Uniformization of Geometric Structures with Applications to Conformal Geometry

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§ 1. Introduction

(1.1) The classical uniformization theory of Riemann surfaces is an outstanding meeting place of the classical function theory and topology. There are diverse aspects of this theory which extend in other set-ups in different ways, cf. [8], [9], [10]. In this paper we shall consider it in the context of "geometric structures" as defined below. This is a direct generalization of the uniformization of Riemann surfaces via Fuchsian and Kleinian groups.

(1.2) Let X be a topological space and G a group of homeomorphisms of X, satisfying the "uniformization condition" (U) : each $g \in G$ is uniquely determined by its action on any nonempty open subset. The pair (X,G) is to be thought of as a <u>model space</u>. An (X,G)-<u>structure</u> on a topological space M is given by

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a covering of M by open sets $\{U_\alpha\}_{\alpha \in A}$ and homeomorphisms $S_\alpha : U_\alpha \hookrightarrow X$ s.t. for all pairs α, β in A with $U_\alpha \cap U_\beta \neq \emptyset$ the mapping $S_\alpha \cdot S_\beta^{-1}|_{S_\beta(U_\alpha \cap U_\beta)}$ is a restriction of an element of G. For example, if X is the standard sphere S^n and G is the full group of Möbius transformations $M(n)$ then by Liouville's theorem for $n \geq 3$ an $(S^n, M(n))$-structure on an n-dimensional manifold M^n is the same as a conformal class of locally conformally Euclidean metrics. The case $n = 2$, with $M(2)$ replaced by its identity component $M_0(2) \approx PSL_2(\mathbb{C})$, plays a central role in the uniformization theory of Riemann surfaces via the Kleinian groups. In Gunning's terminology an $(S^2, M_0(2))$-structure is a $\mathbb{C}\mathbf{P}^1$-<u>structure</u> on a Riemann surface. As an another example of geometric interest consider X = real (resp. complex) projective space and G = the full group of real (resp. complex) projective transformations.

(1.3) A nice class of (X,G)-structures arises as follows. Let Ω be an open subset of X and Γ a subgroup of G which leaves Ω invariant and acts freely and properly discontinuously there. Then $\Gamma \backslash \Omega$ clearly admits an (X,G)-structure. We shall call an (X,G)-structure on M <u>Kleinian</u> if $M \approx \Gamma \backslash \Omega$ as described above.

Of course X has a distinguished (X,G)-structure

σ_0 . An (X,G)-structure σ on a <u>simply connected</u> M is always of the form $\delta^*\sigma_0$ where $\delta : M \longrightarrow X$ is a local homeomorphism. (This is essentially a precise formulation of the "monodromy principle".) Moreover if $\mathrm{Aut}(M,\sigma)$ denotes the automorphism group of this structure then δ determines a homomorphism $\rho : \mathrm{Aut}(M,\sigma) \longrightarrow G$, and δ is ρ-equivariant i.e. for all $\alpha \in \mathrm{Aut}(M,\sigma)$ and $x \in M$ $\delta(\alpha \cdot x) = \rho(\alpha)\delta(x)$. The map δ is unique up to a left-composition by an element of G , and correspondingly ρ is unique up to a conjugation by an element of G .

If M has an (X,G)-structure σ but M is not necessarily simply connected then assuming that it has a universal cover $\tilde{M}$ we see that $\tilde{M}$ has an induced (X,G)-structure $\tilde{\sigma}$ and the deck-transformation group $\Delta \approx \pi_1(M)$ is clearly a subgroup of $\mathrm{Aut}(\tilde{M},\tilde{\sigma})$. Let $\delta : \tilde{M} \longrightarrow X$ be a local homeomorphism s.t. $\tilde{\sigma} = \delta^*\sigma_0$. Then δ is called a <u>development</u> of (M,σ) . If $\rho : \mathrm{Aut}(\tilde{M},\tilde{\sigma}) \longrightarrow G$ is the corresponding homomorphism then $\rho|_\Delta$ is called the <u>holonomy representation</u> of (M,σ) .

It is obvious that if we are in a category where the covering space theory is valid then an (X,G)-structure σ on M is <u>Kleinian</u> iff $\delta : \tilde{M} \longrightarrow \delta(\tilde{M})$ is a covering map and $\rho(\Delta) = \Gamma$ acts freely and properly discontinuously on $\delta(\tilde{M})$. We shall say that (M,σ) is <u>almost Kleinian</u> if only $\delta : \tilde{M} \longrightarrow \delta(\tilde{M})$ is a covering map.

(1.4) A problem of basic geometric interest is to find criteria for an (X,G)-structure to be Kleinian or almost

Kleinian. For the case of $\mathbb{CP}^1$-structures, cf. (1.1), Gunning provided a nice criterion, cf. [9] theorem 7, and (1.5) below. This was proved by another method by Kra [12]. Both proofs use facts special to Riemann surfaces. In this paper we shall re-examine this theorem in the context of general geometric structures. In § 2 and 3 we develop the notions of limit sets and domains of properness for an arbitrary subgroup $\Gamma \leq G$ acting on X and prove the following general

(1.4.1) <u>Uniformization theorem</u> Let M be a compact space with an (X,G)-structure with $\delta : \tilde{M} \longrightarrow X$ a development map, $\rho : \pi_1(M) \longrightarrow G$ the holonomy representation and $\Gamma = \operatorname{im}\rho$. Let N_0 be the union of those components of the domain of normality of Γ which intersect $\operatorname{im}\delta$. Then $\delta|_{\delta^{-1}(N_0)} = \delta^{-1}(N_0) \longrightarrow N_0$ is a covering map.

(1.5) This theorem combined with a theorem of Fried [5] implies a direct extension of Gunning's theorem, cf. (5.3). <u>A compact manifold with a Möbius structure such that the development map is not surjective is almost Kleinian.</u> Conversely of course, except for the manifolds conformal to the spherical space-forms, an almost Kleinian manifold with a Möbius structure has development onto a proper subset of S^n.

Here is another quite different criterion, cf. (5.4). <u>A compact manifold with a Möbius structure so that the domain of properness of its holonomy group is connected</u>

<u>and has finitely generated</u> π_1 <u>is almost Kleinian</u>. If may be remarked that in the proofs of Gunning or Kra the domain of properness plays no direct role.

In [13] it was proved that a connected sum of manifolds with Möbius structures admits a Möbius structure. A convenient source of Kleinian examples is a partial refinement of this statement, cf. (5.6). <u>A connected sum of Kleinian manifolds with a Möbius structure admits a Kleinian Möbius structure</u>. This is an analogue on the "space"-level of the famous Klein-Maskit "combination theorems" cf. [17] which are statements on the "group"-level. This result has been known for some time, cf. Goldman [6] § 5, but no proof is print.

Perhaps it should be pointed out that not every manifold with a Möbius structure is Kleinian or even almost Kleinian. There are some very interesting examples illustrating various phenomena, cf. (5.7). Moreover the abovementioned results are valid in a much greater generality as pointed out in (5.8). In fact the "ideal boundary" of an arbitrary connected, simply connected, complete Riemannian manifold of curvature $\leq -a < 0$ admits many features of the standard conformal geometry of S^n.

The hypothesis of <u>compactness</u> of the space with a geometric structure in Gunning's theorem and also in the theorems proved here is admittedly adhoc. It excludes some geometrically interesting cases, e.g. the noncompact hyperbolic manifolds with finite volume. It is easy to see that the statements of the theorems are no longer valid if compactness is simply dropped. However in replacing compactness by appropriate hypotheses on development, limit sets etc. would bring forth the "geometry" in a more transparent way. This entails some entirely new ideas which so far we have only partially carried out. We shall present these extensions in a subsequent publication.

We wish to thank P. Pansu for explaining to us his ideas on a "coarse conformal geometry", cf. [19]. This significantly extended the validity of our results.

§ 2. Wandering points, twins, and polars

(2.1) The study of dynamics of the holonomy group is an important part of the study of a geometric structure. With this in view we shall develop appropriate notions in a sufficiently general set-up, which were motivated by the notions of the limit set and the domain of discontinuity of a Kleinian group in the classical theory. This discussion also extends that in [14] § 1.

(2.2) Let X be a locally compact, Hausdorff space which has a countable base for topology, and which is locally simply connected, and locally path-connected. Let G be a closed group of homeomorphisms of X with respect to the compact-open topology. The pair (X,G) is to be thought of as "a model space" in the sense described in (1.2). For pairs of spaces X,Y let $C(X,Y)$ denote the space of continuous functions from X to Y again equipped with the compact-open topology. For $A \subseteq X$ let

$$(2.2.1) \qquad G|_A = \{g|_A \mid g \in G\}$$

considered as a subset of $C(A,X)$. Let A^- denote the closure of A in X and

$$(2.2.2) \qquad G(A) = \bigcup_{g \in G} gA$$

A point $x \in X$ is said to be <u>a recurrent point of the G-orbit of</u> A if for every neighborhood V of x the subset $\{g \in G | gA \cap V \neq \emptyset\}$ has a noncompact closure in G. We set

(2.2.3) $G(A)'$ = the set of recurrent points of A.

Clearly this set is a closed G-invariant set. We also set

(2.2.4) $Z(A) = \{g \in G | gA \cap A \neq \emptyset\}$.

(2.3) A point $x \in X$ is said to be <u>wandering</u> (with respect to G) if it has a compact neighborhood U_x such that $Z(U)$ is compact. We set

(2.3.1) L_0 = the set of non-wandering points.

(2.4) Let $p : X \longrightarrow {}_G\backslash X$ be the orbit-space projection so that ${}_G\backslash X$ has a quotient topology. We say that $x, y \in X$ are <u>twins</u> (with respect to G) if $p(x), p(y)$ have no disjoint neighborhoods. This means that for every neighborhood U of x and V of y there exists $g \in G$ such that $gU \cap V \neq \emptyset$. Let

(2.4.1) $\tau(x)$ = the set of twins of x.

Clearly $y \in \tau(x)$ iff $x \in \tau(y)$, and $\tau(x)$ is a closed G-invariant subset.

(2.5) We say that $y \in X$ is a <u>polar</u> of $x \in X$ if y is a recurrent point of every neighborhood of x. Write

(2.5.1) $\mathcal{P}(x)$ = the set of polars of x.

Clearly $\mathcal{P}(x)$ is a closed G-invariant set and $\mathcal{P}(x) \subseteq \tau(x)$.

(2.6) <u>Proposition</u> Let (X,G) be as above and $x \in X$. Then a) $\tau(x) = \bigcap_{U} G(U)^-$, $\mathcal{P}(x) = \bigcap_{U} G(U)'$, where U runs over neighborhoods of x,

b) $G(x)^- \subseteq \tau(x)$,

c) $G(x)' \subseteq L_0 \cap \mathcal{P}(x)$,

d) If x is a wandering point then $\tau(x) = G(x) \cup \mathcal{P}(x)$, and moreover $\mathcal{P}(x) \neq G(x)'$ if x is a recurrent point of a compact subset of $X - L_0$.

<u>Proof</u>. The parts a), b) are clear from definitions. In c) it is again clear from a) that $G(x)' \subseteq \mathcal{P}(x)$. We now show $G(x)' \subseteq L_0$. Let $y \in G(x)'$ so there exist $g_n \in G$ such that $g_n x \longrightarrow y$, and g_n is a divergent sequence in G. Let V be any neighborhood of y. So whenever $g_n x, g_m x \in V$ we see that $g_n g_m^{-1} V \cap V \neq \emptyset$. It is clear that $\mathcal{Z}(V)$ is not compact. Since this holds for every neighborhood of y it follows that $y \in L_0$.

Now we prove d). It follows from a) and b) that for any x we have $\tau(x) \supseteq G(x) \cup P(x)$. Now assume x to be a wandering point and $y \in \tau(x)$. If $y \in G(x)^-$ then $y \in G(x)$ or $y \in G(x)'$ and $G(x)' \subseteq P(x)$ so $y \in G(x) \cup P(x)$. Suppose $y \notin G(x)^-$. Then since x is a wandering point we see that for small neighborhoods V of x we have $y \notin G(V)$. But by a) we see that $y \in P(x)$. This proves the first part of d). Now suppose that x is a recurrent point of a compact subset K of $X-L_0$. Let U_n be a decreasing sequence of neighborhoods of x converging to x . There exist $g_n \in G$ and $k_n \in K$ so that $g_n \cdot k_n \in U_n$, or $k_n \in g_n^{-1}U_n$. Let K_0 be a cluster point of k_n . it is clear that $k_0 \in P(x) \cap K$. But $k_0 \notin G(x)'$ since otherwise by c) k_0 would belong to L_0 , but we chose K to lie in $X-L_0$. This finishes the proof.

q.e.d.

§ 3. Limit sets, properness - and normality - domains

(3.1) Let (X,G) be as in (2.2). We shall use the notations in § 2. We now assume

(U_1) If for a non-empty open subset V of X and $g_1, g_2 \in G$ we have $g_{1|V} = g_{2|V}$ then $g_1 = g_2$.

(U_2) For a non-empty open subset V of X if $G|_V$ has a cluster point g_0 in $C(V,X)$ so that g_0 is injective then there exists $g \in G$ such that $g|_V = g_0$.

These assumptions of course hold for Möbius or projective structures or for the geometric structures defined by an integrable G-structure of finite type, cf. [13] § 2. The assumption (U_1) is the same as the assumption (U) of [13] § 1.

(3.2) We say that G acts <u>locally properly</u> on X if every point $x \in X$ is a wandering point. More stringently, G is said to act <u>properly</u> on X if for every compact subset $K \subseteq X$, we have $Z(K)$ compact. The set

(3.2.1) Ω_{loc} = the set of wandering points = $X - L_0$ is called the <u>domain of local properness of</u> G. The set L_0 is called the 0-<u>limit set of</u> G. Now let

(3.2.2) $\mathbf{L}_1 = \{x \in X \mid x$ is a recurrent point of a compact subset of $\Omega_{1oc}\}$. This set is called the 1-<u>limit set of</u> G , and

(3.2.3) $\Lambda = L_0 \cup L_1$

is called simply the <u>limit set</u> of G . Correspondingly

(3.2.4) $\Omega = X - L$

is called the <u>domain of properness of</u> G . The proof of the following proposition may be left to the reader.

<u>Proposition</u> G acts locally properly on Ω_{1oc} and properly on Ω . Moreover Ω_{1oc} is the largest open subset of X on which G acts locally properly.

It should be remarked that in general Ω need not be a maximal domain on which G acts properly. In fact it may happen that Ω can be extended to more than one maximal open subsets of X on which G acts properly, cf. [14], § 1.

(3.3) A point $x \in X$ is called a <u>point of normality</u> (with respect to G) if it has a neighborhood U_x such that $G|_{U_x}$ is a relatively compact subset in $C(U_x, X)$. Then

(3.3.1) N = the set of points of normality is called the <u>normality domain of</u> G .

(3.4) <u>Theorem</u> $\mathcal{N} \subseteq \Omega$.

<u>Proof</u> Let $x \in \mathcal{N}$. We first show that x is wandering. There exists a neighborhood U_x of x so that $G|_{U_x}$ is relatively compact in $C(U_x,X)$. We claim that $Z(U_x)$ is a relatively compact subset of G . Let $g_n \in Z(U_x)$. Passing to a subsequence we may assume that $g_n|_{U_x}$ and $g_n^{-1}|_{U_x}$ converge to g_0 and h_0 respectively in $C(U_x,X)$. By the continuity of the composition and the fact that $g_n \cdot g_n^{-1} = 1$ we see that g_0 and h_0 are injective. So by the hypothesis (U_2) , cf. (3.1) we have elements $g,h \in G$ such that $g|_{U_x} = g_0$ and $h|_{U_x} = h_0$. So $Z(U_x)$ is relatively compact in G . So x is wandering. Thus $\mathcal{N} \cap L_0 = \emptyset$. Now suppose that we have a sequence $g_n \in G$ so that $g_n|_{U_x} \longrightarrow g_0$ in $C(U_x,X)$. In particular $g_n x \longrightarrow g_0 x = y$, say.

Moreover since g_n converges to g_0 uniformly on a compact subset of U_x we see that for every neighborhood U_y of y there is a neighborhood $V_x \subseteq U_x$ so that $g_n(V_x) \subseteq U_y$ for n sufficiently large. It follows that $\bigcap_{V_x}(G_nV_x)' = y$, where V_x runs over all neighborhoods of x . In the notation of § 2 we see that the polar set $P(x)$ of x coincides with $G(x)'$. So by (2.6), part d), $x \notin L_1$. So $\mathcal{N} \cap L_1 = \emptyset$, and hence $N \subseteq \Omega$.

q.e.d.

(3.5) Remark In general $N \neq \Omega$. It is easy to construct examples when X is non-compact. But in general $N \neq \Omega$ even when X is compact. Here is an example in dimension 3. Consider the group of projective transformations of $\mathbf{R}\,\mathbf{P}^3$ generated by

$$g : (x,y,z,w) \longmapsto (2x,4y,z,w)$$

$$h : (x,y,z,w) \longmapsto (x,y,z+w,w)$$

where (x,y,z,w) are the homogeneous coordinates in $\mathbb{R}\,\mathbb{P}^3$. Here g fixes the line $\lambda : x = 0 = y$ pointwise whereas h fixes the hyperplane $\pi : w = 0$ pointwise. It is easy to see that all points in $\mathbb{R}\,\mathbf{P}^3 - \{\lambda \cup \pi\}$ are wandering. So $L_0 = \lambda \cup \pi$. The recurrent points of any compact set in $\mathbb{R}\,\mathbf{P}^3 - L_0$ are easily seen to lie in L_0 . So $L_1 \subseteq L_0$. Hence $\Lambda = \lambda \cup \pi$ and $\Omega_{loc} = \Omega = \mathbf{R}\,\mathbf{P}^3 - \{\lambda \cup \pi\}$. However looking at the restriction of $\langle g \rangle$ on the line $\mu : y = 0 = z$ we see that the line μ does not lie in the normality domain N . So $N \underset{\neq}{\subset} \Omega$.

(3.6) Remark We have defined the notions of Ω_{loc}, Ω, N etc. with respect to a closed subgroup of the group of homeomorphisms of X . If G is not closed - as indeed may happen when G = the image of the holonomy of an geometric structure - we define Ω_{loc} etc. of G to be that of $\bar{G}$.

§ 4. A uniformization theorem

(4.1) Let (X,G) be a model space satisfying the conditions of (2.2) and the assumptions (U_1) and (U_2) of (3.1). Let M be a topological space with an (X,G)-structure, $p : \tilde{M} \longrightarrow M$ the universal covering projection with deck-transformation group $\Delta \approx \pi_1(M)$, $\delta : \tilde{M} \longrightarrow X$ a development map and $\rho : \Delta \longrightarrow G$ a corresponding holonomy representation. Set $\Omega_M = \operatorname{im} \delta$ and $\Gamma = \operatorname{im} \rho$. Let N_0 be the union of the components of the domain of normality of Γ which have a non-empty intersection with Ω_M. We shall use these notations throughout this section.

(4.2) Theorem Suppose M is a compact space with an (X,G)-structure. Let $\tilde{N} = \delta^{-1}(N_0)$. Then $\delta|_{\tilde{N}} : \tilde{N} \longrightarrow N_0$ is a covering map.

Proof Since δ is a local homeomorphism it suffices to show that $\delta|_{\tilde{N}}$ has a path-lifting property. Fix a point y_0 in $\operatorname{im} \delta|_{\tilde{N}}$, and a path $\beta : [0,1] \longrightarrow N_0$ with $\beta(0) = y_0$. Let $\tilde{x}_0$ be a point in $\tilde{N}$ with $\delta(\tilde{x}_0) = y_0$, and $\tilde{\alpha}$ a maximal lift of β beginning at $\tilde{x}_0$. By way of contradiction assume that β does not lift entirely. Then shrinking the domain of β if necessary and reparametrizing we may assume that $\tilde{\alpha}$ is defined on $[0,1)$. We will show that $\tilde{\alpha}$ has a continuous extension at 1, and so indeed $\tilde{\alpha}$ is not maximal. Let

$\alpha = p \circ \tilde{\alpha} : [0,1) \longrightarrow M$ be the projected path in M. Since M is compact we may choose an increasing sequence $t_n \in [0,1)$ so that $t_n \longrightarrow 1$ and $x_n = \alpha(t_n) \longrightarrow z_0$. Let $\tilde{z}_0 \in \tilde{M}$ be a point lying over z_0, and $\tilde{x}_n = \tilde{\alpha}(t_n)$. Then there exist $g_n \in \Delta$ so that $\tilde{z}_n = g_n \tilde{x}_n \longrightarrow \tilde{z}_0$. Notice that since N_0 is Γ-invariant, we have $\tilde{N}$ Δ-invariant, so $\tilde{z}_n \in \tilde{N}$. Write $\rho(g_n) = \gamma_n$, $y_n = \delta(\tilde{x}_n)$, $w_n = \delta(\tilde{z}_n)$, $w_0 = \delta(\tilde{z}_0)$, and note that

$$\begin{cases} y_n \longrightarrow y_0 \quad \text{and} \\ \\ w_n = \delta(\tilde{z}_n) = \delta(g_n \tilde{x}_n) = \gamma_n \delta(\tilde{x}_n) = \gamma_n y_n \longrightarrow \\ \qquad \delta(z_0) = w_0 . \end{cases}$$

Let $\tilde{V}$ be a neighborhood of $\tilde{z}_0$ and V of w_0 so that $\delta|_{\tilde{V}} : \tilde{V} \longrightarrow V$ is a homeomorphism.

Now choose a compact neighborhood U_{y_0} of y_0 so that $\Gamma|_{U_{y_0}}$ is a relatively compact subset of $C(U_{y_0},X)$. For n sufficiently large $\beta([t_n,1]) \subseteq U_{y_0}$ and by passing to a subsequence if necessary we may assume that $\gamma_n \longrightarrow \gamma_0 \in C(U_{y_0},X)$. Since this convergence is uniform we have for n sufficiently large, $\gamma_n(\beta([t_n,1])) \subseteq V$. Hence the path $\gamma_n \circ \beta|_{[t_n,1]}$ has a lift. This lift would coincide with $g_n \circ \tilde{\alpha}$ on $[t_n,1)$. It is now clear that $\tilde{\alpha}$ itself has a continuation at 1, and the proof is finished.

q.e.d.

§ 5. Applications to conformal geometry

(5.1) We consider the model space $(S^n, M(n))$. An important feature of this structure, called the Möbius structure, is contained in the following proposition

Proposition Let G be a subgroup of $M(n)$, and consider the limit sets etc. w.r.t. G . Then

$$\Lambda = L_0 = L_1^- , \text{ and } \Omega_{loc} = \Omega = N .$$

Proof If $\Omega_{loc} = \emptyset$ there is nothing to prove. Otherwise, let $x \in \Omega_{loc}$ and U_x a small round ball around x contained in Ω_{loc} . Let g_n be a discrete sequence in G . By passing to a subsequence we may assume that $g_n x \longrightarrow y$, so that $y \in L_0$. Also, since Ω_{loc} is G-invariant, we may assume that all $g_n(U_x)$ are pairwise disjoint. It is then obvious that in the spherical metric the radius of the round balls $g_n(U_x)$ goes to zero. So indeed $g_n|_{U_x}$ tends to a constant map c_y in $C(U_x,X)$. We have shown that $G|_{U_x}$ has a compact closure in $C(U_x,X)$. So $x \in N$. Since we always have $N \subseteq \Omega \subseteq \Omega_{loc}$ cf. § 4, it follows that $\Omega_{loc} = \Omega = N$.

q.e.d.

(5.2) Remark It is easy to see that Λ as in (5.1) can be identified with the limit set as defined in the classical situation, cf. [1], [7]. In fact in that case Λ may be identified with $G(x)'$ for any $x \in S^n$, except in the easily analyzed case when $\Lambda = \{2 \text{ points}\}$, each fixed by G and x coincides with one of the fixed points. This may be proved in our set-up by a slight extension of the argument in (5.1), - and in fact the argument applies to any group of quasi-conformal transformations, cf. also (5.8) below. In the following we shall use the well-known properties of the limit set as described e.g. in [7].

(5.3) The following is a direct extension of Gunning's theorem 7 in [9].

Theorem Let M^n be a compact manifold with a Möbius structure, $\delta : \tilde{M}^n \longrightarrow S^n$ its development. $\rho : \Delta \approx \pi_1(M) \longrightarrow \mathcal{M}(n)$ the corresponding holonomy, and $\Gamma = \rho(\Delta)$. Let Ω be the domain of properness of Γ. Suppose δ is not surjective. Then δ is a covering onto the unique component of Ω which intersects $\operatorname{im} \delta$. In particular M is almost Kleinian.

Proof Write $\operatorname{im} \delta = \Omega_M$. Then $\partial\Omega_M$ is Γ-invariant. If $\partial\Omega_M = \{\text{a point}\}$ then regarding this point as ∞, Γ may be considered as a group of similarity transformations in $\mathbf{E}^n = S^n - \{\infty\}$. In this case the result follows from a remarkable theorem of D. Fried [5] (which in fact asserts that M^n has a finite covering which is conformal to a flat space-form or else to a Hopf manifold.) Now suppose that $\partial\Omega_M$ contains at least two points. Then as is well-known $\partial\Omega_M \supseteq \Lambda$. So $\Omega_M \subseteq \Omega$. Since Ω_M is connected it is contained in exactly one component Ω_0 of Ω . Since by (5.1) $\Omega = N$ it follows from the uniformization theorem (4.1) that δ is a covering onto Ω_0 .

q.e.d.

(5.4) We now prove another criterion for a Möbius structure to be Kleinian or almost Kleinian.

Theorem Let $M^n, \delta, \rho, G, \Omega$ be as in (5.2), except that instead of an assumption about δ , we now assume that Ω is connected and $\pi_1(\Omega)$ finitely generated. Then M is almost Kleinian. In particular if $\pi_1(\Omega) = e$, then M is Kleinian.

Proof Let $\Omega_M = \operatorname{im}\delta$. If $\Omega_M \neq S^n$ then the result follows by (5.2). So suppose if possible that $\Omega_M = S^n$.

Case 1 Assume $\pi_1\Omega = e$. By (5.1) we know $\Omega = N$, and if $\tilde{N} = \delta^{-1}(\Omega)$, by (4.1) $\delta|_{\tilde{N}} : \tilde{N} \approx \Omega$. But Λ and hence $\delta^{-1}(\Lambda)$ have no interior, and $\delta : \tilde{M} \longrightarrow S^n$ is a local homeomorphism. Under these conditions it is an easy point-set-topological fact that δ itself is a homeomorphism, and in fact M is conformal to a spherical space-form.

<u>Case 2</u> Assume $\pi_1\Omega \neq 1$, but is finitely generated. Let $\tilde{N} = \delta^{-1}(\Omega)$, $\tilde{L} = \delta^{-1}(\Lambda)$. By (4.1) $\delta|_{\tilde{N}} : \tilde{N} \longrightarrow \Omega$ is a covering. Let $y \in \Lambda$ be an attracting fixed point of an element $g \in \Gamma$. Let $\tilde{x} \in \tilde{M}$ be such that $\delta(\tilde{x}) = y$, and $\tilde{U}$ a neighborhood of $\tilde{x}$ and U a neighborhood of y so that $\delta|_{\tilde{U}}$ is a homeomorphism of $\tilde{U}$ onto U. Since $\pi_1(\Omega)$ is assumed to be finitely generated there is a <u>compact</u> subset $A \subseteq \Omega$ which carries $\pi_1(\Omega)$. Moreover for n sufficiently large $g^n(A) \subseteq U - \Lambda$. So $U - \Lambda$ carries $\pi_1(\Omega)$. But $\delta|_{\tilde{U} - \tilde{L}}$ is a homeomorphism. So the inclusion map $U - \Lambda \longrightarrow \Omega$ which is surjective on π_1 lifts to $\tilde{N}$. It follows that $\delta|_{\tilde{N}}$ must be a homeomorphism. But then again as in case 1 it would follow that δ itself is a homeomorphism, and $\Omega_M = \Omega = S^n$. To summarize: if we assume $\pi_1(\Omega) \neq e$ but finitely generated we must have $\Omega_M \neq S^n$ and so by (5.2) M must be almost Kleinian. q.e.d.

(5.5) <u>Remark</u> In case a compact manifold M^n with a Möbius structure is not almost Kleinian the development map exhibits a rather quaint behavior reminiscent of the behavior of the holomorphic map near an essential singularity. More precisely in the above notation assume that M is not almost Kleinian, so $\Omega_M = S^n$ and suppose Λ has more than 2 points so it is a perfect set. Let $p : \tilde{M} \longrightarrow M$ be the covering projection and $L = p(\delta^{-1}\Lambda)$. (Notice that $\delta^{-1}\Lambda$ is a closed subset invariant under the deck-transformation group so L is a closed subset of M)

<u>Assertion</u> For any open subset U such that $U \cap L \neq \emptyset$ we have $\delta(p^{-1}U) = S^n$.

<u>Proof</u> Indeed let $V = \delta(p^{-1}U)$ and $\Gamma = \rho(\Delta)$. Clearly V is Γ-invariant and contains a small disk D which contains a repelling fixed point of a hyperbolic element $g \in \Gamma$. So V contains $\bigcup_{n=1}^{\infty} (g^n D) = S^n - \{y\}$ where y is the attracting fixed point of g . But D also contains a repelling fixed point of a hyperbolic element $g_1 \in \Gamma$ such that g_1 does not fix y . It is now clear that $V = S^n$. q.e.d.

(5.6) We now point out a convenient construction of Kleinian Möbius structures. It is also useful in other constructions in conformal geometry.

<u>Theorem</u> A connected sum of Kleinian manifolds with a Möbius structure also admits a Kleinian Möbius structure.

<u>Proof</u> Let $M_i = \Gamma_i \backslash \Omega_i$, $i=1,2$ be two Kleinian manifolds with a Möbius structure so that Ω_i are two open connected nonempty subsets of S^n and Γ_i are subgroups of $M(n)$ leaving Ω_i invariant and acting freely and properly

discontinuously there. We shall show that the abstractly defined free product $\Gamma = \Gamma_1 * \Gamma_2$ acts freely and properly discontinuously on a certain region on $\Omega \subseteq S^n$ so that $\Gamma \leq M(n)$ and ${}_\Gamma\backslash\Omega$ is diffeomorphic to a connected sum M of M_1 and M_2. Indeed let $\Delta_i = \pi_1(M_i)$ $i=1,2$. We have projections $p_i : \Delta_i \longrightarrow \Gamma_i$. Let $\Phi_i = \ker p_i$. Let $\{\gamma^i_j\}_{j \in J_i}$ be based loops in M_i so that their homotopy classes $[\gamma^i_j]$ normally generate Φ_i. So Ω_i is a connected covering of M_i which is universal with respect to the property that each lift of γ^i_j is a loop. Consider the complex

$$A = (M_1 \cup M_2 \cup I)/\sim$$

where $I = [0,1]$ and 0 is identified with a point in M_1 and 1 with a point in M_2. Then $\pi_1(A) \approx \Delta_1 * \Delta_2$ and we have a canonical projection $f : \pi_1(A) \longrightarrow \Gamma$ so that $\Phi = \ker p$ is normally generated by $\{\gamma^1{}_j\}_{j \in J_1} \cup \{\gamma^2{}_j\}_{j \in J_2}$. Take the covering B of A corresponding to Φ. This is constructed out of the $|\Gamma/_{\Gamma_2}|$ copies of Ω_i (i.e. to say in 1-1 correspondence with $\Gamma/_{\Gamma_i}$), $i=1,2$ and $|\Gamma|$ copies of I. The copies of I may be considered as the "connecting bonds" between the copies of Ω_1 and those of Ω_2. The main point is that (*) <u>each copy of</u> Ω_i <u>is attached with</u> $|\Gamma_i|$ <u>connecting bonds</u> $i=1,2$ and <u>no copy of</u> Ω_1 <u>is</u>

<u>joined to a copy of</u> Ω_2 <u>by two connecting bonds</u>.

We now thicken I in A and remove the interior so as to obtain the connected sum N of M_1 and M_2 by the process described in [13] so that M has a Möbius structure which restricts to the prescribed Möbius structures on parts of M_i , $i=1,2$ which lie in M . We do the <u>corresponding</u> thickenings etc. in B so as to obtain a manifold Ω with a Möbius structure which covers M with the covering group Γ . We shall now embed Ω into S^n preserving the Möbius structure. In the process of obtaining Ω from B from each copy of Ω_i in B , $|\Gamma_i|$ round disks are removed. We now embed one copy of Ω_1 with $|\Gamma_1|$ round disks removed in S^n preserving the Möbius structure. In each hole of this copy we can insert a copy of Ω_2 (with $|\Gamma_2|$ holes) to which it is attached in Ω . In each hole of a copy of Ω_2 we can insert a copy of Ω_1 (with $|\Gamma_1|$ holes) to which it is attached in Ω . Now the fact (*) we mentioned above implies that we can continue this process to obtain a Möbius structure - preserving embedding of Ω into S^n . We have also used here the existence of inversions and the fact that all round S^{n-1} in S^n are equivalent under $M(n)$. Now the group Γ acts on the image of Ω into S^n preserving the Möbius structure. Since every Möbius transformation defined on a connected open subset of S^n is a restriction of an element of $\mathcal{M}(n)$, we can regard Γ as a subgroup of $\mathcal{M}(n)$. This finishes the proof.

q.e.d.

((5.7) It should be pointed out that the hypotheses in (5.3) and (5.4) cannot be entirely dropped. In fact the method of proof of (5.6) shows that a connected sum of two compact manifolds with a Möbius structure, one of which is almost Kleinian but non-Kleinian and the other $\not\approx S^n$ admits a Möbius structure with surjective development map, so this structure is non-almost Kleinian. Another class of very interesting examples is obtained by conformally deforming a neighborhood of a totally geodesic hypersurface in a compact hyperbolic manifold. For $\mathbb{C}\mathbb{P}^1$-structures they were noticed by Maskit [16] and Hejhal [11] and in a quite different context by Faltings [4]. The non-trivial infinitesimal deformations of the corresponding groups were noticed by Lafontaine [15] and Millson [18]. Looking at their development they were named "Mickey Mouse" examples by Thurston [20], and their geometry (and also their projective analogues) have been beautifully explained by Goldman [6]. We remark that similar deformations can be obtained also for $M^n \times S^p$ by deforming a neighborhood of $N^{n-1} \times S^p$ where M^n is a compact hyperbolic manifold and N^{n-1} is a totally geodesic hypersurface in M^n . Among these we find examples of non-almost Kleinian M^n whose domain of properness is disconnected, although each component is simply connected - thus showing that the hypothesis of connectedness of the domain of properness in (5.4) cannot be dropped. On the other hand a connected sum of non-Kleinian almost Kleinian compact manifolds with limit set $\approx S^{n-2}$ gives an example of a non-almost Kleinian manifold with a connected domain of properness with non-finitely generated π_1 - thus showing that the hypothesis of finite generation of π_1 in (5.4) also cannot be dropped.

(5.8) Further generalizations In this section we have formulated the theorems for the sake of simplicity only in the case of the standard Möbius structures. But it is apparent from the proofs that the strict "angle-preserving" property is not really used in any crucial way and so the results are valid in much more general set-ups. In fact let H^{n+1} be an (n+1)-dimensional complete, connected, simply connected Riemannian manifold with sectional curvatures $\leq -\varepsilon < 0$. In a well-known manner we can attach to it an ideal boundary Σ^n made up of classes of asymptotic geodesic rays. Then Σ^n is homeomorphic to the n-sphere and $H^{n+1} \cup \Sigma^n$ is homeomorphic to a closed disk. Moreover the isometries of H^{n+1} are classified into elliptic, hyperbolic and parabolic types depending on whether there is a fixed point in H^{n+1}, or exactly two fixed points on $\sum^n$, or exactly one fixed point on $\sum^n$: For $p \in H^{n+1}$ the images of round sub-spheres or disks in the unit n-sphere in $T_p(H^{n+1})$ via the exponential map may be considered as "round" sub-spheres or disks of $\sum^n$. (Here p is allowed to vary in H^{n+1}.) This defines a kind of "conformal geometry" in $\sum^n$, cf. [19]. The full group of isometries of H^{n+1} extends to $\sum^n$ and serves as the "Möbius group" of $\sum^n$, and we may consider the structures based on $(\sum^n, G)$. The boundaries of rank-1 non-compact symmetric spaces provide interesting examples of this set-up. The proposition (5.1) is valid for $(\sum^n, G)$-structures. As of this writing we do not know the validity of Fried's theorem quoted in the proof of the theorem (5.3). Otherwise, if we assume in (5.3) that im δ misses two points then (5.3) is valid for $(\sum^n, G)$-structures. Similarly (5.4) with an appropriate modification goes through. For the validity of connected sums we need to assume

1) there exists $g \in G$ which leaves invariant a tame (n-1)-sphere $\sum^{n-1}$ in $\sum^n$ (e.g. a round (n-1)-sphere in $\sum^n$) and intercharges the two components of $\sum^n - \sum^{n-1}$, and

2) G does not fix a point in $\sum^n$ and the fixed points of hyperbolic isometries are dense in $\sum^n$.

For example 2) holds if H^{n+1} covers a manifold M^{n+1} of finite volume; and furthermore 1) holds if M^n admits an isometry $\bar{g}$ such that $\bar{g}^2 = 1$ and the fixed point set is a totally geodesic hypersurface. The conditions 1) and 2) also hold for the boundaries of rank-1 symmetric spaces. The conditions 1) and 2) ensure that there are inversions through sufficiently small spheres, and moreover given any two non-empty open sets U,V of $\sum^n$ there exists $g \in G$, an $(n-1)$-sphere $\sum^{n-1} \subseteq U$ and $g(\sum^{n-1}) \subseteq V$ so that there exists an inversion σ through $\sum^{n-1}$. (Then $g \circ \sigma \circ g^{-1}$ is an inversion through $g \cdot \sum^{n-1})$. This suffices to perform the connected sums of manifolds with a $(\sum,G)$-structure. For the case of the boundary of the complex hyperbolic space this fact was observed by Burns and Shnider [3]. Finally if 1) and 2) hold then the theorem (5.6) is valid for $(\sum,G)$-structures.

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