Applications of the $\bar{\partial}$–technique in

L^2 Hodge theory on complete Kähler manifolds

Takeo Ohsawa

Max–Planck–Institut
für Mathematik
Gottfried–Claren–Straße 26
D–5300 Bonn 3

Federal Republic of Germany

MPI/90-30

Applications of the $\bar{\partial}$–technique in L^2 Hodge theory on complete Kähler manifolds

by

Takeo Ohsawa*

1. We shall survey a recent progress in the L^2 theory for the $\bar{\partial}$–operator on complete Kähler manifolds with emphasis on the extensions of Hodge theory to noncompact manifolds.

2. Let X be a (connected) complex manifold of dimension n, and let $H^r(X)$ and $H^{p,q}(X)$ be respectively the r–th de Rham cohomology group $/\mathbb{C}$ and the Dolbeault cohomology group of type (p,q) of X. First of all we recall two fundamental facts:

Theorem 1 (Hodge–de Rham) If X is compact, $H^r(X)$ and $H^{p,q}(X)$ are finite dimensional vector spaces $/\mathbb{C}$.

Theorem 2 (Hodge) If X is compact and admits a Kähler metric, then there exist canonical $\mathbb{C}$–linear isomorphisms $i^r : \bigoplus_{p+q=r} H^{p,q}(X) \longrightarrow H^r(X)$ such that

$\overline{i^r(H^{p,q}(X))} = i^r(H^{q,p}(X))$.

In case X is noncompact, although neither $H^r(X)$ nor $H^{p,q}(X)$ are finite dimensional in general, there exist some significant cases where the finite dimenality is observed. For instance let V be a complex analytic space of dimension n, and let $x \in V$ be an isolated

*Supported by Max–Planck–Institut für Mathematik

singular point such that the germ of V at x is irreducible. By Hironaka's theorem there exist a Stein neighbourhood $U \ni p$, a locally closed complex submanifold $\tilde{U} \subset \mathbb{P}^N$ for some $N \in \mathbb{N}$, and a proper holomorphic map $\pi : \tilde{U} \longrightarrow U$ such that $\pi | \tilde{U} \setminus \pi^{-1}(x)$ is a biholomorphism. Let $X = \tilde{U}$. Then we have

Proposition 3. Under the above situation,

$$\dim_{\mathbb{C}} H^{p,q}(X) < \infty \quad \text{if} \quad q \geq 1.$$

Moreover $H^{p,q}(X)$ $(q \geq 1)$ depend only on the germ $(X, \pi^{-1}(x))$ in the sense that the restriction maps

$$H^{p,q}(X) \longrightarrow \lim_{W \supset \pi^{-1}(x)} H^{p,q}(W)$$

are isomorphisms for $q \geq 1$, where W runs through the neighbourhoods of $\pi^{-1}(x)$.

In fact, Proposition 3 is a special case of the finiteness theorem of Andreotti–Grauert [1], which we shall recall briefly below. A complex manifold, denoted by X again, is said to be k–convex (resp. k–concave) if there exists a C^2 exhaustion function $\varphi : X \longrightarrow [0,\infty)$ (resp. $\longrightarrow (a,0]$ for some $a \in [-\infty,0)$) such that the complex Hessian $\partial\bar{\partial}\varphi$ has at least $n-k+1$ positive eigenvalues outside a compact subset of X. The function φ is called then a k–convex (resp. k–concave) exhaustion function of X, by an abuse of language.

Theorem 4 Let X be a k–convex (resp. k–concave) manifold. Then

$$\dim_{\mathbb{C}} H^{p,q}(X) < \infty \quad \text{if} \quad q \geq k.$$

Remark It is well known that every noncompact X is n–convex. In fact it is, even the better, n–complete in the sense that X admits an exhaustion function which is everywhere n–convex (cf. [8], [15], [26]). In the situation of Proposition 3, X is 1–convex since the pull–back of any strongly plurisubharmonic exhaustion function of U by π satisfies the requirement of φ for $k = 1$.

Therefore one may naturally look for an analogue of Theorem 2 on k–convex (resp. k–concave) Kähler manifolds, hoping there would exist a relevant generalization of the theory of harmonic forms to noncompact manifolds. In [23,24] we have established a method of comparing L^2 harmonic forms with ordinary cohomology classes, and deduced the following.

Theorem 5 Let X be an n–dimensional Kähler manifold which admits a plurisubharmonic k–convex exhaustion function. Then

$$(1) \qquad \bigoplus_{p+q=r} H^{p,q}(X) \cong H^r(X) \quad \text{if} \quad r \geq n+k$$

and

$$(2) \qquad H^{p,q}(X) \cong \overline{H^{q,p}(X)} \quad \text{if} \quad p+q \geq n+k$$

canonically. Moreover, exterior multiplication by the Kähler form ω induces isomorphisms

$$H_0^{n-s}(X) \xrightarrow{\omega^s \wedge} H^{n+s}(X) \quad \text{for} \quad s \geq k.$$

Here $H_0^{n-s}(X)$ denotes the (n–s)–th de Rham cohomology group with compact support.

Corollary Under the above situation, the natural restriction homomorphisms

$$H^r(X) \longrightarrow \lim_{K \subset\subset X} H^r(X \backslash K)$$

and

$$H^{p,q}(X) \longrightarrow \lim_{K \subset\subset X} H^{p,q}(X \backslash K)$$

are bijective for $r > n - k - 1$ (resp. $p+q < n-k-1$) and surjective for $r = n-k-1$ (resp. $p+q = n-k-1$).

Note That the isomorphisms (1) and (2) do not depend on the metric was not mentioned in our article. The author thanks S. Kosarew for attracting the attention to this point in Bucarest (June 1989).

In 1983, a new L^2 technique was introduced in complex analysis by Donnelly–Fefferman [11], which simplified the proof of Theorem 5 very much (cf. [34]). Moreover, it turned out that one can apply their idea to k–concave cases, too. In the following paragraphs we shall present a framework of an argument for the comparison theorems, several basic L^2 estimates related to it, and finally their applications to Hodge theory by showing the extensions of Theorem 2 after [25], [27], [28] and [31].

3. L^2 cohomology groups of X shall be defined with respect to an arbitrary Hermitian metric ds^2 on X. Let $L^r(X)$ (resp. $L^{p,q}(X)$) be the Hilbert space of L^2 r–forms (resp. L^2(p,q)–forms) on X, and let d (resp. $\bar\partial$) be the maximal closed extension of the exterior derivative (resp. that of the complex exterior derivative of type (0,1)) to $\bigoplus_{r=0}^{2n} L^r(X) \left(= \bigoplus_{p+q=0}^{2n} L^{p,q}(X)\right)$. The L^2 cohomology groups of X are defined by

$$H^r_{(2)}(X) := \mathrm{Ker} d \cap L^r(X)/\mathrm{Im} d \cap L^r(X)$$

and

$$H^{p,q}_{(2)}(X) := \mathrm{Ker}\ \bar\partial \cap L^{p,q}(X)/\mathrm{Im}\ \bar\partial \cap L^{p,q}(X).$$

To simply the notation we put

$$H_{(2)}(X) = \mathrm{Ker} d/\mathrm{Im} d \oplus \mathrm{Ker}\ \bar\partial/\mathrm{Im}\ \bar\partial.$$

If we want to compare L^2 and ordinary cohomology groups, the shortest way seems to look at the inductive limit $\lim\limits_{K} H_{(2)}(X\backslash K)$, where K runs through the compact subsets of X, since there is an exact triplet:

$$\begin{tikzcd} H_0(X) \arrow[rr] & & H_{(2)}(X) \arrow[dl] \\ & \lim\limits_{K} H_{(2)}(X\backslash K) \arrow[ul] & \end{tikzcd}$$

Here we use the same conventional notation $H_0(X)$ for the cohomology with compact support. In particular we obtain the following

<u>Proposition 6</u>

(i) $\quad H^r_0(X) \cong H^r_{(2)}(X)$ if $\lim\limits_{K} H^r_{(2)}(X\backslash K) = \lim\limits_{K} H^{r-1}_{(2)}(X\backslash K) = 0$

(ii) $\quad H^{p,q}_0(X) \cong H^{p,q}_{(2)}(X)$ if $\lim\limits_{K} H^{p,q}_{(2)}(X\backslash K) = \lim\limits_{K} H^{p,q-1}_{(2)}(X\backslash K) = 0,$

for any r and (p,q).

If the metric ds^2 is complete the closed extensions d and $\bar{\partial}$ are also minimal, so that the Hodge's star operator $*$ induces, after composing the complex conjugation, isometrics on $\mathrm{Ker}(d+d^*)$ and $\mathrm{Ker}(\bar{\partial}+\bar{\partial}^*)$. Here d^* and $\bar{\partial}^*$ denote the adjoints of d and $\bar{\partial}$, respectively.

Hence, by the Poincaré duality we obtain

<u>Proposition 7</u> Let (X,ds^2) be an n–dimensional complete Hermitian manifold. Then

(iii) $$H^{2n-r}(X) \cong H^{2n-r}_{(2)}(X) \text{ if } \lim_{K} H^{r}_{(2)}(X\setminus K) = \lim_{K} H^{r-1}_{(2)}(X\setminus K) = 0$$

for any r.

If the cohomology groups are Hausdorff with respect to the natural topology, one can directly apply the Serre duality. More precisely we have

<u>Proposition 8</u> Let (X,ds^2) be an n–dimensional complete Hermitian manifold, and let (p,q) be a pair of nonnegative integers. Suppose that $H^{n-p,n-q}(X)$, $H^{n-p,n-q+1}(X)$ and $H^{n-p,n-q}_{(2)}(X)$ are Hausdorff. Then

(iv) $$H^{n-p,n-q}(X) \cong H^{n-p,n-q}_{(2)}(X)$$

if $\lim_{K} H^{p,q}_{(2)}(X\setminus K) = \lim_{K} H^{p,q-1}_{(2)}(X\setminus K)$.

<u>Proof</u> By the Serre duality, $H^{p,q}_{0}(X)$ is then canonically isomorphic to the topological dual space of $H^{n-p,n-q}(X)$. Since $H^{n-p,n-q}_{(2)}(X)$ is Hausdorff,

$H^{n-p,n-q}_{(2)}(X) \cong \mathrm{Ker}(\bar\partial + \bar\partial^*) \cap L^{n-p,n-q}(X)$ so that it is the topological dual of $H^{p,q}_{(2)}(X)$. Hence (iv) follows from (ii). □

We note that there exist noncomplete algebraic varieties whose Dolbeault cohomology group contain non–Hausdorff (p,q)–components (cf. [18], [20]). Nevertheless, there is the following conjecture of C. Banica (oral communication in Bucarest).

Banica's conjecture: Let X be a Zariski open subset of a Stein manifold. Then $H^{p,q}(X)$ are Hausdorff.

If we restrict ourselves to k–convex or k–concave manifolds, the Hausdorff property of $\bar\partial$–cohomology groups is a consequence of L^2 estimates for $\bar\partial$ as L. Hörmander [16] has pointed out. Here we shall only sketch the argument after [25]. Let E be a holomorphic vector bundle over a k–convex (resp. k–concave) manifold X, and let φ be any k–convex (resp. k–concave) exhaustion function of X. Once for all we fix $c \in \mathbb{R}$ so that $\partial\bar\partial\varphi$ has at least $n-k+1$ positive eigenvalues outside

$$X_{c/2} := \{x \in X; \varphi(x) < \tfrac{c}{2}\} \text{ (resp. outside } X^{c/2} := \{x \in X;\ \varphi(x) > \tfrac{c}{2}\}).$$

The L^2 norm of E–valued forms will be denoted by $\|\ \|$, while the metrics on E and X shall be specified separately.

Estimate I Let h be any C^∞ Hermitian fiber metric of E. If X is k–convex, there exists a complete Hermitian metric ds^2_c on X_c such that, given any continuous function $\mu : [0,c) \longrightarrow \mathbb{R}$ there exists a C^∞ function $\lambda : [0,c) \longrightarrow \mathbb{R}$ for which $\lambda > \mu$ outside $[0,c/2]$ and with respect to the fiber metric $he^{-\lambda(\varphi)}$,

$$\|u\| \le C(\|\bar\partial u\| + \|\bar\partial^* u\| + \|u\|_{(c/2)})$$

for any compactly supported E–valued $C^\infty(n,q)$–form u on X_c with $q \geq k$. Here C is a constant independent of μ and λ, $\|u\|_{(c/2)}$ denotes the L^2 norm of u on $X_{c/2}$, and the adjoint $\bar\partial^*$ is with respect to $he^{-\lambda(\varphi)}$.

For the proof see [2] or [25]. Let T_X denote the tangent bundle of X. Then (p,q)–forms are nothing but $(\overset{n}{\Lambda} T_X \otimes \overset{p}{\Lambda} T_X^*)$–valued (0,q)–forms. Therefore, applying Estimate I to (p,q)–forms in this manner one gets a priori estimates for $\bar\partial$, and by a well known general nonsense (cf. [2], [13], [16]) we obtain

<u>Corollary</u> If $q \geq k$,

$$\dim H^{p,q}(X_c) < \infty$$

$$(H^{p,q}(X_c))^* \cong H_0^{n-p,n-q}(X_c) \qquad \text{(Serre duality)}$$

<u>Estimate I′</u> Let h be any C^∞ Hermitian fiber metric of E. If X is k–concave, there exists a Hermitian metric ds^2 on X such that, given any continuous function $\mu:(c,0] \to \mathbb{R}$ one can find a C^∞ concave decreasing function $\lambda:(c,0] \to \mathbb{R}$ for which $e^{-\lambda} > (1-\lambda'+\lambda'')^n\mu$ outside $[c/2,0]$ and with respect to the modified metrics $\lambda''(\varphi)\partial\varphi\bar\partial\varphi - \lambda'(\varphi)ds^2$ and $he^{\lambda(\varphi)}$,

$$\|u\| \leq C(\|\bar\partial u\| + \|\bar\partial^* u\| + \|u\|_{(c/2)})$$

for any compactly supported E–valued $C^\infty(0,q)$–form u on X^c with $q < n-k$.

For the proof one generalizes the argument of [25] in an obvious manner.

Note The growth condition on λ is imposed so that any prescribed C^∞ form becomes square integrable by adjusting the metrics.

As before we obtain

Corollary If $q < n - k$,

$$\dim H^{p,q}(X^c) < \infty$$

$$H^{p,n-k}(X^c) \text{ is Hausdorff}$$

$$(H^{p,q}(X^c))^* \cong H_0^{n-p,n-q}(X^c) \quad \text{(Serre duality)}$$

By a technique of changing simultaneously the fiber metric and the base metric, we can recover the Runge type approximation theorem for the pairs (X,X_c) and (X,X^c) (cf. [25]). Anyway the $\bar\partial$ method is applicable to prove the following.

Theorem 9 (i) If X is k–convex,

$$H^{p,q}(X) \cong H^{p,q}(X_c) \qquad \text{(approximation)}$$

$$H_0^{p,n-q}(X) \cong H_0^{p,n-q}(X_c) \qquad \text{(Serre duality)}$$

for $q \geq k$.

(ii) If X is k–concave,

$$H^{p,q}(X) \cong H^{p,q}(X^c) \qquad \text{(approximation)}$$

$$H_0^{p,n-q}(X) \cong H_0^{p,n-q}(X^c) \qquad \text{(Serre duality)}$$

for $q < n - k$.

Note For the proof of the second isomorphism of (ii), one uses the Serre duality and the obvious isomorphism

$$H_0^{p,n-q}(X) = \lim_{d<c} H_0^{p,n-q}(X_d)$$

Unfortunately, Estimate I is far from available for the proof of the vanishing of the components of $\lim_{K} H_{(2)}(X\setminus K)$. For the purpose we need another kind of L^2 estimate which amounts to a higher dimensional version of Hardy's inequality $\int_0^1 t^{-2}(\int_0^t f)^2 \leq 4 \int_0^1 f^2$ for $f \in L^2([0,1])$. To illustrate the idea we first describe it in the simplest way after Donnelly-Fefferman [11],

Estimate II Let (X,ds^2) be a Kähler manifold of dimension n, let $\Psi : X \longrightarrow \mathbb{R}$ be a C^2–function, and let $\{\lambda_i\}_{i=1}^n$ be the eigenvalues of $\partial\bar{\partial}\Psi$ counted with multiplicity. Then, for any C^∞ compactly supported (p,q) form u on X,

$$C_{\Psi,u}(\|\bar{\partial}u\| + \|\bar{\partial}^* u\|) \geq \inf_{\text{supp } u} \Gamma_{\Psi,p,q}\|u\|$$

where $C_{\Psi,u} = 2 \sup\{\,|d\Psi(x)|\,;\, x \in \operatorname{supp} u\}$

and
$$\Gamma_{\Psi,p,q} = \operatorname{dist}(0, <\{\sum_{\alpha=1}^{p} \lambda_{i_\alpha} + \sum_{\beta=1}^{q} \lambda_{j_\beta} - \sum_{k=1}^{n} \lambda_k \,|\, ;\, 1 \le i_1 < \dots < i_p \le n$$
$$\text{and } 1 \le j_1 < \dots < j_q \le n\}>)\,,$$

where $\operatorname{dist}(0,<A>)$ denotes the distance between $0 \in \mathbb{R}$ and the convex hull of A.

Proof (Jacobi identity technique) Let Λ be the adjoint of multiplication by the fundamental form of ds^2. Applying the graded Jacobi identity to the operators $\bar\partial$, Λ and left multiplication by $\partial\Psi$ ($\partial := d - \bar\partial$), we have

$$[\bar\partial,[\partial\Psi,\Lambda]] + [[\bar\partial,\Lambda],\partial\Psi] = [[\bar\partial,\partial\Psi^*,\Lambda].$$

Here we identify $\partial\Psi$ with its left multiplication. By using the Kähler identity we obtain

$$[\bar\partial,\partial\Psi^*] + [\partial^*,\partial\Psi] = [\sqrt{-1}\,\partial\bar\partial\Psi,\Lambda].$$

Hence,

$$([\bar\partial,\partial\Psi^*]u,u) + ([\partial^*,\partial\Psi]u,u) = ([\sqrt{-1}\,\partial\bar\partial\Psi,\Lambda]u,u).$$

Integration by parts and application of Cauchy–Schwartz inequality and Kähler identity give the desired estimate. □

Estimate II has already applications to some special cases (cf. [27]).
However, in order to get more general results we need to modify it as follows (cf. [28]).

Estimate III Let (X,ds^2) be a Hermitian manifold of dimension n and let φ_i $(i = 1,2)$ be two real valued C^∞ functions on X. Then there exist a numerical constant β_n such that

$$\|\bar\partial u\|^2_{\varphi_1} + \|\bar\partial^* u\|^2_{\varphi_1}$$

$$\geq \|u\|^2_{\varphi_1} \cdot \inf_{\text{supp } u} (\Gamma_{\varphi_1+\varphi_2,p,q} - \beta_n |d\omega|^2 - 3|d\varphi_2|^2)$$

for any compactly supported $C^\infty(p,q)$ form u. Here ω denotes the fundamental form of ds^2.

Proof is an obvious combination of the Bochner trick and Jacobi–identity technique.

Remark 1. In case $d\omega = 0$, we recover the classical L^2 estimate of Kodaira–Nakano by letting $\varphi_2 = 0$ (generalization to the bundle–valued case is straightforward).

Remark 2. The Jacobi identity technique is a generalization of the Bochner trick, since Kodaira–Nakano identity is nothing but a modification of the Jacobi identity for the operators $\bar\partial,\Lambda$ and the holomorphic connection by the Kähler identities (cf. [7]). We note that there exist real analogue of the above estimates, although they appear only partially in the literature (cf. [12], [30]).

Remark 3. Suitable modifications of Kodaira–Nakano identity of different kind sometimes yield the sharpest control of the growth of holomorphic differential forms and harmonic maps (cf. [9], [10], [29], [33], [38]).

4. In what follows we suppose that X is embedded as a Zariski open subset of a compact complex space $\overline{X}$. First we shall describe the extension of Theorem 2 to the k–concave cases. The notion of k–concavity is related to the embedding $X \hookrightarrow \overline{X}$ in the following way.

Proposition 10 If $\dim(\overline{X} \setminus X) = k$, then X is $(k+1)$–concave.

For the proof see [1] or [28]. In particular, we note that Theorem 9 can be directly applied to X. To extend Theorem 2, we need a proper generalization of Kählerianity to complex spaces.

Definition A Hermitian (resp. Kähler) metric on a (reduced) complex space $\overline{X}$ is a Hermitian (resp. Kähler) metric ds^2 on $\overline{X}_{reg} := \{x \in \overline{X}; \overline{X} \text{ is smooth at } x\}$ such that, for any $x \in \overline{X}$ there exist a neighbourhood $U \ni x$ and a C^∞ function $\varphi : U \longrightarrow \mathbb{R}$ with

$$ds^2 \leq \partial\bar{\partial}\varphi \leq 2ds^2 \ \ (\text{resp. } ds^2 = \partial\bar{\partial}\varphi) \ \text{ on } \ U \cap \overline{X}_{reg}.$$

The following was first discovered by H. Grauert [13]. Our harmonic theory will be based on it.

Proposition 11 Let $\overline{X}$ be a compact complex space with a Hermitian (resp. Kähler) metric ds^2, and let $X \subset \overline{X}_{reg}$ be any Zariski open subset. Then, for $a = -\infty$ and $a = 0$, there exists a C^∞ exhaustion function $\varphi : X \longrightarrow (a,1]$ such that $ds^2 + \partial\bar{\partial}\varphi$ is a complete metric on X.

One can find such φ immediately if $\overline{X} \setminus X$ is discrete. In fact, for each $x \in \overline{X} \setminus X$ there exist a neighbourhood $U \ni x$ and a holomorphic embedding $(U,x) \hookrightarrow (\mathbb{B}^N,0)$, where $\mathbb{B}^N$ denotes the complex unit ball of dimension N centered at the origin 0. Letting z be the coordinate around 0, we have

$$\partial\bar{\partial}(-\log(-\log\|z\|)) = \frac{\partial\bar{\partial}\log\|z\|}{-\log\|z\|} + \frac{\partial\log\|z\|\bar{\partial}\log\|z\|}{(\log\|z\|)^2}.$$

Therefore we obtain our $\varphi : X \longrightarrow (-\infty,1]$ by patching the restrictions of $-\log(-\log\|z\|)$ to $U\setminus\{x\}$ for all x and multiplying a small positive number if necessary. For $a = 0$, one may use the function $(+\log(-\log\|z\|))^{-1}$ instead of $-\log(-\log\|z\|)$, and obtain also a complete metric. In order to generalize this construction to the case $\dim(\bar{X}\setminus X) > 0$, we need to use the property of ds^2 that it is locally equivalent to the restriction of a metric on some nonsingular ambient space. This property of ds^2 is also crucial in the proof of the following refinement of Proposition 11.

Proposition 12 (cf. Proposition 1.1 in [28])

Let the notations be as above, and suppose that $\bar{X}$ admits a Kähler metric. Then, for any $\epsilon > 0$, there exist a complete Kähler metric ds_X^2 on X, a C^∞ exhaustion function Ψ: $X \longrightarrow (-\infty,0]$ and a neighbourhood $W \supset \bar{X}\setminus X$ such that

(*) $\quad |\partial\Psi|^2 < \epsilon$

(**) $\quad |\partial\bar{\partial}\Psi| < 2n$

(***) $\quad$ The eigenvalues $\lambda_1 \geq \ldots \geq \lambda_n$ of $\partial\bar{\partial}\Psi$ with respect to ds_x^2 satisfy

$$1-\epsilon<\lambda_j<1+\epsilon \text{ for } 1\leq j\leq \operatorname{codim}(\bar{X}\setminus X) \text{ on } W\cap X$$

$$-\epsilon<\lambda_j \text{ for } j>\operatorname{codim}(\bar{X}\setminus X) \text{ on } X.$$

Here $|\ |$ denotes the length with respect to ds_x^2.

<u>Note</u> The property (**) is eventually stupid, since ϵ can be arbitrarily small.

We fix an exhaustion function Ψ as above for some small ϵ, say $\epsilon = 1/100\,n$, and choose $c \in \mathbb{R}$ so that $W \supset X_{(c)} := \{x; \Psi(x) < c\}$. Then we are in a good position to apply Estimate III. In fact, we apply it to $X_{(c)}$ with respect to the metric

$$ds^2 = (A(c-\Psi)^{-2}+1)ds_X^2 + 2A(c-\Psi)^{-3}\partial\Psi\bar{\partial}\Psi$$

for some $A > 0$, letting $\varphi_1 = A(c-\Psi)^{-1}$ and $\varphi_2 = \Psi$. By a direct computation we have

$$\Gamma_{\varphi_1+\varphi_2,p,q} - \beta_n|d\omega|^2 - 3|d\varphi_2|^2 > \tfrac{1}{4}$$

for $p+q > n + \dim(\bar{X}\setminus X)$ as long as A is sufficiently large (say $A > 2^{16}\beta_n^2 n^4$) so that

$$\|\bar{\partial}u\|^2_{\varphi_1} + \|\bar{\partial}^* u\|^2_{\varphi_1} \geq \tfrac{1}{4}\|u\|^2_{\varphi_1}$$

for any C^∞ compactly supported (p,q) form u on $X_{(c)}$ if $p+q > n + \dim(\bar{X}\setminus X)$. Since ds^2 is a complete metric, we have then the vanishing of the (p,q) components of the L^2 cohomology with respect to the weighted norm $\|\ \|_{\varphi_1}$. But obviously one has $\|u\| \leq \text{const}\,\|u\|_{\varphi_1}$ for any u, and that two norms $\|\ \|$ and $\|\ \|_{\varphi_1}$ are equivalent near $\bar{X}\setminus X$ since $\lim_{x\to\bar{X}\setminus X}\varphi_1(x) = 0$. Moreover, by the property (*) of Ψ the norms with respect to ds_X^2 and ds^2 are also equivalent near $\bar{X}\setminus X$. Thus we obtain the following

<u>Theorem 13</u> Let X be a nonsingular n–dimensional Zariski open subset of a compact Kähler space $\overline{X}$. Then there exists a complete Kähler metric on X such that

$$\lim_{K} H^{p,q}_{(2)}(X\setminus K) = 0 \quad \text{for } p+q>n+\dim(\overline{X}\setminus X)$$

and

$$H^{p,q}_0(X) \cong H^{p,q}_{(2)}(X) \quad \text{for } p+q > n+\dim(\overline{X}\setminus X)+1.$$

Here the isomorphism is induced by the natural inclusion homomorphism.

<u>Corollary.</u> Under the above situation, there exist canonical isomorphisms

$$\begin{cases} H^r(X) \cong \bigoplus_{p+q=r} H^{p,q}(X) \\ H^{p,q}(X) \cong \overline{H^{q,p}(X)} \end{cases}$$

for $p+q < n-\dim(\overline{X}\setminus X)-1$, and

$$\begin{cases} H^r_0(X) \cong \bigoplus_{p+q=r} H^{p,q}_0(X) \\ H^{p,q}_0(X) \cong \overline{H^{q,p}_0(X)} \end{cases}$$

for $p+q > n+\dim(\overline{X}\setminus X)+1$.

<u>Open question</u> Is Theorem 13 also valid for non–Kähler $\overline{X}$?

Remark In the above proof we heavily use the Kählerianity to absorb the term $\beta_n |d\omega|^2 + 3|d\varphi_2|^2$ into $\Gamma_{\varphi_1+\varphi_2,p,q}$, and the author does not know how to generalize our argument to the non–Kähler case. Of course the generalization is trivial if $\dim(\overline{X}\setminus X) = 0$.

The L^2 cohomology group $H_{(2)}(X)$ with respect to a C^∞ Hermitian metric on $\overline{X}$ obviously does not depend on the choice of the metric, so that it deserves to be studied in detail, too. The following was proved in [27] by regarding the metric on $\overline{X}$ as a limit of complete metrics on X for which the L^2 estimates are uniform.

Theorem 14 If $\dim(\overline{X}\setminus X) = 0$, then

$$\begin{cases} H^{p,q}(X) \cong H^{p,q}_{(2)}(X) \\ \\ H^{n-p,n-q}(X) \cong H^{n-p,n-q}_{(2)}(X) \end{cases}$$

for $p+q>n+1$ and

$$\begin{cases} H^r_0(X) \cong H^r_{(2)}(X) \\ \\ H^r(X) \cong H^r_{(2)}(X) \end{cases}$$

for $r>n+1$, with respect to any Hermitian metric on $\overline{X}$.

Note In [27] the above result is stated only for the Kähler case, but the same proof applies to the general case.

Remark 1 Little is known about the structures of $H^{p,q}_{(2)}(X)$ and $H^{r}_{(2)}(X)$ outside the above ranges, although there seems to exist a relation between $H_{(2)}(X)$ and the intersection cohomology groups of $\bar{X}$ (cf. [6], [17], [21], [36], [37], [39]). See [22] and [35] for the Hodge structure of the intersection cohomology groups.

Remark 2. As for the corollary to Theorem 13, a completely different proof was recently given by Arapura [3]. In case X is projective algebraic, its algebraic version exists (cf. [4,5].

5. Now we turn to discuss the opposite case where X is k–convex. Compared to the k–concave case, the situation seems to be more delicate.

Example ([5], [14]) Let $Y \hookrightarrow \mathbb{P}^N$ be a nonsingular projective surface, and let $E \longrightarrow Y$ be a rank two vector bundle defined as the kernel of a surjective homomorphism

$$3\mathcal{O}_Y(-1) \longrightarrow \mathcal{O}_Y.$$

Let $\bar{X}$ be any (algebraic) compactification of the total space of the dual bundle E^* and let X be the complement of the zero section of E^* in $\bar{X}$. Obviously X is 2–convex, but a computation shows (cf. [5]) that $\dim H^6(X) = 1$ and $\sum\limits_{p+q=6} \dim H^{p,q}(X) \geq 2$. Moreover $H^{4,2}(X) \neq 0$ and $H^{2,4}(X) = 0$, so that the Hodge symmetry doesn't hold either.

Therefore, in order to extend Theorem 2 to k–convex Zariski open subsets $X \subset \bar{X}$, we need to impose an additional condition on the boundary of X. Bauer–Kosarew [4,5] has shown the following.

Theorem 15 Let X be a projective algebraic variety of dimension n, let $Y \subset X$ be an algebraic subset whose ideal sheaf $\mathcal{J}_Y$ is invertible, and let $X = \overline{X} \backslash Y$. Suppose that the line bundle $(\mathcal{J}_Y / \mathcal{J}_Y^2)^* \longrightarrow Y$ is k–ample in the sense of Sommese*. Then

$$\sum_{p+q=r} \dim H^{p,q}(X) = \dim H^r(X)$$

and

$$\dim H^{p,q}(X) = \dim H^{p,q}(X)$$

for r, $p+q > n+k$.

Note It seems to be difficult to see whether there exist canonical isomorphisms

$$\bigoplus_{p+q=r} H^{p,q}(X) \xrightarrow{\sim} H^r(X) \quad \text{s.t.} \quad H^{p,q}(X) = \overline{H^{p,q}(X)}$$

in the above range.

The notion of k–ampleness has a differential geometric counterpart: Let L be a holomorphic line bundle over a reduced complex space Y. L is said to be semipositive of rank ℓ if there exist a C^∞ fiber metric h of L such that the curvature of h/Y_{reg} is

* A holomorphic line bundle $L \longrightarrow Y$ is said to be k–ample if there exist positive integers, ℓ,N and holomorphic sections $s_0, s_1, \ldots, s_N$ of $L^{\otimes \ell}$ such that the ratio $(s_0 : s_1 : \ldots : s_N)$ defines a morpism whose fibers have dimension $\leq k$.

semipositive and, for any C^∞ function $\varphi : Y \longrightarrow \mathbb{R}$ with supp $\varphi \subset\subset Y$, there exists $\epsilon > 0$ such that the curvature of the modified fiber metric $he^{\epsilon\varphi}$ has at least ℓ positive eigenvalues on Y_{reg}.

Theorem 16 Let X be a Zariski open subst of a compact n–dimensional Kähler manifold $\bar{X}$. If $\bar{X}\backslash X$ is a divisor whose normal bundle is semipositive of rank n–k–1, then

$$\bigoplus_{p+q=r} H^{p,q}(X) \cong H^r(X) \qquad \text{if} \quad r > n+k$$

and

$$H^{p,q}(X) \cong \overline{H^{q,p}(X)} \qquad \text{if} \quad p+q > n+k+1.$$

Proof is given in [31]. It is true that the isomorphisms are canonical in the range $r, p+q > n+k+1$ (cf. [32]). We do not know whether $H^{p,q}(X) \cong \overline{H^{q,p}(X)}$ in case $p+q = n+k+1$.

Note Suppose a holomorphic line bundle L over an n–dimensional manifold Y is k–ample, but not (k–1)–ample. Then L is semipositive of rank n–k if and only if rank $d(s_0:\ldots:s_N) \equiv n-k$ for some $s_0,\ldots,s_N \in \Gamma(L^{\otimes \ell})$ ($\ell >> 0$). But clearly

$$\{\text{'semipositive of rank } n-k\text{'–bundles}\} \not\subset \{k\text{–ample bundles}\}$$

if Y is a compact Kähler manifold with $H^1(Y) \neq 0$.

REFERENCES

[1] Andreotti, A. and Grauert, H., Théorème de finitude pour la cohomologie des espaces complexes, Bull. Soc. Math. France 90, 193–259 (1962).

[2] Andreotti, A. and Vesentini, E., Carleman estimates for the Laplace–Beltrami equation on complex manifolds, Publ. Math. IHES 25, 81–130 (1965).

[3] Arapura, D., Local cohomology of sheaves of differential forms and Hodge theory, preprint, Purdue University 1989.

[4] Bauer, I. and Kosarew, S., On the Hodge spectral sequence for some classes of non–complete algebraic manifolds, Math. Ann. 284, 577–593 (1989).

[5] Bauer, I., and Kosarew, S., Some aspects of Hodge theory on non–complete algebraic manifolds, preprint, Mathematica Göttingensis 1989.

[6] Cheeger, J., Goreski, M. and MacPherson, R., L^2–cohomology and intersection homology of singular varieties, Seminar on Differential Geometry, Annales of Math. Studies 102, 303–340 (1982).

[7] Demailly, J.–P., Une preuve simple de la conjecture de Grauert-Riemenschneider, in Séminaire d'Analyse P. Lelong – P. Dolbeault – H. Skoda, L.N. in Math. 1295, p. 24–47, 1987.

[8] Demailly, J.–P., Cohomology of q–convex spaces in top degrees, to appear in Math. Z.

[9] Diederich, K., Herbort, G. and Ohsawa, T., The Bergman kernel on uniformly extendable pseudoconvex domains, Math. Ann. 273, 471–478 (1986).

[10] Diederich, K. and Herbort, G., Extension of holomorphic functions with growth conditions, preprint 1989.

[11] Donnelly, H. and Fefferman, C., L^2–cohomology and index theorem for the Bergman metric, Ann. Math. 118, 593–619 (1983).

[12] Donnelly, H. and Xavier, F., On the differential form spectrum of negatively curved Riemannian manifolds, Amer. J. Math. 106, 169–185 (1984).

[13] Grauert, H., Characterisierung der Holomorphiegebiete durch die vollständige Kählersche Metrik, Math. Ann. 131, 38–75 (1956).

[14] Grauert, H. and Riemenschneider, O., Kähleroche Mannigfaltigkeiten mit hyper–q–konvexem Rand, Problems in Analysis, Symp. in Honor of S. Bochner, p. 61–79, Princeton University Press 1970.

[15] Greene, R.E. and Wu, H., Embedding of open Riemannian manifolds by harmonic functions, Ann. Inst. Fourier, Grenoble 25, 215–235 (1975).

[16] Hörmander, L., L^2–estimates and existence theorems for the $\bar{\partial}$–operator, Acta Math. 113, 89–152 (1965).

[17] Hsian, W.C. and Pati, V., L^2–cohomology of normal algebraic surfaces, Invent. Math. 81, 395–412 (1985).

[18] Kazama, H., $\bar{\partial}$ cohomology of (H,C)–groups, Publ. RIMS, Kyoto Univ. 20, 297–317 (1984).

[19] Looijenga, E., L^2–cohomology of locally symmetric varieties, Comp. Math. 67, 3–20 (1988).

[20] Malgrange, B., La cohomologie d'une variété analytique complexe à bord pseudo-convexe n'est pas nécessairement séparée (English summary) C.R. Acad. Sci. Paris Sér. A–B 208, A93–A95 (1975).

[21] Nagase, M., Remarks on the L^2–cohomology of singular algebraic surfaces, J. Math. Soc. Jap. 41, 97–116 (1989).

[22] Navaro–Aznar, V., Sur la théorie de Hodge des variétés algébriques à singularités isolées, Astérisque 130, 272–307 (1985).

[23] Ohsawa, T., A reduction theorem for cohomology groups of very strongly q–convex Kähler manifolds, Invent. math. 63, 335–354 (1981).

[24] Ohsawa, T., Addendum to: A reduction theorem for cohomology groups of very strongly q–convex Kähler manifolds, Invent. math. 66, 391–393 (1982).

[25] Ohsawa, T., Isomorphism theorems for cohomology groups of weakly 1–complete manifolds, Publ. RIMS 18, 191–232 (1982).

[26] Ohsawa, T., Completeness of noncompact analytic spaces, Publ. RIMS 20, 683–692 (1984).

[27] Ohsawa, T., Hodge spectral sequence on compact Kähler spaces, Publ. RIMS, Kyoto Univ. 23, 265–274 (1987).

[28] Ohsawa, T., Hodge spectral sequence and symmetry on compact Kähler spaces, Publ. RIMS, Kyoto Univ. 23, 613–625 (1987).

[29] Ohsawa, T., On the extension of L^2 holomorphic functions II, Publ. RIMS, Kyoto Univ. 24, 265–275 (1988).

[30] Ohsawa, T., A generalization of the Weitzenböck formula and an analytic approach to Morse theory, RIMS–preprint.

[31] Ohsawa, T., Hodge spectral sequence on pseudoconvex domains II, to appear in Revue ~~Riemanniane~~ Roumaine de Math. Pure et Appl.

[32] Ohsawa, T., in preparation

[33] Ohsawa, T. and Takegoshi, K., On the extension of L^2 holomorphic functions, Math. Z. 195, 197–204 (1987).

[34] Ohsawa, T. and Takegoshi, K., Hodge spectral sequence on pseudoconvex domains, Math. Z. 197, 1–12 (1988).

[35] Saito, Morihiko, Modules de Hodge polarizables, Publ. RIMS, Kyoto Univ. 24, 849–995 (1988).

[36] Saper, L., L^2–cohomology and intersection homology of certain algebraic varieties with isolated singularities, Invent. Math. 82, 207–255 (1985).

[37] Saper, L. and Stern, M., L_2–cohomology of arithmetic varieties, Proc. Nat'l Acad. Sci. USA 84, 5516–5519 (1987), to appear in Ann. of Math.

[38] Takegoshi, K., Energy estimates and Liouville theorems for harmonic maps, to appear in Ann. Ex. Norm. Sup.

[39] Zucker, S., L^2–cohomology of warped products and arithmetic groups, Invent. Math. 70, 169–218 (1982).