Value distribution of cyclotomic polynomial coefficients

Yves Gallot, Pieter Moree and Huib Hommersom

Abstract

Let $a_n(k)$ be the kth coefficient of the nth cyclotomic polynomial $\Phi_n(x)$. As n ranges over the integers, $a_n(k)$ assumes only finitely many values. For any such value v we determine the density of integers n such that $a_n(k) = v$. Also we study the average of the $a_n(k)$. We derive analogous results for the kth Taylor coefficient of $1/\Phi_n(x)$ (taken around $x = 0$). We formulate various open problems.

1 Introduction

Let

$$\Phi_n(x) = \prod_{\substack{j=1 \\ (j,n)=1}}^{n} (x - e^{\frac{2\pi j i}{n}}) = \sum_{k=0}^{\varphi(n)} a_n(k)x^k, \tag{1}$$

denote the nth cyclotomic polynomial and φ Euler's totient function. If $k > \varphi(n)$, we put $a_n(k) = 0$. The coefficients $a_n(k)$ are integers. In this paper we also consider the behaviour of the coefficients $c_n(k)$ in the Taylor series of $1/\Phi_n(x)$ around $x = 0$:

$$\frac{1}{\Phi_n(x)} = \sum_{k=0}^{\infty} c_n(k)x^k.$$

In the 19th century mathematicians were already intrigued by the behaviour of $a_n(k)$, since the $a_n(k)$ seem to be amazingly small. In a nutshell the history of the study of $a_n(k)$ can be described as inspired by conjectures about the $a_n(k)$ being small that were being proved wrong in due course. A 19th century example being the conjecture that $a_n(k) \in \{-1, 0, 1\}$, which turned out to be wrong once it was shown that $a_{105}(7) = -2$. A 21th century example being the recent disproof of Gallot and Moree [7] of the Beiter conjecture (dating back to 1968). This conjecture asserts that if $p < q < r$ are primes, then $|a_{pqr}(k)| \leq (p + 1)/2$. In [7] it is shown to be false for every prime $p \geq 11$ and, moreover, that given any $\delta > 0$ there exist infinitely many triples (p_j, q_j, r_j) with $p_1 < p_2 < \ldots$ consecutive

Mathematics Subject Classification (2000). 11N37, 11B83 primes such that $|a_{p_j q_j r_j}(n_j)| > (2/3-\delta)p_j$ for $j \ge 1$.

On noting that $x^n - 1 = \prod_{d|n} \Phi_d(x)$, we see that

$$\frac{x^n-1}{\Phi_n(x)} = \prod_{d|n,\ d<n} \Phi_d(x)$$

is a polynomial of degree $n - \varphi(n)$ having integer coefficients. From this we infer that the $c_n(k)$ are integers that only depend on the congruence class of k modulo n. Numerics suggest that the $c_n(k)$, like the $a_n(k)$, are surprisingly small. In constrast to the $a_n(k)$ they only seem to have been studied as numbers of independent interest in a paper by the second author [15].

From the equality $x^n - 1 = \prod_{d|n} \Phi_d(x)$ one finds by inclusion and exclusion that $\Phi_n(x) = \prod_{d|n}(x^d-1)^{\mu(n/d)}$, where μ denotes the Möbius function. On using that $\sum_{d|n} \mu(d) = 0$ if $n > 1$, we obtain, for $n > 1$,

$$\Phi_n(x) = \prod_{d|n}(1-x^d)^{\mu(\frac{n}{d})}. \tag{2}$$

Sometimes it will be convenient to write

$$\Phi_n(x) = \prod_{d=1}^{\infty}(1-x^d)^{\mu(\frac{n}{d})},$$

where we have put $\mu(r) = 0$ in case r is not an integer. Thus $a_n(k)$ is the coefficient of x^k in $\prod_{d<k+1}(1-x^d)^{\mu(\frac{n}{d})}$. Since $\mu \in \{-1, 0, 1\}$, we infer that $a_n(k)$ (and likewise $c_n(k)$) assumes only finitely many values as n ranges over the natural numbers.

Put $\mathcal{A}(k) = \{a_n(k) | n \ge 1\}$ and $\mathcal{C}(k) = \{c_n(k) | n \ge 1\}$. Let $\chi_v(m) = 1$ if $m = v$ and 0 otherwise. Our main interest in this paper is to study these sets and some associated quantities such as

$$A(k) := \max\{|a_n(k)| : n \ge 1\} \text{ and } C(k) := \max\{|c_n(k)| : n \ge 1\}.$$

Also we are interested in the averages

$$M(a_n(k)) = \lim_{x\to\infty} \frac{\sum_{n\le x} a_n(k)}{x} \text{ and } M(c_n(k)) = \lim_{x\to\infty} \frac{\sum_{n\le x} c_n(k)}{x}.$$

Furthermore, for a given v we consider the densities

$$\delta(a_n(k)=v) = \lim_{x\to\infty} \frac{1}{x}\sum_{n\le x} \chi_v(a_n(k)) \text{ and } \delta(c_n(k)=v) = \lim_{x\to\infty} \frac{1}{x}\sum_{n\le x} \chi_v(c_n(k)).$$

We note that various authors have considered $A(k)$. In contrast we know of only one paper dealing with $M(a_n(k))$ and sketching how $\delta(a_n(k) = v)$ can be computed. This is a paper due to Herbert Möller [14]. In Section 3 we briefly discuss this previous work and especially the latter paper as we will propose some improvements of it.

The results in this paper suggest that the behaviour of the $c_n(k)$ is so close to that of the $a_n(k)$, making consideration of the $c_n(k)$ hardly worthwhile. This is no longer true if we vary k and keep n fixed. E.g. one can have $|c_{pqr}(k)| = p-1$, whereas the estimate $|a_{pqr}(k)| \le 3p/4$ always holds (see [15]). Furthermore, in our average consideration the quantities $a_n(k)+c_n(k)$ and $a_n(k)-c_n(k)$ show up in a natural way. In the M.Sc. thesis [16], on which this paper is partly based, the notion of $c_n(k)$ is not used, leading to slightly more complicated formulations of some of the results.

2 Basics and preliminaries

Due to the fact that (2) does not hold for $n=1$ various technical complications arise. For this reason it turns out to be helpful to work with the following modified cyclotomic coefficients (with $\epsilon = \pm 1$, by which we mean $\epsilon \in \{-1, 1\}$):

Definition 1 *We let $a_n^{\epsilon}(k)$ be the kth Taylor coefficient (around $x=0$) in the product $\prod_{d|n}(1-x^d)^{\epsilon\mu(n/d)}$, i.e., we have, for $|x|<1$,*

$$\prod_{d|n}(1-x^d)^{\epsilon\mu(\frac{n}{d})} = \sum_{k=0}^{\infty} a_n^{\epsilon}(k)x^k. \tag{3}$$

Sometimes we will use the latter identity in the following form (valid for $|x|<1$):

$$\prod_{d=1}^{\infty}(1-x^d)^{\epsilon\mu(\frac{n}{d})} = \sum_{k=0}^{\infty} a_n^{\epsilon}(k)x^k. \tag{4}$$

Note that the left hand side in (4) equals $\Phi_n^{\epsilon}(x)$ in case $n>1$ and $-\Phi_1^{\epsilon}(x)$ in case $n=1$. From this we see that

$$a_n^1(k) = \begin{cases} a_n(k) & \text{if } n>1; \\ -a_1(k) & \text{if } n=1, \end{cases} \quad \text{and } a_n^{-1}(k) = \begin{cases} c_n(k) & \text{if } n>1; \\ -c_1(k) & \text{if } n=1. \end{cases}$$

The basic properties of $\Phi_n(x)$ and its coefficients given below are quite useful.

Lemma 1 *Let q_1 and q_2 be primes with $k<q_1<q_2$ and $(q_1q_2, n)=1$. Then $a_{nq_1}^{\epsilon}(k) = a_n^{-\epsilon}(k)$ and $a_{nq_1q_2}^{\epsilon}(k) = a_n^{\epsilon}(k)$.*

Proof. An easy consequence of (3) and the properties of the Möbius function. □

By $\gamma(n) = \prod_{p|n} p$ we denote the squarefree kernel of n.

Lemma 2
1) *We have $\Phi_n(x) = \Phi_{\gamma(n)}(x^{n/\gamma(n)})$.*
2) *We have $\Phi_{2n}(x) = \Phi_n(-x)$ if $n>1$ is odd.*
3) *We have $x^{\varphi(n)}\Phi_n(1/x) = \Phi_n(x)$ if $n>1$.*

In terms of the coefficients the three properties of Lemma 2 imply (respectively):

$$a_n^{\epsilon}(k) = \begin{cases} a_{\gamma(n)}^{\epsilon}(\frac{k\gamma(n)}{n}) & \text{if } \frac{n}{\gamma(n)}|k; \\ 0 & \text{otherwise}, \end{cases} \tag{5}$$

$$a^{\epsilon}_{2n}(k) = (-1)^k a^{\epsilon}_n(k) \quad \text{if } 2 \nmid n; \tag{6}$$
$$a^{\epsilon}_n(k) = \epsilon a^{\epsilon}_n(\varphi(n) - k) \quad \text{for } n > 1,\ 0 \le k \le \varphi(n), \tag{7}$$

where to prove (6) in case $n = 1$ we used that $\Phi^{\epsilon}_2(x) = -\Phi^{\epsilon}_1(-x)$. In order to prove (7) in case $\epsilon = -1$, we used the additional observation that

$$\frac{1 - x^n}{\Phi_n(x)} = \sum_{k=0}^{n-\varphi(n)} a^{-1}_n(k) x^k.$$

It is not difficult to derive Lemma 2 from (2), see e.g. Thangadurai [19].

Note that, for $|x| < 1$, we have

$$\prod_{d=1}^{\infty}(1 - x^d)^{\mu(\frac{n}{d})} = \prod_{d=1}^{\infty}\left(1 - \mu(\frac{n}{d})x^d + \frac{1}{2}\mu(\frac{n}{d})(\mu(\frac{n}{d}) - 1)\sum_{j=2}^{\infty} x^{jd}\right), \tag{8}$$

where we used the observation that, for $|x| < 1$,

$$(1 - x^d)^{\mu(\frac{n}{d})} = 1 - \mu(\frac{n}{d})x^d + \frac{1}{2}\mu(\frac{n}{d})(\mu(\frac{n}{d}) - 1)\sum_{j=2}^{\infty} x^{jd}. \tag{9}$$

From (8) it is not difficult to derive a formula for $a^1_n(k)$ for a fixed k; this is just the coefficient of x^k in the right hand side of (8) (this approach seems to be due to D.H. Lehmer [12]). We thus obtain,

$$\begin{cases} a^1_n(1) = -\mu(n); \\ a^1_n(2) = \mu(n)^2/2 - \mu(n)/2 - \mu(n/2); \\ a^1_n(3) = \mu(n)^2/2 - \mu(n)/2 + \mu(n/2)\mu(n) - \mu(n/3). \end{cases}$$

More generally, we have

$$a^1_n(k) = \sum c(k_1, ..., k_s; e_1, \dots, e_s)\mu(\frac{n}{k_1})^{e_1} \cdots \mu(\frac{n}{k_s})^{e_s},$$

where the sum is over all partitions $k_1 + \ldots + k_s$ of all the integers $\le k$ with $k_1 \ge k_2 \ge \cdots \ge k_s$ and over all $e_1, \dots, e_s$ with $1 \le e_j \le 2$ for $1 \le j \le s$. The terms in (10) for which $e_1 + \ldots + e_s$ is even we add together to obtain $\alpha_n(k)$, the *even part* of $a^1_n(k)$. Similarly, we group the terms with $e_1 + \ldots + e_s$ odd together, to form the *odd part*, $\beta_n(k)$, of $a^1_n(k)$. (To the authors knowledge the even and odd part of $a^1_n(k)$ have not been defined and considered before.) For example, $\alpha_n(2) = \mu(n)^2/2$ and $\beta_n(2) = -\mu(n)/2 - \mu(n/2)$. Note that

$$\begin{aligned} a^{\epsilon}_n(k) &= \sum c(k_1, ..., k_s; e_1, \dots, e_s)(\epsilon\mu(\frac{n}{k_1}))^{e_1} \cdots (\epsilon\mu(\frac{n}{k_s}))^{e_s}. \\ &= \alpha_n(k) + \epsilon\beta_n(k). \end{aligned} \tag{10}$$

We have $\alpha_n(k) = (a^1_n(k) + a^{-1}_n(k))/2$ and $\beta_n(k) = (a^1_n(k) - a^{-1}_n(k))/2$. In particular, $2\alpha_n(k), 2\beta_n(k) \in \mathbb{Z}$. From (10) and the properties of the Möbius function it follows that if p and q are two distinct primes exceeding k with $(pq, n) = 1$, then $\alpha_{pn}(k) = \alpha_n(k)$, $\beta_{pn}(k) = -\beta_n(k)$, $\alpha_{pqn}(k) = \alpha_n(k)$ and $\beta_{pqn}(k) = \beta_n(k)$. The reason, as we will see, for distinguishing between the odd and even part, is that the odd part does not contribute to te average, i.e. $M(\beta_n(k))=0$.

The *Ramanujan sum* $r_n(m)$ is defined by

$$r_n(m)=\sum_{\substack{1\le k\le n\\(k,n)=1}} e^{\frac{2\pi imk}{n}}=\sum_{\substack{1\le k\le n\\(k,n)=1}}\zeta_n^{mk}. \tag{11}$$

Alternatively one can write $r_n(m)=\mathrm{Tr}_n(\zeta_n^m)$, where by Tr_n we denote the trace over the cyclotomic field $\mathbb{Q}(\zeta_n)$. It follows at once from the properties of the trace that $r_n(m)=r_n((n,m))$. Since ζ_n^m is an algebraic integer, it follows that $r_n(m)$ is an integer.

The Ramanujan sums have many properties of which we will need only the following two.

Lemma 3 *We have* $r_n(m)=\sum_{d|(n,m)}d\mu(\frac{n}{d})$ *and*

$$r_n(m)=\mu\left(\frac{n}{(n,m)}\right)\frac{\varphi(n)}{\varphi(\frac{n}{(n,m)})}.$$

Nicol [17] showed that Ramanujan sums and cyclotomic polynomials are closely related, by establishing that

$$\Phi_n(x)=\exp\Big(-\sum_{m=1}^{\infty}\frac{r_n(m)}{m}x^m\Big)\text{ and }\sum_{m=1}^{n}r_n(m)x^{m-1}=(x^n-1)\frac{\Phi_n'(x)}{\Phi_n(x)}.$$

2.1 Some sums involving the Möbius function

In order to evaluate $M(a_n(k))$ we will need to evaluate $\sum_{m\le x,\ (m,r)=1}\mu(m)^k$ with $1\le k\le 2$. That is done in Lemmas 4 and 5.

Lemma 4 *Let $r\ge 1$ be an integer. We have*

$$\sum_{\substack{m\le x\\(m,r)=1}}\mu(m)^2=\frac{6x}{\pi^2\prod_{p|r}(1+\frac{1}{p})}+O(\sqrt{x}\varphi(r)),$$

where the implied constant is absolute.

Proof. We have, by inclusion and exclusion,

$$\sum_{\substack{m\le x\\(m,r)=1}}\mu(m)^2=\sum_{\substack{d\le\sqrt{x}\\(d,r)=1}}\mu(d)A_r(\frac{x}{d^2}),$$

where $A_r(x)$ denotes the number of integers $n\le x$ that are coprime with r. Note that

$$\left[\frac{x}{r}\right]\varphi(r)\le A_r(x)\le\left[\frac{x}{r}\right]\varphi(r)+\varphi(r)$$

and hence $A_r(x)=\varphi(r)x/r+O(\varphi(r))$. On using the latter estimate we obtain

$$\begin{aligned}\sum_{\substack{m\le x\\(m,r)=1}}\mu(m)^2 &= x\frac{\varphi(r)}{r}\sum_{\substack{d\le\sqrt{x}\\(d,r)=1}}\frac{\mu(d)}{d^2}+O(\sqrt{x}\varphi(r)).\\ &= x\frac{\varphi(r)}{r}\sum_{(d,r)=1}^{\infty}\frac{\mu(d)}{d^2}+O(\sqrt{x}\varphi(r)).\\ &= \frac{6x}{\pi^2\prod_{p|r}(1+\frac{1}{p})}+O(\sqrt{x}\varphi(r)),\end{aligned}$$

where we used that

$$\frac{\varphi(r)}{r}\sum_{(d,r)=1}^{\infty}\frac{\mu(d)}{d^2}=\frac{\varphi(r)}{r}\prod_{p\nmid r}(1-\frac{1}{p^2})=\frac{\varphi(r)}{\zeta(2)r\prod_{p|r}(1-\frac{1}{p^2})}=\frac{1}{\zeta(2)\prod_{p|r}(1+\frac{1}{p})}$$

and $\zeta(2)=\pi^2/6$. □

Lemma 5 *We have*

$$\sum_{\substack{m\le x\\(m,r)=1}}\mu(m)=o(x),$$

where the implied constant may depend on r.

To prove the lemma, we will apply the Wiener-Ikehara Tauberian theorem in the following form.

Theorem 1 *Let $f(s)=\sum_{n=1}^{\infty}a_n/n^s$ be a Dirichlet series. Suppose there exists a Dirichlet series $F(s)=\sum_{n=1}^{\infty}b_n/n^s$ with positive real coefficients such that*
(a) $|a_n|\le b_n$ *for all n;*
(b) *the series $F(s)$ converges for Re$(s)>1$;*
(c) *the function $F(s)$ can be extended to a meromorphic function in the region Re$(s)\ge 1$ having no poles except for a simple pole at $s=1$.*
(d) *the function $f(s)$ can be extended to a meromorphic function in the region Re$(s)\ge 1$ having no poles except possibly for a simple pole at $s=1$ with residue r.*
Then

$$\sum_{n\le x}a_n=rx+o(x),\ \ x\to\infty.$$

In particular, if $f(s)$ is holomorphic at $s=1$, then $r=0$ and $\sum_{n\le x}a_n=o(x)$ as $x\to\infty$.

Proof of Lemma 5. We apply the Wiener-Ikehara theorem with $F(s)=\zeta(s)$ and

$$f(s)=\sum_{(n,r)=1}\frac{\mu(n)}{n^s}=\frac{1}{\zeta(s)\prod_{p|r}(1-\frac{1}{p^s})}.$$

Of course $F(s)$ satisfies the required properties and has a simple pole at $s=1$ with residue one. Since the finite product in the formula for $f(s)$ is regular for Re$(s)>0$, the result follows on using the well-known fact that $1/\zeta(s)$ can be extended to a meromorphic function in the region Re$(s)\ge 1$ (and hence $r=0$). This completes the proof. □

Landau [11, §173 & §174] gave estimates for $\sum_{n\le x,\ n\equiv l(\text{mod }k)}\mu(n)^r$, with $1\le r\le 2$. Using these, an alternative proof of Lemmas 4 and 5 can be given.

The following result is found on combining Lemma 4 and 5.

Lemma 6 *Let $\epsilon=\pm 1$ and $r\ge 1$. We have, as x tends to infinity,*

$$\sum_{\substack{m\le x,\ \mu(m)=\epsilon\\(m,r)=1}}1\sim\frac{3x}{\pi^2\prod_{p|r}(1+\frac{1}{p})}.$$

3 Previous work

In this section we discuss previous work on the quantities defined in the introduction. Of those $A(k)$ has received quite a bit of attention. H. Möller [14] gave a table for $A(k)$ for $1 \le k \le 20$ which we reproduce below (this before Endo [6], who proved that $A(k) = 1$ for $k \le 6$).

Table 1: $A(k)$ for $1 \le k \le 30$

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$A(k)$	1	1	1	1	1	1	2	1	1	1	2	1	2	2	2
k	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
$A(k)$	2	3	3	3	3	3	3	4	3	3	3	3	4	4	5

The table suggests that $A(k) \le k$ for every $k \ge 1$, this is however very far from being the case as given $r > 1$ we have $A(k) > k^r$ for all k sufficiently large as Möller proved. Bachman [1] extended work of several earlier authors (see the references he gives) and established the best result to date stating that

$$\log A(k) = C_0 \frac{\sqrt{k}}{(\log k)^{1/4}} \left(1 + O\left(\frac{\log\log k}{\sqrt{\log k}}\right)\right), \tag{12}$$

where C_0 can be explicitly given.

In Möller's approach $a_n(k)$ is connected with partitions of k through the basic formula (14). A partition can be identified with a sequence $\{n_j\}_{j=1}^{\infty}$ of non-negative integers satisfying $\sum_j jn_j = m$. Without loss of generalisation we can denote a partition, λ, of k as $\lambda = (k_1^{n_{k_1}} \cdots k_s^{n_{k_s}})$, where $n_{k_1} \ge n_{k_2} \ge \ldots \ge n_{k_s} \ge 1$ (thus the number k_j occurs n_{k_j} times in the partition). The set of all partitions of m will be denoted by $\mathcal{P}(m)$. The number of different partitions of m is denoted by $p(m)$. Hardy and Littlewood in 1918, and Uspensky independently in 1920, proved that

$$p(m) \sim \frac{e^{\pi\sqrt{2m/3}}}{4m\sqrt{3}} \quad \text{as } m \to \infty. \tag{13}$$

In case $\epsilon = 1$ and $n > 1$ the following result is [14, Satz 2].

Lemma 7 *For $n \ge 1$, $k \ge 0$ and $\epsilon = \pm 1$, we have*

$$a_n^{\epsilon}(k) = \sum_{\substack{\lambda = (k_1^{n_{k_1}} \ldots k_s^{n_{k_s}}) \in \mathcal{P}(k) \\ n_{k_1} \ge \ldots \ge n_{k_s} \ge 1}} \prod_{j=1}^{s} (-1)^{n_{k_j}} \binom{\epsilon\mu(\frac{n}{k_j})}{n_{k_j}}, \tag{14}$$

where the sum is over all partitions λ of $\mathcal{P}(k)$.

Proof. The Taylor series of $(1-x)^a$ equals, for $|x| < 1$, $\sum_{j=0}^{\infty}(-1)^j\binom{a}{j}x^j$, where $\binom{a}{j} = a(a-1)\cdots(a-(j-1))/j!$. Using this we infer that

$$(1-x^d)^{\epsilon\mu(\frac{n}{d})} = \sum_{j=0}^{\infty}(-1)^j \binom{\epsilon\mu(\frac{n}{d})}{j} x^{dj}, \ |x| < 1, \tag{15}$$

The proof now follows from (15) and (4). □

Earlier D. Lehmer [12] had used a formula for $a_n(k)$, expressing it in terms of Ramanujan sums. The formula above turns out to be more practical. Our proof above shows that formula (14) is a triviality, whereas Möller's ingenious and rather involved proof of it (he considered only $\epsilon = 1$) obscures this.

Comparison of (9) and (15) yields

$$(-1)^j \binom{\epsilon\mu(\frac{n}{d})}{j} = \begin{cases} 1 & \text{if } j = 0; \\ -\epsilon\mu(n/d) & \text{if } j = 1; \\ \epsilon\mu(n/d)(\epsilon\mu(n/d)-1)/2 & \text{if } j \geq 2. \end{cases} \tag{16}$$

Thus the product appearing in (14) is in $\{-1, 0, 1\}$ and it follows from (14) that $|a_n^{\epsilon}(k)| \leq p(k)$. By the asymptotic formula for $p(m)$ given above it then follows that $\log A(k) \ll \sqrt{k}$ is the trivial upper bound for $\log A(k)$. From Lemma 7 Möller infers that even $|a_n(k)| \leq p(k) - p(k-2)$. To see this note that the partitions having 1 occurring at least twice do not contribute if $\mu(n) \in \{0, 1\}$. If $\mu(n) = -1$, then either $\mu(2n) = 1$ or $\mu(n/2) = 1$. This in combination with (6) allows us then to argue as before and leads us to the same bound. Similarly we have $|c_n(k)| \leq p(k) - p(k-2)$.

Möller uses Lemma 7 to show that

$$M(a_n(k)) = \lim_{x\to\infty} \frac{\sum_{n\leq x} a_n(k)}{x}$$

exists and gives a formula for it. To do so he first computes the Dirichlet series $D_k(s) := \sum_{n=1}^{\infty} a_n(k) n^{-s}$. The required average is then $\lim_{s\to\infty}(s-1)D_k(s)$. The expression so obtained is rather complicated and requires work to be simplified. Here we rederive his result (see Lemma 19) in a more direct way by simply evaluating the averages

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\leq x} \prod_{j=1}^{s} (-1)^{n_{k_j}} \binom{\mu(\frac{n}{k_j})}{n_{k_j}}$$

and then summing over the partitions of k.

Möller's result shows that $M(a_n(k)) = 6e_k/\pi^2$, with e_k a rational number. For $1 \leq k \leq 20$ we give the value of e_k in Table 2 (our table agrees with the one given in [14], except for the incorrect values $e_{10} = 319/1440$ and $e_{16} = 733/2016$ appearing there).

Table 2: Scaled average, $e_k = \zeta(2)M(a_n(k))$, of $a_n(k)$

k	1	2	3	4	5	6	7	8	9	10
e_k	0	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{8}$	$\frac{7}{24}$	$\frac{1}{18}$	$\frac{7}{24}$	$\frac{19}{144}$	$\frac{31}{160}$
k	11	12	13	14	15	16	17	18	19	20
e_k	$\frac{1}{16}$	$\frac{55}{192}$	$\frac{13}{288}$	$\frac{61}{288}$	$\frac{2287}{20160}$	$\frac{733}{4032}$	$\frac{667}{8064}$	$\frac{79}{336}$	$\frac{55}{1344}$	$\frac{221}{960}$

Regarding e_k Möller proposed:

Conjecture 1 [14]. *Let* $k \geq 1$. *Write* $M(a_n(k)) = 6e_k/\pi^2$.
1) *We have* $0 \leq e_k \leq 1/2$.
2) *We have* $(-1)^k(e_k - e_{k+1}) > 0$ *("see-saw conjecture").*

Möller stated that with help of an IBM 7090 he wanted to check his conjecture for further values of k. Had he carried this out, he would have discovered that $(-1)^{34}(e_{34} - e_{35}) = -18059/4626720 < 0$. Other counterexamples occur at $k = 35, 45$ and 94. Indeed, we would not be surprised if part 2 of the Conjecture is violated for infinitely many k. The see-saw conjecture, if true, would imply that $\sum_{k=1}^{m}(-1)^k(e_k - e_{k+1}) > 0$ for every $m \geq 1$. The truth of the latter assertion is still open.

On the other hand, part 1 of the Conjecture is true for $k \leq 100$. The numbers e_k seem to be decreasing to zero and their size seems to be related to the number of prime factors of k, the more prime factors the larger e_k seems to be.

As already pointed out by Möller one could use his method to study the value distribution of $a_n(k)$ in case e.g. $A(k) = 1$ by considering the integer $a_n(k)(a_n(k) - 1)/2$ to determine $\delta(a_n(k) = -1)$ for example. This then yields a sum with $p(k)^2 + p(k)$ terms and this results in an algorithm that has worse complexity than that provided by Theorem 4 below. Aside from this, this seems to be, from the practical point of view, an unwieldy method. A more practical method will be presented in Section 6.

A further result which is of relevance to us, is the following one.

Theorem 2 *Let* $m \geq 1$ *be an integer and* $\epsilon = \pm 1$. *Then*

$$\{a_{mn}^{\epsilon}(k) : n \geq 1,\ k \geq 0\} = \mathbb{Z}.$$

For a proof and the prehistory of this result see Ji, Li and Moree [10].

4 Computation of $\mathcal{A}(k)$ and $\mathcal{C}(k)$

Recall that $\mathcal{A}(k) = \{a_n(k) | n \geq 1\}$ and $\mathcal{C}(k) = \{c_n(k) | n \geq 1\}$. Throughout this section we assume that $k \geq 1$.

Lemma 8 *We have*

$$\{-1, 0, 1\} \subseteq \{a_n(k) : n > 1\} \text{ and } \{-1, 0, 1\} \subseteq \{c_n(k) : n > 1\}.$$

Proof. In formula (7) there is always the term $-\epsilon\mu(n/k)$. Let us take $n = ck\prod_{p \leq k} p$, where c only has prime divisor $> k$. Then all the terms of the form $\mu(n/r)$ with $1 \leq r < k$ are zero (since either $r \nmid n$ or n/r is not squarefree) and we obtain that $a_n^{\epsilon}(k) = -\epsilon\mu(c)(-1)^{\pi(k)}$, where $\pi(x)$ as usual denotes the number of primes $p \leq x$ not exceeding x. In particular, it follows that $a_n^{\epsilon}(k)$ always assumes the values $-1, 0$ and 1. Since $n > 1$ for these examples, $a_n^{\epsilon}(k) = a_n(k)$ if $\epsilon = 1$ and equals $c_n(k)$ if $\epsilon = -1$, and the result follows. □

Lemma 9 *We have* $\mathcal{A}(k) = \{a_n^1(k) | n \geq 1\}$ *and* $\mathcal{C}(k) = \{a_n^{-1}(k) | n \geq 1\}$.

Proof. Using Lemma 8 one infers that

$$\mathcal{A}(k) = \{a_1(k)\} \cup \{a_n(k)|n > 1\} = \{a_n(k)|n > 1\} = \{a_n^1(k)|n > 1\}.$$

Likewise we find $\mathcal{C}(k) = \{a_n^{-1}(k)|n \geq 1\}$. □

The next lemma follows on applying the latter lemma in combination with Lemma 1.

Lemma 10 *We have* $\mathcal{C}(k) = \mathcal{A}(k)$.

Lemma 11 allows one to deduce that $\mathcal{A}(k)$ is a finite set.

Lemma 11 *Put* $N_k = \text{lcm}(1, 2, \cdots, k) \prod_{p \leq k} p$. *We can uniquely decompose* n *as* $n = n_k c_k$ *with* $(c_k, N_k) = 1$ *and* n_k *and* c_k *natural numbers. Let* $\epsilon = \pm 1$. *There exist functions* A_1 *and* B_1 *with as domain the divisors of* N_k *such that*

$$a_n^\epsilon(k) = \begin{cases} A_1(n_k)\mu(c_k)^2 + \epsilon B_1(n_k)\mu(c_k) & \text{if } n_k|N_k; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The assertion regarding the uniqueness of the decomposition $n = n_k c_k$ is trivial. For a given n only those partitions $k_1, k_2, \ldots, k_s$ contribute to (10) for which n/k_i is an integer for $1 \leq i \leq s$. Note that $k_i|n_k$. Thus, we can write

$$\mu(\frac{n}{k_1})^{e_1} \cdots \mu(\frac{n}{k_s})^{e_s} = \mu(\frac{n_k}{k_1})^{e_1} \cdots \mu(\frac{n_k}{k_s})^{e_s} \mu(c_k)^{e_1 + \ldots + e_s}.$$

If $n_k \nmid N_k$, then none of the integers $n_k/k_1, ..., n_k/k_s$ is squarefree and $a_n^\epsilon(k) = 0$, so assume that $n_k|N_k$. On using that $\mu(r)^w$ with $w \geq 1$ either equals $\mu(r)$ or $\mu(r)^2$, it follows from (10) that $a_n^1(k) = A_1(n_k)\mu(c_k)^2 + B_1(n_k)\mu(c_k)$.

Note that $\alpha_n(k) = A_1(n_k)\mu(c_k)^2$ and that $\beta_n(k) = B_1(n_k)\mu(c_k)$. Thus $a_n^{-1}(k) = \alpha_n(k) - \beta_n(k) = A_1(n_k)\mu(c_k)^2 - B_1(n_k)\mu(c_k)$. □

Using formula (5) it is seen that in the latter lemma N_k can be replaced by $k \prod_{p \leq k} p$.

Lemma 12 *Lemma* 11 *holds true also with* N_k *replaced by* $M_k = k \prod_{p \leq k} p$.

The above lemma again shows that $\mathcal{A}(k)$ is a finite set, thus if we fix k, there are only finitely many possibilities for the values of the coefficient of x^k in a cyclotomic polynomial.

The latter lemma in combination with Lemma 9 yields the following result.

Lemma 13 *Let* $M_k = k \prod_{p \leq k} p$. *Then* $\{0, a_d^1(k), a_d^{-1}(k) \mid d|M_k\} = \mathcal{A}(k)$.

We have $|a_n(k)| \leq \max_{n \geq 1} |a_n(k)| = A(k)$. See Table 1 for the values of $A(k)$ for $1 \leq k \leq 30$. Define $\mathcal{A}^\epsilon(k) = \{a_d^\epsilon(k) \mid d|M_k\}$. Numerics show that mostly $\mathcal{A}^\epsilon(k) = \mathcal{A}(k)$. If $\mathcal{A}^\epsilon(k)$ is strictly included in $\mathcal{A}(k)$ (which happens for example for $k = 48, 54$, $\epsilon = \pm 1$ and 66, $\epsilon = -1$), then for the k for which we did the computation ($k \leq 73$), the set $\{a_d^\epsilon(k) \mid d|M_k\}$ equals $\mathcal{A}(k)$ with one element omitted and this element is either $A(k) - 1$ or $-A(k) + 1$.

We next show that the inclusion in Lemma 8 is strict for $k \geq 13$. Our proof rests on the following rather elementary result on prime numbers.

Lemma 14 *For $k \geq 13$ there are consecutive odd primes $p_1 < p_2 < p_3$ such that $p_3 \leq k < p_1 + p_2$.*

Proof. Breusch [3] proved that for $x \geq 48$ there is at least one prime in $[x, 9x/8]$ (this strengthens *Bertrand's Postulate* asserting that there is always a prime between x and $2x$, provided $x \geq 2$). Let $\alpha = 1.32$. A little computation shows that the above result implies that for $x \geq 9$ there is at least one prime in $[x, \alpha x]$. One checks that the assertion is true for $k \in [13, 21)$. Assume that $k \geq 21$ ($\geq 9\alpha^3$). Let p_3 be the largest prime not exceeding k and let p_1 and p_2 be primes such that p_1, p_2 and p_3 are consecutive primes. Then $p_3 \geq k/\alpha$, $p_2 \geq k/\alpha^2$ and $p_1 \geq k/\alpha^3$. On noting that $p_1 + p_2 \geq k(1/\alpha + 1/\alpha^2) > k$, the proof is then completed. □

Lemma 15 *For $k \geq 13$ we have $\{-2, -1, 0, 1\} \in \mathcal{A}(k)$ (and thus $A(k) \geq 2$).*

Proof. Let p_1, p_2 and p_3 be odd primes satisfying the condition of Lemma 14. Using (2) we infer that

$$\Phi_{p_1p_2p_3}(x) \equiv \frac{(1-x^{p_3})}{(1-x)}(1-x^{p_1})(1-x^{p_2}) \equiv (1+x+\cdots+x^{p_3-1})(1-x^{p_1}-x^{p_2}),$$

where we computed modulo x^{k+1}. It follows that $a_{p_1p_2p_3}(k) = -2$. □

The following result shows that if k is odd, then $\mathcal{A}(k)$ is symmetric, that is if $v \in \mathcal{A}(k)$, then also $-v \in \mathcal{A}(k)$.

Lemma 16 *If k is odd, then $\mathcal{A}(k) = -\mathcal{A}(k)$, that is $\mathcal{A}(k)$ is symmetric.*

Proof. Assume that $v \in \mathcal{A}(k)$. If $v = 0$ there is nothing to prove, so assume that $v \neq 0$. Since M_k is odd, it follows by Lemma 13 that $a_d^\epsilon(k) = v$ for some odd integer d and $\epsilon \in \{-1, 1\}$. Then, by (6), $a_{2d}^\epsilon(k) = (-1)^k a_d^\epsilon(k) = -v$. On invoking Lemma 10, the proof is then completed. □

4.1 Numerical evaluation of $a_n(k)$ and $c_n(k)$ for small k

For our purposes it is relevant to be able to numerically evaluate $a_n(k)$ for small k and large n. A computer package like Maple evaluates $a_n(k)$ by evaluating the whole polynomial $\Phi_n(x)$. For large n this is far too costly. Instead it is more efficient to use (2) and expand for every d for which $\mu(n/d) \neq 0$, $(1-x^d)^{\mu(n/d)}$ as a Taylor series up to $O(x^{k+1})$ and multiply all these series together.

The most efficient method to date to compute $a_n(k)$ for small k is due to Grytczuk and Tropak [8]. First they apply formula (5). Thus it is enough to compute $a_n(k)$ with n squarefree. If $\phi(n) < k$, then $a_n(k) = 0$, so we may assume that $\phi(n) \geq k$. Let $d = (n, \prod_{p \leq k} p)$. Put $T_r = \mu(n)\mu((r,d))\varphi((r,d))$. Compute $b_0, \ldots, b_k$ recursively by $b_0 = 1$ and

$$b_j = -\frac{1}{j}\sum_{m=0}^{j-1} b_m T_{j-m} \text{ for } 1 \leq j \leq k.$$

Then $b_k = a_n(k)$. Their proof uses the formula

$$a_n(k) = -\frac{1}{k}\sum_{m=0}^{k-1} a_n(m) r_n(k-m) \text{ for } k \geq 1, \tag{17}$$

which follows by Viète's and Newton's formulae from (1) and it uses the second formula of Lemma 3. However, an alternative proof of (17) is obtained on using the following observation together with the first formula of Lemma 3.

Lemma 17 *Suppose that, as formal power series,*

$$\prod_{d=1}^{\infty}(1-x^d)^{-a_d} = \sum_{d=0}^{\infty} r(d)x^d,$$

then $dr(d) = \sum_{j=1}^{d} r(d-j)\sum_{k|j} ka_k$.

Proof. Taking the logarithmic derivative of $\prod_{d=1}^{\infty}(1-x^d)^{-a_d}$ we obtain

$$\frac{\sum_{d=1}^{\infty} dr(d)x^d}{\sum_{d=0}^{\infty} r(d)x^d} = x\frac{d}{dx}\log\prod_{d=1}^{\infty}(1-x^d)^{-a_d} = \sum_{j=1}^{\infty}(\sum_{k|j} ka_k)x^j,$$

whence the result follows. □

The following result generalizes the Grytczuk-Tropak algorithm to the efficient computation of $a_n^{\epsilon}(k)$ for small k and large n.

Lemma 18 *Let n be squarefree and put $d = (n, \prod_{p\leq k} p)$. Furthermore, we put $T_r = \mu(n)\mu((r,d))\varphi((r,d))$. Compute $b_0, \ldots, b_k$ recursively by $b_0 = 1$ and*

$$b_j = -\frac{\epsilon}{j}\sum_{m=0}^{j-1} b_m T_{j-m} \text{ for } 1 \leq j \leq k.$$

Then $a_n^{\epsilon}(k) = b_k$.

Proof. For $k = 0$ we have $1 = b_0 = a_n^{\epsilon}(k)$ and so we may assume that $k \geq 1$. Apply Lemma 17 with $a_d = -\epsilon\mu(\frac{n}{d})$ (thus $r(d) = a_n^{\epsilon}(d)$). We obtain by part 1 of Lemma 3 that

$$a_n^{\epsilon}(k) = -\frac{\epsilon}{k}\sum_{m=0}^{k-1} a_n^{\epsilon}(m) r_n(k-m) \text{ for } k \geq 1, \tag{18}$$

The proof now follows if we show that $r_n(r) = \mu(n)\mu((r,d))\varphi((r,d))$ for $1 \leq r \leq k$. Since by assumption n is squarefree, part 2 of Lemma 3 implies that $r_n(r) = \mu(n)\mu((n,r))\varphi((n,r))$. In case $1 \leq r \leq k$ this can be rewritten as

$$\begin{aligned} r_n(r) &= \mu(n)\mu((n,r,\prod_{p\leq k} p)\varphi((n,r,\prod_{p\leq k} p)) \\ &= \mu(n)\mu((d,r))\varphi((d,r)). \end{aligned}$$

Thus the proof is completed. □

`Algortihm to compute` $a_n^\epsilon(k)$. If n is not squarefree, then apply (5). Thus we may assume that n is squarefree.
The case $\epsilon = 1$. If $n > \varphi(k)$, then $a_n^1(k) = 0$, otherwise compute $a_n^1(k)$ using Lemma 18.
The case $\epsilon = -1$. Let $0 \le k_1 < n$ be such that $k_1 \equiv k (\bmod n)$. Then $a_n^{-1}(k) = a_n^{-1}(k_1)$. If $k_1 > n - \varphi(n)$, then $a_n^{-1}(k) = 0$, otherwise compute $a_n^{-1}(k)$ using Lemma 18.

In case $n > 1$, $a_n^1(k) = a_n(k)$ and the above algorithm is the Grytczuk-Tropak algorithm.

For every integer v it is a consequence of Theorem 2 that there exists a minimal integer k, $k_{\min}^\epsilon$, such that there exists a natural number n with $a_n(k_{\min}^\epsilon) = v$. Since $\mathcal{A}(k) = \mathcal{C}(k)$ it follows that $k_{\min}^1 = k_{\min}^{-1}$. Put $k_{\min} = k_{\min}^{-1} = k_{\min}^1$. Grytczuk and Tropak [8, Table 2.1] used their method to determine $k_{\min}$ for the integers in the interval $[-9, \ldots, 10]$. Bosma [2] extended this to the range $[-50, 50]$ and we on our turn have extended the range from $[-70, \ldots, 70]$ (in which case $k_{\min} \le 105$).

5 Computation of $M(a_n(k))$ and $M(c_n(k))$

Our starting point is Lemma 7. For each of the $p(k)$ summands we compute the average (in Lemma 19), which turns out to be independent of ϵ, and then find $M(a_n^\epsilon(k))$ by adding these $p(k)$ averages. Note that of course $M(a_n(k)) = M(a_n^1(k))$ and $M(c_n(k)) = M(a_n^{-1}(k))$.

Lemma 19 *Let $\lambda = (k_1^{n_{k_1}} \ldots k_s^{n_{k_s}})$ be a partition with $s \ge 1$, $k_1, \ldots, k_s$ distinct integers and $n_{k_1} \ge n_{k_2} \ge \ldots \ge n_{k_s} \ge 1$. If $n_{k_1} \ge 2$ we let t be the largest integer $\le s$ for which $n_{k_t} \ge 2$, otherwise we let $t = 0$. Let $L = [k_1, \ldots, k_s]$, $G = (k_1, \ldots, k_s)$ and $\epsilon = \pm 1$. We have*

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n \le x} (-1)^{n_{k_1}+\cdots+n_{k_s}} \binom{\epsilon\mu(\frac{n}{k_1})}{n_{k_1}} \cdots \binom{\epsilon\mu(\frac{n}{k_s})}{n_{k_s}} = \frac{6}{\pi^2} \frac{\epsilon_2(\lambda))}{G \prod_{p | \frac{L}{G}} (p+1)},$$

where

$$\epsilon_2(\lambda) = \epsilon_1(\lambda) \mu(\frac{L}{k_{t+1}}) \cdots \mu(\frac{L}{k_s}), \tag{19}$$

and

$$\epsilon_1(\lambda) = \begin{cases} 1 & \text{if } n_{k_1} = 1, \ s \text{ is even and } \mu(L/G) \neq 0; \\ \mu(\frac{L}{k_1})^{s-t}/2 & \text{if } n_{k_1} \ge 2 \text{ and } \mu(L/k_1) = \cdots = \mu(L/k_t) \text{ and } \mu(L/G) \neq 0; \\ 0 & \text{otherwise.} \end{cases}$$

Remark. A case by case analysis from Lemma 19 on using (16) shows that

$$2\epsilon_2(\lambda) = \prod_{j=1}^{s} (-1)^{n_{k_j}} \binom{\mu(\frac{L}{k_j})}{n_{k_j}} + \prod_{j=1}^{s} (-1)^{n_{k_j}} \binom{-\mu(\frac{L}{k_j})}{n_{k_j}}.$$

Proof of Lemma 19. Put

$$S(x)=\sum_{n\le x}(-1)^{n_{k_1}+\cdots+n_{k_s}}\binom{\epsilon\mu(\frac{n}{k_1})}{n_{k_1}}\cdots\binom{\epsilon\mu(\frac{n}{k_s})}{n_{k_s}}.$$

By (16) for $n_{k_i}\ge 2$ the binomial coefficient $\binom{\epsilon\mu(n/k_i)}{n_{k_i}}$ is only non-zero if $\mu(\frac{n}{k_i})=-\epsilon$. Using (16) it follows that a necessary condition for the argument of $S(x)$ to be non-zero is that $L|n$. Now write $n=mL$. Note that $\mu(mL/k_1)\cdots\mu(mL/k_s)=\mu(m)^s\mu(L/k_1)\cdots\mu(L/k_s)$ if $(m,L/k_j)=1$ for $1\le j\le s$ and equals zero otherwise. It is not difficult to show that $[\frac{L}{k_1},\ldots,\frac{L}{k_s}]=\frac{L}{G}$ and using this, that $\mu(L/k_i)\neq 0$ for $1\le i\le s$ iff L/G is squarefree. It follows that if $\mu(L/G)=0$, then $S(x)=0$ and we are done, so next assume that $\mu(L/G)\neq 0$. We infer that

$$S(x)=\sum_{m\le x/L,\ (m,L/G)=1}(-1)^{n_{k_1}+\cdots+n_{k_s}}\binom{\epsilon\mu(mL/k_1)}{n_{k_1}}\cdots\binom{\epsilon\mu(mL/k_s)}{n_{k_s}}.$$

Let us first consider the (easy) case where $n_{k_1}=1$. Then we obtain $S(x)=(-1)^s\mu(\frac{L}{k_1})\cdots\mu(\frac{L}{k_s})\sum_{m\le x/L,\ (m,L/G)=1}(\epsilon\mu(m))^s$. If s is odd, then by Lemma 5 it follows that $\lim_{x\to\infty}S(x)/x=0$ and we are done, so next assume that s is even. Then we apply Lemma 4 and obtain that

$$\lim_{x\to\infty}\frac{S(x)}{x}=\frac{6\mu(\frac{L}{k_1})\cdots\mu(\frac{L}{k_s})}{\pi^2L\prod_{p|\frac{L}{G}}(1+\frac{1}{p})}.$$

The assumption $\mu(L/G)\neq 0$ implies that $L\prod_{p|L/G}(1+1/p)=G\prod_{p|L/G}(p+1)$. Next we consider the case where $n_{k_1}\ge 2$. The corresponding binomial coefficient is only non-zero if $\mu(mL/k_1)=-\epsilon$. Similarly, we must have $\mu(mL/k_j)=-\epsilon$ for $1\le j\le t$. It follows that if it is not true that $\mu(L/k_1)=\ldots=\mu(L/k_t)$, then $S(x)=0$ and hence $\lim_{x\to\infty}S(x)/x=0$ as asserted, so assume that $\mu(L/k_1)=\ldots=\mu(L/k_t)$. We have, on noting that $\mu(mL/k_1)=-\epsilon$ and $(m,L/G)=1$ implies $-\epsilon\mu(m)=\mu(L/k_1)$,

$$\begin{aligned}S(x)&=\sum_{\substack{m\le x/L,\ (m,L/G)=1\\ \mu(mL/k_1)=-\epsilon}}(-\epsilon\mu(m))^{s-t}\mu(\frac{L}{k_{t+1}})\cdots\mu(\frac{L}{k_s})\\ &=(\mu(\frac{L}{k_1}))^{s-t}\mu(\frac{L}{k_{t+1}})\cdots\mu(\frac{L}{k_s})\sum_{\substack{m\le x/L,\ (m,L/G)=1\\ \mu(m)=-\epsilon\mu(L/k_1)}}1.\end{aligned}$$

On invoking Lemma 6 the proof is then completed. □

Theorem 3 *Let $\epsilon=\pm1$. We have*

$$M(a_n^{\epsilon}(k))=\frac{6}{\pi^2}\sum_{\substack{\lambda=(k_1^{n_{k_1}}\ldots k_s^{n_{k_s}})\in\mathcal{P}(k)\\ n_{k_1}\ge\ldots\ge n_{k_s}\ge 1}}\frac{\epsilon_2(\lambda)}{G(\lambda)\prod_{p|\frac{L(\lambda)}{G(\lambda)}}(p+1)},$$

where $\epsilon_2(\lambda)$ is defined in (19), *$L(\lambda)=[k_1,\ldots,k_s]$ and $G(\lambda)=(k_1,\ldots,k_s)$. Moreover, we have $M(\alpha_n(k))=M(a_n(k))=M(c_n(k))$ and $M(\beta_n(k))=0$.*

Proof. The first assertion follows from Lemma 7 together with Lemma 19. It is easy to see that $M(\alpha_n(k))$ and $M(\beta_n(k))$ exist. From

$$M(\alpha_n(k)) + M(\beta_n(k)) = M(a_n(k)) = M(c_n(k)) = M(\alpha_n(k)) - M(\beta_n(k)),$$

the second assertion then follows. □

In Table 3 we demonstrate Theorem 3 in case $k = 4$.

Table 3: Computation of $e_4 = \zeta(2)M(a_n(4))$

partition	λ	n_{k_1}	$L(\lambda)$	$G(\lambda)$	t	s	$\epsilon_2(\lambda)$	contribution to e_4
4	(4^1)	1	4	4	0	1	0	0
$3,1$	$(3^1 1^1)$	1	3	1	0	2	-1	$-1/4$
$2,2$	(2^2)	2	2	2	1	1	$1/2$	$+1/4$
$2,1,1$	$(1^2 2^1)$	2	2	1	1	2	$-1/2$	$-1/6$
$1,1,1,1$	(1^4)	4	1	1	1	1	$1/2$	$+1/2$

It is seen that $e_4 = \zeta(2)M(a_n(4)) = 0 - \frac{1}{4} + \frac{1}{4} - \frac{1}{6} + \frac{1}{2} = \frac{1}{3}$.

The above formula suggests a connection with the group or representation theory of the symmetric group S_k. The conjugacy classes in S_k are in 1-1 correspondence with the partitions of k. If $\lambda = (k_1^{n_{k_1}} \dots k_s^{n_{k_s}})$, then the order of every element in the corresponding conjugacy class equals $L(\lambda)$. In particular, $L(\lambda) \le g(k)$, where $g(k)$ denotes the maximum of all orders of elements in S_k. It was shown by E. Landau in 1903 that $\log g(k) \sim \sqrt{k \log k}$ as k tends to infinity (for a nice account of this see [13], for recent results see [5]), whereas by Stirling's theorem $\log k! \sim k \log k$. The average order of an element in S_k is, not surprisingly, much smaller than $g(k)$: if σ is chosen at random from S_k, define $Z = (\log \mathrm{order}(\sigma) - \frac{1}{2}\log^2 n)/(\log^{3/2} n/\sqrt{3})$, then the distribution of Z is known (see e.g. Nicolas [18]) to converge to the standard normal distribution as $n \to \infty$.

6 Average and value distribution

We give, using Lemma 12, a simpler formula for $M(a_n(k))$ involving $a_n(k)$ for a finite set of n.

Theorem 4 *Let $k \ge 1$ be fixed and $\epsilon = \pm 1$. Put $M_k = k \prod_{p \le k} p$. Then*

$$M(a_n^{\epsilon}(k)) = \frac{3}{\pi^2 \prod_{p \le k}(1 + \frac{1}{p})} \sum_{d | M_k} \frac{a_d^1(k) + a_d^{-1}(k)}{d}.$$

Furthermore, when $v \ne 0$,

$$\delta(a_n^{\epsilon}(k) = v) = \frac{3}{\pi^2 \prod_{p \le k}(1 + \frac{1}{p})} \Big(\sum_{\substack{d | M_k \\ a_d^1(k) = v}} \frac{1}{d} + \sum_{\substack{d | M_k \\ a_d^{-1}(k) = v}} \frac{1}{d} \Big).$$

Proof. Let $r_1 = \prod_{p\le k} p$. We have

$$\begin{aligned}\sum_{n\le x} a_n^\epsilon(k) &= \sum_{d|M_k} \sum_{\substack{dm\le x\\(m,r_1)=1}} (A_1(d)\mu(m)^2 + \epsilon B_1(d)\mu(m))\\ &= \sum_{d|M_k} A_1(d) \sum_{\substack{m\le x/d\\(m,r_1)=1}} \mu(m)^2 + o_k(x),\end{aligned}$$

where we used Lemma 12 and Lemma 5. On invoking Lemma 4 we then obtain that

$$\sum_{n\le x} a_n^\epsilon(k) = \frac{6x}{\pi^2\prod_{p\le k}(1+\frac{1}{p})}\sum_{d|M_k}\frac{A_1(d)}{d} + o_k(x).$$

On noting that $A_1(d) = (a_d^1(k) + a_d^{-1}(k))/2$ (and $B_1(d) = (a_d^1(k) - a_d^{-1}(k))/2$, but this is not needed), the first formula follows.

As to the second identity we notice that, for $v \neq 0$, by Lemma 12

$$\sum_{n\le x,\ a_n^\epsilon(k)=v} 1 = \sum_{\substack{d|M_k\\ a_d^1(k)=v}} \sum_{\substack{md\le x,\ \mu(m)=\epsilon\\(m,r_1)=1}} 1 + \sum_{\substack{d|M_k\\ a_d^{-1}(k)=v}} \sum_{\substack{md\le x,\ \mu(m)=-\epsilon\\(m,r_1)=1}} 1.$$

On invoking Lemma 6, the proof is then completed. □

Using identity (6) we arrive at the following corollary to this theorem:

Corollary 1 *Let $k \ge 3$ be fixed and odd and $M_k = k\prod_{p\le k} p$. Then*

$$M(a_n^\epsilon(k)) = \frac{1}{\pi^2\prod_{2<p\le k}(1+\frac{1}{p})}\sum_{d|M_k/2}\frac{a_d^1(k)+a_d^{-1}(k)}{d}.$$

Furthermore, when $v \neq 0$,

$$\delta(a_n^\epsilon(k)=v) = \frac{3}{\pi^2\prod_{p\le k}(1+\frac{1}{p})}\Big(\sum_{\substack{d|M_k/2\\ a_d^1(k)=v}}\frac{1}{d} + \sum_{\substack{d|M_k/2\\ a_d^1(k)=-v}}\frac{1}{2d} + \sum_{\substack{d|M_k/2\\ a_d^{-1}(k)=v}}\frac{1}{d} + \sum_{\substack{d|M_k/2\\ a_d^{-1}(k)=-v}}\frac{1}{2d}\Big).$$

This result gives an alternative proof of the fact that, with $k \ge 3$ and odd, $\mathcal{A}(k)$ is symmetric. Namely, it shows that for these k we have $\delta(a_n^1(k) = v) > 0$ iff $\delta(a_n^1(k) = -v) > 0$.

In case k is prime, the divisor sum in the previous corollary can be further reduced.

Lemma 20 *Let $k \ge 3$ be a fixed prime. Put $R_k = \prod_{2<p<k} p$. Then*

$$M(a_n^\epsilon(k)) = \frac{1}{\pi^2\prod_{2<p<k}(1+\frac{1}{p})}\sum_{d|R_k}\frac{a_d^1(k)+a_d^{-1}(k)}{d}.$$

Proof. We consider the formula given in the previous corollary. The divisors of $M_k/2$ are either of the form d with $d|kR_k$ or of the form dk^2 with $d|R_k$. For the latter divisors d we find, using (5) that $a_{dk^2}^\epsilon(k) = a_{dk}^\epsilon(1) = -\epsilon\mu(dk)$ and hence

$\sum_{d|M_k/2}(a_d^1(k)+a_d^{-1}(k))/d = \sum_{d|kR_k}(a_d^1(k)+a_d^{-1}(k))/d$. Now suppose that $d|R_k$. Using Lemma 7 we infer that

$$a_{dk}^1(k)+a_{dk}^{-1}(k) = a_d^1(k)+a_d^{-1}(k)-\mu(d)+\mu(d) = a_d^1(k)+a_d^{-1}(k).$$

Using this observation it follows that

$$\sum_{d|kR_k}\frac{a_d^1(k)+a_d^{-1}(k)}{d} = (1+\frac{1}{k})\sum_{d|R_k}\frac{a_d^1(k)+a_d^{-1}(k)}{d},$$

whence the result follows. □

Lemma 21
a) *We have* $e_k 2k\prod_{p\le k}(p+1)\in\mathbb{Z}$.
b) *If* $k\ge 3$ *is a prime, then* $e_k 2\prod_{p<k}(p+1)\in\mathbb{Z}$.

Proof. a) An easy consequence of Theorem 4.
b) An easy consequence of Lemma 20. (It also follows from Theorem 5 below on noting that $\epsilon_2(\lambda)=0$ in case $k|L(\lambda)$.) □

Numerically we observed that actually for $k\le 100$, we have $e_k k\prod_{p\le k}(p+1)\in\mathbb{Z}$. Using Lemma 21 it is seen that $e_k k\prod_{p\le k}(p+1)\in\mathbb{Z}$ if k is an odd prime.

Clearly $\delta(a_n(k)=0) = 1-\sum_{v\ne 0}\delta(a_n(k)=v)$, where the latter sum has only finitely many non-zero values and it is a finite computation to determine those v for which $a_n(k)=v$ for some n. For $1\le k\le 16$ the non-zero values of $\zeta(2)\delta(a_n(k)=v)$ are given in Table 4 (except for $v=0$). By Table 4E we denote the extended version of Table 4, which is Table 11 in [14].

Table 4: Value of $\zeta(2)\delta(a_n(k)=v)$
(**Table 4E**: Extended version of this table, see [16, Table 11])

	$v=-2$	$v=-1$	$v=1$	$v=2$
$k=1$	0	1/2	1/2	0
$k=2$	0	1/12	7/12	0
$k=3$	0	5/24	3/8	0
$k=4$	0	1/6	1/2	0
$k=5$	0	13/80	23/80	0
$k=6$	0	25/144	67/144	0
$k=7$	1/576	577/2688	731/2688	1/1152
$k=8$	0	1/8	5/12	0
$k=9$	0	65/384	347/1152	0
$k=10$	0	161/960	347/960	0
$k=11$	1/2304	8299/50688	11489/50688	1/4608
$k=12$	0	349/2304	1009/2304	0
$k=13$	43/48384	219269/1257984	277171/1257984	43/96768
$k=14$	13/21504	2395/21504	2319/7168	1/2304
$k=15$	13/32256	1345/7168	97247/322560	13/64512
$k=16$	5/21504	12149/64512	1127/3072	5/2688

Let us now look at Theorem 3 and Theorem 4 from the viewpoint of computational complexity. In Theorem 3 the sum has $p(k)$ terms and the estimate (13) yields that $\log p(k) \sim \pi\sqrt{2k/3}$ as k tends to infinity. In Theorem 4 we sum over $t(k)$ terms where $\log t(k) \sim \pi(k)\log 2 \sim k\log 2/\log k$. So Theorem 3 yields the computational superior method. Theorem 4 is, however, much more easily implemented. Using Lemma 7 in combination with Theorem 4 and Lemma 22 below, an alternative proof of Theorem 3 is obtained. If one starts with Lemma 20 and invokes Lemmas 7 and 22, one obtains a sum over partitions of k, where now only odd integers are allowed to occur in the partition. This yields a result superior in complexity to that provided by Theorem 3, since for $p_{\rm odd}(k)$ the number of partitions of m into odd parts we have $\log p_{\rm odd}(k) \sim \pi\sqrt{k/3}$ (see e.g. Bringmann [4]), whereas $\log p(k) \sim \pi\sqrt{2k/3}$.

Theorem 5 *Let $\epsilon = \pm 1$ and $k \geq 3$ be a prime. With the notation from Theorem 3 we have*

$$M(a_n^{\epsilon}(k)) = \frac{2}{\pi^2} \sum_{\substack{\lambda=(k_1^{n_{k_1}}\ldots k_s^{n_{k_s}})\in\mathcal{P}(k) \\ n_{k_1}\geq\ldots n_{k_s}\geq 1}} \frac{\epsilon_2(\lambda)}{\prod_{p|L(\lambda)}(p+1)},$$

where the sum is over all partitions of k into only odd parts having at least one number repeated more than once (i.e. $n_{k_1} \geq 2$).

The restriction that $n_{k_1} \geq 2$ does not yield an extra asymptotical improvement: using a result of Hagis [9], one sees that with $p_1(k)$ the number of partitions of k into only odd parts having at least one number repeated more than once (i.e. $n_{k_1} \geq 2$), we have $p_1(k) \sim p_{\rm odd}(k)$.

Indeed, all results involving $\sum_{d|r} a_d^{\epsilon}(k)/d$ can be turned into a Möller type of result involving partitions of k on invoking the following lemma and Lemma 7.

Lemma 22 *Let $L = [k_1, \ldots, k_s]$ and $G = (k_1, \ldots, k_s)$. The sum*

$$\sum_{d|r} \frac{(-1)^{n_{k_1}+\cdots+n_{k_s}}}{d} \binom{\epsilon\mu(\frac{d}{k_1})}{n_{k_1}} \cdots \binom{\epsilon\mu(\frac{d}{k_s})}{n_{k_s}}$$

equals

$$\frac{(-\epsilon)^s}{L}\mu(\frac{L}{k_{t+1}})\cdots\mu(\frac{L}{k_s}) \prod_{p|\frac{r}{L},\ p\nmid\frac{L}{G}} \Big(1 + \frac{(-1)^s}{p}\Big)$$

if $n_{k_1} = 1$, $\mu(L/G) \neq 0$ and $r|L$, it equals

$$\frac{1}{2L}\mu(\frac{L}{k_1})^{s-t}\mu(\frac{L}{k_{t+1}})\cdots\mu(\frac{L}{k_s})\Big[\prod_{p|\frac{r}{L},\ p\nmid\frac{L}{G}}(1+\frac{1}{p}) - \epsilon\mu(\frac{L}{k_1})\prod_{p|\frac{r}{L},\ p\nmid\frac{L}{G}}(1-\frac{1}{p})\Big]$$

if $n_{k_1} \geq 2$, $\mu(L/k_1) = \cdots = \mu(L/k_t) \neq 0$ and $r|L$, and zero in all other cases.

Corollary 2 *We have*

$$\frac{1}{2}\sum_{d|r} \frac{(-1)^{n_{k_1}+\cdots+n_{k_s}}}{d} \Big(\binom{-\mu(\frac{d}{k_1})}{n_{k_1}} \cdots \binom{-\mu(\frac{d}{k_s})}{n_{k_s}} + \binom{\mu(\frac{d}{k_1})}{n_{k_1}} \cdots \binom{\mu(\frac{d}{k_s})}{n_{k_s}} \Big)$$

$$= \begin{cases} \frac{\epsilon_2(\lambda)}{L(\lambda)} \prod_{p | \frac{r}{L(\lambda)},\ p \nmid \frac{L(\lambda)}{G(\lambda)}} (1+\frac{1}{p}) & \text{if } L(\lambda)|r; \\ 0 & \text{otherwise.} \end{cases}$$

In particular, if r is squarefree, then the sum equals

$$= \begin{cases} \epsilon_2(\lambda) \prod_{p|r}(1+\frac{1}{p}) \prod_{p|L(\lambda)}(p+1)^{-1} & \text{if } L(\lambda)|r; \\ 0 & \text{otherwise.} \end{cases}$$

Proof of Lemma 22. The proof is similar to that of Lemma 19. Indeed, if $S(x)$ now denotes the new sum under consideration, then the proof proceeds as that of Lemma 19. Instead of the sum

$$\sum_{m \le x/L,\ (a,L/G)=1} (\epsilon\mu(m))^s \quad \text{we have } \frac{1}{L} \sum_{mL|r,\ (m,L/G)=1} \frac{(\epsilon\mu(m))^s}{m}.$$

Instead of the sum

$$\sum_{\substack{m \le x/L,\ (m,L/G)=1 \\ \mu(m)=-\epsilon\mu(L/k_1)}} 1 \quad \text{we have } \frac{1}{L} \sum_{\substack{mL|r,\ (m,L/G)=1 \\ \mu(m)=-\epsilon\mu(L/k_1)}} \frac{1}{m}.$$

On relating these sums to Euler products, the result follows. □

Alternative proof of Theorem 3. On combining Theorem 4, Lemma 7 and Corollary 2 (with $r = M_k$), we find that

$$M(a_n^\epsilon(k)) = \frac{6}{\pi^2 \prod_{p \le k}(1+\frac{1}{p})} \sum_{\substack{\lambda=(k_1^{n_{k_1}} \ldots k_s^{n_{k_s}}) \in \mathcal{P}(k) \\ L(\lambda)|M_k}} \frac{\epsilon_2(\lambda)}{L(\lambda)} \prod_{p | \frac{M_k}{L(\lambda)},\ p \nmid \frac{L(\lambda)}{G(\lambda)}} (1+\frac{1}{p}).$$

Now

$$\prod_{p | \frac{M_k}{L(\lambda)},\ p \nmid \frac{L(\lambda)}{G(\lambda)}} (1+\frac{1}{p}) = \frac{\prod_{p | \frac{M_k}{G(\lambda)}} (1+\frac{1}{p})}{\prod_{p | \frac{L(\lambda)}{G(\lambda)}} (1+\frac{1}{p})} = \frac{\prod_{p \le k}(1+\frac{1}{p})}{\prod_{p | \frac{L(\lambda)}{G(\lambda)}} (1+\frac{1}{p})},$$

where we used that

$$\prod_{p|u,\ p \nmid v} (1+\frac{1}{p}) = \prod_{p|uv}(1+\frac{1}{p}) \prod_{p|v}(1+\frac{1}{p})^{-1}$$

and $G(\lambda)|k$. It thus follows that

$$M(a_n^\epsilon(k)) = \frac{6}{\pi^2} \sum_{\substack{\lambda=(k_1^{n_{k_1}} \ldots k_s^{n_{k_s}}) \in \mathcal{P}(k) \\ L(\lambda)|M_k}} \frac{\epsilon_2(\lambda)}{L(\lambda)} \prod_{p | \frac{L(\lambda)}{G(\lambda)}} (1+\frac{1}{p})^{-1}. \tag{20}$$

If $\epsilon_2(\lambda) \ne 0$, then $L(\lambda)/G(\lambda)$ is squarefree and hence

$$G(\lambda) \frac{L(\lambda)}{G(\lambda)} \prod_{p | \frac{L(\lambda)}{G(\lambda)}} (1+\frac{1}{p}) = G(\lambda) \prod_{p | \frac{L(\lambda)}{G(\lambda)}} (p+1)$$

and furthermore $L(\lambda)|M_k$. These two observations in combination with (20) complete the alternative proof of Theorem 3. $\square$

Proof of Theorem 5. On combining Lemma 20, Lemma 7 and Corollary 2 (with $r = R_k$ a squarefree number), we find that

$$M(a_n^{\epsilon}(k)) = \frac{2}{\pi^2 \prod_{2<p<k}(1+\frac{1}{p})} \sum_{\substack{\lambda=(k_1^{n_{k_1}}\ldots k_s^{n_{k_s}})\in\mathcal{P}(k)\\ L(\lambda)|R_k}} \epsilon_2(\lambda)\frac{\prod_{p|R_k}(1+\frac{1}{p})}{\prod_{p|L(\lambda)}(p+1)};$$
$$= \frac{2}{\pi^2} \sum_{\substack{\lambda=(k_1^{n_{k_1}}\ldots k_s^{n_{k_s}})\in\mathcal{P}(k)\\ L(\lambda)|R_k}} \frac{\epsilon_2(\lambda)}{\prod_{p|L(\lambda)}(p+1)}.$$

Since $G(\lambda)|k$ and by assumption k is an odd prime, either $G(\lambda) = 1$ or $G(\lambda) = k$. The latter case only occurs if $\lambda = (k^1)$ in which case $\epsilon_2(\lambda) = 0$, hence we may assume that $G(\lambda) = 1$. Let us assume that $\epsilon_2(\lambda) \neq 0$ and so $\mu(L(\lambda)/G(\lambda)) \neq 0$ and so $L(\lambda)$ must be squarefree. Thus each part k_i of such a partition is squarefree and since $k_i \leq k$ it follows that $k_i|R_k$ iff k_i is odd. We infer that if $2 \nmid L(\lambda)$, then $L(\lambda)|R_k$. If $L(\lambda)$ is even, then $L(\lambda) \nmid R_k$. Thus the sum over all partitions with $L(\lambda)|R_k$, can be restricted to those partitions consisting of only odd parts. If $n_{k_1} = 1$, then the partition consists of distinct odd parts and so the number of parts s must be odd, as by assumption k is odd, and hence $\epsilon_2(\lambda) = 0$ in this case. Thus we can further resctrict our partition sum to the partitions into odd parts only having $n_{k_1} \geq 2$. $\square$

7 Some observations related to Table 4

In this section we make some observations regarding Table 4 (and Table 4E) and prove some results inspired by these observations.

For k is even numerical results suggest that often $\mathcal{A}(k)$ is not symmetric, whereas we have shown (Lemma 16) that for k is odd it is always symmetric. For $k \leq 100$ it is mostly true that if $v \in \mathcal{A}(k)$ and v is negative, then $-v \in \mathcal{A}(k)$. This leads to the question as to whether perhaps $A_+(k) > A_-(k)$ for all k sufficiently large, with $A_+(k) = \max \mathcal{A}(k)$ and $A_-(k) = -\min \mathcal{A}(k)$. It has been shown by Bachman [1] that (12) also holds true if we replace $A(k)$ by $A_+(k)$, or $A_-(k)$. However, this result is not strong enough to decide on the above question.

An other observation that can be made is that for $k \leq 100$ it is true that $\mathcal{A}(k)$ is convex, that is consists of consecutive integers, i.e. if $v_0 < v_1$ are in $\mathcal{A}(k)$, then so are all integers between v_0 and v_1.

Let us define $\mathcal{A}_j(k) = \{a_n(k) : n \equiv j (\text{mod } 2)\}$, for $0 \leq j \leq 1$.

Lemma 23
1) *We have* $0 \in \mathcal{A}_j(k)$.
2) *If* k *is even, then* $\mathcal{A}_1(k) \subseteq \mathcal{A}_0(k) = \mathcal{A}(k)$.
3) *If* k *is odd, then* $v \in \mathcal{A}_1(k)$ *iff* $-v \in \mathcal{A}_0(k)$.

Proof. 1) Consider any integer n_j such that $n_j/\gamma(n_j) > k$ and $n_j \equiv j (\mathrm{mod}\ 2)$. Then, by part 1 of Lemma 2, we have $a_{n_j}(k) = 0$ and hence $0 \in \mathcal{A}_j(k)$.
2) If $v \in \mathcal{A}_1(k)$, then $v = a_d(k)$ for some odd integer d. Then, by (6) we have $a_{2d}(k) = (-1)^k a_d(k) = v$ and hence $v \in \mathcal{A}_0(k)$. We have $\mathcal{A}(k) = \mathcal{A}_0(k) \cup \mathcal{A}_1(k) = \mathcal{A}_0(k)$, since $\mathcal{A}_1(k)$ is included in $\mathcal{A}_0(k)$.
3) Proceding as in part 2 we infer that if $v \in \mathcal{A}_1(k)$, then $-v \in \mathcal{A}_0(k)$. For the reverse implication we make use of Lemma 13. □

Inspection of Table 4E shows that for odd integers k with $A(k) \geq 2$ often $\delta(a_n(k) = A(k))$ and $\delta(a_n(k) = -A(k))$ differ by a factor two or a factor less than two. Regarding this situation we have the following result:

Lemma 24 *Let $k \geq 3$ be odd and $v \neq 0$. We have*

$$\frac{1}{2}\delta(a_n^{\epsilon}(k) = v) \leq \delta(a_n^{\epsilon}(k) = -v) \leq 2\delta(a_n^{\epsilon}(k) = v).$$

Furthermore, we have

$$2\delta(a_n^{\epsilon}(k) = v) = \delta(a_n^{\epsilon}(k) = -v) \text{ iff } v \notin \mathcal{A}_1(k) \text{ (that is iff } -v \notin \mathcal{A}_0(k)).$$

Proof. We write $w_k = \frac{3}{\pi^2}\prod_{p\leq k}(1+\frac{1}{p})^{-1}$ and (with $\epsilon_1 = \pm 1$)

$$\alpha_{\epsilon_1} = \sum_{\substack{d|M_k/2 \\ a_d^1(k)=\epsilon_1 v}} \frac{1}{d} + \sum_{\substack{d|M_k/2 \\ a_d^{-1}(k)=\epsilon_1 v}} \frac{1}{d}.$$

Note that $\alpha_{\epsilon_1} \geq 0$. Then, by Corollary 1, we have

$$\delta(a_n^{\epsilon}(k) = v) = w_k(\alpha_1 + \alpha_{-1}/2) \text{ and } \delta(a_n^{\epsilon}(k) = -v) = w_k(\alpha_{-1} + \alpha_1/2).$$

The first part of the assertion follows on comparing these two formulae. As to the second assertion, the latter formulae imply that it is enough to prove that $\alpha_1 = 0$ iff $v \notin \mathcal{A}_1(k)$. A minor variation of the proof of Lemma 8 shows that $\{-1, 0, 1\} \subseteq \{a_n^1(k) | n > 1,\ 2 \nmid n\}$ and hence $\mathcal{A}_1(k) = \{a_n^1(k) : 2 \nmid n\}$. A minor modification of the proof of Lemma 13 shows that $\{a_n^1(k) : 2 \nmid n\} = \{0, a_d^1(k), a_d^{-1}(k) : d|M_k/2\}$. On noting that $M_k/2$ is odd, we infer that if $v \notin \mathcal{A}_1(k)$, then clearly $\alpha_1 = 0$. On the other hand, if $\alpha_1 = 0$, then v is not in $\{0, a_d^1(k), a_d^{-1}(k) : d|M_k/2\} = \mathcal{A}_1(k)$, completing the proof. □

Table 5: Set theoretic difference $\mathcal{A}(k)\backslash\mathcal{A}_0(k)$ in case $\mathcal{A}(k) \neq \mathcal{A}_0(k)$ and $k \leq 53$

$k = 7$	$\{-2\}$	$k = 11$	$\{-2\}$
$k = 13$	$\{-2\}$	$k = 15$	$\{-2\}$
$k = 17$	$\{-3\}$	$k = 19$	$\{-3\}$
$k = 21$	$\{-3\}$	$k = 23$	$\{-4, -3\}$
$k = 25$	$\{-3\}$	$k = 31$	$\{-4\}$
$k = 35$	$\{5\}$	$k = 37$	$\{5\}$
$k = 39$	$\{5, 6\}$	$k = 43$	$\{-7\}$
$k = 45$	$\{-7\}$	$k = 47$	$\{-9, -8\}$
$k = 51$	$\{8\}$	$k = 53$	$\{9, 10, 11, 12, 13\}$

Example. Inspection of Table 4 shows that $\delta(a_n(7) = -2) = 2\delta(a_n(7) = 2)$. It thus follows by Lemma 24 that there is no even integer n for which $a_n(7) = -2$ (whereas $a_{105}(7) = -2$). Further examples can be derived from Table 5.

For $k \le 100$ it turns out $\mathcal{A}(k)\backslash\mathcal{A}_0(k)$ is always *convex*, i.e. consists of consecutive integers. For part 2 of Lemma 24 to be of some mathematical value we would hope that infinitely often $\mathcal{A}_0(k)$ is strictly contained in $\mathcal{A}(k)$. Note that by Theorem 2 we have $\{\mathcal{A}_0(k) : k \ge 1\} = \{\mathcal{A}(k) : k \ge 1\} = \mathbb{Z}$.

Suppose that $\delta(a_n(k) = v) > 0$. Then the quotient

$$\frac{\delta(a_n(k) = -v)}{\delta(a_n(k) = v)}$$

does not exceed 2 in case k is odd by Lemma 24. Inspection of Table 4E suggests that given *any* real number r, we can find a $v > 0$ and even k such that the latter quotient exceeds r.

8 Some variations

Using the same methods we can easily determine e.g. $M(\mu(n)^2 a_n(k))$, i.e. the average of $a_n(k)$ over all squarefree integers n. We obtain the following result.

Theorem 6 *Let $k \ge 1$ be fixed and put $Q_k = \prod_{p \le k} p$. Then*

$$M(\mu(n)a_n^{\epsilon}(k)) = \frac{3\epsilon}{\pi^2 \prod_{p\le k}(1+\frac{1}{p})} \sum_{d|Q_k} \frac{\mu(d)(a_d^1(k) - a_d^{-1}(k))}{d},$$

and

$$M(\mu(n)^2 a_n^{\epsilon}(k)) = \frac{3}{\pi^2 \prod_{p\le k}(1+\frac{1}{p})} \sum_{d|Q_k} \frac{a_d^1(k) + a_d^{-1}(k)}{d}.$$

This result implies that $M(\mu(n)a_n(k)) = -M(\mu(n)c_n(k))$.

Theorem 7 *Let $\epsilon = \pm 1$. We have*

$$M(\mu(n)^2 a_n^{\epsilon}(k)) = \frac{6}{\pi^2} \sum_{\lambda=(k_1^{n_{k_1}} \ldots k_s^{n_{k_s}}) \in \mathcal{P}(k)} \frac{\epsilon_2(\lambda)\mu(L(\lambda))^2}{\prod_{p|L(\lambda)}(p+1)}.$$

Proof. A simple variation of the proof of Theorem 3. □

Put

$$f_k = \zeta(2)M(\mu(n)a_n(k)) \text{ and } g_k = \zeta(2)M(\mu(n)^2 a_n(k)).$$

Note that

$$g_k = \lim_{x\to\infty} \frac{\sum_{n\le x} \mu(n)^2 a_n(k)}{\sum_{n \le x} \mu(n)^2}.$$

Table 6: Scaled averages, $f_k = \zeta(2)M(\mu(n)a_n(k))$ and $g_k = \zeta(2)M(\mu(n)^2 a_n(k))$

k	1	2	3	4	5	6	7	8	9	10
f_k	-1	$-\frac{1}{6}$	$-\frac{1}{4}$	$-\frac{1}{6}$	$-\frac{5}{24}$	$-\frac{1}{12}$	$-\frac{7}{24}$	$-\frac{7}{72}$	$-\frac{7}{48}$	$-\frac{7}{72}$
g_k	0	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{3}$	$\frac{1}{18}$	$\frac{5}{24}$	$\frac{17}{144}$	$\frac{23}{96}$
k	11	12	13	14	15	16	17	18	19	20
f_k	$-\frac{25}{96}$	$-\frac{31}{576}$	$-\frac{11}{42}$	$-\frac{1}{16}$	$-\frac{11}{84}$	$-\frac{8}{63}$	$-\frac{491}{2688}$	$-\frac{613}{12096}$	$-\frac{2371}{10080}$	$-\frac{173}{4032}$
g_k	$\frac{1}{16}$	$\frac{59}{192}$	$\frac{13}{288}$	$\frac{155}{672}$	$\frac{145}{1344}$	$\frac{425}{4032}$	$\frac{667}{8064}$	$\frac{523}{2016}$	$\frac{55}{1344}$	$\frac{101}{480}$

Lemma 25
1) *If k is a prime, then $g_k = e_k$.*
2) *Let $k = 2q$ with q an odd prime. Then*

$$g_{2q} = e_{2q} + \frac{e_q}{2} - \frac{1}{2q(q+1)}.$$

Proof. 1) Reasoning as in the beginning of the proof of Lemma 20 we infer that in case $k > 2$ is prime, we have

$$M(a_n(k)) = \frac{3}{\pi^2 \prod_{p \le k}(1+\frac{1}{p})} \sum_{d|Q_k} \frac{a_d^1(k) + a_d^{-1}(k)}{d} = M(\mu(n)^2 a_n(k)).$$

Since $e_2 = g_2 = \frac{1}{2}$, the proof is completed.
2) Note that $G(\lambda)|2q$. We consider the contribution of the $\lambda \in \mathcal{P}(q)$ with $G(\lambda) = 1, 2, q$ or $2q$ separately. Denote these by, respectively, $\Sigma_1, \Sigma_2, \Sigma_q$ and Σ_{2q}. In case $G(\lambda) = 2$, we let $k_i' = k_i/2$ for $1 \le i \le s$ and let $\lambda' = (k_1'^{n_{k_1'}} \ldots k_s'^{n_{k_s'}})$ be the associated partition of q. We have $G(\lambda') = 1$, $L(\lambda) = 2L(\lambda')$ and $\epsilon_2(\lambda) = \epsilon_2(\lambda')$. Thus

$$\Sigma_2 = \frac{3}{\pi^2} \sum_{\lambda'=(k_1'^{n_{k_1'}} \ldots k_s'^{n_{k_s'}}) \in \mathcal{P}(q)} \frac{\epsilon_2(\lambda')}{\prod_{p|L(\lambda')}(p+1)} = \frac{1}{2}M(a_n(q)),$$

by Theorem 3. It is easily seen that $\Sigma_3 = \frac{1}{2q\zeta(2)}$ and $\Sigma_{2q} = 0$. On putting everything together we obtain

$$M(a_n(2q)) = \Sigma_1 + \frac{1}{2}M(a_n(q)) + \frac{1}{2q\zeta(2)}. \tag{21}$$

Likewise we write $M(\mu(n)^2 a_n(2q)) = \Sigma_1' + \Sigma_2' + \Sigma_q' + \Sigma_{2q}'$. It is easily seen that

$$M(\mu(n)^2 a_n(2q)) = \Sigma_1 + \Sigma_2' + \frac{1}{2(q+1)\zeta(2)}. \tag{22}$$

We write $\Sigma_2' = \Sigma_{2,1}' + \Sigma_{2,2}'$, with λ contributing to $\Sigma_{2,1}'$ if it contributes to Σ_2' and $4 \nmid L(\lambda)$, and λ contributing to $\Sigma_{2,2}'$ if it contributes to Σ_2' and $4|L(\lambda)$. As before to a $\lambda \in \mathcal{P}(2q)$ with $G(\lambda) = 2$, we associate a partition $\lambda' \in \mathcal{P}(q)$. Note that

$G(\lambda') = 1$, $L(\lambda) = 2L(\lambda')$ and $\epsilon_2(\lambda') = \epsilon_2(\lambda)$. On invoking Theorem 7 we find that

$$\begin{aligned}\Sigma'_{2,1} &= \frac{6}{\pi^2} \sum_{\substack{\lambda=(k_1^{n_{k_1}}\ldots k_s^{n_{k_s}})\in\mathcal{P}(2q) \\ G(\lambda)=2,\ 4\nmid L(\lambda)}} \frac{\epsilon_2(\lambda)\mu(L(\lambda))^2}{\prod_{p|L(\lambda)}(p+1)} \\ &= \frac{2}{\pi^2} \sum_{\substack{\lambda'=(k_1'^{n_{k_1'}}\ldots k_s'^{n_{k_s'}})\in\mathcal{P}(q) \\ G(\lambda')=2,\ 2\nmid L(\lambda')}} \frac{\epsilon_2(\lambda')}{\prod_{p|L(\lambda')}(p+1)} = M(a_n(q)),\end{aligned}$$

by Theorem 5. Since obviously $\Sigma'_{2,2} = 0$, it follows from (22) that

$$M(\mu(n)^2 a_n(2q)) = \Sigma_1 + M(a_n(q)) + \frac{1}{2(q+1)\zeta(2)}.$$

The result follows on equating Σ_1 coming from the latter equality with Σ_1 coming from (21). $\square$

Remark. An alternative proof of part 1 of the latter lemma is obtained on noting that if k is an odd prime, then

$$\frac{\epsilon_2(\lambda)\mu(L(\lambda))^2}{\prod_{p|L(\lambda)}(p+1)} = \frac{\epsilon_2(\lambda)}{G(\lambda)\prod_{p|\frac{L(\lambda)}{G(\lambda)}}(p+1)},$$

and invoking Theorem 7 and Theorem 3.

9 Open problems

For the convenience of the reader we have collected below the open problems arising in this paper.

(**P1**) Is it true that $\mathcal{A}(k)$ is convex ?
(**P2**) Is it true that $\mathcal{A}(k)\backslash\mathcal{A}_0(k)$ is convex ?
(**P3**) Is it true that $\epsilon M(\mu(n)a_n^\epsilon(k)) < 0$?
(**P4**) Is it true that $M(\mu(n)^2 a_n^\epsilon(k)) > 0$ for $k \geq 2$?
(**P5**) Is Möllers conjecture that $0 \leq e(k) \leq 1/2$ true ?
(**P6**) Is $e_k k \prod_{p\leq k}(p+1)$ always an integer ? (Certainly true if k is an odd prime.)
(**P7**) What can one say about the behaviour of $e(k)$ as k gets large, or k has many distinct prime factors ?
(**P8**) What is the smallest integer k_0 such that $A(k_0) > k_0$? Möller [14, (10)] has shown that $k_0 \leq 1820$. Our computations show that $k_0 > 105$.
(**P9**) Find effective estimates for $A(k)$.
(**P10**) Is it true that infinitely often $A(k) > A_0(k)$?
(**P11**) Is it true that $A_+(k) > A_-(k)$ for all k sufficiently large ?
(**P12**) Given any real number r, can we find k and v such that $\delta(a_n(k) = v) \neq 0$ and $\delta(a_n(k) = -v) > r\delta(a_n(k) = v)$?
(**P13**) Determine $\{a_d^\epsilon(k) \mid d|M_k\}$, cf. p. 10.

Acknowledgement. This paper is partly based on the M.Sc. thesis of Huib Hommersom (University of Amsterdam). This was a 'literature M.Sc. thesis', i.e. the student was assigned to get an overview of the literature in a specific topic and report on this. A version of this thesis including research contributions (all by Moree) was posted on the arXiv in 2003 [16]. The extensive numerical work in [16] was carried out by Y. Gallot and partially checked using Maple by Moree.

The present paper was written whilst the second author was working at the Max-Planck-Institute for Mathematics in Bonn. He likes to thank both this institute and the University of Amsterdam for their pleasant and inspiring research atmospheres.

References

[1] G. Bachman, On the coefficients of cyclotomic polynomials, *Mem. Amer. Math. Soc.* **106** (1993), no. 510, vi+80 pp.

[2] W. Bosma, Computation of cyclotomic polynomials with Magma, *Computational algebra and number theory* (Sydney, 1992), 213–225, Math. Appl., 325, Kluwer Acad. Publ., Dordrecht, 1995.

[3] R. Breusch, Zur Verallgemeinerung des Bertrandschen Postulates, dass zwischen x und $2x$ stets Primzahlen liegen, *Math. Z.* **34** (1932), 505–526.

[4] K. Bringmann, Asymptotic formulas for some restricted partition functions, *Ramanujan J.* **12** (2006), 257–266.

[5] M. Deléglise, J.-L. Nicolas and P. Zimmermann, Landau's function for one million billions, arXiv:0803.2160, preprint.

[6] M. Endo, On the coefficients of the cyclotomic polynomials, *Comment. Math. Univ. St. Paul.* **23** (1974/75), 121–126.

[7] Y. Gallot and P. Moree, Ternary cyclotomic polynomials having a large coefficient, arXiv:0712.2365, submitted.

[8] A. Grytczuk and B. Tropak, A numerical method for the determination of the cyclotomic polynomial coefficients, *Computational number theory* (Debrecen, 1989), 15–19, de Gruyter, Berlin, 1991.

[9] P. Hagis, Jr., Some theorems concerning partitions into odd summands, *Amer. J. Math.* **88** (1966), 664–681.

[10] C.-G. Ji, W.-P. Li and P. Moree, Values of coefficients of cyclotomic polynomials II, arXiv:0711.4898, submitted.

[11] E. Landau, *Handbuch der Lehre von der Verteilung der Primzahlen*, Chelsea Publishing Co., New York, 1953.

[12] D. H. Lehmer, Some properties of cyclotomic polynomials, *J. Math. Anal. Appl.* **15** (1966), 105–117..

[13] W. Miller, The maximum order of an element of a finite symmetric group, *Amer. Math. Monthly* **94** (1987), 497–506.

[14] H. Möller, Über die i-ten Koeffizienten der Kreisteilungspolynome, *Math. Ann.* **188** (1970), 26–38.

[15] P. Moree, Reciprocal cyclotomic polynomials, arXiv:0709.1570, submitted.

[16] P. Moree and H. Hommersom, Value distribution of Ramanujan sums and of cyclotomic polynomial coefficients, arXiv:math.NT/0307352, 2003, pp. 61.

[17] C.A. Nicol, Some formulas involving Ramanujan sums, *Canad. J. Math.* **14** (1962), 284–286.

[18] J.L. Nicolas, Distribution statistique de l'ordre d'un élément du groupe symétrique, *Acta Math. Hungar.* **45** (1985), 69–84.

[19] R. Thangadurai, On the coefficients of cyclotomic polynomials, *Cyclotomic fields and related topics* (Pune, 1999), 311–322, Bhaskaracharya Pratishthana, Pune, 2000.

12 bis rue Perrey,
31400 Toulouse, France.
e-mail: galloty@orange.fr

Max-Planck-Institut für Mathematik,
Vivatsgasse 7, D-53111 Bonn, Germany.
e-mail: moree@mpim-bonn.mpg.de

Bestevaerstraat 46, 1056 HP Amsterdam, The Netherlands
e-mail: hjhom48@hotmail.com