Some remarks on
holomorphic vector bundles over non–Kähler manifolds

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Abstract. We compare some moduli spaces of holomorphic structures on a given smooth vector bundle over an arbitrary complex manifold.

If we consider an SU(2) vector bundle E over a Kähler surface S, then the moduli space of stable holomorphic structures on E is equal to the moduli space of anti-self-dual SU(2) connections on E if and only if $b_1(S) = 0$. This fact has a generalization for non-Kähler cases (2.4), (2.6), (2.7), (3.3). A modification of vanishing theorem is stated (1.10), which can be used to get a generalized Atiyah-Hitchin-Singer's elliptic complex on non-Kähler manifolds.

From now on our basic reference is [**Kob**]. Let M be a compact connected complex n-manifold with a hermitian metric $g_{\mu\bar{\nu}}$ $(1 \leq \mu, \nu \leq n)$. The associated fundamental form will be denoted by $\Phi = \sqrt{-1} \sum g_{\mu\bar{\nu}} dz^\mu \wedge d\bar{z}^\nu$. We do *not* assume that Φ is a Kähler form, but we may and will assume that

$$d'd''(\Phi^{n-1}) = 0 \tag{0.1}$$

after a conformal change of the metric, if necessary [**Gau**]. Such a metric will be called a *Gaudochon metric.*

1. Degree of bundles. For a holomorphic vector bundle $\mathcal{E}$ over M, we define [**Buc**], [**LY**] the *degree* of $\mathcal{E}$ relative to Φ by

$$\deg(\mathcal{E}) = \deg_\Phi(\mathcal{E}) = \int_M c_1(\mathcal{E}, h) \wedge \Phi^{n-1} = \frac{1}{2n\pi} \int_M (\operatorname{tr} K)\Phi^n,$$

where $c_1(\mathcal{E}, h)$ is the first Chern form associated to a hermitian metric h on $\mathcal{E}$, $\operatorname{tr} K$ is the *scalar curvature* and K is the *mean curvature* [**Kob**]. The condition (0.1) implies that the degree is independent of the choice of h. Obviously $\deg(\mathcal{E}) = \deg(\det \mathcal{E})$ and isomorphic bundles have the same degree. Thus we have a group homomorphism

$$\deg : H^1(M, \mathcal{O}^\times) \to \mathbb{R}.$$

On Kähler manifolds degree is a topological invariant, but in non-Kähler case this is no longer true, i.e., there exists a hermitian manifold (M, Φ) with a holomorphic line bundle $\mathcal{L}$ such that $c_1(\mathcal{L}) = 0 \in H^2(M; \mathbb{Z})$ and $\deg \mathcal{L} \neq 0$. In particular, $H^1(M, \mathcal{O}) \neq 0$ and the isomorphism class $[\mathcal{L}]$ of $\mathcal{L}$ generates an infinite cyclic subgroup in

$$\operatorname{Pic}^0(M) = \{\ell \in H^1(M, \mathcal{O}^\times) \mid c_1(\ell) = 0 \in H^2(M; \mathbb{Z})\}.$$

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For example, let λ be a nonzero complex number with $|\lambda| \neq 1$. Then on the Hopf manifold $M = (\mathbb{C}^n - \{0\})/(z \mapsto \lambda z)$, we consider the 'metric'

$$\Phi = \frac{\sqrt{-1}}{|z|^2}(dz^1 \wedge d\bar{z}^1 + \cdots + dz^n \wedge d\bar{z}^n),$$

which satisfies $d'd''(\Phi^{n-1}) = 0$ (and $d'd''(\Phi^{n-2}) \neq 0$ for $n > 2$. cf. (1.2)). Then it is easy to see that the mean curvature K of the *Chern connection* on the holomorphic tangent bundle $\mathcal{T}$ of M is identically equal to $n - 1$. Thus M is an *Einstein-Hermitian manifold* and $\deg \mathcal{T} > 0$. It follows (cf. (1.9)) that $H^0(M, \Omega^p) = 0$ $(1 \leq p \leq n)$, where Ω^p is the sheaf of holomorphic p-forms. Of course this can be obtained easily since there is no isolated singularity of a holomorphic function in dim > 1.

When $n = 2$, Buchdahl [**Buc**] found a necessary and sufficient condition for degree to be a topological invariant. In general we have the following. Let

$$\mathrm{Pic}^0(M)_{\mathbb{R}} = \{\ell \in H^1(M, \mathcal{O}^\times) \mid c_1(\ell)_{\mathbb{R}} = 0 \in H^2(M; \mathbb{R})\}.$$

1.2. PROPOSITION. *Consider the following statements.*

(1) $b_1(M) = 2\dim_{\mathbb{C}} H^1(M, \mathcal{O})$
(2) $\deg(\mathrm{Pic}^0(M)) = 0$
(3) $\deg(\mathrm{Pic}^0(M)_{\mathbb{R}}) = 0$
(4) *degree is a topological invariant.*

Then (1) implies (2). (2), (3) and (4) are equivalent. If $d'd''(\Phi^{n-2}) = 0$, then (4) implies (1).

PROOF: For the proof, we identify

$$\mathrm{Pic}^0(M) \simeq H^1(M; \mathcal{O})/H^1(M; \mathbb{Z}) \simeq Z^{0,1}/B, \tag{1.3}$$

where

$$Z^{0,1} = \{\alpha \in A^{0,1}(M) \mid d''\alpha = 0\}$$

and

$$B = \{-d''g \cdot g^{-1} \mid g \in C^\infty(M, \mathbb{C}^\times)\} \simeq C^\infty(M, \mathbb{C}^\times)/\mathbb{C}^\times. \tag{1.4}$$

Note that B is a subgroup of $Z^{0,1}$ containing

$$B^{0,1} = \{d''f \mid f \in C^\infty(M, \mathbb{C})\}.$$

Also we have

$$B/B^{0,1} \simeq H^1(M; \mathbb{Z}). \tag{1.5}$$

Now the deg $|\mathrm{Pic}^0(M)$ is defined by

$$\deg[\alpha] = \frac{\sqrt{-1}}{2\pi}\int_M (d'\alpha - d''\bar{\alpha}) \wedge \Phi^{n-1} \tag{1.6}$$

for $[\alpha] \in \mathrm{Pic}^0(M),\ \alpha \in Z^{0,1}$.

Now suppose (1) is true. Then $\mathrm{Pic}^0(M)$ is a compact group and hence we get (2), which is obviously equivalent to (4).

Suppose (2) is true. Let $\mathcal{L}$ be a holomorphic line bundle with $c_1(\mathcal{L})_{\mathbb{R}} = 0 \in H^2(M;\mathbb{R})$. Then for any hermitian metric h on $\mathcal{L}$, $c_1(\mathcal{L}, h)$ is a closed real (1,1)-form and hence there exists a $\beta = \beta' + \beta'' \in A^{1,0} \oplus A^{0,1}$ such that $c_1(\mathcal{L}, h) = \frac{\sqrt{-1}}{2\pi} d\beta$. Then $d''\beta = 0,\ \beta' = -\overline{\beta''}$ and

$$c_1(\mathcal{L}, h) = \frac{\sqrt{-1}}{2\pi}(d'\beta'' - d''\overline{\beta''}).$$

Thus $\deg(\mathcal{L}) = \deg[\beta''] = 0$. This implies (3).

Obviously, (3) implies (2).

Finally, suppose $d'd''(\Phi^{n-2}) = 0$ and (4) is true. By (1.5), for any $\alpha \in Z^{0,1}$

$$\int_M d'\alpha \wedge \Phi^{n-1} = 0.$$

Then as in [**Buc**], there exists a unique $\beta \in B^{0,1}$ such that

$$\Lambda d'(\alpha + \beta) = 0$$

for each $\alpha \in Z^{0,1}$. Then by the next observation, we have $d'(\alpha + \beta) = 0$.

OBSERVATION. *Let $\alpha \in Z^{0,1}$ and $\Lambda d'\alpha = 0$. Then $d'\alpha = 0$ if $d'd''(\Phi^{n-2}) = 0$.*

(For this observation, we do not need the assumption (0.1). This can be extended to "flat" holomorphic hermitian vector bundles.)

Now we obtain a map

$$\alpha \mapsto \overline{\alpha + \beta}$$

of $Z^{0,1}$ into the space $H^0(M, d\mathcal{O})$ of d-closed holomorphic 1-forms. This map induces an isomorphism

$$H^{0,1}(M) \simeq H^0(M, d\mathcal{O}).$$

This implies (1) [**Kod**]. ∎

1.7 COROLLARY. *On Kählerian manifolds, the degree relative to any Gauduchon metric is a topological invariant.*

REMARK. The condition $d'd''(\Phi^{n-2}) = 0$ implies that, for instance,

$$\int_M c_2(\mathcal{E}, h) \wedge \Phi^{n-2}$$

is independent of the choice of h [**BC**]. Hence one can obtain Lübke inequality [**L1**] and the lower bound for the Yang-Mills functional.

Next proposition is trivial.

1.8. PROPOSITION. *If degree is a topological invariant on M and $b_2(M) = 0$, then there are no stable bundles of* rk > 1. *Every holomorphic vector bundle is semi-stable and every Einstein-Hermitian vector bundle is a direct sum of line bundles with the same degree.*

The following vanishing theorem indicates a role of degree.

1.9. VANISHING THEOREM [**Kob**]. *Let $(\mathcal{E}, h)$ be an Einstein-Hermitian vector bundle over a Hermitian manifold (M, Φ). If* $\deg(\mathcal{E}) < 0$, *then $\mathcal{E}$ has no holomorphic section. If* $\deg(\mathcal{E}) = 0$, *then every section of $\mathcal{E}$ is parallel.*

Since every holomorphic line bundle admits an Einstein-Hermitian metric, the vanishing theorem applies to any holomorphic line bundle. This vanishing theorem has a following generalization.

1.10. PROPOSITION. *Let $(\mathcal{E}, h)$ be a hermitian holomorphic vector bundle over (M, Φ). Let $D = D' + D''$ be the Chern connection on $(\mathcal{E}, h)$ and u be a smooth section of $\mathcal{E}$.*

(1) *If $K \leq 0$ and $\Lambda D'D''u = 0$, then $Du = 0$. If, moreover, $K < 0$ at some point of M, then $u = 0$.*
(2) *If $K \geq 0$ and $\Lambda D''D'u = 0$, then $Du = 0$. If, moreover, $K > 0$ at some point of M, then $u = 0$.*

PROOF: Observe that if $\Lambda D'D''u = 0$,

$$\sqrt{-1}\Lambda d'd''h(u,u) = |Du|^2 - h(Ku, u).$$

Then the maximum principle of E. Hopf applies. (2) is similarly proved. ∎

This vanishing theorem can be used to get a generalized Atiyh-Hitchin-Singer's elliptic complex (cf. [**AHS**], [**K2**]) for an Einstein-Hermitian connection on a hermitian manifold.

2. Holomorphic structures. Now we fix a smooth complex vector bundle E over M of rank r. There are three important concepts on E, namely, holomorphic structures, unitary structures and connections. The sets of these structures will be denoted by $\mathrm{Hol}(E)$, $\mathrm{Herm}(E)$ and $\mathrm{Con}(E)$, respectively. Then there is a *Chern map*

$$\mathrm{Hol}(E) \times \mathrm{Herm}(E) \to \mathrm{Con}(E).$$

The group $\mathrm{GL}(E)$ of smooth bundle automorphisms of E acts naturally on these spaces and the Chern map is equivariant. The Chern map is *natural* in the sense that for any vector bundle $\rho(E)$ *associated* to E, the diagram

$$\begin{array}{ccc}
\mathrm{Hol}(E) \times \mathrm{Herm}(E) & \longrightarrow & \mathrm{Con}(E) \\
\downarrow & & \downarrow \\
\mathrm{Hol}(\rho(E)) \times \mathrm{Herm}(\rho(E)) & \longrightarrow & \mathrm{Con}(\rho(E))
\end{array}$$

commutes equivariantly. We consider only the case $\rho(E) = \det E$, since we have a complete understanding in that situation. A different point of view is considered in [**New**], [**OV**], [**L2**].

From now on we will assume that $\mathrm{Hol}(E) \neq \emptyset$. Then there is a commutative diagram

$$\begin{array}{ccc} \mathrm{Hol}(E) & \xrightarrow{\det} & \mathrm{Hol}(\det E) \\ \downarrow & & \downarrow \\ \mathcal{M}(E) & \longrightarrow & \mathcal{M}(\det E) \end{array} \tag{2.1}$$

where $\mathcal{M}(E) = \mathrm{Hol}(E)/\operatorname{GL}(E)$, which we may call the *moduli space of holomorphic structures* on E. We identify ([**Gri**], [**AHS**], [**AB**], [**Qui**], [**Kob**], [**K2**]) a holomorphic structure with the corresponding *Cauchy-Riemann operator* $D'' : A^0(E) \to A^{0,1}(E)$, $D'' \circ D'' = 0$. They form a subset of an affine space, of which the model space is $A^{0,1}(\operatorname{End} E)$. Thus $\mathrm{Hol}(E)$ and hence $\mathcal{M}(E)$ is canonically equipped with a smooth topology [**Pal**]. Note that there is a simple transitive action of the group $\mathrm{Pic}^0(M)$ on $\mathcal{M}(\det E)$ and hence $\mathcal{M}(\det E)$ is (noncanonically) isomorphic to $\mathrm{Pic}^0(M)$. The *surjective* map

$$\det : \mathrm{Hol}(E) \to \mathrm{Hol}(\det E) \tag{2.2}$$

is a trivial fiber bundle. Once a holomorphic structure or equivalently a Cauchy-Riemann operator D'' is chosen, a trivialization of $\mathrm{Hol}(E)$ over $\mathrm{Hol}(\det E)$ is given by

$$\mathrm{Hol}(E) \simeq \mathrm{Hol}(\det E) \times \{\beta \in A^{0,1}(\operatorname{End} E) : \operatorname{tr}\beta = 0, D''(\beta) + \beta \circ \beta = 0\}.$$

The fiber of (2.2) at $\mathcal{L} \in \mathrm{Hol}(\det E)$ is denoted by

$$\mathrm{Hol}(E, \mathcal{L}) = \{\mathcal{E} \in \mathrm{Hol}(E) \mid \det \mathcal{E} = \mathcal{L}\},$$

and

$$\mathcal{M}(E, \mathcal{L}) := \mathrm{Hol}(E, \mathcal{L})/\operatorname{SL}(E),$$

where

$$\operatorname{SL}(E) = \{g \in \operatorname{GL}(E) \mid \det g = 1\}.$$

The fiber bundle

$$\mathcal{M}(E) \to \mathcal{M}(\det E) \tag{2.3}$$

becomes trivial after it is divided by a finite group (2.4). The group $\mathrm{Pic}^0(M)$ also acts on $\mathcal{M}(E)$, by tensoring, and the induced action on $\mathcal{M}(E)$ of the r-torsion subgroup

$$T := T_r = \{\ell \in \mathrm{Pic}^0(M) \mid r\ell = 0\}$$

commutes with the projection $\mathcal{M}(E) \to \mathcal{M}(\det E)$. Note that T is a finite group isomorphic to $(\mathbb{Z}/r\mathbb{Z})^{b_1}$, where b_1 is the first Betti number of M. Although the stabilizers in T are not simply described, we have

2.4. PROPOSITION. *$\mathcal{M}(E)/T$ is isomorphic to the product $\mathcal{M}(\det E) \times (\mathcal{M}(E)/\mathrm{Pic}^0(M))$ as spaces over $\mathcal{M}(\det E)$.*

PROOF: Probably, the proof using the Cauchy-Riemnann operators might be more clear. But here is the direct proof. The isomorphism $\mathcal{M}(E)/T \to \mathcal{M}(\det E) \times (\mathcal{M}(E)/\mathrm{Pic}^0(M))$ is given by

$$[\mathcal{E}]_T \mapsto [\det \mathcal{E}] \times [\mathcal{E}]_{\mathrm{Pic}^0(M)}.$$

Obviously this is a well-defined continuous map. To see the injectivity, suppose

$$[\det \mathcal{E}_1] \times [\mathcal{E}_1]_{\mathrm{Pic}^0(M)} = [\det \mathcal{E}_2] \times [\mathcal{E}_2]_{\mathrm{Pic}^0(M)}.$$

Then $[\det \mathcal{E}_1] = [\det \mathcal{E}_2]$ and $[\mathcal{E}_1]_{\mathrm{Pic}^0(M)} = [\mathcal{E}_2]_{\mathrm{Pic}^0(M)}$. Thus there exists a $[\mathcal{L}] \in \mathrm{Pic}^0(M)$ such that $\mathcal{E}_1 \otimes \mathcal{L} \simeq \mathcal{E}_2$. Then $\det \mathcal{E}_1 \otimes \mathcal{L}^r \simeq \det \mathcal{E}_2$. Thus $\mathcal{L}^r \simeq \mathcal{O}$, i.e., $[\mathcal{L}] \in T$. Hence $[\mathcal{E}_1]_T = [\mathcal{E}_2]_T$.

For the surjectivity, let $[\mathcal{L}] \times [\mathcal{E}_1] \in \mathcal{M}(\det E) \times \mathcal{M}(E)$ be given. Then

$$[\mathcal{L}] = [\det \mathcal{E}_1] + \ell$$

for some unique $\ell \in \mathrm{Pic}^0(M)$. Since $\mathrm{Pic}^0(M)$ is a divisible group, there exists a ℓ_1 such that $\ell = r\ell_1$. Locally, this ℓ_1 can be chosen continuously. Then we put $[\mathcal{E}] = [\mathcal{E}_1] \otimes \ell_1$. Then $[\mathcal{E}]_T \in \mathcal{M}(E)/T$ is independent of the choice of ℓ_1 and $[\mathcal{E}]_T$ maps to $[\mathcal{L}] \times [\mathcal{E}_1]_{\mathrm{Pic}^0(M)}$. This establishes the isomorphism. ∎

2.5. LEMMA. *The followings are equivalent.*

(1) $b_1(M) = 0$
(2) $C^\infty(M, \mathbb{C}^\times)$ *is a divisible group*
(3) $C^\infty(M, \mathbb{C}^\times)$ *is connected.*
(4) $C^\infty(M, \mathbb{C}^\times)/\mathbb{C}^\times$ *is a divisible group*
(5) $C^\infty(M, \mathbb{C}^\times)/\mathbb{C}^\times$ *is connected.*
(6) $\mathrm{Pic}^0(M)$ *has no torsion*
(7) $\mathrm{Pic}^0(M) \simeq H^1(M, \mathcal{O})$.

Moreover these imply that $\mathrm{Pic}^0(M)$ *acts freely on* $\mathcal{M}(E)$.

Now we get (cf. **[K3]**, **[OV]** **[L2]**)

2.6. COROLLARY. (1) *If $b_1 = 0$, then $\mathcal{M}(E, \mathcal{L}) \simeq \mathcal{M}(E)/\mathrm{Pic}^0(M)$ for any $\mathcal{L} \in \mathrm{Hol}(\det E)$.*
(2) If $H^1(M, \mathcal{O}) = 0$, then $\mathcal{M}(E, \mathcal{L}) \simeq \mathcal{M}(E)$ for any $\mathcal{L} \in \mathrm{Hol}(\det E)$.

PROOF: (1) Since $b_1 = 0, T = 0$ and hence by (2.4) $\mathcal{M}(E) \simeq \mathcal{M}(\det E) \times (\mathcal{M}(E)/\mathrm{Pic}^0(M))$ as spaces over $\mathcal{M}(\det E)$. In particular, the fiber $\mathcal{M}(E)_{[\mathcal{L}]}$ of $\mathcal{M}(E) \to \mathcal{M}(\det E)$ at

$[\mathcal{L}] \in \mathcal{M}(\det E)$ is isomorphic to $\mathcal{M}(E)/\mathrm{Pic}^0(M)$. Thus it suffices to show that $\mathcal{M}(E)_{[\mathcal{L}]} \simeq \mathcal{M}(E,\mathcal{L})$. From the commutative diagram (2.1), we have an injection

$$\mathcal{M}(E,\mathcal{L}) \to \mathcal{M}(E)_{[\mathcal{L}]}.$$

To see the surjectivity of this map, let $[D''] \in \mathcal{M}(E)$ and $[\det D''] = [\mathcal{L}]$ (i.e., $[D''] \in \mathcal{M}(E)_{[\mathcal{L}]}$). Then $\det D'' = \mathcal{L} - \beta_1$ for some $\beta_1 \in B \simeq \mathcal{C}^\infty(M, \mathbb{C}^\times)/\mathbb{C}^\times$ (cf. (1.4)). Since B is divisible, $\beta_1 = r\beta$ for some (unique) $\beta \in B$. Now

$$[D'' + \beta 1_E] = [D'']$$

and $\det(D'' + \beta 1_E) = \det D'' + \mathrm{tr}(\beta 1_E) = \mathcal{L}$. This establishes a homeomorphism.

(2) follows from (1). ∎

2.7. REMARKS. (1) If we consider *stable* structures ([**Buc**], [**LY**]), then we have propositions similar to (2.4) and (2.6) with $\mathcal{M}^s(E) := \mathrm{Hol}^s(E)/\mathrm{GL}(E)$ and $\mathcal{M}^s(E,\mathcal{L}) := \mathrm{Hol}^s(E,\mathcal{L})/\mathrm{SL}(E)$.

(2) If the map $\mathcal{M}(E,\mathcal{L}) \hookrightarrow \mathcal{M}(E)$ is surjective, then obviously $\mathrm{Pic}^0(M) = 0$, i.e., $H^1(M,\mathcal{O}) = 0$.

3. Einstein-Hermitian connections. From now on we fix a unitary structure h on E. The space of irreducible Einstein h-connections on E is denoted by $\mathcal{C}^s(E)$ and

$$\mathcal{N}^s(E) := \mathcal{C}^s(E)/\mathrm{U}(E),$$

where $\mathrm{U}(E)$ is the group of smooth isometries of (E,h). We assume that $\mathcal{C}^s(E) \neq \emptyset$. Then as in the previous section we have the following commutative diagram.

$$\begin{array}{ccc} \mathcal{C}^s(E) & \xrightarrow{\det} & \mathcal{C}^s(\det E) \\ \downarrow & & \downarrow \\ \mathcal{N}^s(E) & \longrightarrow & \mathcal{N}^s(\det E) \end{array}$$

The map $\det : \mathcal{C}^s(E) \to \mathcal{C}^s(\det E)$ is a trivial fibration. Once a point $D \in \mathcal{C}^s(E)$ is chosen, the trivialization is given by

$$\mathcal{C}^s(E) \simeq \mathcal{C}^s(\det E) \times \{A \in A^1(uE) \mid D''(A'') + A'' \circ A'' = 0,\ \Lambda(D(A) + A \circ A) = 0\},$$

where uE is the real vector bundle of skew-hermitian endomorphisms of (E,h). We put for $\nabla \in \mathcal{C}^s(\det E)$,

$$\mathcal{C}^s(E,\nabla) = \{D \in \mathcal{C}^s(E) \mid \det D = \nabla\}$$

and

$$\mathcal{N}^s(E,\nabla) = \mathcal{C}^s(E,\nabla)/\mathrm{SU}(E),$$

where $\mathrm{SU}(E) = \mathrm{U}(E) \cap \mathrm{SL}(E)$. Then

(3.1) $\mathcal{N}^s(E) \simeq \mathcal{M}^s(E)$ ([**LY**]) and hence $\mathcal{M}^s(E)$ is an open subset of $\mathcal{M}(E)$ ([**K1**], cf. [**Kob**]).

(3.2) Let $\mathcal{L}$ be the holomorphic structure on $\det E$ defined by $\nabla \in \mathcal{C}^s(\det E)$. Then $\mathcal{N}^s(E, \nabla) \simeq \mathcal{M}^s(E, \mathcal{L})$ and hence $\mathcal{M}^s(E, \mathcal{L})$ is an open subset of $\mathcal{M}(E, \mathcal{L})$.

(3.3) When $r = n = 2$, $\mathcal{M}^s(E)$ is the ordinary moduli space $\mathcal{M}(c_1, c_2)$ considered in algebraic geometry and $\mathcal{N}^s(E, d)$ is, if $c_1(E) = 0$ and $\nabla = d$, the moduli space of anti-self-dual $SU(2)$-connections. These two spaces are equal if and only if $H^1(M, \mathcal{O}) = 0$ (cf. (2.6) and (2.7)).

(3.4) If

$$\mathcal{M}^s_*(E) = \{[\mathcal{E}] \in \mathcal{M}^s(E) \mid H^2(M, sl\mathcal{E}) = 0\}$$

and

$$\mathcal{M}^s_*(E, \mathcal{L}) = \{[\mathcal{E}] \in \mathcal{M}^s(E, \mathcal{L}) \mid H^2(M, sl\mathcal{E}) = 0\}$$

then $\mathcal{M}^s_*(E)$ (resp. $\mathcal{M}^s_*(E, \mathcal{L})$) is a Kähler manifold and the tangent space at $[\mathcal{E}]$ is isomorphic to $H^1(M, \operatorname{End} \mathcal{E})$ (resp. $H^1(M, sl\mathcal{E})$) (cf. [**K2**], [**Kob**]), where $sl\mathcal{E}$ is the bundle of trace-free endomorphisms of $\mathcal{E}$.

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