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0 Introduction

Let G be a topological group, which is typically assumed to satisfy some nice properties, such as local compactness, σ-compactness, normality etc. The usual examples include finitely generated discrete groups and finite dimensional Lie groups. For such group one defines continuous cohomology as cohomology of the cochain complex $C^\bullet(G, V)$ with values in some topological vector space V. If in addition vector space V has a notion of boundedness, for example given by a norm, or a collection of seminorms or more generally a bornology, then one can consider subspaces of continuous bounded cochains $C_b^\bullet(G, V)$. Under certain boundedness conditions on the action of G on V the bounded cochains form a subcomplex of all continuous cochains. The cohomology of this complex is known as continuous bounded cohomology and we denote it $H_b^\bullet(G,V)$, and we have a natural comparison maps $\Psi_n\colon H_b^n(G,V) \to H^n(G,V)$.

We are interested in the properties of this comparison map, in particular whether it is monomorphism and epimorphism. This question has been mostly resolved in the case of finitely generated discrete groups ([Mo]).

Theorem 0.0.1 *If G is a finitely generated discrete such that $H^1(G,V)=0$ then Ψ_n is an isomorphism for all $n \geqslant 0$ if and only if G is finite.*

The proof relates cohomological and geometrical properties of G. Mineyev in [Mi] showed that surjectivity of Ψ_n, for $n \geqslant 2$ is equivalent to the fact that G is a hyperbolic group. And in [Mo] Monod shows that injectivity of Ψ under assumption of vanishing of the first cohomology is equivalent to certain stronger version of property (T). It has been shown [EF] that these two properties are "mutually exclusive" in the sense that only finite groups possess both of them.

The situation with Lie groups is much more complicated. Even for semisimple Lie groups the properties of Ψ are generally not known. For SL_n it has been shown by Goncharov ([Go]) that continuous cohomology is generated by Borel classes, that can be explicitly written using polylogarithms, and that Ψ_n is surjective for $n \leqslant 5$.

On the other hand instead of requiring boundedness of cochains one could impose some other restriction on their growth ([Me2]). For example, consider cochains of polynomial growth with respect to some metric on G. Typically one takes a word metric for discrete groups or in the case of Lie groups the metric generated by a metric form on its Lie algebra $\mathfrak{g}$. In this case under some growth conditions on the group G, we have

Theorem 0.0.2 *If G has polynomial growth, the (polynomial) comparison map*

$$\Psi^{\mathrm{poly}}\colon H^\bullet_{\mathrm{poly}}(G,V) \to H^\bullet_c(G,V)$$

is an isomorphism.

One of the proofs of this theorem uses the fact that the Schwartz space $\mathbb{S}$, which is dual to the polynomial growth functions, is a nuclear space, and therefore the map $\mathbb{C}[x] \to \mathbb{S}$ has properties similar to localization maps.

Another way to generalize the original question, is to consider cochains which are bounded only on some class of subsets of G, instead of everywhere on G. We introduce a notion of a bornological manifold which provides a suitable category to handle this question. The bornological site of such a manifold with respect to a natural Grothendieck topology captures information about geometric and dynamic properties of group G, and the category of sheaves of completely bornological spaces incorporates analytical properties of topological vector spaces.

The shortcoming of bornological topological spaces introduces in this paper is that they only prescribe "where" functions can grow, but not "how fast". In particular the polynomial growth cohomology is not encompassed by this construction. To include it as well one could work with bornological differentiable manifolds instead.

In section 1 we define bornological topological spaces and introduce some compatibility conditions between topology and bornology. Then we give a few elementary properties of sheaves on the bornological site, and finish by describing the underlying sober space of the bornological site and its relation to the Stone-Čech compactification. In section 2 we recall the descent spectral sequence associated to a simplicial site and apply it to the case of bounded group cohomology with discrete coefficients.

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1 Bornological Topological Spaces

We begin by recalling the definition of bornological set.

Definition 1.0.1 *Bornology $\mathcal{B}$ on a set X is an ideal of subsets of X, containing every point $x \in X$. Explicitly, $\mathcal{B}$ satisfies the following conditions:*

- *for every $x \in X$, $\{x\} \in \mathcal{B}$,*
- *if $B \in \mathcal{B}$ and $A \subset B$, then $A \in \mathcal{B}$,*
- *if $A, B \in \mathcal{B}$, then $A \cup B \in \mathcal{B}$.*

Subsets $B \in \mathcal{B}$ are called bounded subsets of X. We say that a map $f\colon (X, \mathcal{B}_X) \to (Y, \mathcal{B}_Y)$ is bounded if $f(B) \in \mathcal{B}_Y$ for every $B \in \mathcal{B}_X$.

1.0.2 First let us define category **BTop** of general bornological topological spaces without imposing any compatibility between bornology and topology. Objects of **BTop** are triples $(X, \mathcal{T}, \mathcal{B})$, where X is a set equipped with topology $\mathcal{T}$ and bornology $\mathcal{B}$, and the morphisms are bounded continuous maps.

If $\mathcal{B}_0$ is an arbitrary collection of subsets of X, we denote by $\mathcal{B} = \langle \mathcal{B}_0 \rangle$ the minimal bornology containing $\mathcal{B}_0$, and say that $\mathcal{B}$ is *generated* by $\mathcal{B}_0$. A collection $\mathcal{B}_1$ of bounded subsets is called a *base* of bornology $\mathcal{B}$ if every bounded subset is contained in some $B \in \mathcal{B}_1$. For a map $f\colon X \to Y$ in **BTop**, the *restriction* of bornology from Y to X, denoted by $f^{-1}(\mathcal{B}_Y)$ or $\mathcal{B}_Y|_X$ when f is injective, is the maximal bornology on X such that f is bounded. Clearly $f^{-1}(\mathcal{B}_Y)$ is generated by preimages of bounded subsets of Y.

1.0.3 Category **BTop** is complete and cocomplete. Limits and colimits can be taken in the underlying category of topological spaces and equipped with the following bornologies. The limit bornology is generated by intersections

$$\varprojlim \mathcal{B}_\alpha = \left\langle \bigcap_\alpha p_\alpha^{-1}(B_\alpha), \quad B_\alpha \in \mathcal{B}_\alpha \right\rangle,$$

where $p_\alpha \colon \varprojlim X_\alpha \to X_\alpha$ are the natural maps.

Similarly, the colimit bornology is generated by the images of $\mathcal{B}_\alpha$ in $\varinjlim X_\alpha$.

Example 1.0.4 a) Collection of all subsets of a topological space X is the maximal bornology, we will call it *trivial bornology* and denote by $\mathcal{B}_t$ and the space X equipped with this bornology by X_t.

b) Collection of all finite subsets of X is the minimal bornology. We will call it *discrete bornology* and denote it by $\mathcal{B}_d$, and the bornological space by X_d.

c) Collection of all subsets $B \subset X$, such that there is a compact K containing B form *compact bornology*. We denote it $\mathcal{B}_c$ and the space by X_c.

1.0.5 Let $U \colon \mathbf{BTop} \to \mathbf{Top}$ be the functor of underlying topological space. Since U preserves both limits and colimits, it has left and right adjoint functors. It is clear from the definition of bounded maps that right adjoint is given by $U_r(X) = X_t$ and left adjoint by $U_l(X) = X_d$.

1.1 Compatibility of Bornology and Topology

Definition 1.1.1 *a) We say that bornology $\mathcal{B}$ is* supercompact *if $\mathcal{B}_c \subset \mathcal{B}$.*

b) We say that bornology $\mathcal{B}$ is closed *if for every $B \in \mathcal{B}$ the closure $\overline{B} \in \mathcal{B}$.*

c) The bornological topological space is locally bounded *if every point has a bounded neighborhood.*

Lemma 1.1.2 *If $(X, \mathcal{B})$ is locally bounded, then $\mathcal{B}$ is supercompact. Moreover, if X is locally compact, then the converse is also true.*

Proof: Let $K \subset X$ be a compact subset, consider open covering of K by $\{U_x, x \in K\}$. We may assume that each U_x is bounded. Since K is compact, it is covered by finite number of U_x, hence it is also bounded.

Conversely, if V_x is a compact neighborhood of x and $\mathcal{B}_c \subset \mathcal{B}$, then V_x is bounded.

□

1.1.3 Bornological locally convex vector spaces. Here we briefly recall basic constructions from the theory of bornological vector spaces, for more details we refer to [HN]. Let $(E, \mathcal{T}, \mathcal{B})$ be a bornological locally convex vector space, E' it's topological dual, i.e., the space of continuous linear maps $E \to \mathbb{R}$. For any subset $S \in E$ we denote by S° the *polar* subset of E' defined as the set of all $f \in E'$ such that $|f(x)| \leqslant 1$ for all $x \in S$.

Let $\{U_i\}$ be a base of neighborhoods of $0 \in E$, then the set of polars $\{U_i^\circ\}$ is a base of bornology on E' called *equicontinuous* bornology. Dually, for a base $\{V_i\}$ of bounded subsets containing 0, the set of polars $\{V_i^\circ\}$ is a base of neighborhoods of $0 \in E'$, that defines the $\mathcal{B}$*-topology* on E', which is topology of uniform convergence on bounded subsets of E.

Collection of subsets $F \subset E'$, consisting of maps $f \in E'$, such that $\bigcup_{f \in F} f(B)$ is bounded for all $B \in \mathcal{B}$, form $\mathcal{B}$*-bornology* on E', that will be denoted by $\mathcal{B}'$. It is also called *equibounded* bornology.

For a locally convex vector space $(E, \mathcal{T})$ we can define *von Neumann* bornology $\mathcal{B}_N(\mathcal{T})$ consisting of subsets absorbed by every neighborhood of 0. Bornology $\mathcal{B}$ and topology $\mathcal{T}$ are said to be compatible is $\mathcal{B} \subset \mathcal{B}_N(\mathcal{T})$. The von Neumann bornology of a $\mathcal{B}$-topology is the corresponding $\mathcal{B}$-bornology.

1.1.4 Let $\mathcal{K}(\mathcal{T})$ be the equicontinuous bornology on E' and $\mathcal{B}_f$ be the finite-dimensional bornology (smallest vector bornology) on E. In general we have the following relation between bornologies on the dual space E'.

$$\mathcal{K}(\mathcal{T}) \subset \mathcal{B}_N(\mathcal{T})' \subset \mathcal{B}_f'$$

A locally convex space $(E, \mathcal{T})$ is called *barreled* if all these bornologies coincide. Any Frechet space (a complete metrizable locally convex vectorspace) is barreled.

1.1.5 Since we are interested in spaces of continuous functions, we will assume from now on, that X satisfies appropriate separation axioms. To be precise, we assume that X is *completely normal*, i.e., any two closed subsets of X can be separated by a continuous function, although some of the statements are true under weaker assumptions.

For a bornological topological space $(X, \mathcal{B})$, we write $C(X)$ for the space of continuous functions on X and $C_b(X) = C_b(X, \mathcal{B})$ for subspace of bounded continuous functions with respect to bornology $\mathcal{B}$. Vector space $C_b(X)$ can be equipped with associated topology and bornology. Let $\mathcal{U}_\mathcal{B}$ denote the $\mathcal{B}$-topology on $C_b(X)$, i.e. the topology of uniform convergence on bounded subsets $B \in \mathcal{B}$. The corresponding von Neumann bornology $\mathcal{B}_N$ is the equibounded bornology with respect to $\mathcal{B}$. A subset of functions $F \subset C_b(X)$ is bounded in $\mathcal{B}_N$ if and only if for every $B \in \mathcal{B}$ the union $\bigcup_{f \in F} f(B)$ is bounded in $\mathbb{R}$.

Let $C_b(X)'$ be the topological dual of $C_b(X)$ with respect to $\mathcal{U}_\mathcal{B}$. Since every point is a bounded subset in $\mathcal{B}$, we have a natural map $X \hookrightarrow C(X)'$, that sends $x \in X$ to the evaluation map $\mathrm{ev}_x(f) = f(x)$. It is injective, because by our assumption functions separate points. We denote by $\mathcal{K} = \mathcal{K}(\mathcal{U}_\mathcal{B})$ the equicontinuous bornology on $C_b(X)'$ and its restriction to X. Similarly, for any bornology $\mathcal{B}_1$ on $C_b(X)$ we denote by $\mathcal{B}_1'$ the corresponding equibounded bornology on $C_b(X)'$ and its restriction to X.

Lemma 1.1.6 *Assume that X is locally compact, and let $\mathcal{B}_f$ be the finite-dimensional bornology on $C(X)$, then $\mathcal{B}'_f = \mathcal{B}_c$.*

Proof: First of all, since image of a compact is compact and hence bounded in $\mathbb{R}$, we see that bornology $\mathcal{B}'_f$ is supercompact. Now, let us take $B \in \mathcal{B}'_f$ and show that it is compact. Suppose B is not compact, then we need to construct a function $\varphi \in C(X)$ unbounded on B. Pick a point $x_0 \in B$ and an open neighborhood U_0 with compact closure. By separability assumptions there exists function $\varphi_0 \in C(X)$, such that $\varphi_0(x_0) = 1$ and $\varphi_0|_{X-U_0} = 0$. Since $B - \overline{U}_0$ is not compact, we may pick another point $x_1 \in B - \overline{U}_0$.

Inductively, we find sequence of points x_n, their respective neighborhoods U_n, such that each $\overline{U}_n$ is compact and disjoint from all other $\overline{U}_m$, and functions φ_n, supported on U_n, such that $\varphi_n(x_n) = 1$. The sum $\varphi = \sum n\varphi_n$ is a continuous function, unbounded on B.

□

1.1.7 I follows from the definition of equibounded bornology, that $\mathcal{B} \subset \mathcal{K}$. Combining this with (1.1.4) we see that in general we have inclusions of bornologies on X:

$$\mathcal{B} \subset \mathcal{K} \subset \mathcal{B}'_N \subset \mathcal{B}'_f.$$

Here $\mathcal{B}_N$ and $\mathcal{B}_f$ are respectively the von Neumann and finite-dimensional bornologies on $C_b(X, \mathcal{B})$.

Definition 1.1.8 *We say that bornology $\mathcal{B}$ is* saturated *if $\mathcal{B} = \mathcal{B}'_f$.*

Proposition 1.1.9 *Assume that X is locally compact, then $\mathcal{B}$ is saturated if and only if it is supercompact and closed.*

Proof: ($\Rightarrow$) We have already seen that bornology $\mathcal{B}'_f$ is always supercompact. Take $B \subset X$ such that every function $f \in C_b(X)$ is bounded on B. Since f is continuous it is also bounded on the closure $\overline{B}$, hence $\mathcal{B} = \mathcal{B}'_f$ is closed.

($\Leftarrow$) We argue similar to proof of lemma (1.1.6). Take $B \in \mathcal{B}'_f$ and suppose that it doesn't belong to $\mathcal{B}$. By assumption B is non-empty, so we can pick a point $x_0 \in B$. Since $(X, \mathcal{B})$ is locally bounded (1.1.2), we can pick an open neighborhood $U_0 \ni x_0$, such that $U_0 \in \mathcal{B}$. Find $\varphi_0 \in C_b(X)$ such that $\varphi_0(x_0) = 1$ and supp $\varphi_0 \subset U_0$.

Since $\overline{U}_0 \in \mathcal{B}$, we find that $B - \overline{U}_0 \notin \mathcal{B}$ and therefore is non-empty. Thus we can repeat the inductive argument from (1.1.6) and conclude that $B \notin \mathcal{B}'_f$, which contradicts the initial assumption.

□

Corollary 1.1.10 *For any locally compact bornological space $(X, \mathcal{B})$, bornology $\overline{\mathcal{B}} := \mathcal{B}'_f$ is saturated, hence $\overline{\overline{\mathcal{B}}} = \overline{\mathcal{B}}$.*

1.1.11 In light of previous considerations, saturated bornologies are sufficient for the purpose of studying spaces of bounded functions. We denote by $\mathbf{BTop}_c$ the full subcategory of **BTop** consisting of locally compact completely normal saturated bornological topological spaces.

As before, we denote the forgetful functor $U\colon \mathbf{BTop}_c \to \mathbf{Top}_{lc}$, where $\mathbf{Top}_{lc}$ is the full subcategory of locally compact, completely normals spaces. The right adjoint functor U_r equips space X with the trivial bornology, and the left adjoint U_l equips it with the compact bornology.

1.2 Bornological Site

Definition 1.2.1 *A collection of maps* $\{X_i \to X\}$ *in* $\mathbf{BTop}_c$ *is a* bornological cover *if* $\{U(X_i) \to U(X)\}$ *is an open cover in* **Top** *and images of* $\mathcal{B}_{X_i}$ *generate* $\mathcal{B}_X$*.*

Proposition 1.2.2 *Bornological covers form a subcanonical Grothendieck topology.*

Proof: (a) Clearly $\mathrm{id}\colon X \to X$ is a cover, and composition of covers is again a cover. Let $f\colon Y \to X$ be a bounded continuous map, denote by Y_i the fibered product

$$\begin{array}{ccc} Y_i & \xrightarrow{f_i} & X_i \\ {\scriptstyle g_i}\downarrow & & \downarrow \\ Y & \xrightarrow[f]{} & X \end{array}$$

Bornology on Y_i is generated by intersections $g_i^{-1}(B) \cap f_i^{-1}B_i$, where B and B_i are bounded subsets of Y and X_i respectively. Since $f(B)$ is bounded, it is covered by finitely many B_i, hence B is also covered by finitely many of the above intersections, and therefore $\mathcal{B}_Y = \langle \mathcal{B}_{Y_i} \rangle$.

(b) Denote by $\widetilde{Z}$ the presheaf on bornological site of X represented by Z. We want to show that $\widetilde{Z}$ is a sheaf, i.e., the natural map

$$\mathrm{Map}_b(X, Z) \to \mathrm{Ker}\left(\prod \mathrm{Map}_b(X_i, Z) \rightrightarrows \prod \mathrm{Map}_b(X_{ij}, Z)\right)$$

is an isomorphism. The injectivity follows from the fact that the forgetful functor U is injective on morphisms. For surjectivity we need to show that the map $f\colon X \to Z$ obtained by descent is bounded. Let $B \in \mathcal{B}_X$, then it is contained in a finite union $B \subset \bigcup B_i$, $B_i \in \mathcal{B}_{X_i}$. Therefore $f(B)$ is bounded in Z.

□

Lemma 1.2.3 *Let* $(X, \mathcal{B}) \in \mathbf{BTop}_c$*, and* $\{X_i \to X\}$ *an open cover of the underlying topological space. Write* $\mathcal{B}_i$ *for restrictions of* $\mathcal{B}$ *to* X_i*. Collection* $\{(X_i, \mathcal{B}_i) \to (X, \mathcal{B})\}$ *is a bornological cover if and only if every bounded subset* $B \subset X$ *is covered by finitely many* X_i*.*

Proof: The "only if" part follows immediately from the definition. For the other direction, observe that for every $B \in \mathcal{B}$ we can write $B = \bigcup (B \cap X_i)$, the union can be taken finite by assumption and each intersection is bounded in restricted bornologies.

□

For an object $(X, \mathcal{B}) \in \mathbf{BTop}_c$ we write $\mathcal{S}(X, \mathcal{B})$ for the bornological site, and also $\mathcal{S}(X)$ for the topological site of the underlying space.

Proposition 1.2.4 *The forgetful function U induces map of sites $\alpha\colon \mathcal{S}(X) \to \mathcal{S}(X, \mathcal{B})$ and the left adjoint $U_l\colon X \mapsto X_c$ induces map $\gamma\colon \mathcal{S}(X_c) \to \mathcal{S}(X)$. The composition*

$$\mathcal{S}(X) \xrightarrow{\alpha} \mathcal{S}(X_c) \xrightarrow{\gamma} \mathcal{S}(X)$$

is identity.

Proof: By definition of bornological cover functor U sends covers to covers, hence it induces morphism α. Now, let us show that U_l sends topological covers to bornological covers. Suppose $\{X_i \to X\}$ is a topological cover of X. Take a compact subset $K \subset X$, it is covered by finite number of X_i, and we denote this finite set of indices I. However, the intersections $K \cap X_i$ are not necessarily bounded in $(X_i)_c$. We need to find compacts $K_i \subset X_i$, such that $K \subset \bigcup_{i \in I} K_i$.

We shall proceed by induction on the size of the set I. If $|I| = 1$, then there is nothing to prove, so assume that $|I| > 1$. We may also assume that for some $i \in I$ the set

$$M_i = X_i - \bigcup_{\substack{j \in I \\ j \neq i}} X_j$$

is not empty, otherwise we could remove the corresponding index from I. The complement $N_i = X - X_i$ is a closed subset disjoin from M_i, hence we can find disjoint open neighborhoods $V_i \supset M_i$ and $W_i \supset N_i$. Now, $X - W_i$ is a closed subset of X, and $K_i = K \cap (X - W_i)$ is compact. Denote $D_i = K \cap (X - V_i)$, we have $K = K_i \cup D_i$ and since $D_i \subset \bigcup_{I - \{i\}} X_i$ we reduce to the case of a smaller I. This shows that U_l induces morphism of sites $\mathcal{S}(X_c) \to \mathcal{S}(X)$.

Finally, since the unit of adjunction $\mathrm{Id} \to UU_l$ is an isomorphism, we find that composition $\gamma \circ \alpha = \mathrm{id}$.

□

1.2.5 Cohomology of X_c. In general sites $\mathcal{S}(X_c)$ and $\mathcal{S}(X)$ are not isomorphic, however as we will see in terms of cohomology they are equivalent. Let $\mathrm{Sh}(X)$ and $\mathrm{Sh}(X, \mathcal{B})$ be the categories of sheaves of abelian groups on the topological and bornological sites respectively. For a map $f\colon (X, \mathcal{B}_X) \to (Y, \mathcal{B}_Y)$ we denote f^* and f_* the pullback and pushforward functors between categories of sheaves.

Lemma 1.2.6 *There is a natural isomorphism of functors $\gamma_* \xrightarrow{\sim} \alpha^*\colon \mathrm{Sh}(X_c) \to \mathrm{Sh}(X)$.*

Proof: Let $\eta\colon \gamma^*\gamma_* \to \mathrm{Id}$ be the counit of the adjunction. Composing it with α^* we obtain a map

$$\gamma_* \simeq \alpha^*\gamma^*\gamma_* \xrightarrow{\eta} \alpha^*.$$

We want to show that $\gamma_*(F) \simeq \alpha^*(F)$ for any $F \in \mathrm{Sh}(X_c)$. Consider stalks of both sheaves at $x \in X$.

$$\gamma_*(F)_x = \varinjlim_{x \in V_i} F((V_i)_c),$$

where colimit is taken over all open neighborhoods of x. For the other sheaf we have

$$\alpha^*(F)_x = \varinjlim_{x \in (V, \mathcal{B})} F(V, \mathcal{B}),$$

where colimit is over all open neighborhoods V of x and all bornologies $\mathcal{B}$ on V such that inclusion $(V, \mathcal{B}) \to (X, \mathcal{B}_X)$ is bounded. Notice that subcategory consisting of $((V_i)_c)$ is cofinal, therefore the two stalks are isomorphic for any $x \in X$.

□

Proposition 1.2.7 *For any sheaf $F \in \mathrm{Sh}(X_c)$ the pullback $\alpha^* \colon H^\bullet(X_c, F) \to H^\bullet(X, \alpha^* F)$ is an isomorphism.*

Proof: Since γ_* has exact left adjoint, it preserves injectives. Also from $\gamma_* = \alpha^*$ we see that it is exact itself, therefore $R\Gamma(X_c, F) = R\Gamma(X, \gamma_* F) = R\Gamma(X, \alpha^* F)$.

□

1.3 Sheaves of $\mathcal{O}_{(X,\mathcal{B})}$-modules

Let $(V, \mathcal{T}, \mathcal{B})$ be a locally convex vector space with topology $\mathcal{T}$ and bornology $\mathcal{B}$. It is called *completely bornological* if the following holds:

a) topology $\mathcal{T} = \mathcal{T}_{\mathcal{B}}$ is the finest topology compatible with bornology $\mathcal{B}$,

b) bornology has a base consisting of closed disks $D \subset V$, such that linear subspace V_D spanned by D with seminorm p_D induced by D is a Banach space.

For example, any Fréchet space is completely bornological.

Let V, W, P be three completely bornological spaces. Any bounded linear map between two completely bornological spaces is continuous. The space $\mathcal{L}(V, W)$ of all such maps, equipped with $\mathcal{B}$-topology and $\mathcal{B}$-bornology is a completely bornological space as well. Denote $\mathbf{B}(V, W; P)$ the space of bi-bounded bilinear maps $b \colon V \times W \to P$, and $V \widehat{\otimes} W$ the completed bornological tensor product, i.e., the universal completely bornological space, representing bi-bounded bilinear maps. In other words we have isomorphisms

$$\mathrm{Hom}_b(V \widehat{\otimes} W, P) \;\simeq\; \mathbf{B}(V, W; P) \;\simeq\; \mathrm{Hom}_b(V, \mathcal{L}(W, P)).$$

Lemma 1.3.1 *For any $X \in \mathbf{BTop}_c$ and a completely bornological vector space V, the space of continuous bounded maps $\mathrm{Map}(X, V)$ with $\mathcal{B}_X$-topology and $\mathcal{B}_X$-bornology is also completely bornological.*

Proof: As was mentioned before $\mathcal{B}$-bornology of $\mathrm{Map}(X, V)$ is the von Neumann bornology of the $\mathcal{B}$-topology. Now, to show that is satisfies property (b) let $\mathcal{B}_1$ be a suitable base of $\mathcal{B}_V$, and consider a map $\psi \colon \mathcal{B}_X \to \mathcal{B}_1$. To every such map we associate a disk in $\mathrm{Map}(X, V)$

$$D_\psi = \{ f \in \mathrm{Map}(X, V) \mid f(B) \subset \psi(B), \text{ for any } B \in \mathcal{B}_X \}.$$

Clearly p_{D_ψ} is a norm, since all $p_{\psi(B)}$ are norms. Using local boundedness of X (lemma 1.1.2) we see that the limit of a sequence of continuous functions is again continuous, therefore the linear span of D_ψ is a Banach space.

□

We will write $\widetilde{V} = \widetilde{V}_X$ for the sheaf of completely bornological spaces on the site $\mathcal{S}(X,\mathcal{B})$ represented by V, and $\mathcal{O}_{(X,\mathcal{B})} := \widetilde{\mathbb{R}_c}$ for the *structure sheaf* on $\mathcal{S}(X,\mathcal{B})$. A sheaf of $\mathcal{O}_{(X,\mathcal{B})}$-modules $\mathcal{F}$ is called *quasi-coherent* if for every $(U,\mathcal{B}_U) \in \mathcal{S}(X,\mathcal{B})$

$$\mathcal{F}(U,\mathcal{B}_U) = \mathcal{O}(U,\mathcal{B}_U) \underset{\mathcal{O}(X,\mathcal{B})}{\widehat{\otimes}} \mathcal{F}(X,\mathcal{B}).$$

1.3.2 Recall that a locally convex vector space V is said to have $\mathcal{B}$-*approximation property* if for every locally convex space W the algebraic tensor product $V' \otimes W$ is dense in $\mathcal{L}(V,W)$ equipped with $\mathcal{B}$-topology.

Proposition 1.3.3 *If V is a completely bornological space with $\mathcal{B}$-approximation property, then the sheaf $\widetilde{V}_X$ is quasi-coherent and for any bounded map $f\colon (X,\mathcal{B}_X) \to (Y,\mathcal{B}_Y)$ we have a natural isomorphism $\widetilde{V}_X = f^*\widetilde{V}_Y$.*

Proof: Both statements of the proposition immediately follow from isomorphisms

$$C_b(Z) \widehat{\otimes} V \xrightarrow{\sim} \mathrm{Map}(Z,V),$$

for any bornological space Z.

If V is a finite dimensional space then the map is clearly an isomorphism. In general case, apply approximation property to the identity map $\mathrm{id}\colon V \to V$ to see that every bounded subset $B \subset V$ can be approximated by a finite dimensional bounded subset B_1. Therefore, the image of algebraic tensor product $C_b(Z) \otimes V$ is dense in $\mathrm{Map}(Z,V)$ and since the latter space is completely bornological the map in question is an isomorphism.

□

Proposition 1.3.4 *For any quasi-coherent sheaf $\widetilde{V}$ on $\mathcal{S}(X,\mathcal{B})$, and any $i \geqslant 1$,*

$$H^i(\mathcal{S}(X,\mathcal{B}),\widetilde{V}) = 0.$$

Proof: The standard proof using partition of unity applies here as well, since all functions in the partition can be assumed to be bounded with respect to trivial bornology, and hence for any other bornology $\mathcal{B}$.

□

1.4 Underlying Sober Space

We begin by recalling the definition of a locale and duality between locales and sober topological spaces. For further details we refer to [Jo]. Let $(P, \leqslant)$ be a partially ordered set.

Definition 1.4.1 *a) A poset $(P, \leqslant)$ is a* lattice *if for any $a, b \in P$ there exist least upper bound denoted $a \vee b$ and greatest lower bound denoted $a \wedge b$.*

b) A lattice $(P, \leqslant)$ is a locale *if any subset (not necessarily finite) $B \subset P$ has least upper bound $\bigvee B$, and it satisfies infinite distributive law*

$$a \wedge \bigvee B = \bigvee \{a \wedge b \mid b \in B\}.$$

Let $\mathbf{P} = (P, \leqslant)$, $\mathbf{Q} = (Q, \leqslant)$ be two locales, a morphism of locales $f\colon \mathbf{P} \to \mathbf{Q}$ is a map of posets $f^{\mathrm{op}}\colon (Q, \leqslant) \to (P, \leqslant)$ preserving finite $\wedge$ and arbitrary $\vee$. We denote the category of locales by **Loc**.

For a topological space X we denote by $\Omega(X)$ the lattice of open subsets of X. A continuous map $f\colon X \to Y$ induces a map of locales $\Omega f\colon \Omega(X) \to \Omega(Y)$ given by preimage of open subsets: $\Omega f^{\mathrm{op}} = f^{-1}$.

1.4.2 Points of a locale. The locale of a single point space $\Omega(\text{point})$ consists of two elements $\{0 \leqslant 1\}$, where 0 corresponds to an empty subset and 1 to the entire space. A point of a locale $\mathbf{P}$ is a morphism $p\colon \Omega(\text{point}) \to \mathbf{P}$. Explicitly, such map is determined by either one of the following subsets of $\mathbf{P}$:

$$I = (p^{\mathrm{op}})^{-1}(0), \quad F = (p^{\mathrm{op}})^{-1}(1).$$

Subsets I and F are respectively a prime ideal and a prime filter in $\mathbf{P}$. Moreover, since p^{op} preserves arbitrary $\vee$, I is a principal ideal generated by $\bigvee I$, i.e., $x \in I$ if and only if $x \leqslant \bigvee I$.

We denote the set of points of $\mathbf{P}$ by $\mathrm{pt}(\mathbf{P})$. We can endow this set with topology induced by $\mathbf{P}$. For any $x \in \mathbf{P}$, denote

$$\varphi(x) = \{p \in \mathrm{pt}(\mathbf{P}) \mid p(x) = 1\}.$$

The collection of subsets $\{\varphi(x) \mid x \in \mathbf{P}\}$ form a topology on $\mathrm{pt}(\mathbf{P})$.

Functor of points pt considered as a functor from locales to topological spaces is right adjoint to Ω, and map φ induces counit of adjunction $\Omega(\mathrm{pt}(\mathbf{P})) \to \mathbf{P}$. Moreover, the image of pt consists of sober spaces (recall that topological space X is called *sober* if every irreducible closed subset of X has a generic point). We denote the full subcategory of sober spaces by **Sob**. Functors pt and Ω restricted to **Sob** provide an adjoint equivalence of **Loc** and **Sob**.

1.4.3 For a bornological topological space $(X, \mathcal{B}) \in \mathbf{BTop}_c$ we denote by $\Omega(X, \mathcal{B})$ the set of objects $(U, \mathcal{B}_U) \in \mathbf{BTop}_c$ such that U is an open subset of X and the inclusion $(U, \mathcal{B}_U) \hookrightarrow (X, \mathcal{B})$ is bounded. We define a partial order on $\Omega(X, \mathcal{B})$ by saying that $(U, \mathcal{B}_U) \leqslant (V, \mathcal{B}_V)$ if $U \subset V$ and the inclusion map is bounded.

Proposition 1.4.4 *The poset $\Omega(X, \mathcal{B})$ is a locale.*

Proof: It follows directly from definitions that

$$(U, \mathcal{B}_U) \wedge (V, \mathcal{B}_V) = (U, \mathcal{B}_U) \times_{(X,\mathcal{B})} (V, \mathcal{B}_V),$$
$$\bigvee (V_i, \mathcal{B}_i) = \left(\bigcup V_i, \langle \mathcal{B}_i \rangle \right).$$

The bornology generated on the union $\bigcup V_i$ is closed, since closures of finite unions are unions of closures, and it is supercompact in virtue of proposition (1.2.4). The distributive law follows from the distributive law for bounded subsets:

$$B_U \cap \bigcup_{i=1}^{n} B_{V_i} = \bigcup_{i=1}^{n} (B_U \cap B_{V_i}).$$

□

Now we will proceed to describe the sober space $\mathrm{pt}(X, \mathcal{B}) := \mathrm{pt}(\Omega(X, \mathcal{B}))$. Since $\mathrm{pt}(X, \mathcal{B})$ is an open subset in $\mathrm{pt}(X_t)$ it is enough to consider only trivial bornology.

Lemma 1.4.5 *Let $(U, \mathcal{B}_U)$ be a prime element of $\Omega(X, \mathcal{B})$, then $X - U$ consists of at most one point.*

Proof: Suppose there are two distinct point $z_1, z_2 \in X - U$. We can separate them with neighborhoods V_1 and V_2 with disjoint closures. Equip V_1 and V_2 with trivial bornologies and denote

$$(U_1, \mathcal{B}_1) = (U \cup V_1, \langle \mathcal{B}_U, \mathcal{B}_t(V_1) \rangle),$$
$$(U_2, \mathcal{B}_2) = (U \cup V_2, \langle \mathcal{B}_U, \mathcal{B}_t(V_2) \rangle).$$

It is easy to see that $(U_1, \mathcal{B}_1) \wedge (U_2, \mathcal{B}_2) = (U, \mathcal{B}_U)$ since bounded subsets are of the form

$$(B \cup B_1) \cap (B' \cup B_2) \subset B \cup B' \in \mathcal{B}_U,$$

where $B, B' \in \mathcal{B}_U$, and $B_i \in \mathcal{B}_t(V_i)$, $i = 1, 2$.

□

We will call prime element $(U, \mathcal{B}_U)$ and the corresponding point of $\Omega(X, \mathcal{B})$ of *type 0*, if $X - U$ is empty, and of *type 1* if $X - U$ is a single point.

1.4.6 Points of type 0. Clearly $(X, \mathcal{B})$ is prime element of $\Omega(X_t)$ if and only if $\mathcal{B}$ is a prime (non-trivial) bornology, i.e., prime as an element of lattice of bornologies on X. This is equivalent to saying that closed elements in $\mathcal{B}$ form a prime ideal of closed subsets of X containing ideal of compacts, or that their complements in X form a prime filter of open subsets of X, containing all cocompact subsets. Since X is locally compact, such filters are non-principal.

1.4.7 Points of type 1. First, consider intersections $(U, \mathcal{B}_U) = (U, \mathcal{B}_1) \wedge (U, \mathcal{B}_2)$. As before, we conclude that $\mathcal{B}_U$ is either trivial or a prime bornology. In addition to that we have $(U, \mathcal{B}_U) = (X, \mathcal{B}) \wedge (U, \mathcal{B}_t)$ implies $\mathcal{B}_U = \mathcal{B}_t$. Therefore, for $(U, \mathcal{B}_U)$ to be prime, we must have either $\mathcal{B}_U = \mathcal{B}_t$ or $\mathcal{B}_U$ is not restriction of a bornology on X.

Lemma 1.4.8 *Bornology $\mathcal{B}_U$ is a restriction from X if and only if for some neighborhood V of $x = X - U$, we have $U \cap V \in \mathcal{B}_U$.*

Proof: Indeed, the "only if" part if obvious since all spaces are locally bounded. To show the other implication, consider bornology $\mathcal{B}'$ on X generated by closures in X of all $B \in \mathcal{B}_U$. It is closed by definition and locally bounded by assumption, hence it is supercompact. Clearly $\mathcal{B}_U \subset \mathcal{B}'|_U$. For the converse, let $B \in \mathcal{B}'$ closed and disjoint from x, then it is separated by an open neighborhood of x, therefore being a closure of an element of $\mathcal{B}_U$ we find that $B \in \mathcal{B}_U$. Now, if $x \in B$, we write $B \cap U = (B - V) \cup (B \cap V \cap U)$, the first part is in $\mathcal{B}_U$ by the previous argument and the second one by the assumption $U \cap V \in \mathcal{B}_U$.

□

Finally, pick a neighborhood $V \ni x$, and let $\mathcal{B}'$ be the bornology on X generated by $\mathcal{B}_U$ and V. Also, denote

$$\widetilde{\mathcal{B}} = \langle \mathcal{B}_U|_V, \mathcal{B}_t(U - V) \rangle .$$

We have $(X, \mathcal{B}') \wedge (U, \widetilde{\mathcal{B}}) = (U, \mathcal{B}_U)$, hence $\mathcal{B}_U = \widetilde{\mathcal{B}}$. In other words, $\mathcal{B}_U$ must contain complements of all neighborhoods on x.

It is straightforward to check that these conditions are also sufficient for $(U, \mathcal{B}_U)$ to be a prime element. Passing to the complements, we see that a point of type 1 corresponds to a point $x \in X$ and either a trivial bornology on $X - x$ or a prime filter of open subsets in $X - x$ containing all neighborhoods of x. Notice that since compact subsets in $X - x$ can be separated from x the latter condition implies that the filter also contains all cocompacts.

1.4.9 To summarize, $\mathrm{pt}(X_t)$ as a set is the union

$$\mathrm{pt}(X_t) \quad = \quad X \quad \sqcup \quad \bigsqcup_{x \in X} \left\{ \begin{array}{c} \text{prime filters} \\ \text{of open subsets of } X - x \\ \text{containing all} \\ \text{neighborhoods of } x \end{array} \right\} \quad \sqcup \quad \left\{ \begin{array}{c} \text{prime filters} \\ \text{of open subsets of } X \\ \text{containing all cocompacts} \end{array} \right\}.$$

Open subset $\varphi(Y, \mathcal{B}_Y)$ consists of primes $(U, \mathcal{B}_U)$ such that $(Y, \mathcal{B}_Y) \not\leq (U, \mathcal{B}_U)$. In terms of decomposition above, it consists of $U \subset X$ in the first component, and filters F such that there is $B \in \mathcal{B}_Y$ that intersects every $V \in F$. For example, $\varphi(X_c)$ is the subset of points of type 1, and $\varphi(X - x, \mathcal{B}_t) = \mathrm{pt}(X_t) - x$.

1.5 Stone-Čech compactification

Let $\mathbf{P}$ be a locale, for $a, b \in \mathbf{P}$ we will write $a \prec b$, if there is $c \in \mathbf{P}$, such that $b \vee c = 1$ and $a \wedge c = 0$. We say that $\mathbf{P}$ is *regular* if for every $a \in \mathbf{P}$ we have

$$a = \bigvee \{ b \mid b \prec a \}.$$

An ideal $I \subset \mathbf{P}$ is said to be *regular* if for every $a \in I$ there is $b \in I$ with $a \prec b$. The set of such ideals forms a locale that will be denoted by $\mathcal{R}(\mathbf{P})$. Similarly, a filter $F \subset \mathbf{P}$ is *regular* if for every $a \in F$ there is $b \in F$ with $b \prec a$.

Also, we say that $\mathbf{P}$ is *normal* if for any $a, b \in \mathbf{P}$, such that $a \vee b = 1$, there exist $c, d \in \mathbf{P}$, so that

$$a \vee c = 1, \quad b \vee d = 1, \quad c \wedge d = 0.$$

For a completely normal topological space X the lattice $\Omega(X)$ is regular and normal, however, it is easy to see that for a bornological space $(X, \mathcal{B}) \in \mathbf{BTop}_c$, lattice $\Omega(X, \mathcal{B})$ is not even Hausdorff.

1.5.1 Stone-Čech compactification. The inclusion functor of compact regular locales into arbitrary locales has left adjoint functor called *Stone-Čech compactification* and denoted by β. Let $\mathbf{P}$ be a normal locale, then the compactification $\beta\mathbf{P}$ can be identified with the locale of regular ideals $\mathcal{R}(\mathbf{P})$.

Moreover, if in addition locale $\mathbf{P}$ is regular, then the set of points of $\beta\mathbf{P}$, i.e., the prime elements of $\beta\mathbf{P}$, is isomorphic to the set of regular ultrafilters of $\mathbf{P}$. The topology on $\mathrm{pt}(\beta\mathbf{P})$ consists of subsets

$$\varphi(a) = \{F \mid F - \text{regular ultrafilter},\ a \in F\}\,.$$

In this case the unit of adjunction $\mathbf{P} \to \beta\mathbf{P}$ is a monomorphism. For a topological space X we write $\beta X := \mathrm{pt}(\beta\Omega(X))$.

1.5.2 Wallman compactification. Let $\mathbf{W}$ be a sublattice of $\Omega(X)$, such that

a) $\mathbf{W}$ is a base of topology on X,

b) for any $x \in X$ and $U \in \mathbf{W}$ containing x, there exists $V \in \mathbf{W}$, such that $U \cup V = X$ and $x \notin V$.

Denote $\omega(X, \mathbf{W})$ the set of maximal ideals of $\mathbf{W}$, it is called the *Wallman compactification* of X relative to $\mathbf{W}$. For a normal X the entire lattice $\Omega(X)$ satisfies these conditions, and we will write $\omega X := \omega(X, \Omega(X))$. Equivalently, ωX is the set of ultrafilters of closed subsets of X, the closed subsets of ωX are of the form

$$\{F \mid Z \in F,\ \text{for some closed } Z \subset X\}.$$

If X is completely normal, then the natural map $\beta X \to \omega X$ is an isomorphism.

Before exploring relation between spaces $\mathrm{pt}(X_t)$ and βX, let us introduce another property of bornologies.

Definition 1.5.3 *We say that bornology $\mathcal{B}$ is* open *if for every $B \in \mathcal{B}$ there is open $U \in \mathcal{B}$, containing B.*

Let $\mathcal{B}$ be an open and closed bornology, clearly such bornology is necessarily supercompact. Also, it follows from definitions that subset of $\mathcal{B}$ consisting of open subsets of X is a regular ideal in $\Omega(X)$, containing a neighborhood of every point $x \in X$. To see regularity, notice that for any open $B \in \mathcal{B}$, we have a chain of bounded subsets $B \subset \overline{B} \subset U$, hence $B \prec U$.

Denote by $\mathbf{BTop}_{oc}$ the full subcategory of $\mathbf{BTop}_c$, consisting of spaces with open bornology. We will write $\Omega(X, \mathcal{B})_{\mathrm{op}}$ for the sublattice of $\Omega(X, \mathcal{B})$ consisting of spaces in $\mathbf{BTop}_{oc}$. The inclusion induces an epimorphism of locales $q \colon \Omega(X, \mathcal{B}) \twoheadrightarrow \Omega(X, \mathcal{B})_{\mathrm{op}}$.

Proposition 1.5.4 *There is a natural epimorphism of locales* $p \colon \Omega(X_t)_{\mathrm{op}} \to \mathcal{R}\Omega(X)$, *and a section* s *of* p, *i.e., we have the following diagram*

$$\begin{array}{c} \Omega(X_t) \\ \big\downarrow{\scriptstyle q} \\ \Omega(X_t)_{\mathrm{op}} \\ {\scriptstyle s}\big\uparrow\ \big\downarrow{\scriptstyle p} \\ \mathcal{R}\Omega(X) = \beta\Omega(X). \end{array}$$

Moreover, p *induces bijection between the set of points of* $\Omega(X_t)_{\mathrm{op}}$ *of type 0 and* $\beta X - X$.

Proof: (a) Let I be a regular ideal of open subsets of X and $(U, \mathcal{B}) \in \Omega(X_t)_{\mathrm{op}}$. Define p^{op} and s^{op} as follows

$$p^{\mathrm{op}}(I) = \left(\bigcup_{U \in I} U,\ \langle U \mid U \in I \rangle \right),$$

$$s^{\mathrm{op}}(U, \mathcal{B}) = \left\{ V \mid V \in \Omega(X),\ V \prec U,\ V \in \mathcal{B} \right\}.$$

Regularity of I implies that $p^{\mathrm{op}}(I)$ is a space with an open bornology. Now, clearly $s^{\mathrm{op}}(U, \mathcal{B})$ is an ideal, to show that it is regular let V' be an open bounded subset of U containing $\overline{V}$, it exists by openness of $\mathcal{B}$. Since X is normal there also exists open $V'' \prec U$, containing $\overline{V}$, the intersection $V' \cap V'' \in s^{\mathrm{op}}(U, \mathcal{B})$ and contains $\overline{V}$.

(b) Let $J := s^{\mathrm{op}} p^{\mathrm{op}}(I)$, we want to show that $J = I$. For any $V \in J$ we have $V \in \mathcal{B}$, hence $V \in I$. Conversely, for $W \in I$, we have $W \in \mathcal{B}$ and from regularity of I we find $W \prec U$, therefore $W \in J$.

(c) From identification of open bornologies and covering regular ideals mentioned above we immediately conclude that p is a morphism of locales, and by (b) it is an epimorphism.

(d) It is straightforward to check that s^{op} preserves finite $\wedge$. We will show that it preserves arbitrary $\vee$. Since all three conditions in the definition of s^{op} respect finite unions, we see that $\bigvee s^{\mathrm{op}}(U_i, \mathcal{B}_i) \subset s^{\mathrm{op}}(\bigvee(U_i, \mathcal{B}_i))$. For the other inclusion, take $V \in \bigvee \mathcal{B}_i$, hence $V \subset \overline{V} \subset \bigcup_{i=1}^{N} B_i$. By openness of bornologies $\mathcal{B}_i$, subsets B_i may be assumed to be open. By assumption $V \prec \bigcup U_i$, and by normality of X we may shrink B_i so that $B_i \prec U_i$, without changing their union. Therefore, $V \in \bigvee s^{\mathrm{op}}(U_i, \mathcal{B}_i)$.

(e) The sober space $\mathrm{pt}(\Omega(X_t)_{\mathrm{op}})$ admits decomposition similar to (1.4.9), where all filters are required to be regular. Points of type 0 correspond to non-principal prime regular filters of open subsets in X, and such filters are necessarily maximal. These are exactly points in the boundary of the Stone-Čech compactification $\beta X - X$.

□

1.5.5 From the proof of the proposition we see that for any open $U \subset X$, there is an embedding $\beta U \hookrightarrow \mathrm{pt}(\Omega(X_t)_{\mathrm{op}})$. The fiber $(p^{-1}(x) - x)$ is Hausdorff for all $x \in X$ and isomorphic to the fiber over x of the map $\beta(X - x) \to \beta X$.

In terms of Wallman compactification, map p sends a prime open bornology $\mathcal{B}$ on $U \subset X$ to the ultrafilter of closures in X of complements to open bounded subsets in $\mathcal{B}$.

2 Bounded Group Cohomology

2.1 Simplicial Sites

Consider a simplicial site $\mathcal{S}_\bullet$, i.e., a functor from the category opposite to the category of finite ordered sets $\mathbf{\Delta}$ to the category of sites. We can construct a *total site* $\mathrm{Tot}\,\mathcal{S}$ associated to the simplicial site $\mathcal{S}_\bullet$ as follows.

a) The objects of $\mathrm{Tot}\,\mathcal{S}$ are the union of objects of all $\mathcal{S}_n$,

b) a morphism $(u, f)\colon U \to V$, where $U \in \mathcal{S}_n$ and $V \in \mathcal{S}_m$ is given by a pair consisting of a map $u\colon [m] \to [n]$ in $\mathbf{\Delta}$, and a map $f\colon U \to u^{-1}(V)$ in $\mathcal{S}_n$,

c) the coverings of $U \in \mathrm{Tot}\,\mathcal{S}$ are of the form $\{(\mathrm{id}, f_i)\}$, where $\{f_i\colon U_i \to U\}$ is a covering in $\mathcal{S}_n$.

We define categories of sheaves of sets, abelian groups, their derived category etc. on a simplicial site $\mathcal{S}_\bullet$ as the corresponding category on its total site. In particular we have identification of the category $\mathrm{Sh}(\mathrm{Tot}\,\mathcal{S})$ and the category of collections $\{F_n \in \mathrm{Sh}(\mathcal{S}_n)\}$ and binding morphisms $F(u)\colon F_m \to u_* F_n$, for all $u\colon [m] \to [n]$ in $\mathbf{\Delta}$, satisfying certain compatibility conditions.

At the level of derived categories we have a spectral sequence

$$E_1^{pq} = H^q(\mathcal{S}_p, C^\bullet) \;\Rightarrow\; H^{p+q}(\mathrm{Tot}\,\mathcal{S}, C^\bullet),$$

where $C^\bullet \in D^+(\mathrm{Tot}\,\mathcal{S})$.

We say that a sheaf $F \in \mathrm{Sh}(\mathrm{Tot}\,\mathcal{S})$ (respectively $C^\bullet \in D^+(\mathrm{Tot}\,\mathcal{S})$) is *cartesian* if the binding pullback maps $u^*F_m \to F_n$ (respectively $Lu^*C^\bullet_m \to C^\bullet_n$) are isomorphisms.

2.1.1 Descent spectral sequence. Recall that for any map of sites $a\colon \mathcal{S} \to \mathcal{S}'$ we have the simplicial site $\mathrm{cosk}_0\,(\mathcal{S} \to \mathcal{S}')$ defined by

$$\mathrm{cosk}_0\,(\mathcal{S} \to \mathcal{S}')_n = \underbrace{\mathcal{S} \times_{\mathcal{S}'} \mathcal{S} \times_{\mathcal{S}'} \ldots \times_{\mathcal{S}'} \mathcal{S}}_{n+1 \text{ times}},$$

with obvious simplicial maps. We denote by $a_n\colon \mathrm{cosk}_0\,(\mathcal{S} \to \mathcal{S}')_n \to \mathcal{S}'$ the map induced by a.

The map a is said to be a *(cohomological) descent map* if the pullback functor

$$a^*\colon D^+(\mathcal{S}') \to D^+(\mathrm{Tot}\,\mathrm{cosk}_0\,(\mathcal{S} \to \mathcal{S}'))$$

is a fully faithful functor with essential image consisting of cartesian objects. In particular we have the descent spectral sequence

$$E_1^{pq} = H^q(\mathcal{S}_p, a_p^*C^\bullet) \;\Rightarrow\; H^{p+q}(\mathcal{S}', C^\bullet).$$

Lemma 2.1.2 *Let $f\colon (X, \mathcal{B}_X) \to (Y, \mathcal{B}_Y)$ be a fibration in $\mathbf{BTop}_c$, then f is a descent map in $\mathcal{O}$-modules.*

Proof: By assumption, there exists a cover of bornological site $\mathcal{S}(Y, \mathcal{B}_Y)$ trivializing f, so we may assume that $(X, \mathcal{B}_X) = (Y, \mathcal{B}_Y) \times (F, \mathcal{B}_F)$ for some bornological space $(F, \mathcal{B}_F)$. For a completely bornological $\mathcal{O}_Y$-module M we have

$$f^*M = \mathcal{O}_X \mathop{\widehat{\otimes}}_{\mathcal{O}_Y} M = \mathcal{O}_F \,\widehat{\otimes}\, M.$$

Therefore, $\mathcal{O}_Y \to \mathcal{O}_X$ is a faithfully flat map, and one can use the classical argument to show that it is a descent map.

□

2.2 Classifying stack of a group

Let G be a locally compact completely normal group. Customarily, we denote $BG_\bullet$ the simplicial space with components $BG_n = G^n$, where face and degeneracy maps are given by

$$\sigma_i(g_1, \ldots g_n) = (\ldots g_{i-1}, 1, g_{i+1}, \ldots), \quad \text{for } 0 \leqslant i \leqslant n;$$

$$\delta_i(g_1, \ldots g_n) = (\ldots, g_i g_{i+1}, \ldots), \quad \text{for } 0 < i < n;$$

$$\delta_0(g_1, \ldots g_n) = (g_2, \ldots, g_n), \quad \text{and} \quad \delta_n(g_1, \ldots g_n) = (g_1, \ldots, g_{n-1}).$$

Given a representation $\rho\colon G \to \mathrm{End}(V)$ in a topological (locally convex) vector space V, we denote by $C^\bullet(G, V)$ the cochain complex associated by Dold-Kan equivalence to the cosimplicial space $\mathrm{Hom}(BG_\bullet, V)$. Since sheaves of continuous functions $C(G^n)$ are acyclic, we have $H^\bullet(G, V) := H^\bullet(BG_\bullet, \widetilde{V}) \simeq H^\bullet(C^\bullet(G, V))$.

2.2.1 Observe, that functors U_r and U_l considered in the first section both preserve arbitrary products. Indeed, U_r has a left adjoint, and for U_l it follows from the fact that arbitrary products of compact spaces are compact. So unlike the Stone-Čech compactification βG that in general is not a group or even an associative monoid (see [FV]), both $G_c = U_l(G)$ and $G_t = U_r(G)$ are group objects in the category $\mathbf{BTop}_c$, as well as their underlying sober spaces of points.

Therefore, we have simplicial bornological spaces $BG_{c\bullet}$ and $BG_{t\bullet}$ obtained by applying U_l and U_r to the simplicial space $BG_\bullet$, and a natural map $BG_{c\bullet} \to BG_{t\bullet}$.

Remark 2.2.2 In general G_c is not a normal subgroup in G_t, as can be seen from the following example. Let $G = GL_2(\mathbb{R})$, and take X to be the unit interval $(0,1) \subset \mathbb{R}$ with trivial bornology. Consider maps $g\colon X \to G_t$ and $h\colon X \to G_c$, given by

$$g(x) = \begin{pmatrix} x^{-1} & 0 \\ 0 & x \end{pmatrix}, \qquad h(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}.$$

Then one can check that the conjugate

$$ghg^{-1} = \begin{pmatrix} 1 & x^{-1} \\ 0 & 1 \end{pmatrix} \notin G_c(X).$$

2.2.3 Let $(V, \mathcal{B}_V)$ be a bornological topological vector space, and equip $\mathrm{End}(V)$ with the $\mathcal{B}_V$-bornology. A map $\rho\colon G_t \to \mathrm{End}(V)$ in $\mathbf{BTop}_c$ turns V into a *uniformly bounded* representation of G. For example, if V is a Banach space with the norm bornology, then $\mathrm{End}(V)$ is given the operator norm bornology, and uniformly bounded representations are those for which

$$\sup_{g \in G} \|\rho(g)\| < \infty.$$

Assume in addition, that V has $\mathcal{B}$-approximation property, so that the sheaf $\widetilde{V} \in \mathrm{Sh}(G_t^n)$ represented by $(V, \mathcal{B}_V)$ is a quasi-coherent sheaf. According to proposition (1.3.4) $\widetilde{V}$ is acyclic, and we have

$$H_b^\bullet(G, V) := H^\bullet(BG_{t\bullet}, \widetilde{V}) = H^\bullet(C_b^\bullet(G_t, V)),$$

i.e., the bounded continuous cohomology in the sense of Borel ([Bo]). Also, from proposition (1.2.7) we see that

$$H^\bullet(BG_{c\bullet}, \widetilde{V}) \simeq H^\bullet(BG, \widetilde{V}) = H^\bullet(G, V).$$

So the comparison map $\Psi\colon H_b^\bullet(G, V) \to H^\bullet(G, V)$ that we are interested in, is induced by the inclusion $BG_{c\bullet} \hookrightarrow BG_{t\bullet}$.

2.2.4 Descent spectral sequence. Applying delooping construction to $G_c \to G_t \to G_t/G_c$ we obtain a fibration of simplicial bornological spaces

$$G_t \longrightarrow [G_t/G_c] \xrightarrow{\ p\ } BG_c.$$

Here G_t is the constant simplicial object with components G_t and $[G_t/G_c]$ the quotient stack, i.e., the simplicial space with $[G_t/G_c]_n = G_c^n \times G_t$. The descent spectral sequence for p starts with

$$E_1^{pq} = H^q(G_t^p \times [G_t/G_c]), \widetilde{V}) \;\Rightarrow\; H^{p+q}(BG_c, \widetilde{V}).$$

And since G_t is a group object in $\mathbf{BTop}_c$ we have

$$E_2^{pq} = H^p(G_t, H^q([G_t/G_c], \widetilde{V})) \;\Rightarrow\; H^{p+q}(BG_c, \widetilde{V}).$$

Notice that since G is a locally compact, completely normal group, it acts freely on the Stone-Čech compactification of G and therefore on $\mathrm{pt}(G_t)$ as well. Hence, the quotient $\mathrm{pt}(G_t)/G$ is a topological space. In virtue of proposition (1.2.7), the cohomology of $[G_t/G_c]$ is isomorphic to the cohomology of $\mathrm{pt}(G_t)/G$. In particular we see that vanishing of $H^i(\mathrm{pt}(G_t)/G, \widetilde{V})$ for $i \geqslant 1$ is a sufficient condition for the comparison map Ψ to be an isomorphism.

2.3 Discrete coefficients case

As an illustration of these methods we consider the case of discrete coefficients. Let A be an abelian group considered as a bornological space with discrete topology and compact bornology. We will show that in this case the comparison map $\Psi\colon H_b^\bullet(G, A) \to H^\bullet(G, A)$ is an isomorphism. The map $p\colon [G_t/G_c] \to BG_c$ is a fibration, hence it is a descent map for the sheaves of abelian groups as well, therefore from the spectral sequence in (2.2.4) it is enough to show vanishing of $H^i(\mathrm{pt}(G_t)/G, A)$, for $i \geqslant 1$.

Lemma 2.3.1 *For a discrete abelian group A and all $i \geqslant 1$, the cohomology*

$$H^i(\mathrm{pt}(G_t)/G, A) = 0.$$

Proof: The orbit of $G \subset \mathrm{pt}(G_t)$ is the generic point of $\mathrm{pt}(G_t)/G$. A constant sheaf on a space with a generic point is necessarily flasque, hence it is acyclic.

□

Corollary 2.3.2 *The comparison map $\Psi\colon H_b^\bullet(G, A) \to H^\bullet(G, A)$ is an isomorphism.*

Notice however, that this bounded cohomology is understood as the cohomology of the classifying space BG, rather than cohomology of the complex $C_b^\bullet(G, A)$ of bounded functions on G. Since we don't have partition of unity for discrete coefficients these two are generally different.

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