POLYTOPAL LINEAR RETRACTIONS

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Abstract. We investigate graded retracts of polytopal algebras (essentially the homogeneous rings of affine cones over projective toric varieties) as polytopal analogues of vector spaces. In many cases we show that these retracts are again polytopal algebras and that codimension 1 retractions factor through retractions preserving the semigroup structure. We expect that these results hold in general.

This paper is a part of the project started in [BG1, BG2], where we have investigated the graded automorphism groups of polytopal algebras. Part of the motivation comes from the observation that there is a reasonable 'polytopal' generalization of linear algebra (and, subsequently, that of algebraic K-theory).

1. Introduction

The category Vect_k of finite-dimensional vector spaces over a field k has a natural extension that we call the *polytopal k-linear category* Pol_k. The objects of Pol_k are the *polytopal semigroup algebras* (or just *polytopal algebras*) in the sense of [BGT]. That is, an object $A \in |\mathrm{Pol}_k|$ is (up to graded isomorphism) a standard graded k-algebra $k[P]$ associated with a lattice polytope $P \subset \mathbb{R}^n$. The morphisms of Pol_k are the homogeneous k-algebra homomorphisms. Here, as usual, 'standard graded' means 'graded and generated in degree 1', and a lattice polytope is a finite *convex* polytope whose vertices are lattice points of the integral lattice $\mathbb{Z}^n \subset \mathbb{R}^n$.

The generators of $k[P]$ correspond bijectively to the lattice points in P, and their relations are the binomials representing the affine dependencies of the lattice points. More precisely, let L_P denote the set of lattice points in P. Then S_P is the subsemigroup of $\mathbb{Z}^{n+1}$ generated by

$$\mathrm{E}_P = \{(x,1) \mid x \in \mathrm{L}_P\}.$$

and $k[P] = k[S_P]$ is the semigroup algebra of S_P with coefficients in k (in [BGT, BG1, BG2] we have always used the notation $k[S_P]$). Unless specified otherwise, we will identify the sets L_P and E_P.

The category Pol_k contains Vect_k as a full subcategory. In fact, we can identify a vector space with the degree 1 component of its symmetric algebra, which, upon the choice of a basis, can be considered as a polynomial ring $k[X_1, \dots, X_n]$. This polynomial ring is isomorphic to the polytopal algebra $k[\Delta_{n-1}]$ defined by the $(n-1)$-simplex Δ_{n-1}. Vector space homomorphisms extend to homomorphisms of

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symmetric algebras, and thus to homomorphisms of the corresponding polytopal algebras.

Observe that polytopal algebras are essentially the homogeneous coordinate rings of projective normal toric varieties. Actually, one needs the extra condition of *very ampleness* for the underlying polytope to get such a homogeneous coordinate ring (see [BG1,§5] for details).

Our investigation of polytopal algebras is motivated by two closely related goals: (1) to find the connections between the combinatorial structure of P and the algebraic structure of $k[P]$, and (2) to extend theorems valid in Vect_k to Pol_k.

It follows from [Gu] that an algebra isomorphism of $k[P]$ and $k[Q]$ implies the isomorphism of P and Q as lattice polytopes. This result identifies the objects of the category Pol of lattice polytopes with the objects of Pol_k (up to isomorphism), but there remains the question to which extent the morphisms in Pol_k are determined by those in Pol, namely the $\mathbb{Z}$-affine maps between lattice polytopes. Similarly one must ask whether certain classes of morphisms in Pol_k can be described in the same way as the corresponding classes in Vect_k.

As we have shown in [BG1], there is a total analogy with the linear situation for the automorphism groups in Pol_k (called *polytopal linear groups* in [BG1]): they are generated by elementary automorphisms (generalizing elementary matrices), toric automorphisms (generalizing diagonal invertible matrices) and automorphisms of the underlying polytope; moreover, there are normal forms for such representations of arbitrary automorphisms. (In [BG2] this analogy has been extended to automorphisms of so-called polyhedral algebras.)

In the present paper we study *retractions* in Pol_k, that is, idempotent homogeneous endomorphisms of polytopal algebras and their images. Our results support the following two conjectures:

Conjecture A. *The images of retractions in* Pol_k *are polytopal algebras.*

Conjecture B. *A codimension 1 retraction factors through either a facet projection or an affine lattice retraction of the underlying lattice polytope.*

These conjectures generalize the standard facts that every finitely generated vector space has a basis and that an idempotent matrix is conjugate to a sub-unit matrix (ones and zeros on the main diagonal and zeros anywhere else). Conjecture B must be restricted to codimension 1 since there exist counterexamples for higher codimension.

In the following we say that a retraction h of a polytopal algebra is *polytopal* if $\mathrm{Im}(h)$ is a polytopal algebra, and codimension 1 retractions satisfying the stronger condition of Conjecture B are called *tame* (the notion of tameness has a natural generalization to higher codimension). We will provide evidence supporting Conjectures A and B by proving them in number of special cases (we always assume that the field k is algebraically closed):

(1) If $\dim \mathrm{Im}(h) \leq 2$, then h is polytopal (Theorem 2.2).
(2) Retractions of normal polytopal algebras C that arise from a tensor product decomposition $A \otimes B = C$ of graded k-algebras are polytopal; in other words,

graded tensor factors of normal polytopal algebras are again polytopal (Theorem 3.1).

(3) If $\mathrm{Im}(h) = k[Q]$ for a lattice subpolytope Q of P such that Q meets the interior of P, then h is tame (Theorem 7.2).
(4) If $\mathrm{Im}(h) = k[Q]$, but the subpolytope Q is contained in the boundary of P, then h need not be tame (Examples 5.2 and 5.3) in general. However, it is tame if $\mathrm{codim}(h) = 1$ (Theorem 7.3).
(5) If h is a codimension 1 retraction such that $\mathrm{gr.\,aut}_k(\mathrm{Im}(h))$ contains a torus of dimension $\dim \mathrm{Im}(h)$ whose action can be extended algebraically to $k[P]$, then h is tame (Theorem 7.5).
(6) If $\dim(P) \le 2$, then all codimension 1 retractions of $k[P]$ are tame (Theorem 8.1).

In Section 6 we discuss the class of *segmentonomial* ideals, that is, ideals generated by polynomials f whose Newton polytope has dimension ≤ 1. Section 4 contains a review of the results of [BG1].

Conventions and terminology. The basic facts about affine semigroup rings can be found in [BH, Chap. 6] and [Gi]. For toric varieties we refer the reader to [Fu, Oda]. Newton polytopes are treated in [GKZ].

By k we will always denote a field, and we set $k^* = k \setminus \{0\}$. $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ are the rational, integer and real numbers and $\mathbb{Z}_+$, $\mathbb{Q}_+$, $\mathbb{R}_+$ are the sub-semigroups of non-negative elements.

An *affine* semigroup is a finitely generated subsemigroup of a free abelian group. The group of differences of a semigroup S will be denoted by $\mathrm{gp}(S)$. An affine semigroup is *positive* if its maximal subgroup is trivial. A positive affine semigroup can be embedded into a free commutative semigroup $\mathbb{Z}_+^m$, thus the semigroup ring $k[S]$ has a (non-unique) grading $k[S] = k \oplus A_1 \oplus A_2 \oplus \cdots$ making all nontrivial elements of S homogeneous of positive degree. The uniquely determined minimal generating set of a positive affine semigroup S coincides with its set of indecomposable elements.

In $\mathbb{R} \otimes \mathrm{gp}(S)$ the semigroup S spans the cone $C(S)$.

An affine semigroup S is called *homogeneous* if S does not have nontrivial invertibles and its irreducible elements are contained in some proper hyperplane of $\mathbb{R} \otimes \mathrm{gp}(S)$. This condition is equivalent to the requirement that the semigroup ring $k[S]$ is a standard graded ring in which S consists of homogeneous elements. Alternatively, there is a lattice polytope P and a subset $X \subset \mathrm{L}_P$ such that $k[S]$ is isomorphic to the subalgebra of $k[P]$ generated by X.

We put $\mathrm{rank}(S) = \mathrm{rank}(\mathrm{gp}(S))$. An affine semigroup S is called *normal* if

$$\forall x \in \mathrm{gp}(S),\ (c \in \mathbb{N},\ cx \in S) \Rightarrow (x \in S).$$

The normality of S is equivalent to that of $k[S]$.

The elements of an affine semigroup S will be called *terms* of $k[S]$, while the elements of type $as \in k[S]$, $a \in k$, $s \in S$, will be called *monomials*.

For an affine semigroup S we have the embeddings

$$S \subset \mathrm{gp}(S) \subset \mathbb{R} \otimes \mathrm{gp}(S) \approx \mathbb{R}^{\mathrm{rank}(S)}.$$

Therefore, we can speak of the *convex hull* $\operatorname{conv}(X)$ of a subset $X \subset S$ in $\mathbb{R} \otimes \operatorname{gp}(S)$. For an element $f \in k[S]$ there is a unique canonical representation

$$f = \sum_{s \in S} a_s s, \qquad a_s \in k.$$

The polytope

$$\operatorname{conv}(\{s \in S \mid a_s \neq 0\}) \subset \mathbb{R} \otimes \operatorname{gp}(S)$$

is called the *Newton polytope* of f, which we denote by $\mathrm{N}(f)$. We will frequently use the formula

$$\mathrm{N}(fg) = \mathrm{N}(f) + \mathrm{N}(g),$$

where '+' denotes the *Minkowski sum* of convex polytopes:

$$P + Q = \{x + y \mid x \in P,\ y \in Q\} \subset \mathbb{R}^n, \quad P \subset \mathbb{R}^n,\ Q \subset \mathbb{R}^n.$$

As remarked already, a lattice polytope $P \subset \mathbb{R}^n$ is a finite convex polytope whose vertices belong to the integral lattice $\mathbb{Z}^n$. It is called *normal* if $k[P]$ is normal.

Let $S \subset \mathbb{Z}^n$ be an affine semigroup. The semigroup operation in S is written additively when we do linear algebra in $\mathbb{Z}^n$, and multiplicatively when S is considered as the set of terms in $k[S]$.

Finally, we adopt the blanket assumption that the field k is *algebraically closed*, unless specified otherwise.

2. Retracts of dimension two

A *retract* of a k-algebra A is an algebra B such that there exist k-homomorphisms $f : B \to A$ and $g : A \to B$ with $g \circ f = 1_B$. This is equivalent to saying that there is an endomorphism $h : A \to A$ such that $h^2 = h$ and $\operatorname{Im}(h) \approx B$. We will call such g and h *retractions* and will frequently make passages between the two equivalent definitions. Moreover, all the retractions considered below are supposed to be graded. For a retraction h as above we put

$$\operatorname{codim}(h) = \dim(A) - \dim(B).$$

The arguments used in the sequel need k to be algebraically closed. We do not know whether the following implication holds for a retraction $h : A \to A$, $A \in |\operatorname{Pol}_k|$, over a not necessarily algebraically closed field k:

$$\bar{k} \otimes h \text{ is polytopal (tame)} \implies h \text{ is polytopal (tame)}$$

($\bar{k}$ is the algebraic closure).

That polytopality is in general not an invariant property under scalar extension/restriction is exhibited by the following

Example 2.1. Consider the standard graded $\mathbb{R}$-algebra

$$A = \mathbb{R}[X, Y, Z]/(X^2 + Y^2 + Z^2).$$

Then A is a factorial non-polytopal algebra over $\mathbb{R}$ while $\mathbb{C} \otimes A$ is isomorphic to the polytopal algebra $\mathbb{C}[2\Delta_1]$ defined by a lattice segment $2\Delta_1$ of length 2.

The factoriality of A is proved in [Fo,§11]. But the only factorial polytopal algebras (over any field) are polynomial algebras – an easy observation. Hence A is not polytopal because it is singular at the irrelevant maximal ideal. But we have the isomorphism

$$\alpha : \mathbb{C}[U^2, UV, V^2] = \mathbb{C}[2\Delta_1] \to \mathbb{C} \otimes A$$

defined by $U^2 \mapsto X + iY$, $V^2 \mapsto X - iY$, $UV \mapsto iZ$.

Conjecture A holds in Krull dimension ≤ 2:

Theorem 2.2. *A retract B of a polytopal algebra A is polytopal if* $\dim B \le 2$.

The crucial step in the proof is

Proposition 2.3. *Let k be an algebraically closed field and A a standard graded k-algebra of dimension 2. If A is a normal domain and the class group $\mathrm{Cl}(A)$ is finitely generated, then A is isomorphic to $k[c\Delta_1]$ as a graded k-algebra for some $c \in \mathbb{N}$ (as usual, Δ_1 is the unit segment).*

Proof. We have the projectively normal embedding of $\mathrm{Proj}(A)$ given by A. Therefore, the projective curve $\mathrm{Proj}(A)$ is normal and thus smooth. Consider the exact sequence

$$0 \to \mathbb{Z} \to \mathrm{Cl}(\mathrm{Proj}(A)) \to \mathrm{Cl}(A) \to 0.$$

of Weil divisors arising from viewing $\mathrm{Spec}(A)$ as a cone over $\mathrm{Proj}(A)$ ([Ha, Ex. II.6.3(b)]). Since $\mathrm{Cl}(A)$ is finitely generated, so is $\mathrm{Cl}(\mathrm{Proj}(A))$. In particular the Jacobian $\mathcal{J}(\mathrm{Proj}(A)) \approx \mathrm{Cl}^0(\mathrm{Proj}(A))$ is trivial (Cl^0 denotes degree zero divisor classes). Therefore the genus of $\mathrm{Proj}(A)$ is 0, or equivalently $\mathrm{Proj}(A) \approx \mathbb{P}^1_k$. Using the normality of A once again we get

$$A \approx \bigoplus_{i=0}^{\infty} H^0(\mathbb{P}^1_k, \mathcal{L}^{\otimes i})$$

for some very ample line bundle $\mathcal{L}$ on $\mathbb{P}^1_k$. But due to the equality $\mathrm{Pic}(\mathbb{P}^1_k) = \mathbb{Z}$ such a line bundle is a positive multiple of $\mathcal{O}(1)$, and hence A is the Veronese subalgebra of the polynomial algebra $k[\Delta_1]$ of some level $c \in \mathbb{N}$. $\square$

Proof of Theorem 2.2. In case $\dim(B) = 1$ it is easy to see that $B \approx k[X]$.

Consider the case $\dim(B) = 2$. We write $A = k[P]$ and denote the retraction $A \to B \subset A$ by g. Consider the set $(S_P \cap \mathrm{Ker}(g)) \subset k[P]$ of monomials. There is a unique face $F \subset P$ such that $(S_P \cap \mathrm{Ker}(g)) = (S_P \setminus S_F)$ and $k[P]/(S_P \cap \mathrm{Ker}(g))$ is naturally isomorphic to $k[F]$. Then g is a composite of the two retractions

$$k[P] \xrightarrow{\pi} k[F] \xrightarrow{\rho} B$$

where ρ is the homomorphism induced by g. Observe that ρ is in fact a retraction as it is split by $\pi|_B$.

Therefore we can from the beginning assume that $(S_P \cap \mathrm{Ker}(g)) = 0$. In this situation g extends (uniquely) to the normalizations

$$\overline{k[P]} = k[\bar{S}_P] \xrightarrow{\bar{g}} \bar{B}. \tag{1}$$

This extension is given by

$$\forall z \in \bar{S}_P, \quad \bar{g}(z) = \frac{g(x)}{g(y)}, \quad \text{where } x, y \in S_P \text{ and } z = \frac{x}{y}.$$

It is known that the semigroup S_{nP} is normal for all natural numbers $n \geq \dim(P)-1$ [BGT]. Therefore, by restricting the retraction (1) to the nth Veronese subalgebra for such a number n, we get the retraction

$$k[nP] \xrightarrow{\bar{g}_n} \bar{B}_{(n)}. \tag{2}$$

Let us show that $\mathrm{Cl}(\bar{B}_{(n)})$ is finitely generated for $n \geq \dim(P)-1$. We choose a lattice point x of S_{nP} that is in the interior of the cone $C(S_P)$. By localization (2) gives rise to the retraction

$$(x\bar{g}_n(x))^{-1}k[nP] \to (\bar{g}_n(x))^{-1}\bar{B}_{(n)}. \tag{3}$$

Since $(x\bar{g}_n(x))^{-1}k[nP]$ is a localization of the Laurent polynomial ring $x^{-1}k[nP] = k[\mathrm{gp}(S_{nP})]$, it is a factorial ring. Then its retract $(\bar{g}_n(x))^{-1}\bar{B}_{(n)}$ is factorial as well (for example, see [Cos]). By Nagata's theorem [Fo] $\mathrm{Cl}(\bar{B}_{(n)})$ is generated by the classes of the height 1 prime ideals of $\bar{B}_{(n)}$ containing $\bar{g}_n(x)$ – a finite set.

It is also clear from (2) that $\bar{B}_{(n)}$ is generated in degree 1. Consequently, by Proposition 2.3 for each $n \geq \dim(P)-1$ there is a natural number c_n and an isomorphism

$$\varphi_n : \bar{B}_{(n)} \to k[c_n\Delta_1].$$

We now fix such a number n. Restricting φ_n and φ_{n+1} to the iterated Veronese subalgebra $\bar{B}_{(n(n+1))} = (\bar{B}_{(n)})_{(n+1)} = (\bar{B}_{(n+1)})_{(n)}$ we obtain two isomorphisms of $\bar{B}_{(n(n+1))}$ with $k[c_{n(n+1)}\Delta_1]$. It follows that there exists $c \in \mathbb{N}$ with $c_n = cn$ and $c_{n+1} = c(n+1)$, and furthermore the restrictions of φ_n and φ_{n+1} differ by an automorphism of $k[c_{n(n+1)}\Delta_1]$. However, each automorphism of $k[c_{n(n+1)}\Delta_1]$ can be lifted to an automorphism of $k[\Delta_1]$, and then restricted to all Veronese subrings of $k[\Delta_1]$. (This follows from the main result of [BG1].) Therefore we can assume that the restrictions of φ_n and φ_{n+1} coincide. Then they define an isomorphism of the subalgebra V of $\bar{B}$ generated by the elements in degree n and $n+1$ to the corresponding subalgebra of $k[\Delta_1]$; see Lemma 2.4 below. Taking normalizations yields an isomorphism $\bar{B} \approx k[c\Delta_1]$. But then $B = k[c\Delta_1]$ as well, because $\bar{B}$ and B coincide in degree 1 (being retracts of algebras with this property). □

Lemma 2.4. *Let A and B be $\mathbb{Z}$-graded rings. Suppose that B is reduced. If the homogeneous homomorphisms $\varphi : A_{(n)} \to B_{(n)}$ and $\psi : A_{(n+1)} \to B_{(n+1)}$ coincide on $A_{(n(n+1))}$, then they have a common extension to a homogeneous homomorphism $\chi : V \to B$, where V is the subalgebra of A generated by $A_{(n)}$ and $A_{(n(n+1))}$. If, in addition, A is reduced and φ and ψ are injective, then χ is also injective.*

Proof. One checks easily that one only needs to verify the following: if $uv = u'v'$ for homogeneous elements $u, u' \in A_{(n)}$, $v, v' \in A_{(n+1)}$, then $\varphi(u)\psi(v) = \varphi(u')\psi(v')$. As B is reduced, it is enough that $(\varphi(u)\psi(v) - \varphi(u')\psi(v'))^{n(n+1)} = 0$. Since

$$u^p(u')^{n(n+1)-p},\ v^p(v')^{n(n+1)-p} \in A_{(n(n+1))}, \qquad p \in [0, n(n+1)],$$

this follows immediately from the hypothesis that φ and ψ coincide on $A_{(n(n+1))}$.

If A is reduced, then every non-zero homogeneous ideal in A intersects $A_{(n(n+1))}$ non-trivially, and this implies the second assertion. □

3. Tensor factors

In this section we examine a special case of conjecture A. Suppose

$$A = k \oplus A_1 \oplus A_2 \oplus \cdots \quad \text{and} \quad A = k \oplus B_1 \oplus B_2 \oplus \cdots .$$

are (standard) graded. Then, as usual, their tensor product (over k) is the (standard) graded algebra

$$A \otimes_k B = k \oplus C_1 \oplus C_2 \oplus \cdots , \quad C_h = \bigoplus_{i+j=h} (A_i \otimes_k B_j).$$

Clearly, A is a graded retract of $A \otimes B$ – one just considers the retraction

$$A \otimes B \xrightarrow{1_A \otimes \pi_B} A \otimes k = A,$$

where $\pi_B : B \to k$ is the augmentation with $\operatorname{Ker}(\pi_B)$ the irrelevant maximal ideal.

In the special case of polytopal algebras $A = k[P]$ and $B = k[Q]$ we have the following easily verified formula

$$k[P] \otimes k[Q] \approx k[\operatorname{join}(P, Q)].$$

We recall from [BG2] that a lattice polytope R is a join of two lattice polytopes P and Q if R has two faces isomorphic to P and Q as lattice polytopes, the affine hulls of these faces do not intersect each other in the affine hull of R, and $\mathrm{L}_R = \mathrm{L}_P \cup \mathrm{L}_Q$. Clearly, in this situation R is the convex hull of these faces. (Note that R is unique up to an isomorphism of lattice polytopes.)

Theorem 3.1. *If A and B are two standard graded k-algebras such that $A \otimes B \approx k[P]$ for some normal lattice polytope P, then both A and B are polytopal.*

In particular, we obtain the *splitting off property* of polytopal extensions:

$$\big(A \otimes k[Q] \approx k[P]\big) \Rightarrow (A \approx k[R] \text{ for some } R)$$

if P is a normal lattice polytope.

An essential step in the proof of Theorem 3.1 is the special case when one of the factors collapses into the cone over a point, that is a polynomial ring in one variable.

Proposition 3.2. *Let A be as in Theorem 3.1 and P be any lattice polytope. If $A[X] \approx k[P]$ as graded algebras (X a variable) then there is a polytope R for which $A \approx k[R]$.*

Remark 3.3. Let us call an ideal I *binomial* if it is generated by monomials αT and binomials $T - \beta T'$ where $\alpha, \beta \in k^*$ and T, T' denote terms. The arguments in the proof of Proposition 3.2, presented below, yield the following more general result: for a not necessarily algebraically closed field k and an ideal $I \subset k[X_1, \dots , X_n]$ the ideal

$$Ik[X_1, \dots , X_n, X] \subset k[X_1, \dots , X_n, X]$$

can be transformed into a binomial ideal by a linear automorphism of $k[X_1, \dots , X_n, X]$ if and only if this is possible for I in $k[X_1, \dots, X_n]$.

Several times we will use the following theorem [ES, 2.6] characterizing binomial prime ideals in affine semigroup rings over algebraically closed fields. (In [ES] the theorem is given only for polynomial rings, but the generalization is immediate.)

Theorem 3.4. *Let k be an algebraically closed field. A binomial ideal I in an affine semigroup ring $k[S]$ is prime if and only if the residue class ring $k[S]/I$ contains a (multiplicative) affine semigroup S' such that $k[S]/I = k[S']$ and, moreover, the natural epimorphism $k[S] \to k[S']$ maps the monomials in $k[S]$ to those in $k[S']$.*

Proof of Proposition 3.2. Since polytopal algebras are precisely the standard graded affine semigroup rings coinciding with their normalizations in degree 1 [BGT], we only need to show that A is a semigroup ring.

Theorem 3.4 reduces the proposition to the assertion on binomial ideals I discussed in Remark 3.3 where the ideal I to be considered is the kernel of a surjection $k[X_1, \dots, X_n] \to A$.

We set $X_{n+1} = X$. By hypothesis there exists a linear transformation Ψ,

$$\Psi(X_i) = t_{i1}X_1 + \dots + t_{in}X_n + t_{in+1}X_{n+1}, \qquad i \in [1, n+1], \quad t_{ij} \in k,$$

such that $\Psi(Ik[X_1, \dots, X_{n+1}])$ is binomial. For all $j \in [1, n]$ we consider the surjection

$$\pi_j : k[X_1, \dots, X_{n+1}] \to k[X_1, \dots, X_n], \quad \pi_j(X_i) = X_i, \ \pi_j(X_{n+1}) = X_j.$$

We define the linear endomorphisms ε_0 and ε_j, $j \in [1, n]$ of $k[X_1, \dots, X_n]$ by

$$\begin{aligned} \varepsilon_0(X_i) &= t_{i1}X_1 + \dots + t_{in}X_n, \quad i \in [1, n] \\ \varepsilon_j(X_i) &= t_{i1}X_1 + \dots + t_{in}X_n + t_{in+1}X_j, \quad i \in [1, n]. \end{aligned}$$

There exists j for which ε_j is an automorphism. In fact, let T denote the $(n+1) \times (n+1)$ matrix (t_{ij}). If ε_0 is not an automorphism, then the $n \times n$ submatrix T' of T consisting of the intersection of the first n rows with the first n columns of T has rank $n-1$ and the larger submatrix T'' consisting of the first n rows has rank n (otherwise $\operatorname{rank} T < n+1$).) There exists $j \in [1, n]$ such that the jth column of T' lies in the vector space U spanned by the remaining $n-1$ columns of T', but the $(n+1)$th column of T'' does not belong to U. If we replace the jth column of T' by its sum with $(n+1)$st column of T'', we obtain an $n \times n$ matrix of rank n, and this matrix defines the automorphism ε_j.

There exists a unique epimorphism $\nu : k[X_1, \dots, X_n, X_{n+1}] \to k[X_1, \dots, X_n]$ making the diagram

$$\begin{array}{ccc} k[X_1, \dots, X_n, X_{n+1}] & \xrightarrow{\Psi} & k[X_1, \dots, X_n, X_{n+1}] \\ \nu \big\downarrow & & \big\downarrow \pi_j \\ k[X_1, \dots, X_n] & \xrightarrow{\varepsilon_j} & k[X_1, \dots, X_n], \end{array}$$

commutative. Since $\nu(X_i) = X_i$ for $i \in [1, n]$, we have $\nu(Ik[X_1, \dots, X_n, X_{n+1}]) = I$, and π_j maps binomial ideals to binomial ideals. $\square$

Two more auxiliary facts:

Lemma 3.5. *Let S be a positive affine semigroup. Then the graded ring*

$$\operatorname{gr}_{\mathfrak{m}}(k[S]) = \operatorname{gr}_{\mathfrak{m}}(k[S]_{\mathfrak{m}})$$

is a domain if and only if S is homogeneous.

Proof. If S is not homogeneous, then among the relations between the irreducible elements of S there occurs an equation

$$s_1^{a_1}\cdots s_r^{a_r} = s_1^{b_1}\cdots s_r^{b_r}, \quad a_1+\cdots+a_r < b_1+\cdots+b_r,$$

for some $a_i, b_i \geq 0$. But then $\overline{s_1}^{a_1}\cdots\overline{s_r}^{a_r} = 0$ in $\operatorname{gr}_{\mathfrak{m}}(k[S])$, where $\bar{s}_i$ denotes the residue class of s_i in $\mathfrak{m}/\mathfrak{m}^2$. Since these are nonzero classes we get a contradiction.

The other implication is clear. □

Lemma 3.6. *Let S be a normal positive affine semigroup. For an edge E of the polyhedral cone $C(S)$ spanned by S in $\mathbb{R}\otimes\operatorname{gp}(S)$ let x_E be the generator of the subsemigroup $E\cap S \subset S$ (isomorphic to $\mathbb{Z}_+$) and let $x_E^{-1}S$ denote the sub-semigroup of* $\operatorname{gp}(S)$ *generated by $S\cup\{x_E^{-1}, x_E^{-2},\dots\}$. Then*

$$x_E^{-1}S \approx \mathbb{Z}\times S'$$

for some normal positive affine semigroup S'.

For example, this is proved in [Gu, §2].

Proof of Theorem 3.1. As A and B are retracts of $k[P]$ they have to be normal domains.

We denote by X and Y the affine normal toric varieties corresponding to A and B. The two irrelevant maximal ideals will be denoted by $\mathfrak{m}_A$ and $\mathfrak{m}_B$ respectively.

Let $y\in Y$ be any smooth point. Then for the local ring of the point $(\mathfrak{m}_A, y)\in X\times Y$ we have

$$\operatorname{gr}(\mathcal{O}_{(\mathfrak{m}_A,y)}) \approx A[Y_1,\dots,Y_{\dim(B)}].$$

as graded algebras (the Y_i are indeterminates over A).

On the other hand, identifying $X\times Y$ with the variety $\operatorname{maxSpec}(k[P])$ we see that there is a maximal ideal $\mathfrak{n}\subset k[P]$ such that

$$\operatorname{gr}(\mathcal{O}_{(\mathfrak{m}_A,y)}) \approx \operatorname{gr}_{\mathfrak{n}}(k[P]_{\mathfrak{n}}).$$

By an iterated use of Lemma 3.6 we get

$$k[P]_{\mathfrak{n}} \approx (k[S][T_1,T_1^{-1},\dots,T_s,T_s^{-1}])_{\mathfrak{r}}$$

for some positive normal affine semigroup S, an integer $s\geq 0$, and a maximal ideal $\mathfrak{r}\subset k[S][T_1,T_1^{-1},\dots,T_s,T_s^{-1}]$, containing $S\setminus\{1\}$. Since k is algebraically closed there are elements $a_1,\dots,a_s\in k^*$ such that $\mathfrak{r}$ is generated by the maximal monomial ideal $\mathfrak{r}_0\subset k[S]$ and $\{T_1-a_1,\dots,T_s-a_s\}$. Therefore one has a natural graded isomorphism

$$\operatorname{gr}_{\mathfrak{n}}(k[\bar{S}_P]_{\mathfrak{n}}) \approx (\operatorname{gr}_{\mathfrak{r}_0}(k[S]_{\mathfrak{r}_0}))[\tau_1,\dots,\tau_s]$$

for $\tau_i = T_i - a_i$.

Consequently,

$$A[Y_1,\dots,Y_{\dim(B)}] \approx (\operatorname{gr}_{\mathfrak{r}_0}(k[S]_{\mathfrak{r}_0}))[\tau_1,\dots,\tau_s]$$

as graded algebras. By Lemma 3.5 S is a homogeneous affine semigroup. Then, clearly, $S \approx S_Q$ for some normal polytope Q and we get the graded isomorphism

$$A[Y_1, \dots, Y_{\dim(B)}] \approx k[Q][\tau_1, \dots, \tau_s] \approx k[\operatorname{join}(Q, \Delta_{s-1}]]$$

(Δ_{s-1} denotes the $(s-1)$-unit simplex). By an iterated use of Proposition 3.2 we can split off the polynomial extension on the left. That is, A is polytopal. By symmetry B is also polytopal. □

Remark 3.7. As a consequence of 3.2 one can show that any retract of a polytopal algebra defined by a single equation is polytopal. In fact, suppose that $k[P] = k[X_1, \dots, X_n]/(b)$ where b is a binomial of degree $d > 0$ and denote the natural epimorphism from $k[X_1, \dots, X_n]$ onto $k[P]$ by π. Let h be a retraction of $k[P]$, $y_1, \dots, y_m$ a k-basis of $\operatorname{Im}(h)_1$ and $z_1, \dots, z_p$ a k-basis of $\operatorname{Ker}(h)_1$. Let $g : k[Y_1, \dots, Y_m] \to \operatorname{Im}(h)$ be given by $g(Y_i) = y_i$, and let γ be the (unique) epimorphism $k[X_1, \dots, X_n] \to k[Y_1, \dots, Y_m]$ with $g \circ \gamma = h \circ \pi$. Then $\operatorname{Ker}(g)$ is generated by $\pi(b)$. If $\pi(b) = 0$, then $\operatorname{Im}(h)$ is a polynomial ring. Otherwise $b' = \pi(b)$ is of degree d. Now we extend g to the epimorphism $g' : (k[Y_1, \dots, Y_m]/(b'))[Z_1, \dots, Z_p] \to k[P]$ with $g'(Z_i) = z_i$. Comparing Hilbert functions we see that g' is an isomorphism, and Proposition 3.2 shows that $\operatorname{Im}(h) \approx k[Y_1, \dots, Y_m]/(b')$ is polytopal.

4. Polytopal linear groups

Here we survey the relevant part of [BG1] that will be used in the sequel. Throughout this section k is a general (not necessarily algebraically closed) field.

For a lattice polytope P the group $\Gamma_k(P) = \operatorname{gr.aut}(k[P])$ is a linear k-group in a natural way. It coincides with $\mathrm{GL}_n(k)$ in the case P is the unit $(n-1)$-simplex Δ_{n-1}. The groups $\Gamma_k(P)$ are called *polytopal linear groups* in [BG1].

An element $v \in \mathbb{Z}^d$, $v \neq 0$, is a *column vector* (for P) if there is a facet $F \subset P$ such that $x + v \in P$ for every lattice point $x \in P \setminus F$. The facet F is called *the base facet* of v. The set of column vectors of P is denoted by $\operatorname{Col}(P)$. A pair (P, v),

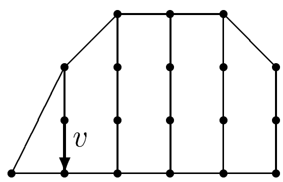

Figure 1. A column structure

$v \in \operatorname{Col}(P)$, is called a *column structure.* Let (P, v) be a column structure and $P_v \subset P$ be the base facet for $v \in \operatorname{Col}(P)$. Then for each element $x \in S_P$ we set $\operatorname{ht}_v(x) = m$ where m is the largest non-negative integer for which $x + mv \in S_P$. Thus $\operatorname{ht}_v(x)$ is the 'height' of x above the facet of the cone $C(S_P)$ corresponding to P_v in direction $-v$. It is an easy observation that $x + \operatorname{ht}_v(x) \cdot v \in S_{P_v} \subset S_P$ for any $x \in S_P$.

Let (P, v) be a column structure and $\lambda \in k$. We identify the vector v, representing the difference of two lattice points in P, with the degree 0 element $(v, 0) \in \operatorname{gp}(S_P) \subset$

$k[\mathrm{gp}(S_P)]$. Then the assignment

$$x \mapsto (1+\lambda v)^{\mathrm{ht}_v(x)} x.$$

gives rise to a graded k-algebra automorphism e_v^λ of $k[P]$. Observe that e_v^λ becomes an elementary matrix in the special case when $P = \Delta_{n-1}$, after the identifications $k[\Delta_{n-1}] = k[X_1, \dots, X_n]$ and $\Gamma_k(P) = \mathrm{GL}_n(k)$. Accordingly e_v^λ is called an *elementary automorphism.*

In the following $\mathbb{A}_k^s$ denotes the additive group of the s-dimensional affine space.

Proposition 4.1. *Let $v_1, \dots, v_s$ be pairwise different column vectors for P with the same base facet $F = P_{v_i}$, $i = 1, \dots, s$.*

(a) *The mapping*

$$\varphi : \mathbb{A}_k^s \to \Gamma_k(P), \qquad (\lambda_1, \dots, \lambda_s) \mapsto e_{v_1}^{\lambda_1} \circ \cdots \circ e_{v_s}^{\lambda_s},$$

is an embedding of algebraic groups. In particular, $e_{v_i}^{\lambda_i}$ and $e_{v_j}^{\lambda_j}$ commute for all $i, j \in \{1, \dots, s\}$ and $(e_{v_1}^{\lambda_1} \circ \cdots \circ e_{v_s}^{\lambda_s})^{-1} = e_{v_1}^{-\lambda_1} \circ \cdots \circ e_{v_s}^{-\lambda_s}$.

(b) *For $x \in \mathrm{L}_P$ with $\mathrm{ht}_{v_1}(x) = 1$ one has*

$$e_{v_1}^{\lambda_1} \circ \cdots \circ e_{v_s}^{\lambda_s}(x) = (1 + \lambda_1 v_1 + \cdots + \lambda_s v_s)x.$$

The image of the embedding φ given by Lemma 4.1 is denoted by $\mathbb{A}(F)$. Of course, $\mathbb{A}(F)$ may consist only of the identity map of $k[P]$, namely if there is no column vector with base facet F.

Put $n = \dim(P)+1$. The n-torus $\mathbb{T}_n = (k^*)^n$ acts naturally on $k[P]$ by restriction of its action on $k[\mathrm{gp}(S_P)]$ that is given by

$$(\xi_1, \dots, \xi_{n+1})(e_i) = \xi_i e_i, \quad i \in [1, n].$$

Here e_i is the i-th element of a fixed basis of $\mathrm{gp}(S_P) = \mathbb{Z}^n$. This gives rise to an algebraic embedding $\mathbb{T}_n \subset \Gamma_k(P)$, whose image we denote by $\mathbb{T}_k(P)$. It consists precisely of those automorphisms of $k[P]$ which multiply each monomial by a scalar from k^*.

The (finite) automorphism group $\Sigma(P)$ of the semigroup S_P is also a subgroup of $\Gamma_k(P)$.

Theorem 4.2. *Let P be a convex lattice n-polytope and k a field. Every element $\gamma \in \Gamma_k(P)$ has a (not uniquely determined) presentation*

$$\gamma = \alpha_1 \circ \alpha_2 \circ \cdots \circ \alpha_r \circ \tau \circ \sigma,$$

where $\sigma \in \Sigma(P)$, $\tau \in \mathbb{T}_k(P)$, and $\alpha_i \in \mathbb{A}(F_i)$ such that the facets F_i are pairwise different and $\#L_{F_i} \le \#L_{F_{i+1}}$, $i \in [1, r-1]$.

For any positive affine semigroup S and any grading

$$k[S] = k \oplus A_1 \oplus A_2 \oplus \cdots,$$

making all non-unit monomials homogeneous of positive degree, the automorphisms of $k[S]$ that multiply monomials by scalars from k^* constitute a closed $\mathrm{rank}(S)$-torus

$$\mathbb{T}_k(S) \subset \mathrm{gr.\,aut}_k(k[S]), \quad \mathbb{T}_k(S) \approx (k^*)^{\mathrm{rank}(S)}.$$

This torus is naturally identified with the set of k-rational points of the open subscheme

$$\operatorname{Spec}(k[\operatorname{gp}(S)]) \subset \operatorname{Spec}(k[S]).$$

and is called the *embedded torus*. In case k is infinite, the embedded torus is maximal (by literally the same arguments as at the end of §3 in [BG1]).

5. The structure of retractions

Now we first consider Conjecture B in detail and then observe that it does not admit a direct extension to codimension ≥ 2.

Let $P \subset \mathbb{R}^n$ be a lattice polytope of dimension n and $F \subset P$ a face. Then there is a uniquely determined retraction

$$\pi_F : k[P] \to k[F], \quad \pi_F(x) = 0 \quad \text{for } x \in \mathrm{L}_P \setminus F.$$

Retractions of this type will be called *face retractions* and *facet retractions* if F is a facet or, equivalently, $\operatorname{codim}(\pi_F) = 1$.

Now suppose there are an affine subspace $H \subset \mathbb{R}^n$ and a vector subspace $W \subset \mathbb{R}^n$ with $\dim W + \dim H = n$, such that

$$\mathrm{L}_P \subset \bigcup_{x \in \mathrm{L}_P \cap H} (x + W).$$

(Observe that $\dim(H \cap P) = \dim H$.) The triple (P, H, W) is called a *lattice fibration of codimension* $c = \dim W$, whose *base polytope* is $P \cap H$; its *fibers* are the maximal lattice subpolytopes of $(x + W) \cap P$, $x \in \mathrm{L}_P \cap H$ (the fibers may have smaller dimension than W). P itself serves as a *total polytope* of the fibration. If $W = \mathbb{R}w$ is a line, then we call the fibration *segmental* and write (P, H, w) for it. Note that the

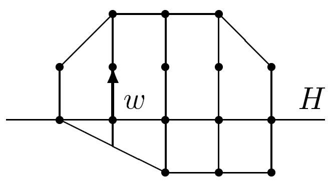

Figure 2. A lattice segmental fibration

column structures introduced in Section 4 give rise to lattice segmental fibrations in a natural way.

For a lattice fibration (P, H, W) let $L \subset \mathbb{Z}^n$ denote the subgroup spanned by L_P, and let H_0 be the translate of H through the origin. Then one has the direct sum decomposition

$$L = (L \cap W) \oplus (L \cap H_0).$$

Equivalently,

$$\operatorname{gp}(S_P) = L \oplus \mathbb{Z} = \big(\operatorname{gp}(S_P) \cap W_1\big) \oplus \operatorname{gp}(S_{P \cap H_1}).$$

where W_1 is the image of W under the embedding $\mathbb{R}^n \to \mathbb{R}^{n+1}$, $w \mapsto (w, 0)$, and H_1 is the vector subspace of $\mathbb{R}^{n+1}$ generated by all the vectors $(h, 1)$, $h \in H$.

For a fibration (P, H, W) one has the naturally associated retraction:

$$\rho_{(P,H,W)} : k[P] \to k[P \cap H];$$

it maps L_P to $\mathrm{L}_{P\cap H}$ so that fibers are contracted to their intersection points with the base polytope $P \cap H$.

Clearly, if $f : k[P] \to k[P]$ is a retraction, then for any graded automorphism α of $k[P]$ the composite map $f^\alpha = \alpha \circ f \circ \alpha^{-1}$ is again a retraction and $\mathrm{Im}(f^\alpha) = \alpha(\mathrm{Im}(f))$ and $\mathrm{Ker}(f^\alpha) = \alpha(\mathrm{Ker}(f))$. Now the exact formulation of Conjecture B is as follows.

Conjecture B. *For a codimension 1 retraction $f : k[P] \to k[P]$ there is $\alpha \in \Gamma_k(P)$ such that $f^\alpha = \iota \circ g$ for a retraction g of type either π_F or $\rho_{(P,H,w)}$ and $\iota : \mathrm{Im}(g) \to k[P]$ a graded k-algebra embedding.*

In other words this conjecture claims that any codimension 1 retraction can be 'modified' by an automorphism so that the corrected retraction factors through a retraction preserving the monomial structure.

The explicit description of the embeddings ι is not addressed in this paper.

A necessary condition for Conjecture B is that any codimension 1 retraction f can be modified by a graded automorphism α so that f^α has either a homogeneous binomial of degree 1 or a monomial of degree 1 in its kernel. A weaker condition is that $\mathrm{Ker}(f^\alpha)$ contains a homogeneous binomial of degree ≥ 1 (evidently this holds if there is a monomial in $\mathrm{Ker}(f^\alpha)$).

We remark that even an example of just an endomorphism in Pol_k, such that $\mathrm{Ker}(f^\alpha)$ contains no (homogeneous) binomial for any α, is not readily found. However, such exists, even in the class of codimension 2 retractions.

The examples below are constructed from joins of polytopes. The following lemma enables us to describe $\Gamma_k(R)$ for $R = \mathrm{join}(P, Q)$ under a mild assumption on P and Q. We identify them with the corresponding faces of R.

Lemma 5.1. *Let P and Q be lattice polytopes, both having interior lattice points. Then* $\mathrm{Col}(\mathrm{join}(P, Q)) = \mathrm{Col}(P) \cup \mathrm{Col}(Q)$.

Proof. That each of the column vectors of the polytopes serves as a column vector for $\mathrm{join}(P, Q)$ is clear.

Now let $v \in \mathrm{Col}(\mathrm{join}(P, Q))$. If v is parallel to either P or Q then either $v \in \mathrm{Col}(P)$ or $v \in \mathrm{Col}(Q)$ since $\mathrm{L}_{\mathrm{join}(P,Q)} = \mathrm{L}_P \cup \mathrm{L}_Q$. So without loss of generality we can assume that v is parallel neither to P nor to Q. Since P and Q span $\mathrm{join}(P, Q)$ they cannot be contained simultaneously in the base facet of v. But then either $p+v \in \mathrm{join}(P, Q)$ or $q + v \in \mathrm{join}(P, Q)$ for suitable vertices $p \in P$ and $q \in Q$. We get a contradiction because one of the points $r+v$ or $s+v$ is outside $\mathrm{join}(P, Q)$ for interior lattice points $r \in P$, $s \in Q$. $\square$

Example 5.2. Let Q be the lattice triangle spanned by $(0, -1)$, $(-1, 0)$, and $(1, 1)$. Then Q contains only one more lattice point, namely $(0, 0)$. Identifying U with $(0, 0)$,

V with $(0,-1)$, and W with $(-1,0)$ we see that the polynomial ring $k[U,V,W]$ can be embedded into $k[Q]$ such that the indeterminates correspond to lattice points. Moreover, $k[\operatorname{gp}(S_Q)]$ is then just the Laurent polynomial ring $k[\mathbb{Z}^3]=k[U^{\pm 1},V^{\pm 1},W^{\pm 1}]$.

Let $h' : k[X,Y]\to k[U,V,W]$ be defined by $h'(X)=U+V$, $h'(Y)=U+W$. Then h' induces a retraction h of $k[U,V,W,X,Y]$, namely the retraction mapping X and Y to $h'(X)$ and $h'(Y)$ respectively and leaving U,V,W invariant. This retraction extends in a natural way to retraction of $k[U^{\pm 1},V^{\pm 1},W^{\pm 1},X,Y]$, and can then be restricted to

$$k[Q]\otimes k[\Delta_1]\subset k[U^{\pm 1},V^{\pm 1},W^{\pm 1},X,Y]$$

where we identify $k[X,Y]$ with the polytopal ring $k[\Delta_1]$ of the unit segment. It can further be restricted to $k[\operatorname{join}(2Q,2\Delta_1)]$ which is embedded into $k[Q]\otimes k[\Delta_1]$ as the tensor product of the second Veronese subalgebras of the normal algebras $k[Q]$ and $k[\Delta_1]$.

We claim that the just constructed retraction h of $k[P]$, $P=\operatorname{join}(2Q,2\Delta_1)$, $\dim P=4$, has no conjugate h^α by an automorphism $\alpha\in\Gamma_k(P)$ such that the kernel of h^α contains a binomial.

The polytope Q has no column structures, a property inherited by $2Q$. Moreover, both $2Q$ and $2\Delta_1$ have interior points. Therefore the only column structures on P are those it gets from $2\Delta_1$ (see Lemma 5.1). Then every element $\alpha\in\Gamma_k(P)$ is of the form $\tau\circ\beta$, where τ is a toric automorphism and $\beta=1\otimes\beta'$ for some $\beta'\in\Gamma_k(2\Delta_1)$. Since τ does not affect the monomial structure, we can assume $\tau=1$. Furthermore the graded automorphisms of $k[2\Delta_1]$ are all restrictions of automorphisms of $k[\Delta_1]=k[X,Y]$ so that we have to take into account all automorphisms of $k[P]$ induced by a substitution

$$X\mapsto a_{11}X+a_{12}Y,\quad Y\mapsto a_{21}X+a_{22}Y,\quad U\mapsto U,\quad V\mapsto V,\quad W\mapsto W$$

with $\det(a_{ij})\neq 0$. Then h^α is induced by the substitution

$$a_{11}X+a_{12}Y\mapsto U+V,\qquad a_{21}X+a_{22}Y\mapsto U+W,$$

leaving U,V,W invariant. Also h^α extends to a retraction of $k[U^{\pm 1},V^{\pm 1},W^{\pm 1},X,Y]$ and then restricts to $k[U,V,W,X,Y]$. This shows that the kernel of the extension cannot contain a monomial; otherwise it would contain a monomial in X and Y, but h^α is injective on $k[X,Y]$. If the kernel contains a binomial b, we can assume that $b\in k[U,V,W,X,Y]$. In other words, we can find a binomial in the ideal $\mathfrak{p}$ of $k[U,V,W,X,Y]$ generated by

$$a_{11}X+a_{12}Y-(U+V),\qquad a_{21}X+a_{22}Y-(U+W).$$

Since the prime ideal $\mathfrak{p}$ contains no monomials, we can assume that the two terms of b are coprime. But then b reduces to a monomial modulo one of the variables, and since $\mathfrak{p}$ reduces to an ideal generated by linear forms, it reduces to a prime ideal. The reduction of $\mathfrak{p}$ modulo any of the variables cannot contain another variable.

In view of what has been said above it is natural to introduce the following classes of *tame retractions* (for arbitrary codimension) and *tame morphisms* in Pol_k: a retraction $f : k[P]\to k[P]$ is called tame if there are a sequence of lattice polytopes

$P = P_m, P_{m-1} \dots, P_1, P_0 = Q$, automorphisms $\alpha_i \in \Gamma_k(P_i)$ and monomial structure preserving retractions $\pi_i : k[P_i] \to k[P_{i-1}]$ such that

$$f = \iota \circ \pi_1 \circ \alpha_1 \circ \cdots \circ \pi_{m-1} \circ \alpha_{m-1} \circ \pi_m \circ \alpha_m$$

for a k-algebra embedding $\iota : k[Q] \to k[P]$.

The definition of a tame morphism is literally the same with the word 'retraction' changed to 'morphism'. Notice that the image of a tame retraction is again a polytopal ring, whereas that of a tame morphism is a semigroup ring of some homogeneous semigroup.

In this terminology Conjecture B merely says that all codimension 1 retractions are tame, while Example 5.2 shows that there are polytopal codimension 2 retractions, which are not even tame morphisms. The following example gives a wild retraction that as a morphism is tame.

Example 5.3. Let P be the join of $7\Delta_1$ and $2\Delta_1$, and thus $\dim P = 3$ (see Figure 3). It is straightforward to check that the assignment

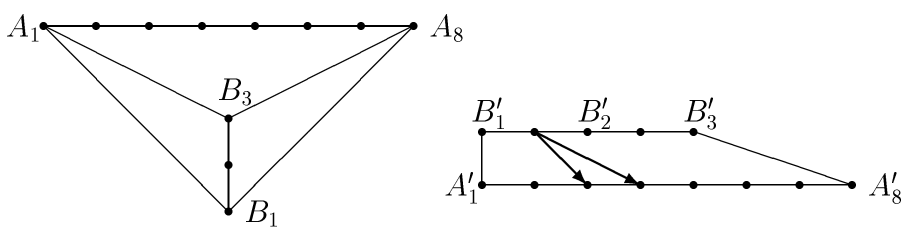

FIGURE 3. The polytopes P and P'

$$B_1 \mapsto A_2 + A_3, \quad B_2 \mapsto A_4 + A_5, \quad B_3 \mapsto A_6 + A_7, \quad A_i \mapsto A_i, \quad i \in [1, 8]$$

defines a retraction h of $k[P]$, and it is easily checked that $h = \iota \circ g \circ f$, where ι is the natural embedding of $k[7\Delta_1]$ into $k[P]$, f is the morphism $k[P] \to k[P']$ given by the assignment $A_i \mapsto A_i'$, $B_i \mapsto B_i'$, and $g : k[P'] \to k[7\Delta_1]$ is a tame surjection. In fact, one can choose $g = \pi_{7\Delta_1} \circ \alpha$, where $\alpha \in \mathbb{A}(7\Delta_1)$ is an automorphism of $k[P']$. (The column vectors used for α are indicated in the figure.)

The kernel of h is generated by $B_1 - (A_2 + A_3)$, $B_2 - (A_4 + A_5)$, $B_3 - (A_6 + A_7)$. Assume there is an automorphism $\gamma \in \Gamma_k(P)$ such that the image of this ideal contains a binomial or monomial of degree 1. Then

$$b = \gamma(c_1(B_1 - (A_2 + A_3)) + c_2(B_2 - (A_4 + A_5)) + c_3(B_3 - (A_6 + A_7))$$

is a binomial or monomial for suitable c_1, c_2, c_3, $c_i \neq 0$ for at least one i. In view of the definition of P as the join of two lattice segments with interior points, we can write γ as the composite of two automorphisms induced by automorphisms γ_1 of $k[7\Delta_1]$ and γ_2 of $k[2\Delta_1]$, respectively. In particular, if a polynomial contains a term 'on the edge $7\Delta_1$', then so does its image under γ, and the same holds true for the edge $2\Delta_1$. Thus b is not a monomial. But neither can it be a binomial because then the polynomial

$$\gamma_1(c_1(A_2 + A_3) + c_2(A_4 + A_5) + c_3(A_6 + A_7))$$

would be a term a with

$$\gamma_1^{-1}(a) = c_1(A_2 + A_3) + c_2(A_4 + A_5) + c_3(A_6 + A_7).$$

The right hand side is an 'interior polynomial' of $k[7\Delta_1]$. On the other hand, an automorphism mapping a term to an 'interior polynomial' must preserve the monomial structure by [BG1, Lemma 4.1].

6. Segmentonomial ideals

For an affine semigroup S an element $f \in k[S]$ will be called *segmentonomial* if the Newton polytope $\mathrm{N}(f) \subset \mathbb{R} \otimes \mathrm{gp}(S)$ has dimension ≤ 1. (Clearly, monomials as well as binomials are segmentonomials.) An ideal $I \subset k[S]$ is called segmentonomial if it is generated by a system of segmentonomials.

It is proved in [ES] that every minimal prime ideal over a binomial ideal of $k[X_1, \dots, X_n]$ (a polynomial ring) is again binomial. In this section we derive the same result for segmentonomial ideals in arbitrary affine semigroup rings.

Theorem 6.1. *Let S be an affine semigroup and $I \subset k[S]$ be a segmentonomial ideal.*

(a) *A minimal prime overideal $I \subset \mathfrak{p} \subset k[S]$ is binomial, and $k[S]/I$ is again an affine semigroup ring.*
(b) *Suppose that $\mathrm{ht}(I) = 1$, $f \in I$, $\dim(\mathrm{N}(f)) = 1$, and $\mathfrak{p}$ is as above. Then for every system of pairwise distinct lattice points $x_1, \dots, x_m \in \mathrm{L}_P$, such that none of the pairs (x_i, x_j), $i \neq j$, spans a line in $\mathbb{R} \otimes \mathrm{gp}(S)$ parallel to $\mathrm{N}(f)$, the residue classes $\bar{x}_1, \dots, \bar{x}_m$ constitute a k-linearly independent subset of $k[S]/\mathfrak{p}$.*

Proof. We prove claim (a) by induction on $r = \mathrm{rank}(S)$. Claim (b) will follow automatically from the description of $\mathfrak{p}$ derived below.

For $r = 0$ there is nothing to show. Assume the theorem is proved for semigroups of rank $< r$ and choose a segmentonomial $f \in I$. Then $\mathfrak{p}$ contains a minimal prime $\mathfrak{p}_0$ over the principal ideal (f). Assume that $\mathfrak{p}_0$ is a binomial ideal. By Theorem 3.4, $k[S]/\mathfrak{p}_0 \approx k[S_1]$ for some affine semigroup S_1 and such that monomials in $k[S]$ go to monomials in $k[S_1]$. But then segmentonomials in $k[S]$ are likewise mapped to segmentonomials in $k[S_1]$. This holds true because affinely independent terms lift to affinely independent terms. By induction hypothesis the image of $\mathfrak{p}$ in $k[S_1]$ is binomial. Since binomials can be lifted to binomials in $k[S]$, we conclude that $\mathfrak{p}$ is binomial.

The general situation thus reduces to the case in which $I = (f)$ for some segmentonomial $f \in k[S]$ and $\mathrm{ht}(I) = \mathrm{ht}(\mathfrak{p}) = 1$.

If $\mathfrak{p}$ contains a monomial, then $\mathfrak{p}$ is a height 1 monomial prime ideal, and we are done.

Otherwise $S \cap \mathfrak{p} = \emptyset$. Consider the localization $\mathfrak{p}\, k[\mathrm{gp}(S)]$. It is a height 1 prime ideal in the Laurent polynomial ring $k[\mathrm{gp}(S)]$. Therefore, $\mathfrak{p}\, k[\mathrm{gp}(S)] = f_0 k[\mathrm{gp}(S)]$ for some prime element $f_0 \in k[\mathrm{gp}(S)]$.

Also f_0 is segmentonomial. In fact, we have $f = f_0 f_1$ for some $f_1 \in k[\mathrm{gp}(S)]$ implying the equality $\mathrm{N}(f) = \mathrm{N}(f_0) + \mathrm{N}(f_1)$ for the corresponding Newton polytopes. Since $\dim N(f_0) = 0$ is excluded, $\dim(\mathrm{N}(f_0)) = 1$. Multiplying f_0 by a suitable term from $\mathrm{gp}(S)$ we can achieve that the origin $0 \in \mathbb{R} \otimes \mathrm{gp}(S)$ is one of the end-points of $\mathrm{N}(f_0)$.

Let $\ell \subset \mathbb{R} \otimes \mathrm{gp}(S)$ denote a rational line containing $\mathrm{N}(f_0)$. In a suitable basis of the free abelian group $\mathrm{gp}(S)$ the line ℓ becomes a coordinate direction. Therefore, we can assume that

$$k[\mathrm{gp}(S)] = k[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]$$

and that f_0 is a monic polynomial in X_1. Since k is algebraically closed, if follows that $f_0 = X_1 - a$ for some $a \in k$. Since $\mathfrak{p}$ does not contain a monomial, one has

$$\mathfrak{p} = \mathfrak{p}\, k[\mathrm{gp}(S)] \cap k[S] = (X_1 - a)k[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}] \cap k[S].$$

Thus $\mathfrak{p}$ is the kernel of the composite homomorphism

$$k[S] \hookrightarrow k[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}] \xrightarrow{X_1 \mapsto a} k[X_2, X_2^{-1}, \dots, X_n, X_n^{-1}]$$

This is a homomorphism mapping the elements of S to Laurent monomials in $X_2, X_2^{-1}, \dots, X_n, X_n^{-1}$, and therefore $\mathfrak{p}$ is generated by binomials. $\square$

7. Based retractions

Throughout this section we suppose that $h : k[P] \to k[P]$ is a retraction and that $A = \mathrm{Im}(h)$. We also assume $P \subset \mathbb{R}^n$, $\dim(P) = n$, $\mathrm{gp}(S_P) = \mathbb{Z}^{n+1}$ (and that k is algebraically closed.)

Lemma 7.1. *The following conditions are equivalent:*

(a) *there is a subset $X \subset \mathrm{L}_P$ such that the restriction $h : k[S_X] \to A$ is an isomorphism, where $k[S_X] \subset k[P]$ is the subalgebra generated by the semigroup $S_X = \langle X \rangle \subset S_P$,*

(b) *there is a $(\dim(A) - 1)$-dimensional cross section Q of P by a linear subspace H such that Q is a lattice polytope (i. e. the vertices of Q are lattice points) and*

$$h|_{k[Q]} : k[Q] \to A$$

is an isomorphism. In particular, A is a polytopal algebra.

Proof. We only need to derive (b) from (a). Let H be an affine hull of X in $\mathbb{R}^n$. We have to show that $Q = H \cap P$ is a lattice polytope with $L_Q = X$. Consider the subsemigroup

$$S'_Q = \{x \in S \mid x \neq 0 \text{ and } \mathbb{R}_+ x \cap P \subset H\} \cup \{0\}.$$

Then $h(k[S'_Q]) = A$ as well. On the other hand

$$\dim k[S'_Q] = \dim H + 1 = \dim k[S_X] = \dim A.$$

Thus the restriction $h : k[S'_Q] \to A$ is also an isomorphism. It follows that $X = L_Q$, and every element in S'_Q is a product of elements of X. Furthermore $Q = \mathrm{conv}(X)$ since any rational point of the complement $Q \setminus \mathrm{conv}(X)$ gives rise to elements in $S'_Q \setminus S_X$. $\square$

A subpolytope $Q \subset P$ as in Lemma 7.1(b) (if it exists) will be called a *base* of h and h is a *based retraction.* Notice that a base is not necessarily uniquely determined.

Theorem 7.2. *Suppose a retraction $h : k[P] \to k[P]$ has a base Q that intersects the interior of P. Then $h^\tau = \iota \circ \rho_{(P,H,W)}$ for some toric automorphism $\tau \in \mathbb{T}_k(P)$, a lattice fibration (P,H,W) and a k-algebra embedding $\iota : k[H \cap P] \to k[P]$. In particular, h is tame.*

Proof. It is not hard to check that there is no restriction in assuming that $k[Q] = \operatorname{Im}(h)$.

Note that $\operatorname{Ker}(h) \cap S_P = \emptyset$. In fact, if a monomial is mapped to 0 by h, then $\operatorname{Ker}(h)$ contains a monomial prime ideal $\mathfrak{p}$ of height 1. Since $\mathfrak{p}$ in turn contains all monomials in the interior of S_P, it must also contain monomials from S_Q, which is impossible. Thus h can be extended to the normalization $k[\bar{S}_P]$; on $k[\bar{S}_Q] \subset k[\bar{S}_P]$ the extension is the identity.

Set $L = \mathbb{Z}^{n+1}$, and let U be the intersection of the $\mathbb{Q}$-vector subspace of $\mathbb{Q}^{n+1}$ generated by S_Q with L. Choose a basis $v_1, ..., v_m$ of a complement of U in L. Since S_Q contains elements of degree 1 (given by the last coordinate), we can assume that $\deg v_i = 0$ for $i \in [1,m]$. In sufficiently high degree we can find a lattice point x in $\bar{S}_Q$ such that xv_i, $xv_i^{-1} \in \bar{S}_P$. We have the relation $(xv_i)(xv_i^{-1}) = x^2$.

It follows that $h(xv_i) = a_i x_i$, equivalently $h(x(a_i^{-1}v_i)) = x_i$, for some $x_i \in \bar{S}_Q$ and $a_i \in k^*$. After a toric 'correction' leaving $k[\bar{S}_Q]$ fixed we can assume $a_i = 1$ for all i.

After the inversion of the elements of S_P, we can further extend the homomorphism h to a map defined on the Laurent polynomial ring $k[L]$. Then we have

$$h(v_i x x_i^{-1}) = 1.$$

The vectors $v_i + x - x_i$ are also a basis of a complement of U, and thus part of a basis of L. Therefore the elements

$$v_i x x_i^{-1} - 1, \qquad i = [1,m],$$

generate a prime ideal of height m in $k[L]$.

It is now clear that h (after the toric correction) is just the retraction $\rho_{(P,H,W)}$ where H is the affine hull of Q in $\mathbb{R}^n$ and W is the sublattice of $\mathbb{Z}^n$ generated by the vectors $v_i + x - x_i$ upon the identification of $\mathbb{Z}^n$ with the degree 0 sublattice of $\mathbb{Z}^{n+1}$. ◻

Example 5.2 shows that even a based retraction h of $k[P]$ need not be tame if the base does not intersect the interior of P and h has codimension ≥ 2. However, in codimension 1 all based retraction are tame, as follows from Theorem 7.2 and

Theorem 7.3. *Suppose the codimension 1 retraction $h : k[S_P] \to k[S_P]$ has a base F not intersecting the interior of P. Then F is a facet of P and $h^\varepsilon = \iota \circ \pi_F$ for some $\varepsilon \in \mathbb{A}(F)$ and a k-algebra embedding $\iota : k[S_F] \to k[S_P]$.*

In the proof we will use a general fact on pyramids. Recall that a *pyramid* $\Pi \subset \mathbb{R}^n$ is a polytope which is spanned by a point v and a polytope B such that the affine hull of B does not contain v. In this situation v is called an *apex* and B is called a *base* of Π.

Lemma 7.4. *Let $\Pi \subset \mathbb{R}^n$ be a pyramid and $\Pi = \Pi_1 + \Pi_2$ be a Minkowski sum representation by polytopes $\Pi_1, \Pi_2 \subset \mathbb{R}^n$. Then both Π_1 and Π_2 are homothetic images of Π (with respect to appropriate centers and non-negative factors).*

Proof. The case $\dim(\Pi) = 2$ is an easy exercise.

Now we use induction on $\dim(\Pi)$. Assume $\dim(\Pi) = n$ and assume the claim has been shown for pyramids of dimension $\dim(\Pi) - 1$. Consider any $(n-1)$-dimensional subspace $\Lambda \subset \mathbb{R}^n$ perpendicular to the base $B \subset \Pi$. For a polytope $R \subset \mathbb{R}^n$ let R_Λ denote the image of R in Λ under the orthogonal projection $\mathbb{R}^n \to \Lambda$. Then Π_Λ is an $(n-1)$-dimensional pyramid and we have the Minkowski sum representation

$$\Pi_\Lambda = (\Pi_1)_\Lambda + (\Pi_2)_\Lambda.$$

By induction hypothesis there are homothetic transformations of Λ transforming Π_Λ into $(\Pi_1)_\Lambda$ and $(\Pi_2)_\Lambda$ respectively. Considering all the possible subspaces $\Lambda \subset \mathbb{R}^n$ we conclude that

(i) both Π_1 and Π_2 are n-pyramids (provided none of them is just a point – in this situation the lemma is obvious) such that the cones they span at corresponding vertices are parallel shifts of the cone spanned by Π at its apex v,
(ii) the corresponding bases of Π_1 and Π_2 are parallel to B.

That is exactly what we wanted to show. □

Proof of Theorem 7.3. As in the proof of 7.2 we can assume $k[F] = \mathrm{Im}(h)$, and, furthermore, $S_P \cap \mathrm{Ker}(h) = \emptyset$, for otherwise h itself passes through a facet retraction. Thus h can be extended to the Laurent polynomial ring $k[L]$, $L = \mathbb{Z}^{n+1}$, and in particular to a retraction of $k[\bar{S}_P]$ with image $k[\bar{S}_Q]$. The latter restricts to retractions $k[iP] \to k[iQ]$ for all i. The kernel of the extension h' is a height 1 prime ideal and thus principal; $\mathrm{Ker}(h') = \varphi K[L]$ and $\mathrm{Ker}(h) = (\varphi k[L]) \cap k[P]$ for some element $\varphi \in k[L]$.

Since F is a base of h, $\mathrm{Ker}(h)$ contains the elements $x - \ell$, $x \in \mathrm{L}_P \setminus F$, $\ell = h(x)$, and ℓ is a linear form on the points of L_F. Then $\mathrm{N}(\varphi)$ is a Minkowski summand of the pyramid $\mathrm{N}(x - \ell)$ with vertex at x. One can shift $\mathrm{N}(\varphi)$ by an integer vector into $\mathrm{N}(x - h(x)) \subset P$ such that the image R satisfies

$$(**) \qquad R \subset P \text{ and } R \cap F \neq \emptyset.$$

Evidently R is the Newton polytope of $y\varphi$ for some $y \in \mathbb{Z}^{n+1}$. Replacing φ by $y\varphi$, we can assume that $\mathrm{N}(\varphi)$ satisfies $(**)$.

By Lemma 7.4 $\mathrm{N}(\varphi)$ is homothetic to $\mathrm{N}(x - \ell)$. Clearly, $F \cap \mathrm{N}(\varphi)$ is a base of $\mathrm{N}(\varphi)$. The corresponding apex of $\mathrm{N}(\varphi)$ is some $z \in \mathrm{L}_P \setminus F$.

Consider the valuation

$$v_F : \mathbb{Z}^{n+1} \to \mathbb{Z}$$

determined by the conditions:

$$\mathrm{Im}(v_F) = \mathbb{Z},\ v_F(\mathrm{L}_F) = 0,\ v_F(\mathrm{L}_P) \geq 0$$

We claim: $v_F(z) = 1$ and $y + b - z \in P$ for any $b \in \mathrm{L}_{F \cap \mathrm{N}(\varphi)}$ and $y \in \mathrm{L}_P \setminus F$.

In fact, for $i \in \mathbb{N}$ big enough there is an element $z' \in \mathrm{L}_{iP}$ such that $v_F(z') = 1$. Since iF is a base of the induced retraction $\bar{h}_i : k[iP] \to k[iP]$ there exists a linear form ℓ' on L_{iF} such that $z' - \ell' \in \mathrm{Ker}(h)$. Thus $\mathrm{N}(\varphi)$ is a Minkowski summand of $\mathrm{N}(z' - \ell')$. Because of the condition $(**)$ we conclude $v_F(z) \le v_F(z')$. Hence $v_F(z) = 1$.

Now choose $y \in \mathrm{L}_P \setminus F$. Since F is a base of h, we can write $y - \ell'' \in \mathrm{Ker}(h)$ for some linear form ℓ'' on the points of L_F. Therefore, the pyramid $\mathrm{N}(\varphi)$ is a Minkowski summand of the pyramid $\mathrm{N}(y - \ell'')$ which has its apex at y. By Lemma 7.4 the cones spanned by these pyramids at their vertices are the same modulo a parallel shift. This observation in conjunction with the already established equality $v_F(z) = 1$ makes the claim clear.

We have shown that the vectors $b - z \in \mathbb{Z}^n$, $b \in \mathrm{L}_{F\cap \mathrm{N}(\varphi)}$, are column vectors for P. Now, by Lemma 4.1(b) there exists $\varepsilon \in \mathbb{A}(F))$ such that $\varepsilon(\varphi) = cz$ for some $c \in k^*$. Therefore, $\mathrm{Ker}(h^\varepsilon)$ is the monomial prime ideal $(\mathrm{L}_P \setminus F)k[P] \subset k[P]$, and this finishes the proof of Theorem 7.3. $\square$

Using Theorem 7.2 and Theorem 7.3 and Borel's theorem on maximal tori in linear groups [Bor] we now derive another sufficient condition for a codimension 1 retraction to be tame.

Theorem 7.5. *A codimension* 1 *retraction* $h : k[P] \to k[P]$ *is tame if the graded automorphism group of the k-algebra* $\mathrm{Im}(h)$ *contains a closed* $\dim(P)$*-torus whose action on* $\mathrm{Im}(h)$ *extends to an algebraic action on* $k[P]$.

Proof. Let $\mathbb{T} \subset \mathrm{gr.aut}(\mathrm{Im}(h))$ be such a closed torus. Then there is an algebraic embedding $\mathbb{T} \subset \Gamma_k(P)$ and a maximal intermediate torus $\mathbb{T} \subset \mathbb{T}' \subset \Gamma_k(P)$. By Borel's theorem

$$\mathbb{T}' = \mathbb{T}_k(P)^\alpha = \{\alpha \circ \tau \circ \alpha^{-1} |\ \tau \in \mathbb{T}_k(P)\}$$

for some $\alpha \in \Gamma_k(P)$. Let us show that h^α is a based retraction. By the previous theorems this completes the proof. We can assume that $\mathrm{Ker}(h^\alpha) \subset k[P]$ is not a monomial ideal, or, equivalently, $\mathrm{Ker}(h^\alpha)$ contains no monomials. Otherwise h factors through a facet projection, and we are done.

Claim. $\mathrm{Ker}(h^\alpha) \subset k[P]$ is a binomial ideal.

We know that $\mathrm{Ker}(h^\alpha)$ is stabilized by the codimension 1 subtorus $\mathbb{T}^\alpha \subset \mathbb{T}_k(P)$. Since $\mathrm{Ker}(h^\alpha)$ a is non-monomial height 1 prime ideal of $k[P]$ there is an element $\varphi \in k[\mathrm{gp}(S_P)]$ such that

$$\mathrm{Ker}(h^\alpha) = k[P] \cap (\varphi k[\mathrm{gp}(S_P)]).$$

(Compare with the proof of Theorem 6.1.) In this situation for any $\tau \in \mathbb{T}^\alpha$ we have $\tau(\varphi)k[\mathrm{gp}(S_P)] = \varphi k[\mathrm{gp}(S_P)]$. Since $\mathbb{T}^\alpha$ is a subtorus of the embedded torus, the latter equation is equivalent to the existence of an element $a_\tau \in k^*$ such that $\tau(\varphi) = a_\tau \varphi$. ($\mathbb{T}^\alpha$ acts naturally on coordinate ring $k[\mathbb{T}_k(P)] = k[\mathrm{gp}(S_P)]$.) This means that for any $\tau \in \mathbb{T}^\alpha$ the terms in the canonical k-linear expansion of φ are multiplied by the same scalar from k^* when τ is applied, a condition equivalent to to the segmentonomiality of φ. In fact, if $\dim \mathrm{N}(\varphi) \ge 2$ we would have three non-collinear terms T_1, T_2 and T_3 in the k-linear expansion of φ, and the condition

$$\frac{\tau(T_1)}{T_1} = \frac{\tau(T_2)}{T_2} = \frac{\tau(T_3)}{T_3}$$

on $\tau \in \mathbb{T}_k(P)$ would be equivalent to the condition that each τ is a solution to *two independent* binomial equations on $\mathbb{T}_k(P) = (k^*)^{\dim(P)+1}$. Hence, $\mathbb{T}^\alpha \subset \mathbb{T}_k(P)$ would be a sub-torus of codimension at least 2 – a contradiction. By this argument we have shown that φ is a segmentonomial and, hence, a binomial by Theorem 6.1(a). Therefore, by Theorem 3.4 we can assume $\operatorname{Im}(h) = k[S]$, $S = h(S_P)$. In particular,

$$\sigma_*(\mathbb{T}^\alpha) = \mathbb{T}_k(S),$$

where $\sigma_* : \operatorname{gr.aut}(\operatorname{Im}(h)) \to \operatorname{gr.aut}(k[S])$ is the natural map induced by σ and $\mathbb{T}_k(S)$ is the embedded torus of $\operatorname{Spec}(k[S])$ corresponding to S. The equality holds true because monomial preserving automorphisms are mapped to monomial preserving automorphisms.

Identifying $\operatorname{Im}(h)$ and $k[S]$ via σ and $\mathbb{T}^\alpha$ and $\mathbb{T}_k(S)$ via σ_* we get a $\mathbb{T}^\alpha$-equivariant embedding

$$\iota : k[S] \to k[P].$$

Assume $\{s_1, \dots, s_m\}$ be the set of degree 1 terms in $k[S]$. Then for any $\tau \in \mathbb{T}^\alpha$ there exist $a_\tau^{(i)} \in k^*$ such that

$$\tau(\iota(s_i)) = a_\tau^{(i)} \iota(s_i), \qquad i \in [1, m].$$

Arguments similar to those in the proof of the claim above show that the polytopes $\mathrm{N}(\iota(s_i))$, $i \in [1, m]$, are segments parallel to $\mathrm{N}(\varphi)$, maybe some of them degenerated into points.

Let $z \in \mathbb{R} \otimes \operatorname{gp}(S_P) \setminus \{0\}$ be such that the line $\mathbb{R}z \subset \mathbb{R} \otimes \operatorname{gp}(S_P)$ is parallel to all the $\mathrm{N}(\iota(s_i))$. Denote by t_i, $i \in [1, m]$ the upper end-points of $\mathrm{N}(\iota(s_i))$ in the direction $\mathbb{R}_+ z$. Then standard arguments with Newton polytopes ensure that the t_i are subject to the same affine *binomial* dependencies as the s_i. It is now clear that the subpolytope

$$\operatorname{conv}(\{t_1, \dots, t_m\}) \subset P$$

is a base for the retraction $h : k[P] \to k[S]$. □

8. Retractions of polygonal algebras

Throughout this section P denotes a lattice polygon (i. e. $\dim(P) = 2$).

Theorem 8.1. *Any codimension 1 retraction $h : k[P] \to k[P]$ is based and, therefore, either $h^\tau = \iota \circ \rho_{(P,H,w)}$ for some lattice segmental fibrations (P, H, w), $\tau \in \mathbb{T}_k(P)$ and a k-embedding $\iota : k[H \cap P] \to k[P]$, or $h^\varepsilon = \iota \circ \pi_F$ for a facet $F \subset P$, $\varepsilon \in \mathbb{E}_k(P)$ and a k-embedding $\iota : k[F] \to k[P]$. In particular, h is tame.*

By Theorems 7.2 and 7.3 it is enough to find a base for h. The first step in its construction is given by

Proposition 8.2. *A multiple $c\Delta_1$, $c \in \mathbb{N}$, of the unit segment Δ_1 can be embedded as a lattice polytope into P if and only if there is a k-algebra embedding of $k[c\Delta_1]$ into $k[P]$.*

Proof. Clearly, without loss of generality we can assume $c \geq 2$.

Let $\varepsilon : k[c\Delta_1] \to k[P]$ be an embedding. We write

$$\mathrm{L}_{c\Delta_1} = \{x_0, x_1, \dots, x_c\}.$$

Thus we have the equations $\varepsilon(x_{i-1})\varepsilon(x_{i+1}) = \varepsilon(x_i)^2$ for $i \in [1, c-1]$. Put

$$\varphi = \frac{\varepsilon(x_1)}{\varepsilon(x_0)} = \frac{\varepsilon(x_2)}{\varepsilon(x_1)} = \cdots .$$

In the Laurent polynomial ring $k[\operatorname{gp}(S_P)] = k[\mathbb{Z}^3]$ we can write $\varphi = \varphi_1/\varphi_2$ with coprime φ_1, φ_2. The equality $\varphi_2^c\varepsilon(x_c) = \varphi_1^c\varepsilon(x_0)$ (and the factoriality of $k[\mathbb{Z}^3]$) imply that φ_1^c divides $\varepsilon(x_c)$ and φ_2^c divides $\varepsilon(x_0)$.

Case (a). Both φ_1 and φ_2 are monomials in $k[\mathbb{Z}^3]$. In this situation the Newton polygon $\mathrm{N}(\varepsilon(x_c))$ is the parallel shift of $\mathrm{N}(\varepsilon(x_0))$ by the c-th multiple of the vector representing the support term of the monomial φ. But then the existence of the desired embedding $c\Delta_1 \to \operatorname{conv}(\mathrm{N}(\varepsilon(x_0)), \mathrm{N}(\varepsilon(x_c)) \subset P$ is obvious.

Case (b). At least one of φ_1 and φ_2, say φ_1, is not a monomial. Then $c\Delta_1$ can be embedded in any of the edges of the polygon $\mathrm{N}(\varphi_1^c) = c\,\mathrm{N}(\varphi_1)$. Since $\varepsilon(x_c) = \psi\varphi_1^c$ for some $\psi \in k[\mathbb{Z}^3]$, we get

$$\mathrm{N}(\varepsilon(x_c)) = \mathrm{N}(\psi) + \mathrm{N}(\varphi_1^c)$$

and the existence of an embedding $c\Delta_1 \to \mathrm{N}(\varepsilon(x_c)) \subset P$ is evident. □

Remark 8.3. We expect that Proposition 8.2 holds without any restrictions: for lattice polytopes P and Q a k-algebra embedding $k[Q] \to k[P]$ should only exist if Q can be embedded into P (as a lattice subpolytope).

Below we will need the following notion. For a lattice polygon $P \subset \mathbb{R}^2$ its *lattice width* $\operatorname{width}_\ell(P)$ with respect to the line $\ell = z' + \mathbb{R}z$, $z, z' \in \mathbb{Z}^2$, $z \neq 0$, is defined as

$$\operatorname{width}_\ell(P) = \max_{x \in P \cap \mathbb{Z}^2} \varphi(x) - \min_{y \in P \cap \mathbb{Z}^2} \varphi(y)$$

where $\varphi : \mathbb{Z}^2 \to \mathbb{Z}$ is a $\mathbb{Z}$-linear form with $\varphi(z) = 0$ and $\varphi(\mathbb{Z}) = \mathbb{Z}$. In other words, $\operatorname{width}_\ell(P)$ is the number of integral translates of ℓ intersecting P minus 1.

Proof of Theorem 8.1. Let $h : k[P] \to k[P]$ be a codimension 1 retraction. By Theorem 2.2 there is an isomorphism $\operatorname{Im}(h) \approx k[c\Delta_1]$ for some $c \in \mathbb{N}$, and Proposition 8.2 then yields an embedding $c\Delta_1 \to P$. Let Q denote its image.

If the restriction of h to the subalgebra $k[Q] \subset k[P]$ is injective then it is evidently bijective, and we are done by Lemma 7.1. Therefore we can assume

$$\operatorname{Ker}(h) \cap k[Q] \neq \{0\}.$$

Clearly, $I = \operatorname{Ker}(h) \cap k[Q] \subset k[Q]$ is segmentonomial. Since $\operatorname{ht}(\operatorname{Ker}(h)) = 1$ the ideal $\operatorname{Ker}(h)$ is a minimal prime ideal of I. If $\operatorname{Ker}(h)$ contains a monomial, then h factors through a facet projection π_F, and then F is a base for h. In the other case it is enough to show there is yet *another* embedding $c\Delta_1 \to P$, say with image R, such that Q and R are not parallel. Theorem 6.1(b) shows that we then have found a base.

Let ℓ denote the line spanned by Q. By Theorem 6.1(b) we have

$$(*) \qquad \operatorname{width}_\ell(P) = c.$$

We use the notation

$$P_i = \mathrm{N}(\varepsilon(x_i)), \qquad i \in [0, c].$$

(As in the proof of Proposition 8.2 we write $\mathrm{L}_{c\Delta_1} = \{x_0, x_1, \dots, x_c\}$.) Evidently there is no loss in generality in assuming that

$$P = \operatorname{conv}(P_0 \cup \cdots \cup P_c)$$

since, roughly speaking, the retraction can be restricted to the lattice polytope on the right hand side.

Case (a): $P_0 \neq P_1$. For every linear form $\lambda : \mathbb{R}^2 \to \mathbb{R}$ we consider all triples (λ, v_0, v_1) such that $v_0 \in P_0$, $v_1 \in P_1$ and

$$\lambda(v_i) > \lambda(x) \qquad \text{for all } x \in P_i, \quad x \neq v_i, \quad i = 0, 1.$$

In particular v_0 and v_1 are vertices of P_0 and P_1 respectively. We have

$$iP_1 = (i-1)P_0 + P_i, \qquad i \in [2, c].$$

It follows that λ has a unique maximum on P_i, necessarily taken at $v_i = iv_1 - (i-1)v_0$, $i \in [0, c]$. Thus we get the system

$$\{v_0, v_1, \dots, v_c\} \subset P$$

of collinear lattice points.

Subcase (a_1). There is a triple (λ, v_0, v_1) such that $[v_0, v_1]$ is *not parallel to* ℓ, and in particular $v_0 \neq v_1$. Then the points $v_0, v_1, \dots, v_c$ are pairwise different, and so we find $c+1$ successive lattice points in P on a line not parallel to ℓ.

Subcase (a_2). In the other case one notes first that the edges of P_0 and P_1 are pairwise parallel under an orientation preserving correspondence (in other words, P_0 and P_1 have the same normal fans; see [GKZ] for this notion). Moreover, if v_0 is a vertex of P_0 and v_1 is the corresponding vertex of P_1, then $v_0 = v_1$ or the segment $[v_0, v_1]$ is parallel to ℓ.

Let us choose $z \in \mathbb{R}^2$, $z \neq 0$ such that ℓ is parallel to $\mathbb{R}z$, and let ξ be a non-zero linear form with $\xi(z) = 0$.

Furthermore we consider the *upper* and *lower boundary* of P_0 in direction z. (A point y on ∂P_0 belongs to the lower boundary if the ray in direction z through y enters P at y, and the upper boundary is defined analogously; see Figure 4.) Each of these subsets has two endpoints u_0^-, u_0^+ and l_0^-, l_0^+ respectively (which may coincide). We choose the indices such that $\xi(u_0^-) \le \xi(u_0^+)$ and $\xi(l_0^-) \le \xi(l_0^+)$. It is clear that ξ takes its minimum on P_0 in u_0^- and l_0^- and its maximum in the other two points. Applying the same construction to P_1 we obtain the points u_1^-, u_1^+ and l_1^-, l_1^+. The correspondence between the vertices of P_0 and those of P_1 pairs u_0^- with u_1^- etc.

If $u_0^- = u_1^-$ and $l_0^- = l_1^-$, then $P_0 = P_1$ by our initial observation in this subcase. But we are assuming that $P_0 \neq P_1$ in Case (a).

For example, suppose that $u_0^- \neq u_1^-$. Then we can find a linear form λ such that u_0^- and u_1^- are the unique maxima of λ on P_0 and P_1 respectively. So the triple (λ, u_0^-, u_1^-) defines $c+1$ lattice points $v_0, \dots, v_c$ in P on a line ℓ' parallel to ℓ, and one

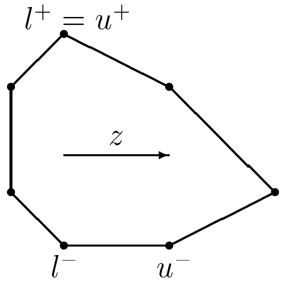

FIGURE 4. Lower and upper boundary

sees easily that ℓ' intersects P in an edge: it is impossible that $\xi(t) < \xi(u_0^-) = \xi(u_1^-)$ for some $t \in P_i$, $i \in [0, c]$.

Since $\operatorname{width}_\ell(P) = \operatorname{width}_{\ell'}(P) = c$ there is a lattice point $t \in P$ such that the triangle $T \subset P$ spanned by t and $c+1$ successive lattice points

$$w_0, w_1, \dots, w_c \in \ell' \cap \operatorname{conv}(\{v_0, v_1, \dots, v_c\}) \subset P$$

has $\operatorname{width}_\ell(T) = c$. We apply an integral affine transformation $\psi : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\psi(t) = (0,0)$, and

$$\psi(\operatorname{conv}(\{w_0, v_1, \dots, w_c\})) = \operatorname{conv}((c, r), (c, r+c))$$

for some $r \in \mathbb{Z}$. There exists $d \in \mathbb{Z}$ such that $dc \in [r, r+c]$. The segment $[(0,0), (c, dc)]$ clearly contains exactly $c+1$ equidistant lattice points and is not parallel to the image of ℓ under the transformation ψ. Again we are done.

Case (b): $P_0 = P_1$. Then it follows from the equation $iP_1 = (i-1)P_0 + P_i$, $i \in [2, c]$ that all the P_i coincide and are therefore identical with P. If P has an edge containing at least $c+1$ lattice points, then either this edge is not parallel to Q or there is a triangle T as in subcase (a_2), and in both cases we are done.

Only the case in which every edge of P has lattice length strictly less than c is left. In this situation we resort to the general result below (Proposition 8.4) which we have singled out because of its independent interest. □

Proposition 8.4. *Assume the edges of the lattice polygon $P \subset \mathbb{R}^2$ have lattice lengths strictly less than c and $\operatorname{width}_\ell(P) \le c$ for some line $\ell = \mathbb{R}x \subset \mathbb{R}^2$, $x \in \mathbb{Z}^2$, $x \neq 0$. Then there is no k-algebra embedding $\varepsilon : k[c\Delta_1] \to k[P]$ such that*

$$P = \mathrm{N}(\varepsilon(x_0)) = \mathrm{N}(\varepsilon(x_1)) = \cdots = \mathrm{N}(\varepsilon(x_c))$$

where $x_0, \dots, x_c$ denote the successive lattice points of $c\Delta_1$.

Proof. Assume to the contrary that such ε exists. For a subset $W \subset P$ and an element

$$f = a_1 y_1 + \cdots + a_r y_r \in k[P], \quad a_j \in k, \quad y_j \in \mathrm{L}_P,$$

we put

$$f_W = a_{j_1} y_{j_1} + \cdots + a_{j_t} y_{j_t}, \qquad \{y_{j_1}, \dots, y_{j_t}\} = W \cap \{y_1, \dots, y_r\}.$$

Let $E \subset P$ be an edge. It is impossible to find a graded k-algebra embedding of $k[c\Delta_1]$ into $k[E]$ (by Hilbert function reasons or Proposition 8.2). In particular, the homomorphism

$$\pi_E \circ \varepsilon : k[c\Delta_1] \to k[E], \qquad x_i \mapsto \varepsilon(x_i)_E,$$

is not injective. (As usual, π_E is the facet projection $k[P] \to k[E]$.) It follows that

$$(\pi_E \circ \varepsilon)(k[c\Delta_1]) \approx k[\mathbb{Z}_+].$$

as graded k-algebras. Thus there is an element $\xi \in k^*$ such that

$$\varepsilon(x_0)_E = \xi\varepsilon(x_1)_E = \xi^2\varepsilon(x_2)_E = \cdots = \xi^c\varepsilon(x_c)_E. \tag{1}$$

Consider the toric automorphism $\tau : k[c\Delta_1] \to k[c\Delta_1]$ determined by

$$x_i \mapsto \xi^i x_i, \qquad i \in [0, c].$$

Then

$$\varepsilon'(x_0)_E = \varepsilon'(x_1)_E = \cdots = \varepsilon'(x_c)_E,$$

where $\varepsilon' = \varepsilon \circ \tau$.

Applying the same arguments to the embedding $\varepsilon' : k[c\Delta_1] \to k[P]$ and an edge $E' \subset P$ sharing an end-point with E we conclude

$$\varepsilon'(x_0)_{E'} = \varepsilon'(x_1)_{E'} = \cdots = \varepsilon'(x_c)_{E'}.$$

(Because of the common end-point there is no need to further 'correct' ε' by a toric automorphism.) We can assume $\varepsilon = \varepsilon'$.

Going once around P we finally arrive at the equations

$$\varepsilon(x_0)_{\partial P} = \varepsilon(x_1)_{\partial P} = \cdots = \varepsilon(x_c)_{\partial P}, \tag{2}$$

where ∂P denotes the boundary of P.

After an integral affine change of coordinates we may assume $\ell = \mathbb{R}(0, 1)$ and that $P \subset \mathbb{R}^2_+$. Then $k[P]$ can be considered as a subalgebra of $k[X, Y, Z]$ generated by monomials X^iY^jZ. Moreover, after a parallel translation of P we can assume that P contains a lattice point $(0, j)$, or equivalently, $k[P]$ contains a monomial Y^jZ.

For any element $a \in k^*$ we have the retraction

$$\pi^a : k[X, Y, Z] \to k[X, Z], \qquad \pi_a(X^iY^jZ^m) \mapsto a^jX^iZ^m$$

for all i, j, m. The image of $k[P]$ under π^a is a polytopal algebra $k[c'\Delta_1]$ generated by the elements $Z, XZ, \ldots, X^{c'}Z$ with $c' = \mathrm{width}_\ell(P)$. Due to the infinity of the

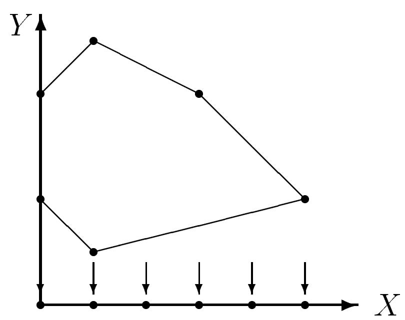

FIGURE 5. The map π^a

field k there is $a \in k^*$ such that the elements

$$\pi^a(\varepsilon(x_i)) \qquad i \in [0,c],$$

are pairwise different. In fact, it is clear that for any finite set of polynomials $\mathcal{P} \subset k[X,Y,Z]$ there is an element $a \in k^*$ such that π^a maps $\mathcal{P}$ to a set of $\#\mathcal{P}$ pairwise different polynomials from $k[X,Y]$.

Set

$$\pi^a(\varepsilon(x_j)) = \sum_{i=0}^{c'} \alpha_{ij} X^i Z.$$

In view of the fact that the elements $\varepsilon(x_j)$ coincide 'along ∂P', we see that the coefficients α_{0j} coincide for all j, and the same holds for the coefficients $\alpha_{c'j}$. The $(c'+1)\times(c+1)$ matrix (α_{ij}) has two linear dependent rows, and since $c' \le c$, its rank is $< c+1$.

It follows that $\pi^a \circ \varepsilon$ is not injective. Thus we see as above that there exists η in k^* with

$$\pi^a(\varepsilon(x_0)) = \eta\pi^a(\varepsilon(x_1)) = \eta^2\pi^a(\varepsilon(x_2)) = \cdots = \eta^c\pi^a(\varepsilon(x_c)).$$

Then however $\alpha_{00} = \eta\alpha_{01} = \cdots = \eta^c\alpha_{0c}$. By a suitable choice of a we can furthermore achieve that $\alpha_{00} \ne 0$. Then it follows immediately that $\eta = 1$. This is a contradiction since we have chosen a such that the $\pi^a(\varepsilon(x_i))$ are pairwise different. $\square$

Remark 8.5. Essentially the same arguments as we have used in §7 and §8 imply the analogous results for graded retractions of the semigroup algebras corresponding to affine positive *normal* semigroups (we only assume all monomials are homogeneous of positive degree) – first one observes that there is the appropriate notion of a lattice fibration for rational polyhedral pointed cones, and then the arguments are applied to all Veronese subalgebras. (They are essentially the same as polytopal semigroup rings.)

This remark is analogous to Remark 3.3(c) in [BG1], saying that the description of graded automorphism groups for graded affine normal semigroup rings is essentially the same as that for polytopal semigroup rings.

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