Simple Singularities and Symplectic Fillings

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Abstract: It is proved that the diffeomorphism type of the minimal symplectic fillings of the link of a simple singularity is unique. In the proof the uniqueness of the diffeomorphism type of $\mathbf{C}P^2 \setminus D$, where D is a pseudo holomorphic rational curve of degree 3 and with one $(2,3)$-cusp, is discussed.

§0. Introduction.

The link L of an isolated singularity O on an algebraic variety V carries a natural contact structure given by the maximal complex tangency ξ. Associated to (V, O), we have two kinds of objects. One is a resolution of the singularity (V, O) and the other is so-called the Milnor fiber, which exists when (V, O) has a smoothing. Both of them provide examples of compact manifold bounding L. From symplectic viewpoint, their boundaries are "symplectically convex". Such manifolds are called symplectic fillings of (L, ξ). In general, the two symplectic fillings produced by the minimal resolution and the Milnor fiber (if it exists) of an isolated singularity on a complex surface have different topologies. In a previous paper [O-O2], we studied topological constraints for minimal symplectic fillings of links of

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simple singularities on complex surfaces. Here “minimal” means that the symplectic 4-manifold does not contain any symplectically embedded sphere of self intersection number -1, which we may call a symplectic (-1)-curve. For simple singularities, Brieskorn [B1,2,3] proved that the minimal resolution and the Milnor fiber are diffeomorphic as a consequence of existence of simultaneous resolution. Of course, we note that they are not isomorphic as complex manifolds. This result suggests us uniqueness of the diffeomorphism type for minimal symplectic fillings in these cases. The goal of this paper is to prove the following uniqueness theorem for the diffeomorphism type of minimal symplectic fillings of the link of simple singularities.

Main Theorem. *Let X be any minimal symplectic filling of the link of a simple singularity. Then the diffeomorphism type of X is unique. Hence, it must be diffeomorphic to the Milnor fiber.*

Moreover, we can also find that the symplectic deformation type of the filling is unique in the proof.

This theorem implies the following Brieskorn's theorem.

Corollary. *The minimal resolution of a simple singularity is diffeomorphic to the Milnor fiber of the singularity.*

For other classes of isolated singularities in a complex surface, we cannot, in general, expect the uniqueness result on the diffeomorphism type of the minimal symplectic fillings of the link of the singularities. But, for class of *simple elliptic singularities*, we can determine the diffeomorphism types of the minimal symplectic fillings of the link of these singularities [O-O3]. McDuff [M1] also obtained uniqueness result for other cyclic quotient singularities. After submitting this paper, there appeared a preprint “On Lens Spaces and Their Symplectic Fillings” by P. Lisca, which announces the classification of symplectic fillings of lens spaces up to diffeomorphisms.

§1. Preliminaries.

Firstly, we recall the notion of simple singularities and prepare some notations. Let $\Gamma \subset SU_2$ be a finite subgroup of SU_2. It is well known that Γ is classified by the Dynkin diagrams of type A_n $(n \geq 1)$, D_n $(n \geq 4)$, E_6, E_7 and E_8. According to these types, Γ is isomorphic to the cyclic group $\mathbf{Z}/(n+1)\mathbf{Z}$, the binary dihedral group $\mathbf{D}^*_{2(n-2)}$, the binary tetrahedral group $\mathbf{T}^*$, the binary octahedral group $\mathbf{O}^*$ and the binary icosahedral group $\mathbf{I}^*$, respectively. The Dynkin diagrams for above types are diagrams with three branches of length p, q and r. The numbers p, q and r are listed Table 1.1 below, while the rank of the Cartan matrix corresponding to Γ is given by $p + q + r - 2$ (see also Table 1.1).

Since Γ acts on $\mathbf{C}^2$ holomorphically, the quotient space $\mathbf{C}^2/\Gamma$ is a complex surface with a quotient singularity at the origin, which is called a simple singularity. The simple singularities can be also realized as isolated hypersurface singularities in $\mathbf{C}^3$ as follows. Let $f_\Gamma(x, y, z)$ be the polynomial in the following table (see [Sai2], for example).

Group Γ	Polynomial $f_\Gamma(x, y, z)$	(p, q, r)	rank Γ
A_n	$x^{n+1} + yz$	$(1, k, l)$	n
D_n	$x^2y + y^{n-1} + z^2$	$(2, 2, n-2)$	n
E_6	$x^4 + y^3 + z^2$	$(2, 3, 3)$	6
E_7	$x^3y + y^3 + z^2$	$(2, 3, 4)$	7
E_8	$x^5 + y^3 + z^2$	$(2, 3, 5)$	8

Table 1.1

Here (k, l) in the A_n-case is an arbitrary pair of positive integers satisfying $k + l = n + 1$. The polynomial f_Γ generates the ideal R_Γ in the polynomial ring $\mathbf{C}[x, y, z]$. It turns out that $\mathbf{C}[x, y, z]/R_\Gamma$ is isomorphic to the ring of Γ-invariant polynomials on $\mathbf{C}^2$. Thus the simple singularity of type Γ can be realized as an isolated singularity at the origin on the surface defined by $f_\Gamma(x, y, z) = 0$.

The link L of a simple singularity is defined by

$$L = \{(x, y, z) \in \mathbf{C}^3 \mid f_\Gamma(x, y, z) = 0,\ |x|^2 + |y|^2 + |z|^2 = \epsilon\},$$

where ϵ is a small positive real number. We define the contact distribution ξ on L by

$$\xi = \{v \in TL \mid J_0 v \in TL\},$$

where J_0 is the standard complex structure on $\mathbf{C}^3$. Note that ξ carries a globally defined contact form λ, i.e., $\xi = \ker \lambda$. A compact symplectic 4-manifold (X, ω) is called a weak symplectic filling of (L, ξ), if the following conditions are satisfied:

1) L is the boundary of X as oriented manifolds, where L is oriented by $\lambda \wedge d\lambda$ and X is oriented by ω^2,

2) ω is positively proportional to $d\lambda$ on ξ.

Moreover, if there exists an extension $\tilde{\lambda}$ of λ on a collar neighborhood of $L = \partial X$ such that $\omega = d\tilde{\lambda}$, we call (X, ω) a strong symplectic filling of (L, ξ). If ω is exact, e.g., L is a homology sphere, the symplectic form ω of a weak symplectic filling can be modified to another symplectic form such that X becomes a strong symplectic filling (see e.g., Proposition 3.1 in [E], Lemma 2.1 in [O-O2]). Hence, for the link of a simple singularity, we may assume that our symplectic filling is a strong symplectic filling. If we replace the condition 1) above by 1^-) below, we get a notion of "(strong) concave fillings".

1^-) L is the boundary of X and the orientations induced by the contact structure and the symplectic structure, respectively, are opposite.

If (X, ω) is a strong symplectic filling of (L, ξ), a collar neighborhood of ∂X can be embedded in the symplectization of (L, ξ) such that the outward normal vector of X points positive side. For strong concave fillings, we have a similar embedding, but the outward normal vector points negative side. Hence, we can glue a strong symplectic filling X and a strong concave filling Y to get a closed symplectic manifold possibly after modifying the symplectic structures.

§2. Outline of the proof of Main Theorem.

Our strategy is similar to the one in [O-O2], but we investigate the situation more precisely here. Let M_Γ be the Milnor fiber defined by $\{(x, y, z) \in \mathbf{C}^3 \mid f_\Gamma(x, y, z) = 1\}$, where $f_\Gamma(x, y, z)$ is the polynomial in Table 1.1. We regard $f_\Gamma(x, y, z)$ as a weighted homogeneous polynomial. The weights and degrees are given by the following table.

Group Γ	Weights (a, b, c)	degree h
A_n	$(1, k, l)$	$n+1$
D_n	$(n-2, 2, n-1)$	$2(n-1)$
E_6	$(3, 4, 6)$	12
E_7	$(4, 6, 9)$	18
E_8	$(6, 10, 15)$	30

Table 2.1

We recall a compactification of the Milnor fiber of a simple singularity after Saito. For more information, see [Sai1]. Consider the weighted projective space $\mathbf{P}(a,b,c,1) = (\mathbf{C}^4 - 0)/\sim$, where $(x,y,z,w) \sim (t^a x, t^b y, t^c z, tw)$ for $t \in \mathbf{C}^*$. By a map ι from $\mathbf{C}^3$ to $\mathbf{P}(a,b,c,1)$ defined by $\iota(x,y,z) = [x,y,z,1]$, $\mathbf{C}^3$ can be embedded as an open dense subset so that $\mathbf{P}(a,b,c,1) = \mathbf{C}^3 \cup \mathbf{P}(a,b,c)$. Here $\mathbf{P}(a,b,c) = \{[x,y,z,0] \in \mathbf{P}(a,b,c,1)\}$. Denote by $\overline{M_\Gamma}$ the closure of $\iota(M_\Gamma)$ in $\mathbf{P}(a,b,c,1)$, which is the hypersurface defined by $f_\Gamma(x,y,z) = w^h$ in $\mathbf{P}(a,b,c,1)$. It is singular, when Γ is not of the A_1-type. The singular points on $\overline{M_\Gamma}$ lie on $C = \overline{M_\Gamma} \setminus \iota(M_\Gamma) = \overline{M_\Gamma} \cap \mathbf{P}(a,b,c)$, which is a smooth rational curve. These singularities turns out cyclic quotient singularities of type $A_{n,1}$, where $n = p,q,r$. Taking the minimal resolution $\pi : \widetilde{M_\Gamma} \to \overline{M_\Gamma}$, we have a compact projective smooth surface $\widetilde{M_\Gamma}$, which contains the Milnor fiber M_Γ as an open dense subset. A little modification is necessary for A_n-singularities. The equations of type A_n in Table 1.1 are different from what Saito used in [Sai1], but the same argument works. The reason why we use $x^{n+1} + yz$ is two-fold. Firstly, we discuss cases both for n even and odd, while the cases with even n are not discussed in [Sai1]. Secondly, it seems more convenient in the argument in Section 3 with this equation and we choose the weights $(1,k,l) = (1,2,n-1)$ when $n \geq 2$. Anyway, in the A_n-case, some of p,q,r in Table 1.1 are 1 and the number of singularities on $\overline{M_\Gamma}$ is less than 3. To deal all the cases systematically, in the A_n-case, we also blow-up at an appropriate number of non-singular points on C and denote it by $\widetilde{M_\Gamma}$ to get the following configuration of the compactifying divisor for any Γ.

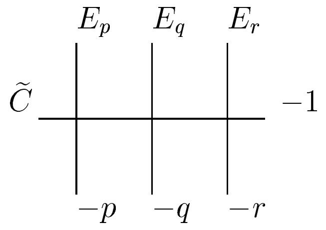


Figure 2.2

We denote by $\widetilde{C}$ the proper transform of C, which has self intersection number -1 and by E_p, E_q and E_r the exceptional set for the resolution or blow-up with the self-intersection number $E_n^2 = -n$ for $n = p,q,r$ respectively. Here the integers p,q and r are given by Table 1.1. Each of curves

E_p, E_q and E_r intersects with $\widetilde{C}$ at one point transversally. Saito also showed that

$$K_{\widetilde{M_\Gamma}} = -(2\widetilde{C} + E_p + E_q + E_r). \tag{2.4}$$

In particular, $-K_{\widetilde{M_\Gamma}}$ is an effective divisor. Write

$$M_\Gamma^{\mathrm{out}} = \widetilde{M_\Gamma} \setminus \iota(M_\Gamma \cap B^6),$$

which is a regular neighborhood of $\widetilde{C} \cup E_p \cup E_q \cup E_r$. Here B^6 is the closed unit disc around the origin in $\mathbf{C}^3$. Note that M_Γ^{out} is a strong concave filling in the sense of Section 1. Thus we can glue the symplectic filling X and M_Γ^{out} to obtain a closed symplectic manifold $(Z(\Gamma), \omega)$:

$$Z(\Gamma) := X \cup_{S^3/\Gamma} M_\Gamma^{\mathrm{out}}, \tag{2.5}$$

which we call a compactification of the filling X. We apply the following theorem in [O-O1] to $Z(\Gamma)$.

Theorem 2.6. *Let (M, ω) be a closed symplectic 4-manifold such that*

$$\int_M c_1(M) \wedge \omega > 0.$$

Then M is a rational or ruled symplectic 4-manifold up to blow-up and down.

We briefly recall the argument in [O-O1]. Using Taubes' result, we find a symplectically embedded 2-sphere with non-negative self intersection number in (M, ω). Then a theorem by McDuff [M1] states that (M, ω) must be a rational or ruled symplectic 4-manifold, the symplectic deformation type of which is also uniquely determined. (As for classification of symplectic structures on such manifolds, see [G], [M1], [L-M].) Namely, after blowing-down a family of disjoint symplectic (-1)-curves, we get a minimal symplectic 4-manifold, which is symplectic deformation equivalent either to the complex projective plane or to a minimal ruled symplectic 4-manifold. Note that Theorem 2.6 was also independently proved by A. Liu [Liu], see also [L-L]. In section 3, we will describe the symplectic deformation type, in particular, the diffeomorphism type of $Z(\Gamma)$ explicitly (Theorem 3.1). The symplectic filling X is the complement of a regular neighborhood of the compactifying divisor, as in Figure 2.2, in $Z(\Gamma)$. In other words, our remaining task is to determine how the compactifying divisor $\widetilde{C} \cup E_p \cup E_q \cup E_r$ is embedded in $Z(\Gamma)$. We proceed as follows. Firstly, we make blow-up and down operations on $Z(\Gamma)$ and deduce the problem to the one concerning the image D of the compactifying divisor in $\mathbf{C}P^2$ or $\mathbf{C}P^1 \times \mathbf{C}P^1$. We will see that D is a pseudo-holomorphic cubic or bidegree $(2,2)$ curve with one $(2,3)$-cusp singularity in $\mathbf{C}P^2$ or $\mathbf{C}P^1 \times \mathbf{C}P^1$ with respect to a certain compatible almost complex structure (Section 4). Then we will show a uniqueness result of the diffeomorphism type of the complements $\mathbf{C}P^2 \setminus D$ and $\mathbf{C}P^1 \times \mathbf{C}P^1 \setminus D$ (Section 5). This is an analog of symplectic isotopy problem for singular pseudo-holomorphic curves.

§3. Description of the compactification $Z(\Gamma)$.

In this section, we show the following

Theorem 3.1. *The compactification $Z(\Gamma)$ is a rational or ruled symplectic 4-manifold up to blow up and down. More explicitly, $Z(\Gamma)$ is symplectomorphic to $n+3$-point blow-up of the complex projective plane. In particular, it is diffeomorphic to the following rational symplectic 4-manifold according to the type of Γ.*

Type of Γ	Diffeomorphism type of $Z(\Gamma)$
A_n	$\mathbf{C}P^2 \# (n+3)\overline{\mathbf{C}P^2}$
D_n	$\mathbf{C}P^2 \# (n+3)\overline{\mathbf{C}P^2}$
E_6	$\mathbf{C}P^2 \# 9\overline{\mathbf{C}P^2}$
E_7	$\mathbf{C}P^2 \# 10\overline{\mathbf{C}P^2}$
E_8	$\mathbf{C}P^2 \# 11\overline{\mathbf{C}P^2}$

Proof. To prove the first part of Theorem 3.1, it suffices to show that $Z(\Gamma)$ satisfies the assumption in Theorem 2.6. Recall that $2\widetilde{C} + E_p + E_q + E_r$ is an anti-canonical divisor of $\widetilde{M_\Gamma}$, which is contained in M_Γ^{out}. On the other hand, Theorem 2 in [O-O2] and Theorem 10.1 in [K] state that the canonical bundle K_X of any minimal symplectic filling X of S^3/Γ with the quotient contact structure is trivial. Since S^3/Γ is a rational homology 3-sphere, we conclude that $2\widetilde{C} + E_p + E_q + E_r$ is also an anti-canonical divisor of $Z(\Gamma)$. Note that it is written as a positive combination of symplectically embedded surfaces. Hence we have

$$\int_{Z(\Gamma)} c_1(TZ(\Gamma)) \wedge \omega = 2\int_{\widetilde{C}} \omega + \int_{E_p} \omega + \int_{E_q} \omega + \int_{E_r} \omega > 0.$$

This is the inequality in the assumption of Theorem 2.6.

Next we shall prove the second part. To do this, we need more detailed information on the topology of the symplectic filling X. The following proposition is an improvement of a result in [O-O2].

Proposition 3.2. *Let X be any minimal symplectic filling of the link of the simple singularity of type Γ. Then the intersection form of X is negative definite and $b_2^-(X) = \mathrm{rank}\ \Gamma$. Here* rank Γ *is the rank of the Cartan matrix corresponding to the Dynkin diagram of Γ.*

Remark 3.3. The fact that X is negative definite is the conclusion of Theorem 1 in [O-O2] (see also Theorem 1.4 in [Lis]). Moreover, from Theorem 2 in [O-O2] and Theorem 10.1 in [K], we know that the canonical bundle K_X is always trivial. In particular, X is a spin manifold. Then, using Froyshov's result [F], we obtain the following estimate

$$b_2^-(X) \leq \mathrm{rank}\ \Gamma. \tag{3.4}$$

This estimate was already obtained in [O-O2]. What we want to show here is that the equality exactly holds in (3.4). We would also like to mention that we do *not* use Froyshov's result to prove it in this paper.

We prepare a couple of lemmas.

Lemma 3.5. *We have* $b_1(M_\Gamma^{\mathrm{out}}) = 0$.

Proof. Note that M_Γ^{out} is a regular neighborhood of the configuration of rational curves as in Figure 2.2. Then the conclusion is obvious. ∎

Next, we prove the following.

Lemma 3.6. *For any symplectic filling X of the link of the simple singularity of type Γ, we have $b_1(X) = 0$.*

Proof. Suppose that $b_1(X) \neq 0$, then we have a non trivial abelian covering $p : \widetilde{X} \to X$ of any degree $d > 1$. Let us take d bigger than the order of Γ, which is certainly isomorphic to the fundamental group of ∂X. Then the boundary $\partial\widetilde{X}$ consists of at least two connected components, each of which is isomorphic to S^3/Γ_i' for some finite subgroup $\Gamma' \subset SU_2$. By pulling back the symplectic from ω on X to $\widetilde{X}$ by p, $\widetilde{X}$ is also a symplectic 4-manifold with non-connected boundary $\bigsqcup_i S^3/\Gamma_i'$. But Theorem 3 in [O-O2] (also Theorem 1.4 in [Lis]) claims that $\partial\widetilde{X}$ should be connected, which is a contradiction. ∎

Remark 3.7. From the proof of Lemma 3.6, we can also show that $\pi_1(X)$ has no finite index subgroup. Namely, if $\pi_1(X)$ contains a subgroup of index d, there corresponds a d-fold covering $\overline{X}$ of X. It carries the pull-back symplectic form. If $\overline{X}$ has at least two boundary, which are contactomorphic to the link of some simple singularities, which contradicts to connectedness of the boundary established in [O-O2]. If the boundary of $\overline{X}$ is connected, we can glue with some $M_{\Gamma'}^{\mathrm{out}}$ to get a closed symplectic 4-manifold with $b_1 = 0$ and $b_2^+ = 1$. Since $\overline{X} \to X$ is a d-fold covering, the Euler number and the signature satisfy $\chi(\overline{X}) = d\chi(X)$ and $\sigma(\overline{X}) = d\sigma(X)$, from which we can easily derive a contradiction.

Now we show the following.

Lemma 3.8. *For the closed symplectic 4-manifold $Z(\Gamma)$ in (2.5), we have $b_1(Z(\Gamma)) = 0$.*

Proof. We apply the Mayer-Vietoris exact sequence (over $\mathbf{Q}$-coefficients) to the decomposition in (2.5). Then by Lemma 3.5 and Lemma 3.6 imply Lemma 3.8. ∎

Proof of Proposition 3.2. From the formula (2.4) and the fact that $K_X = 0$ for the minimal symplectic filling X, we have

$$c_1^2(K_{Z(\Gamma)}) = c_1^2(K_{M_\Gamma^{\mathrm{out}}}) = c_1^2(K_{\widetilde{M_\Gamma}}) = 8 - (p + q + r).$$

On the other hand, we have $b_2(Z(\Gamma)) = b_2(X) + 4$, and $\sigma(Z(\Gamma)) = \sigma(X) + \sigma(M_\Gamma^{\mathrm{out}}) = \sigma(X) - 2$ (Novikov additivity for signatures). It follows that

$$2\chi(Z(\Gamma)) + 3\sigma(Z(\Gamma)) = 6 - 4b_1(Z(\Gamma)) + 5b_2^+(X) - b_2^-(X).$$

Since $b_2^+(X) = 0$ and $b_1(Z(\Gamma)) = 0$ by Lemma 3.8, we obtain the following equality

$$2\chi(Z(\Gamma)) + 3\sigma(Z(\Gamma)) = 6 - b_2^-(X).$$

Note that $c_1^2(K_{Z(\Gamma)}) = 2\chi(Z(\Gamma)) + 3\sigma(Z(\Gamma))$. Combining these equalities, $b_2^-(X)$ can be calculated as

$$b_2^-(X) = p + q + r - 2. \tag{3.12}$$

Hence we obtain Proposition 3.2 (with values p, q, r in Table 1.1). ∎

Proof of Theorem 3.1, continued. Now, Lemma 3.8 exclude the case that $Z(\Gamma)$ is diffeomorphic to any ruled surface over a Riemann surface of genus ≥ 1. There are two symplectic deformation types of minimal rationally ruled symplectic 4-manifolds, i.e., $\mathbf{C}P^2$ and $\mathbf{C}P^1 \times \mathbf{C}P^1$. Note that the one point blow-up of $\mathbf{C}P^1 \times \mathbf{C}P^1$ has the same symplectic deformation type as the two point blow-up of $\mathbf{C}P^2$. Furthermore, we have $b_2^+(Z(\Gamma)) = b_2^+(M_\Gamma^{\text{out}}) = 1$ and

$$b_2^-(Z(\Gamma)) = b_2^-(X) + b_2^-(M_\Gamma^{\text{out}}) = p + q + r + 1.$$

Hence we can determine the symplectic deformation type, in particular, the diffeomorphism types of $Z(\Gamma)$, which completes the proof of Theorem 3.1. ∎

§4. Contraction of exceptional rational curves.

In this section, we make blow-up and down operations on the symplectic 4-manifold $Z(\Gamma)$ constructed in section 3 to transform the compactifying divisor $\widetilde{C} \cup E_p \cup E_q \cup E_r$ to a pseudo holomorphic singular cubic curve in a rational surface. We shall describe the singular curve explicitly.

We call a 2-dimensional homology class e a symplectic (-1)-class, if e contains a symplectic (-1)-curve, i.e., a symplectically embedded sphere of self intersection number -1.

First of all, we show the following useful lemma.

Lemma 4.1. *Let D be an irreducible J-holomorphic curve in a symplectic 4-manifold M. Suppose that D represents the anti-canonical class, i.e., the homology class $[D]$ is the Poincaré dual of the first Chern class $c_1(M)$. Then there exists a compatible almost complex structure J' such that J' coincides J in a neighborhood of D and any symplectic (-1)-class e is represented by a unique J'-holomorphic representative E. Moreover, E intersects D at exactly one regular point transversally.*

Remark 4.2. It is shown in [M1] that any symplectic (-1)-class is represented by a J-holomorphic embedded sphere for a generic compatible almost complex structure J. structure J. In Lemma 4.1, the almost complex structure J' is not generic in the whole space of compatible almost complex structure, since J' is prescribed in a neighborhood of D.

Proof. We pick a compatible almost complex structure J' satisfying the following conditions:

1) J' coincides J in a neighborhood of D, in particular, D is J'-holomorphic,

2) there are no J'-holomorphic sphere with self intersection number less than -1, except possibly D, in particular there are no embedded J'-holomorphic sphere with self intersection number less than -1.

This is possible, since the virtual dimension of the moduli space of pseudo-holomorphic curves as in 2) is negative.

Note that the class e has a unique pseudo-holomorphic representative for a generic compatible almost complex structure, which is necessarily embedded. We pick a sequence of generic compatible almost complex structure J_n converging to J'. Denote by E_n the J_n-holomorphic representative of the class e. Suppose that E_n converges to some J'-holomorphic sphere E. Since the self intersection number of e is -1, E is not multiply covered. Then the fact that the virtual genus of the class e is 0 implies that E is an embedded J'-holomorphic sphere representing the class e, which is the desired J'-holomorphic (-1)-curve. (Here we call the number $(e \cdot e - c_1(M) \cdot e + 2)/2$ the virtual genus of the class e.) Suppose that E_n converges as stable maps to $f : \cup \mathbf{C}P^1_{(i)} \to M$ consisting of at least two components and write $f(\mathbf{C}P^1_{(i)}) = B_i$. We may assume that at least one of B_i is a multiple of D. Otherwise, the genericity argument by McDuff [M1] guarantees existence of symplectically embedded 2-sphere representing the class e. We denote by $k[D]$ the sum of all homology classes $[B_i]$, which are multiples of $[D]$.

For other B_i, we write $B_i = l_i C_i$ with $l_i \geq 1$ and C_i being the images of simple (i.e., not multiply covered) J'-holomorphic maps. Since all C_i's are not contained in a neighborhood of D, the genericity of J' assures transversality for somewhere injective J'-holomorphic spheres C_i. Hence, for a sufficiently large n, the homology classes $\{C_i\}$ are represented by J_n-holomorphic spheres $C_{i,n}$. This implies that the intersection numbers $C_i \cdot e = C_{i,n} \cdot E_n$ are non negative:

$$C_i \cdot (\sum l_i C_i + kD) \geq 0.$$

On the other hand, e is a symplectic (-1)-class, hence we have $c_1(M)e = 1$. Since D represent the anti-canonical class of M, we find

$$D \cdot (\sum l_i C_i + kD) = 1.$$

Since k and l_i are positive integers, these two inequalities implies that

$$e^2 = (\sum l_i C_i + kD)^2 > 0,$$

which contradicts to $e^2 = -1$.

Let E be the J'-holomorphic (-1)-sphere representing the class e. Since $D\cdot E = 1$ and E and D are J'-holomorphic curves, E and D intersects exactly at one point transversally. ∎

From now on, we pick a compatible almost complex structure on $Z(\Gamma)$, which is *integrable* near the compactifying divisor.

Case: E_n with $(p, q, r) = (2, 3, n-3)$, $n = 6, 7, 8$.

We recall that we have $\widetilde{C}$ in $M^{\mathrm{out}}_{E_n} \subset Z(E_n)$ is a (-1)-curve. We can blow down $Z(E_n)$ along $\widetilde{C}$ to get a 4-manifold $Z(E_n)_{(1)}$. Then the curve E_p becomes a (-1)-curve E'_p. We contract the (-1)-curve E'_p to get a 4-manifold $Z(E_n)_{(2)}$. Then we can see that (-1)-curve E''_q and $(-n+5)$-curve E''_r, (note that $-r+2 = -n+5$), intersect as in the following picture.

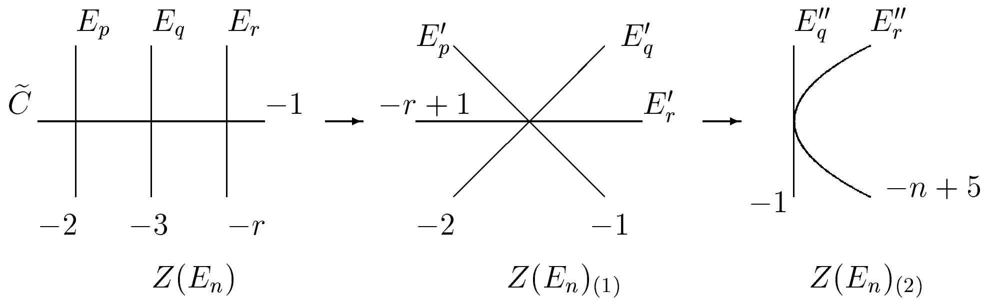


Figure 4.3

Furthermore, we contract the E''_q to get a 4-manifold $Z(E_n)_{(3)}$ which contains a singular curve E'''_r. Its self intersection number is $-n+9$. From Theorem 3.1, it is easy to see that $Z(E_n)_{(3)}$ is symplectic deformation equivalent to $\mathbf{C}P^2\#n\overline{\mathbf{C}P^2}$. Thus we can blow down $Z(E_n)_{(3)}$ along further n (-1)-curves to get a $\mathbf{C}P^2$. It is, however, necessary to know how E'''_r is transformed under the blow-down process. For this purpose, we use Lemma 4.1 to make exceptional curves of the blow-down process and E'''_r pseudo-holomorphic, with respect to a common compatible almost complex structure, simultaneously. Up to this stage $Z(E_n)_{(3)}$, the blow-down is carried out also as complex blow-down and the complex structure near the compactifying divisor descends to the one in a neighborhood of $Z(E_n)_{(3)}$. Hence we may assume that the almost complex structure is integrable in a neighborhood of E'''_r, the effect to E'''_r under these blow-down is same as in complex geometry. Note also that

McDuff showed that any closed symplectic 4-manifold becomes minimal by blow-down a maximal family of symplectic (-1)-curves simultaneously [M1].

More precisely, we show the following.

Proposition 4.4. *Let e_j, $j = 1, \cdots, n$, be a maximal family of disjoint (-1)-rational curves in $Z(E_n)_{(3)}$, which is symplectic deformation equivalent to $\mathbf{C}P^2 \# n\overline{\mathbf{C}P^2}$. Let $\pi : Z(E_n)_{(3)} \to \mathbf{C}P^2$ be a smooth mapping corresponding to the blowing down along the e_j's. Then $\pi(E_r''')$ is represented by a pseudo holomorphic curve of degree 3 with one cusp singularity of type $(2,3)$.*

Proof. Choose a maximal family of disjoint symplectic (-1)-curves in $Z(\Gamma)$ to get $\mathbf{C}P^2$. Applying Lemma 4.1 with $D = E_r'''$, then we may assume that e_j and E_r''' are pseudo-holomorphic with respect to a common almost complex structure. Moreover, e_j and E_r''' intersect only at one point and transversally. Therefore the only singularity of $\pi(E_r''')$ is the image of the one in E_r'''. Since we choose our almost complex structure integrable near this curve, the type of the singularity is the same as in complex geometry, i.e., a $(2,3)$-cusp. We also find that the self intersection number of $\pi(E_r''')$ must be $(-n+9)+n = 9$, which implies that its homology class is $3[\mathbf{C}P^1] \in \mathrm{H}_2(\mathbf{C}P^2)$. Thus we get the conclusion. ∎

Case: D_n with $(p,q,r) = (2,2,n-2)$.

We recall the compactification in section 2 (also Figure 2.2) and blow-down (-1)-curves in it. We first contract the central (-1)-curve $\widetilde{C}$ and then contract the image E_q' of E_q (see the first two steps in Figure 4.5). We denote the blown down manifold at this stage by $Z(D_n)_{(2)}$. (One rational curve E_p'' with self intersection number 0 and the other rational curve E_r'' with self intersection number $-n+4$, which are tangent each other)

We blow up at a point on E_p'', which becomes a (-1)-curve $\widetilde{E}_p''$. We contract $\widetilde{E}_p''$ next (see the last three steps in Figure 4.5). We denote the manifold at this stage by $Z(D_n)_{(3)}^{(1)}$, where we have one rational curve $A = e_1'$ with self intersection number 0 and $D = E_r'''$ a singular rational curve with a $(2,3)$-cusp with self intersection number $-n+8$, which intersect at the cusp point.

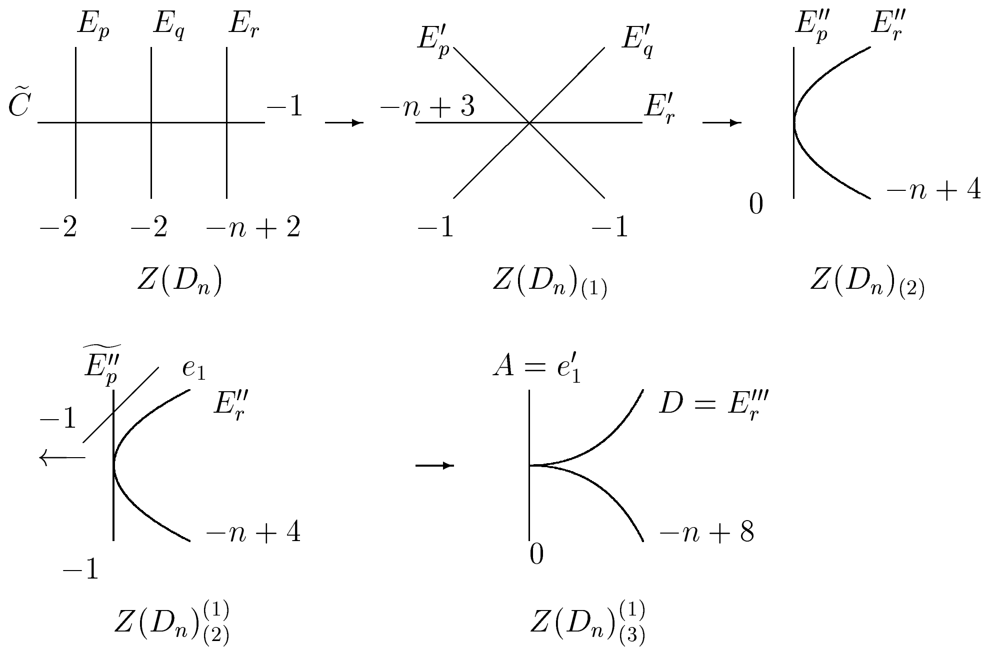


Figure 4.5

Finally, we contract a maximal family of disjoint (-1)-curves away from A. We shall show the following:

Proposition 4.6. *There is a family of disjoint (-1)-curves in $Z(D_n)^{(1)}_{(3)}$ such that $Z(D_n)^{(1)}_{(3)}$ becomes $\mathbf{C}P^1 \times \mathbf{C}P^1$ after contracting those (-1)-curves and A is identified with $\mathbf{C}P^1 \times \{pt\}$.*

Proof. We contract a maximal family of disjoint (-1)-curves ϵ_k, which do not intersect A.

Then the resulting manifold is a ruled symplectic 4-manifold with A a fiber of the ruling structure (see [M1]). Since the first Betti number is 0, the base of the fibration must be a 2-sphere. There are two symplectic deformation types. They are $\mathbf{C}P^1 \times \mathbf{C}P^1$ and F_1, the Hirzebruch surface of degree 1, which is diffeomorphic to $\mathbf{C}P^2 \# \overline{\mathbf{C}P^2}$. Both has the second Betti number 2, hence we find that the number of (-1)-curves contracted is n, which is positive. (Recall that $b_2(Z(D_n)^{(1)}_{(3)} = n + 2$.) If it is $\mathbf{C}P^1 \times \mathbf{C}P^1$, we get the conclusion of the proposition. Otherwise it is F_1. Pick one of

(-1)-curves, say ϵ_1, which is contracted to a point p. We make a blowing-up at the point p. Then the proper transform $\tilde{f}_p$ of the fiber f_p passing through the point p is a (-1)-curve, which is disjoint from A and other (-1)-curves $\{\epsilon_k\}_{2\leq k\leq n}$. So $\{\tilde{f}_p\}\cup\{\epsilon_k\}_{2\leq k\leq n}$ is a disjoint family of symplectic (-1)-curves. Blowing down these symplectic (-1)-curves, we arrive at $\mathbf{C}P^1\times\mathbf{C}P^1$. Since $\tilde{f}_p$ and ϵ_k are disjoint from A, a tubular neighborhood of A is not affected by the blowing-down process. Hence A has selfintersection 0 in $\mathbf{C}P^1\times\mathbf{C}P^1$ and it is a fiber of a ruling structure. ∎

Remark 4.7. The process of blowing up at p and blowing down f is called an elementary transformation in algebraic geometry.

Theorem 4.8. *The image of the divisor $A\cup D$ in $\mathbf{C}P^1\times\mathbf{C}P^1$ consists of $\mathbf{C}P^1\times\{pt\}$ and a pseudo holomorphic rational curve with one $(2,3)$-cusp singularity, which represents the homology class $[2,2]$. Moreover, they intersect at the cusp point.*

Proof. In a previous proposition we described the blown-down manifold. Note that the singular curve D represents the anti-canonical class. Thus Lemma 4.1 implies that all contracted (-1)-curves intersect D at a regular point transversally. Hence the image of D has self-intersection number $-n+8+n=8$. Since the intersection number with $A=\mathbf{C}P^1\times\{pt\}$ is 2, we conclude that the image of D represents the homology class $[2,2]$. ∎

Case: A_n with $(p,q,r)=(1,1,1)$ for $n=1$ and $(p,q,r)=(1,2,n-1)$ else.

We discuss the A_1-case separately. In this case we contract three (-1)-curves E_p, E_q, E_r. Then the central divisor $\widetilde{C}$ becomes a non-singular rational curve $\overline{C}$ of self-intersection number 2.

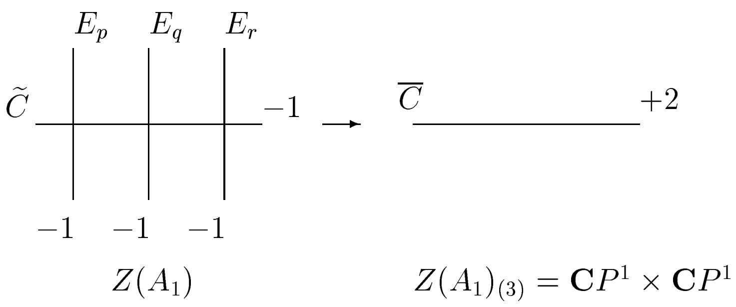
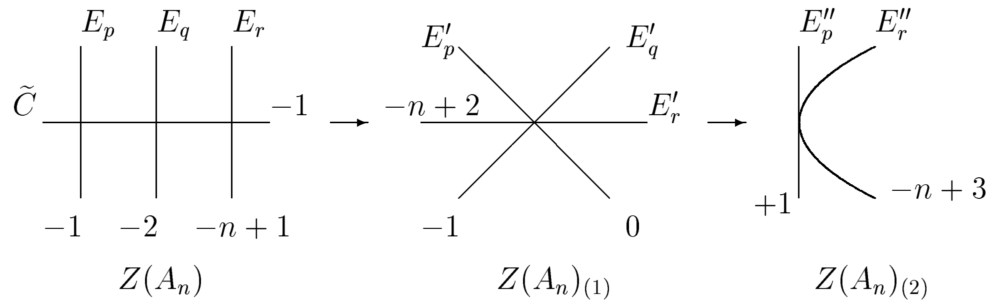


Figure 4.9

Since there are no (-1)-curves in the complement of $\overline{C}$, a result of McDuff [M1] asserts that the manifold is $\mathbf{C}P^1 \times \mathbf{C}P^1$ and $\overline{C}$ is isotopic to the diagonal. Hence X is diffeomorphic to a disk bundle over the 2-sphere with Euler number -2.

Remark 4.10. The anti-canonical class is represented by $2\overline{C}$, hence the manifold is spin. This also leads to the assertion that the manifold is $\mathbf{C}P^1 \times \mathbf{C}P^1$, since it is the only rational surface with a spin structure.

We next consider the case of A_n with $n \geq 2$. We use the result in section 2 with $(p, q, r) = (1, 2, n-1)$. Firstly we contract the central (-1)-curve $\tilde{C}$ and then, contract the image E'_q of E_q. The configuration consists of a rational curve E''_p with self-intersection number 1 and a rational curve of self-intersection number $-n+3$, which are tangent each other. Denote by $Z(A_n)_{(2)}$ the blown-down manifold at this stage. We blow-up $Z(A_n)_{(2)}$ at two points on E''_p and, then, blow-down the proper transform $\tilde{E}''_p$ of E''_p. Then E''_r becomes a singular rational curve $D = E'''_r$ with one $(2,3)$-cusp point and self intersection number $-n+7$, and two exceptional curves e_1 and e_2 become two symplectically embedded spheres $A = e'_1$ and $B = e'_2$ with self intersection number 0, which intersects transversally at the cusp point of D. We denote by $Z(A_n)^{(2)}_{(3)}$ the ambient manifold at this stage.



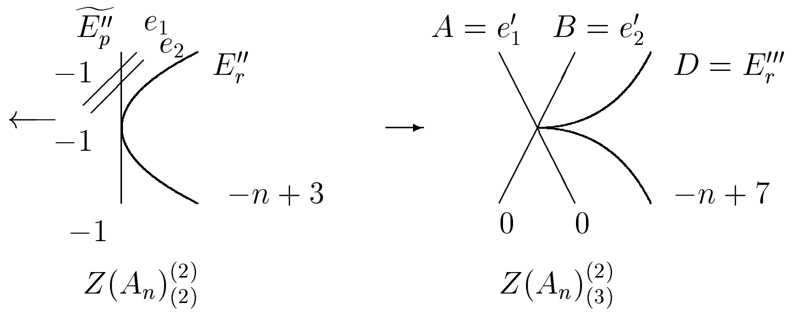


Figure 4.11

Next we contract a maximal family of disjoint (-1)-curves away from A. The following proposition is proved in a way similar to that of Proposition 4.6.

Proposition 4.12. *There is a family of disjoint (-1)-curves in $Z(A_n)^{(2)}_{(3)}$ such that $Z(A_n)^{(2)}_{(3)}$ becomes a ruled symplectic 4-manifold over a 2-sphere and A a fiber of the ruling structure.*

In this process, the contracted (-1)-curves ϵ_k may intersect the other rational curve B. However, changing the family of disjoint (-1)-curves, we have the following:

Theorem 4.13. *There is a family of disjoint (-1)-curves such that after blowing down those (-1)-curves $Z(A_n)^{(2)}_{(3)}$ becomes $\mathbf{C}P^1 \times \mathbf{C}P^1$ with A and B being identified with $\mathbf{C}P^1 \times \{pt\}$ and $\{pt\} \times \mathbf{C}P^1$ respectively. Moreover, the image of D is a singular rational curve with one $(2,3)$-cusp point representing the class $[2,2]$.*

Proof. From Proposition 4.12, after blowing-down a certain family of (-1)-curves ϵ_k, some of which may intersect B, $Z(A_n)^{(2)}_{(3)}$ becomes a ruled symplectic 4-manifold N over a 2-sphere.

We note the following:

Lemma 4.14. *For any ϵ_k, B and ϵ_k are disjoint or they intersect transversally at one point.*

Proof. Since A is one of the fiber of the ruling structure, the intersection number of any fiber and the image of B is $+1$. Hence, the image of B has no singular points, which means that ϵ_k intersects B at most once counting with multiplicity. ∎

We return to the proof of Theorem 4.13. Suppose that some of ϵ_k intersect B. By a small smooth perturbation of the ruling structure, we may assume that the images of all intersection points of B and $\cup\epsilon_k$ lie on distinct fibers on N. Then we blow-up these intersection points and blow-down the proper transforms $\{\tilde{f}_*\}$ of the fibers $\{f_*\}$ passing those points. As in the previous theorem, this corresponds to changing a family of disjoint (-1)-curves. After blowing-down the new family of (-1)-curves which do not intersect A and B at all, A and B becomes a symplectically embedded spheres of self intersection number 0 in a ruled symplectic 4-manifold. This implies that the ambient manifold is $\mathbf{C}P^1\times\mathbf{C}P^1$ and A and B are $\mathbf{C}P^1\times\{pt\}$ and $\{pt\}\times\mathbf{C}P^1$ respectively. See Figure 4.15 below.

The description of the image of D can be obtained in a similar way as in a previous theorem. ∎

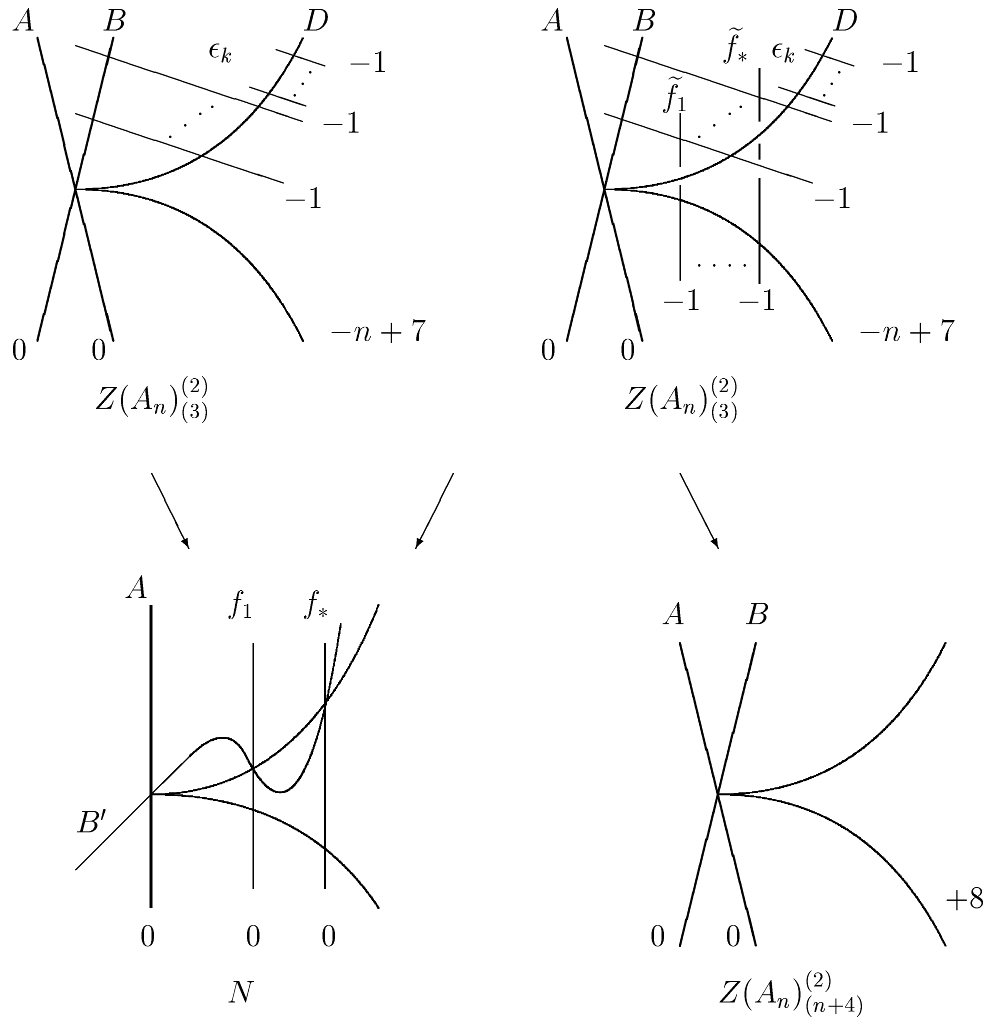


Figure 4.15

§5. Diffeomorphism type of $\mathbf{C}P^2 \setminus D$.

For symplectically embedded surfaces in $\mathbf{C}P^2$ with small degrees, there are uniqueness results on symplectic isotopy types. (See [G] for degree 1 and 2 cases, [Sik] for the degree 3 case and [I-S] for more investigation. See also [Ba1] for a related result.) Recently, Siebert and Tian proved uniqueness of symplectic isotopy type for symplectically embedded surfaces up to degree 17 [S-T]. We need a similar uniqueness result for singular pseudo holomorphic curves.

It is neccessary to discuss compactness property for singular pseudo holomorphic curves in order to show the uniqueness of isotopy type of such pseudo holomorphic curves. We give a description of stable map compactification and prove stratum-wise transversality. Barraud proved local manifold structure for moduli spaces of singular pseudo holomorphic curves under certain conditions [Ba2]. Here we only need to show transversality for pseudo holomorphic curve equation for a restricted class of stable maps and give a rather straightforward proof of transversality for reader's convenience.

Let D_0 be a singular rational cubic curve, with one $(2,3)$-cusp point, in $\mathbf{C}P^2$ with respect to the standard complex structure. The goal of this section is to show the following

Theorem 5.1. *Let D be a pseudo-holomorphic rational cubic curve with one $(2,3)$-cusp point in $\mathbf{C}P^2$. Then $\mathbf{C}P^2 \setminus D$ is diffeomorphic to $\mathbf{C}P^2 \setminus D_0$.*

The following theorem is obtained as a corollary of Theorem 5.1.

Theorem 5.2 *Let D' be a pseudo-holomorphic rational curve with one $(2,3)$-cusp point, which represents the class $[2,2]$ in $\mathbf{C}P^1 \times \mathbf{C}P^1$. Then $\mathbf{C}P^1 \times \mathbf{C}P^1 \setminus D'$ is diffeomorphic to $\mathbf{C}P^1 \times \mathbf{C}P^1 \setminus D'_0$, where D'_0 is a rational curve with one $(2,3)$-cusp point, which is holomorphic with respect to the standard complex structure and represents the class $[2,2]$. In particular, the diffeomorphism type is unique.*

Proof. Pick a non-singular point P on D'. We make a blow-up at P and then blow-down the proper transforms of fibers f_1, f_2 passing through P. (Here f_1 and f_2 are fibers with respect to the two ruling structures on the product.) Then the problem reduces to the case of cubic curve in $\mathbf{C}P^2$. ∎

Our strategy to prove Theorem 5.1 is as follows. Pick a 1-parameter family of compatible almost complex structures $\{J_t\}$ joining arbitrary J and the standard J_0 and try to find a 1-parameter family of J_t-holomorphic singular curves with singularities of $(2,3)$-cusp type. Fix a point p on $\mathbf{C}P^2$. For later purpose, we assume that J_t coincide, on a neighborhood of p, with an integrable complex structure. We use the continuing method. The openness for $\{t\}_{0 \leq t \leq s}$ follows from transversality of the linearized operator. In dimension 4, Hofer, Lizan and Sikorav [H-L-S] established a nice transversality result only depends on the topological type of the normal bundle, when the pseudo holomorphic curve is immersed. We extend the argument to a singular curve case as in [I-S], but with a smaller domain of the linearization operator. Since we need a family of J_t-holomorphic curves with the prescribed type of singularities, we consider the transversality problem in the space of mappings with a certain condition on the differential at a fixed point (see below). The closedness concerns compactness. Since a sequence of singular cubic curves with one $(2,3)$-cusp may converges to a multiple covered curve, possibly with bubbling components. Therefore we have to consider all possible degenerations. For this purpose, we use the stable map compactification. (For detailed discussion on stable maps, see e.g., [F-O].)

Let $\mathcal{M}_0(3, J)$ be the moduli space of J-stable maps of genus 0 representing the class $3[\mathbf{C}P^1]$ in $\mathbf{C}P^2$. A stable map of genus 0 means that the domain semi-stable curve is of genus 0. (Note that non-singular cubic curve is of genus 1. Singular cubic curves can be also seen as stable maps of genus 1. But we do *not* treat in that way.)

We stratify $\mathcal{M}_0(3, J)$ according to the following types.

1) The image of the pseudo holomorphic map from the Riemann sphere is irreducible.

1-1) It is an immersion with one double point;

1-2) It is not an immersion and an image of a simple pseudo holomorphic map from the Riemann sphere;

1-3) It is not an immersion and an image of a triple covered pseudo holomorphic map from the Riemann sphere;

2) The image of the stable map from the semi-stable curve of genus 0 consists of two irreducible components, whose degrees are 1 and 2 respectively.

2-1) The both components are simple, that is, embedded;

2-2) The degree 2 component is a double cover of a degree 1 curve;

3) The image of the stable map from the semi-stable curve of genus 0 consists of three irreducible components, whose degrees are 1.

3-1) The domain of the stable map consists of three copies of the Riemann sphere;

3-2) The domain of the stable map consists of four copies of the Riemann sphere;

In type 3), the domain of the stable map is a semi-stable curve of genus 0, i.e., a tree of copies of the Riemann sphere. The stability condition implies that the holomorphic map is not constant on the components corresponding to the exterior vertices of the tree. Hence there are at most three exterior vertices. In case 3), there are two possibilities. Namely, 3-1) and 3-2). We remark that the stable maps of type 3-2) are ones from the semi-stable curve consisting of four irreducible components of genus 0 (see Figure 5.3), three curves of degree 1 meet, in the image, at a common point, to which the central fourth component is mapped constantly. Further, we remark that in type 1-1) the number of the double points is necessary one because the adjunction formula implies that the virtual genus of the image is 1 which is bigger than the genus of the domain of the stable map by just 1.

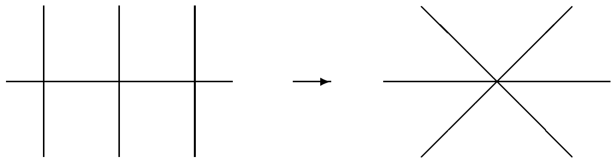


Figure 5.3

Although we can stratify $\mathcal{M}_0(3, J)$ more finely, the stratification above is enough for our purpose.

We note that the pseudo holomorphic rational curve with $(2, 3)$-cusp point in $\mathbf{C}P^2$, which we are interested in, belongs to the stratum of type 1-2) by the following lemma.

Lemma 5.3. *The image of the pseudo holomorphic map of type* 1-2) *has only one singular point with multiplicity* 2.

Proof. We denote the image by D. Since it is not an immersion, it has a singular point. Since the virtual genus of D is 1 and the genus of the domain of the pseudo holomorphic map is 0, we find that D has only one singular point p. To prove the lemma, it is enough to show that

$$\min_l \; \{l \cdot_p D \mid l \text{ is a pseudo holomorphic line in } \mathbf{C}P^2 \text{ through } p\} = 2,$$

where $l \cdot_p D$ is the intersection number of l and D at p. Since the intersection number of two pseudo holomorphic curves in an almost complex 4-manifold is always positive [M2], we find that the number in the left hand side above is at most 3. On the other hand, $l \cdot_p D = 1$ implies that l and D intersect transversally at p by [M2]. This contradicts to that p is the singular point. If it is 3, then $l \cdot_p D = 3$ for all pseudo holomorphic line l through p. We pick a point q on D which is different from the singular point p. By Gromov's theorem [G], there exists a pseudo holomorphic line l through p and q. In this case we have $l \cdot D \geq 4$, which is a contradiction. Thus Lemma 5.3 is proved. ∎

Although we only have to deal with the stratum of type 1-2), there is no guarantee that the closure of this stratum is a nice topological space such as an orbifold. Therefore, we firstly study the ambient moduli space $\mathcal{M}_0(3, J)$. In fact, we will show that the linearized operator, with the constraints corresponding to the stratification, is surjective. In particular, the parametrized moduli space is a product orbifold cobordism. Namely we prove the following

Theorem 5.4. *The moduli space $\mathcal{M}_0(3, J)$ of stable J-holomorphic curves of degree 3 and genus 0 is a compact orbifold. For a 1-parameter family $\{J_t\}_{0 \leq t \leq 1}$ of compatible almost complex structures, the parametrized moduli space $\cup\{t\} \times \mathcal{M}_0(3, J_t)$ is a product cobordism. Moreover, the linearization operator of the pseudo holomorphic curve equation is surjective along each stratum.*

Proof. It is enough to show surjectivity of the linearized operator on each stratum. We discuss each case separately, although the basic strategy is same as in [H-L-S]. In particular, note that the linearization operator maps sections of a subline bundle, which is generated by the image of the differential of the pseudo holomorphic map, to $(0,1)$-forms with coefficients in th subline bundle.

Case 1-1). The argument in [H-L-S] implies transversality.

Case 2-1). The both components are embedded. Although, the argument in [H-L-S] can be applied to them separately, there is a constraint coming from the coincidence condition at the double point in the domain.

Consider the tangent space of the mapping space of pairs (f_1, f_2) such that $f_1(0) = f_2(\infty)$, where f_1 and f_2 are mapping, from $\mathbf{C}P^1$, of degree 1 and 2, respectively. (Here we use the $W^{1,p}$-completion, $p > 2$, as usual.) The tangent space at (f_1, f_2) is given by

$$\Gamma((f_1 \vee f_2)^* T\mathbf{C}P^2) = \{(\xi_1, \xi_2) \in \Gamma(f_1^* T\mathbf{C}P^2) \times \Gamma(f_2^* T\mathbf{C}P^2) \mid \xi_1(0) = \xi_2(\infty)\}.$$

The linearization operator of the pseudo holomorphic curve equation $D\overline{\partial}$ is an operator from $\Gamma((f_1 \vee f_2)^* T\mathbf{C}P^2)$ to the L^p-completion of $\Omega^{0,1}(f_1^* T\mathbf{C}P^2) \times \Omega^{0,1}(f_2^* T\mathbf{C}P^2)$.

The surjectivity of $D\overline{\partial}$ follows from the argument in [H-L-S] and the following lemma.

Lemma 5.5. *Let L be a holomorphic line bundle over $\mathbf{C}P^1$ with $c_1(L) \geq 0$ and D a first order linear differential operator with the same symbol as the Dolbeault operator. Denote by p a point on $\mathbf{C}P^1$. Then D is surjective from $\{\xi \in \Gamma(L) \mid \xi(p) = 0\}$ to $\Omega^{0,1}(L)$.*

Proof. Since the index of the real Fredholm operator $D|_{\{\xi \in \Gamma(L) \ \mid \ \xi(p)=0\}}$ from $\{\xi \in \Gamma(L) \mid \xi(p) = 0\}$ to $\Omega^{0,1}(L)$ is $2c_1(L)$, it is sufficient to show that the dimension of the kernel does not exceed $2c_1(L)$. Let us assume the contrary. Pick $d = c_1(L)$ distinct points $q_1, \ldots, q_d$ on $\mathbf{C}P^1 \setminus \{p\}$. By assumption, there are at least $2d+1$ linearly independent sections $s_1, \ldots, s_{2d+1}$ satisfying $Ds_i = 0$ and $s_i(p) = 0$. Then we can find real numbers $\lambda_1, \ldots, \lambda_{2d+1}$, which contains non-zero numbers, such that the linear combination $s = \sum \lambda_i s_i$ vanishes at $q_1, \ldots, q_d$ as well as p. Note that s is a non-zero section and all zeros contributes positively to the intersection number of s and the zero section [H-L-S]. Hence the self intersection number, which is $c_1(L)$ must be at least $d+1$. This is a contradiction. ∎

Case 2-2). Suppose that the degree 2 component f_2 is a double cover of a degree 1 curve. Then f_2 factors into $\psi \circ \pi$, where π is a branched double covering of the Riemann sphere and ψ is a degree 1 mapping. Then we have $f_2^* T\mathbf{C}P^2 = \pi^* \psi^* T\mathbf{C}P^2$. Note that $\psi^* T\mathbf{C}P^2$ contains the tangent bundle $T\mathbf{C}P^1$ with the quotient bundle being the hyperplane section bundle H. Hence the bundle $f_2^* T\mathbf{C}P^2$ contains a complex line bundle of degree 4 with the quotient bundle of degree 2. Then the argument [H-L-S] applied to these line bundles derives the surjectivity of the linearization operator. The rest of the argument is same as above.

Case 3-1). The argument goes as in Case 2.

Case 3-2). Note that the component of the domain, which is mapped to a constant, is $\mathbf{C}P^1$ and also that the linearization operators on other components (of degree 1) are surjective even after restricting the domains of sections vanishing at one point (Lemma 5.5). The argument goes as in Case 2.

Case 1-3). The argument goes as in Case 2.

Case 1-2). Firstly we note that the nonconstant simple pseudo holomorphic map $f : \mathbf{C}P^1 \to \mathbf{C}P^2$ can be locally expanded around the singular point in the following form:

$$f(z) = (z^k, z^m) + O(m+1),$$

where $m > k$, k does not divide m and $O(m+1)$ denotes a function of z and $\overline{z}$ which vanishes to order m. (See Proposition 2.6 in [M2].) Therefore, Lemma 5.3 implies that the order of f at the singular point is 2, thus the order of df at the singular point is 1.

We consider the space of mappings $f : \mathbf{C}P^1 \to \mathbf{C}P^2$ such that $df_p(e) = 0$. Here p is a point on $\mathbf{C}P^1$ and e is a non-zero tangent vector at p, which are once chosen and fixed. (We use, here, the $W^{2,p}$-completion as in [L-O].) Note that if f with $df_p(e) = 0$ is J-holomorphic, $df_p = 0$. Hence this is a reasonable mapping space dealing with Case 2.

The tangent space at f is given by

$$W = \{\xi \in \Gamma(f^*T\mathbf{C}P^2) \mid \nabla_e \xi = 0\}.$$

To show the surjectivity of $D\overline{\partial}$ from W to $\Omega^{0,1}(f^*T\mathbf{C}P^2)$, we prove the surjectivity from a smaller domain

$$W_0 = \{\xi \in \Gamma(f^*T\mathbf{C}P^2) \mid \xi(p) = 0,\ \nabla_e \xi = 0\},$$

which is the tangent space of f with the fixed $f(p)$ and $df_p(e) = 0$.

We start proving the surjectivity. We need a slight modification of the argument in [H-L-S]. Let $f : \mathbf{C}P^1 \to \mathbf{C}P^2$ be a J-holomorphic map of degree 3 with one singular point, i.e., df vanishes at one point $p \in \mathbf{C}P^1$. Since f is not an immersion, the image of $T\mathbf{C}P^1$ under df is not a subbundle of $f^*T\mathbf{C}P^2$. Pick a complex linear connection on $f^*T\mathbf{C}P^2$. Since the base space is a Riemann surface, the connection induces a structure of a holomorphic vector bundle. Recall that the order of df at p is 1. Then the argument in page 516 in [G-H][1] implies that $f^*T\mathbf{C}P^2$ contains a line bundle of degree 3, $T\mathbf{C}P^1 \otimes \mathcal{O}(p)$ with a quotient line bundle of degree 6. The conditions that p is mapped to a prescribed point and $df_p = 0$ impose complex codimension 2 conditions both for tangential and normal directions. Namely, we consider

$$\{\xi \in \Gamma(L) \mid \xi(p) = 0,\ \nabla_e \xi = 0\},$$

[1]See also [S-Y], which deals with a similar situation arising from holomorphic maps.

where L is the line bundle of degree 3 or 6 above. Here is a lemma analogous to Lemma 5.5.

Lemma 5.6. *Let L be a holomorphic line bundle with $c_1(L) \geq 1$ and D a first order differential operator with the same symbol as the Dolbeault operator. Let p be a point on $\mathbf{C}P^1$ and e a non-zero tangent vector at z. Then D is surjective from $\{\xi \in \Gamma(L) \mid \xi(p) = 0,\ \nabla_e \xi = 0\}$ to $\Omega^{0,1}(L)$.*

Proof. Let ξ be a solution of $D\xi = 0$ such that $\xi(p) = 0$ and $\nabla_e \xi = 0$. Note that these conditions implies that the 1-jet of ξ vanishes at p.

Pick a complex linear connection on L and write the operator D as

$$D = \overline{\partial} + B_l + B_a,$$

where B_l and B_a are the complex linear part and the anti-complex linear part of the 0-th order operator $D - \overline{\partial}$, respectively.

We shall show that the local intersection index of ξ with the zero section at p is at least 2. Denote by w a local holomorphic coordinate of $\mathbf{C}P^1$ with $w(p) = 0$. Since the difference of D and the Dolbeault operator is a 0-th order operator, we have $\overline{\partial}\xi$ vanishes at p. Hence, ξ/w is locally a $W^{1,p}$-function. Write $\eta = \xi/w$, which satisfies the following differential equation.

$$\overline{\partial}\eta + B_l(\eta) + \frac{\overline{w}}{w} B_a(\eta) = 0$$

Note that $\overline{w}/w$ is locally L^∞. Therefore we can apply the Carleman similarity principle (Appendix A.6 in [H-Z]) to this situation and conclude that the intersection index of η and the zero section at p is positive. Recall that the 1-jet of ξ vanishes at p, hence η vanishes at p. This implies that the intersection index of ξ, which equals $w\eta$ near p, and the zero section at p must be at least 2.

As in the proof of Lemma 5.5, we can show that ξ is divisible by w^2 to get a local $W^{1,p}$-function. The rest of the proof continues as in the proof of Lemma 5.5. ∎

The argument in [H-L-S] and Lemma 5.6 derives surjectivity of $D\overline{\partial}$ with fixing $f(p)$ and requiring that $df(p) = 0$. Therefore we have finished the proof of Theorem 5.4. ∎

Obviously, the closure of the stratum of type 1-2) does not intersect the stratum of type 1-1). Note that the three components in type 3-1) are mapped to three pseudo holomorphic lines with three double points. Hence, after smoothing two of three double points, the glued pseudo holomorphic curve must have one double point, which never occurs in type 1-2). Thus the closure of the stratum of type 1-2) does not intersect the stratum of type 3-1). Since the closure of the stratum of type 1-2) intersects strata of type 1-3), 2-1), 2-2) and 3-2), we will require some constraints on pseudo holomorphic curves in $\mathcal{M}_3(3, J)$ to control pseudo holomorphic curves in the stratum of type 1-2).

Lemma 5.7. *Let D and D_0 be pseudo holomorphic curves as in Theorem 5.1. Then there is a one-parameter family of singular pseudo-holomorphic curves D_t joining D and D_0 such that the singular point on D_t is a fixed point p in $\mathbf{C}P^2$.*

Proof. Firstly, we consider the situation with the standard complex structure J_0. Choose a rational cubic curve C with a cusp at p. Pick distinct 6 points $p = q_1, \dots, q_6$ on $C \subset \mathbf{C}P^2$ such that

1) there is no collection of three lines with a triple intersection point, whose union contains $q_1, \dots, q_6$;

2) there is no pair of a line and a conic, which are tangent at p and contain $q_2, \dots, q_6$.

We consider the moduli space $\mathcal{M}_0(3, J_0; p; q_2, \dots, q_6)$ of rational cubic curves with a cusp at $p = q_1$ and passing through $q_2, \dots, q_6$. To do so, we consider the moduli space $\mathcal{M}_{0,6}(3, J_0)$ of stable maps of genus zero, with 6 marked points $z_1, \dots, z_6$ and of degree 3. We consider the following evaluation map:

$$\mathbf{ev} : \mathcal{M}_{0,6}(3, J_0) \to T^*\mathbf{C}P^2 \times \mathbf{C}P^2 \times \dots \times \mathbf{C}P^2$$

given by $(\phi, (z_1, z_2, \dots, z_6)) \mapsto (d\phi(z_1), \phi(z_2), \dots, \phi(z_6))$. Then we have

$$\mathcal{M}_0(3, J_0; p; q_2, \dots, q_6) = \mathbf{ev}^{-1}((p, 0), q_2, \dots, q_6).$$

Because of the conditions 1) and 2) for 6 points $p = q_1, \dots, q_6$, all elements in $\mathcal{M}_0(3, J_0; p; q_2, \dots, q_6)$ belong to the case 1-2) in the list after Theorem 5.2 after forgetting marked points. Note that there are configuration of a line and a conic, whose union contains $q_1, \dots, q_6$, but the condition 2) above implies that the point z_1 is an immersive point. Therefore we only need to work with the stratum in the case 1-2). We write $\mathcal{M}_{0,(1,5)}(3, J_0; p) = (pr_1 \circ \mathbf{ev})^{-1}(p, 0)$, where pr_1 is the first factor projection to $T^*\mathbf{C}P^2$. Since the moduli space

$\mathcal{M}_{0,6}(3, J_0)$ is compact by construction (see [F-O]), $\mathcal{M}_{0,(1,5)}(3, J_0; p)$ is also compact. The transversality can be shown in a similar way as in the proof of Theorem 5.4 (the case 1-2). It is sufficient to show the transversality for $\mathbf{ev}' : \mathcal{M}_{0,(1,5)}(3, J_0; p) \to \mathbf{C}P^2 \times \ldots \times \mathbf{C}P^2$ given by $\mathbf{ev}'(\phi, (z_1, \ldots, z_6)) = (\phi(z_2), \ldots, \phi(z_6))$. Since $z_2, \ldots, z_6$ are immersive points of ϕ and they can move independently, we only need to show surjectivity of the linearization of $\mathbf{ev}'$ in the normal directions to $d\phi(T_{z_i}\mathbf{C}P^1)$.

For $\phi \in \mathcal{M}_0(3, J_0; p; q_2, \ldots, q_6)$, the pull back bundle $\phi^* T\mathbf{C}P^2$ contains $\mathcal{O}(3)$, which is the extension of $d\phi(T\mathbf{C}P^1)$ as before and its quotient bundle is isomorphic to $\mathcal{O}(6)$. Since the Dolbeault operator is surjective from the space of $W^{2,p}$-sections of $\mathcal{O}(6)$, which vanishes at z_1 of the second order and $z_2, \ldots, z_6$ of the first order, with the target $W^{1,p}(\mathbf{C}P^1, T^*_{(0,1)}\mathbf{C}P^1 \otimes \mathcal{O}(6))$, $p > 2$, we got the desired surjectivity of the linearization of $\mathbf{ev}'$ in the normal directions. It, particularly, implies that $\phi \in \mathcal{M}_0(3, J_0; p; q_2, \ldots, q_6)$ is compact 0-dimensional manifold. Note that compactness follows from compactness of $\mathcal{M}_{0,(1,5)}(3, J_0; p)$ and continuity of $\mathbf{ev}'$. Because J_0 is an integrable complex structure, transversality implies that all elements in $\mathcal{M}_0(3, J_0; p; q_2, \ldots, q_6)$ have positive sign with respect to the natural orientation. Therefore it is not null oriented-cobordant.

Now we vary the almost complex structure and points $\{q_2, \ldots, q_6\}$. Since J-holomorphic curves of genus zero and of degree 1 or 2 are transversal for any compatible almost complex structure J [H-L-S], we can find a one-parameter family of compatible almost complex structure $\{J_t\}$ joining J_0 to J and paths $q_i(t)$, $i = 2, \ldots, 6$, with $q_i(0) = q_i$ such that

1') there is no collection of three J_t-holomorphic spheres with a triple intersection point, whose union contains $q_1(t), \ldots, q_6(t)$;

2) there is no pair of a J_t-holomorphic sphere of degree 1 and a J_t-holomorphic sphere of degree 2, which are tangent at p and contain $q_2(t), \ldots, q_6(t)$.

Transversality of J_t-holomorphic curve equation can be shown in a similar way as in the standard complex structure J_0, once we have the following lemma, which can be proven as in Lemmas 5.5 and 5.6.

Lemma 5.8. *Let L be a holomorphic line bundle with $c_1(L) \geq 6$ and D a first order differential operator with the same symbol as the Dolbeault operator. Let $z_1, \ldots, z_6$ be distinct points on $\mathbf{C}P^1$ and e a non-zero tangent vector at z_1. Then D is surjective from $\{\xi \in \Gamma(L) \mid \xi(z_i) = 0,\ i = 1, \ldots, 6, \ \text{and } \nabla_e \xi = 0\}$ to $\Omega^{0,1}(L)$.*

Combining the arguments above, we conclude that the first factor projection from $\bigcup_{0\le t\le 1}\{t\}\times\mathcal{M}_0(3,J_t;p;q_2(t),\ldots,q_6(t))$ to $[0,1]$ is submersion. Since we noticed that the boundary at $t=0$ is not null oriented-cobordant, it is not an empty set, which derives the conclusion of Lemma 5.7. ∎

Proof of Theorem 5.1. Since $\bigcup_{0\le t\le 1}\{t\}\times\mathcal{M}_0(3,J_t;p;q_2(t),\ldots,q_6(t))$ is a product cobordism, there is a one-parameter family of singular pseudo-holomorphic curves D_t joining D and D_0. We may assume that the singular points are a fixed point p in $\mathbf{C}P^2$.

We may assume that the restrictions of $\{J_t\}$ to a neighborhood of p are independent of t and integrable. We blow-up $\mathbf{C}P^2$ at p. Hence the J_t-holomorphic curves D_t can be lifted to pseudo holomorphic curves D_t' for some almost complex structure $\widetilde{J}_t$, which are integrable in a neighborhood of the exceptional divisor E. Note that E and D_t' are tangent. After making a blow-up at the point p', where E and D_t' are tangent, the proper transforms of those curves intersects transversally at a point p''. Then we get a configuration of four embedded spheres intersecting each other transversally with at most double points. We denote by $\widetilde{D_t}$ the proper transform of D_t, and E_1, E_2 and E_3 are (proper transforms of) exceptional divisors with self intersection number -1, -2 and -3, respectively.

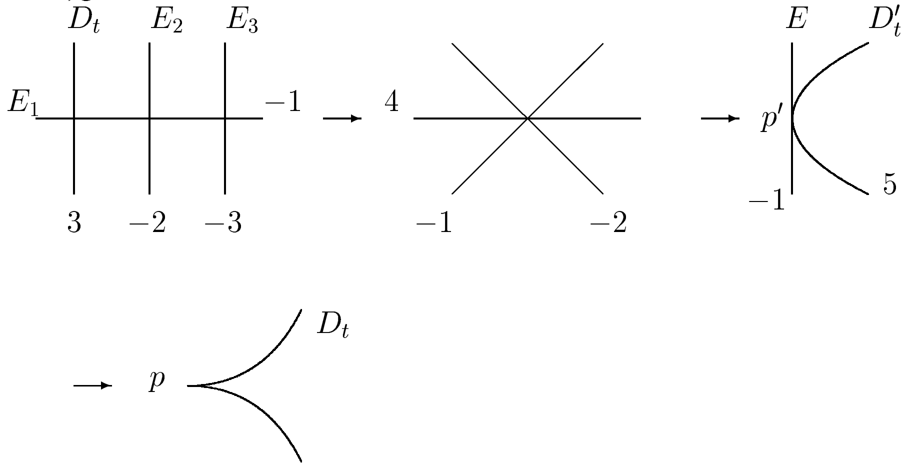


Figure 5.9

Since J_t coincide on a neighborhood of p, E_1, E_2 and E_3 depend smoothly on t. Note also that each E_i intersects $\widetilde{D}_t$ transversally exactly at one point. Since $\widetilde{D}_t$ is a smooth family of embedded surface, we can find an isotopy Ψ_t of the blown-up manifold $\widetilde{M}$ such that $\Psi_t(\widetilde{D}_0) = \widetilde{D}_t$ and preserving E_1, E_2 and E_3. Restricting Ψ_t to $\widetilde{M} \setminus (\widetilde{D}_t \cup E_1 \cup E_2 \cup E_3)$, we get a one parameter family of diffeomorphisms among $\mathbf{C}P^2 \setminus D_t$. ∎

§6. Proof of Main Theorem.

Recall that the minimal symplectic filling is the complement of $\widetilde{C} \cup E_p \cup E_q \cup E_r$ in $Z(\Gamma)$. Therefore, it is diffeomorphic to the complement of a regular neighborhood of the total transform of $\widetilde{C} \cup E_p \cup E_q \cup E_r$ in $Z(A_n)^{(2)}_{(3)}$, $Z(D_n)^{(1)}_{(3)}$ and $Z(E_n)_{(3)}$, respectively. After these stages, we started contracting (-1)-curves, which are not explicitly described.

We, however, presented a blowing down process to obtain a minimal rational symplectic 4-manifold. The final situation is explicitly described in Theorem 5.1 and Theorem 5.2.

Now we keep track of the opposite process. Namely, we blow up $\mathbf{C}P^2$ or $\mathbf{C}P^1 \times \mathbf{C}P^1$ at a number of points on D to obtain $Z(A_n)^{(2)}_{(3)}$, $Z(D_n)^{(1)}_{(3)}$ and $Z(E_n)_{(3)}$, respectively. In this converse process, we can describe the divisors $E_r''' \cup A \cup B$, $E_r''' \cup A$ and E_r''', respectively. The diffeomorphism types of the complement of these divisors are uniquely determined by our argument. Hence we get uniqueness of the diffeomorphism type of minimal symplectic fillings for each type of simple singularities.

Remark 6.1. We treat cases of A_n, D_n and E_n in a similar way based on Theorem 5.1 and 5.2. Here we would like to comment other approaches for cases of A_n and D_n. For A_n-case, we can contract a maximal family of symplectic (-1)-curves in $Z(A_n)_{(2)}$, which are disjoint from E_p'' to get the complex projective plane. The compactifying divisor is transformed to the union of a pseudo holomorphic line and a pseudo holomorphic conic, which are tangent each other. For D_n case, we can contract a maximal family of symplectic (-1)-curves in $Z(D_n)_{(2)}$, which are disjoint from E_p'' to get the product of two copies of the complex projective line. The compactifying divisor is transformed to the union of two pseudo holomorphic spheres of bidegree $(1,0)$ and $(1,2)$, respectively, which are tangent each other. The rest of the argument is based on uniqueness of isotopy type of these divisors in the projective plane or the product of two copies of the complex projective line.

Appendix.

Theorem. *Let M be a closed symplectic 4-manifold containing a pseudo-holomorphic rational curve C with a $(2,3)$-cusp point. Suppose that C is non-singular away from the $(2,3)$-cusp point. If the self-intersection number C^2 of C is positive, then M must be a rational symplectic 4-manifold and C^2 is at most 9.*

Proof. If $M \setminus C$ is not minimal, we contract all pseudo-holomorphic (-1)-rational curves which do not intersect C to get a pair (M', C), where $M' \setminus C$ is minimal. Write $k = C^2$. We blow-up M' at $(k-1)$ points on the non-singular part of C to get a closed symplectic 4-manifold $\widetilde{M}$. We denote the set of the exceptional curves by $\{e_i\}$. The proper transform D of C is a pseudo-holomorphic rational curve with one $(2,3)$-cusp point and $D^2 = 1$. The following lemma enables us to assume that D is a pseudo-holomorphic curve with respect to an almost complex structure, which is integrable near the $(2,3)$-cusp point.

Lemma A1. *D can be perturbed in a neighborhood of the cusp point so that the perturbed curve is a pseudo-holomorphic rational curve with one $(2,3)$-cusp point with respect to an almost complex structure, which is integrable near the cusp point.*

Then blowing up $\widetilde{M}$ at the cusp point and taking the converse process in §4, we have a divisor whose configuration of the irreducible components is given by the first picture in Figure 4.3 (in $Z(E_8)$). The complement of a regular neighborhood of the divisor is a symplectic filling of the link of the E_8-singularity. Therefore, the complement of a regular neighborhood of D in $\widetilde{M}$ is a symplectic filling of the link of the E_8-singularity.

Lemma A2. *$\widetilde{M} \setminus D$ is minimal.*

Since $\widetilde{M} \setminus D$ is minimal, our uniquness result shows that $\widetilde{M}$ is a rational symplectic 4-manifold and D is the anti-canonical divisor. By Proposition 4.4 ($n = 8$ case), we find that we can blow down $\widetilde{M}$ along disjoint eight (-1)-rational curves to obtain $\mathbf{C}P^2$ and D becomes a pseudo-holomorphic rational curve of degree 3 with one $(2,3)$-cusp point. It follows that M' is symplectic deformation eqivalent to $\mathbf{C}P^2 \# (9-k)\overline{\mathbf{C}P^2}$ (for $1 \le k \le 9$) or $\mathbf{C}P^1 \times \mathbf{C}P^1$ (only when $k = 8$). In particular, we obtain $k \le 9$. Moreover, C

correponds either to the proper transform of the singular cubic curve under the blow-up at $(9-k)$ points of the non-singular part of the cubic curve or to the singular $(2,2)$-curve in $\mathbf{C}P^1 \times \mathbf{C}P^1$. ∎

Proof of Lemma A1. This lemma follows from the argument in the proof of Lemma 4.3 in [M2]. We choose local complex-valued coordinates (w_1, w_2) around the cusp point P such that the symplectic form equals

$$-\frac{1}{2}\mathrm{Im}(dw_1 \wedge d\overline{w}_1 + dw_2 \wedge d\overline{w}_2).$$

By Proposition 2.6 in [M2], the local behaviour of D near P is described by a map from the unit disc so that $F(z) = (az^2, bz^3) + (f_1(z), f_2(z))$ with $\deg f_i \geq 4$. For a sufficient small ϵ, we take a cut-off function $\rho(|z|)$ such that $\rho(|z|) = 0$ for $|z| \leq \epsilon/2$, and $\rho(|z|) = 1$ for $|z| \geq \epsilon$. We perturb $F(z)$ to $G(z) = (az^2, bz^3) + \rho(|z|)(f_1(z), f_2(z))$. Then it is holomorphic in the $\epsilon/2$ neighborhood of P and the restriction of the symplectic form is non-degenerate away from the cusp point. It is easy to see that the perturbed curve is pseudo-holomorphic with respect to a compatible almost complex structure, which is integrable near P. This is the desired perturbation of D. ∎

Remark. The almost complex structure in the proof is not generic among compatible almost complex structures.

Proof of Lemma A2. Assume that it is not minimal. We contract pseudo-holomorphic (-1)-rational curves f_j in $\widetilde{M} \setminus D$ to obtain $\pi : \widetilde{M} \to \widetilde{M}'$ such that $\widetilde{M}' \setminus D$ is minimal. Then by our uniqness result in this paper, we find that $\widetilde{M}'$ is a rational symplectic 4-manifold and D is the anti-canonical divisor of $\widetilde{M}'$. Note that $f_j \cdot e_i \geq 1$ for some i, because $M' \setminus C$ is minimal. On the other hand, we have

$$K_{\widetilde{M}} = \pi^* K_{\widetilde{M}'} + \sum f_j = -[D] + \sum f_j.$$

Since e_i is a pseudo-holomorphic (-1)-rational curve, $e_i \cdot K_{\widetilde{M}} = -1$ by the adjunction formula. Thus we have

$$-1 = e_i \cdot K_{\widetilde{M}} = e_i \cdot (-[D]) + \sum_j e_i \cdot f_j \geq 0,$$

which is a contradiction. ∎

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