Vector fields of Bryant type for foliations and harmonic morphisms *

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Abstract

We find some new Bryant type vector fields for foliations on a Riemannian manifold and investigate the geometry of generalized Bryant type vector fields and bi-minimal distributions. We give necessary and sufficient conditions for a conformal foliation with 1-dimensional leaves in a constant curvature space to produce harmonic morphisms. Also we obtain a classification of p-harmonic morphisms with 1-dimensional fibres from a constant curvature Riemannian manifold for all $p \geq 2$.

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1 Introduction

Harmonic morphisms between Riemannian manifolds are maps which preserve germs of harmonic functions. By a basic result of Fuglede and Ishihara [6, 8] harmonic morphisms are harmonic maps which are horizontally weakly conformal. Hence harmonic morphisms are a class of harmonic maps with a strong geometric flavour and thier fibres are conformal foliations.

A very fruitful idea, due to J.C.Wood, [17], was to place the study of harmonic morphisms into the framework of conformal foliations. In order to capture all the harmonic morphisms which can be defined (maybe just locally) on a given Riemannian manifold it is thus of importance to find conditions on a foliation which ensure its leaves can be given locally as

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fibres of (submersive) harmonic morphisms [4,5,12,14,15,17], i.e. *foliations which produce harmonic morphisms.* In [5], R.L. Bryant showed that (we call simply it Bryant's criterion) any conformal foliation $\mathcal{V}$ of codimension not equal to two produces harmonic morphisms if and only if the vector field

$$W := (n-2)\mathrm{trace}({}^{\mathcal{H}}B) - n\mathrm{trace}({}^{\mathcal{V}}B) \tag{1.1}$$

is locally a gradient vector field where $\mathcal{H}$ is the orthogonal complement of $\mathcal{V}$, n is the codimension of $\mathcal{V}$, and ${}^{\mathcal{H}}B$ (resp.${}^{\mathcal{V}}B$) is the second fundamental form of $\mathcal{H}$ (resp. $\mathcal{V}$)[17]. An alternative description is that $\mathcal{V}$ produces harmonic morphisms if and only if the 1-form W^b is closed where ${}^b : TM \to T^*M$ is the musical isomorphism[4].

On a two-dimensional Riemannian manifold a foliation (of dimension one) is locally generated by conformal vector fields if and only if it produces harmonic morphisms. This follows from the fact that a harmonic morphism to a one-dimensional Riemannian manifold is essentially a harmonic function and, if the domain is two-dimensional, then any harmonic function is locally the real part of a conformal map. Together with Bryant's beautiful criterion we have that a foliation on a Riemannian surface is locally generated by conformal vector fields if and only if the 1-form $(\mathrm{trace}({}^{\mathcal{H}}B) + \mathrm{trace}({}^{\mathcal{V}}B))^b$ is closed. In this paper we consider *any* dimensional Riemannian manifold and show that

Proposition 1.1 (Observed also by R. Pantilie[16]). *Let $\mathcal{V}$ be a one-dimensional foliation on the Riemannian manifold (M, g) and let $\mathcal{H}$ be its orthogonal complement. Then $\mathcal{V}$ is locally generated by conformal vector fields if and only if it is conformal and the 1-form U^b is closed where*

$$U := \mathrm{trace}({}^{\mathcal{H}}B) + n\mathrm{trace}({}^{\mathcal{V}}B) \tag{1.2}$$

and $n = \mathrm{codim}\mathcal{V}$.

In the general case, let $\mathcal{V}$ be a distribution on a Riemannian manifold (M, g) and κ a non-zero real number, we define the κ*-form of $\mathcal{V}$ with respect to g* by

$$\theta_\kappa = (\kappa\mathrm{trace}({}^{\mathcal{H}}B) + \mathrm{trace}({}^{\mathcal{V}}B))^b.$$

It gives a measure of the failure of $\mathcal{V}$ to be a *bi-minimal distribution*, i.e. both $\mathcal{V}$ and $\mathcal{H}$ are minimal. In fact, $\theta_\kappa = 0$ if and only if $\mathrm{trace}({}^{\mathcal{H}}B) = \mathrm{trace}({}^{\mathcal{V}}B) = 0$. Recall that $\mathcal{V}$ is said to be *minimal* if $\mathrm{trace}({}^{\mathcal{V}}B) = 0$[17]. Let $n = \dim\mathcal{H}$ and $m = \dim M$. We show that (see Section 4)

Proposition 1.2. *For any smooth function f on M, $\theta_\kappa + df$ is $\kappa-$form of $\mathcal{V}$ with respect to $\tilde{g}$ where*

$$\tilde{g} := e^{-\frac{2f}{m-n}} g|_{\mathcal{V}} + e^{-\frac{2f}{\kappa n}} g|_{\mathcal{H}}$$

and $g|_{\mathcal{H}}$ (resp. $g|_{\mathcal{V}}$) is the horizontal (resp. the vertical) component of g.

We say that $\tilde{g}$ is obtained by *$\kappa-$deformation of g*. From Proposition 1.2, we get the following (see Section 4):

Corollary 1.3. *let $\mathcal{V}$ be a distribution on the Riemannian manifold (M, g). Then the $\kappa-$form θ_κ is closed if and ony if $\mathcal{V}$ is locally bi-minimal with respect to some $\kappa-$deformed metric on M. In particular, if M has zero first Betti number then $\mathcal{V}$ is (globally) bi-minimal with respect to some $\kappa-$deformed metric on M.*

In the special case that (M, g) is a constant curvature space and $\mathcal{V}$ a one-dimensional conformal foliation, we have following (see Section 5):

Theorem 1.4. *Let $n \geq 3$ and (M^{n+1}, g) be a Riemannian manifold with constant sectional curvature and $\mathcal{V}$ a one-dimensional conformal foliation on (M, g). If θ_κ is closed for some $\kappa \neq \frac{1}{n}$ then $\mathcal{V}$ or $\mathcal{H}$ is totally geodesic where $\mathcal{H}$ is the orthogonal complement of $\mathcal{V}$.*

This result can be viewed as an extension of Bryant's observation that for $n \geq 3$ a harmonic morphism from an $(n+1)$-dimensional constant curvature space with one-dimensional fibres has either geodesic fibres or totally geodesic horizontal distribution[5]. Together with Bryant's criterion[5] we get (see Section 5)

Theorem 1.5. *Let $n \geq 3$ and $\mathcal{V}$ be a conformal foliation with 1-dimemsional leaves in a constant curvature space (M^{n+1}, g). Then, the following assertions are equivalent:*

(i) $\mathcal{V}$ produces harmonic morphisms,

(ii) the $\kappa-$form of $\mathcal{V}$ is closed for some $\kappa \neq \frac{1}{n}$,

(iii) the $\kappa-$form of $\mathcal{V}$ is closed for all κ.

Note that Theorem 1.5 fails if (M^{n+1}, g) is an Einstein manifold, for example a foliation arises from the type (iii) harmonic morphisms ([13]) are not homothetic foliation (please see Section 2 for the definition of homothetic foliation).

Viewing harmonic morphisms as maps which pull back (local) harmonic funcations to (local) harmonic functions, their most natural generalisation is the notion of p-harmonic morphisms, i.e., maps which pull back (local) p-harmonic funcations to (local) p-harmonic functions. Recently a Fuglede-Ishihara type characterisation for p-harmonic morphisms has been obtained in [10]. A foliation on a Riemannian manifold is said to *produce p-harmonic morphisms* if its leaves can be given locally as fibres of (submersive) p-harmonic morphisms. The following characterization is a natural extension of Bryant criterion (see Proposition 6.1 in Section 6):

Proposition 1.6. *Let $\mathcal{V}$ be a conformal foliation on (M, g) of codim$\mathcal{V} \neq p$ and $\mathcal{H}$ its orthogonal complement. Then $\mathcal{V}$ produces p-harmonic morphisms if and only if the vector field*

$$W_p := (n-p)\mathrm{trace}({}^{\mathcal{H}}B) - n\mathrm{trace}({}^{\mathcal{V}}B)$$

is locally a gradient vector field where $n =$ codim $\mathcal{V}$.

Combine with Proposition 1.1 we get the following:

Theorem 1.7. *Let $\mathcal{V}$ be a one-dimensional foliation on an $(n+1)$-dimensional Riemannian manifold (M, g). Then fibres of $\mathcal{V}$ are locally given by the integral curves of conformal fields if and only if $\mathcal{V}$ produces $(n+1)$-harmonic morphisms.*

Notice that for $n = 1$ we obviously recover harmonic morphisms, and get the above-mentioned basic fact. We also show minimal foliations can be characterized by n-harmonic morphisms which extend a result due to J.C.Wood[17] (see Proposition 7.2). Together with Theorem 1.5, Proposition 1.6, Theorem 1.7 and a classification theorem due to R.Bryant[5] we have the following classification result:

Theorem 1.8. *Let $n \geq 3$ and (M, g) be an $(n+1)$-dimensional Riemannian manifold with constant sectional curvature and $\mathcal{V}$ a one-dimensional conformal foliation on (M, g). Then $\mathcal{V}$ produces p-harmonic morphism if and only if*

(i) $\mathcal{V}$ has geodesic leaves for $p = n$,

(ii) $\mathcal{V}$'s fibres are locally given by the integral curves of conformal vector fields for $p = n+1$,

(iii) $\mathcal{V}$ is a Riemannian foliation locally generated by Killing vectors fields or homothetic foliations by geodesics orthogonal to an umbilical foliation by hypersurfaces for p not equal to n and $n+1$.

This paper is organized as follows: In Section 2 and 6 we construct some new Bryant type vector fields. Section 3 we study the properties of Riemannian foliations (please see Section 2 for the definition of Riemannian foliation). In Section 4 we discuss bi-minimal distribution by using generalized Bryant type vector fields. Section 5 we mainly investigate one dimensional conformal foliation on a constant curvature space. Finaly, in Section 6 and 7 we characterize conformal actions and minimal foliations by using p-harmonic morphisms.

2 Vector fields of Bryant type (I)

In this section we establish vector fields of Bryant type for one-dimensional foliations which are locally given by conformal vector fields.

Lemma 2.1. *Let $\mathcal{V}$ be a one-dimensional foliation on (M^{n+1}, g) and let $\mathcal{H}$ be its orthogonal complement. Then U defined in (1.2) can be written by*

$$U = \nabla \log \lambda^n - n\lambda^2 \Sigma_\alpha (\mathcal{L}_V g)(V, e_\alpha) e_\alpha + \frac{\lambda^2}{2}[n\lambda^2 (\mathcal{L}_V g)(V, V) - \Sigma_\alpha (\mathcal{L}_V g)(e_\alpha, e_\alpha)]V \quad (2.1)$$

where λ is a locally smooth positive function, V a local vertical field such that $g(V, V) = \lambda^{-2}$ and $\{e_\alpha\}$ a local orthonormal frame for $\mathcal{H}$.

Proof. A simple calculation (see [17]) gives the following formula:

$$(\mathcal{L}_V g)(X, X) = -2g({}^{\mathcal{H}}B(X, X), V) \quad (2.2)$$

where X is a horizontal vector field. By reversing roles of $\mathcal{V}$ and $\mathcal{H}$ we have

$$(\mathcal{L}_X g)(V, V) = -2g({}^{\mathcal{V}}B(V, V), X). \quad (2.3)$$

By simple calculations, we have

$$(\mathcal{L}_V g)(V, V) = V(\lambda^{-2}) \quad (2.4)$$

and

$$(\mathcal{L}_V g)(V, X) = -g(V, [V, X]). \quad (2.5)$$

From (2.4) we have

$$\begin{aligned} {}^{\mathcal{V}}(\nabla \log \lambda) &= \tfrac{1}{2\lambda^2}{}^{\mathcal{V}}(\nabla \lambda^2) \\ &= \tfrac{1}{2} g(\nabla \lambda^2, V)V \\ &= \tfrac{1}{2} V(\lambda^2) V \\ &= -\tfrac{\lambda^4}{2} V(\lambda^{-2})V = -\tfrac{\lambda^4}{2}(\mathcal{L}_V g)(V, V)V. \end{aligned} \quad (2.6)$$

Using (2.5) we get

$$
\begin{aligned}
[\mathcal{L}_X(\lambda^2 g)](V,\,V) &= [\mathcal{L}_X(\lambda^2)]g(V,\,V)+\lambda^2(\mathcal{L}_X g)(V,\,V)\\
&= [X(\lambda^2)]\tfrac{1}{\lambda^2}+\lambda^2[\mathcal{L}_X(g(V,\,V))-2g(\mathcal{L}_X V,\,V)]\\
&= \tfrac{1}{\lambda^2}X(\lambda^2)+\lambda^2 X(\tfrac{1}{\lambda^2})+2\lambda^2 g(V,\,[V,\,X])\\
&= 2\lambda^2(\mathcal{L}_V g)(V,\,X).
\end{aligned}
\tag{2.7}
$$

It follows that

$$
\begin{aligned}
\mathrm{trace}({}^{\mathcal{V}}B) &= \lambda^{2\mathcal{V}}B(V,\,V)\\
&= \lambda^2\Sigma g({}^{\mathcal{V}}B(V,\,V),\,e_\alpha)e_\alpha\\
&= -\tfrac{\lambda^2}{2}\Sigma(\mathcal{L}_{e_\alpha}g)(V,\,V)e_\alpha\\
&= -\tfrac{1}{2}\Sigma[\mathcal{L}_{e_\alpha}(\lambda^2 g)](V,\,V)e_\alpha+\tfrac{1}{2}\Sigma e_\alpha(\lambda^2)g(V,\,V)e_\alpha\\
&= \tfrac{1}{2\lambda^2}\Sigma e_\alpha(\lambda^2)e_\alpha-\lambda^2\Sigma(\mathcal{L}_V g)(V,\,e_\alpha)e_\alpha\\
&= \tfrac{1}{2}{}^{\mathcal{H}}(\nabla\log\lambda^2)-\lambda^2\Sigma(\mathcal{L}_V g)(V,\,e_\alpha)e_\alpha\\
&= \nabla\log\lambda+\tfrac{\lambda^4}{2}(\mathcal{L}_V g)(V,\,V)V-\lambda^2\Sigma(\mathcal{L}_V g)(V,\,e_\alpha)e_\alpha
\end{aligned}
\tag{2.8}
$$

from (2.3),(2.6) and (2.7). On the other hand from (2.1) it is easy to see that

$$
\mathrm{trace}({}^{\mathcal{H}}B)=-\frac{\lambda^2}{2}\Sigma[(\mathcal{L}_V g)(e_\alpha,\,e_\alpha)]V. \tag{2.9}
$$

Now our conclusion can be obtained from (2.8) and (2.9) immediately.

Say that a distribution $\mathcal{V}$ is (a) *Riemannian*, (resp. (b) *conformal*) according as

$$
\begin{array}{lrcl}
 & (a)\quad (\mathcal{L}_V g)(X,\,Y) &=& 0\\
(\text{resp.} & (b)\quad (\mathcal{L}_V g)(X,\,Y) &=& \mu(V)g(X,\,Y))
\end{array}
$$

for all $x\in M,\ X,\,Y\in\mathcal{H}_x,\ V\in\mathcal{V}_x$ where $\mu(V)$ is a non-negative number which depends only on V and not on X and Y. Suppose that $\mathcal{V}$ is integrable and hence determines a foliation $\mathcal{F}$. The $\mathcal{F}$ is *Riemannian*, (resp. *conformal*) if $\mathcal{V}$ is Riemannian, (resp. conformal) or equivalently if its leaves are locally the fibres of a Riemannian (resp. horizontally conformal) submersion. We say that a distribution $\mathcal{V}$ is *homothetic* if it is conformal and the mean curvature of its orthogonal complement is locally a gradient field[12]. A distribution $\mathcal{V}$ is said to be (i) *umbilic* if, for each $x\in M$, the normal curvature in direction X is independent of $X\in\mathcal{V}_x,\ g(X,\,X)=1$; (ii)*totally geodesic* if all normal curvatures are zero. Then

Proposition 2.2 (cf.[17]). *A distrbution is (1) Riemannian, (2) conformal if and only if its orthogonal complement distribution is (1) totally geodesic, (2) umbilic.*

Proof of Proposition 1.1. Let V be a local conformal vector field tangent to the leaves. Then

$$\mathcal{L}_V g = \mu(V) g \tag{2.10}$$

for some local function $\mu(V)$ which depends only on V. It is clear that $\mathcal{L}_V g|_{\mathcal{H}\times\mathcal{H}}$ is linear with respect to V. This implies that $\mathcal{V}$ is comformal. Furthermore from (2.10) we have

$$(\mathcal{L}_V g)(V,\, e_\alpha) = 0, \tag{2.11}$$

$$(\mathcal{L}_V g)(V,\, V) = \mu(V)\lambda^{-2}, \tag{2.12}$$

$$(\mathcal{L}_V g)(e_\alpha,\, e_\alpha) = \mu(V) \tag{2.13}$$

for a local orthonormal frame $\{e_\alpha\}$ for $\mathcal{H}$. Substitute (2.11)-(2.13) into (2.1) we have that $U = \nabla \log \lambda^n$ where $\lambda = [g(V,\, V)]^{-2}$.

Conversely, we assume that locally $U = \nabla\xi$. Let V be a local vertical vector field such that $g(V,\, V) = e^{-\frac{2\xi}{n}}$. Then

$$U = \nabla \log[g(V,\, V)]^{-2n}.$$

Compare it with (2.1) we get

$$\mu(V) := \frac{1}{g(V,\, V)}(\mathcal{L}_V g)(V,\, V) = \frac{1}{n}\Sigma(\mathcal{L}_V g)(e_\alpha,\, e_\alpha) \tag{2.14}$$

and

$$(\mathcal{L}_V g)(V,\, e_\alpha) = 0 \tag{2.15}$$

for a local orthonormal frame $\{e_\alpha\}$ for $\mathcal{H}$. Now (2.14) and (2.15) implies that $\mathcal{L}_V g = \mu(V) g$ because $\mathcal{V}$ is conformal.

Definition 2.3. Let $\mathcal{V}$ be a foliation on the (connected) Riemannian manifold $(M,\, g)$. $\mathcal{V}$ is said to *produce harmonic morphisms on* $(M,\, g)$ if each point of M has an open neighbourhood O which is the domain of a submersive harmonic morphism $\phi : (O,\, g|_O) \to (N,\, h)$ whose fibres are open subsets of the leaves of $\mathcal{V}$.

By Proposition 1.1 and Bryant's criterion we obtain the following immediately:

Proposition 2.4. ([12, Proposition 2.5]) *For $n \geq 3$, let $\mathcal{V}$ be a one-dimensional foliation on $(M^{n+1},\, g)$. Then any two of the following assertions imply the remaing assertion.*

(i) *$\mathcal{V}$ produces harmonic morphisms,*

(ii) *$\mathcal{V}$ (or $\mathcal{H}$) is homothetic,*

(iii)*$\mathcal{V}$ is locally generated by conformal vector fields.*

3 Riemannian foliations

Let U and W be vector fields defined in (1.2) and (1.1) respectively. Notice that for $\dim \mathcal{V} = \mathrm{codim}\mathcal{V} = 1$ we have the following:

$$U + W = 0. \tag{3.1}$$

In general, we get

Proposition 3.1. *On a Riemannian manifold a conformal folition $\mathcal{V}$ satisfies* (3.1) *if and only if $\mathcal{V}$ is Riemannian or* $\mathrm{codim}\mathcal{V} = 1$.

Proof. Our conclusion is an immediate consequence of Proposition 2.2 and the following equation

$$U + W = (\mathrm{codim}\mathcal{V} - 1)\mathrm{trace}({}^{\mathcal{H}}B).$$

Combine Proposition 1.2, Proposition 3.1 and Bryant's criterion we have

Corollary 3.2. *For $n \neq 2$, let $\mathcal{V}$ be a one-dimensional Riemannian foliation on (M^{n+1}, g) and let $\mathcal{H}$ be its orthogonal complement. Then $\mathcal{V}$ produces harmonic morphisms (cf. Definition 2.3) if and only if it is locally generated by conformal vector fields.*

It is easy to show that (cf. Proposition 3.3 below) any Riemannain foliation locally generated by conformal vector fields is locally generated by Killing fields. The converse also holds. Hence Corollary 3.2 is a different expression of Bryant and Pantilie' observation[5][12, Proposition 2.3]. Our method gives an alternative proof of Bryant and Pantilie' result.

Proposition 3.3. *Let $\mathcal{V}$ be a Riemannian foliation on the Riemannian manifold (M, g). Then $\mathcal{V}$ locally generated by Killing vector fields (i.e. in the neighbourhood of each point a local frame for $\mathcal{V}$ made up of Killing vector fields can be found) if and only if*

(i) *$\mathcal{V}$ is Riemannian, and*

(ii) *$\mathcal{V}$ is locally generated by conformal vector fields.*

Furthermore, if $\dim\mathcal{V} = 1$ then (ii) *above can be replaced by*

(ii') *$\mathcal{H}$ is homothetic.*

Proof. We need to prove just the "Furthermore" part.

In the case of $\dim \mathcal{V} = 1$, $\mathcal{H}$ is automatically conformal. Moreover $\mathcal{V}$ is Riemannian implies that $\mathcal{H}$ is minimal. Hence $dU^b = 0$ if and only if $d[\mathrm{trace}({}^{\mathcal{V}}B)]^b = 0$.

4 Bi-minimal distributions

Let (M, g) be an m-dimensional Riemannian manifold and $\mathcal{V}$ an $(m-n)$-dimensional distribution on it. Let $\mathcal{H}$ be the orthogonal complement of $\mathcal{V}$. As before we shall denote the orthogonal projections onto $\mathcal{H}$ and $\mathcal{V}$ by the same letters ${}^{\mathcal{H}}$ and ${}^{\mathcal{V}}$. Similarly for their dual distributions $\mathcal{H}^*$ and $\mathcal{V}^*$.

In this section, we agree to use the following ranges of indices:

$$1 \leq \alpha, \beta, \gamma, \cdots \leq n, \quad 1 \leq A, B, C, \cdots \leq m, \quad n+1 \leq i, j, k, \cdots \leq m.$$

Let $r, s : M \to (0, \infty)$ be smooth functions. Denote by g_{rs} the Riemannian metric on M defined by $g_{rs} = r^2 g|_{\mathcal{V}} + s^2 g|_{\mathcal{H}}$ where $g|_{\mathcal{V}}$ and $g|_{\mathcal{H}}$ denote the vertical and the horizontal parts of g.

Lemma 4.1. *Set $\tilde{g} = g_{rs}$ then*

$$r^{2\mathcal{V}}B = s^2 \widetilde{{}^{\mathcal{V}}B} + r^{2\mathcal{H}}(\nabla \log r) g|_{\mathcal{V}} \tag{4.1}$$

where $\widetilde{{}^{\mathcal{V}}B}$ is the second fundamental form of $\mathcal{V}$ with respect to $\tilde{g}$.

Proof. Let $\{e_\alpha\}$ and $\{e_i\}$ denote frames of $\mathcal{H}$ and $\mathcal{V}$, respectively, orthonormal with respect to g, and let $\{\omega_\alpha\}$ and $\{\omega_i\}$ denote their dual frames. We denote the corresponding objects with respect to $\tilde{g}$ by adding a tilde ˜. Thus

$$\tilde{\omega}_\alpha = s\omega_\alpha, \qquad \tilde{\omega}_i = r\omega_i. \tag{4.2}$$

From the first structure equation we have

$$d\omega_A = -\Sigma \omega_{AB} \wedge \omega_B \tag{4.3}$$

where $\{\omega_{AB}\}$ is the Riemannian connection $1-$form of g. By using (4.2) and (4.3) it is easy to show that

$$\Sigma(\delta_{\alpha\beta} d\log s + \tilde{\omega}_{\alpha\beta} - \omega_{\alpha\beta}) \wedge \omega_\beta + \Sigma(\frac{r}{s}\tilde{\omega}_{\alpha i} - \omega_{\alpha i}) \wedge \omega_i = 0. \tag{4.4}$$

By reversing roles of $\mathcal{V}$ and $\mathcal{H}$ we have

$$\Sigma(\delta_{ij} d\log r + \tilde{\omega}_{ij} - \omega_{ij}) \wedge \omega_j + \Sigma(\frac{s}{r}\tilde{\omega}_{i\alpha} - \omega_{i\alpha}) \wedge \omega_\alpha = 0. \tag{4.5}$$

From (4.4) and (4.5) ones get

$$\Sigma_B \psi_{AB} \wedge \omega_B = 0$$

where

$$(\psi_{AB}) = \begin{pmatrix} \delta_{\alpha\beta} d\log s + \tilde{\omega}_{\alpha\beta} - \omega_{\alpha\beta} & \frac{r}{s}\tilde{\omega}_{\alpha i} - \omega_{\alpha i} \\ \frac{s}{r}\tilde{\omega}_{i\beta} - \omega_{i\beta} & \delta_{ij} d\log r + \tilde{\omega}_{ij} - \omega_{ij} \end{pmatrix}. \tag{4.6}$$

By Cartan's lemma, we have

$$\psi_{AB} = \Sigma_C \xi_{ABC} \omega_C, \qquad \xi_{ABC} = \xi_{ACB}. \tag{4.7}$$

Using (4.2) and (4.6) we have

$$\begin{aligned} \widetilde{{}^{\mathcal{V}}B} &\equiv \tfrac{1}{2}\Sigma_{i,\alpha}[\tilde{\omega}_{\alpha i} \otimes \tilde{\omega}_i + \tilde{\omega}_i \otimes \tilde{\omega}_{\alpha i}] \otimes \tilde{e}_\alpha \quad \text{mod } \tilde{\omega}_\alpha \\ &= -\tfrac{1}{2}\Sigma_{i,\alpha}[\tilde{\omega}_{i\alpha} \otimes \tilde{\omega}_i + \tilde{\omega}_i \otimes \tilde{\omega}_{i\alpha}] \otimes \tilde{e}_\alpha \\ &= -\tfrac{1}{2}\Sigma_{i,\alpha}[\tfrac{r}{s}(\omega_{i\alpha} + \psi_{i\alpha}) \otimes r\omega_i + r\omega_i \otimes \tfrac{r}{s}(\omega_{i\alpha} + \psi_{i\alpha})] \otimes \tfrac{1}{s} e_\alpha \\ &= \tfrac{r^2}{s^2}{}^{\mathcal{V}}B + (I) \end{aligned} \tag{4.8}$$

where

$$\begin{aligned} (I) &\equiv -\tfrac{1}{2}\Sigma_{i,\alpha}\tfrac{r^2}{s^2}[\psi_{i\alpha} \otimes \omega_i + \omega_i \otimes \psi_{i\alpha}] \otimes e_\alpha \quad \text{mod } \omega_\alpha \\ &= -\tfrac{r^2}{2s^2}\Sigma_{i,\alpha}[\Sigma\xi_{i\alpha j}\omega_j \otimes \omega_i + \omega_i \otimes \Sigma\xi_{i\alpha j}\omega_j] \otimes e_\alpha \\ &= -\tfrac{r^2}{2s^2}\Sigma_{i,j,\alpha}[\xi_{i\alpha j} + \xi_{j\alpha i}]\omega_i \otimes \omega_j \otimes e_\alpha \\ &= -\tfrac{r^2}{2s^2}\Sigma_{i,j,\alpha}[\xi_{ij\alpha} + \xi_{ji\alpha}]\omega_i \otimes \omega_j \otimes e_\alpha \end{aligned} \tag{4.9}$$

by using (4.7). Note that both $\tilde{\omega}_{ij}$ and ω_{ij} are skew-symmetric with respect to i, j, (4.6) and (4.7) deduce that

$$\begin{aligned} \xi_{ij\alpha} + \xi_{ji\alpha} &= (\psi_{ij} + \psi_{ji})(e_\alpha) \\ &= (\delta_{ij} + \delta_{ji})(d\log r)(e_\alpha) \\ &= 2\delta_{ij} e_\alpha(\log r). \end{aligned} \tag{4.10}$$

Substitute it into (4.9) we get

$$(I) = -\frac{r^2}{s^2}{}^{\mathcal{H}}(\nabla \log r) g|_{\mathcal{V}}. \tag{4.11}$$

Now (4.1) is an immediate conclusion of (4.8) and (4.11).

Proposition 4.2. *Let $\mathcal{V}$ a distribution on a Riemannian manifold (M, g). Denote by g_{rs} the Riemannian metric on M defined by $g_{rs} = r^2 g|_{\mathcal{V}} + s^2 g|_{\mathcal{H}}$ where r, $s : M \to (0, \infty)$ be smooth functions. Then*

(i) *$\mathcal{V}$ is an umbilical foliation on (M, g) if and only if $\mathcal{V}$ is an umbilical foliation on (M, g_{rs}),*

(ii) $\mathcal{V}$ *is a conformal foliation on* (M, g) *if and only if* $\mathcal{V}$ *is a conformal foliation on* (M, g_{rs}).

Proof. (i) Let $\tilde{V}$ be a vertical vector in T_xM such that $g_{rs}(\tilde{V}, \tilde{V}) = 1$. Put $V := r\tilde{V}$. Then $g(V, V) = 1$. By using (4.1) we get

$$\zeta(V, V) := {}^{\mathcal{V}}B(V, V) = s^2\tilde{\zeta}(\tilde{V}, \tilde{V}) + {}^{\mathcal{H}}(\nabla \log r)$$

where $\zeta(V, V)$ denotes the normal curvature of V [17].

(ii) By reversing roles of $\mathcal{V}$ and $\mathcal{H}$ and using Proposition 2.2.

Lemma 4.3.

$$\mathrm{trace}({}^{\mathcal{V}}B) = s^2\widetilde{\mathrm{trace}({}^{\mathcal{V}}B)} + {}^{\mathcal{H}}(\nabla \log r^{m-n}). \tag{4.12}$$

Proof. It is an immediate conclusion of (4.1) and (4.2).

Call a smooth map $\phi : (M, g) \to (N, h)$ between Riemannian manifolds *horizontally (weakly) conformal* if for any point $x \in M$ which is not contained in the critical set $C_\phi = \{x \in M | d\phi_x = 0\}$ of ϕ, the restriction of $d\phi_x$ to the orthogonal complement

$$\{X \in T_xM | g(X, Y) = 0 \text{ for all} \, Y \in \mathrm{Ker}\ d\phi_x\}$$

of $\mathrm{Ker}\, d\phi_x$ is surjective and conformal onto tangent space $T_{\phi(x)}N$.

Proposition 4.4. *Let* $\phi : (M^m, g) \to (N^n, h)$ *be a horizontally conformal submersion with dilation* λ, *i.e.* $\phi^*h = \lambda^2 g|_{\mathcal{H}}$. *Then we have*

$$\tau(\phi) + s^2\widetilde{\mathrm{trace}({}^{\mathcal{V}}B)} = 0 \tag{4.13}$$

for any positive smooth function $s : M \to \mathbb{R}$ *where*

$$\tilde{g} := \lambda^{\frac{4-2n}{m-n}} g|_{\mathcal{V}} + s^2 g|_{\mathcal{H}}$$

and $\tau(\phi) := \mathrm{trace}\nabla d\phi$ is the tension field of ϕ.

Proof. Set $r := \lambda^{\frac{2-n}{m-n}}$. Then, from (4.12)

$$\mathrm{trace}({}^{\mathcal{V}}B) = s^2\widetilde{\mathrm{trace}({}^{\mathcal{V}}B)} + (2-n){}^{\mathcal{H}}(\nabla \log \lambda).$$

Substitute it into the *fundamental equation* (see (6.3) below) we get (4.13).

Proposition 4.5. *Let $\phi : (M^m, g) \to (N^n, h)$ be a horizontally conformal submersion with dilation λ. Then ϕ is a harmonic morphism if and only if the fibres of ϕ are minimal submanifolds of $(M, g_{\lambda^{\frac{4-2n}{m-n}} s})$ for some smooth positive function s on M if and only if the fibres of ϕ are minimal submanifolds of $(M, g_{\lambda^{\frac{4-2n}{m-n}} s})$ for any smooth positive function s on M.*

In particular, when dim$N = 2$, we take $s \equiv 1$ and get the following([2])

Corollary 4.6. *Let $\phi : (M, g) \to (N, h)$ be a horizontally conformal submersion to a Riemannian surface. Then ϕ is a harmonic morphism if and only if the fibres of ϕ are minimal.*

Corollary 4.7 [12]. *Let $\phi : (M, g) \to (N, h)$ be a horizontally conformal submersion with dilation λ. Then ϕ is a harmonic morphism if and only if the parallel displacement defined by the horizontal distribution preserves the mass of the fibres, where the fibres are given the mass density λ^{2-n}.*

Prooof From remark (3.6) in [17], for any basic vector field X on M we have

$$^{\mathcal{V}^*}(\mathcal{L}_X \tilde{\omega}) = -g(X, \widetilde{\mathrm{trace}(^{\mathcal{V}}B)})\tilde{\omega} \tag{4.14}$$

where $\tilde{\omega}$ is local volume forms of the fibres of ϕ with respect to $g_{\lambda^{\frac{4-2n}{m-n}} s}$. On the other hand, substituting $r = \lambda^{\frac{4-2n}{m-n}}$ into (4.2) we get

$$\tilde{\omega} = \lambda^{2-n}\omega$$

where ω is local volume forms of the fibres of ϕ with respect to g.

Lemma 4.8. *Let $\omega^{\mathcal{V}}$ be the mean curvature form of $\mathcal{V}$, i.e.* $(\mathrm{trace}(^{\mathcal{V}}B))^b$. *Then*

$$\tilde{\omega}^{\mathcal{V}} = \omega^{\mathcal{V}} - {}^{\mathcal{H}^*}d\log r^{m-n}. \tag{4.15}$$

Proof. A direct calculation shows that

$$\left(^{\mathcal{H}}(\nabla f)\right)^b = {}^{\mathcal{H}^*}df, \quad X^{\tilde{b}} = s^2 X^b \tag{4.16}$$

where f is a function on M and X a horizontal vector. Using (4.12) and (4.16) we have

$$\begin{aligned}\tilde{\omega}^{\mathcal{V}} &= [\widetilde{\mathrm{trace}({}^{\mathcal{V}}B)}]^{\tilde{b}} \\ &= \tfrac{1}{s^2}[\mathrm{trace}({}^{\mathcal{V}}B) - {}^{\mathcal{H}}(\nabla \log r^{m-n})]^{\tilde{b}} \\ &= \omega^{\mathcal{V}} - {}^{\mathcal{H}^*}d\log r^{m-n}.\end{aligned}$$

Definition 4.9. Let $\mathcal{V}$ be a distribution on (M, g). For a non-zero real number κ we define the *κ−form of $\mathcal{V}$ with respect to g* by

$$\theta_\kappa := \kappa\omega^{\mathcal{H}} + \omega^{\mathcal{V}} \tag{4.17}$$

where, $\omega^{\mathcal{H}}$ (resp. $\omega^{\mathcal{V}}$) is the mean curvature form of $\mathcal{H}$ (resp. $\mathcal{V}$) and $\mathcal{H}$ is the orthogonal complement of $\mathcal{V}$.

Remark 4.10. Let $n = \dim\mathcal{H}$. Then $W^b = -n\theta_{\frac{2-n}{n}}$ for $n \neq 2$ and $U^b = n\theta_{\frac{1}{n}}$ from (1.1) and (1.2).

Proof of Proposition 1.2. In (4.15), by reversing roles of $\mathcal{V}$ and $\mathcal{H}$ we have

$$\tilde{\omega}^{\mathcal{H}} = \omega^{\mathcal{H}} - {}^{\mathcal{V}^*}d\log s^n. \tag{4.18}$$

Take $r = e^{-\frac{f}{m-n}}$, and $s = e^{-\frac{f}{\kappa n}}$ then, form (4.15) and (4.18)

$$\tilde{\omega}^{\mathcal{V}} = \omega^{\mathcal{V}} + {}^{\mathcal{H}^*}df, \quad \tilde{\omega}^{\mathcal{H}} = \omega^{\mathcal{H}} + \frac{1}{\kappa}{}^{\mathcal{V}^*}df.$$

Hence $\tilde{\theta}_\kappa = \theta_\kappa + df$.

Proof of Corollary 1.3. Suppose that θ_κ is closed, then locally

$$\theta_\kappa = -df$$

where f is a function on an open subset O of M. By using Proposition 1.2 we have $\tilde{\theta}_\kappa = 0$ on O where

$$\tilde{g} := e^{-\frac{2f}{m-n}}g|_{\mathcal{V}} + e^{-\frac{2f}{\kappa n}}g|_{\mathcal{H}}.$$

Converse is obvious.

If M has zero first Betti number, then each closed form on M is excat, hence $\mathcal{V}$ is (globally) bi-minimal with respect to some κ−deformed metric on M.

Remark 4.11. When θ_κ is bounded, that is,

$$sup_{g(Y,\,Y)=1}\theta_\kappa(Y) < 1$$

Then Finsler metric $F : TM \to R$ defined by $F(Y) := \sqrt{g(Y,\,Y)} + \theta_\kappa(Y)$ is a Runders metric. Furthermore θ_κ is closed if and only if Runders space $(M,\,F)$ has the same geodesics as $(M,\,g)$ [9].

We will use the following conclusions in the next section, they are easy to prove.

Proposition 4.12. *Let $\phi : (M,\,g) \to (N,\,h)$ be a horizontally conformal submersion with dilation λ and θ_κ is the $\kappa-$form of its fibres. Then any two of the following assertions imply the remaining assertion:*
(i) *θ_κ is exact for some κ,*
(ii) *ϕ has minimal fibres,*
(iii) *λ is constant along horizontal curves.*

Proposition 4.13. *Let $\mathcal{V}$ be a conformal foliation with minimal leaves. Then θ_κ is closed for some κ if and only if $\mathcal{V}$ is homothetic.*

5 One-dimensional foliations

In this section we discuss one dimensional foliations. In particular, we will establish some criterions for deciding whether or not a one dimensional foliation in a constant curvature space produces harmonic morphisms. Let $\mathcal{V}$ be a conformal foliation on the Riemannian manifold $(M,\,g)$. A smooth positive function $\lambda : O \to \mathbb{R}$ on an open subset O of M is called a *local dilation* of $\mathcal{V}$ if $\mathcal{V}|_O$ is a Riemannian foliation on $(O,\,\lambda^2 g|_O)$[12].

Proposition 5.1. Suppose that $\mathcal{V}$ is a one-dimensional foliation on a Riemannian manifold $(M,\,g)$ and V a unit vector field which is tangent to $\mathcal{V}$. Then for any basic vector field X we have

$$\mathcal{L}_X V = g(X,\,\mathrm{trace}(^{\mathcal{V}}B))V.$$

Proof. It is easy to see that

$$^{\mathcal{V}^*}(\mathcal{L}_X V^b) = -g(X,\,\mathrm{trace}(^{\mathcal{V}}B))V^b \tag{5.1}$$

from (4.14). On the other hand, $\mathcal{L}_X V$ is vertical because X is basic. Hence it is sufficient to show that

$$(\mathcal{L}_X V)^b(V) = g(X, \, \mathrm{trace}(^{\mathcal{V}}B)).$$

In fact, by using (5.1)

$$(\mathcal{L}_X V)^b(V) = -^{\mathcal{V}^*}(\mathcal{L}_X V^b)(V) = g(X, \, \mathrm{trace}(^{\mathcal{V}}B)).$$

Corollary 5.2. *Let $\mathcal{V}$ is a one-dimensional foliation on a Riemannian manifold $(M, \, g)$ and $\mathcal{H}$ the orthogonal complement of $\mathcal{V}$. Then $\mathcal{H}$ is locally principal connection for $\mathcal{V}$ if and only if $\mathcal{H}$ exists local dilations.*

Proof. $\mathcal{H}$ is automatically conformal because $\dim \mathcal{V} = 1$. By definition, $\mathcal{H}$ exists local dilations if and only if for each $x \in M$, there exists a smooth positive function λ on an open subset $O(\ni x)$ of M such that

$$\mathrm{trace}(^{\mathcal{V}}B) = {}^{\mathcal{H}}(\nabla \log \lambda) \tag{5.2}$$

(cf. [12] page 461). By using (4.12) it is equivalent to that

$$\widetilde{\mathrm{trace}(^{\mathcal{V}}B)} = 0$$

where $\tilde{g} = g_{\lambda s}$ for $s : O \to R^+$. Take a vertical field V, such that $g_{\lambda s}(V, \, V) = 1$ then from Proposition 5.1 we have

$$\mathcal{L}_X V = 0$$

for any basic vector field X. Hence $\mathcal{H}$ is a principal connection for $\mathcal{V}$ with respect to V on O (cf. [13, Definition 1.3]).

Proposition 5.3. *Let $\mathcal{V}$ is a conformal one-dimensional foliation on the Riemannian manifold $(M^{n+1}, \, g)$. Then the following assertions are equivalent:*

(i) *$d\theta_\kappa = 0$ for some κ,*

(ii) *each point has a neighborhood on which a local dilation λ of $\mathcal{V}$, can be found, such that, if V is a vertical field with $g(V, \, V) = \lambda^{-2\kappa n}$, then $\mathcal{L}_X V = 0$ for any basic field X.*

Proof. Note that the following relation holds:

$$\kappa \mathrm{trace}(^{\mathcal{H}}B) + \mathrm{trace}(^{\mathcal{V}}B) = \nabla \log \lambda^{\kappa n}$$

if and only if

$$\mathrm{trace}({}^{\mathcal{H}}B) = n^{\mathcal{V}}(\nabla \log \lambda), \tag{5.3}$$

$$\mathrm{trace}({}^{\mathcal{V}}B) = {}^{\mathcal{H}}(\nabla \log \lambda^{\kappa n}). \tag{5.4}$$

By Remark 1.4(3) in [12], (5.3) holds if and only if λ is a local dilation of $\mathcal{V}$. By Lemma 4.3, (5.4) holds if and only if $\mathrm{trace}(\widetilde{{}^{\mathcal{V}}B}) = 0$ where corresponding $\tilde{g} = g_{\lambda^{\kappa n} 1}$. On the other hand, for a vertical field V, $\tilde{g}(V, V) = 1$ if and only if $g(V, V) = \lambda^{-2\kappa n}$. This together with Proposition 5.1 implies our conclusion.

Corollary 5.4 (cf. [12, Lemma 2.8]) *Let $\phi : (M, g) \to (N, h)$ be a horizontally conformal submersion with dilation λ and V a (local) vertical field on M such that $g(V, V) = \lambda^{-2\kappa n}$. Suppose that $d\theta_\kappa = 0$ for some κ. Then the following assertions are equivalent:*

(i) *the fibres of ϕ form a homothetic foliation at least away of the interior of set $\{x \in M | d(V(\sigma))(x) = 0 \neq (V(\sigma))(x)\}$.*

(ii) *for any basic field X, we have $V(X(\sigma)) = 0$ where $\sigma = \log \lambda$.*

Proof of Theorem 1.4. Because $d\theta_\kappa = 0$ we have for each $x \in M$, there exists an open subset O of x in M and $\lambda : O \to R$ such that $\theta_\kappa = d \log \lambda^{\kappa n}$. Taking a vertical vector field V with $g(V, V) = \lambda^{-2\kappa n}$. By Proposition 5.3, $\mathcal{L}_X V = 0$ for any basic field X. Writing $\sigma = \log \lambda$ and $\tilde{g} := e^{2\sigma\kappa n} g|_{\mathcal{V}} + e^{2\sigma} g|_{\mathcal{H}}$, then $g(V, V) = e^{-2\kappa n \sigma}$ and $\mathcal{V}$ is a Riemannian foliation with geodesic leaves on $(O, \tilde{g})$ by Corollary 1.3. By using the following Lemma 5.9, Lemma 5.10 and Lemma 5.11 we get $\mathcal{V}$ is Riemannian or geodesic.

Proof of Theorem 1.5. This is an immediate consequence of Theorem 1.4, Remark 4.10 and Bryant's criterion [5].

From now (in this section) M will always denote an $(n+1)-$dimensional constant curvature space with $n \geq 3$, $\mathcal{V}$ a one-dimensional conformal foliation with $d\theta_\kappa = 0$ for some $\kappa \neq \frac{1}{n}$ and λ a local dilation of $\mathcal{V}$. As before $\tilde{g} := e^{2\sigma\kappa n} g|_{\mathcal{V}} + e^{2\sigma} g|_{\mathcal{H}}$ will denote the associated (local) metric on M with respect to which $\mathcal{V}$ is Riemannian with geodesic leaves. Let $\sigma = \log \lambda$, and let V a (local) vertical field on M such that $g(V, V) = e^{-2\kappa n \sigma}$, and θ its dual vertical 1-from. Writing $\Omega = d\theta$ then Ω is basic by using Proposition 5.3 and $\Omega = 0$ if and only if $\mathcal{H}$ is integrable. The following lemma can be obtained by a straightforward computation.

Lemma 5.6. *For any basic X and Y we have:*

$$\begin{aligned}
{}^{\mathcal{H}}(\nabla_X Y) &= {}^{\mathcal{H}}(\tilde{\nabla}_X Y) - X(\sigma)Y - Y(\sigma)X + {}^{\mathcal{H}}(\tilde{\nabla}\sigma)\tilde{g}(X, Y), \\
{}^{\mathcal{V}}(\nabla_X Y) &= \{e^{2\sigma(\kappa n-1)}V(\sigma)\tilde{g}(X, Y) - \tfrac{1}{2}\Omega(X, Y)\}V, \\
{}^{\mathcal{H}}(\nabla_V X) &= -V(\sigma)X + \tfrac{1}{2}e^{-2\sigma(\kappa n-1)}(i_X\Omega)^{\tilde{b}^{-1}}, \\
{}^{\mathcal{V}}(\nabla_V X) &= -\kappa n X(\sigma)V, \\
{}^{\mathcal{H}}(\nabla_V V) &= \kappa n e^{-2\sigma(\kappa n-1)}{}^{\mathcal{H}}(\tilde{\nabla}\sigma), \\
{}^{\mathcal{V}}(\nabla_V V) &= -\kappa n V(\sigma)V
\end{aligned}$$

where ∇ and $\tilde{\nabla}$ are the Riemannian connections of g and $\tilde{g}$ respectively.

By using Proposition 5.3, Lemma 5.6 and S.Gudmundsson's fundamental equations of a horizontally conformal submersion (5) in [7], the following lemma can be obtained by a straightforward computation.

Lemma 5.7. *For any horizontal vectors X, Y, Z and H we have:*

$$\begin{aligned}
R(X, V, Y, V) &= \tfrac{1}{2}\kappa n e^{-2\sigma\kappa n}(\mathcal{L}_{\mathcal{H}(\tilde{\nabla}\sigma)}\tilde{g})(X, Y) \\
&\quad + \kappa n e^{-2\sigma\kappa n}\{(2-\kappa n)X(\sigma)Y(\sigma) - |{}^{\mathcal{H}}\tilde{\nabla}\sigma|^2_{\tilde{g}}\tilde{g}(X, Y)\} \\
&\quad + e^{-2\sigma}\left\{V(V(\sigma)) + (\kappa n-1)V(\sigma)^2\right\}\tilde{g}(X, Y) \\
&\quad + \tfrac{1}{4}e^{2\sigma(1-2\kappa n)}h((i_X\Omega)^{\tilde{b}^{-1}}, (i_Y\Omega)^{\tilde{b}^{-1}}),
\end{aligned}$$

$$\begin{aligned}
R(X, Y, Z, V) &= -\tfrac{1}{2}e^{-2\sigma\kappa n}(\tilde{\nabla}\Omega)(X, Y, Z) \\
&\quad + \tfrac{1}{2}(1-\kappa n)e^{-2\sigma\kappa n}\{X(\sigma)\Omega(Y, Z) + Y(\sigma)\Omega(Z, X) - 2Z(\sigma)\Omega(X, Y)\} \\
&\quad - e^{-2\sigma}\{X(V(\sigma)) + \kappa n X(\sigma)V(\sigma)\}\tilde{g}(Y, Z) \\
&\quad + e^{-2\sigma}\{Y(V(\sigma)) + \kappa n Y(\sigma)V(\sigma)\}\tilde{g}(X, Z) \\
&\quad + \tfrac{1}{2}e^{-2\sigma\kappa n}\{\Omega(X, \tilde{\nabla}\sigma)\tilde{g}(Y, Z) - \Omega(Y, \tilde{\nabla}\sigma)\tilde{g}(X, Z)\},
\end{aligned}$$

$$\begin{aligned}
&R(X, Y, Z, H) = e^{-2\sigma N}R(\phi_*X, \phi_*Y, \phi_*Z, \phi_*H) \\
&-\tfrac{1}{4}e^{-2\sigma\kappa n}\{\Omega(H, X)\Omega(Y, Z) + \Omega(H, Y)\Omega(Z, X) - 2\Omega(H, Z)\Omega(X, Y)\} \\
&+\tfrac{1}{2}e^{-2\sigma}V(\sigma)\{\Omega(Y, H)\tilde{g}(X, Z) - \Omega(X, H)\tilde{g}(Y, Z) + \Omega(X, Z)\tilde{g}(Y, H) - \Omega(Y, Z)\tilde{g}(X, H)\} \\
&-e^{-2\sigma}\{X(\sigma)H(\sigma)\tilde{g}(Y, Z) - X(\sigma)Z(\sigma)\tilde{g}(Y, H) - Y(\sigma)H(\sigma)\tilde{g}(X, Z) + Y(\sigma)Z(\sigma)\tilde{g}(X, H)\} \\
&+e^{-2\sigma}\{\tilde{g}(X, Z)\tilde{g}(\tilde{\nabla}_Y{}^{\mathcal{H}}\tilde{\nabla}\sigma, H) - \tilde{g}(Y, Z)\tilde{g}(\tilde{\nabla}_X{}^{\mathcal{H}}\tilde{\nabla}\sigma, H) \\
&+\tilde{g}(Y, H)\tilde{g}(\tilde{\nabla}_X{}^{\mathcal{H}}\tilde{\nabla}\sigma, Z) - \tilde{g}(X, H)\tilde{g}(\tilde{\nabla}_Y{}^{\mathcal{H}}\tilde{\nabla}\sigma, Z)\} \\
&-e^{-2\sigma}\{\tilde{g}(X, Z)\tilde{g}(Y, H) - \tilde{g}(X, H)\tilde{g}(Y, Z)\}\{e^{2(\kappa n-2)\sigma}V(\sigma)^2 + |{}^{\mathcal{H}}\tilde{\nabla}\sigma|^2_{\tilde{g}}\}
\end{aligned}$$

where ${}^{N}R$ is the Riemannian curvature tensor of the codomain of the horizontally conformal submersion $\phi : (O, g|_O) \to (N, h)$ produced by $\mathcal{V}$.

Lemma 5.8. *Let $\mathcal{H}$ be integrable. Then $\mathcal{V}$ either is Riemannian or geodesic.*

Proof. Since $\mathcal{H}$ is integrable, we have $\Omega = 0$. From Lemma 5.7, for any triple $\{X, Y, Z\}$ of orthogonal basic fields with $\tilde{g}(Z, Z) = 1$ we have

$$\frac{1}{2}(\mathcal{L}_{\mathcal{H}(\tilde{\nabla}\sigma)}\tilde{g})(X, Y) + (2 - \kappa n)X(\sigma)Y(\sigma) = 0, \tag{5.5}$$

$$X(V(\sigma)) = -\kappa n X(\sigma)V(\sigma), \tag{5.6}$$

$$-{}^N R(\phi_* X, \phi_* Z, \phi_* Y, \phi_* Z) = X(\sigma)Y(\sigma) + \tilde{g}(\tilde{\nabla}_X{}^{\mathcal{H}}\tilde{\nabla}\sigma, Y). \tag{5.7}$$

Note that

$$(\mathcal{L}_{\mathcal{H}(\tilde{\nabla}\sigma)}\tilde{g})(X, Y) = \tilde{g}(\tilde{\nabla}_X{}^{\mathcal{H}}\tilde{\nabla}\sigma, Y) + \tilde{g}(\tilde{\nabla}_Y{}^{\mathcal{H}}\tilde{\nabla}\sigma, X), \tag{5.8}$$

$${}^N R(\phi_* X, \phi_* Z, \phi_* Y, \phi_* Z) = {}^N R(\phi_* Y, \phi_* Z, \phi_* X, \phi_* Z). \tag{5.9}$$

From (5.7)-(5.9) we get

$$-{}^N R(\phi_* X, \phi_* Z, \phi_* Y, \phi_* Z) = X(\sigma)Y(\sigma) + \frac{1}{2}(\mathcal{L}_{\mathcal{H}(\tilde{\nabla}\sigma)}\tilde{g})(X, Y).$$

Together with (5.5) we yield

$${}^N R(\phi_* X, \phi_* Z, \phi_* Y, \phi_* Z) = (1 - \kappa n)X(\sigma)Y(\sigma). \tag{5.10}$$

Since $\kappa \neq \frac{1}{n}$ and the left-hand of (5.10) is a basic function we get that $X(\sigma)Y(\sigma)$ is a basic function. Combine with (5.6) and Proposition 5.3 we have

$$\begin{aligned} 0 &= V(X(\sigma)Y(\sigma)) \\ &= V(X(\sigma))Y(\sigma) + V(Y(\sigma))X(\sigma) \\ &= X(V(\sigma))Y(\sigma) + Y(V(\sigma))X(\sigma) \\ &= -2\kappa n V(\sigma)X(\sigma)Y(\sigma). \end{aligned}$$

If, at a point x, $V(\sigma)(x) \neq 0$, then this holds in an open neighbourhood O of x. It follows that $X(\sigma)Y(\sigma) = 0$ on O. By using Lemma 5.6 of [12], $\nabla\sigma|_O$ is vertical, and hence, from Proposition 4.12, $\mathcal{V}|_O$ have geodesic leaves. From Lemma 5.17 of [12], locally either $\mathcal{V}$ is Riemannian or geodesic. Since M has constant sectional curvature and $d\theta_\kappa = 0$, so σ can be considered as a global density and it is analytic. Hence globally either $\mathcal{V}$ is Riemannian or geodesic.

Lemma 5.9. *Let $\mathcal{H}$ be homothetic. Then $\mathcal{V}$ either is Riemannian or geodesic.*

Proof. Let
$$F = \{x \in M | \mathcal{V} \text{ is Riemannian}\}.$$
By Lemma 2.7 of [12] $\mathcal{H}$ is integrable on $M\backslash F$. Lemma 5.8 implies that on $M\backslash F$ either $\mathcal{V}$ is Riemannian or geodesic.

Lemma 5.10. $X(V(\sigma)) = 0$ *for any horizonal* X.

Proof. Let $\{X, Y, Z\}$ be orthogonal basic fields. Substitute them into the second formula in Lemma 5.7 and multiply by $e^{2\sigma\kappa n}$. Then the first term from the right-hand is basic and the terms in the last three lines are zero. We conclude that the second term from the right-hand is basic. Since $\kappa \neq \frac{1}{n}$ we get
$$V(X(\sigma))\Omega(Y, Z) + V(Y(\sigma))\Omega(Z, X) = 2V(Z(\sigma))\Omega(Y, X). \tag{5.11}$$
Simialrly,
$$V(Y(\sigma))\Omega(Z, X) + V(Z(\sigma))\Omega(X, Y) = 2V(X(\sigma))\Omega(Z, Y). \tag{5.12}$$
Substracting (5.12) from (5.11) gives the following:
$$V(X(\sigma))\Omega(Y, Z) = V(Z(\sigma))\Omega(X, Y). \tag{5.13}$$
Let $\dim M \geq 5$. If $\{H, X, Y, Z\}$ are orthogonal basic fields we have
$$V(H(\sigma))V(X(\sigma))\Omega(Y, Z) = 0. \tag{5.14}$$
by using (5.13). Put $\mathcal{S} = \{x \in M | \Omega_x \neq 0\}$. Then (5.14) implies that on $\mathcal{S}$ we have $V(X(\sigma))V(Y(\sigma)) = 0$ for any pair $\{X, Y\}$ of orthogonal basic fields. By Lemma 5.6 in [12] it follows that on $\mathcal{S}$ we have $V(X(\sigma)) = 0$ for any basic field X. On the other hand by Lemma 5.8 and (5.6) it is clear that on $M\backslash\mathcal{S}$ for any basic field X we also have $V(X(\sigma)) = 0$.

Now we suppose that $\dim M = 4$. Then it suffices to show that $V(X(\sigma)) = 0$ for any basic field X on $\mathcal{S}$ and on the interior of $M\backslash\mathcal{S}$ where $\mathcal{S}$ is defined as above. Let $\{X, Y\}$ be a pair of orthogonal basic fields defined on the interior of $M\backslash\mathcal{S}$. By using (5.10) and $\kappa \neq \frac{1}{n}$ we obtain that $X(\sigma)Y(\sigma)$ is a basic function. If X and Y have the same length then $X + Y$ and $X - Y$ is a pair of orthogonal basic fields. Hence $X(\sigma)^2 - Y(\sigma)^2$ is a basic function and thus $X(\sigma)$ and $Y(\sigma)$ are basic. We have $V(X(\sigma)) = 0$.

By using Proposition 5.3, we have $i_V\Omega = 0$. Together with $\dim M = 4$, each point from $\mathcal{S}$ has an open neighbourhood on which a unique basic vector field Z can be defined such that
$$i_Z\Omega = 0 \tag{5.15}$$

and $\tilde g(Z, Z) = 1$. Choose basic vector fields X and Y such that $\{X, Y, Z\}$ is orthonormal local frame for $\mathcal{H}$ with respect to $\tilde g$. (5.15) implies that on $\mathcal{S}$, we have $\Omega(X, Y) \neq 0$. Together with (5.13), (5.15) we get $V(Z(\sigma))\Omega(X, Y) = 0$. Hence $V(Z(\sigma)) = 0$. By using (5.15) we have

$$h((i_X\Omega)^{\tilde b^{-1}}, (i_Y\Omega)^{\tilde b^{-1}}) = 0, \tag{5.16}$$

$$h((i_X\Omega)^{\tilde b^{-1}}, (i_X\Omega)^{\tilde b^{-1}}) = h((i_Y\Omega)^{\tilde b^{-1}}, (i_Y\Omega)^{\tilde b^{-1}}). \tag{5.17}$$

The first equation in Lemma 5.7 together with (5.16) gives that

$$\frac{1}{2}(\mathcal{L}_{\mathcal{H}(\tilde\nabla\sigma)}\tilde g)(X, Y) + (2 - \kappa n)X(\sigma)Y(\sigma) = 0. \tag{5.18}$$

By using the last equation in Lemma 5.7 and (5.15) we have

$$-{}^N R(\phi_* X, \phi_* Z, \phi_* Y, \phi_* Z) = \frac{1}{2}V(\sigma)\Omega(X, Y) + X(\sigma)Y(\sigma) + \tilde g(\tilde\nabla_X{}^{\mathcal{H}}\tilde\nabla\sigma, Y). \tag{5.19}$$

Note that $\Omega(X, Y) = -\Omega(Y, X)$. Together with (5.8) and (5.9) we have

$$-{}^N R(\phi_* X, \phi_* Z, \phi_* Y, \phi_* Z) = X(\sigma)Y(\sigma) + \frac{1}{2}(\mathcal{L}_{\mathcal{H}(\tilde\nabla\sigma)}\tilde g)(X, Y). \tag{5.20}$$

From (5.18) and (5.20) we obtain that

$${}^N R(\phi_* X, \phi_* Z, \phi_* Y, \phi_* Z) = (1 - \kappa n)X(\sigma)Y(\sigma). \tag{5.21}$$

Since $\kappa \neq \frac{1}{n}$ and the left-hand of (5.21) is a basic function we get that $X(\sigma)Y(\sigma)$ is a basic function. Similarly using (5.17) we have $X(\sigma)^2 - Y(\sigma)^2$ is a basic function. Thus $X(\sigma)$ and $Y(\sigma)$ are basic. We have $V(X(\sigma)) = 0$.

Lemma 5.11. *Suppose that for any horizontal X, $X(V(\sigma)) = 0$. Then $\mathcal{V}$ is homothetic.*

Proof. Let S is the interior of the set $\{x \in M | d(V(\sigma))(x) = 0 \neq (V(\sigma))(x)\}$. By Corollary 5.4, it is sufficient to prove that $\mathcal{V}$ is homothetic on S. By the second equation of Lemma 5.7 on S we have

$$\begin{aligned} 0 &= R(Y, X, Y, V) \\ &= -\tfrac{1}{2}e^{-2\sigma\kappa n}(\tilde\nabla\Omega)(Y, X, Y) + \tfrac{3}{2}(1-\kappa n)e^{-2\sigma\kappa n}Y(\sigma)\Omega(X, Y) \\ &\quad +\kappa n e^{-2\sigma}X(\sigma)V(\sigma) - \tfrac{1}{2}e^{-2\sigma\kappa n}\Omega(X, \tilde\nabla\sigma) \end{aligned}$$

for any pair $\{X, Y\}$ of orthogonal basic fields with $\tilde g(Y, Y) = 1$. Hence we get

$$e^{2\sigma(1-\kappa n)}\left\{\frac{1}{2}(\tilde\nabla\Omega)(Y, X, Y) - \frac{3}{2}(1-\kappa n)Y(\sigma)\Omega(X, Y) + \frac{1}{2}\Omega(X, \tilde\nabla\sigma)\right\} = \kappa n X(\sigma)V(\sigma). \tag{5.22}$$

Since $d(V(\sigma)) = 0$ on S the right-hand side above is a basic function on S. Using $X(V(\sigma)) = 0$ we have the second factor from the left-side of (5.22) is basic and thus, if this second factor is nonzero, together $\kappa \neq \frac{1}{n}$ we get σ is a basic function. But this is impossible on S. Hence the second factor from the left-side of (5.22) is zero on S and also for $X(\sigma)$. It follows that $\mathcal{V}$ has geodesic fibres. By using Proposition 4.13, $\mathcal{V}$ is homothetic on S.

6 Vector fields of Bryant type (II)

In this section we discuss vector fields of Bryant type for p-harmonic morphisms. Let $\mathcal{V}$ be a foliation on a (connected) Riemannian manifold (M, g). $\mathcal{V}$ is said to *produce p-harmonic morphisms on* (M, g) if each point of M has an open neighbourhood O which is the domain of a submersive p-harmonic morphism $\phi : (O, g|_O) \to (N, h)$ whose fibres are open subsets of the leaves of $\mathcal{V}$.

Let $\phi : (M, g) \to (N, h)$ be a smooth map and $p \in \mathbb{R}$ such that $p \geq 2$. The Euler-Lagrange equation associated with the p-energy E_p is (cf.[3, Lemma 1.1])

$$\tau_p(\phi) = \|d\phi\|^{p-2}[\tau(\phi) + d\phi(\nabla(\log \|d\phi\|^{p-2}))] \tag{6.1}$$

where $\tau(\phi)$ is the tension field of ϕ. Let ϕ be a horizontally conformal submersion with dilation λ, i.e. $\phi^* h = \lambda^2 g|_{\mathcal{H}}$. Then we have

$$\|d\phi\|^2 = n\lambda^2 \tag{6.2}$$

and the *fundamental equation* [2] (see [4] for a different proof)

$$\tau(\phi) + \mathrm{trace}(^{\mathcal{V}}B) + (n-2)^{\mathcal{H}}(\nabla(\log \lambda)) = 0. \tag{6.3}$$

Substitute (6.2) and (6.3) into (6.1) we have

$$\frac{\tau_p(\phi)}{n^{\frac{p}{2}-1}\lambda^{p-2}} + \mathrm{trace}(^{\mathcal{V}}B) + (n-p)^{\mathcal{H}}(\nabla(\log \lambda)) = 0. \tag{6.4}$$

We now characterize conformal foliations which produce p-harmonic morphisms.

Proposition 6.1 *Let $\mathcal{V}$ be a conformal foliation on (M, g) of codim$\mathcal{V} \neq p$ and $\mathcal{H}$ its orthogonal complement. Then $\mathcal{V}$ produces p-harmonic morphisms if and only if the vector field*

$$W_p := (n-p)\mathrm{trace}(^{\mathcal{H}}B) - n\mathrm{trace}(^{\mathcal{V}}B)$$

is locally a gradient vector field where $n =$ codim $\mathcal{V}$.

Note that when $p = 2$, Proposition 6.1 is R.L.Bryant's criterion. From (1.1) and (4.17) we have $W_2 = W$ and $W_p^b = -n\theta_{\frac{p-n}{n}}$ for $n \neq p$.

Proof of Proposition 6.1 Similar to the proof of $p = 2$ (cf.[4][5][12]) and using (6.4) and Theorem 3 in [10].

Proof of Theorem 1.7 Taking $p = n + 1$ then codim$\mathcal{V} < p$. Now our conclusion is an immediate consequence of the following equation

$$W_{n+1} + U = 0$$

where U is defined in (1.2).

7 Minimal foliations

In the last section we characterize conformal foliations which are minimal.

Lemma 7.1. *Let $\phi : (M^m, g) \to (N^n, h)$ be a horizontally conformal submersion with dilation λ and let $r, s : M \to (0, \infty)$ be smooth functions. Denote by g_{rs} the Riemannian metric on M defined by $g_{rs} = r^2 g|_{\mathcal{V}} + s^2 g|_{\mathcal{H}}$. Then $\phi : (M, g_{rs}) \to (N, h)$ is a p-harmonic morphism if and only if*

$$\mathrm{trace}({}^{\mathcal{V}}B) + {}^{\mathcal{H}}\nabla[\log(\lambda^{n-p} r^{n-m} s^{p-n})] = 0.$$

Proof By a straightforward computation ones obtain

$${}^{\mathcal{H}}(\tilde{\nabla} f) = s^{2\mathcal{H}}(\nabla f) \tag{7.1}$$

for any smooth function f on M (cf. Lemma 3 in [11]). Notice that $\phi : (M, g_{rs}) \to (N, h)$ is a horizontally conformal submersion with dilation $\frac{\lambda}{s}$, from (6.4) we get

$$\frac{\widetilde{\tau_p(\phi)}}{n^{\frac{p}{2}-1}(\frac{\lambda}{s})^{p-2}} + \widetilde{\mathrm{trace}({}^{\mathcal{V}}B)} + (n-p)^{\mathcal{H}}(\tilde{\nabla} \log \frac{\lambda}{s}) = 0 \tag{7.2}$$

where corresponding $\tilde{g} = g_{rs}$. By using (4.12)(7.1) and (7.2) we have

$$\frac{\widetilde{\tau_p(\phi)}}{n^{\frac{p}{2}-1}\lambda^{p-2}s^{-p}} + \mathrm{trace}({}^{\mathcal{V}}B) + {}^{\mathcal{H}}\nabla[\log(\lambda^{n-p} r^{m-n} s^{p-n})] = 0. \tag{7.3}$$

Now our lemma is an immediate conclusion of (7.3) and Theorem 3 in [10].

Proposition 7.2. *Let $\mathcal{V}$ be a conformal foliation on a Riemannian manifold (M, g) and $\mathcal{H}$ its orthogonal complement. Then $\mathcal{V}$ is minimal if and only if it produces n-harmonic morphisms where $n = \mathrm{codim}\mathcal{V}$.*

Proof. Since $\mathcal{V}$ is conformal, its leaves locally are the fibres of a horizontally conformal submersion ϕ (cf. [1, Proposition 3.1]). Suppose that $\mathcal{V}$ is minimal then we have that ϕ is an n-harmonic map from (6.4), hence, it is an n-harmonic morphism[10]. Conversely, denote the dilation of ϕ by λ. In Lemma 7.1 take smooth positive functions $s = \lambda$ and $r = 1$ we obtain that $\mathcal{V}$ is minimal.

Note that when $n = 2$ we recover J.C.Wood's result[17].

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