Einstein-Weyl structures on complex manifolds

Liviu Ornea and Misha Verbitsky[1]

Abstract
A Hermitian Einstein-Weyl manifold is a complex manifold admitting a Ricci-flat Kähler covering $\widetilde{M}$, with the deck transform acting on $\widetilde{M}$ by homotheties. We show that a Hermitian Einstein-Weyl structure on a compact complex manifold is unique, if it exists. This result is a conformal analogue of Calabi's theorem stating the uniqueness of Calabi-Yau metrics in a given Kähler class.

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1 Introduction

E. Calabi ([C]) has shown that a compact manifold of Kähler type with vanishing first Chern class can admit at most one Kähler-Einstein metric in a given Kähler class (see [B] for details and implications of this extremely influential work).

In this note, we generalize this result to conformal setting. Recall that a locally conformally Kähler (LCK) manifold is a complex manifold admitting a Kähler covering $\widetilde{M}$, with the deck transform acting on $\widetilde{M}$ by holomorphic homotheties. If $\widetilde{M}$ is, in addition, Ricci-flat, M is called Hermitian Einstein-Weyl, or locally conformally Kähler Einstein-Weyl.[1]

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[1]Normally, one defines Hermitian Einstein-Weyl differently, and then this definition becomes a theorem; see Claim 3.2.

Since the deck transform group acts on $\widetilde{M}$ conformally, the LCK-structure defines a conformal class of Hermitian metrics on M. A metric in this class is called **an LCK-metric**. In the literature, the distinction between "LCK-metrics" and "LCK-structures" is often ignored.

We give an introduction to LCK-geometry in Section 2, and explain the properties of Einstein-Weyl structures in Section 3.

Theorem 1.1. Let (M, J) be a compact complex manifold. Then it admits at most one Einstein-Weyl locally conformally Kähler structure, up to a constant multiplier.

We prove Theorem 1.1 in Section 4.

Remark 1.2. For a Calabi-Yau manifold, the metric is uniquely determined by the complex structure and the Kähler class in cohomology. In a conformal setting, the Einstein-Weyl LCK-structure is defined uniquely. This happens because a relevant cohomology group is $H^2(M, L)$, where L is the weight bundle of the conformal structure (see Definition 2.2). It is easy to show that all cohomology of the local system L vanish, cf. [L, Remark 6.4].

The compatibility between a complex structure and a Weyl structure naturally leads to the LCK-condition. This was observed by I. Vaisman (see also [PPS]). Moreover, as shown by P. Gauduchon ([G]), a compact Einstein-Weyl locally conformally Kähler manifold is necessarily Vaisman (see Theorem 3.4). Then Theorem 1.1 is translated into the uniqueness of an Einstein-Weyl Vaisman metric on a given compact complex manifold.

The Vaisman manifolds are intimately related to Sasakian geometry (see e.g. [OV1]). Given a Sasakian manifold X, the product $S^1 \times X$ has a natural Vaisman structure. Conversely, any Vaisman manifold admits a canonical Riemannian submersion to S^1, with fibers which are isometric and equipped with a natural Sasakian structure.

Under this correspondence, the Einstein-Weyl Vaisman manifolds correspond to Sasaki-Einstein manifolds. The Sasaki-Einstein manifolds recently became a focus of much research, due to a number of new and unexpected examples constructed by string physicists (see [MSY], [CLPP], [GMSW1], [GMSW2], and the references therein). For a physicist, Sasaki-Einstein manifolds are interesting because of AdS/CFT correspondence in string theory. From the mathematical point of view, these examples are as mysterious as the Mirror Symmetry conjecture 15 years ago.

The Sasakian manifolds, being transverse Kähler[2], can be studied by the means of algebraic geometry. One might hope to obtain and study the Sasaki-Einstein metrics by the same kind of procedures as used to study the Kähler-Einstein metrics in algebraic geometry. However, this analogy is not perfect. In particular, it is possible to show that the Sasaki-Einstein structures on CR-manifolds are not unique. We shall address this problem in a forthcoming paper.

One may hope to approach the classification of Sasaki-Einstein structures using the Einstein-Weyl geometry.

2 Vaisman manifolds

We first review the necessary notions of locally conformally Kähler geometry. See [DO], [OV1], [OV2], [OV3], [Ve] for details and examples.

Let (M, J, g) be a complex Hermitian manifold of complex dimension n. Denote by ω its fundamental two-form $\omega(X, Y) = g(X, JY)$.

Definition 2.1. A Hermitian metric g on (M, J) is **locally conformally Kähler** (LCK for short) if

$$d\omega = \theta \wedge \omega.$$

for a closed 1-form θ.

Clearly, for any function $f: M \longrightarrow \mathbb{R}^{>0}$, $f\omega$ is also an LCK-metric. A conformal class of LCK-metrics is called **an LCK-structure**.

The form θ is called **the Lee form of the LCK-metric**, and the dual vector field $\theta^\sharp$ is called **the Lee field**.

The one-form θ can be interpreted as a (flat) connection one-form in the bundle of densities of weight 1, usually denoted L. This is the real line bundle associated to the representation

$$A \mapsto | \det(A) |^{\frac{1}{2n}}, \quad A \in \mathrm{GL}(2n, \mathbb{R})$$

Definition 2.2. The bundle L, equipped with a connection $\nabla_0 + \theta$, is called **the weight bundle of a locally conformally Kähler structure**. One

[2]This viewpoint was systematically developed in the work of C.P. Boyer, K. Galicki and collaborators. See *e.g.* [BG].

could consider the form ω as a closed, positive $(1,1)$-form, taking values in L^2.

Remark 2.3. Passing to a covering, we may assume that the flat bundle L is trivial. Then ω can be considered as a closed, positive $(1,1)$-form taking values in a trivial vector bundle, that is, a Kähler form. Therefore, any LCK-manifold admits a covering $\widetilde{M}$ which is Kähler. The deck transform acts on $\widetilde{M}$ by homotheties. This property can be used as a definition of LCK-structures (see Section 1).

Definition 2.4. A **Vaisman manifold** is an LCK manifold whose Lee form is parallel with respect to the Levi-Civita connection of g.

Definition 2.5. Let $(\mathcal{C}, g, \omega)$ be a Kähler manifold. Assume that ρ is a free, proper action of $\mathbb{R}^{>0}$ on $\mathcal{C}$, and g and ω are homogeneous of weight 2:

$$\operatorname{Lie}_v \omega = 2\omega, \quad \operatorname{Lie}_v g = 2g,$$

where v is the tangent vector field of ρ. The quotient $\mathcal{C}/\rho$ is called **a Sasakian manifold**. If $N = \mathcal{C}/\rho$ is given, $\mathcal{C}$ is called **the Kähler cone of** N. As a Riemannian manifold, $\mathcal{C}$ is identified with the **Riemannian cone** of (N, g_N), $\mathcal{C}(N) = (N \times \mathbb{R}^{>0}, t^2 g_N + dt^2)$.

The Sasakian manifolds are discussed in [BG], in great detail.

The following characterization of *compact* Vaisman manifolds is known (see [OV1]):

Remark 2.6. A compact complex manifold (M, J) is Vaisman if it admits a Kähler covering $(\widetilde{M}, J, h) \to (M, J)$ such that:

- The monodromy group $\Gamma \cong \mathbb{Z}$ acts on $\widetilde{M}$ by holomorphic homotheties with respect to h (this means that (M, J) is equipped with an LCK-structure).

- $(\widetilde{M}, J, h)$ is isomorphic to a Kähler cone over a compact Sasakian manifold S. Moreover, there exists a Sasakian automorphism φ and a positive number $q > 1$ such that Γ is isomorphic to the cyclic group generated by $(x, t) \mapsto (\varphi(x), tq)$.

Remark 2.7. In these assumptions, denote by $\theta^\sharp$ the vector field $t\frac{d}{dt}$ on $\widetilde{M} = (S \times \mathbb{R}^{>0}, g_S t^2 + dt^2)$. Chose the metric $g = g_S + dt^2$ on $M = \widetilde{M}/\Gamma$. Clearly, $\theta^\sharp$ descends to a Lee field on M, denoted by the same letter. Then $J(\theta^\sharp)$ is tangent to the fibers of the natural projection $\widetilde{M} \longrightarrow \mathbb{R}^{>0}$, hence belongs to TS. This vector field is called **the Reeb field** of the Sasakian manifold S. Clearly, the orbits of $J(\theta^\sharp)$ on $\widetilde{M}$ are precompact (contained in a compact set).

Remark 2.8. It will be important for us to note that the Kähler metric h on the covering $\widetilde{M} = \mathcal{C}(S) = S \times \mathbb{R}^{>0}$ has a global Kähler potential ψ, which is expressed as $\psi(x,t) = t^2$. The metric $\psi^{-1} \cdot h$ projects on N into the LCK metric g. Moreover, $\psi = \mid \theta \mid^2$, the norm being taken with respect to the lift of g.

On a Vaisman manifold, the Lee field $\theta^\sharp$ is Killing, parallel and holomorphic. One easily proves that $\mathcal{L}_{\theta^\sharp}\omega = 2\omega$.

Recall from [To] the notion of transverse geometry:

Definition 2.9. Consider a manifold endowed with a foliation $\mathcal{F}$ with tangent bundle F and normal bundle Q. A differential, or Riemannian, form α on X is **basic** (or **transverse**) if $X \rfloor \alpha = 0$ and $\mathrm{Lie}_X \alpha = 0$ for every $X \in F$. A **transverse geometry** of $\mathcal{F}$ is a geometry defined locally on the leaf space of $\mathcal{F}$. A **Kähler transverse structure** on $(M, \mathcal{F})$ is a complex Hermitian structure on Q defined by a pair $g_{\mathcal{F}}, \omega_{\mathcal{F}}$ of transverse forms, in such a way that the induced almost complex structure defined locally on the leaf space $M/\mathcal{F}$ is integrable and Kähler.

Example 2.10: Let (M, J, ω) be a Vaisman manifold, $\theta^\sharp$ its Lee field. Consider the holomorphic foliation $\mathcal{F}$, generated by $\theta^\sharp$ and $J\theta^\sharp$. The form $\omega - \theta \wedge J\theta$ is transverse Kähler. Hence the Vaisman manifolds provide examples of transverse Kähler foliations ([Va], [Ts1]). Similarly, a Sasakian manifold has a transverse Kähler geometry associated to the foliation generated by the Reeb field.

A compact complex manifold of Vaisman type can have many Vaisman structures, still the Lee field is unique up to homothety:

Proposition 2.11. If g_1, g_2 are Vaisman metrics on the same compact manifold (M, J), then $\theta_1^{\sharp g_1} = c\theta_2^{\sharp g_2}$, for some real constant c.

Proof. The result was proven by Tsukada in [Ts2]. Here we include an alternative proof. Recall from [Ve] that for a Vaisman structure (g, J), the two-form

$$\eta := \omega - \theta \wedge J\theta$$

is exact and positive, with the null-space generated by $\langle \theta^\sharp, J\theta^\sharp \rangle$. It is the transverse Kähler form of $(M, \mathcal{F})$ (see Example 2.10). Let g_1, g_2 be Vaisman metrics, ω_1, ω_2 the corresponding Hermitian forms, θ_i and $\theta_i^\sharp$ the corresponding Lee forms and Lee fields. Consider the $(1,1)$-forms η_1, η_2, defined as above,

$$\eta_i := \omega_i - \theta_i \wedge J\theta_i.$$

Unless their null-spaces coincide, the sum $\eta_1 + \eta_2$ is strictly positive. Then

$$\int_M (\eta_1 + \eta_2)^{\dim M} > 0.$$

This is impossible, because η_i are exact. We obtained that the 2-dimensional bundles generated by $\theta_i^\sharp, J\theta_i^\sharp$ are equal:

$$\langle \theta_1^\sharp, J\theta_1^\sharp \rangle = \langle \theta_2^\sharp, J\theta_2^\sharp \rangle$$

This implies that $\theta_1^\sharp$, considered as a vector in $T^{1,0}(M)$, is proportional to $\theta_2^\sharp$ over $\mathbb{C}$.

$$\theta_1^\sharp = a\theta_2^\sharp + bJ\theta_2^\sharp, \quad a, b \in \mathbb{R}. \tag{2.1}$$

Since $\theta_i^\sharp$ is holomorphic, the proportionality coefficient is constant.

To finish the proof of Proposition 2.11, it remains to show that this proportionality coefficient is real. Here we use Remark 2.7: the orbits of $J\theta_1^\sharp$ should be pre-compact. From (2.1) we obtain

$$J\theta_1^\sharp = aJ\theta_2^\sharp - b\theta_2^\sharp.$$

But $aJ\theta_2^\sharp - b\theta_2^\sharp$ acts on the metric by a homothety, with a coefficient which is proportional to e^{-b}. Therefore, an orbit of this vector field is contained in a compact set if and only if $b = 0$.

■

Remark 2.12. Let $L_{\mathbb{C}} = L \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification of the weight bundle of the Vaisman manifold (M, J, g). The Lee form then is the connection form of the standard Hermitian connection in $L_{\mathbb{C}}$, and one can prove (see [Ve]) that its curvature can be identified with the above form $\eta = \omega - \theta \wedge J\theta$, hence it is exact.

3 Einstein-Weyl LCK manifolds

Einstein-Weyl structures are defined and studied for their own, see *e.g.* [CP]. Here we specialize the definitions to LCK structures.

The Levi-Civita connection ∇^g of g is not the best tool to study the conformal properties of an LCK manifold. Instead, the **Weyl connection** defined by

$$\nabla = \nabla^g - \frac{1}{2}\{\theta \otimes Id + Id \otimes \theta + g \otimes \theta^\sharp\}$$

is torsion-free and satisfies $\nabla g = \theta \otimes g$.

The Ricci tensor of the Weyl connection is not symmetric. Hence, to obtain the analogue of the Einstein condition one gives:

Definition 3.1. An LCK-manifold is **Einstein-Weyl** if the symmetric part of the Ricci tensor of the Weyl connection is proportional to the metric. An Einstein-Weyl LCK-manifold is also called **Hermitian Einstein-Weyl**.

Let ∇ be a Weyl connection on an LCK-manifold. One can see that ∇ is the covariant derivative associated to the connection one-form θ in the weight bundle L. Since θ is closed, we can take a covering $\widetilde{M}$ of M, with $\theta = df$, for some function f on $\widetilde{M}$. The Weyl connection becomes the Levi-Civita connection for the metric $e^{-f}g$ on $\widetilde{M}$. Since $\nabla(e^{-f}g) = \nabla(J) = 0$, $e^{-f}g$ is a Kähler metric. This way one obtains a Kähler covering of an LCK-manifold, starting from a Weyl connection. The converse construction is also clear: The Levi-Civita connection on a Kähler covering $\widetilde{M}$ of an LCK-manifold M is independent from homotheties, hence descends to M, and satisfies the conditions for Weyl connection.

This gives the following claim.

Claim 3.2. Let ∇ be a Weyl connection on a complex Hermitian manifold. Then ∇ satisfies the Einstein-Weyl condition if and only if ∇ is Ricci-flat on the Kähler covering of M.

■

Remark 3.3. Claim 3.2 also follows from Proposition 3.5 (below). Indeed, a trivialization of the weight bundle $L_{\mathbb{C}}$ induces a trivialization of canonical class $K = L_{\mathbb{C}}^{-n}$.

From a deep result of Gauduchon in [G], it follows that:

Theorem 3.4. Let (M, J, g) be a compact Einstein-Weyl LCK manifold. Then the Ricci tensor of the Weyl connection vanishes identically and the Lee form is parallel. In particular, (M, J, g) is Vaisman.

■

From Theorem 3.4, we obtain that all Kähler coverings of an Einstein-Weyl LCK-manifold are Ricci-flat. This property can be used as a definition of Einstein-Weyl LCK-manifolds.

The locally conformally Kähler Einstein-Weyl structures can be expressed in terms of the complexified weight bundle.

Proposition 3.5. ([Ve, Proposition 5.6]) Let M be an Einstein-Weyl LCK-manifold, K its canonical class, $L_{\mathbb{C}}$ its weight bundle. Consider K, $L_{\mathbb{C}}$ as Hermitian holomorphic bundles, with the metrics induced from M. Then $L_{\mathbb{C}}^n \cong K^{-1}$.

■

Let (M, J, g) be an Einstein-Weyl Vaisman LCK-manifold, and $\widetilde{M}$ its Kähler covering, which trivialises L. From Proposition 3.5, it is clear that $\widetilde{M}$ has trivial canonical class. Let Ω be a section of canonical class of $\widetilde{M}$ which is equivariant under the monodromy action. Such a section is unique up to a constant. Indeed, if Ω_1, Ω_2 are two equivariant sections of canonical class, the quotient $\frac{\Omega_1}{\Omega_2}$ is a holomorphic function on $\widetilde{M}$ which is invariant under monodromy, hence descends to a global holomorphic function on M. Therefore $\frac{\Omega_1}{\Omega_2} = const$. Rescaling Ω such that $|\Omega| = 1$, we obtain

$$\Omega \wedge \overline{\Omega} = \frac{1}{n!\, 2^n}\omega^n,$$

where $n = \dim_{\mathbb{C}} M$. In particular, given two Einstein-Weyl structures ω_1 and ω_2, we always have $\omega_1^n = \lambda\omega_2^n$, where λ is a positive constant. After rescaling, we may also assume that

$$\det \omega_1 = \det \omega_2, \tag{3.1}$$

where $\det \omega_i = \omega_i^n$, $n = \dim_{\mathbb{C}} M$.

4 Uniqueness of Einstein-Weyl structures

In this Section, we prove Theorem 1.1. Clearly, Theorem 1.1 follows from (3.1) combined with the following proposition.

Proposition 4.1. Let (M, J) be a compact complex manifold admitting two Vaisman metrics ω_1 and ω_2, such that $\det \omega_1 = \det \omega_2$. Then $\omega_1 = \omega_2$.

Proof: We start with the following claim, which is implied by Tsukada's theorem (Proposition 2.11).

Claim 4.2. In these assumptions, denote the corresponding Lee fields by $\theta_i^\sharp$, $i = 1, 2$. Then

$$\theta_1^\sharp = \theta_2^\sharp.$$

Proof: By Proposition 2.11, $\theta_1^\sharp = c\theta_2^\sharp$. Denote by $\widetilde{\omega}_i$ the Kähler forms on $\widetilde{M}$ corresponding to ω_i. By construction, $\operatorname{Lie}_{\theta_i^\sharp} \omega_i = 2\omega_i$, where Lie denotes the Lie derivative. Therefore,

$$\operatorname{Lie}_{\theta_i^\sharp} \omega_i^n = 2n\omega_i^n$$

Using (3.1), we obtain that

$$2n\omega_1^n = \operatorname{Lie}_{\theta_1^\sharp} \omega_1^n = c \operatorname{Lie}_{\theta_2^\sharp} \omega_1^n = 2nc\omega_1^n$$

Therefore, $c = 1$. We proved Claim 4.2. ■

Return to the proof of Proposition 4.1. Consider a form

$$\eta_i := \omega_i - \theta_i \wedge J\theta_i. \tag{4.1}$$

This is a positive, exact $(1,1)$-form on M, which can be interpreted as a curvature of the weight bundle (see the proof of Proposition 2.11). First of all, we deduce from $\eta_1 = \eta_2$ the statement of Proposition 4.1.

Lemma 4.3. In the assumptions of Proposition 4.1, assume that $\eta_1 = \eta_2$, where η_i are $(1,1)$-forms defined in (4.1). Then $\omega_1 = \omega_2$.

Proof. As follows from (4.1), to prove $\omega_1 = \omega_2$ it suffices to show $\theta_1 = \theta_2$. Let $\widetilde{M}$ be the Kähler $\mathbb{Z}$-covering of M, which is a cone over a compact

Sasakian manifold, and φ_1, φ_2 the corresponding Kähler potentials, obtained as in Remark 2.8. It is easy to see that $\theta_i = d\log\varphi_i$ and $\eta_i = d^c\theta_i$ ([Ve]). Therefore,

$$\eta_1 - \eta_2 = d^c d \log\left(\frac{\varphi_1}{\varphi_2}\right) \tag{4.2}$$

The functions φ_i are automorphic under the deck transform action on $\widetilde{M}$, with the same factors of monodromy. Therefore, their quotient $\frac{\varphi_1}{\varphi_2}$ is well defined on M. By (4.2), $0 = \eta_1 - \eta_2 = d^c d\log\left(\frac{\varphi_1}{\varphi_2}\right)$, hence $\psi := \log\frac{\varphi_1}{\varphi_2}$ is pluriharmonic on a compact complex manifold M. Therefore ψ is constant. This gives $\theta_1 - \theta_2 = d\psi = 0$. Lemma 4.3 is proven. ■

Return to the proof of Proposition 4.1. Note that η_i are transverse Kähler forms. Since

$$\det\eta_i = (\theta^\sharp \wedge J\theta^\sharp)\rfloor \det\omega_i,$$

it follows that

$$\det\eta_1 = \det\eta_2.$$

Let ρ be a transverse form, defined as $\rho = \sum_{k+l=n-2}\eta_1^k\wedge\eta_2^l$. Then

$$(\eta_1 - \eta_2)\wedge\rho = 0. \tag{4.3}$$

As η_i are both positive, ρ is strictly positive, transversal $(n-2, n-2)$-form. It is well known that on a complex manifold X, any positive $(\dim X - 1, \dim X - 1)$-form is an $(\dim X - 1)$-st power of a Hermitian form. Therefore, there exists a transverse form α such that $\rho = \alpha^{n-2}$. Then (4.3) gives

$$(\eta_1 - \eta_2)\wedge\alpha^{n-2} = 0.$$

From (4.2), we obtain

$$\eta_1 - \eta_2 = dd^c\psi,$$

where $\psi := \log\left(\frac{\varphi_1}{\varphi_2}\right)$ is a smooth, transversal function on M.

We now associate to α a second-order differential operator $\mathcal{D}$ acting on transverse $\mathcal{C}^\infty$ functions, which is defined as follows. For any transverse function f, $dd^cf\wedge\alpha^{n-2}$ is a transverse top $(n-1, n-1)$ form, and hence there exists a unique transverse function g such that $dd^cf\wedge\alpha^{n-2} = g\cdot\alpha^{n-1}$. We define

$$\mathcal{D}(f) = g, \quad \text{where } dd^cf\wedge\alpha^{n-2} = g\cdot\alpha^{n-1}.$$

In other words,

$$\mathcal{D}(f) = \frac{dd^c f \wedge \alpha^{n-2}}{\alpha^{n-1}}.$$

From the definition, we have $D(\psi) = 0$. Obviously $\mathcal{D}$ has positive symbol on the ring of transverse functions, identified locally with functions on a space of leaves of $\mathcal{F}$.[1] This allows us to apply the generalized maximum principle:

Proposition 4.4. ([PW]) Let $\mathcal{D}$ be a second order differential operator on $\mathbb{R}^n$ with positive symbol, satisfying $\mathcal{D}(const.) = 0$, and let $f \in \ker \mathcal{D}$ be a function in its kernel. Assume that f has a local maximum. Then f is constant. ∎

Return to the proof of Theorem 1.1. Recall that from (4.2), we have

$$\eta_1 - \eta_2 = d^c d \log(\psi), \quad \psi \in \ker \mathcal{D}.$$

To show that $\eta_1 = \eta_2$ it is enough to prove that the kernel of $\mathcal{D}$ contains only constant functions. As follows from the generalized maximum principle, a function in $\ker \mathcal{D}$ which has a local maximum is necessarily constant. Since M is compact, any continuous function on M must have a maximum. Therefore, $\psi \in \ker \mathcal{D}$ is constant, and $\eta_1 - \eta_2 = dd^c\psi = 0$. The proof of Theorem 1.1 is finished.

All locally conformal Kähler structures underlying a locally conformally hyperkähler structure on a compact hypercomplex manifold are necessarily Einstein-Weyl. This gives

Corollary 4.5. Let (M, I_1, I_2, I_3) be a compact hypercomplex manifold. Then it can admit at most one locally conformally hyperkähler structure. ∎

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[1] In fact, the symbol of $\mathcal{D}$ is equal to the symmetric, positive definite 2-form associated with α.

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