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# SMOOTH LOCUS OF TWISTED AFFINE SCHUBERT VARIETIES AND TWISTED AFFINE DEMAZURE MODULES

MARC BESSON AND JIUZU HONG

Abstract. Let $\mathscr{G}$ be an absolutely special parahoric group scheme over $\mathbb{C}$. When $\mathscr{G}$ is not $E_6^{(2)}$, using methods and results of Zhu, we prove a duality theorem for $\mathscr{G}$: there is a duality between the level one twisted affine Demazure modules and function rings of certain torus fixed point subschemes in twisted affine Schubert varieties for $\mathscr{G}$. As a consequence, we determine the smooth locus of any twisted affine Schubert variety in affine Grassmannian of $\mathscr{G}$, which confirms a conjecture of Haines and Richarz, when $\mathscr{G}$ is of type $A_{2\ell-1}^{(2)}, D_{\ell+1}^{(2)}, D_4^{(3)}$. Some partial results for $A_{2\ell}^{(2)}$ and $E_6^{(2)}$ are also obtained.

Additionally, we give geometric descriptions of the Frenkel-Kac isomorphism for twisted affine Lie algebras, and the fusion product for twisted affine Demazure modules.

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## 1. Introduction

Let $G$ be an almost simple algebraic group over $\mathbb{C}$ and let $\mathrm{Gr}_G$ be the affine Grassmannian of $G$. The geometry of the affine Grassmannian is related to integral highest weight representations of Kac-Moody algebras via affine Borel-Weil theorem. Similarly, the geometry of affine Schubert varieties are closely related to affine Demazure modules.

Let $T$ be a maximal torus in $G$ and let $X_*(T)^+$ be the set of dominant coweights. For any $\lambda \in X_*(T)^+$, let $\overline{\mathrm{Gr}}^{\lambda}_{G}$ be the associated affine Schubert variety in $\mathrm{Gr}_G$, which is the closure of the $G(\mathcal{O})$-orbit $\mathrm{Gr}^{\lambda}_{G}$, where $\mathcal{O} = \mathbb{C}[[t]]$. Evens-Mirković [EM] and Malkin-Ostrik-Vybonov [MOV] proved that the smooth locus of $\overline{\mathrm{Gr}}^{\lambda}_{G}$ is exactly the open Schubert cell $\mathrm{Gr}^{\lambda}_{G}$. Zhu [Zh1] proved that there is a duality between the affine Demazure modules and the coordinate ring of the $T$-fixed point subschemes of affine Schubert varieties when $G$ is of type $A$ and $D$, and many cases for type $E_6$. As a consequence, this gives another approach to determine the smooth locus of $\overline{\mathrm{Gr}}^{\lambda}_{G}$ for type $A, D$ and many cases of type $E$.

In this paper, we study a connection between the geometry of twisted affine Schubert varieties and twisted affine Demazure modules. Following the method of Zhu in [Zh1], we will use the weight multiplicities of twisted affine Demazure modules to determine the smooth locus of twisted affine Schubert varieties.

Let $G$ be an almost simple algebraic group of **simply-laced** or **adjoint** type with the action of an absolutely special automorphism $\sigma$ of order $m$, defined in Section 2.1. Assume that $\sigma$ acts on $\mathcal{O}$ by rotation of order $m$. Let $\mathscr{G}$ be the $\sigma$-fixed point subgroup scheme of the Weil restriction group $\mathrm{Res}_{\mathcal{O}/\bar{\mathcal{O}}}(G_{\mathcal{O}})$, where $\bar{\mathcal{O}} = \mathbb{C}[[t^m]]$. Then $\mathscr{G}$ is an absolutely special parahoric group scheme over $\bar{\mathcal{O}}$, in the sense of Haines-Richarz [HR]. One may define the affine Grassmannian $\mathrm{Gr}_{\mathscr{G}}$ of $\mathscr{G}$. Following [PR, Zh2], we will call it a twisted affine Grassmannian. For any $\bar{\lambda}$ the image of a dominant coweight $\lambda$ in the set $X_*(T)_\sigma$ of $\sigma$-coinvariants of $X_*(T)$, the twisted affine Grassmannian $\mathrm{Gr}_{\mathscr{G}}$ and twisted affine Schubert varieties $\overline{\mathrm{Gr}}^{\bar{\lambda}}_{\mathscr{G}}$ share many similar properties with the usual affine Grassmannian $\mathrm{Gr}_G$ and affine Schubert varieties. For instane, when $\mathscr{G}$ is special, a version of the geometric Satake isomorphism was proved by Zhu in [Zh3].

Following [HR], when $\mathscr{G}$ is not of type $A^{(2)}_{2\ell}$, any special parahoric group scheme is absolutely special. When $\mathscr{G}$ is of type $A^{(2)}_{2\ell}$, there are two special parahoric group schemes, and only one of them is absolutely special. We prove Theorem 4.5 in Section 4, which asserts that

**Theorem 1.1.** *When the parahoric group scheme $\mathscr{G}$ is absolutely special but not of type $E^{(2)}_6$, the following restriction is an isomorphism:*

$$H^0(\overline{\mathrm{Gr}}^{\bar{\lambda}}_{\mathscr{G}}, \mathscr{L}) \to H^0((\overline{\mathrm{Gr}}^{\bar{\lambda}}_{\mathscr{G}})^{T^\sigma}, \mathscr{L}|_{(\overline{\mathrm{Gr}}^{\bar{\lambda}}_{\mathscr{G}})^{T^\sigma}}),$$

*where $\mathscr{L}$ is the level one line bundle on $\mathrm{Gr}_{\mathscr{G}}$, $T^\sigma$ is the $\sigma$-fixed point subgroup of a $\sigma$-stable maximal torus $T$ in $G$ and $(\overline{\mathrm{Gr}}^{\bar{\lambda}}_{\mathscr{G}})^{T^\sigma}$ is the $T^\sigma$-fixed point subsheme of $\overline{\mathrm{Gr}}^{\bar{\lambda}}_{\mathscr{G}}$.*

This theorem extends Zhu's duality to the setting of absolutely special parahoric group schemes. The dual $H^0(\overline{\mathrm{Gr}}^{\bar{\lambda}}_{\mathscr{G}}, \mathscr{L})^\vee$ is a twisted affine Demazure module, see Theorem 3.10. Hence, Theorem 1.1 is a duality between twisted affine Demazure modules and the coordinate rings of the $T^\sigma$-fixed point subschemes of twisted affine Schubert varieties. One of the motivations of the work of Zhu [Zh1] is to give a geometric realization of

Frenkel-Kac vertex operator construction for untwisted simply-laced affine Lie algebras. The analogue of Frenkel-Kac construction for twisted affine Lie algebras also exists in literature, see [BT, FLM]. In fact, our Theorem 1.1 implies a geometric Frenkel-Kac isomorphism (including the case $E_6^{(2)}$), see Theorem 4.10.

As a conseqence of Theorem 1.1, we obtain Theorem 4.11, which asserts that

**Theorem 1.2.** *Assume that $\mathscr{G}$ is of type $A_{2\ell-1}^{(2)}, D_{\ell+1}^{(2)}$ or $D_4^{(3)}$. For any $\bar{\lambda} \in X_*(T)_\sigma$, the smooth locus of the twisted affine Schubert variety $\overline{\mathtt{Gr}}_{\mathscr{G}}^{\bar{\lambda}}$ is exactly the open cell $\mathtt{Gr}_{\mathscr{G}}^{\bar{\lambda}}$.*

This theorem confirms a conjecture of Haines-Richarz [HR, Conjecture 5.4] when $\mathscr{G}$ is of type $A_{2\ell-1}^{(2)}, D_{\ell+1}^{(2)}$ or $D_4^{(3)}$. In fact, we also get partial results on the smooth locus when $\mathscr{G}$ is of type $A_{2\ell}^{(2)}$ and $E_6^{(2)}$, see Theorem 4.12. The main reason that Theorem 1.1 does not cover the case of $A_{2\ell}^{(2)}$, is that map $\iota$ defined in Section 3.1 does not necessarily preserve dominance relation in this case, see Lemma 3.3. As for $E_6^{(2)}$, the main reason is that the duality theorem for $E_6$ is still not fully established, see Remark 4.9. In fact, one can define absolutely parahoric group schemes over any base field k of any characteristic, and the twisted Schubert variety $\overline{\mathtt{Gr}}_{\mathscr{G}}^{\bar{\lambda}}$ over the field k. By the works of Haines-Richarz [HR] and Lourenço [Lo], Theorem 1.2 remains true for the twisted Schubert variety over any field k, see Remark 4.13. The problem of determining the smooth locus of twisted affine Schubert varieties was first studied by Richarz, and some results in the cases of $A_{2\ell-1}^{(2)}, A_{2\ell}^{(2)}$ were obtained in [Ri2]. A weaker question, the smoothness of twisted Schubert varieties for special parahoric group schemes have been fully answered by Haines-Richarz in [HR].

To prove Theorem 1.1, one of the main ingredients is Theorem 4.2 in Section 4, which asserts that the $T^\sigma$-fixed point ind-subscheme $(\mathtt{Gr}_{\mathscr{G}})^{T^\sigma}$ is isomorphic to the affine Grassmannian $\mathtt{Gr}_{\mathscr{T}}$, where $\mathscr{T}$ is the $\sigma$-fixed point subscheme of the Weil restriction group $\mathrm{Res}_{O/\bar{O}}(T_O)$.

Let $\pi : \mathbb{P}^1 \to \bar{\mathbb{P}}^1$ be the map given by $t \mapsto t^m$, where $\bar{\mathbb{P}}^1$ is a copy of $\mathbb{P}^1$. Another main ingredient of the proof of Theorem 1.1 is the construction of the level one line bundle $\mathcal{L}$ on the moduli stack $\mathrm{Bun}_{\mathcal{G}}$ of $\mathcal{G}$-torsors, where $\mathcal{G}$ is the parahoric Bruhat-Tits group scheme obtained as the $\sigma$-fixed subgroup scheme of the Weil restriction group $\mathrm{Res}_{\mathbb{P}^1/\bar{\mathbb{P}}^1}(G_{\mathbb{P}^1})$ with $G$ being simply-connected. This is achieved in Section 3. It is known that the level one line bundle on $\mathrm{Bun}_{\mathcal{G}}$ does not necessarily exist for an arbitary parahoric Bruhat-Tits group scheme $\mathcal{G}$ over a smooth projective curve, for example when $\mathcal{G}$ is of type $A_{2\ell}$, cf. [He, Remark 19 (4)] [Zh2, Proposition 4.1]. In Theorem 3.13, when $\sigma$ is absolutely special, we prove that there exists a level one line bundle $\mathcal{L}$ on the moduli stack $\mathrm{Bun}_{\mathcal{G}}$ of $\mathcal{G}$-torsors. Following the method of Sorger in [So], we use the non-vanishing of twisted conformal blocks to construct this line bundle on $\mathrm{Bun}_{\mathcal{G}}$, where the general theory of twisted conformal blocks was recently developed by Hong-Kumar in [HK].

With the level one line bundle on $\mathrm{Bun}_{\mathcal{G}}$ when $\mathcal{G}$ is simply-connected, we can construct the level one line bundle on the global affine Schubert variety $\overline{\mathtt{Gr}}_{\mathcal{G}}^{\lambda}$ for $\mathcal{G}$ being either simply-connected or adjoint. This is a flat family of Schubert varieties over $\mathbb{P}^1$. Over a generic point, the fiber is just a usual affine Schubert variety $\overline{\mathtt{Gr}}_G^{\lambda}$, and over the origin

$o$, we get the twisted affine Schubert variety $\overline{\mathrm{Gr}}^{\bar\lambda}_{\mathscr{G}}$. The main idea of this paper is that, our duality theorem for twisted affine Schubert varieties can follow from Zhu's duality theorem for usual affine Schubert varieties via the level one line bundle on the global affine Schubert variety $\overline{\mathrm{Gr}}^{\lambda}_{\mathcal{G}}$.

In Section 5, we make connection between our geometric approach to affine Demazure module for absolutely special parahoric group schemes, and the affine Demazure modules for twisted current algebras studied in the literature. The fusion product of affine Demazure modules was studied by Fourier-Littelmann [FL], and Zhu gave a geometric description in [Zh1]. When $\sigma$ is a diagram automorphism on $\mathfrak{g}$ and $\mathfrak{g}$ is not $A_{2\ell}$, the fusion product for the twisted current algebra $\mathfrak{g}[t]^\sigma$ was proved by Fourier-Kus in [FK]. Chari-Ion-Kus introduced the hyperspecial current algebra $\mathfrak{C}\mathfrak{g}$ for $A^{(2)}_{2\ell}$ and studied the twisted affine Demazure modules in [CIK], where they presented this algebra by using a basis. The twisted affine demazure modules of this hyperspecial current algebra were further studied by Kus-Venkatesh in [KV]. In Theorem 5.5, we prove that their hyperspecial current algebra $\mathfrak{C}\mathfrak{g}$ for $A^{(2)}_{2\ell}$ can be identified with the twisted current algebra $\mathfrak{g}[t]^\sigma$ via a composition of Kac isomorphism and Cartan involution. For any absolutely special automorphism $\sigma$, we give a geometric description of fusion product for twisted affine Demazure modules of $\mathfrak{g}[t]^\sigma$ in Theorem 5.1.

For the convenience of the readers, we use the following notations frequently:

$C$: complex projective curve $\mathbb{P}^1$.
$G$: almost-simple algebraic group of adjoint or simply-connected type.
$\mathscr{G}$: parahoric group scheme over the ring of formal power series.
$\mathcal{G}$: parahoric Bruhat-Tits group scheme over curve.
$\mathrm{Gr}_G$: affine Grassmannian of $G$.
$\mathrm{Gr}_{\mathscr{G}}$: affine Grassmanian of $\mathscr{G}$.
$\mathrm{Gr}_{\mathcal{G},C}$: global affine Grassmannian of $\mathcal{G}$.
$\overline{\mathrm{Gr}}^{\lambda}_G$: affine Schubert variety.
$\overline{\mathrm{Gr}}^{\bar\lambda}_{\mathscr{G}}$: twisted affine Schubert variety.
$\overline{\mathrm{Gr}}^{\lambda}_{\mathcal{G}}$: global affine Schubert variety for $\mathcal{G}$.
L: level one line bundle on $\mathrm{Gr}_G$.
$\mathscr{L}$: level one line bundle on $\mathrm{Gr}_{\mathscr{G}}$.
$\mathcal{L}$: level one line bundle on $\mathrm{Bun}_{\mathcal{G}}$ and $\mathrm{Gr}_{\mathcal{G},C}$.

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## 2. Main definitions

Let $G$ be an almost simple algebraic group over $\mathbb{C}$ of adjoint or simply-connected type. We choose a maximal torus and Borel subgroup $T \subset B \subset G$. We denote by $X^*(T)$

the lattice of weights of $T$, and by $X_*(T)$ the lattice of coweights. Their natural pairing is denoted by $\langle , \rangle$. Let $\Phi$ denote the set of roots of $G$, and denote by $\Phi^+$ the set of positive roots of $G$ with respect to $B$. Let $\check{\Phi}$ denote the set of coroots, so $(\Phi, X^*(T), \check{\Phi}, X_*(T))$ is a root datum for $G$, and write $W$ for the Weyl group of $G$. Let $Q$ denote the root lattice of $G$, and $\check{Q}$ the coroot lattice.

We follow the labelling of the vertices of the Dynkin diagram in [Ka, Table Fin, p.53]. We denote by $\{\alpha_i \,|\, i \in I\}$ (respectively $\{\check{\alpha}_i \,|\, i \in I\}$ the set of simple roots in $\Phi$ (respectively coroots in $\check{\Phi}$), where $I$ is the set of vertices of the associated Dynkin diagram of $G$. Let $\{\omega_i \,|\, i \in I\}$ be the set of fundamental weights of $G$, and let $\{\check{\omega}_i \,|\, i \in I\}$ be the set of fundamental coweights of $G$. We also choose a pinning $\{x_{\alpha_i}, y_{\alpha_i} \,|\, i \in I\}$ of $G$ with respect to $B$ and $T$.

Let $\mathfrak{g}, \mathfrak{b}, \mathfrak{t}$ denote the Lie algebras of $G, B, T$ respectively. Let $\{e_i, f_i \,|\, i \in I\}$ denote the set of Chevalley generators associated to the pinning $\{x_{\alpha_i}, y_{\alpha_i} \,|\, i \in I\}$. Let $e_\theta$ (resp. $f_\theta$) be the highest (resp. lowest ) root vector in $\mathfrak{g}$, such that $[e_\theta, f_\theta]$ is the coroot $\theta^\vee$ of $\theta$.

2.1. **Absolutely special automorphisms.** Let $\sigma$ be an automorphism of order $m$ on $G$ preserving $B$ and $T$. Let $\tau$ be a diagram automorphism preserving $B, T$ and a pinning $\{x_{\alpha_i}, y_{\alpha_i} \,|\, i \in I\}$. Let $r$ be the order of $\tau$.

When $\mathfrak{g}$ is not $A_{2\ell}$, we take $\sigma$ to be $\tau$. When $\mathfrak{g}$ is $A_{2\ell}$, by [Ka, Theorem 8.6] there exists a unique automorphism $\sigma$ of order $m = 4$ such that

$$\begin{cases} \sigma(e_i) = e_{\tau(i)}, & \text{if } i \neq \ell, \ell+1; \\ \sigma(e_i) = \mathrm{i} e_{\tau(i)}, & \text{if } i \in \{\ell, \ell+1\}; \\ \sigma(f_\theta) = f_\theta, \end{cases} \tag{1}$$

where i is a square root of $-1$. One can check that

$$\begin{cases} \sigma(f_i) = f_{\tau(i)}, & \text{if } i \neq \ell, \ell+1; \\ \sigma(f_i) = -\mathrm{i} f_{\tau(i)}, & \text{if } i \in \{\ell, \ell+1\}; \\ \sigma(e_\theta) = e_\theta \end{cases} . \tag{2}$$

In fact, $\sigma = \tau \circ \mathrm{i}^h$, where $h \in \mathfrak{t}$ such that

$$\alpha_i(h) = \begin{cases} 0, & \text{if } i \neq \ell, \ell+1 \\ 1, & \text{if } i = \ell, \ell+1 \end{cases} .$$

This automorphism induces a unique automorphism on $G$. We still call it $\sigma$.

We call these automorphisms on $G$ or $\mathfrak{g}$ **absolutely special**. Throughout this paper, we will only consider absolutely special automorphisms.

The following table describe the fixed point Lie algebras for all absolutely special automorphisms:

(3)

| $(\mathfrak{g}, m)$ | $(A_{2\ell-1}, 2)$ | $(A_{2\ell}, 4)$ | $(D_{\ell+1}, 2)$ | $(D_4, 3)$ | $(E_6, 2)$ |
|---|---|---|---|---|---|
| $\mathfrak{g}^\sigma$ | $C_\ell$ | $C_\ell$ | $B_\ell$ | $G_2$ | $F_4$ |

,

where by convention $C_1$ is $A_1$ and $\ell \geq 3$ for $D_{\ell+1}$. When $(\mathfrak{g}, m) \neq (A_{2\ell}, 4)$, the fixed point Lie algebra $\mathfrak{g}^\sigma$ is well-known as listed in the above table. When $(\mathfrak{g}, m) = (A_{2\ell}, 4)$, the fixed Lie algebra $\mathfrak{g}^\sigma$ is of type $C_\ell$, which can follow from the twisted Kac-Moody theory, cf. [Ka, §8].

Recall that we follow the labelling of the vertices of the Dynkin diagram of $\mathfrak{g}$ in [Ka, Table Fin, p.53]. Set

(4)

$$\begin{cases} \beta_i = \alpha_i|_{\mathfrak{t}^\sigma}, \text{ for } i = 1, 2, \cdots, \ell, \quad \text{when } (\mathfrak{g}, m) = (A_{2\ell-1}, 2), \text{ or } (D_{\ell+1}, 2) \\ \beta_1 = \alpha_2|_{\mathfrak{t}^\sigma}, \beta_2 = \alpha_1|_{\mathfrak{t}^\sigma}, \quad \text{when } (\mathfrak{g}, m) = (D_4, 3) \\ \beta_1 = \alpha_6|_{\mathfrak{t}^\sigma}, \beta_2 = \alpha_3|_{\mathfrak{t}^\sigma}, \beta_3 = \alpha_2|_{\mathfrak{t}^\sigma}, \beta_4 = \alpha_1|_{\mathfrak{t}^\sigma}, \quad \text{when } (\mathfrak{g}, m) = (E_6, 2) \\ \beta_i = \alpha_i|_{\mathfrak{t}^\sigma}, \text{ for } i = 1, 2, \cdots, \ell-1; \beta_\ell = (\alpha_\ell + \alpha_{\ell+1})|_{\mathfrak{t}^\sigma} = 2\alpha_\ell|_{\mathfrak{t}^\sigma}, \quad \text{when } (\mathfrak{g}, m) = (A_{2\ell}, 4). \end{cases}$$

Let $I_\sigma$ be the set of all subscript indices of $\beta_i$. Then for each case, the set $\{\beta_j \mid j \in I_\sigma\}$ gives rise to the set of simple roots of $\mathfrak{g}^\sigma$. One can see easily that this labelling will coincide with the labelling of non simply-laced Dynkin diagrams in [Ka, Table Fin, p.53].

We now define a map $\eta : I \to I_\sigma$. When $(\mathfrak{g}, m) \neq (A_{2\ell}, 4)$, $\eta$ is defined such that $\beta_{\eta(i)} = \alpha_i|_{\mathfrak{t}^\sigma}$ for any $i \in I$. When $(\mathfrak{g}, m) = (A_{2\ell}, 4)$, set

$$\eta(i) = \eta(2\ell + 1 - i) = i, \text{ for any } 1 \leq i \leq \ell.$$

Let $\{\check{\beta}_j \mid j \in I_\sigma\}$ be the set of simple coroots of $\mathfrak{g}^\sigma$. We can describe $\check{\beta}_j$ as follows:

$$\check{\beta}_j = \sum_{i \in \eta^{-1}(j)} \check{\alpha}_i. \tag{5}$$

Let $\{\lambda_j \mid j \in I_\sigma\}$ be the set of fundamental weights of $\mathfrak{g}^\sigma$, and let $\{\check{\lambda}_j \mid j \in I_\sigma\}$ be the set of fundamental coweights of $\mathfrak{g}^\sigma$. The fundamental weights can be described as follows:

$$\lambda_j = \omega_i|_{\mathfrak{t}^\sigma}, \quad \text{for some } i \text{ with } \eta(i) = j. \tag{6}$$

In the case of fundamental coweights, we need to describe them separately. When $(\mathfrak{g}, m) \neq (A_{2\ell}, 4)$,

$$\check{\lambda}_j = \sum_{i \in \eta^{-1}(j)} \check{\omega}_i. \tag{7}$$

When $(\mathfrak{g}, m) = (A_{2\ell}, 4)$, we have

$$\check{\lambda}_j = \begin{cases} \check{\omega}_j + \check{\omega}_{2\ell+1-j}, & j = 1, 2, \cdots, \ell-1 \\ \frac{1}{2}(\check{\omega}_\ell + \check{\omega}_{\ell+1}), & j = \ell \end{cases}. \tag{8}$$

2.2. **Affine Grassmannian of absolutely special parahoric group schemes.** Let $\mathcal{K}$ denote the field of formal Laurent series in $t$ with coefficients in $\mathbb{C}$. Let $\mathcal{O}$ denote the ring of formal power series in $t$ with coefficients in $\mathbb{C}$. By abuse of notation, we still use $\sigma$ to denote the automorphism of order $m$ on $\mathcal{K}$ and $\mathcal{O}$ such that $\sigma$ acts on $\mathbb{C}$ trivially, and $\sigma(t) = \epsilon t$, where we fix a primitive $m$-th root of unity $\epsilon$. Set $\bar{\mathcal{K}} = \mathcal{K}^\sigma$ and $\bar{\mathcal{O}} = \mathcal{O}^\sigma$. Then $\bar{\mathcal{K}} = \mathbb{C}((\bar{t}))$ and $\bar{\mathcal{O}} = \mathbb{C}[[\bar{t}]]$, where $\bar{t} = t^m$.

Let $\mathscr{G}$ be the smooth group scheme $\mathrm{Res}_{\mathcal{O}/\bar{\mathcal{O}}}(G_{\mathcal{O}})^\sigma$ over $\bar{\mathcal{O}}$, which represents the following group functor

$$R \mapsto G(\mathcal{O} \otimes_{\bar{\mathcal{O}}} R)^\sigma, \quad \text{for any } \bar{\mathcal{O}} - \text{algebra } R,$$

where the $G(O\otimes_{\bar{O}} R)$ denotes the group of $\sigma$-equivariant morphisms from Spec $(O\otimes_{\bar{O}} R)$ to $G$. Then, $\mathscr{G}$ is an absolutely special parahoric group scheme in the sense of Haines-Richarz [HR], as we choose $\sigma$ to be absolutely special. In fact, up to isomorphism, this construction exhausts all absolutely special parahoric subgroups in $\mathscr{G}(\mathcal{K})$ that are defined in [HR].

We can similarly define the smooth group scheme $\mathscr{T} := \mathrm{Res}_{O/\bar{O}}(T_O)^\sigma$, which is a maximal torus in $\mathscr{G}$. Note that, for general almost simple algebraic group $G$, we can still define $\mathscr{G}$ and $\mathscr{T}$, but we need to take the neutral components of $\mathrm{Res}_{O/\bar{O}}(G_O)^\sigma$ and $\mathrm{Res}_{O/\bar{O}}(T_O)^\sigma$ respectively. For convenience, throughout this paper we only work with $G$ being adjoint or simply-connected.

Let $L^+\mathscr{G}$ denote the jet group and $L\mathscr{G}$ be the loop group of $\mathscr{G}$ over $\mathbb{C}$, that is, for all $\mathbb{C}$-algebras $R$, we set $L^+\mathscr{G}(R) = \mathscr{G}(R[[t]])$ and $L\mathscr{G}(R) = \mathscr{G}(R((t)))$. We denote by $\mathtt{Gr}_{\mathscr{G}}$ the affine Grassmannian of $\mathscr{G}$, which is defined as the fppf quotient $L\mathscr{G}/L^+\mathscr{G}$. In particular, we have

$$\mathtt{Gr}_{\mathscr{G}}(\mathbb{C}) = G(\mathcal{K})^\sigma/G(O)^\sigma.$$

It is known that $\mathtt{Gr}_{\mathscr{G}}$ is a projective ind-variety, cf. [PR]. Following [PR, Zh2], we will call it a twisted affine Grassmannian of $\mathscr{G}$. We can also attach the twisted affine Grassmannian $\mathtt{Gr}_{\mathscr{T}} := L\mathscr{T}/L^+\mathscr{T}$ of $\mathscr{T}$. This is a highly non-reduced ind-scheme. Moreover,

$$\mathtt{Gr}_{\mathscr{T}}(\mathbb{C}) = T(\mathcal{K})^\sigma/T(O)^\sigma.$$

Note that the actions of $\sigma$ on $T, T(O)$ and $T(\mathcal{K})$ agree with the action of its diagram automorphism part $\tau$. For any $\lambda \in X_*(T)$, we can naturally attach an element $t^\lambda \in T(\mathcal{K})$. We now define the norm $n^\lambda \in T(\mathcal{K})^\sigma$ of $t^\lambda$,

$$n^\lambda := \prod_{i=0}^{m-1} \sigma^i(t^\lambda) = (\prod_i \sigma^i(\lambda)(\epsilon))t^{\sum \sigma^i(\lambda)}. \tag{9}$$

There exists a natural bijection

$$T(\mathcal{K})^\sigma/T(O)^\sigma \simeq X_*(T)_\sigma, \tag{10}$$

where $X_*(T)_\sigma$ denotes the set of $\sigma$-coinvariants in $X_*(T)$. Any $\bar{\lambda} \in X_*(T)_\sigma$ corresponds to the coset $n^\lambda T(O)^\sigma$, where $\lambda$ is a representative of $\bar{\lambda}$. By Theorem [PR, Theorem 0.1], the components of $\mathtt{Gr}_{\mathscr{G}}$ can be parametrized by elements in $\pi_1(G)_\sigma$, where $\pi_1(G) \simeq X_*(T)/\check{Q}$, and $(X_*(T)/\check{Q})_\sigma$ is the the set of coinvariants of $\sigma$ in $X_*(T)/\check{Q}$.

When $G$ is of adjoint type, we describe $(X_*(T)/\check{Q})_\sigma$ in the following table.

(11)

| $(G,m)$ | $(A_{2\ell-1},2)$ | $(A_{2\ell},4)$ | $(D_{2\ell+1},2)$ | $(D_{2\ell},2)$ | $(D_4,3)$ | $(E_6,2)$ |
|---|---|---|---|---|---|---|
| $X_*(T)/\check{Q}$ | $\mathbb{Z}_{2\ell}$ | $\mathbb{Z}_{2\ell+1}$ | $\mathbb{Z}_4$ | $\mathbb{Z}_2\times\mathbb{Z}_2$ | $\mathbb{Z}_2\times\mathbb{Z}_2$ | $\mathbb{Z}_3$ |
| $(X_*(T)/\check{Q})_\sigma$ | $\mathbb{Z}_2$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | 0 | 0 |

2.3. **Twisted affine Schubert varieties.** Let $e_0$ be the base point in $\mathtt{Gr}_{\mathscr{G}}(\mathbb{C})$. For any $\bar{\lambda} \in X_*(T)$, let $e_{\bar{\lambda}}$ denote the point $n^{\lambda} e_0 \in \mathtt{Gr}_{\mathscr{G}}(\mathbb{C})$. The point $e_{\bar{\lambda}}$ only depends on $\bar{\lambda} \in X_*(T)_\sigma$. Let $X_*(T)^+_\sigma$ denote the set of images of $X_*(T)^+$ in $X_*(T)_\sigma$ via the projection $X_*(T) \to X_*(T)_\sigma$. Then, we have the following Cartan decomposition for $\mathtt{Gr}_{\mathscr{G}}$ (cf. [Ri1]),

$$\mathtt{Gr}_{\mathscr{G}}(\mathbb{C}) = \bigsqcup_{\bar{\lambda} \in X_*(T)^+_\sigma} \mathtt{Gr}^{\bar{\lambda}}_{\mathscr{G}}, \tag{12}$$

where $\mathtt{Gr}^{\bar{\lambda}}_{\mathscr{G}} := G(O)^\sigma e_{\bar{\lambda}}$. The Schubert variety $\overline{\mathtt{Gr}}^{\bar{\lambda}}_{\mathscr{G}}$ is defined to be the reduced closure of $\mathtt{Gr}^{\bar{\lambda}}_{\mathscr{G}}$ in $\mathtt{Gr}_{\mathscr{G}}$. Moreover,

$$\dim \overline{\mathtt{Gr}}^{\bar{\lambda}}_{\mathscr{G}} = 2\langle \lambda, \rho \rangle,$$

where $\rho$ is the sum of all fundamental coweights of $\mathfrak{g}$. It is easy to see that the dimension is independent of the choice of $\lambda$.

For any $\bar{\lambda}, \bar{\mu} \in X_*(T)^+_\sigma$, we write $\bar{\mu} \leq \bar{\lambda}$ if $\mathtt{Gr}^{\bar{\mu}}_{\mathscr{G}} \subseteq \overline{\mathtt{Gr}}^{\bar{\lambda}}_{\mathscr{G}}$. For any $i \in I$, let $\bar{\check{\alpha}}_i$ denote the image of $\check{\alpha}_i$ in $X_*(T)_\sigma$. For any $j \in I_\sigma$, set

$$\gamma_j = \bar{\check{\alpha}}_i, \quad \text{if } j = \eta(i). \tag{13}$$

It is clear that $\gamma_j$ is well-defined.

The following lemma follows from [Ri1, Corollary 2.10].

**Lemma 2.1.** *$\bar{\mu} \leq \bar{\lambda}$ if and only if $\bar{\lambda} - \bar{\mu}$ is a non-negative integral linear combination of $\{ \gamma_j \mid j \in I_\sigma \}$.*

By the ramified geometric correspondence [Zh3], the set $X_*(T)_\sigma$ can be realized as the weight lattice of the reductive group $H := (\check{G})^\tau$, where $\check{G}$ is the Langlands dual group of $G$ and $\tau$ is a diagram autorphism on $\check{G}$ corresponding to the one on $G$, and $\{ \gamma_j \mid j \in I_\sigma \}$ is the set of simple roots for $H$. Moreover, $X_*(T)^+_\sigma$ is the set of dominant weights of $H$, and the partial order $\leq$ is exactly the standard partial order for dominant weights of $H$.

We now assume $G$ is of adjoint type. From the perspective of the geometric Satake, we can determine the minimal elements in $X_*(T)^+_\sigma$, in other words the minimal Schubert variety in each connected component of $\mathtt{Gr}_{\mathscr{G}}$. From the table (11), we see that when $(G, m) = (A_{2\ell-1}, 2)$, $\mathtt{Gr}_{\mathscr{G}}$ has two components, where $\mathtt{Gr}^{\bar{\check{\omega}}_1}_{\mathscr{G}}$ is the minimal Schubert variety in the non-neutral component, since $\bar{\check{\omega}}_1$ gives the miniscule dominant weight of $H \simeq \mathrm{Sp}_{2\ell}$. When $(G, m) = (D_{\ell+1}, 2)$, $\mathtt{Gr}_{\mathscr{G}}$ also has two components and $\mathtt{Gr}^{\bar{\check{\omega}}_\ell}_{\mathscr{G}}$ is the minimal Schubert variety in the non-neutral component, since $\bar{\check{\omega}}_\ell$ is the miniscule dominant weight of $H \simeq \mathrm{Spin}_{2\ell+1}$. Otherwise, $\mathtt{Gr}_{\mathscr{G}}$ has only one component. In fact, when $(G, m) = (A_{2\ell}, 4)$, $H \simeq \mathrm{SO}_{2\ell+1}$, in which case the weight lattice $X_*(T)_\sigma$ concides with the root lattice of $H$.

Let $S$ denote the following set

$$S = \begin{cases} \{0\} & \text{if } (G, r) \neq (A_{2\ell-1}, 2), (D_{\ell+1}, 2) \\ \{0, \check{\omega}_1\} & \text{if } (G, r) = (A_{2\ell-1}, 2) \\ \{0, \check{\omega}_\ell\} & \text{if } (G, r) = (D_{\ell+1}, 2) \end{cases} . \tag{14}$$

For any $\kappa \in S$, let $\mathrm{Gr}_{\mathscr{G},\kappa}$ be the component of $\mathrm{Gr}_{\mathscr{G}}$ containing the Schubert variety $\mathrm{Gr}_{\mathscr{G}}^{\bar{\kappa}}$, or equivalently containing the point $e_{\bar{\kappa}}$. Then,

$$\mathrm{Gr}_{\mathscr{G}} = \sqcup_{\kappa \in S} \mathrm{Gr}_{\mathscr{G},\kappa} .$$

2.4. **Global affine Grassmannian of parahoric Bruhat-Tits group schemes.** Let $C$ be a complex projective line $\mathbb{P}^1$ with a coordinate $t$, and with the action of $\sigma$ such that $t \mapsto \epsilon t$. Let $\bar{C}$ be the quotient curve $C/\sigma$, and let $\pi : C \to \bar{C}$ be the projection map. Then $\bar{C}$ is also isomorphc to $\mathbb{P}^1$. Let $\mathcal{G} = \mathrm{Res}_{C/\bar{C}}(G \times C)^\sigma$ be the group scheme over $\bar{C}$, which is the $\sigma$-fixed point subgroup scheme of the Weil restriction $\mathrm{Res}_{C/\bar{C}}(G \times C)$ of the constant group scheme $G \times C$ from $C$ to $\bar{C}$. Then, $\mathcal{G}$ is a parahoric Bruhat-Tits group scheme over $\bar{C}$ in the sense of Heinloth [He] and Pappas-Rapoport [PR]. Let $o$ (resp.$\bar{o}$) be the origin of $C$ (resp.$\bar{C}$), and let $\infty$ (resp. $\bar{\infty}$) be the infinite point in $C$ (resp. $\bar{C}$).

The group scheme $\mathcal{G}$ has the following properties:

(1) For any $y \in \bar{C}$, if $y \neq \bar{o}, \bar{\infty}$, the fiber $\mathcal{G}_{|_y}$ over $y$ is isomorphic to $G$; the restriction $\mathcal{G}_y$ to the formal disc $\mathbb{D}_y$ around $y$ is isomorphic to the constant group scheme $G_{\mathbb{D}_y}$ over $\mathbb{D}_y$.

(2) When $y = \bar{o}$ or $\bar{\infty}$ in $\bar{C}$, $\mathcal{G}_{|_y}$ has a reductive quotient $G^\sigma$; the restriction $\mathcal{G}_y$ to $\mathbb{D}_y$ is isomorphic to the parahoric group scheme $\mathscr{G}$.

Similarly, we can define the parahoric Bruhat-Tits group scheme $\mathcal{T} := \mathrm{Res}_{C/\bar{C}}(T \times C)^\sigma$.

Given an $R$-point $p \in C(R)$ we denote by $\Gamma_p \subset C_R$ the graph of $p$ where $C_R := C \times \mathrm{Spec}(R)$, and denote by $\hat{\Gamma}_p$ the formal completion of $C_R$ along $\Gamma_p$, and let $\hat{\Gamma}_p^\times$ be the punctured formal completion along $\Gamma_p$. Let $\bar{p}$ be the image of $p$ in $\bar{C}$. We similarly define $\bar{C}_R$, $\Gamma_{\bar{p}}$, $\hat{\Gamma}_{\bar{p}}$ and $\hat{\Gamma}_{\bar{p}}^\times$.

For any $\mathbb{C}$-algebra $R$, we define

$$\mathrm{Gr}_{\mathcal{G},C}(R) := \left\{ (p, \mathcal{P}, \beta) \;\middle|\; \begin{array}{l} p \in C(R) \\ \mathcal{P} \text{ a } \mathcal{G}\text{-torsor on } \bar{C} \\ \beta : \mathcal{P}|_{\bar{C}_R \setminus \Gamma_{\bar{p}}} \simeq \mathring{\mathcal{P}}|_{\bar{C}_R \setminus \Gamma_{\bar{p}}} \end{array} \right\}, \tag{15}$$

where $\mathring{\mathcal{P}}$ is the trivial $\mathcal{G}$-bundle.

The functor $\mathrm{Gr}_{\mathcal{G},C}$ is represented by an ind-scheme which is ind-proper over $C$. We call it the global affine Grassmannian $\mathrm{Gr}_{\mathcal{G},C}$ of $\mathcal{G}$ over $C$.

For any $p \neq o, \infty \in C$, the fiber $\mathrm{Gr}_{\mathcal{G},p} := \mathrm{Gr}_{\mathcal{G},C}|_p$ is isomorphic to the usual affine Grassmannian $\mathrm{Gr}_G$, and the fiber $\mathrm{Gr}_{\mathcal{G},p}$ over $p = o, \infty$ is isomorphic to the twisted affine Grassmannian $\mathrm{Gr}_{\mathscr{G}}$ of the parahoric group scheme $\mathscr{G}$.

**Remark 2.2.** *One can define the global affine Grassmannian $\mathrm{Gr}_{\mathcal{G}}$ over $\bar{C}$, see [Zh2, Section 3.1]. The global affine Grassmannian defined above is actually the base change of $\mathrm{Gr}_{\mathcal{G}}$ along $\pi : C \to \bar{C}$.*

We can also define the jet group scheme $L^+\mathcal{G}_C$ over $C$ as follows,

$$L^+\mathcal{G}_C(R) := \left\{ (p, \gamma) \;\middle|\; \begin{array}{l} p \in C(R) \\ \gamma \text{ is a trivialization of the trivial } \mathcal{G}\text{-torsor on } \bar{C} \text{ along } \hat{\Gamma}_{\bar{p}} \end{array} \right\} \tag{16}$$

Again, $L^+\mathcal{G}_C$ is the base change of the usual jet group scheme $L^+\mathcal{G}$ of $\mathcal{G}$ along $\pi : C \to \bar{C}$. For any $p \neq o, \infty \in C$, the fiber $L^+\mathcal{G}_C|_p$ is isomorphic to the jet group scheme $L^+G$ of $G$, and the fiber $L^+\mathcal{G}_C|_p$ over $p = o, \infty$ is isomorphic to jet group scheme $L^+\mathscr{G}$.

We have a left action of $L^+\mathcal{G}_C$ on $\mathtt{Gr}_{\mathcal{G},C}$ given by

$$((p,\gamma),(p,\mathcal{P},\beta)) \mapsto (p,\mathcal{P}',\beta), \tag{17}$$

where $\mathcal{P}'$ is obtained by choosing a trivialization of $\mathcal{P}$ along $\hat{\Gamma}_{\bar{p}}$ and then composing this trivialization with $\gamma$ and regluing with $\beta$.

We also can define the global loop group $L\mathcal{G}_C$ of $\mathcal{G}$ over $C$,

$$L\mathcal{G}_C(R) := \left\{ (p,\gamma) \,\middle|\, \begin{array}{l} p \in C(R) \\ \gamma \text{ is a trivialization of the trivial } \mathcal{G}\text{-torsor on } \bar{C} \text{ along } \hat{\Gamma}^{\times}_{\bar{p}} \end{array} \right\}. \tag{18}$$

Then $\mathtt{Gr}_{\mathcal{G},C}$ is isomorphic to the fppf quotient $L\mathscr{G}_C/L^+\mathscr{G}_C$. We can also define $L^+\mathcal{T}_C$ and $L\mathcal{T}_C$ similarly. Then,

$$L\mathcal{T}_C|_p \simeq \begin{cases} T_{\mathcal{K}_p} & \text{if } p \neq o,\infty \\ \mathscr{T} & \text{if } p = o,\infty \end{cases},$$

where $\mathcal{K}_p$ is the field of formal Laurant series of $C$ at $p$.

2.5. **Global Schubert varieties.** For each $p \in C$, we can attach a lattice $X_*(T)_p$,

$$X_*(T)_p = \begin{cases} X_*(T) & \text{if } p \neq o,\infty \\ X_*(T)_\sigma & \text{if } p = o,\infty \end{cases}.$$

By [Zh2, Proposition 3.4], for any $\lambda \in X_*(T)$ there exists a section $s^\lambda : C \to L\mathcal{T}_C$, such that for any $p \in C$, the image of $s^\lambda(p)$ in $X_*(T)_p$ is given by

$$\begin{cases} \lambda \in X_*(T) & \text{if } p \neq o,\infty \\ \bar{\lambda} \in X_*(T)_\sigma & \text{if } p = o,\infty \end{cases}.$$

This naturally gives rise to $C$-points in $\mathtt{Gr}_{\mathcal{T},C}$ and $\mathtt{Gr}_{\mathcal{G},C}$, which will still be denoted by $s^\lambda$. Following [Zh2, Definition 3.1], for each $\lambda \in X_*(T)$ we define the global Schubert variety $\overline{\mathtt{Gr}}^\lambda_{\mathcal{G},C}$ to be the minimal $L^+\mathcal{G}_C$-stable irreducible closed subvariety of $\mathtt{Gr}_{\mathcal{G},C}$ that contains $s^\lambda$. Then, [Zh2, Theorem 3] asserts that

**Theorem 2.3.** *The global Schubert variety $\overline{\mathtt{Gr}}^\lambda_{\mathcal{G},C}$ is flat over C, and for any $p \in C$ the fiber $\overline{\mathtt{Gr}}^\lambda_{\mathcal{G},p}$ is reduced and*

$$\overline{\mathtt{Gr}}^\lambda_{\mathcal{G},p} \simeq \begin{cases} \overline{\mathtt{Gr}}^\lambda_G & \textit{if } p \neq o,\infty \\ \overline{\mathtt{Gr}}^{\bar{\lambda}}_{\mathscr{G}} & \textit{if } p = o,\infty \end{cases}.$$

**Remark 2.4.** *In fact, we can construct the section $s^\lambda : C \to \mathtt{Gr}_{\mathcal{T},C}$ explicitly. For any $\mathcal{T}$-torsor $\mathcal{P}$ over $\bar{C}$, the pull-back $\pi^*(\mathcal{P})$ is a $(\sigma,T)$-torsor on $C$, i.e. $\pi^*(\mathcal{P})$ carries an action of $\sigma$ of order $m$ and compatible with the action on $T$. Then, the section $s^\lambda$ amounts to the* *triple* $(\mathrm{Id}: C \to C, \mathscr{F}, \beta)$*, where* $\mathrm{Id}: C \to C$ *is the identity map,* $\mathscr{F}$ *is the* $(\sigma, T)$*-torsor with the trivialization* $\mathscr{F}|_{C^2\setminus\Delta C} \simeq \mathring{\mathscr{F}}$ *as* $(\sigma, T)$*-torsor, such that for any weight* $\nu \in X^*(T)$*,*

$$\beta_\nu : \mathscr{F} \times_T \mathbb{C}^\nu \simeq \mathscr{O}_{C^2}\Big(\sum_{i=0}^{m-1} \langle \sigma^i(\lambda), \nu\rangle \Gamma_{\sigma^i}\Big),$$

*where* $\Gamma_{\sigma^i}$ *represents the graph of* $\sigma^i : C \to C$*.*

## 3. Construction of level one line bundle

In this section, we keep the assumption that $G$ is of adjoint type with the action of an absolutely special automorphism $\sigma$.

3.1. **Borel-Weil-Bott theorem on $\mathrm{Gr}_{\mathscr{G}}$.** We define the twisted affine Lie algebra $\hat{L}(\mathfrak{g}, \sigma) := \mathfrak{g}(\mathcal{K})^\sigma \oplus \mathbb{C}K$ with the canonical center $K$ as follows,

$$[x[f] + zK, x'[f'] + z'K] = [x, x'][ff'] + m^{-1}\mathrm{Res}_{t=0}\left((df)f'\right)(x, x')K, \tag{19}$$

for $x[P], x'[P'] \in \mathfrak{g}(\mathcal{K})^\sigma$, $z, z' \in \mathbb{C}$; where $\mathrm{Res}_{t=0}$ denotes the coefficient of $t^{-1}dt$, and $(,)$ is the normalized Killing form on $\mathfrak{g}$, i.e. $(\check{\theta}, \check{\theta}) = 2$.

We use $P(\sigma, c)$ to denote the set of highest weights of $\mathfrak{g}^\sigma$ which parametrizes the integrable highest weight modules of $\hat{L}(\mathfrak{g}, \sigma)$ of level $c$, see [HK, Section 2]. For each $\lambda \in P(\sigma, c)$, we denote by $\mathscr{H}_c(\lambda)$ the associated integrable highest weight module of $\hat{L}(\mathfrak{g}, \sigma)$.

Recall that $\{\lambda_i \,|\, i \in I_\sigma\}$ be the set of fundamental weights of $\mathfrak{g}^\sigma$, where we follow the labellings in (4). Also, $\{\check{\beta}_i \,|\, i \in I_\sigma\}$ is the set of simple coroots of $\mathfrak{g}^\sigma$.

**Lemma 3.1.** *For an absolutely special automorphism* $\sigma$*, we have*

$$P(\sigma, 1) = \begin{cases} \{0\} & \text{if } (\mathfrak{g}, m) \neq (A_{2\ell-1}, 2), (D_{\ell+1}, 2) \\ \{0, \lambda_1\} & \text{if } (\mathfrak{g}, m) = (A_{2\ell-1}, 2) \\ \{0, \lambda_\ell\} & \text{if } (\mathfrak{g}, m) = (D_{\ell+1}, 2) \end{cases}.$$

*Proof.* We first consider the case when $(\mathfrak{g}, m) \neq (A_{2\ell}, 4)$. We can read from [HK, Lemma 2.1], for any $\lambda \in (\mathfrak{t}^\sigma)^*$, $\lambda \in P(\sigma, 1)$ if and only if

$$\langle \lambda, \check{\beta}_i \rangle \in \mathbb{Z}_{\geq 0} \quad \text{for any } i \in I_\sigma,$$

and $\langle \lambda, \check{\theta}_0 \rangle \leq 1$, where $\theta_0$ is the highest short root of $\mathfrak{g}^\sigma$ and $\check{\theta}_0$ is the coroot of $\theta_0$, and hence $\check{\theta}_0$ is the highest coroot of $\mathfrak{g}^\sigma$. In this case, $\lambda \in P(\sigma, 1)$ if and only if $\lambda = 0$ or a miniscule dominant weight of $\mathfrak{g}^\sigma$ (cf. [BH, Lemma 2.13]). Following the labellings in [Ka, Table Fin,p53], when $\mathfrak{g}^\sigma$ is of type $C_\ell$, $\lambda_1$ is the only miniscule weight; when $\mathfrak{g}^\sigma$ is of type $B_\ell$, $\lambda_\ell$ is the only miniscule weight. Any other non simply-laced Lie algebra has no miniscule weight. This finishes the argument of the lemma when $(\mathfrak{g}, m) \neq (A_{2\ell}, 4)$.

Now, we assume that $(\mathfrak{g}, m) = (A_{2\ell}, 4)$. In this case, it is more convenient to choose a different set of simple roots for $\mathfrak{g}^\sigma$, rather than the one described in (4). Namely, we can also choose

$$\{\alpha_i|_{\mathfrak{t}^\sigma} \,|\, i = 1, 2, \cdots, \ell - 1\} \cup \{-\theta|_{\mathfrak{t}^\sigma}\}$$

as the set of simple roots of $\mathfrak{g}^\sigma$. With this set of simple root, we can also read from [HK, Lemma 2.1], for any $\lambda \in (\mathfrak{t}^\sigma)^*$, $\lambda \in P(\sigma, 1)$ if and only if $\lambda = 0$.

□

**Remark 3.2.** *It is not true that* $0 \in P(\sigma, 1)$ *for any automorphism* $\sigma$. *For example,* $0 \notin P(\tau, 1)$*, when* $\mathfrak{g} = A_{2\ell}$ *and* $\tau$ *is a diagram automorphism; instead* $0 \in P(\tau, 2)$.

We define the following map

$$\iota : X_*(T) \to (\mathfrak{t}^\sigma)^*, \tag{20}$$

such that for any $\lambda \in X_*(T)$, $\iota(\lambda)(h) = (\lambda, h)$, where we regard $\lambda$ as an element in $\mathfrak{t}$ and $(,)$ is the normalized Killing form on $\mathfrak{t}$. It is clear that $\iota(0) = 0$. This map naturally descends to a map $X_*(T)_\sigma \to (\mathfrak{t}^\sigma)^*$. By abuse of notation, we still call it $\iota$.

Recall some terminilogy introduced in Section 2.1. $I_\sigma$ is the set parametrizing simple roots of $\mathfrak{g}^\sigma$, and we also defined a map $\eta : I \to I_\sigma$. The set $\{\check{\lambda}_j \,|\, j \in I_\sigma\}$ is the set of fundamental coweights of $\mathfrak{g}^\sigma$, and $\{\lambda_j \,|\, j \in I_\sigma\}$ is the set of fundamental weights of $\mathfrak{g}^\sigma$. We also recall that $\check{\alpha}_i$ is a simple coroot of $\mathfrak{g}$ for each $i \in I$, and $\gamma_j$ is the image of $\check{\alpha}_i$ in $X_*(T)_\sigma$. The following lemma already appears in [Ha, Lemma 3.2] in a slighly different setting.

**Lemma 3.3.** *For any* $j \in I_\sigma$*, we have*

$$\iota(\gamma_j) = \begin{cases} \beta_j, & \text{if } (\mathfrak{g}, m) \neq (A_{2\ell}, 4), \text{ or }, (\mathfrak{g}, m) = (A_{2\ell}, 4) \text{ and } j \neq \ell \\ \frac{1}{2}\beta_\ell, & \text{if } (\mathfrak{g}, m) = (A_{2\ell}, 4) \text{ and } j = \ell \end{cases}.$$

*Proof.* By the definition of $\iota$, for any $\gamma_j = \overline{\check{\alpha}_i}$ with $j = \eta(i)$, and $k \in I_\sigma$ we have the following equalities:

$$\langle \check{\lambda}_k, \gamma_j \rangle = \langle \check{\lambda}_k, \iota(\overline{\check{\alpha}_i}) \rangle = (\check{\lambda}_k, \check{\alpha}_i) = \langle \check{\lambda}_k, \alpha_i \rangle.$$

Then, this lemma can readily follows from the description of fundamental coweights of $\mathfrak{g}^\sigma$ in (7) and (8). □

Recall the set $S$ defined in (14).

**Lemma 3.4.** *For any* $i \in I$*, we have* $\iota(\check{\omega}_i) = \lambda_{\eta(i)}$*. As a consequence,* $\iota$ *maps* $X_*(T)_\sigma^+$ *bijectively into the set of dominant weights of* $\mathfrak{g}^\sigma$*. Furthermore,* $\iota$ *maps* $S$ *bijectively into* $P(\sigma, 1)$.

*Proof.* For any $i \in I$ and $j \in I_\sigma$, we have

$$\langle \iota(\check{\omega}_i), \check{\beta}_j \rangle = (\check{\omega}_i, \check{\beta}_j) = (\check{\omega}_i, \sum_{a \in \eta^{-1}(i)} \check{\alpha}_a) = \delta_{\eta(i), j}.$$

Hence, $\iota(\check{\omega}_i) = \lambda_{\eta(i)}$.

In view of Lemma 3.1, $\iota$ maps $S$ bijectively into $P(\sigma, 1)$. □

**Remark 3.5.** *In view of Lemma 3.3 and Lemma 3.4, when* $(G, m) \neq (A_{2\ell}, 4)$*, the root systems of* $\mathfrak{g}^\sigma$ *and* $H := (\check{G})^\tau$ *can be naturally identified, where* $H$ *is discussed in Section 2.3. Namely,* $\{\overline{\check{\omega}_i} \,|\, i \in I\}$ *is a set of fundmental weights of* $H$ *corresponding to* $\{\lambda_j \,|\, j \in I_\sigma\}$ *of* $\mathfrak{g}^\sigma$*, and the set of simple roots* $\{\gamma_j \,|\, j \in I_\sigma\}$ *corresponds to* $\{\beta_j \,|\, j \in I_\sigma\}$ *of* $\mathfrak{g}^\sigma$.

For any $g \in G(\mathcal{K})^\sigma$, we can define a Lie algebra automorphism

$$\widehat{\mathrm{Ad}}_g(x[f]) := \mathrm{Ad}_g(x[f]) + \frac{1}{m}\mathrm{Res}_{t=0}(g^{-1}dg, x[f])K, \tag{21}$$

for any $x[f] \in \mathfrak{g}(\mathcal{K})^\sigma$, where $(,)$ is the normalized Killing form on $\mathfrak{g}$. By Lemma 3.4, $\iota(\kappa) \in P(\sigma, 1)$ for any $\kappa \in S$. Thus, $c\iota(\kappa) \in P(\sigma, c)$ for any level $c \geq 1$.

Set

$$\mathscr{H}_c := \oplus_{\kappa\in S}\mathscr{H}_c(c\iota(\kappa)). \tag{22}$$

Let $\tilde{\mathfrak{g}} := \mathfrak{g}\otimes\mathcal{K}\oplus\mathbb{C}K'\oplus\mathbb{C}d'$ be the untwisted Kac-Moody algebra associated to $\mathfrak{g}$, where $K'$ is the canonical center and $d'$ is the scaling element. We may define an automorphism $\sigma$ on $\tilde{\mathfrak{g}}$ as follows,

$$\sigma(x[f(t)]) = \sigma(x)[f(\epsilon t)], \quad \sigma(K') = K', \quad \sigma(d') = d',$$

for any $x[f] \in \mathfrak{g}\otimes\mathcal{K}$. Then the fixed point Lie algebra $\tilde{\mathfrak{g}}^\sigma$ is exactly the twisted Kac-Moody alegbra $\tilde{L}(\mathfrak{g}, \sigma)$ containing $\hat{L}(\mathfrak{g}, \sigma)$ as the derived algebra. Following from [Ka, Theorem 8.7,§8], in this realization the canonical center $K$ in $\tilde{L}(\mathfrak{g}, \sigma)$ is equal to $mK'$, and the scaling element $d$ in $\tilde{\mathfrak{g}}$ is equal to $d'$ when $\tilde{\mathfrak{g}}^\sigma$ is not $A^{(2)}_{2\ell}$, and $d = 2d'$when $\tilde{\mathfrak{g}}^\sigma = A^{(2)}_{2\ell}$.

For any $g \in G(\mathcal{K})$, one can define an automorphism $\widehat{\mathrm{Ad}}_g$ on $\tilde{\mathfrak{g}}$ as in [Ku, Section 13.2.3]. From the formula *loc.cit*, it is clear that if $g \in G(\mathcal{K})^\sigma$, then $\widehat{\mathrm{Ad}}_g$ commutes with $\sigma$. In particular, it follows that $\widehat{\mathrm{Ad}}_g$ restricts to an automorphism on $\tilde{L}(\mathfrak{g}, \sigma)$. One may observe easily that, restricting further to $\widehat{L}(\mathfrak{g}, \sigma)$, this is exactly the automorphism defined in (21).

By demanding that $d\cdot v_\kappa = 0$ for each $\kappa \in S$, the action $\hat{L}(\mathfrak{g}, \sigma)$ on $\mathscr{H}$ extends uniquely to an action of $\tilde{L}(\mathfrak{g}, \sigma)$.

**Lemma 3.6.** *For any $g \in G(\mathcal{K})^\sigma$, there exists an intertwining operator $\rho_g : \mathscr{H}_c \simeq \mathscr{H}_c$ such that*

$$\rho_g(x[f]\cdot v) = \widehat{\mathrm{Ad}}_g(x[f])\cdot\rho_g(v), \tag{23}$$

*for any $x[f] \in \mathfrak{g}(\mathcal{K})^\sigma$ and $v \in \mathscr{H}_c$. In particular, for any $\kappa \in S$,*

$$\widehat{\mathrm{Ad}}_{n^{-\kappa}}(\mathscr{H}_c(0)) = \mathscr{H}_c(c\iota(\kappa)), \text{ and } \widehat{\mathrm{Ad}}_{n^{-\kappa}}(\mathscr{H}_c(c\iota(\kappa))) = \mathscr{H}_c(0). \tag{24}$$

*Proof.* Let $G'$ be the simply-connected cover of $G$, and let $p : G'(\mathcal{K})^\sigma \to G(\mathcal{K})^\sigma$ be the induced map. Then,

$$G(\mathcal{K})^\sigma = \sqcup_{\kappa\in S} n^{-\kappa}\overline{G'(\mathcal{K})^\sigma}, \tag{25}$$

where $\overline{G'(\mathcal{K})^\sigma} = p(G'(\mathcal{K})^\sigma)$. By twisted analogue of Faltings Lemma (cf. [HK, Proposition 10.2]), for any element $g \in \overline{G'(\mathcal{K})^\sigma}$, there exists an operator $\rho_g$ which maps $\mathscr{H}_c(c\iota(\kappa))$ to $\mathscr{H}_c(c\iota(\kappa))$ with the desired property (23), for any $\kappa \in S$. By decomposition (25), it suffices to show that, for nonzero $\kappa$, $n^{-\kappa}$ satisfies property (24).

Assume $\kappa \neq 0$ in $S$. From the table (11), the group $(X_*(T)/\check{Q})_\sigma$ is at most of order 2. Therefore, $n^{-2\kappa} \in \overline{G'(\mathcal{K})^\sigma}$. For each $\mathscr{H}_c(c\iota(\kappa))$, we denote the action by $\pi_{c,\kappa} : \hat{L}(\mathfrak{g}, \sigma) \to$

$\mathrm{End}(\mathscr{H}_c(c\iota(\kappa)))$. Then the property (23) for $n^{-2\kappa}$, is equivalent to the existence of an isomorphism of representations,

$$\rho_{n^{-2\kappa}} : (\mathscr{H}_c(c\iota(\kappa)), \pi_{c,\kappa}) \simeq (\mathscr{H}_c(c\iota(\kappa)), \pi_{c,\kappa} \circ \widehat{\mathrm{Ad}}_{n^{-2\kappa}}). \tag{26}$$

Let $v_\kappa$ be the highest weight vector in $\mathscr{H}_c(c\iota(\kappa))$. Then $v_\kappa$ is of $\mathfrak{t}^\sigma$-weight $c\iota(\kappa)$. We regard $\check{\beta}_i$ as elements in $\mathfrak{t}^\sigma$. By formula (21),

$$\widehat{\mathrm{Ad}}_{n^{-\kappa}}(\check{\beta}_i) = \check{\beta}_i - (\kappa, \check{\beta}_i)c = \check{\beta}_i - \langle \iota(\kappa), \check{\beta}_i \rangle c.$$

Hence, $v_\kappa$ is of $\mathfrak{t}^\sigma$-weight 0 and a highest weight vector in the representation

$$(\mathscr{H}_c(c\iota(\kappa)), \pi_{c,\kappa} \circ \widehat{\mathrm{Ad}}_{n^{-\kappa}}).$$

By Schur lemma, there exists an intertwining operator $\rho_{0\kappa}$,

$$\rho_{0\kappa} : (\mathscr{H}_c(0), \pi_{c,0}) \simeq (\mathscr{H}_c(c\iota(\kappa)), \pi_{c,\kappa} \circ \widehat{\mathrm{Ad}}_{n^{-\kappa}}). \tag{27}$$

We also can regard $\rho_{0\kappa}$ as the following intertwining operator

$$\rho_{0\kappa} : (\mathscr{H}_c(0), \pi_{c,0} \circ \widehat{\mathrm{Ad}}_{n^{-\kappa}}) \simeq (\mathscr{H}_c(c\iota(\kappa)), \pi_{c,\kappa} \circ \widehat{\mathrm{Ad}}_{n^{-2\kappa}}) \tag{28}$$

Combining isomorphisms (26),(28), we get

$$(\mathscr{H}_c(c\iota(\kappa)), \pi_{c,\kappa}) \xrightarrow{\rho_{n^{-2\kappa}}} (\mathscr{H}_c(c\iota(\kappa)), \pi_{c,\kappa} \circ \widehat{\mathrm{Ad}}_{n^{-2\kappa}}) \xrightarrow{(\rho_{0\kappa})^{-1}} (\mathscr{H}_c(0), \pi_{c,0} \circ \widehat{\mathrm{Ad}}_{n^{-\kappa}}).$$

We define $\rho_{n^{-\kappa}}$ to be the following operator

$$\rho_{n^{-\kappa}} = (\rho_{0\kappa}, (\rho_{0\kappa})^{-1} \circ \rho_{n^{-2\kappa}}) : \mathscr{H}_c(0) \oplus \mathscr{H}_c(c\iota(\kappa)) \simeq \mathscr{H}_c(0) \oplus \mathscr{H}_c(c\iota(\kappa)).$$

The map $\rho_{n^{-\kappa}}$ satisfies property (23). □

As discussed in Section 2.2, the components of $\mathtt{Gr}_{\mathscr{G}}$ are parametrized by elements in $(X_*(T)/\check{Q})_\sigma$. Moreover, $\mathtt{Gr}_{\mathscr{G}} = \sqcup_{\kappa \in S} \mathtt{Gr}_{\mathscr{G},\kappa}$, where $S$ is defined in (14).

Let $\mathscr{G}'$ be the parahoric group scheme $\mathrm{Res}_{O/\bar{O}}(G'_O)^\sigma$, and let $L^+\mathscr{G}'$ (resp. $L\mathscr{G}'$) denote the jet group scheme (resp. loop group scheme) of $\mathscr{G}'$. The group $L\mathscr{G}$ acts on $L\mathscr{G}'$ by conjugation. Set

$$L^+\mathscr{G}'_\kappa := \mathrm{Ad}_{n^{-\kappa}}(L^+\mathscr{G}').$$

Then, $L^+\mathscr{G}'_\kappa$ is a subgroup scheme of $L\mathscr{G}'$. We have

$$\mathtt{Gr}_{\mathscr{G},\kappa} \simeq L\mathscr{G}'/L^+\mathscr{G}'. \tag{29}$$

By the twisted analogue of Faltings lemma (cf. [HK, Proposition 10.2]), there exists a group homomorphism $L\mathscr{G}' \to \mathrm{PGL}(\mathscr{H}_1(0))$. Consider the central extension

$$1 \to \mathbb{G}_m \to \mathrm{GL}(\mathscr{H}_1(0)) \to \mathrm{PGL}(\mathscr{H}_1(0)) \to 1. \tag{30}$$

The pull-back of (30) to $L\mathscr{G}'$ defines the following canonical central extension of $L\mathscr{G}'$:

$$1 \to \mathbb{G}_m \to \widehat{L\mathscr{G}'} \to L\mathscr{G}' \to 1. \tag{31}$$

It is known that $\widehat{L\mathscr{G}'}$ is a Kac-Moody group of twisted type (up to a scaling multiplicative group) in the sense of Kumar and Mathieu, see [PR]. Let $\widehat{L^+\mathscr{G}'_\kappa}$ denote the preimage of $L^+\mathscr{G}'_\kappa$ in $\widehat{L\mathscr{G}'}$ via the projection map $\widehat{L\mathscr{G}'} \to L\mathscr{G}'$. As the same proof as in [BH, Lemma 2.19], $\widehat{L^+\mathscr{G}'_\kappa}$ is a parabolic subgroup in $\widehat{L\mathscr{G}'}$, moreover

$$\mathrm{Gr}_{\mathscr{G},\kappa} \simeq \widehat{L\mathscr{G}'}/\widehat{L^+\mathscr{G}'_\kappa}, \tag{32}$$

i.e. $\mathrm{Gr}_{\mathscr{G},\kappa}$ is a partial flag variety of the Kac-Moody group $\widehat{L\mathscr{G}'}$.

**Proposition 3.7.** *There exists a line bundle $\mathscr{L}$ on $\mathrm{Gr}_{\mathscr{G}}$ such that $\mathscr{L}$ is of level one on each component of $\mathrm{Gr}_{\mathscr{G}}$.*

*Proof.* We first consider the simply-connected cover $G'$ of $G$. By [HK, Theorem 10.7 (1)], there exists a canonical splitting of $\widehat{L\mathscr{G}'} \to L\mathscr{G}'$ in the central extension (30) over $L^+\mathscr{G}'$. We may define a line bundle $\mathscr{L}$ on $\mathrm{Gr}_{\mathscr{G}'} = \widehat{L\mathscr{G}'}/\widehat{L^+\mathscr{G}'}$ via the character $\widehat{L^+\mathscr{G}'} := \mathbb{G}_m \times L^+\mathscr{G}' \to \mathbb{G}_m$ defined via the first projection. In fact, as the argument in [LS, Lemma 4.1], this line bundle is the ample generator of $\mathrm{Pic}(\mathrm{Gr}_{\mathscr{G}'})$ of level 1. This finishes the proof of part (1).

We now consider the case when $G$ is of adjoint type. Since the neutral component $\mathrm{Gr}_{\mathscr{G},\circ}$ is isomorphic to $\mathrm{Gr}_{\mathscr{G}'}$, we get the level one line bundle on $\mathrm{Gr}_{\mathscr{G},\circ}$ induced from the one on $\mathrm{Gr}_{\mathscr{G}'}$. For any other component $\mathrm{Gr}_{\mathscr{G},\kappa}$, by (32) we have an isomorphism $\mathrm{Gr}_{\mathscr{G},\circ} \simeq \mathrm{Gr}_{\mathscr{G},\kappa}$. Therefore, this gives rise to the level one line bundle on $\mathrm{Gr}_{\mathscr{G},\kappa}$. □

The line bundle $\mathscr{L}$ on $\mathrm{Gr}_{\mathscr{G}}$ naturally has a $\widehat{L\mathscr{G}'}$-equivariant structure, since $\mathscr{L}$ admits a unique $\widehat{L\mathscr{G}'}$-equivariant structure on each component of $\mathrm{Gr}_{\mathscr{G}}$ as a partial flag variety of $\widehat{L\mathscr{G}'}$. Now, by the standard Borel-Weil-Bott theorem for Kac-Moody group (cf. [Ku]), we get the following theorem.

**Theorem 3.8.** *As representations of $\hat{L}(\mathfrak{g},\sigma)$, we have $H^0(\mathrm{Gr}_{\mathscr{G}},\mathscr{L}^c)^\vee \simeq \mathscr{H}_c$, where $\mathscr{L}^c$ is the c-power of $\mathscr{L}$.*

Let $v_0$ be the highest weight vector in $\mathscr{H}_0$. For any $\bar{\lambda} \in X_*(T)_\sigma$, we define

$$v_{\bar{\lambda}} := \rho_{n^\lambda}(v_0), \tag{33}$$

where $\rho_{n^\lambda}$ in defined in Lemma 3.6. Then $v_{\bar{\lambda}}$ is independent of the choice of the representative $\lambda$ in $X_*(T)$ and is well-defined up to a nonzero scalar.

**Lemma 3.9.** *The $\mathfrak{t}^\sigma$-weight of the vector $v_{\bar{\lambda}}$ is $-c\iota(\bar{\lambda})$.*

*Proof.* For any $h \in \mathfrak{t}^\sigma$, by Lemma 3.6,

$$h \cdot v_{\bar{\lambda}} = h \cdot \rho_{n^\lambda}(v_0) = \rho_{n^\lambda}(\widehat{\mathrm{Ad}}_{n^{-\lambda}}(h)v_0).$$

By the formula (21), we have

$$\widehat{\mathrm{Ad}}_{n^{-\lambda}}(h) = h - \langle\lambda, h\rangle K.$$

It follows that

$$h \cdot v_{\bar{\lambda}} = -\langle\lambda, h\rangle c v_{\bar{\lambda}} = -c\iota(\lambda)(h)v_{\bar{\lambda}}.$$

This concludes the proof of the lemma. □

**Definition 3.10.** For any dominant $\bar{\lambda} \in X_*(T)^+_\sigma$, we define the twisted affine Demazure module $D(c, \bar{\lambda})$ as the following $\mathfrak{g}[t]^\sigma$-module,

$$D(c, \bar{\lambda}) := U(\mathfrak{g}[t]^\sigma) v_{\bar{\lambda}}.$$

In view of Lemma 3.9, $D(c, \bar{\lambda})$ contains an irreducible representation $V(-c\iota(\lambda))$ of $\mathfrak{g}^\sigma$ of lowest weight $-c\iota(\lambda)$. The following theorem follows from [Ku, Theorem 8.2.2 (a)].

**Theorem 3.11.** *As $\mathfrak{g}[t]^\sigma$-modules, $H^0(\overline{\mathrm{Gr}}^{\bar{\lambda}}_{\mathscr{G}}, \mathscr{L}^c)^\vee \simeq D(c, \bar{\lambda})$.*

3.2. **Construction of level one line bundles on** $\mathrm{Bun}_{\mathcal{G}}$**.** In this subsection, we consider the parahoric Bruhat-Tits group scheme $\mathcal{G} := \mathrm{Res}_{C/\bar{C}}(G \times C)^\Gamma$ over $\bar{C}$ as in the setting of Section 2.4.

Let $\mathrm{Bun}_{\mathcal{G}}$ be the moduli stack of $\mathcal{G}$-torsors on $\bar{C}$. It is known that $\mathrm{Bun}_{\mathcal{G}}$ is a smooth Artin stack (cf. [He]). By [He, Theorem 3], the Picard group $\mathrm{Pic}(\mathrm{Bun}_{\mathcal{G}})$ of $\mathrm{Bun}_{\mathcal{G}}$ is isomorphc to $\mathbb{Z}$, since the group $X^*(\mathcal{G}|_y)$ of characters for $\mathcal{G}|_y$ is trivial for any $y \in \bar{C}$. In this subsection, we will construct the ample generator $\mathcal{L} \in \mathrm{Pic}(\mathrm{Bun}_{\mathcal{G}})$ when $G$ is simply-connected, and we will construct a level one line bundle on every component of $\mathrm{Gr}_{\mathcal{G},C}$ when $G$ is of adjoint type.

By Lemma 3.1, we have $0 \in P(\sigma, 1)$ for any absolutely special automorphism $\sigma$. Recall that $\mathscr{H}_1(0)$ is the basic representation of level one associated to $0 \in P(\sigma, 1)$.

We now define the following space of twisted covacua of level one,

$$\mathscr{V}_{C,\sigma}(0) := \frac{\mathscr{H}_1(0)}{\mathfrak{g}[t^{-1}]^\sigma \cdot \mathscr{H}_1(0)}, \tag{34}$$

where $\mathfrak{g}[t^{-1}]^\sigma$ is the Lie subalgebra of $\hat{L}(\mathfrak{g}, \sigma)$.

**Lemma 3.12.** *The dimension of the vector space $\mathscr{V}_{C,\sigma}(0)$ is 1.*

*Proof.* Let $v_0$ be the highest weight vector in $\mathscr{H}_1(0)$. Then

$$\mathscr{H}_1(0) = U((t^{-1}\mathfrak{g}[t^{-1}])^\sigma) \cdot v_0 = U((t^{-1}\mathfrak{g}[t^{-1}])^\sigma)(t^{-1}\mathfrak{g}[t^{-1}])^\sigma v_0 \oplus \mathbb{C}v_0,$$

where $U((t^{-1}\mathfrak{g}[t^{-1}])^\sigma)$ denotes the universal enveloping algebra of $(t^{-1}\mathfrak{g}[t^{-1}])^\sigma$. We can write $\mathfrak{g}[t^{-1}]^\sigma = \mathfrak{g}^\sigma \oplus (t^{-1}\mathfrak{g}[t^{-1}])^\sigma$. Hence,

$$\mathfrak{g}[t^{-1}]^\sigma \cdot \mathscr{H}_1(0) = \mathfrak{g}^\sigma \cdot U((t^{-1}\mathfrak{g}[t^{-1}])^\sigma)(t^{-1}\mathfrak{g}[t^{-1}])^\sigma v_0 + U((t^{-1}\mathfrak{g}[t^{-1}])^\sigma)(t^{-1}\mathfrak{g}[t^{-1}])^\sigma v_0 \tag{35}$$

$$= U((t^{-1}\mathfrak{g}[t^{-1}])^\sigma)(t^{-1}\mathfrak{g}[t^{-1}])^\sigma v_0, \tag{36}$$

where the first equality holds since $\mathfrak{g}^\sigma \cdot v_0 = 0$, and the second equality holds since $\mathfrak{g}^\sigma$ normalizes $(t^{-1}\mathfrak{g}[t^{-1}])^\sigma$ under the Lie bracket. Therefore, $\dim \mathscr{V}_{C,\sigma}(0) = 1$. □

Let $G'$ be the simply-connected cover of $G$. Recall the Heinloth uniformization theorem for $\mathcal{G}' := \mathrm{Res}_{C/\bar{C}}(G' \times C)^\Gamma$over the affine line $\bar{C}\backslash\bar{o}$ (cf. [He]),

$$\mathrm{Bun}_{\mathscr{G}'} \simeq G'[t^{-1}]^\sigma \backslash \mathrm{Gr}_{\mathscr{G}'},$$

where $\mathrm{Gr}_{\mathscr{G}'}$ denotes the affine Grassmannian of $\mathscr{G}' := \mathrm{Res}_{O/\bar{O}}(G'_O)^\sigma$, and $G'[t^{-1}]^\sigma \backslash \mathrm{Gr}_{\mathscr{G}'}$ denotes the fppf quotient.

**Theorem 3.13.** *The line bundle $\mathscr{L}$ descends to a line bundle $\mathcal{L}$ on* $\mathrm{Bun}_{\mathscr{G}'}$ .

*Proof.* Let $\mathscr{L}$ be the level one line bundle on $\mathrm{Gr}_{\mathscr{G}'}$ constructed from Proposition 3.7. To show that the line bundle $\mathscr{L}$ can descend to $\mathrm{Bun}_{\mathcal{G}'}$, as in the argument in [So], it suffices to show that there is a $G'[t^{-1}]^\sigma$-linearization on $\mathscr{L}$. This is equivalent to the splitting of the central extension (31) over $G'[t^{-1}]^\sigma$. We use the same argument as in [So, Proposition 3.3], since the vector space $\mathscr{V}_{C,\sigma}(0)$ is nonvanishing by Lemma 3.12, the central extension (31) splits over $G'[t^{-1}]^\sigma$. □

We consider the projection map $\mathrm{pr} : \mathrm{Gr}_{\mathcal{G}',C} \to \mathrm{Bun}_{\mathcal{G}'}$. By abuse of notation, we still denote by $\mathcal{L}$ the line bundle on $\mathrm{Gr}_{\mathcal{G}',C}$ pulling-back from $\mathcal{L}$ on $\mathrm{Bun}_{\mathcal{G}'}$.

**Corollary 3.14.** *The restriction of the line bundle $\mathcal{L}$ to the fiber $\mathrm{Gr}_{\mathcal{G}',p}$ is the ample generator of* $\mathrm{Pic}(\mathrm{Gr}_{\mathcal{G}',p})$*, for any $p \in C$.*

*Proof.* It follows from Theorem 3.13 and [Zh2, Proposition 4.1]. □

The following theorem is interesting by itself, but will not be used in this paper.

**Theorem 3.15.** *There is a natural isomorphism*

$$H^0(\mathrm{Bun}_{\mathcal{G}'}, \mathcal{L}) \simeq \mathscr{V}_{C,\sigma}(0)^\vee,$$

*where $\mathscr{V}_{C,\sigma}(0)^\vee$ denotes the dual of $\mathscr{V}_{C,\sigma}(0)$. In particular,*

$$\dim H^0(\mathrm{Bun}_{\mathcal{G}'}, \mathcal{L}) = 1.$$

*Proof.* The theorem follows from the same argument as in [HK, Theorem 12.1]. □

Now, we would like to construct the line bundle $\mathcal{L}$ of level one on $\mathrm{Gr}_{\mathcal{G},C}$, where $\mathcal{G} = \mathrm{Res}_{C/\bar{C}}(G_C)^\sigma$ with $G$ of adjoint type.

**Theorem 3.16.** *There exists a line bundle $\mathcal{L}$ on $\mathrm{Gr}_{\mathcal{G},C}$ such that the restriction of $\mathcal{L}$ to the fiber $\mathrm{Gr}_{\mathcal{G},p}$ is the level one line bundle on $\mathrm{Gr}_{\mathcal{G},p}$, for any $p \in C$.*

*Proof.* Let $X$ be a component of $\mathrm{Gr}_{\mathcal{G},C}$. Fix any point $x \in X$, $x \in \mathrm{Gr}_{\mathcal{G},p}$ for a unique $p \in C$. If $p = o$, then $X$ contains at least one component of $\mathrm{Gr}_{\mathscr{G}}$. If $p \neq o, \infty$, then $x$ is a point in an affine Schubert variety $\overline{\mathrm{Gr}}^{\lambda}_{\mathcal{G},p}$ for some $\lambda \in X_*(T)^+$. By Theorem 2.3, $\overline{\mathrm{Gr}}^{\lambda}_{\mathcal{G},p}$ admits a flat degeneration to $\overline{\mathrm{Gr}}^{\bar{\lambda}}_{\mathscr{G}}$. If $p = \infty$, $x$ is a point in a twisted affine Schubert variety $\overline{\mathrm{Gr}}^{\bar{\lambda}}_{\mathcal{G},\infty}$. Similarly, there is a flat family connecting $\overline{\mathrm{Gr}}^{\bar{\lambda}}_{\mathcal{G},\infty}$ and $\overline{\mathrm{Gr}}^{\bar{\lambda}}_{\mathscr{G}}$. This concludes that $X$ must contain at least one component of $\mathrm{Gr}_{\mathscr{G}}$. In other words, $\mathrm{Gr}_{\mathscr{G}}$ has as many or more components than $\mathrm{Gr}_{\mathcal{G},C}$.

Recall that the components of $\mathrm{Gr}_{\mathscr{G}}$ are parametrized by $(X_*(T)/\check{Q})_\sigma$. On the other hand, by [He, Theorem 2], $\pi_0(\mathrm{Bun}_{\mathcal{G}})$ can also be identified with $(X_*(T)/\check{Q})_\sigma$. We have a natural projection $\mathrm{pr} : \mathrm{Gr}_{\mathcal{G},C} \to \mathrm{Bun}_{\mathcal{G}}$. In view of Heinloth's uniformization theorem [He, Theorem 3], the map pr is surjective. Hence $\mathrm{Gr}_{\mathcal{G},C}$ has as many or more components than $\mathrm{Bun}_{\mathcal{G}}$. It forces that $\mathrm{Gr}_{\mathscr{G}}$, $\mathrm{Gr}_{\mathcal{G},C}$ and $\mathrm{Bun}_{\mathcal{G}}$ have the same number of components. In particular, it follows that there is a natural bijection bertween components of $\mathrm{Gr}_{\mathscr{G}}$ and $\mathrm{Gr}_{\mathcal{G},C}$.

It is well-known that the neutral component $\mathrm{Gr}_{\mathcal{G},C,\circ}$ of $\mathrm{Gr}_{\mathcal{G},C}$ is isomorphic to $\mathrm{Gr}_{\mathcal{G}',C}$. Thus, we naturally get the level one line bundle $\mathcal{L}$ on the neutral component $\mathrm{Gr}_{\mathcal{G},C,\circ}$. Recall the set $S$ in (14) that parametrizes the components of $\mathrm{Gr}_{\mathscr{G}}$. For any nonzero $\kappa \in S$

(if it exists), the component $\mathrm{Gr}_{\mathscr{G},\kappa}$ of $\mathrm{Gr}_{\mathscr{G}}$ contains $e_{\bar{\kappa}}$. Thus, the associated component $\mathrm{Gr}_{\mathcal{G},C,\kappa}$ is exactly the one containing $s^\kappa$. The component $\mathrm{Gr}_{\mathscr{G},\kappa}$ is isomorphic to

$$L\mathcal{G}_C/\mathrm{Ad}_{s^\kappa}(L^+\mathcal{G}_C),$$

where is $s^\kappa$ is a $C$-point in $\mathrm{Gr}_{\mathcal{G},C}$ as defined in Section 2.5. Then there exists a natural isomorphism

$$\mathrm{Gr}_{\mathcal{G}',C} = L\mathcal{G}'_C/L^+\mathcal{G}'_C \simeq L\mathcal{G}'_C/\mathrm{Ad}_{s^\kappa}(L^+\mathcal{G}'_C),$$

given by $gL^+\mathcal{G}'_C \mapsto \mathrm{Ad}_{s^{-\kappa}}(g)\mathrm{Ad}_{s^\kappa}(L^+\mathcal{G}'_C)$. Therefore, the line bundle of level one on the non-neutral component can be realized as the pull-back from the line bundle $\mathcal{L}$ on the neutral component $\mathrm{Gr}_{\mathcal{G},C,\circ}$ via this isomorphism. □

## 4. Smooth locus of twisted affine Schubert varieties

In this section, we always assume that $\sigma$ is an absolutely special automophism on $G$, and $G$ is of adjoint type.

**4.1. $\mathrm{Gr}_{\mathscr{T}}$ as a fixed-point ind-subscheme of $\mathrm{Gr}_{\mathscr{G}}$.** We first recall a theorem in [Zh1, Theorem 1.3.4].

**Theorem 4.1.** *The natural morphism $\mathrm{Gr}_T \to \mathrm{Gr}_G$ identifies $\mathrm{Gr}_T$ as the $T$-fixed point ind-subscheme $(\mathrm{Gr}_G)^T$ of $\mathrm{Gr}_G$.*

The original proof of this theorem is not correct (communicated to us by Richarz and Zhu independently), also see [HR2, Remark 3.5]. A correct proof can be found in [HR2, Proposition 3.4], and a similar proof was known to Zhu earlier.

It is clear that $T^\sigma$ is a subgroup scheme of $L\mathscr{T}$ and $L\mathscr{G}$. Hence there is a natural action of $T^\sigma$ on $\mathrm{Gr}_{\mathscr{G}}$. We now prove an analogue of Theorem 4.1 in the setting of absolutely special parahoric group schemes.

**Theorem 4.2.** *The natural morphism $\mathrm{Gr}_{\mathscr{T}} \to \mathrm{Gr}_{\mathscr{G}}$ identifies $\mathrm{Gr}_{\mathscr{T}}$ as the $T^\sigma$-fixed point ind-subscheme $(\mathrm{Gr}_{\mathscr{G}})^{T^\sigma}$ of $\mathrm{Gr}_{\mathscr{G}}$.*

*Proof.* Let $L^{--}G$ be the ind-group scheme represented by the following functor, for any $\mathbb{C}$-algebra $R$,

$$L^{--}G(R) := \ker(\mathrm{ev}_\infty : G(R[t^{-1}]) \to G(R)),$$

where $\mathrm{ev}_\infty$ is the evaluation map sending $t^{-1}$ to 0. Let $L^{--}\mathscr{G}$ be the ind-group scheme which represents the following functor , for any $\mathbb{C}$-algebra $R$,

$$L^{--}\mathscr{G}(R) := \ker(\mathrm{ev}_\infty : G(R[t^{-1}])^\sigma \to G(R)^\sigma).$$

We can similarly define $L^{--}T$ and $L^{--}\mathscr{T}$.

By the similar argument as in [Zh4, Lemma 2.3.5] or [HR2, Lemma 3.1], we have an open embedding

$$L^{--}\mathscr{G} \hookrightarrow \mathrm{Gr}_{\mathscr{G}}$$

given by $g \mapsto ge_0$, where $e_0$ is the base point in $\mathrm{Gr}_{\mathscr{G}}$. Let $I$ be the Iwahori subgroup of $L^+\mathscr{G}$, which is the preimage of $B^\sigma$ via the evaluation map $\mathrm{ev} : L^+\mathscr{G} \to G^\sigma$ for a $\sigma$-stable Borel subgroup $B$ in $G$. We have the following decomposition

$$\mathrm{Gr}_{\mathscr{G}} = \bigsqcup_{\bar{\lambda}\in X_*(T)_\sigma} Ie_{\bar{\lambda}}. \tag{37}$$

For each $\bar{\lambda} \in X_*(T)_\sigma$, we choose a representative $\lambda \in X_*(T)$. The twisted Iwahori Schubert cell

$$Ie_{\bar{\lambda}} = n^\lambda \mathrm{Ad}_{n^{-\lambda}}(I)e_0$$

is contained in $n^\lambda L^{--}\mathscr{G}e_0$. Then by the decomposition (37), $\bigcup_{\bar{\lambda}\in X_*(T)_*} n^\lambda L^{--}\mathscr{G}e_0$ is an open covering of $\mathtt{Gr}_\mathscr{G}$. We may naturally regard $\mathtt{Gr}_\mathscr{T}$ as an ind-subscheme of $\mathtt{Gr}_\mathscr{G}$. Hence, we may regard $e_0$ as the base point in $\mathtt{Gr}_\mathscr{T}$. Under this convention,

$$\bigcup_{\lambda\in X_*(T)_\sigma} n^\lambda L^{--}\mathscr{T}e_0 = \bigcup_{\lambda\in X_*(T)_\sigma} L^{--}\mathscr{T}n^\lambda e_0$$

is an open covering of $\mathtt{Gr}_\mathscr{T}$. Therefore, it suffices to show that for each $\bar{\lambda} \in X_*(T)_\sigma$,

$$(n^\lambda L^{--}\mathscr{G}e_0)^{T^\sigma} \simeq n^\lambda L^{--}\mathscr{T}e_0.$$

Further, it suffices to show that $(L^{--}\mathscr{G})^{T^\sigma} \simeq L^{--}\mathscr{T}$, where the action of $T^\sigma$ on $L^{--}\mathscr{G}$ is by conjugation. From the proof of [HR2, Proposition 3.4], one may see that $(L^{--}G)^{T^\sigma} \simeq L^{--}T$. This actually implies that $(L^{--}\mathscr{G})^{T^\sigma} \simeq L^{--}\mathscr{T}$. Hence, this finishes the proof of the theorem. □

An immediate consequence of Theorem 4.2 is the following corollary.

**Corollary 4.3.** *The $T^\sigma$-fixed $\mathbb{C}$-point set in $\mathtt{Gr}_\mathscr{G}$ is $\{e_{\bar{\lambda}} \mid \lambda \in X_*(T)_\sigma\}$.*

4.2. **A duality isomorphism for twisted Schubert varieties.** Let $\mathtt{Gr}_G$ be the affine Grassmannian of $G$, and let L be the line bundle on $\mathtt{Gr}_G$ that is of level one on every component of $\mathtt{Gr}_G$. For any $\lambda \in X_*(T)$, let $\overline{\mathtt{Gr}}_G^\lambda$ denote the closure of $G(\mathcal{O})$-orbit at $L_\lambda := t^\lambda G(\mathcal{O}) \in \mathtt{Gr}_G$. Let $(\overline{\mathtt{Gr}}_G^\lambda)^T$ denote the $T$-fixed point subscheme of $\overline{\mathtt{Gr}}_G^\lambda$. Zhu [Zh1] proved that

**Theorem 4.4.** *When G is simply-laced and not of type E, the resriction map $H^0(\overline{\mathtt{Gr}}_G^\lambda, \mathrm{L}) \to H^0((\overline{\mathtt{Gr}}_G^\lambda)^T, \mathrm{L}|_{(\overline{\mathtt{Gr}}_G^\lambda)^T})$ is an isomorphism.*

It was proved by Evens-Mirković [EM] and Malkin-Ostrik-Vybonov [MOV], that the smooth locus of $\overline{\mathtt{Gr}}_G^\lambda$ is the open cell $\mathtt{Gr}_G^\lambda$ for any reductive group $G$. In fact, this theorem can also be deduced from Theorem 4.4 when $G$ is of type $A, D$ , and in many cases when $G$ is of type $E$.

We will prove a twisted version of Theorem 4.4, and as a consequence we get the similar result of Evans-Mirković and Malkin-Ostrik-Vybonov in twisted setting. This confirms a conjecture of Haines-Richarz [HR], when $\mathscr{G}$ is of type $A^{(2)}_{2\ell-1}, D^{(2)}_{\ell+1}, D^{(3)}_4$.

From Theorem 4.2, we have the identification $\mathtt{Gr}_\mathscr{T} \xrightarrow{\simeq} \mathtt{Gr}_\mathscr{G}^{T^\sigma}$. Let $\mathscr{I}^{\bar{\lambda}}$ denote the ideal sheaf of the $T^\sigma$-fixed subscheme $(\overline{\mathtt{Gr}}_\mathscr{G}^{\bar{\lambda}})^{T^\sigma}$ of $\overline{\mathtt{Gr}}_\mathscr{G}^{\bar{\lambda}}$. Then we have a short exact sequence of sheaves

$$0 \to \mathscr{I}^{\bar{\lambda}} \to \mathscr{O}_{\overline{\mathtt{Gr}}_\mathscr{G}^{\bar{\lambda}}} \to \mathscr{O}_{(\overline{\mathtt{Gr}}_\mathscr{G}^{\bar{\lambda}})^{T^\sigma}} \to 0. \tag{38}$$

Recall that $\mathscr{L}$ is the line bundle on $\mathtt{Gr}_\mathscr{G}$ which is of level one on every component. Tensoring the above short exact sequence with $\mathscr{L}$ and taking the functor of global sections, we obtain the following exact sequence

$$(39)\qquad 0 \to H^0(\overline{\mathrm{Gr}}^{\bar\lambda}_{\mathscr{G}}, \mathscr{I}^{\bar\lambda}\otimes\mathscr{L}) \to H^0(\overline{\mathrm{Gr}}^{\bar\lambda}_{\mathscr{G}}, \mathscr{L}) \xrightarrow{r} H^0((\overline{\mathrm{Gr}}^{\bar\lambda}_{\mathscr{G}})^{T^\sigma}, \mathscr{L}|_{(\overline{\mathrm{Gr}}^{\bar\lambda}_{\mathscr{G}})^{T^\sigma}}) \to \cdots,$$

where $r$ is the restriction map.

**Theorem 4.5.** *When $\mathscr{G}$ is not of type $E_6^{(2)}$, the restriction map*

$$H^0(\overline{\mathrm{Gr}}^{\bar\lambda}_{\mathscr{G}}, \mathscr{L}) \xrightarrow{r} H^0((\overline{\mathrm{Gr}}^{\bar\lambda}_{\mathscr{G}})^{T^\sigma}, \mathscr{L}|_{(\overline{\mathrm{Gr}}^{\bar\lambda}_{\mathscr{G}})^{T^\sigma}})$$

*is an isomorphism.*

This theorem will follow from the following proposition and Lemma 4.8. In fact, this theorem holds for many twisted affine Schubert varieties of $E_6^{(2)}$, see Remark 4.9. The following proposition does not exclude $E_6^{(2)}$.

**Proposition 4.6.** *The map $r$ is a surjection.*

*Proof.* It is well-known that any twisted affine Schubert varietiety $\overline{\mathrm{Gr}}^{\bar\lambda}_{\mathscr{G}}$ is a usual Schubert variety in a partial affine flag variety of Kac-Moody group. See the identification (29) and an argument for untwisted case in [BH, Proposition 2.21]. By [Ku, Theorem 8.2.2 (d)], we have that for any $\bar\lambda \geq \bar\mu$ in $X_*(T)^+_\sigma$, the following restriction map

$$(40)\qquad H^0(\overline{\mathrm{Gr}}^{\bar\lambda}_{\mathscr{G}}, \mathscr{L}) \to H^0(\overline{\mathrm{Gr}}^{\bar\mu}_{\mathscr{G}}, \mathscr{L})$$

is surjective, and

$$(41)\qquad H^0(\mathrm{Gr}_{\mathscr{G}}, \mathscr{L}) = \varprojlim H^0(\overline{\mathrm{Gr}}^{\bar\lambda}_{\mathscr{G}}, \mathscr{L}|_{\overline{\mathrm{Gr}}^{\bar\lambda}_{\mathscr{G}}}).$$

We also have the following surjective map

$$(42)\qquad H^0((\overline{\mathrm{Gr}}^{\bar\lambda}_{\mathscr{G}})^{T^\sigma}, \mathscr{L}) \to H^0((\overline{\mathrm{Gr}}^{\bar\mu}_{\mathscr{G}})^{T^\sigma}, \mathscr{L})$$

for all $\bar\lambda \geq \bar\mu$, since these $T^\sigma$-fixed closed subschemes are affine and the morphism $(\overline{\mathrm{Gr}}^{\bar\mu}_{\mathscr{G}})^{T^\sigma} \hookrightarrow (\overline{\mathrm{Gr}}^{\bar\lambda}_{\mathscr{G}})^{T^\sigma}$ is a closed embedding. Moreover,

$$H^0((\mathrm{Gr}_{\mathscr{G}})^{T^\sigma}, \mathscr{L}|_{(\mathrm{Gr}_{\mathscr{G}})^{T^\sigma}}) = \varprojlim H^0((\overline{\mathrm{Gr}}^{\bar\lambda}_{\mathscr{G}})^{T^\sigma}, \mathscr{L}|_{(\overline{\mathrm{Gr}}^{\bar\lambda}_{\mathscr{G}})^{T^\sigma}}).$$

Therefore, for any $\bar\lambda \in X_*(T)^+_\sigma$ we have the following surjective maps

$$H^0(\mathrm{Gr}_{\mathscr{G}}, \mathscr{L}) \to H^0(\overline{\mathrm{Gr}}^{\bar\lambda}_{\mathscr{G}}, \mathscr{L}), \quad H^0((\mathrm{Gr}_{\mathscr{G}})^{T^\sigma}, \mathscr{L}) \to H^0((\overline{\mathrm{Gr}}^{\bar\lambda}_{\mathscr{G}})^{T^\sigma}, \mathscr{L}).$$

Then to prove the map

$$H^0(\overline{\mathrm{Gr}}^{\bar\lambda}_{\mathscr{G}}, \mathscr{L}) \to H^0((\overline{\mathrm{Gr}}^{\bar\lambda}_{\mathscr{G}})^{T^\sigma}, \mathscr{L}|_{(\overline{\mathrm{Gr}}^{\bar\lambda}_{\mathscr{G}})^{T^\sigma}})$$

is surjective, it is sufficient to prove that the map

$$(43)\qquad H^0(\mathrm{Gr}_{\mathscr{G}}, \mathscr{L}) \to H^0((\mathrm{Gr}_{\mathscr{G}})^{T^\sigma}, \mathscr{L}|_{(\mathrm{Gr}_{\mathscr{G}})^{T^\sigma}})$$

is surjective, since we will have the following commutative diagram, for all $\bar\lambda$,

(44)
$$\begin{array}{ccc} H^0(\mathtt{Gr}_{\mathscr{G}},\mathscr{L}) & \longrightarrow & H^0((\mathtt{Gr}_{\mathscr{G}})^{T^\sigma},\mathscr{L}|_{(\mathtt{Gr}_{\mathscr{G}})^{T^\sigma}}) \\ \downarrow & & \downarrow \\ H^0(\overline{\mathtt{Gr}}^{\bar\lambda}_{\mathscr{G}},\mathscr{L}|_{\overline{\mathtt{Gr}}^{\bar\lambda}_{\mathscr{G}}}) & \xrightarrow{r} & H^0((\overline{\mathtt{Gr}}^{\bar\lambda}_{\mathscr{G}})^{T^\sigma},\mathscr{L}|_{(\overline{\mathtt{Gr}}^{\bar\lambda}_{\mathscr{G}})^{T^\sigma}}). \end{array}$$

By Theorem 4.2, we have $\mathtt{Gr}_{\mathscr{T}} \simeq (\mathtt{Gr}_{\mathscr{G}})^{T^\sigma}$. Therefore, the surjectivity of the map (43) follows from the following Lemma 4.7. □

We first make a digression on Heisenberg algebras and their representations. The subspace $\hat{\mathfrak{t}}^\sigma := (\mathfrak{t}_{\mathcal{K}})^\sigma \oplus \mathbb{C}K \hookrightarrow \hat{L}(\mathfrak{g},\sigma)$ is a Lie subalgebra. In fact, $\hat{\mathfrak{t}}^\sigma$ is an extended (completed) Heisenberg algebra with center $\mathfrak{t}^\sigma \oplus \mathbb{C}K$. Therefore, any integrable irreducible highest weight representation of $\hat{\mathfrak{t}}^\sigma$ is parametrized by an element $\mu \in (\mathfrak{t}^\sigma)^*$ and the level $c$, i.e. $K$ acts by the scalar $c$ on this representation. We denote this representation by $\pi_{\mu,c}$. By the standard construction,

$$\pi_{\mu,c} = \mathrm{ind}^{\hat{\mathfrak{t}}^\sigma}_{(\mathfrak{t}_{\mathcal{O}})^\sigma\oplus\mathbb{C}K}\mathbb{C}_{\mu,c}, \tag{45}$$

where ind is the induced representation in the sense of univeral enveloping algebras ,and $\mathbb{C}_{\mu,c}$ is the 1-dimensional module over $(\mathfrak{t}_{\mathcal{O}})^\sigma \oplus \mathbb{C}K$ where the action of $(\mathfrak{t}_{\mathcal{O}})^\sigma$ factors through $\mathfrak{t}^\sigma$.

**Lemma 4.7.** *The restriction map* $H^0(\mathtt{Gr}_{\mathscr{G}},\mathscr{L}^c) \to H^0(\mathtt{Gr}_{\mathscr{T}},\mathscr{L}^c|_{\mathtt{Gr}_{\mathscr{T}}})$ *is surjective.*

*Proof.* Proving surjectivity here is equivalent to proving injectivity for the dual modules,

$$0 \to H^0(\mathtt{Gr}_{\mathscr{T}},\mathscr{L}^c|_{\mathtt{Gr}_{\mathscr{T}}})^\vee \to H^0(\mathtt{Gr}_{\mathscr{G}},\mathscr{L}^c)^\vee.$$

Note that both of these spaces are modules for the Heisenberg algebra $\hat{\mathfrak{t}}^\sigma$; the morphism is a $\hat{\mathfrak{t}}^\sigma$-morphism. Since $\mathscr{T}$ is discrete, we naturally have the following decomposition

$$H^0(\mathtt{Gr}_{\mathscr{T}},\mathscr{L}^c|_{\mathtt{Gr}_{\mathscr{T}}}) \simeq \bigoplus_{\bar\lambda\in X_*(T)_\sigma} \mathscr{O}_{\mathtt{Gr}_{\mathscr{T}},e_{\bar\lambda}} \otimes \mathscr{L}^c|_{e_{\bar\lambda}},$$

where $\mathscr{O}_{\mathtt{Gr}_{\mathscr{T}},e_{\bar\lambda}}$ is the structure sheaf of the component of $\mathtt{Gr}_{\mathscr{T}}$ containing $e_{\bar\lambda}$. We also notice that, the identify component of $\mathtt{Gr}_{\mathscr{T}}$ is naturally the formal group with Lie algebra $(\mathfrak{t}_{\mathcal{K}})^\sigma/(\mathfrak{t}_{\mathcal{O}})^\sigma$. In view of the construction (45), we have

$$H^0(\mathtt{Gr}_{\mathscr{T}},\mathscr{L}^c|_{\mathtt{Gr}_{\mathscr{T}}})^\vee = \bigoplus_{\bar\lambda\in X_*(T)_\sigma} \pi_{-c\iota(\bar\lambda),c};$$

where the map $\iota : X_*(T)_\sigma \to (\mathfrak{t}^\sigma)^*$ is defined in (20). Since each $\pi_{-c\iota(\bar\lambda),c}$ is irreducible, and generated by a $-c\iota(\bar\lambda)$-weight vector $w_{-c\iota(\bar\lambda)}$, it suffices to show that the morphism

$$\pi_{-c\iota(\bar\lambda),c} \to H^0(\mathtt{Gr}_{\mathscr{G}},\mathscr{L}^c)^\vee$$

sends $w_{-c\iota(\bar\lambda)}$ to a nonzero vector.

By Theorem 3.8, we may define a Plücker embedding

$$\phi : \mathtt{Gr}_{\mathscr{G}} \to \mathbb{P}(\mathscr{H}_c)$$

given by $ge_0 \mapsto [\rho_g(v_0)]$ for any $ge_0 \in \mathtt{Gr}_{\mathscr{G}}$, where $\rho_g$ is defined in Lemma 3.6, and $[\rho_g(v_0)]$ represents the line in $\mathscr{H}_c$ that contains $\rho_g(v_0)$. Then we may pick a linear form $f_{\bar{\lambda}}$ on $\mathscr{H}_c$ which is nonzero on $[v_{\bar{\lambda}}]$, and which is 0 on other weight vectors, where $v_{\bar{\lambda}}$ is defined in (33). The restriction $f_{\bar{\lambda}}|_{\phi(\mathtt{Gr}_{\mathscr{G}})}$ produces a nontrivial element in $H^0(\mathtt{Gr}_{\mathscr{G}}, \mathscr{L})$, since $\phi(e_{\bar{\lambda}}) = v_{\bar{\lambda}}$.

Observe that the map $\pi_{-c\iota(\bar{\lambda}),c} \to H^0(\mathtt{Gr}_{\mathscr{G}}, \mathscr{L}^c)^\vee$ sends $w_{-c\iota(\bar{\lambda})}$ to a nonzero scalar of $v_{\bar{\lambda}}$. Thus the map $\pi_{-c\iota(\bar{\lambda}),c} \to H^0(\mathtt{Gr}_{\mathscr{G}}, \mathscr{L}^c)^\vee$ is nontrivial and thus injective. □

By Lemma 4.7, we obtain the following short exact sequence

$$0 \to H^0(\overline{\mathtt{Gr}}^{\bar{\lambda}}_{\mathscr{G}}, \mathscr{I}^{\bar{\lambda}} \otimes \mathscr{L}) \to H^0(\overline{\mathtt{Gr}}^{\bar{\lambda}}_{\mathscr{G}}, \mathscr{L}) \xrightarrow{r} H^0(\overline{\mathtt{Gr}}^{\bar{\lambda}}_{\mathscr{G}}, \mathscr{L} \otimes \mathscr{O}_{(\overline{\mathtt{Gr}}^{\bar{\lambda}}_{\mathscr{G}})}) \to 0.$$

Thus, the obstruction to the map $r$ being an isomorphism is the vanishing of the first term $H^0(\overline{\mathtt{Gr}}^{\bar{\lambda}}_{\mathscr{G}}, \mathscr{I}^{\bar{\lambda}} \otimes \mathscr{L})$.

Let $\mathrm{I}^\lambda$ denote the ideal sheaf of the $T$-fixed subscheme on $\overline{\mathtt{Gr}}^{\lambda}_{G}$. We will show that the vanishing of the first term can be deduced from the vanishing of $H^0(\overline{\mathtt{Gr}}^{\lambda}_{G}, \mathrm{I}^\lambda \otimes \mathrm{L})$.

Recall that $\overline{\mathtt{Gr}}^{\lambda}_{\mathcal{G},C}$ is a global Schubert variety defined in Section 2.5. The constant group scheme $T^\sigma \times C$ over $C$ is naturally a closed subgroup scheme of $\mathcal{T}$. Hence $T^\sigma$ acts on $\overline{\mathtt{Gr}}^{\lambda}_{\mathcal{G},C}$ naturally. Let $(\overline{\mathtt{Gr}}^{\lambda}_{\mathcal{G},C})^{T^\sigma}$ be the $T^\sigma$-fixed subscheme of $\overline{\mathtt{Gr}}^{\lambda}_{\mathcal{G},C}$, and let $\mathcal{I}^\lambda$ be the ideal sheaf of $(\overline{\mathtt{Gr}}^{\lambda}_{\mathcal{G},C})^{T^\sigma}$. Then, $\mathcal{I}^\lambda|_p$ is the ideal sheaf of $(\overline{\mathtt{Gr}}^{\lambda}_{\mathcal{G},C}|_p)^{T^\sigma}$. Recall that,

$$\overline{\mathtt{Gr}}^{\lambda}_{\mathcal{G},o} = \overline{\mathtt{Gr}}^{\bar{\lambda}}_{\mathscr{G}}, \quad \overline{\mathtt{Gr}}^{\lambda}_{\mathcal{G},\infty} \simeq \overline{\mathtt{Gr}}^{\bar{\lambda}}_{\mathscr{G}}, \quad \overline{\mathtt{Gr}}^{\lambda}_{\mathcal{G},p\neq o,\infty} \simeq \overline{\mathtt{Gr}}^{\lambda}_{G}.$$

In particular, we have

$$\mathcal{I}^\lambda|_o = \mathscr{I}^\lambda, \quad \mathcal{I}^\lambda|_\infty \simeq \mathscr{I}^\lambda, \quad \mathcal{I}^\lambda|_{p\neq o,\infty} \simeq \mathrm{I}^\lambda.$$

**Lemma 4.8.** *Assume that $G$ is not of type $E_6$. Then the ideal $\mathcal{I}^\lambda$ is flat over $C$.*

*Proof.* Consider $\overline{\mathtt{Gr}}^{\lambda}_{\mathcal{G},C\setminus\{o,\infty\}}$ and the $T^\sigma$-fixed subscheme $(\overline{\mathtt{Gr}}^{\lambda}_{\mathcal{G},C\setminus\{o,\infty\}})^{T^\sigma}$. We denote by $Z^\lambda$ the flat closure of $(\overline{\mathtt{Gr}}^{\lambda}_{\mathcal{G},C\setminus\{o,\infty\}})^{T^\sigma}$ in $\mathtt{Gr}_{\mathcal{G},C}$. Since $Z$ is the closure of a $T^\sigma$-fixed subscheme, we see that $Z^\lambda|_o \subset \overline{\mathtt{Gr}}^{\lambda}_{\mathcal{G},C}|_o$, and $Z^\lambda|_\infty \subset \overline{\mathtt{Gr}}^{\lambda}_{\mathcal{G},C}|_\infty$.

To show $\mathcal{I}^\lambda$ is flat over $C$, it is sufficient to show that $(\overline{\mathtt{Gr}}^{\lambda}_{\mathcal{G},C})^{T^\sigma}$ is flat over $C$. This is equivalent to showing $Z^\lambda = (\overline{\mathtt{Gr}}^{\lambda}_{\mathcal{G},C})^{T^\sigma}$. In particular, it suffices to show the fibers $Z^\lambda|_o$ and $Z^\lambda|_\infty$ are isomorphic to $(\overline{\mathtt{Gr}}^{\bar{\lambda}}_{\mathscr{G}})^{T^\sigma}$. Since the fiber $Z^\lambda|_\infty$ at $\infty$ is similar to the fiber $Z^\lambda|_o$ at $o$, it suffices to show that $Z^\lambda|_o = (\overline{\mathtt{Gr}}^{\bar{\lambda}}_{\mathscr{G}})^{T^\sigma}$. Note that both of these are finite schemes, we can compare the dimensions of their structure sheaves as follows:

$$
\begin{aligned}
\dim \mathscr{O}_{(\overline{\mathrm{Gr}}^{\bar\lambda}_{\mathscr{G}})^{T^\sigma}} \geq \dim \mathscr{O}_{Z^\lambda|_o} &= \dim \mathscr{O}_{(\overline{\mathrm{Gr}}^{\lambda}_{\mathcal{G},p\neq o,\infty})^{T^\sigma}} \\
&= \dim \mathscr{O}_{(\overline{\mathrm{Gr}}^{\lambda}_{\mathcal{G},p\neq o,\infty})^{T}} \\
&= \dim H^0(\overline{\mathrm{Gr}}^{\lambda}_{\mathcal{G}}, \mathcal{L}|_{p\neq o,\infty}) \\
&= \dim H^0(\overline{\mathrm{Gr}}^{\bar\lambda}_{\mathscr{G}}, \mathscr{L}) \\
&\geq \dim \mathscr{O}_{(\overline{\mathrm{Gr}}^{\bar\lambda}_{\mathscr{G}})^{T^\sigma}},
\end{aligned}
$$

where the first equality follows from the flatness of $Z^\lambda$ over $C$, the third equality follows from Theorem 4.4, the fourth equality follows since $\overline{\mathrm{Gr}}^{\lambda}_{\mathcal{G},C}$ is flat over $C$ (cf. Theorem 2.3), and the last inequality follows from Proposition 4.6 . From this comparision, it follows that $\dim \mathscr{O}_{Z^\lambda|_o} = \dim \mathscr{O}_{(\overline{\mathrm{Gr}}^{\lambda}_{\mathcal{G},p\neq o})^{T^\sigma}}$. Hence, $\mathscr{O}_{Z^\lambda|_o} = \mathscr{O}_{(\overline{\mathrm{Gr}}^{\lambda}_{\mathcal{G},p\neq 0})^{T^\sigma}}$. This concludes the proof of the lemma. □

*Proof of Theorem 4.5.* By Lemma 4.8, if $H^0(\overline{\mathrm{Gr}}^{\lambda}_{G}, \mathrm{I}^\lambda \otimes \mathrm{L}) = 0$, then $H^0(\overline{\mathrm{Gr}}^{\bar\lambda}_{\mathscr{G}}, \mathscr{I}^{\bar\lambda} \otimes \mathscr{L}) = 0$. When $G$ is not of type $E_6$, from [Zh1, Section 2.2] it is known that $H^0(\overline{\mathrm{Gr}}^{\lambda}_{G}, \mathrm{I}^\lambda \otimes \mathrm{L}) = 0$ for any $\lambda \in X_*(T)$. Hence, when $\mathscr{G}$ is not of type $E_6^{(2)}$, $H^0(\overline{\mathrm{Gr}}^{\bar\lambda}_{\mathscr{G}}, \mathscr{I}^{\bar\lambda} \otimes \mathscr{L}) = 0$ for any $\bar\lambda \in X_*(T)^+_\sigma$. Therefore, the theorem follows from Lemma 4.7 and the long exact sequence (39). □

**Remark 4.9.** *In [Zh1], Zhu also proved Theorem 4.4 for many cases of affine Schubert varieties when G is type E. In particular, when G is type $E_6$ and for any $\lambda$ which is a non-negative summation of fundamental coweights $\check\omega_1, \check\omega_2, \check\omega_4, \check\omega_5, \check\omega_6$ following the labelling in [Ka, Table Fin, p.53], Theorem 4.4 holds. Therefore, it follows that when $\bar\lambda \in X_*(T)_\sigma$ is a non-negative summation of $\overline{\check\omega}_1, \overline{\check\omega}_2, \overline{\check\omega}_6 \in X_*(T)^+_\sigma$, our Lemma 4.8 and Theorem 4.5 hold. Note that $\overline{\check\omega}_1 = \overline{\check\omega}_5$ and $\overline{\check\omega}_2 = \overline{\check\omega}_4$. To fully prove the case of $E_6$, by the method in [Zh1], it suffices to prove Theorem 4.4 when $\lambda = \check\omega_3$. Due to the complexity of this method for exceptional groups, this case is still open.*

As an application of Theorem 4.5, we get a geometric Frenkel-Kac isomorphism for twisted affine algebras.

**Theorem 4.10.** *For any absolutely special $\mathscr{G}$, the restriction map*

$$H^0(\mathrm{Gr}_{\mathscr{G}}, \mathscr{L}) \to H^0(\mathrm{Gr}_{\mathscr{T}}, \mathscr{L}|_{\mathrm{Gr}_{\mathscr{T}}})$$

*is an isomoprhism, via the embedding $\mathrm{Gr}_{\mathscr{T}} \to \mathrm{Gr}_{\mathscr{G}}$.*

*Proof.* By Theorem 4.2, it suffices to show that the restriction map $r : H^0(\mathrm{Gr}_{\mathscr{G}}, \mathscr{L}) \to H^0(\mathrm{Gr}_{\mathscr{T}}, \mathscr{L}|_{(\mathrm{Gr}_{\mathscr{G}})^{T^\sigma}})$ is an isomorphism. In view of (41) and (42) and as a consequence of Theorem 4.5, the restriction map $r$ is an isomorphism when $\mathscr{G}$ is not $E_6^{(2)}$.

When $\mathscr{G}$ is of type $E_6^{(2)}$, the element $\overline{\check\omega}_1 \in X_*(T)^+_\sigma$ corresponds to the highest root of $H := (\check{G})^\vee$, see Section 2.3. Thus, for any $\bar\lambda \in X_*(T)_\sigma$, there exists $k \in \mathbb{N}$ such that

$\bar{\lambda} \leq k\bar{\omega}_1$. It follows that

$$H^0(\mathtt{Gr}_{\mathscr{G}}, \mathscr{L}) = \varprojlim_k H^0(\overline{\mathtt{Gr}}_{\mathscr{G}}^{k\bar{\omega}_1}, \mathscr{L}|_{\overline{\mathtt{Gr}}_{\mathscr{G}}^{k\bar{\omega}_1}}),$$

and

$$H^0((\mathtt{Gr}_{\mathscr{G}})^{T^\sigma}, \mathscr{L}|_{(\mathtt{Gr}_{\mathscr{G}})^{T^\sigma}}) = \varprojlim_k H^0((\overline{\mathtt{Gr}}_{\mathscr{G}}^{k\bar{\omega}_1})^{T^\sigma}, \mathscr{L}|_{(\overline{\mathtt{Gr}}_{\mathscr{G}}^{k\bar{\omega}_1})^{T^\sigma}}).$$

Now, by Remark 4.9, we see that the restriction map $r$ is also an isomorphism when $\mathscr{G}$ is $E_6^{(2)}$.

□

4.3. **Application: Smooth locus of twisted affine Schubert varieties.** We now wish to investigate the smooth locus of the Schubert variety $\overline{\mathtt{Gr}}_{\mathscr{G}}^{\bar{\lambda}}$.

**Theorem 4.11.** *Assume that $\mathscr{G}$ is of type $A_{2\ell-1}^{(2)}, D_{\ell+1}^{(2)}, D_4^{(3)}$. For any $\lambda \in X_*(T)_\sigma^+$, the smooth locus of $\overline{\mathtt{Gr}}_{\mathscr{G}}^{\bar{\lambda}}$ is precisely the open Schubert cell $\mathtt{Gr}_{\mathscr{G}}^{\bar{\lambda}}$.*

*Proof.* For any $\bar{\mu} \in X_*(T)_\sigma^+$, if $e_{\bar{\mu}} = n^\mu e_0$ is a smooth point in $\overline{\mathtt{Gr}}_{\mathscr{G}}^{\bar{\lambda}}$, then by [Zh1, Lemma 2.3.3] $\dim \mathscr{O}_{(\overline{\mathtt{Gr}}_{\mathscr{G}}^{\bar{\lambda}})^{T^\sigma}, e_{\bar{\mu}}} = 1$.

By Theorem 3.11, we have $H^0(\overline{\mathtt{Gr}}_{\mathscr{G}}^{\bar{\lambda}}, \mathscr{L})^\vee \simeq D(1, \bar{\lambda})$, where $D(1, \bar{\lambda})$ is the Demazure module defined in Definition 3.10. Then by Theorem 4.5, we have

$$\dim D(1, \bar{\lambda})_{-\iota(\bar{\mu})} = \operatorname{length} \mathscr{O}_{(\overline{\mathtt{Gr}}_{\mathscr{G}}^{\bar{\lambda}})^{T^\sigma}, e_{\bar{\mu}}},$$

where $D(1, \bar{\lambda})_{-\iota(\bar{\mu})}$ is the $-\iota(\bar{\mu})$-weight space in $D(1, \bar{\lambda})$. We will prove that for any $\bar{\mu} \lneq \bar{\lambda}$, $\dim D(1, \bar{\lambda})_{-\iota(\bar{\mu})} \geq 2$, which would imply that $e_{\bar{\mu}}$ is not a smooth point in $\overline{\mathtt{Gr}}_{\mathscr{G}}^{\bar{\lambda}}$. From the surjectivity of (40), we have an embedding $D(1, \bar{\mu}) \hookrightarrow D(1, \bar{\lambda})$. On the the other hand, $V(-\iota(\bar{\lambda})) \hookrightarrow D(1, \bar{\lambda})$, where $V(-\iota(\bar{\lambda}))$ is the irreducible representation of $\mathfrak{g}^\sigma$ of lowest weight $-\iota(\bar{\lambda})$. In view of Lemma 2.1, Lemma 3.3 and Lemma 3.4, when $G$ is not of type $A_{2\ell}$, the relation $\bar{\mu} \lneq \bar{\lambda}$ implies that $\iota(\bar{\mu}) \lneq \iota(\bar{\lambda})$. Hence, $V(-\iota(\bar{\lambda}))_{-\iota(\bar{\mu})} \neq 0$. Furthermore, as subspaces in $D(1, \bar{\lambda})$,

$$D(1, \bar{\mu}) \cap V(-\iota(\bar{\lambda})) = 0.$$

It follows that $\dim D(1, \bar{\lambda})_{-\iota(\bar{\mu})} \geq 2$. This concludes the proof of the theorem. □

From the proof of the above theorem, we see that our technique does not fully apply to the Schubert variety $\overline{\mathtt{Gr}}_{\mathscr{G}}^{\bar{\lambda}}$ when $\mathscr{G}$ is of type $A_{2\ell}^{(2)}$. Also, since the duality theorem is not fully established yet for $E_6^{(2)}$, we can only get some partial result in this case.

Recall the group $H = (\check{G})^\tau$ mentioned in Section 2.3. By the ramified geometric Satake, $(X_*(T)_\sigma, X_*(T)_\sigma^+, \gamma_j, j \in I_\sigma)$ can be regarded as the weight lattice, the set of dominant weights, and simple roots of $H$. When $(G, m) = (E_6, 2)$, $H$ is $F_4$; when $(G, m) = (A_{2\ell}, 4)$, $H$ is $B_\ell$ of adjoint type. We follow the labelling of Dynkin diagram in [Ka, Table Fin, p 53]. Let $\{ \varpi_j \mid j \in I_\sigma \}$ be the set of fundamental dominant weights of $H$.

**Theorem 4.12.** (1) *Let $\mathscr{G}$ be of type $E_6^{(2)}$. If $\bar{\lambda}$ is a non-negative linear combination of the fundmental weights $\varpi_1, \varpi_2, \varpi_4$ of $H$, then the smooth locus of $\overline{\mathrm{Gr}}^{\bar{\lambda}}_{\mathscr{G}}$ is the open cell $\mathrm{Gr}^{\bar{\lambda}}_{\mathscr{G}}$.*

(2) *Let $\mathscr{G}$ be of type $A_{2\ell}^{(2)}$. For any $\bar{\lambda}, \bar{\mu} \in X_*(T)^+_\sigma$ with $\bar{\mu} \lneq \bar{\lambda}$, the Schubert cell $\mathrm{Gr}^{\bar{\mu}}_{\mathscr{G}}$ is contained in the singular locus of $\overline{\mathrm{Gr}}^{\bar{\lambda}}_{\mathscr{G}}$, except if $\bar{\mu} \prec \bar{\lambda}$ is a cover relation such that the simple short root $\gamma_\ell$ appears in $\bar{\lambda} - \bar{\mu}$.*

*Proof.* Part (1) of the theorem follows from Remark 3.5 and Remark 4.9, where under the map $\eta : I \to I_\sigma$, $\varpi_1 = \bar{\check{\omega}}_6, \varpi_3 = \bar{\check{\omega}}_2, \varpi_4 = \bar{\check{\omega}}_1$.

For part (2) of the theorem, we will prove this part by several setps. Let $c_\ell$ be the coefficient of $\gamma_\ell$ in $\bar{\lambda} - \bar{\mu}$.

Step 1. Observe from the proof of Theorem 4.11, when the coefficient $c_\ell$ is even, we have $\dim D(1, \bar{\lambda})_{-\iota(\bar{\mu})} \geq 2$. Thus, $e_{\bar{\mu}}$ is singular in $\overline{\mathrm{Gr}}^{\bar{\lambda}}_{\mathscr{G}}$.

Step 2. Assume that the coefficient $c_\ell > 1$. There exists sequence of dominant elements in $X_*(T)^+_\sigma$,

$$\bar{\mu} = \bar{\lambda}_k \prec \bar{\lambda}_{k-1} \prec \cdots \prec \bar{\lambda}_1 \prec \bar{\lambda}_0 = \bar{\lambda}, \tag{46}$$

such that each $\prec$ is a cover relation. Then, by a theorem of Stembridge [St, Theorem 2.8], for each $i$, $\bar{\lambda}_i - \bar{\lambda}_{i+1}$ is a positive root of $H$, for any $0 \leq i \leq k-1$, and the coefficient of $\gamma_\ell$ in each $\bar{\lambda}_i - \bar{\lambda}_{i+1}$ is either 0 or 1. Let $j$ be the least integer such that the coefficient of $\gamma_\ell$ in $\bar{\lambda}_{j-1} - \bar{\lambda}_j$ is 1. Such $j$ exists, since $c_\ell \neq 1$. Then the coefficient of $\gamma_\ell$ in $\bar{\lambda}_j - \bar{\mu}$ is even. By Step 1, we have $\dim D(1, \bar{\lambda}_j)_{-\iota(\bar{\mu})} \geq 2$. On the other hand, we have the inclusion $D(1, \bar{\lambda}_j) \subset D(1, \bar{\lambda})$. It follows that $\dim D(1, \bar{\lambda})_{-\iota(\bar{\mu})} \geq 2$. Hence, the variety $\overline{\mathrm{Gr}}^{\bar{\lambda}}_{\mathscr{G}}$ is singular at the point $e_{\bar{\mu}}$.

Step 3. We now assume that the coefficient $c_\ell = 1$. By assumption, $\bar{\mu} \prec \bar{\lambda}$ is not a cover relation. Then, in the sequence of cover relations in (46), either the coefficient of $\gamma_\ell$ in $\bar{\lambda}_{k-1} - \bar{\lambda}_k$ is 0, or the coefficient of $\gamma_\ell$ in $\bar{\lambda}_0 - \bar{\lambda}_1$ is 0. If the coefficient of $\gamma_\ell$ in $\bar{\lambda}_{k-1} - \bar{\lambda}_k$ is 0, by Step 1 $\dim D(1, \bar{\lambda}_{k-1})_{-\iota(\bar{\mu})} \geq 2$, implying that $\dim D(1, \bar{\lambda})_{-\iota(\bar{\mu})} \geq 2$. Hence $e_{\bar{\mu}}$ is singular in $\overline{\mathrm{Gr}}^{\bar{\lambda}}_{\mathscr{G}}$. If the coefficient of $\gamma_\ell$ in $\bar{\lambda}_0 - \bar{\lambda}_1$ is 0, then by Step 1 again, $e_{\bar{\lambda}_1}$ is a singular point in $\overline{\mathrm{Gr}}^{\bar{\lambda}}_{\mathscr{G}}$. Since the singular locus of $\overline{\mathrm{Gr}}^{\bar{\lambda}}_{\mathscr{G}}$ is closed, the point $e_{\bar{\mu}}$ is also singular. □

Let $\mathscr{G}$ be of type $A_{2\ell}^{(2)}$. We now explicitly describe the cover relation $\bar{\mu} \prec \bar{\lambda}$ such that $\gamma_\ell$ appears in $\bar{\lambda} - \bar{\mu}$. Note that $X_*(T)_\sigma$ is a root lattice of $H \simeq SO_{2n+1}$. In fact, $X_*(T)_\sigma$ is spanned by $\varpi_1, \varpi_2, \cdots, \varpi_{\ell-1}, 2\varpi_\ell$. Reading more carefully from [St, Theorem 2.8], we can see that, $\bar{\mu} \prec \bar{\lambda}$ is a cover relation such that $\gamma_\ell$ appears in $\bar{\lambda} - \bar{\mu}$, if and only if one of the followings holds:

(1) $\bar{\lambda} - \bar{\mu} = \gamma_\ell$.
(2) $\bar{\lambda} - \bar{\mu} = \sum_{j=i}^{\ell} \gamma_j$ and $\mu = \sum_{k=1}^{i-1} a_k \varpi_k$ with all $a_k > 0$, for some $1 \leq i \leq \ell - 1$.

**Remark 4.13.** (1) *When $\mathscr{G}$ is a special but not absolutely special parahoric group scheme of type $A_{2\ell}^{(2)}$, i.e. when $\sigma$ is the diagram automorphism and $G$ is of type $A_{2\ell}$, there is a counter-example that $\overline{\mathrm{Gr}}^{\bar{\lambda}}_{\mathscr{G}}$ is smooth but $\overline{\mathrm{Gr}}^{\bar{\lambda}}_{\mathscr{G}} \neq \mathrm{Gr}^{\bar{\lambda}}_{\mathscr{G}}$, cf. [HR,* *Section 5.1] or [Zh3, page 3]. This phenomenon is somewhat related to Remark 3.2.*

(2) *One can define the affine Grassmannian $\mathrm{Gr}_{\mathscr{G}}$ and twisted affine Schubert varieties $\overline{\mathrm{Gr}}^{\bar\lambda}_{\mathscr{G}}$ of the absolutely special parahoric group scheme $\mathscr{G}$ with the base field* k *of characteristic p. In [HR, Section 6], when $p \neq r$, Haines and Richarz reduced the question of the smooth locus of the $\overline{\mathrm{Gr}}^{\bar\lambda}_{\mathscr{G}}$ over characteristic p to characteristic zero case. In fact, by the work of Lourenço [Lo], one may construct a global twisted affine Schubert variety over $\mathbb{Z}$ so that the base change to the field* k *of characteristic p (including $p = r$) is the given twisted affine Schubert variety defined over* k*. Then the argument of Haines and Richarz can still apply to the case of characteristic $p = r$. Therefore, Theorem 4.11 also holds for any base field* k.

## 5. Fusion product for twisted affine Demazure modules

5.1. **Fusion product via the geometry of affine Grassmannian.** In this section, for convenience we assume $G$ to be simply connected. We describe a geometric formulation of fusion product for twisted affine Demazure modules. This closely follows the approach in [Zh1, Section 1.2].

Keeping the same set-up as in Section 2.4, we first introduce the Beilinson-Drinfeld (BD for short) Grassmanian over $C = \mathbb{P}^1$. For any $\mathbb{C}$-algbra $R$, we define

$$\mathrm{Gr}_{\mathcal{G},C^n}(R) := \left\{ (p_1, \dots p_n, \mathcal{F}, \beta) \,\middle|\, \begin{array}{l} p_i \in C(R) \\ \mathcal{F} \text{ a } \mathcal{G}\text{-torsor on } \bar{C}_R \\ \beta : \mathcal{F}|_{\bar{C}_R \setminus \cup \hat{\Gamma}_{\bar{p}_i}} \simeq \mathring{\mathcal{F}}|_{\bar{C}_R \setminus \cup \hat{\Gamma}_{\bar{p}_i}} \end{array} \right\}, \tag{47}$$

where $\mathring{\mathcal{F}}$ is the trivial $\mathcal{G}$-torsor. When $n = 1$, this is the global affine Grassmannian $\mathrm{Gr}_{\mathcal{G}}$ over $C$. Let $\mathrm{Gr}_{\mathcal{G},p,q}$ denote the fiber of $\mathrm{Gr}_{\mathcal{G},C^2}$ over the point $(p, q) \in C^2$. The BD Grassmannian $\mathrm{Gr}_{\mathcal{G},C^2}$ satisfies the following property

$$\mathrm{Gr}_{\mathcal{G},p,q} = \begin{cases} \mathrm{Gr}_{\mathcal{G},p} \times \mathrm{Gr}_{\mathcal{G},q} & \text{if } p \neq q \\ \mathrm{Gr}_{\mathcal{G},p} & \text{if } p = q \end{cases}.$$

There is a projection morphism $\pi_2 : \mathrm{Gr}_{\mathcal{G},C^2} \to \mathrm{Bun}_{\mathcal{G}}$, given by $(p_1, p_2, \mathcal{F}, \beta) \mapsto \mathcal{F}$. We get the level one line bundle $\mathcal{L}$ on $\mathrm{Gr}_{\mathcal{G},C^2}$ which is the pull-back (via $\pi_2$) of the level one line bundle on $\mathrm{Bun}_{\mathcal{G}}$ constructed in Theorem 3.13. This line bundle satisfies the following property:

$$\mathcal{L}|_{p,q} = \begin{cases} \mathcal{L}_p \boxtimes \mathcal{L}_q & \text{if } p \neq q \\ \mathcal{L}_p & \text{if } p = q \end{cases},$$

where $\mathcal{L}_p$ is the level one line bundle on the affine Grassmannian $\mathrm{Gr}_{\mathcal{G},p}$ for any $p \in C$. When $p \neq o, \infty$, the affine Grassmannian $\mathrm{Gr}_{\mathcal{G},p}$ is isomorphic to $\mathrm{Gr}_G$ and $\mathcal{L}_p$ can be identified with L as discussed in Section 4; when $p = o$ or $\infty$, $\mathrm{Gr}_{\mathcal{G},p}$ is isomorphic to $\mathrm{Gr}_{\mathscr{G}}$ and $\mathcal{L}_p$ can be identified with $\mathscr{L}$.

In the proof of the following theorem, we adpat an argument of [Zh1, Theorem 1.2.2] in the setting of parahoric Bruhat-Tits group scheme $\mathcal{G}$.

**Theorem 5.1.** *For any $\lambda \in X_*(T)^+$ and $\bar{\mu} \in X_*(T)^+_\sigma$, we have the following isomorphism*

$$H^0(\overline{\mathrm{Gr}}^{\lambda}_{G}, \mathrm{L}^c) \otimes H^0(\overline{\mathrm{Gr}}^{\bar{\mu}}_{\mathscr{G}}, \mathscr{L}^c) \simeq H^0(\overline{\mathrm{Gr}}^{\bar{\lambda}+\bar{\mu}}_{\mathscr{G}}, \mathscr{L}^c),$$

*as $G^\sigma$-modules, where $\mathrm{L}^c$ (resp. $\mathscr{L}^c$) is c-th power of the line bundle* L*(resp. $\mathscr{L}$) with $c \geq 0$.*

*Proof.* We first introduce the global convolution Grassmannian $\mathrm{Gr}_{\mathcal{G},C}\tilde{\times}\mathrm{Gr}_{\mathscr{G}}$ over $C$. For any $\mathbb{C}$-algebra $R$, we define

$$\mathrm{Gr}_{\mathcal{G},C}\tilde{\times}\mathrm{Gr}_{\mathscr{G}}(R) := \left\{ (p, \mathcal{F}_1, \mathcal{F}_2, \beta_1, \beta_2) \left| \begin{array}{l} p \in C(R) \\ \mathcal{F}_1, \mathcal{F}_2 : \mathcal{G}\text{-torsors on } \bar{C}_R \\ \beta_1 : \mathcal{F}_1|_{\bar{C}_R \backslash \Gamma_{\bar{p}}} \simeq \mathring{\mathcal{F}}|_{\bar{C}_R \backslash \Gamma_{\bar{p}}} \\ \beta_2 : \mathcal{F}_2|_{(\bar{C}\backslash \bar{o})_R} \simeq \mathcal{F}_1|_{(\bar{C}\backslash \bar{o})_R} \end{array} \right. \right\}. \tag{48}$$

We have the projection

$$\mathrm{pr} : \mathrm{Gr}_{\mathcal{G},C}\tilde{\times}\mathrm{Gr}_{\mathscr{G}} \to \mathrm{Gr}_{\mathcal{G},C},$$

given by $(p, \mathcal{F}_1, \mathcal{F}_2, \beta_1, \beta_2) \mapsto (p, \mathcal{F}_1, \beta_1)$. This projection is a fibration over $\mathrm{Gr}_{\mathcal{G},C}$ with the fibers isomorphic to $\mathrm{Gr}_{\mathscr{G}}$. We also introduce the following $L^+\mathscr{G}$-torsor $\mathcal{P}$ over $\mathrm{Gr}_{\mathcal{G},C}$,

$$\mathcal{P}(R) = \left\{ (p, \mathcal{F}, \beta_1, \beta_2) \left| \begin{array}{l} p \in C(R) \\ \mathcal{F} : \mathcal{G}\text{-torsor} \\ \beta_1 : \mathcal{F}|_{\bar{C}_R \backslash \Gamma_{\bar{p}}} \simeq \mathring{\mathcal{F}}|_{\bar{C}_R \backslash \Gamma_{\bar{p}}} \\ \beta_2 : \mathcal{F}|_{\hat{\Gamma}_{\bar{p}}} \simeq \mathring{\mathcal{F}}|_{\hat{\Gamma}_{\bar{p}}} \end{array} \right. \right\}.$$

Then $\mathcal{P} \times_{L^+\mathscr{G}} \mathrm{Gr}_{\mathscr{G}} \simeq \mathrm{Gr}_{\mathcal{G},C}\tilde{\times}\mathrm{Gr}_{\mathscr{G}}$.

Recall that for any $\lambda \in X_*(T)^+$, we have the global affine Schubert variety $\overline{\mathrm{Gr}}^{\lambda}_{\mathcal{G},C}$. Let $\mathcal{P}|_{\overline{\mathrm{Gr}}^{\lambda}_{\mathcal{G},C}}$ be the restriction of $\mathcal{P}$ on $\overline{\mathrm{Gr}}^{\lambda}_{\mathcal{G},C}$. For any $\lambda \in X_*(T)^+$ and $\mu \in X_*(T)^+_\sigma$, we define

$$\overline{\mathrm{Gr}}^{\lambda}_{\mathcal{G},C}\tilde{\times}\overline{\mathrm{Gr}}^{\bar{\mu}}_{\mathscr{G}} := \mathcal{P}|_{\overline{\mathrm{Gr}}^{\lambda}_{\mathcal{G},C}} \times_{L^+\mathscr{G}} \overline{\mathrm{Gr}}^{\bar{\mu}}_{\mathscr{G}}.$$

Therefore, $\overline{\mathrm{Gr}}^{\lambda}_{\mathcal{G},C}\tilde{\times}\overline{\mathrm{Gr}}^{\bar{\mu}}_{\mathscr{G}}$ is a fibration over $\overline{\mathrm{Gr}}^{\lambda}_{\mathcal{G},C}$ with fibers isomorphc to $\overline{\mathrm{Gr}}^{\bar{\mu}}_{\mathscr{G}}$. In particular $\overline{\mathrm{Gr}}^{\lambda}_{\mathcal{G},C}\tilde{\times}\overline{\mathrm{Gr}}^{\bar{\mu}}_{\mathscr{G}}$ is flat over $C$. The variety $\overline{\mathrm{Gr}}^{\lambda}_{\mathcal{G},C}\tilde{\times}\overline{\mathrm{Gr}}^{\bar{\mu}}_{\mathscr{G}}$ has the following properties:

$$\overline{\mathrm{Gr}}^{\lambda}_{\mathcal{G},C}\tilde{\times}\overline{\mathrm{Gr}}^{\bar{\mu}}_{\mathscr{G}}|_{p\neq o,\infty} \simeq \overline{\mathrm{Gr}}^{\lambda}_{G} \times \overline{\mathrm{Gr}}^{\bar{\mu}}_{\mathscr{G}}$$

and

$$\overline{\mathrm{Gr}}^{\lambda}_{\mathcal{G},C}\tilde{\times}\overline{\mathrm{Gr}}^{\bar{\mu}}_{\mathscr{G}}|_{p=o} \simeq \overline{\mathrm{Gr}}^{\bar{\lambda}}_{\mathscr{G}}\tilde{\times}\overline{\mathrm{Gr}}^{\bar{\mu}}_{\mathscr{G}} := p^{-1}(\overline{\mathrm{Gr}}^{\bar{\lambda}}_{\mathscr{G}}) \times_{L^+\mathscr{G}} \overline{\mathrm{Gr}}^{\bar{\mu}}_{\mathscr{G}},$$

where $p$ is the projection $p : L\mathscr{G} \to \mathrm{Gr}_{\mathscr{G}}$.

Note that we also have a natural morphism

$$m : \mathrm{Gr}_{\mathcal{G},C}\tilde{\times}\mathrm{Gr}_{\mathscr{G}} \to \mathrm{Gr}_{\mathcal{G},C^2}|_{C\times o},$$

given by $(p, \mathcal{F}_1, \mathcal{F}_2, \beta_1, \beta_2) \mapsto (p, o, \mathcal{F}_2, \beta_1 \circ \beta_2)$. This is an isomorphism away from $o$ and over $o$, we get the usual convolution morphism

$$m_o : \mathrm{Gr}_{\mathscr{G}} \tilde{\times} \mathrm{Gr}_{\mathscr{G}} \to \mathrm{Gr}_{\mathscr{G}}.$$

This morphism restricts to the following morphism

$$m_o : \overline{\mathrm{Gr}}^{\bar\lambda}_{\mathscr{G}} \tilde{\times} \overline{\mathrm{Gr}}^{\bar\mu}_{\mathscr{G}} \to \overline{\mathrm{Gr}}^{\bar\lambda+\bar\mu}_{\mathscr{G}}.$$

This is a partial Bott-Samelson resolution. The line bundle $m^*\mathcal{L}^c$ on $\mathrm{Gr}_{\mathcal{G},C}\tilde{\times}\mathrm{Gr}_{\mathscr{G}}|_{p\neq o,\infty}$ is isomorphic to $\mathrm{L}^c \boxtimes \mathscr{L}^c$, where we identify $\mathrm{Gr}_{\mathcal{G},C}\tilde{\times}\mathrm{Gr}_{\mathscr{G}}|_{p\neq o,\infty}$ with $\mathrm{Gr}_G \times \mathrm{Gr}_{\mathscr{G}}$. The line bundle $m^*\mathcal{L}^c$ on $\mathrm{Gr}_{\mathcal{G},C}\tilde{\times}\mathrm{Gr}_{\mathscr{G}}|_o \simeq \mathrm{Gr}_{\mathscr{G}} \tilde{\times} \mathrm{Gr}_{\mathscr{G}}$ is exactly $(m_o)^*\mathscr{L}^c$. Since $m_o$ is a partial Bott-Samelson resoluton, we have

$$H^0(\overline{\mathrm{Gr}}^{\bar\lambda}_{\mathscr{G}} \tilde{\times} \overline{\mathrm{Gr}}^{\bar\mu}_{\mathscr{G}}, (m_o)^*\mathscr{L}^c) \simeq H^0(\overline{\mathrm{Gr}}^{\bar\lambda+\bar\mu}_{\mathscr{G}}, \mathscr{L}^c), \tag{49}$$

as $L^+\mathscr{G}$-modules. Note that $G^\sigma \times C$ is naturally a subgroup scheme of $L^+\mathscr{G}$ over $C$.

The variety $\overline{\mathrm{Gr}}^{\lambda}_{\mathcal{G},C} \tilde{\times} \overline{\mathrm{Gr}}^{\bar\mu}_{\mathscr{G}}$ is flat over $C$ and it connects $\overline{\mathrm{Gr}}^{\lambda}_{G} \times \overline{\mathrm{Gr}}^{\bar\mu}_{\mathscr{G}}$ and $\overline{\mathrm{Gr}}^{\bar\lambda}_{\mathscr{G}} \tilde{\times} \overline{\mathrm{Gr}}^{\bar\mu}_{\mathscr{G}}$. Moreover, $\overline{\mathrm{Gr}}^{\lambda}_{\mathcal{G},C} \tilde{\times} \overline{\mathrm{Gr}}^{\bar\mu}_{\mathscr{G}}$ admits an action of $G^\sigma$ and $\mathcal{L}^c$ is also $G^\sigma$-equivariant (by construction of $\mathcal{L}$ as the pull-back from the line bundle from $\mathrm{Bun}_{\mathcal{G}}$), we get

$$H^0(\overline{\mathrm{Gr}}^{\bar\lambda}_{\mathscr{G}} \tilde{\times} \overline{\mathrm{Gr}}^{\bar\mu}_{\mathscr{G}}, (m_o)^*\mathscr{L}^c) \simeq H^0(\overline{G}^{\lambda}_{G} \times \overline{\mathrm{Gr}}^{\bar\mu}_{\mathscr{G}}, \mathrm{L}^c \boxtimes \mathscr{L}^c) \simeq H^0(\overline{G}^{\lambda}_{G}, \mathrm{L}^c) \otimes H^0(\overline{\mathrm{Gr}}^{\bar\mu}_{\mathscr{G}}, \mathscr{L}^c). \tag{50}$$

as $G^\sigma$-modules. Combining isomorphisms (49) (50), we conclude the theorem. □

**Remark 5.2.** *When $G$ is of adjoint type, Theorem 5.1 should also hold, but we need to replace $G$ by its simply-connected cover $G'$. One needs some extra argument to show that the line bundle $\mathcal{L}$ is $(G')^\sigma$-equivariant on non-neutral components of $\mathrm{Gr}_{\mathcal{G},C^2}$.*

Set

$$D(c,\lambda) := H^0(\overline{\mathrm{Gr}}^{\lambda}_{G}, \mathrm{L}^c)^\vee.$$

This is an affine Demazure module of the current algebra $\mathfrak{g}[t]$, cf. [Ku, Theorem 8.2.2 (a)]. As a consequence of Theorem 5.1, we have

**Corollary 5.3.** *For any $\lambda \in X_*(T)^+$ and $\bar\mu \in X_*(T)^+_\sigma$, we have*

$$D(c,\lambda) \otimes D(c,\bar\mu) \simeq D(c,\bar\lambda+\bar\mu)$$

*as $G^\sigma$-modules.*

Fourier-Kus proved the fusion for twisted affine demazure modules of the twisted current algebra $\mathfrak{g}[t]^\sigma$ when $\mathfrak{g}$ is not $A_{2\ell}$ and $\sigma$ is a diagram automorphism. Chari-Ion-Kus introduced the hyperspecial current algebra $\mathfrak{C}\mathfrak{g}$ for $A^{(2)}_{2\ell}$ and studied the twisted affine Demazure modules in [CIK], where they presented this algebra by using a basis. The fusion product of the twisted affine demazure modules of this hyperspecial current algebra were further studied by Kus-Venkatesh in [KV]. In Section 5.2, we will prove that the hyperspecial current algebra $\mathfrak{C}\mathfrak{g}$ can be identified with the twisted current algebra $\mathfrak{g}[t]^\sigma$ where $\mathfrak{g}$ is $A_{2\ell}$ and $\sigma$ is the absolutely special automorphism. With all this said, our Corollary 5.3 is a consequence of the works of [FK, CIK, KV].

5.2. **Matching with the hyperspecial current algebras in case of $A_{2\ell}^{(2)}$.** We define an element $h \in \mathfrak{t}$ by the condition that $\alpha_i(h) = 0$ for all $i \neq \ell, \ell+1$, and $\alpha_\ell(h) = \alpha_{\ell+1}(h) = 1$. Then $\sigma = \tau \mathrm{i}^h$, where $\mathrm{i} = \sqrt{-1}$ .

Let $\alpha_{ij}$ denote the positive root $\alpha_i + \cdots + \alpha_j$, for any $1 \leq i \leq j \leq 2\ell$. Notice that $\tau(\alpha_{ij}) = \alpha_{2\ell+1-j,2\ell+1-i}$. For each root $\alpha$, let $e_\alpha$ denote the standard basis in $\mathfrak{g} = sl_{2\ell+1}$.

**Lemma 5.4.** *We have the following formula*

$$\sigma(e_{\pm\alpha_{ij}}) = (-1)^{j-i}\mathrm{i}^{\pm\alpha_{ij}(h)} e_{\alpha_{2\ell+1-j,2\ell+1-i}}, \quad \textit{for any } 1 \leq i \leq j \leq 2\ell+1, \tag{51}$$

*and* $\sigma(h_i) = h_{2\ell+1-i}$, *for any* $1 \leq i \leq 2\ell+1$.

*Proof.* In [Ka, §7.10], the formula for the action of $\tau$ on every basis vector in $\mathfrak{g}$ is given. Combining this formula and $\sigma = \tau \mathrm{i}^h$, we can deduce the formula for $\sigma$. □

Following [KV, Section 1.8], the hyperspecial current algebra $\mathfrak{C}\mathfrak{g} \subset \mathfrak{g}[t, t^{-1}]^\tau$ consists of the following basis elements:

(1) $e_{\pm\alpha} \otimes t^k + (-1)^{i+j} e_{\pm\tau(\alpha)} \otimes (-t)^k$, $\alpha = \alpha_{ij}$ with $1 \leq i \leq j < \ell$, and $k \geq 0$;

(2) $e_{\pm\alpha} \otimes t^{t\pm 1} + (-1)^{i+j} e_{\pm\tau(\alpha)} \otimes (-t)^{k\pm 1}$, $\alpha = \alpha_{i,2\ell-j}$ with $1 \leq i \leq j < \ell$, and $k \geq 0$;

(3) $e_{\pm\alpha} \otimes t^{2k\pm 1}$, $\alpha = \alpha_{i,2\ell+1-i}$ with $1 \leq i \leq \ell$, and $k \geq 0$;

(4) $e_{\pm\alpha} \otimes t^{(2k+1\pm 1)/2} + (-1)^{\ell+i} e_{\pm\tau(\alpha)} \otimes (-t)^{(2k+1\pm 1)/2}$, $\alpha = \alpha_{i\ell}$ with $1 \leq i \leq \ell$, and $k \geq 0$;

(5) $h_i \otimes t^k + h_{2\ell+1-i} \otimes (-t)^k$, $1 \leq i \leq \ell$, and $k \geq 0$.

There exists an isomorphism $\eta_k : \hat{L}(\mathfrak{g}, \tau) \simeq \hat{L}(\mathfrak{g}, \sigma)$ due to Kac (cf. [HK, p8] and [Ka, §8.5]), which is defined as follows

$$\eta_k(x[t^j]) = x[t^{2j+k}],$$

for any $x$ a simultaneous $(-1)^j$-eigenvector of $\tau$, and a $k$-eigenvector of ad$h$.

Let $\phi$ be the Cartan involution on $\mathfrak{g}$ such that $\phi(e_i) = -f_i$ and $\phi(h_i) = -h_i$. Notice that for any root $\alpha$, $\phi(e_\alpha) = -e_{-\alpha}$. Now, we define an automorphism $\eta_c : \hat{L}(\mathfrak{g}, \tau) \to \hat{L}(\mathfrak{g}, \tau)$ as follows,

$$\eta_c(x[t^k]) = \phi(x)[t^k], \quad \text{for any } x[t^k] \in \hat{L}(\mathfrak{g}, \tau).$$

It is easy to verify that $\eta_c$ is well-defined, since $\eta_c \circ \tau = \tau \circ \eta_c$, where $\tau$ is regarded as the twisted automorphism on the untwisted affine Lie algebra $\hat{\mathfrak{g}}$. We define the following isomorphism of twisted affine algebras

$$\eta := \eta_k \circ \eta_c : \hat{L}(\mathfrak{g}, \tau) \simeq \hat{L}(\mathfrak{g}, \sigma). \tag{52}$$

**Theorem 5.5.** *The map $\eta$ restricts to the following isomorphism of Lie algebras*

$$\eta : \mathfrak{C}\mathfrak{g} \xrightarrow{\sim} \mathfrak{g}[t]^\sigma$$

*Proof.* From the definition of (52), we have the following calculations:

(1) When $\alpha = \alpha_i + \cdots + \alpha_j$ with $1 \leq i \leq j < \ell$, and $k \geq 0$, we have

$$\eta(e_{\pm\alpha} \otimes t^k + (-1)^{i+j} e_{\pm\tau(\alpha)} \otimes (-t)^k) = -(e_{\mp\alpha} \otimes t^{2k} + (-1)^{i+j} e_{\mp\tau(\alpha)} \otimes (\mathrm{i}t)^{2k}).$$

(2) When $\alpha = \alpha_{i,2\ell-j}$ with $1 \le i \le j < \ell$,, and $k \ge 0$, we have

$$\eta(e_{\pm\alpha} \otimes t^{k\pm 1} + (-1)^{i+j} e_{\pm\tau(\alpha)} \otimes (-t)^{k\pm 1}) = -(e_{\mp\alpha} \otimes t^{2k} + (-1)^{i+j} e_{\mp\tau(\alpha)} \otimes (\mathrm{i}t)^{2k}).$$

(3) When $\alpha = \alpha_{i,2\ell+1-i}$ with $1 \le i \le \ell$, and $k \ge 0$, we have

$$\eta(e_{\pm\alpha} \otimes t^{2k\pm 1}) = -e_{\mp\alpha} \otimes t^{4k}.$$

(4) When $\alpha = \alpha_{i\ell}$ with $1 \le i \le \ell$, and $k \ge 0$, we have

$$\eta(e_{\pm\alpha} \otimes t^{(2k+1\pm 1)/2} + (-1)^{\ell+i} e_{\pm\tau(\alpha)} \otimes (-t)^{(2k+1\pm 1)/2}) = -(e_{\mp\alpha} \otimes t^{2k+1} + (-1)^{\ell+i} e_{\mp\tau(\alpha)} \otimes (\mathrm{i}t)^{2k+1}).$$

(5) For any $1 \le i \le \ell$, and $k \ge 0$, we have

$$\eta(h_i \otimes t^k + h_{2\ell+1-i} \otimes (-t)^k) = -(h_i \otimes t^{2k} + h_{2\ell+1-i} \otimes (\mathrm{i}t)^{2k}).$$

Then, it is easy to check that the image of the described basis in $\mathfrak{C}\mathfrak{g}$ under the map $\eta$ is exactly a basis in $\mathfrak{g}[t]^\sigma$. This completes the proof of the theorem. □

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