R M \otimes_R R'$ is an isomorphism.

(b) If $M \in Ob\Delta_R^{(n)}$, i.e. if M is a differential $R^{\mathfrak{e}}$-module of n-th order, then the $R'^{\mathfrak{e}}$-module M' has the same order: $M' \in Ob\Delta_{R'}^{(n)}$.

(c) Let $\varphi : R \longrightarrow A$ be a differential algebra (i.e. φ is a k-algebra morphism turning A into a differential $R^{\mathfrak{e}}$-module. Then $R' \otimes_R A$ has a unique k-algebra structure such that the canonical maps $A \longrightarrow R' \otimes_R A \longleftarrow R'$ are k-algebra morphisms. And $R' \otimes_R A$ is a differential $R'^{\mathfrak{e}}$-module.

Proof. (a) Consider the full subcategory Ξ of $R^{\mathfrak{e}} - mod$ generated by all modules M such that the canonical (R', R)-bimodule morphism

$$R' \otimes_R M \longrightarrow R' \otimes_R M \otimes_R R' \tag{1}$$

is an isomorphism. It follows from the exactness of the functors $R'\otimes_R$ and $\otimes_R R'$ that Ξ is a Serre subcategory of the category $R^{\mathfrak{e}} - mod$. Since Ξ contains the $R^{\mathfrak{e}}$-module R, it contains the Serre subcategory Δ_R^-. According to the part 1) of the proof of Proposition 6.3.2, the functor $M\otimes_R$ is compatible with the localization $R'\otimes_R$ if and only if the morphism (1) is an isomorphism. This proves the assertion *(a)*.

The assertions *(b)* and *(c)* are proved by the same argument as the corresponding assertions of Proposition 6.3.2. ∎

6.5.2. Proposition. *Let $R \longrightarrow R'$ be an algebra morphism such that the functor*

$$Q = R'\otimes_R : R - mod \longrightarrow R' - mod$$

is an exact localization and the R' is flat as a left R-module. Let M' be a differential $R'^{\mathfrak{e}}$-module. And let $M := Q^{\wedge}(M')_{diff}$ (i.e. M is the differential part of the $R^{\mathfrak{e}}$-module M'). Then the canonical morphism $\varphi : R'\otimes_R M \longrightarrow M'$ is an isomorphism of $R^{\mathfrak{e}}$-modules. Moreover, the isomorphism φ induces, for any $n \geq 0$, an isomorphism $R' \otimes_R M_n \longrightarrow M'_n$, where M'_n (resp. M_n) denotes the $\Delta_{R'}^{(n+1)}-$ (resp.$\Delta_R^{(n+1)}-$) torsion of M' (resp. of M).

Proof. (i) First we shall prove the fact for artinian $R^{\mathfrak{e}}$-modules.

Let M' be any artinian $R'^{\mathfrak{e}}$-module; i.e. there exists an $R'^{\mathfrak{e}}$-module epimorphism $(\nu)R' \longrightarrow M'$ for some ordinal ν (as usual, $(\nu)R'$ denotes the direct sum of ν copies of R'). We can include this epimorphism into a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K' & \longrightarrow & (\nu)R' & \longrightarrow & M' & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & K & \longrightarrow & (\nu)R & \longrightarrow & M & \longrightarrow & 0 \end{array} \tag{1}$$

Here the upper row is regarded as a sequence of $R^{\mathfrak{e}}$-module morphisms; K is the pull-back of the corresponding morphisms. Thus M is an artinian $R^{\mathfrak{e}}$-module, and in the corresponding commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K' & \longrightarrow & (\nu)R' & \longrightarrow & M' & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & R'\otimes_R K & \longrightarrow & R'\otimes_R (\nu)R & \longrightarrow & R'\otimes_R M & \longrightarrow & 0 \end{array} \qquad (2)$$

the central vertical arrow is, obviously, an isomorphism. Since $R'\otimes_R$ is an exact localization, for any R'-module L, the canonical epimorphism $R'\otimes_R L \longrightarrow L$ is an isomorphism, and $R'\otimes_R$ sends universal squares into universal squares. Therefore the left vertical arrow is an isomorphism too. This implies, since both rows of (2) are exact, that the right vertical arrow is an isomorphism.

(ii) Let now K' be any $R'^{\mathfrak{e}}$-module from the diagonal $\Delta_{R'}$. According to Proposition 5.11.4.1, K is a submodule of an artinian $R'^{\mathfrak{e}}$-module M'. By (i), there exists an artinian $R^{\mathfrak{e}}$-module M and an $R^{\mathfrak{e}}$-module monomorphism $M \longrightarrow M'$ such that the canonical $R'^{\mathfrak{e}}$-module morphism $R'\otimes_R M \longrightarrow M'$ is an isomorphism. Let K be a pull-back of the $R^{\mathfrak{e}}$-module morphisms $K' \longrightarrow M' \longleftarrow M$. Then K is an $R^{\mathfrak{e}}$-submodule of M, hence $K \in Ob\Delta_R$; and the canonical (R', R)-bimodule morphism $R'\otimes_R K \longrightarrow K'$ is an isomorphism (cf. the argument in (i)).

(iii) Assume now that the fact is true for all $L' \in Ob\Delta_{R'}^{(n)}$. Let M' belong to $Ob\Delta_{R'}^{(n+1)}$; i.e. there exists an exact sequence of $R'^{\mathfrak{e}}$-modules

$$0 \longrightarrow K' \longrightarrow M' \longrightarrow L' \longrightarrow 0 \qquad (3)$$

with $L' \in Ob\Delta_{R'}^{(n)}$ and $K' \in Ob\Delta_{R'}$. Consider the diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K' & \longrightarrow & M' & \longrightarrow & L' & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & K & \longrightarrow & M & \longrightarrow & L & \longrightarrow & 0 \end{array} \qquad (4)$$

where K is the $\Delta_R^{(n)}$-torsion of the $R^{\mathfrak{e}}$-module K', and L is the Δ_R-torsion of the $R^{\mathfrak{e}}$-module L'. Finally, M is the pull-back of the morphisms $M'/K \longrightarrow L' \longleftarrow K'$. We have the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K' & \longrightarrow & M' & \longrightarrow & L' & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & R'\otimes_R K & \longrightarrow & R'\otimes_R M & \longrightarrow & R'\otimes_R L & \longrightarrow & 0 \end{array} \qquad (5)$$

Since the right and the left vertical arrows are isomorphisms, the central vertical arrow is an isomorphism too.

(iv) Let $M' \in Ob\Delta_{R'}^{\infty}$. Set $M = \sup\{M_n \mid n \geq 0\}$, where M_n is the $\Delta_R^{(n+1)}$-torsion of the $R^{\mathfrak{e}}$-module M'; and let M'_n is the $\Delta_{R'}^{(n+1)}$-torsion of M'. Then, by (iii), the canonical morphism $R'\otimes_R M_n \longrightarrow M'_n$ is an isomorphism for all n. This implies that the canonical morphism

$$R'\otimes_R M = \text{colim}\{R'\otimes_R M_n \mid n \geq 0\} \longrightarrow \text{colim}\{M'_n \mid n \geq 0\} = M'$$

is an isomorphism. ∎

The following version of Proposition 6.5.2 for strongly differential bimodules does not require R' to be a flat left R-module.

6.5.2.1. Proposition. *Let $R \longrightarrow R'$ be an algebra morphism such that the functor*

$$Q = R' \otimes_R : R - mod \longrightarrow R' - mod$$

is an exact localization. Let M' be a strongly differential $R'^{\mathfrak{e}}$-module. And let $M := Art_R^{\infty}(Q^{\hat{}}(M'))$. Then the canonical morphism $\varphi : R' \otimes_R M \longrightarrow M'$ is an isomorphism of $R^{\mathfrak{e}}$-modules. Moreover, the isomorphism φ induces, for any $n \geq 0$, an isomorphism $R' \otimes_R M_n \longrightarrow M'_n$, where M'_n is the $Art_{R'}^{(n+1)}$-torsion of M' and, similarly, M_n is the $Art_R^{(n+1)}$-torsion of M.

Proof. The assertion follows from the part (i) and a simplified version of the parts (iii) and (iv) of the argument of Proposition 6.5.2. ∎

6.5.2.2. Note. If $f : R \longrightarrow R'$ is any morphism of commutative algebras, then, for any differential $R'^{\mathfrak{e}}$-module M', the $R^{\mathfrak{e}}$-module $M = f_{\#}M'$ obtained by restriction of scalars is differential too. More generally, for any $R'^{\mathfrak{e}}$-module M' and for any nonnegative n, $f_{\#}(\Delta_{R'}^{(n)}M) \subseteq \Delta_R^{(n)}(f_{\#}M)$ and, therefore, $f_{\#}(M'_{diff}) \subseteq f_{\#}(M)_{diff}$. This follows from the observation that, as a set, $\Delta_{R'}^{(n)}M = \{z \in M \mid K_{R'}^n \cdot z = 0\}$, where $K_{R'}$ is the kernel of the multiplication $R'^{\mathfrak{e}} \longrightarrow R'$, and $f \otimes_k f(K_R) \subseteq K_{R'}$.

Clearly Proposition 6.5.2 (for a commutative R) is a consequence of this fact. ∎

6.5.3. Proposition. *Let $R \longrightarrow R'$ be an algebra morphism such that the functor*

$$Q = R' \otimes_R : R - mod \longrightarrow R' - mod$$

is an exact localization and R' is flat as a left R-module.

Let M be an $R^{\mathfrak{e}}$-module, $M' := R' \otimes_R M \otimes_R R'$. If the natural morphism $M \longrightarrow M'$ is injective, then, for any $n \geq 0$, the morphism $R' \otimes_R \Delta_R^{(n)}M \longrightarrow \Delta_{R'}^{(n)}M'$ is an isomorphism. In particular, $R' \otimes_R M_{diff} \longrightarrow M'_{diff}$ is an $R^{\mathfrak{e}}$-module isomorphism.

Proof. By Proposition 6.5.2, $R' \otimes_R \Delta_R^{(n)}(Q^{\hat{}}(M')) \longrightarrow \Delta_{R'}^{(n)}M'$ is an isomorphism for any n. Let $\mathcal{M}$ be the image of the canonical morphism $M \longrightarrow R' \otimes_R M \otimes_R R' = M'$. Clearly $\Delta_R^{(n)}(\mathcal{M}) = \mathcal{M} \cap \Delta_R^{(n)}(Q^{\hat{}}(M'))$. Note that the functor $\mathfrak{Q} : L \mapsto R' \otimes_R L \otimes_R R'$, being exact, respects pull-backs. In particular, it respects intersections. Note that

$$R' \otimes_R \mathcal{M} \otimes_R R' \cap R' \otimes_R \Delta_R^{(n)}(Q\hat{}(M')) = R' \otimes_R \Delta_R^{(n)}(Q\hat{}(M'))$$

It follows that in the commutative diagram

$$\begin{array}{ccc} R' \otimes_R \Delta_R^{(n)}(\mathcal{M}) \otimes_R R' & \longrightarrow & R' \otimes_R \Delta_R^{(n)}((Q\hat{}(M')) \\ \uparrow & & \downarrow \\ R' \otimes_R \Delta_R^{(n)}(\mathcal{M}) & \xrightarrow{\phi_n} & \Delta_{R'}^{(n)}M' \end{array}$$

both vertical arrows and the upper horizontal arrow are isomorphisms. Therefore, for all n, the morphism $\phi_n : R' \otimes_R \Delta_R^{(n)} \mathcal{M} \longrightarrow \Delta_{R'}^{(n)} M'$ is an isomorphism. This implies that the canonical morphism $\phi : R' \otimes_R M_{diff} \longrightarrow M'_{diff}$ is an isomorphism. ∎

A similar argument proves the following version of 6.5.5 for strongly differential parts of bimodules.

6.5.3.1. Proposition. *Let $R \longrightarrow R'$ be an algebra morphism such that the functor*

$$Q = R' \otimes_R : R - mod \longrightarrow R' - mod$$

is an exact localization and R' is flat as a left R-module.

Let M be an R^e-module, $M' := R' \otimes_R M \otimes_R R'$. If the natural morphism $M \longrightarrow M'$ is injective, then, for any $n \geq 0$, the morphism $R' \otimes_R Art_R^{(n)} M \longrightarrow Art_{R'}^{(n)} M'$ is an isomorphism. In particular, the map $R' \otimes_R Art_R^{\infty}(M) \longrightarrow Art_{R'}^{\infty}(M')$ is an R'^e-module isomorphism.

6.5.4. Proposition. *Let $R \longrightarrow R'$ be an algebra morphism such that the functor*

$$Q = R' \otimes_R : R - mod \longrightarrow R' - mod$$

is an exact localization. Let L be a coherent R-module (i.e. there exists an exact sequence $F_1 \longrightarrow F_0 \longrightarrow L \longrightarrow 0$, where F_i are free modules of finite type). Then, for any R-module N, the natural R^e-module morphism

$$\mathrm{Hom}_k(L, N) \longrightarrow \mathrm{Hom}_k(R' \otimes_R L, R' \otimes_R N)$$

induces, for all $n \geq 0$, isomorphisms

$$R' \otimes_R Diff_n^s(L, N) \longrightarrow Diff_n^s(R' \otimes_R L, R' \otimes_R N).$$

In particular, we have an R'^e-module isomorphism

$$R' \otimes_R Diff^s(L, N) \longrightarrow Diff^s(R' \otimes_R L, R' \otimes_R N).$$

Here $Diff^s$ (resp. $Diff_n^s$) denotes strongly differential operators (resp. strongly differential operators of order no greater than n).

Proof. Thanks to Proposition 6.5.3.1, it suffices to show that the strongly differential part of the R^e-module $\mathrm{Hom}_k(R' \otimes_R L, R' \otimes_R N)$ is contained in the image of the morphism

$$R'^e \otimes \mathrm{Hom}_k(L, N) \longrightarrow \mathrm{Hom}_k(R' \otimes_R L, R' \otimes_R N). \qquad (1)$$

(a) Let M be an artinian R^e-module. And let $\phi : M \longrightarrow \mathrm{Hom}_k(R' \otimes_R L, R' \otimes_R N)$ be an R^e-module morphism. We claim that the image of ϕ is contained in the image of the canonical map (1).

In fact, since there is an R^e-module epimorphism of a direct sum of a set of copies of R onto M, we can assume that $M = R$. And any R^e-module morphism from R to

$\mathrm{Hom}_k(R'\otimes_R L, R'\otimes_R N)$ is uniquely defined by the value, f, of the identity of R^e, and, therefore, $rf = fr$ for all $r \in R$; i.e. $f \in \mathrm{Hom}_R(R'\otimes_R L, R'\otimes_R N)$. Since $R'\otimes_R$ is a localization, the canonical map

$$\mathrm{Hom}_{R'}(R'\otimes_R L, R'\otimes_R N) \longrightarrow \mathrm{Hom}_R(R'\otimes_R L, R'\otimes_R N)$$

is an isomorphism; and $\mathrm{Hom}_{R'}(R'\otimes_R L, R'\otimes_R N)$ is isomorphic to $\mathrm{Hom}_R(L, R'\otimes_R N)$. Since L is a finitely generated R-module, the morphism

$$R'\otimes_R \mathrm{Hom}_R(L, N) \longrightarrow \mathrm{Hom}_R(L, R'\otimes_R N)$$

is an epimorphism.

(b) Let $0 \longrightarrow M' \xrightarrow{\iota} M \xrightarrow{e} M'' \longrightarrow 0$ be an exact sequence in $R^e - mod$ such that M'' is artinian and $M' \in ObArt_R^{(n)}$. Let $f : M \longrightarrow \mathrm{Hom}_k(R'\otimes_R L, R'\otimes_R N)$ be an R^e-module morphism and the composition $f\circ\iota$ takes values in the image of (1). This means that f induces a (unique) R^e-module morphism, f'', from M'' to the cokernel, $\mathfrak{C}$, of (1). Since M'' is artinian, it is generated by its central elements. Fix any central element $x' \in M''$. Let $x \in M$ be a preimage of x' and set $f_x := f(x)$. The morphism f_x has the property: $f_x r - r f_x \in Im(f\circ\iota)$ for all $r \in$R.

($b1$) Let L be a free R-module of finite type with free generators $\{e_i \mid i \in J\}$. Let g_x denote the R'-module morphism $R'\otimes_R L \longrightarrow R'\otimes_R N$ such that $g_x(e_i) = f_x(e_i)$ for all $i \in$J. Then $f_x \in g_x + Im(f\circ\iota)$. Since the image of $f\circ\iota$ is contained in the image of the map (1) and g_x belongs to the image of (1) (cf. (a)) , f_x belongs to the image of (1).

This proves the proposition in the case when L is a free R-module of a finite rank.

($b2$) *The general case.* Let L be an arbitrary coherent R-module; i.e. there exists an exact sequence of R-module morphisms $F_1 \longrightarrow F_0 \longrightarrow L \longrightarrow 0$, where F_i are free modules of finite type. Then we have the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0\to & Hom_k(R'\otimes_R L, R'\otimes_R N) & \to & Hom_k(R'\otimes_R F_0, R'\otimes_R N) & \to & Hom_k(R'\otimes_R F_1, R'\otimes_R N) & \to 0 \\
 & \uparrow & & \uparrow & & \uparrow & \\
0\to & R'^e\otimes_{R^e} Hom_k(L, R'\otimes_R N) & \to & R'^e\otimes_{R^e} Hom_k(F_0, R'\otimes_R N) & \to & R'^e\otimes_{R^e} Hom_k(F_1, R'\otimes_R N) & \to 0
\end{array} \qquad (1)$$

Since, according to (b1), $Diff^s(R'\otimes_R F_i, R'\otimes_R N)$ is contained in the image of $R'^e\otimes_{R^e} \mathrm{Hom}_k(F_i, N)$, $i = 0, 1$, the same is true for L; i.e. $Diff^s(R'\otimes_R L, R'\otimes_R N)$ is contained in the image of $R'^e\otimes_{R^e} \mathrm{Hom}_k(L, N)$. ■

6.5.4.1. Proposition. *Let $R \longrightarrow R'$ be an algebra morphism such that the functor*

$$R'\otimes_R : R - mod \longrightarrow R' - mod$$

is an exact localization (say the ring R' is the localization of R at a left Ore set). Then

(a) The action of $D^s(R)$ on R extends naturally to an action on R' giving a canonical ring homomorphism $D^s(R) \longrightarrow D^s(R')$ which induces a left R'-module isomorphism $R'\otimes_R^s D(R) \longrightarrow D^s(R')$.

(b) For any $D^s(R)$-module M, the R'-module $R'\otimes_R M$ has a natural, in particular compatible with $D^s(R) \longrightarrow D^s(R')$, structure of a $D^s(R')$-module.

(c) If the ring R' is such that the functor $\otimes_R R' : mod-R \longrightarrow mod-R'$ is a localization (e.g. R' is the localization of R at a left and right Ore set), then we also get an induced right R'-module isomorphism $D^s(R) \otimes_R R' \longrightarrow D^s(R')$.

Proof. The assertion *(a)* follows from Proposition 6.5.4.

(b) The assertion *(b)* follows from *(a)*.

(c) The assertion *(c)* is the right hand side version of *(a)*. ■

Complementary facts.

C.1. Differential operators and Spec.

In the commutative case, the 'local nature' of differential operators implies that the localization of a D-module at a point of a scheme is a D-module at this point (i.e. on the affine scheme of the local ring at this point). This fact is, of course, quite useful, since it allows to study D-modules 'locally', at points of underlying topological space (see, for instance, the local criterion for holonomicity of a complex of D-modules).

A remarkable consequence of our localization theorems is that the fact is true in the general, noncommutative setting: localizations of (complexes of) D-modules at points of the spectrum are (complexes of) D-modules at the points. Note however that this important fact cannot be even formulated in the language of rings and modules. The reason is that the category of modules over a noncommutative ring localized at a point of the spectrum is not, usually, equivalent to a category of modules. As a result, we need to switch from categories of modules to more general abelian categories and replace algebras by monads and modules over algebras by modules over monads.

For the reader's convenience, we remind first what is the spectrum of an abelian category and mention a couple of its basic properties. After that we shall formulate the fact.

C.1.0. Preliminaries on Spec. Recall that, for any two objects X, Y of $\mathcal{A}$, we write $X \succ Y$ if Y is a subquotient of a finite direct sum of copies of X (cf. Note 2.5.1). For any $X \in Ob\mathcal{A}$, denote by $\langle X \rangle$ the full subcategory of $\mathcal{A}$ such that $Ob\langle X \rangle = Ob\mathcal{A} - \{Y \in Ob\mathcal{A} \mid Y \succ X\}$. It is easy to check that $X \succ Y$ iff $\langle Y \rangle \subseteq \langle X \rangle$. This observation provides a convenient realization of the quotient of $(Ob\mathcal{A}, \succ)$ with respect to the equivalence relation induced by $\succ$: $X \approx Y$ if $X \succ Y \succ X$. Namely, $(Ob\mathcal{A}, \succ)/\approx$ is isomorphic to $(\{\langle X \rangle \mid X \in Ob\mathcal{A}\}, \supseteq)$.

Set $Spec\mathcal{A} = \{P \in Ob\mathcal{A} \mid P \neq 0$, and for any nonzero subobject X of $P, X \succ P\}$. *The spectrum,* **Spec**$\mathcal{A}$, *of* the category $\mathcal{A}$ is the preordered set of equivalence (with respect to $\succ$) classes of objects of Spec$\mathcal{A}$.

C.1.0.1. Note. It follows from the definition of $Spec\mathcal{A}$ that any simple object (i.e. a nonzero object without proper nonzero subobjects) belongs to the spectrum. Moreover, it is easy to see that if $M \succ L$ and M is a simple object, then either $L = 0$, or L is isomorphic to a direct sum of a finite number of copies of M; in the latter case $L \approx M$. In particular, if both L and M are simple objects, then $M \succ L$ iff M is isomorphic to L. ■

The canonical realization of $(Ob\mathcal{A}, \succ)/\approx$ induces a canonical realization of **Spec**$\mathcal{A}$: $(\mathbf{Spec}\mathcal{A} = \{\langle P \rangle \mid P \in Spec\mathcal{A}\}, \supseteq)$.

C.1.0.2. Proposition. *For any $P \in Spec\mathcal{A}$, the subcategory $\langle P \rangle$ is a Serre subcategory of $\mathcal{A}$. If $\mathcal{A}$ is a category with the property (sup), then the converse is true: if X is an object of $\mathcal{A}$ such that $\langle X \rangle$ is a Serre subcategory of $\mathcal{A}$, then X is equivalent (in the sense of $\succ$) to* a *$P \in Spec\mathcal{A}$; i.e. $\langle X \rangle = \langle P \rangle$.*

Proof. See Proposition 2.3.3 and 2.4.7 in [R]. ∎

A nonzero object X of a category $\mathcal{A}$ is called *quasifinal* if, for any nonzero object Y of $\mathcal{A}$, $Y \succ X$. The category $\mathcal{A}$ having quasifinal objects is called *local.*

One can check that all simple objects of a local category (if any) are isomorphic to each other. In particular, the category of left modules over a commutative ring R is local iff the ring R is local.

C.1.0.3. Proposition. *For any $P \in Spec\mathcal{A}$, the quotient category $\mathcal{A}/\langle P \rangle$ is local.*

Proof. See Proposition 3.3.1 and Corollary 3.3.2 in [R]. ∎

For more details on the spectrum of abelian categories see [R1] or [R], Chapter III.

C.1.1. Differential monads and the spectrum. The following proposition is a consequence of Proposition 6.2.4.

C.1.1.1. Proposition. *Suppose that, for any $\mathbf{P} \in$**Spec**$\mathcal{A}$, the localization $Q_{\mathbf{P}} : \mathcal{A} \longrightarrow \mathcal{A}/\mathbf{P}$ at $\mathbf{P}$ has a right adjoint.*

*(a) Let F be an exact D^--functor (i.e. $F \in Ob\Delta^-$). Then, for any $\mathbf{P} \in$**Spec**$\mathcal{A}$, there exists a unique functor $F_{\mathbf{P}} : \mathcal{A}/\mathbf{P} \longrightarrow \mathcal{A}/\mathbf{P}$ such that $Q_{\mathbf{P}} \circ F = F_{\mathbf{P}} \circ Q_{\mathbf{P}}$. The functor $F_{\mathbf{P}}$ belongs to $\Delta^-_{\mathcal{A}/\mathbf{P}}$ for all $\mathbf{P} \in$**Spec**$\mathcal{A}$.*

*If $F \in Ob\Delta^{(n)}$ for some positive n, then $F_{\mathbf{P}} \in Ob\Delta^{(n)}_{\mathcal{A}/\mathbf{P}}$ for all $\mathbf{P} \in$**Spec**$\mathcal{A}$. Similarly, if $F \in Ob\Delta^{\infty}$, then $F_{\mathbf{P}} \in Ob\Delta^{\infty}_{\mathcal{A}/\mathbf{P}}$.*

*(b) Let $\mathbb{F} = (F, \mu)$ be a D^--monad such that the functor F is exact. Then, for any $\mathbf{P} \in$**Spec**$\mathcal{A}$, the monad $\mathbb{F}$ induces a D^- -monad $\mathbb{F}_{\mathbf{P}} = (F_{\mathbf{P}}, \mu_{\mathbf{P}})$ on $\mathcal{A}/\mathbf{P}$. And for all $\mathbf{P}$, the canonical functor*

$$\Psi_{\mathbf{P}} : \mathbb{F} - mod/\mathfrak{F}^{-1}(\mathbf{P}) \longrightarrow \mathbb{F}_{\mathbf{P}} - mod$$

is an equivalence of categories.

*If the monad $\mathbb{F}$ is differential, then the quotient monad $\mathbb{F}_{\mathbf{P}}$ is differential for all $\mathbf{P} \in$**Spec**$\mathcal{A}$.*

C.1.1.2. Note. If the functor F in Proposition C.1.1.1 is only right exact, then we should switch to derived categories of categories of modules and use analogs of results of Section 6.4. We leave details to the reader. ∎

C.1.2. Remark. Proposition C.1.1.1 shows in particular that in noncommutative geometry monads and modules over monads are inavoidable substitute of algebras and modules over algebras. In fact, even if $\mathcal{A} = R - mod$ for some associative ring R, the quotient category $\mathcal{A}/\mathbf{P}$ is not usually equivalent to the category of modules over any ring. ∎

C.2. D-affinity for monads.

Working with categories of endofunctors, one can get, using quite elementary tools, some suggestive and useful versions of important facts. One of them we have considered in Sections 6.1, 6.2 – the compatibility of differential actions with localizations. Here we shall discuss an elementary prototype of D-affinity.

Let $Q : \mathcal{A} \longrightarrow \mathcal{B}$ be a functor having a right adjoint, $Q^\wedge : \mathcal{B} \longrightarrow \mathcal{A}$; and let $\eta = \eta_Q$ and $\epsilon = \epsilon_Q$ be adjunction arrows, $\eta : Id_{\mathcal{A}} \longrightarrow Q^\wedge \circ Q$, $\epsilon : Q \circ Q^\wedge \longrightarrow Id_{\mathcal{B}}$.

To any functor $G : \mathcal{B} \longrightarrow \mathcal{B}$, there corresponds a functor $G'' : \mathcal{A} \longrightarrow \mathcal{A}$ defined by the formula: $G'' := Q^\wedge \circ G \circ Q$. The map $G \mapsto G''$ defines a functor $\mathcal{I}_Q : End(\mathcal{B}) \longrightarrow End(\mathcal{A})$.

C.2.1. Lemma. *The functor $\mathcal{I}_Q : End(\mathcal{B}) \longrightarrow End(\mathcal{A}), G \mapsto Q^\wedge \circ G \circ Q$, extends naturally to a monoidal functor $(\mathcal{I}_Q, \phi) : (End(\mathcal{B}), \circ, Id) \longrightarrow (End(\mathcal{A}), \circ, Id)$. If the functor $Q^\wedge$ is fully faithful, then ϕ is an isomorphism.*

Proof. For any $G, G' \in ObEnd(\mathcal{B})$, the morphism

$$\phi(G,G') : \mathcal{I}_Q(G) \circ \mathcal{I}_Q(G') = (Q^\wedge \circ G \circ Q) \circ (Q^\wedge \circ G' \circ Q) \longrightarrow \mathcal{I}_Q(G \circ G') = Q^\wedge \circ G \circ G' \circ Q$$

equals to $Q^\wedge \circ G\epsilon_Q G' \circ Q$, where $\epsilon_Q : Id_{\mathcal{A}} \longrightarrow Q^\wedge \circ Q$ is an adjunction arrow. One can check that $(\mathcal{I}_Q, \phi)$ is a monoidal functor. We leave details to a reader.

If the functor $Q^\wedge$ is fully faithful, ϵ_Q is an isomorphism; hence φ, is an isomorphism.

■

C.2.2. Corollary. *Let $\mathbb{G} = (G, \mu)$ be a monad on $\mathcal{B}$. Then $\mathcal{I}_Q\mathbb{G} := (\mathcal{I}_Q(G), \mu')$, where μ' equals to $\mathcal{I}_Q\mu \circ \phi(G,G)$, is a monad on $\mathcal{A}$.*

C.2.3. Proposition. *(a) Assume the setting of Lemma C.2.1. Let $\mathbb{G} = (G, \mu)$ be a monad in $\mathcal{B}$. Then the pair of adjoint functors $(Q, Q^\wedge)$ induces a functor, Q'', from the category $\mathcal{I}_Q\mathbb{G} - mod$ of $\mathcal{I}_Q\mathbb{G}$-modules to $\mathbb{G} - mod$.*

(b) If the functor $Q^\wedge$ is fully faithful (i.e. Q is a localization), then the functor $Q'' : \mathcal{I}_Q\mathbb{G} - mod \longrightarrow \mathbb{G} - mod$ is an equivalence of categories.

Proof. Fix adjunction arrows $\eta : Id_{\mathcal{A}} \longrightarrow Q^\wedge \circ Q$ and $\epsilon : Q \circ Q^\wedge \longrightarrow Id_{\mathcal{B}}$.

(a) The functor $Q^\wedge$ induces a functor $Q'' : \mathbb{G} - mod \longrightarrow \mathcal{I}_Q\mathbb{G} - mod$ assigning to any $\mathbb{G}$-module $\mathcal{M} = (M, m : G(M) \longrightarrow M)$ the $\mathcal{I}_Q\mathbb{G}$-module $\mathcal{I}_Q\mathcal{M} := (Q^\wedge(M), m')$, where $m' = Q^\wedge(m \circ G\epsilon(M))$, and to any $\mathbb{G}$-module morphism f the morphism $Q^\wedge f$.

(b) Suppose now that the functor $Q^\wedge$ is fully faithful. Recall that this means exactly that the adjunction arrow ϵ is an isomorphism. Therefore we can assign to any $\mathcal{I}_Q\mathbb{G}$-module $\mathcal{N} = (N, \nu)$ the $\mathbb{G}$-module $Q'(\mathcal{N}) := (Q(N), \nu')$, where the action ν' is the composition of $\epsilon^{-1}G \circ Q(N) : G \circ Q(N) \longrightarrow Q \circ Q^\wedge \circ G \circ Q(N) = Q \circ \mathcal{I}_Q G(N)$ and $Q\nu$. The map Q' is functorial. And one can check that $\epsilon' = \{\epsilon'(\mathcal{N}) := \epsilon(N) \mid \mathcal{N} \in Ob\mathcal{I}_Q\mathbb{G} - mod\}$ is a functor isomorphism from $Q' \circ Q''$ to $Id_{\mathbb{G}-mod}$, and $\eta' = \{\eta'(\mathcal{M}) := \eta(M) \mid \mathcal{M} \in Ob\mathbb{G} - mod\}$ is a functor morphism from $Id_{\mathcal{I}\mathbb{G}_Q - mod}$ to $Q'' \circ Q$. It follows that ϵ' and η' are adjunction arrows between functors Q' and Q''. Since ϵ is an isomorphism, ϵ' is an isomorphism which means that the functor Q'' is fully faithful. Therefore Q' is a localization which makes invertible exactly those $\mathcal{I}_Q\mathbb{G}$-module morphisms $s : (N, \nu) \longrightarrow (N', \nu')$ the localization Q makes invertible.

We claim that $Q's$ is invertible only if s is invertible.

In fact, let $s : (N, \nu) = \mathcal{N} \longrightarrow \mathcal{N}' = (N', \nu')$ be an $\mathcal{I}_Q\mathbb{G}$-module morphism such that Qs is invertible. Note that the action $\nu : G(N) \longrightarrow N$ is a coequalizer of the pair of morphisms $G\nu, \mu(N) : G \circ G(N) \longrightarrow G(N)$. Thus we have a commutative diagram with exact rows:

$$\begin{array}{ccccccc} G \circ G(N) & \xrightarrow{\mu(N)-G\nu} & G(N) & \xrightarrow{\nu} & N & \longrightarrow 0 \\ {\scriptstyle \mathcal{J}_Q G \circ \mathcal{J}_Q G(s)} \downarrow & & {\scriptstyle \mathcal{J}_Q G(s)} \downarrow & & s \downarrow & \\ G \circ G(N') & \xrightarrow{\mu(N')-G\nu'} & G(N') & \xrightarrow{\nu'} & N' & \longrightarrow 0 \end{array} \qquad (1)$$

Note that the vertical arrows of (1) are invertible, since $\mathcal{I}_Q Gs := Q^{\wedge} \circ G \circ Qs$ and Qs is invertible. This implies that s is invertible.

It follows from the universal property of localizations that a localization is an equivalence of categories iff it makes invertible only invertible arrows. ∎

C.2.3.1. Note. Proposition C.2.3 is true without any restriction on the categories involved. For instance, they might be non-additive. In the latter case, the left horizontal arrows in the diagram (1) should be replaced by the corresponding double arrows. ∎

C.2.4. Interpretations. The functor $Q : \mathcal{A} \longrightarrow \mathcal{B}$ having a right adjoint $Q^{\wedge}$ can be regarded as an inverse image functor of a morphism from $\mathcal{B}$ to $\mathcal{A}$ (then $Q^{\wedge}$ is a direct image functor of this morphism; cf. [R], Ch.VII). Note that the pair Q, $Q^{\wedge}$ determines a pair of adjoint functors

$$Q' : End(\mathcal{A}) \longrightarrow End(\mathcal{B}),\ F \mapsto Q \circ F \circ Q^{\wedge},\ Q'^{\wedge} : End(\mathcal{B}) \longrightarrow End(\mathcal{A}),\ G \mapsto Q^{\wedge} \circ G \circ Q. \qquad (1)$$

Thus the functor $Q'^{\wedge}$ might be viewed as a direct image functor of a morphism from $End(\mathcal{B})$ to $End(\mathcal{A})$. The monad $\mathcal{I}_Q\mathbb{G} = (\mathcal{I}_Q G, \mu') := (Q'^{\wedge}(G), \mu')$ corresponding to a monad $\mathbb{G} = (G, \mu)$ on $\mathcal{B}$ (cf. Proposition C.2.3) can be regarded as the direct image (sometimes as global sections) of the monad $\mathbb{G}$. This way Proposition 6.5.3 might be interpreted as "$\mathbb{G}$-affinnity of $\mathcal{B}$ over $\mathcal{A}$".

Unfortunately, we do not work usually with the whole category of endomorphisms. Instead, we are interested in endomorphisms which have right adjoint. And, in general, the restriction of the 'direct image functor' $Q'^{\wedge}$ (see (1)) to the full subcategory $\mathfrak{End}(\mathcal{B})$ of $End(\mathcal{B})$ generated by functors having a right adjoint does not take values in $\mathfrak{End}(\mathcal{A})$. It does, however, if the functor $Q^{\wedge}$ has a right adjoint, as one can see from (1). In this case we call the morphism $\mathcal{B} \longrightarrow \mathcal{A}$ *affine.*

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