LINKS BETWEEN GEOMETRY AND PHYSICS

Second meeting

Schloß Ringberg, April 1989

Max–Planck–Institut
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List of participants

L.	Alvarez-Gaumé	CERN Theory Division, Genève
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R.S.	Ward	University of Durham, Department of Mathematical Sciences
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R.	Zucchini	MPI für Physik und Astrophysik, München

INTRODUCTORY LECTURE

Michael Atiyah

Oxford University

§ 1 Review of last Ringberg meeting

At the meeting in Ringberg two years ago I ended by summarizing in a brief table the main topics relating Geometry and Physics at that time. The table listed the topics under the relevant dimension as follows:

dimension	Theory	Physical background
4	Donaldson	self–dual Yang–Mills
3	Floer	Chern–Simons
3	Jones (Knots)	
2	Conformal Field Theory	
1	Loop Groups,	Virasoro
0	Lie Groups	

Donaldson and Floer theory are intimately related, but there appeared to be a major gap between gauge theories in 3 and 4 dimensions on the one hand and theories arising from lower dimensional ideas.

§ 2 Topological Quantum Field Theories

A major breakthrough since last time has been made by E. Witten who has shown:

(1) Donaldson/Floer theory can be formulated, via a suitable Lagrangian, as a quantum field theory which is *topological* in the sense that the Hamiltonian is zero.

(2) The Jones theory can be understood (and generalized) by interpreting it as a topological QFT in $2+1$ dimensions, with the Chern–Simons function as Lagrangian.

This Witten–Jones theory, in its Hamiltonian version, assigns finite–dimensional vector spaces to Riemann surfaces. Work of N. Hitchin suggests that it may be possible to understand these spaces (for a non–abelian group G) in terms of the abelian theory of its maximal torus and an appropriate generalization of the Weyl group.

§ 3 Dimensional Reduction

For some time R. Ward and I have advocated the view that all 2–dimensional integrable systems may be obtained by dimensional reduction from the self–dual Yang–Mills equations in dimension 4. Recently L. Mason and G. Sparling have explicitly shown that both the KdV and the Non–Linear Schrödinger equations arise in this way. Moreover the twistor theory in 4 dimensions explains the main features of the 2–dimensional reductions.

This suggests that, in our table, we should start at the top and work down – even at the quantum level.

§ 4 Witten's Euler characteristic

A few years ago, in connection with strings on orbifolds Witten introduced a numerical invariant for a finite group G acting on a compact manifold X. This invariant, denoted say by $W_G(X)$, is defined by

$$W_G(X) = \frac{1}{|G|} \sum_{g_1, g_2} \chi(X^{g_1, g_2})$$

where the summation is over pairs g_1, g_2 of *commuting* elements of G, X^{g_1,g_2} denotes the common fixed point set of g_1, g_2 and χ is the usual Euler characteristic.

Recently G.B. Segal and I discovered that

$$W_G(X) = \chi K_G(X)$$

when $K_G(X)$ is the equivariant K–theory of X, and

$$\chi = \mathrm{rk}\, K^0 - \mathrm{rk}\, K^1 .$$

This formula was suggested by the idea that the S^1–equivariant cohomology of the free loop space of X (as defined by Witten), which is a mod 2 graded theory, should be viewed as the K–theory of X.

§ 5 Topics for the conference

I hope that at least some of the topics I have mentioned will be treated in detailed lectures during the conference. There are of course other important topics to be covered and I have only concentrated on the most geometrical aspects.

QUANTUM FIELD THEORY IN TWO AND THREE DIMENSIONS

Jürg Fröhlich

Theoretical Physics

ETH–Zürich

The purpose of this lecture is to discuss some features of two–dimensional conformal field theory and three–dimensional gauge theory, describe the mathematical connections between these theories and show how Yang–Baxter representations of the braid groups and representations of quantum groups appear in their study. Underlying this presentation is work of Witten; Tsuchiya and Kanie; Kohno; Moore and Seiberg; Fredenhagen, Rehren and Schroer; Woronowicz, Faddeev et al., Jimbo, Drinfel'd, Reshetikhin; Jones, Ocneanu and Wenzl; Myrheim, Wilczek and Zee, Wu, ... ; Laughlin, Girvin, Wiegmann, ... ; and work which has been carried out in collaboration with G. Felder, G. Keller, Chr. King and P.–A. Marchetti [1–3, 6–8]. Work by Witten [5] and by Felder [4] is of particular importance.

It has been known for a number of years that there are connections between gauge theories in $d+1$ dimensions and scalar field theories in d dimensions. This has been exploited at a classical and quantized level. A particularly beautiful connection between three–dimensional Chern–Simons gauge theory and two–dimensional conformal field theory (W–Z–W models) has recently been discovered by Witten [5]. Both, Chern–Simons theory and the equivalent W–Z–W models, have an intimate connection with the representation theory of (Kac–Moody) current algebras and of quantum groups, and with complex analysis. The representation theories of current algebra and of quantum groups, based on classical Lie groups, are essentially *identical* in structure. The relation between the two theories are uncovered by studying associated Yang–Baxter representations of braid groups. These representations can be used to construct invariants for knots and links imbedded in a general class of three–manifolds [5,6]. The invariants are generalizations of the *Jones polynomial*.

The appearance of quantum groups and of braid group representations in two – and three – dimensional quantum field theory is quite fundamental: The structure of superselection sectors in such theories is coded into certain Yang–Baxter representations of braid groups which, in examples, turn out to generate the commutant of a tensor product representation of some quantum group. The quantum group plays the role of a global internal symmetry of the quantum field theory; the representations of the braid groups describe the statistics of superselection sectors [1,7,8]. These are "*invariants*" of the quantum field theory in the sense that they are locally *independent* of coupling constants and masses: Quantum groups as symmetries and representations of braid groups as a description of the statistics of superselection sectors appear in a general class of two–dimensional quantum field theories, *not* just of conformal field theories, and of three–dimensional gauge theories, *not* just of topological theories.

These ideas might have important applications to real *physics*: Excitations carrying fractional charge and fractional spin and exhibiting braid group statistics could play an important role in theories of the fractional quantum Hall effect and in certain models of high–temperature superconductivity, [7,8].

Briefly, the four parts of the lecture can be summarized in the following way:

Mathematical aspects.

1) 3–dim. Chern–Simons theory

| ≀

2–dim W–Z–W. models

⟷

Reps. of braid groups and of quantum groups; link polynomials ;new invariants for framed three–manifolds.

2) Representation theory of chiral algebras

↕

complex analysis: vector bundles over Riemann sur–faces; flat connections on such bundles

← reps. of braid groups →

Representation theory of quantum groups.

Physics.

3) General 3–dim. gauge theories with Chern–Simons term in the action (parity breaking!) ⟶ describe charged particles with arbitrary real spin and braid group statistics (intermediate between Fermi– and Bose statistics).

↕

General theory of superselection sectors, as developed by Doplicher, Haag and Roberts, Buchholz and Fredenhagen, and others. (See [1,7,8] and refs. quoted there.)

4.) Applications to condensed matter physics: Fractional quantum Hall effect; models of high–T_c superconductivity. *Main problem*: Derivation of *effective* gauge theory from *microscopic* quantum–mechanical theory.

References

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2. G. Felder, J. Fröhlich and G. Keller, "On the Structure of Unitary Conformal Field Theory, I & II", Commun. Math. Phys., to appear.

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6. J. Fröhlich and Chr. King, "The Chern–Simons Theory and Knot Polynomials", submitted to Commun. Math. Phys.; J. Fröhlich and Chr. King, "Two–Dimensional Conformal Field Theory and Three–Dimensional Topology", Preprint ETH–TH/89–9.

7. J. Fröhlich and P.–A. Marchetti, "Quantum Field Theory of Vortices and Anyons", Commun. Math. Phys. 1988/89.

8. J. Fröhlich, F. Gabbiani, and P.–A. Marchetti, "Three–Dimensional Relativistic Theories with Braid Group Statistics", Preprint ETH 1989.

Note: Precise references to work by the colleagues mentioned at the beginning of these notes, and others, were not available to the author at the place of writing (Schloss Ringberg), but can be found in refs. 1 through 8.

SUPERSYMMETRIC QUANTUM FIELD THEORY, DISCRETE GROUPS, AND ENTIRE CYCLIC COHOMOLOGY

Alain Connes

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We describe an analogy between the theory of discrete groups of exponential growth and quantum field theory. Both theories involve analysis in infinite dimension and require the same tool provided by entire cyclic cohomology. It allows to give a cohomological meaning to Wightman funtionals of boson fields in the presence of supersymmetry, and to develop successfully higher index theory for non simply connected manifolds.

We explain how the machine works in the case of discrete groups.

1) The finite dimensional "toy" case.

If we start with a non commutative algebra $\mathcal{A}$ playing the role of functions on a space X, the most obvious commutative notions to extend to this case are the notion of vector bundle on X yielding the group $K_0(\mathcal{A})$ and of K–homology cycle on X yielding the notion of Fredholm module $(\mathcal{H},F,\gamma)$ on $\mathcal{A}$. Then the replacement of the formula

$$\int_X f^0 df^1 \wedge \dots \wedge df^n$$

by its analogue:

(*) $$\mathrm{Tr}_s(f^0[F,f^1] \dots [F,f^n])$$

yields the theory of cyclic cohomology $HC^*(\mathcal{A})$ which is a direct generalization of the de Rham theory. Among its distinctive features let us retain only the following:

Lemma 1

a) If $\mathcal{A}$ is an algebra, τ a 2q cyclic cocycle on $\mathcal{A}$ then the formula $P \longrightarrow \tau(P, \dots, P)$ gives a pairing with K theory (i.e. is unchanged by a homotopy of P's)

b) the equality (*) defines a cyclic cocycle if $n = 2q \geq$ degree of summability of $(\mathcal{H}, F, \gamma)$ and the pairing with K_0 is the index pairing.

As applications of this lemma to the non commutative case let us quote the integrality of the trace on algebra of free groups, and the integrality of the conductivity in the Quantum Hall effect. But the real power of cyclic cohomology beyond this integrality result is the possibility of constructing *cyclic cocycles* on $\mathcal{A}$ without the presence of a Fredholm module, thus:

Lemma 2 Let $\mathcal{A} = \mathbb{C}\Gamma$ be the algebra of a discrete group, and $c(g_1, \dots, g_n)$ be a group cocycle, then the following defines a cyclic cocycle on $\mathcal{A}$:

$\tau_c(g_0, \dots, g_n) = 0$ if $\prod g_i \neq 1$, $\tau_c(g_0, \dots, g_n) = c(g_1, \dots, g_n)$ if $\prod g_i = 1$.

It turns out moreover that the K theory of $\mathbb{C}\Gamma \otimes R$ where R is the ring of ∞ matrices with rapid decay is extremely rich and contains elements associated with *Index theory of manifolds with* $\pi_1 = \Gamma$. Thus in particular to any elliptic operator D on such a manifold M_{ev} there corresponds an element $\mathrm{Ind}_\Gamma(D) \in K_0(R\Gamma)$; and one has the following generalization of the Atiyah Singer index theorem.

Theorem 3 (with Henri Moscovici) With the above notations one has:

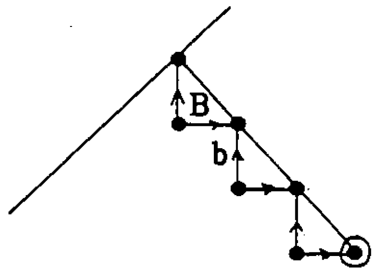
$$< \mathrm{Ind}_\Gamma(D), \tau_c > = \frac{1}{(2\pi i)^q} \frac{q!}{(2q)!} < \mu(D)\varphi^*(c), [M] > .$$

This theorem and some hard analysis on discrete hyperbolic groups allows to prove the Novikov conjecture for these groups.

2) The infinite dimensional case

The theory of cyclic cohomology and its computation is based on a bicomplex (b,B) which is quite simple to define by

$b\varphi(a^0, \dots, a^{n+1}) = \Sigma (-1)^j \varphi(a^0, \dots, a^j a^{j+1}, \dots a^{n+1})$ and $B\varphi$ = Cyclic antisymmetrization of $\varphi(1, a^0, \dots, a^{n-1}) - (-1)^n \varphi(a^0, \dots, a^{n-1}, 1)$. In fact it is slightly better to normalize by $\frac{1}{n} B = d_2$ and $(n+1)b = d_1$. It is a quite important lemma of the theory that any cocycle can be assembled at one point



and that it then gives cyclic cohomology – this follows technically from the vanishing of the 1st spectral sequence – this gives for the pairing with K theory the formula

$$\Sigma\,(-1)^n/n!\;\varphi_{2n}(e,\dots,e)$$

and makes it obvious that one should try to understand the ∞ dim cycles with some growth condition.

For this one needs good examples of K–cycles which are not finitely summable but have yet some summability property, and also to extend the formula (*) above. My examples come from discrete groups and lead to the following notion:

<u>Definition</u> A θ–summable (unbounded Fredholm module) over $\mathcal{A}$ is given by Hilbert space representation $\mathcal{H}$ of $\mathcal{A}$ with $\mathbb{Z}/2$–grading γ and an unbounded operator $D = D^*$, $\gamma D = -D\gamma$ such that $\mathrm{Tr}(e^{-\beta D^2}) < \infty\ \forall \beta$ and $[D,a]$ bounded $\forall a \in \mathcal{A}$.

For $\mathcal{A} = \mathbb{C}\Gamma$ where Γ is a discrete subgroup of a Lie group, the crucial example is obtained as the Dirac dual Dirac operator in the symmetric space G/K which was considered by Hörmander, Miščenko, Kasparov and G. Luke for instance, and can be reformulated à la Witten as $d_\tau + (d_\tau)^* = D$. In general the construction of the entire cocycle character of such a module relies on the following ansatz, analogue of (*):

(**) $$\varphi_{2n}(x^0, \ldots, x^{2n}) = \lambda_n \text{"Tr"}(Fx^0[F,x^1] \ldots [F,x^{2n}])$$

where "Tr" is a suitable trace on the formal algebra obtained by adjoining F to $\mathcal{A}$, and λ_n are constants which are uniquely determined by the cocycle property and turn out to be given by the following *generating function*:

$$\Sigma \frac{(-1)^n}{n!} \varphi_{2n}(x, \ldots, x) = \text{"Tr"} \left[\frac{Fx}{\sqrt{1-[F,x]^2}} \right].$$

The larger algebra turns out to be the convolution algebra of distributions T on $[0,+\infty[$ with operator values in $\mathcal{H}$ and such that T(s) is holomorphic in s for $s > 0$ and belongs to the Schatten class $\mathcal{L}^{1/s}$, then F is in Laplace transform given by

$$F = \frac{D + \gamma\, \text{"}\lambda^{1/2}\text{"}}{\sqrt{D^2 + \lambda}}.$$

Here $\lambda^{1/2}$ is a *local* square root of the derivative δ_0' of the Dirac mass at the origin. (It plays a role in supersymmetry). Fortunately Jaffe, Lesniewsky and Osterwalder wrote down a more direct and equivalent formula as:

$$\int_{s_1 \le s_2 \le \ldots \le s_n \le 1} \prod ds_i \quad \mathrm{Tr}_s \left[a^0 e^{-s_1 Q^2} [Q,a^1] e^{-s_2 Q^2} [Q,a^2] \ldots \right]$$

The theory develops then as in the finite toy case, but again it takes all its power from a procedure which allows to manufacture entire cocycles on group rings out of very little data which combines Quillen's superconnection formalism and the formula of Jaffe, Lesniewsky and Osterwalder and the geometry of ∞ dimensional manifolds of negative curvature.

CONSTRUCTIVE FIELD THEORY AND ENTIRE CYCLIC COHOMOLOGY *

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I. Introduction

Alain Connes described in his report how entire cyclic cohomology provides a tool to study the analysis and the geometry of infinite dimensional manifolds. I propose a related method, constructive quantum field theory. These methods provide a powerful analytic tool, developed over the past 25 years, for the study of analysis over spaces of infinite dimension. In particular, they can be used in the study of analysis on certain loop spaces, and they lead to the mathematical definition of function space integrals in a number of quantum field settings, see for example [GJ, JL2]. In fact, the existence of functional integrals relies on two techniques of constructive quantum field theory: phase cell analysis and cluster expansions. The former involves analyzing degrees of freedom localized simultaneously in space-time and in Fourier variables. The latter combines these ideas with decoupling of different phase cell regions in performing function space integrals.

One example is the generalization of Gårding's inequality to the infinite dimensional situation. This coercive estimate establishes the exponential growth of the eigenvalues for the Hamiltonian (the Laplace operator on loop space) in the Wess-Zumino example. This

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model describes the coupling of a boson field ϕ coupled to a fermion ψ. That Hamiltonian is

$$H = H_0 + \int_{S^1} |V'(\phi)|^2 - \int_{S^1} \bar{\psi} V'' \psi \ .$$

It satisfies the estimate, for ϵ sufficiently small [JL1],

$$N_\tau \leq (H + I) \ , \qquad \text{for } \tau < 1 \ .$$

Since N_τ is a modified form of the free Hamiltonian, with the one particle operator μ^τ replacing $\mu = (\partial^2 + 1)^{1/2}$, the eigenvalues of H grow at least as fast as $(k^2+1)^{1/2}$, $k \in Z$. Hence

$$\mathrm{Tr}\, e^{\beta H} < \infty \ , \qquad \text{for all } \beta > 0. \tag{1}$$

Entire cyclic cohomology allows us to compute invariants and to give a cohomological interpretation to Wightman functionals. Here I describe some work in this direction done in collaboration with A. Lesniewski, K. Osterwalder, and several students. Entire cyclic cohomology makes contact with classical statistical mechanics. This version of the theory enables one to drop the assumption of θ-summability in order to construct a Chern character or to define the index of an operator. Some main references for this talk are [C2, JLO1, K, JLO2, JLWis, JL3].

II. Quantum Algebra

A quantum algebra is a Z_2-graded algebra $\mathcal{A}$, with a continuous one-parameter automorphism group α_t which commutes with the grading and with taking adjoints. For example, if α_t arises from a Hamiltonian flow, such as one generated by H above, then

$$\alpha_t = e^{itH} a e^{-itH} , \qquad t \in R. \tag{2}$$

The infinitesimal generator $D = -id/dt\alpha_t|_{t=0}$ of α_t is a derivation, namely

$$D(ab) = (Da)b + aDb.$$

The fundamental assumption is that $D = d^2$, where d is a super-derivation,

$$d(ab) = (da)b + a^\gamma db,$$

where $\gamma{:}a \to a^\gamma$ denotes the action of the grading. Continuity of α_t ensures the existence of a dense, invariant subalgebra $\mathcal{A}_\alpha$ of $\mathcal{A}$ on which α_t extends to an entire function of t.

III. Super-KMS Functionals

The theory of KMS states is a central feature of the abstract treatment of statistical mechanics on the one hand, and of Tomita's theory of modular automorphisms on the other. Here we study a super-version of this condition in which the KMS functional is not necessarily positive (a property of a state).

Connes spoke about systems for which

$$\mathrm{Tr}\,(e^{-H}) < \infty. \tag{3}$$

This is the θ-summability condition. In that case, one can give a standard example of a super-KMS functional as a supertrace,

$$\omega(a) = \mathrm{Str}\,(ae^{-H}) = \mathrm{Tr}\,(\gamma a e^{-H}). \tag{4}$$

In the statistical mechanics approach we describe, we can also study cases where (3) is not valid, such as when H may have continuous spectrum, or when H may not even exist. However, we can think of our functionals as arising as limits of funtionals of the form (4).

We define a super-KMS functional as a continuous linear functional ω on $\mathcal{A}$, such that for all elements of $\mathcal{A}_\alpha$,

$$\omega(da) = 0, \qquad \text{super} - \text{translation invariance} \tag{5}$$

and

$$\omega(ab) = \omega(b^\gamma \alpha_i(a)), \qquad \text{super} - \text{KMS property}. \tag{6}$$

The super-property is reflected in the action of the grading, and for this reason ω is not in general positive.

Proposition 1. *If ω is a super-KMS functional on $\mathcal{A}$, then the function*

$$f(t_1, \ldots, t_n) = \omega(a_0, \alpha_{t_1}(a_1), \ldots, \alpha_{t_n}(a_n)) \tag{7}$$

extends to an analytic function in the strip $0 \leq \mathrm{Im}t_1 \leq \cdots \leq \mathrm{Im}t_n \leq 1$. *Furthermore, for t in the strip,*

$$|f| \leq const. \prod_{j=0}^{j=n} \|a_j\| \,. \tag{8}$$

Proof. One starts on the subalgebra $\mathcal{A}_\alpha$. Using the sKMS condition and a Pfragmén-Lindelöf type of theorem, one can establish (8) inductively in n. The passage to analyaticity on the full algebra $\mathcal{A}$ follows by the Weierstrass approximation theorem.

IV. Entire Cyclic Cohomology

The double complex of entire cyclic cohomology is determined by the coboundary operators b and B introduced by Connes. These operators define the creation-annihilation complex

$$b : \mathcal{C}^n(\mathcal{A}) \to \mathcal{C}^{n+1}(\mathcal{A}) \ , \qquad B : \mathcal{C}^{n+1}(\mathcal{A}) \to \mathcal{C}^n(\mathcal{A}) \tag{9}$$

and satisfy

$$b^2 = 0, \qquad B^2 = 0, \qquad Bb + bB = 0 \ . \tag{10}$$

The coboundary operator $\partial = b + B$ is used to define entire cyclic cohomology. Explicitly

$$(Bf_n)(a_0, \ldots, a_{n-1}) = \sum_{j=0}^{n-1} (-1)^{(n-1)j} \Big(f_n\left(1, a^\gamma_{n-j}, \ldots, a^\gamma_{n-1}, a_0, \ldots, a_{n-j-1}\right) + (-1)^{n-1} f_n\left(a^\gamma_{n-j}, \ldots, a^\gamma_{n-1}, a_0, \ldots, a_{n-j-1}, 1\right)\Big) \ , \tag{11}$$

and

$$(bf_n)(a_0, \ldots, a_{n+1}) = \sum_{j=0}^{n} (-1)^j f_n(a_0, \ldots, a_j a_{j+1}, \ldots, a_{n+1}) + (-1)^{n+1} f_n(a^\gamma_{n+1} a_0, a_1, \ldots, a_n) \ . \tag{12}$$

Here

$$a^\gamma = \begin{cases} a^\Gamma, & \text{if } f_n \in \mathcal{C}^n_+(\mathcal{A}) \\ a, & \text{if } f_n \in \mathcal{C}^n_-(\mathcal{A}) \ . \end{cases} \tag{13}$$

By a computation we then establish

Theorem 2. *The functionals*

$$\tau_n(a_0, \ldots, a_n) = i^{n \ mod 2} \int_{0 \le t_1 \le \cdots \le t_n \le 1} \omega(a_0, \alpha_{it_1}(da^\gamma_1), \ldots, \alpha_{it_n}(da^{\gamma^n}_n)) \tag{14}$$

are a cocycle for the entire cyclic cohomology.

In order to identify τ, let us consider an example: Suppose that $Q = \begin{pmatrix} 0 & Q_- \\ Q_+ & 0 \end{pmatrix}$ is an odd, self-adjoint, Fredholm operator. Assume further that the super-derivation d is given by the graded commutator $da = Qa - a^\gamma Q = \delta_Q(a)$. In this case $\tau(I) = \text{index}(Q_+)$.

This example suggests that we define

$$\text{index}(d_+) = \tau(I) ,$$

where d_+ is the part of d which maps the even subspace of $\mathcal{A}$ into the odd subspace.

V. The Chern Character as an Index

If e is an even idempotent in the algebra $\mathcal{A}$, then Connes' pairing of τ with K_0 extends to this framework. In particular let $\omega_e(\,\cdot\,) = \omega(e \cdot e)$ and let $d^e = e(da)e$. Then we have [JL3],

Theorem 3. *The pairing*

$$\langle \tau, e\rangle = \sum_{n=0}^{\infty} (-1)^n \frac{(2n)!}{n!} \tau_{2n}(e, e, \ldots, e)$$

is equal to the index(d_+^e).

VI. Stability

Let δ_q be a bounded derivation, defined by an odd element of $\mathcal{A}$, and let

$$d_q = d + \delta_q .$$

Based on perturbation theory, we can establish the stability of the index under such bounded deformations. This justifies our calling $\langle \tau, e\rangle$ a geometric invariant. First we establish the existence of a continuous, even, automorphism α_t^q generated by $(d_q)^2$ and a functional ω^q which is sKMS with respect to α_t^q and $(d_q)^2$.

Theorem 4. *The cocycle τ^q defined by ω^q is cohomologous to τ.*

VII. Conclusions

Let me mention some results related to open problems:

i. The general case of constructing infinite volume limits of the models described here should yield an interesting example of functionals which are super-KMS, but are not θ-summable.

ii. In the case of Wess-Zumino models on a cylinder, it is possible to compute the index, and it is non-trivial. See [JLWel, JL2]. In the case of the infinite volume limits, one would like to analyze the index and the question of whether the Dirac operator Q exists. In case super-symmetry is broken, one expects that Q does not exist. In the case of phase transitions (as one expects when the finite volume theory has a non-zero index), it is not clear whether an operator Q exists for which $H = Q^2$.

iii. In the complex case, the Wess-Zumino models require no renormalization; however the real case does require "Wick ordering" type of renormalization of the super-potential V. **Theorem 5.** *These renormalizations are purely cohomological, in the sense that the cocycle τ^V for a given superpotential V is independent of the counterterms.* It is possible that the cohomological interpretation of the Wightman functions given by τ could provide a topological explaination for asymptotic freedom; asymptotically free models would have unchanged cohomology under renormalization, *i.e.* they would be cohomologically stable. On the other hand, non-asymptotically free models would have renormalization of the leading interaction term and hence would be cohomologically unstable. One should investigate carefully the class of unbounded perturbations which are cohomologically stable.

iv. The analysis of multi-component Wess-Zumino models along the lines already carried out for one-component models is being done by Ernst [E]. One would like to use these models as starting points for construction of non-linear σ-models as the family of potentials λV ranges over $\lambda \to \infty$.

v. One can investigate an equivariant form of entire cyclic cohomology [F].

vi. Can one construct a similar cocycle for a "topological" setup where $Q^2 = 0$?

vii. It is interesting to investigate whether the global automorphism group α_t can be replaced by a local object, perhaps α_{t_1,t_2}, suitable for functional on a Riemann surface.

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SOLVABLE CLASSICAL FIELD THEORIES

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Simple Prototype. On the space LM of loops in a manifold M , there is a natural set of differential equations for real-valued functions $\phi : LM \longrightarrow \mathbb{R}$. These may be described by exhibiting their solutions. To construct a solution, take a function

$$f : M \times S^1 \longrightarrow \mathbb{C}$$

and put

$$\varphi(x) = \frac{1}{2\pi}\int_0^{2\pi} f(x(\theta),\theta)d\theta\ , \quad x \in LM\ . \tag{1}$$

To understand better what the equations are that this field φ satisfies, let us take $M = \mathbb{R}$, and decompose $x(\theta)$ into its Fourier components:

$$x(\theta) = \sum_{\mathbb{Z}} x_k e^{ik\theta}\ .$$

In terms of the coordinates x_k , the equations in question are

(2) $$\partial_j \partial_k \varphi = \partial_{j-1} \partial_{k+1} \varphi \,, \quad \forall\, j,k \in \mathbb{Z} \,.$$

The integral formula (1) gives the general solution of these equations (2).

An even simpler version is obtained by considering only polynomial loops. For example, we may set all the x_k to be zero, apart from x_0, x_1 & x_2. Then the only equation which remains is

$$(\partial_2 \partial_0 - \partial_1^2)\varphi = 0 \,,$$

which is the wave equation in $2+1$ dimensions. The integral formula (1) for solutions of this essentially goes back to Whittaker, at the turn of the century.

The polynomial case has a neat interpretation in terms of complex geometry. Think of $e^{i\theta}$ as the equator $|\lambda| = 1$ on the Riemann sphere $\mathbb{P}_1$; then polynomials $x = \sum_{k=0}^{n} x_k \lambda^k$ correspond to holomorphic sections of the holomorphic line bundle L^n, of Chern number n, over $\mathbb{P}_1$. The function f should really be thought of as an element of the cohomology group $H^1(L^n, \mathcal{O}(-2))$.

Whether one uses this geometric description, or discards it by adopting a more analytic approach, is partly a matter of taste. Historically, the geometric picture came first, and has proved particularly useful.

The Nonlinear Version. In order to get something less trivial, one needs to "nonlinearize" the integral formula (1). This is achieved by using the Birkhoff factorization of loops in $GL(N,\mathbb{C})$. The function $f : \mathbb{R} \times S^1 \longrightarrow \mathbb{C}$ is replaced by a matrix–valued function $F : \mathbb{R} \times S^1 \longrightarrow GL(N,\mathbb{C})$. If $x(\theta)$ is a loop in $\mathbb{R}$, then $F(x(\theta),\theta)$ is a loop in $GL(N,\mathbb{C})$. For generic F and x , this can be factorized as $\hat{H} H^{-1}$, where the matrices H and $\hat{H}$ are functions of (x_k,λ) , and are holomorphic and nonsingular for $|\lambda| \le 1$ and $|\lambda| \ge 1$ respectively. Then the matrix–valued function

$$J(x_k) = H(x_k,0)\, \hat{H}(x_k,\infty)^{-1}$$

is a solution of the equations

$$\partial_j(J^{-1}\partial_k J) = \partial_{k-1}(J^{-1}\partial_{j+1}J)\,, \quad \forall\, j,k\,. \tag{3}$$

These are a set of nonlinear equations that generalizes (2).

Remarks.

(a) If $N = 1$, then (3) reduces to (2), the correspondence being $F = \exp(f)$, $J = \exp(\varphi)$.

(b) In the polynomial case, the matrix F determines a holomorphic vector bundle over the complex manifold L^n , and everything is encoded into this vector bundle.

(c) The "linear system" associated with (3) (solvable equations always have linear systems associated with them) is $D_k\psi = 0$, where

$$D_k = \partial_k - \lambda\partial_{k-1} + J^{-1}\partial_k J\,.$$

The vanishing of the commutators $[D_j,D_k]$ is equivalent to the equations (3).

(d) These equations encompass a large variety of well–known solvable classical systems. The simplest examples come from the polynomial case $x = x_0 + x_1\lambda + x_2\lambda^2$, when (3) becomes

$$\partial_0(J^{-1}\partial_2 J) - \partial_1(J^{-1}\partial_1 J) = 0 . \tag{4}$$

Imposing a special dependence on x_1, for example, reduces (4) to various two–dimensional integrable systems. These include the Sine–Gordon equation and the other Toda field equations, and nonlinear sigma and chiral models.

(e) One may generalize by allowing F to depend on more variables. For example, if $F : \mathbb{R}^2 \times S^1 \longrightarrow GL(N,\mathbb{C})$, one can play the same game with $F(x(\theta),y(\theta),\theta)$. One equation dealt with in this way is the self–dual Yang–Mills equation in four dimensions, which historically was the first of these nonlinear equations to be handled by the sort of procedure described here.

The NLS and KdV hierachies. Recently L. Mason and G. Sparling observed that these hierarchies of integrable soliton equations emerge naturally from the scheme described above. Briefly, one takes $N = 2$, restricts to loops with $x_k = 0$ for $k < 0$, and imposes a special dependence on x_0. Namely, $J^{-1}\partial_0 J$ is assumed to be a constant matrix. Two nontrivial possibilities arise, depending on whether the rank of $J^{-1}\partial_0 J$ is 2 or 1. In the former case, one gets the NLS hierarchy, and in the latter case, the KdV hierarchy.

Two Questions.

(a) Certain three–dimensional systems, such as the KP equation, appear not to fit into this scheme. Is there some way in which they can be incorporated? What appears to be involved is that the spectral parameter λ gets replaced by an operator $\partial / \partial z$, where z is an "auxiliary" coordinate.

(b) Do these methods have any relevance to integrable quantum systems or integrable lattice (statistical) models?

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ABELIANIZATION OF BUNDLES OVER RIEMANN SURFACES

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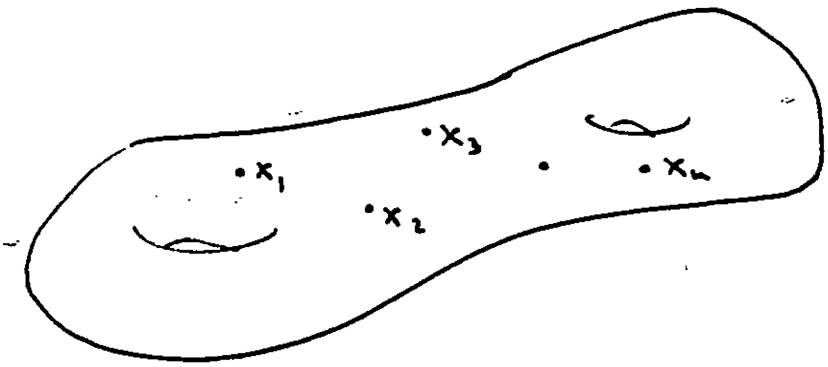


Σ genus g

The moduli space of stable G–bundles over a Riemann surface Σ plays an important role in aspects of conformal field theory, as does the moduli space of parabolically stable bundles – bundles with a reduction to a Borel subgroup at n marked points $x_1, \dots, x_n$.

Here we shall describe a systematic way of relating bundles with non–abelian structure group to line bundles and consider some possible applications of the idea.

The starting point is the cotangent space at a point m of the moduli space. For stable bundles this is

$$T_m^* = H^0(\Sigma; \mathfrak{g} \otimes K)$$

where $\mathfrak{g}$ is the (complex) bundle of Lie algebras associated to the principal bundle represented by m .

If $p_1, \dots, p_\ell$ ($\ell = \text{rank } G$) are a basis for the invariant polynomials on the Lie algebra, with degrees $d_1, \dots, d_\ell$ respectively, then evaluating them on a cotangent vector gives a map

$$p : T^*M \longrightarrow \bigoplus_{i=1}^{\ell} H^0(\Sigma; K^{d_i}) = W .$$

By Riemann–Roch and a well-known identity,

$$\dim W = \sum_{i=1}^{\ell} (2d_i - 1)(g - 1) = \dim G(g - 1) .$$

In the case of marked points, with divisor $D = x_1 + \dots + x_n$

$$T^*_m = \{\alpha \in H^0(\Sigma; \mathfrak{g} \otimes K(D)) \,|\, \alpha(x_i) \in \mathfrak{b}_i \text{ and is nilpotent}\}$$

where $\mathfrak{b}_i$ is the Borel subalgebra at $x_i \in \Sigma$. Applying an invariant polynomial p_i we get a section of $K^{d_i} D^{d_i}$ which vanishes at $x_1, \dots, x_n$ by the nilpotency and so a map to $W = \bigoplus_{i=1}^{\ell} H^0(\Sigma; K^{d_i} D^{d_i - 1})$. Here,

$$\dim W = \Sigma(2d_i - 1)(g-1) + n\,\Sigma(d_i - 1)$$

$$= \dim G(g-1) + n\,\frac{(\dim G - \ell)}{2}$$

$$= \dim G(g-1) + n\,\dim(G/B)\,.$$

In both cases we have <u>$\dim W = \dim M$</u> .

By a general argument (see [1]), these functions Poisson commute and are functionally independent, making

$$p : T^*M \longrightarrow W$$

a completely integrable Hamiltonian system.

The symplectic manifold can actually be embedded as an open set $T^*M \subset \mathscr{M}$ in a bigger sympletic manifold (the moduli space of stable "Higgs bundles") such that

$$p : \mathscr{M} \longrightarrow W$$

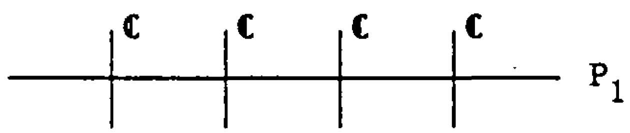
is proper. The generic fibre is then an abelian variety, in general a "Prym—Tyurin variety".

<u>Example</u> The moduli space of parabolically semi–stable $SL(2,\mathbb{C})$ bundles on P^1 with 4 marked points is P^1 .The map $p : T^*P^1 \longrightarrow \mathbb{C}$ is of the form

$$p(\eta,\varphi) = \eta^2 q(\varphi) \text{ with } q(\varphi) \text{ a quartic polynomial.}$$

If $c \neq 0$, then $\eta^2 q(\varphi) = c$ is an elliptic curve minus its four branch points over P_1.

If $c = 0$, then $\eta^2 q(\varphi) = 0$ consists of the zero section of T^*P^1 with 4 fibres:



$\mathcal{M}$ is formed by adding four lines to complete the elliptic curves and the fibres and give a familiar elliptic surface.

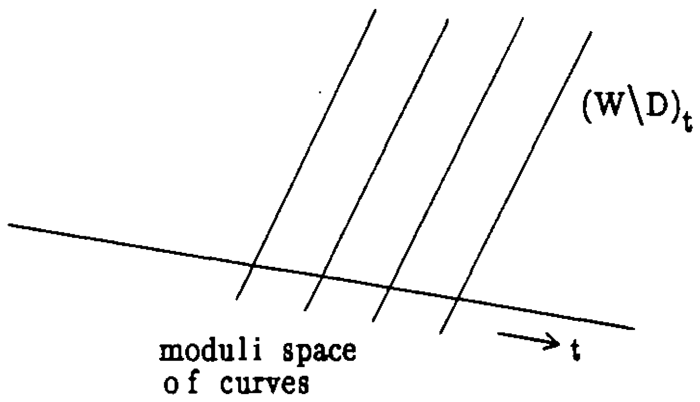
The general case is similar – the singular fibre $p^{-1}(0)$ consists of M and other components.

The integrable system of the above example is a case of Euler's spinning top equation.

In W lies the discriminant locus D – the divisor of singular values of p. We have then associated to each (Σ,G) a family of abelian varieties parametrized by $W\setminus D$, and therefore have a representation

$$\pi_1(W\setminus D) \longrightarrow Sp(2 \dim G(g-1),\mathbb{Z}) .$$

The invariant vectors of $\pi_1(W\setminus D)$ acting through representations of this symplectic group give vector spaces associated to the Riemann surface. Moreover, by letting the conformal structure of the Riemann surface change we obtain a bigger family of abelian varieties.



The covariant constant sections of a flat connection along $(W \backslash D)_t$ have an induced flat connection along the moduli space of curves.

Conjecture 1: The representation of $Sp(2n,\mathbb{Z})$ on theta–functions defines this way a (projectively) flat connection on holomorphic sections of the determinant bundles of M.

Conjecture 2: The cohomology of the abelian variety which is invariant under the action of $\pi_1(W \backslash D)$ provides a suitable setting for the Casson invariant of 3–manifolds.

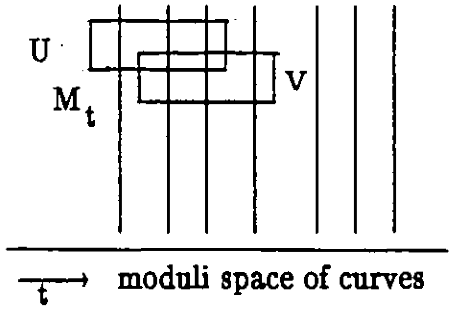
Apart from these questions the Poisson–commuting functions corresponding to the Killing form p_1 provide a means of obtaining the flat connection on the bundle of holomorphic sections of the determinant bundle, considered as a bundle over the moduli space of curves. Since we have the quadratic function

$$p_1 : T^*M \longrightarrow H^0(\Sigma;K^2)$$

then dually, we have a map

$$(*) \qquad H^1(\Sigma;K^{-1}) \longrightarrow H^0(M;S^2T)$$

and hence $(3g-3)$ symmetric tensors on M . These can be used to define local heat equations in the following way.



Choose coordinates on U , V and a trivialization of the determinant line bundle L , using $t_1, \dots, t_{3g-3}$ as part of the coordinate system. Then

$$\left.\frac{\partial}{\partial t_i}\right|_u - \left.\frac{\partial}{\partial t_i}\right|_v \quad \text{represents a class in } H^1(M_t, \mathscr{D}^1(L))$$

where $\mathscr{D}^1(L)$ is the sheaf of linear differential operators on L . The symbol map $\mathscr{D}^1(L) \longrightarrow T$ gives the Kodaira–Spencer class in $H^1(M;T)$.

Take a section $G = \Sigma\, G^{ij} \frac{\partial}{\partial z_i} \otimes \frac{\partial}{\partial z_j} \in H^0(M;S^2T)$ and on U and V define second order differential operators on L with G as symbol using the given coordinates:

$$D_u = \Sigma\, G^{ij} \frac{\partial^2}{\partial z_i \partial z_j} .$$

Then $D_u - D_v$ defines a class in $H^1(M_t, \mathscr{D}^1(L))$ which via the map (*) is a multiple of the deformation class. Thus (with some global assumptions) we have well–defined operators

$$\left.\frac{\partial}{\partial t_i}\right|_u - D_{iu} = \left.\frac{\partial}{\partial t_i}\right|_v - D_{iv} \text{ on } U \cap V.$$

Moreover,

$$\left[\frac{\partial}{\partial t_i} - D_i, \frac{\partial}{\partial t_j} - D_j\right] = -\frac{\partial D_j}{\partial t_i} + \frac{\partial D_i}{\partial t_j} + [D_i, D_j]$$

is a globally defined *second-order* operator since the symbols of D_i, D_j Poisson–commute. However, under the appropriate assumptions all such operators are constant scalars. Thus, defining covariant differentiation on sections of L by the heat equation

$$\left.\frac{\partial s}{\partial t_i}\right|_u = D_{iu} s$$

we have a (projectively) flat connection.

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SUPERSELECTION SECTORS WITH BRAID GROUP STATISTICS

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Braid group representations and link invariants which have been observed in 2–dimensional conformal field theory and in 3–dimensional topological field theory actually occur as invariants in generic quantum field theory in low dimensional space time. This follows by a generalization of the theory of superselection sectors developed by Borchers [2], Doplicher, Haag and Roberts [3]. The basic result of this theory is the intrinsic notion of statistics. To each sector (i.e. charge quantum number) there corresponds an up to equivalence unique unitary representation of the symmetric group. If one applies this theory to 2–dimensional field theory then one finds instead a unitary representation of the braid group.

The Doplicher–Haag–Roberts theory treats sectors which differ only locally from the vacuum. This includes all sectors which one can reach by applying fields to the vacuum which are relatively local to the observables; and recently Buchholz, Mack and Todorov [4] have shown that in conformal field theory all conformally covariant sectors with positive energy are of this type. On the other hand sectors of a more general type are known (or expected) to occur in gauge theories. There is a theorem due to Buchholz

* (based on joint work with K.H. Rehren and B. Schroer [1])

and myself [5] which states that in massive theories sectors containing one particle states differ from the vacuum only in a region which extends to spacelike infinity along some path. The DHR theory has been generalized to such sectors and yields a representation of the symmetric group in $d \geq 4$ dimensions. In $d = 3$ dimensions one finds a representation of the braid group.

The analysis of superselection sectors can best be carried out in the algebraic framework of quantum field theory. There the basic structure is the algebra of observables $\mathcal{A}$ together with its subalgebras $\mathcal{A}(\mathcal{O})$ of observables measurable in the spacetime region $\mathcal{O}$. The net $\mathcal{O} \longrightarrow \mathcal{A}(\mathcal{O})$ satisfies locality,

$$[\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2)] = 0 \quad \text{for} \quad \mathcal{O}_1 < \mathcal{O}_2'$$

$\mathcal{O}_2'$ denoting the spacelike complement of $\mathcal{O}_2$, and translation covariance

$$\alpha_x(\mathcal{A}(\mathcal{O})) = \mathcal{A}(\mathcal{O}+x) \ , \quad \alpha_x \in \mathrm{Aut}(\mathcal{A}) \, .$$

The aim is now, as formulated first by Borchers, to analyse the positive energy representations of $\mathcal{A}$. In the Doplicher–Haag–Roberts analysis one considers a more special class of representations π namely those which are equivalent to some fixed vacuum representation π_0 on the algebra of the spacelike complement of some bounded region $\mathcal{O}$. Using the corresponding unitary intertwiner one may realize π in the same Hilbert space as π_0, and

$$\pi(A) = \pi_0(A) \quad \text{for} \quad A \in \mathcal{A}(\mathcal{O}') \, .$$

One now can show that $\pi_0(A) \longrightarrow \pi(A)$, $A \in \mathcal{A}$, defines an endomorphism ρ of $\pi_0(\mathcal{A})$. The composition of endomorphisms yields new representations

$$\pi_1 \times \pi_2 = \rho_1 \rho_2 \pi_0 \quad , \quad \pi_i = \rho_i \pi_0 \,.$$

Thus, similar to quantum groups, the representations can be composed. In the following we shall omit the symbol π_0 and identify $\mathcal{A}$ will $\pi_0(\mathcal{A})$.

Now let us look at the implications of locality. The localization region $\mathcal{O}$ of ρ may be shifted to $\mathcal{O} + x \subset \mathcal{O}'$

$$\rho_x = \alpha_x \rho \, \alpha_{-x}$$

and ρ_x is equivalent to ρ

$$\rho_x(A) U_x = U_x \rho(A) \,.$$

Now $\rho\rho_x = \rho_x \rho$ because of locality, and

$$\rho^2(A) \rho \, (U_x)^{-1} U_x = \rho \, (U_x)^{-1} \rho \rho_x(A) U_x = \rho \, (U_x)^{-1} \rho_x \rho(A) U_x = \rho \, (U_x)^{-1} U_x \rho^2(A)$$

Hence

$$\varepsilon_\rho = \rho \, (U_x)^{-1} U_x$$

commutes with $\rho^2(\mathcal{A})$. ε_ρ is called the statistics operator. ε_ρ is locally constant in x .

Hence it is constant in $d \geq 3$ spacetime dimensions and may have two values in 2 spacetime dimensions.

ε_ρ satisfies the equation

$$\varepsilon_\rho \rho(\varepsilon_\rho)\varepsilon_\rho = \rho(\varepsilon_\rho)\varepsilon_\rho \rho(\varepsilon_\rho) .$$

Thus $\sigma_i \longrightarrow \rho^{i-1}(\varepsilon_\rho)$ generates a unitary representation of the braid group. In $d \geq 3$ dimensions one has in addition the relation

$$\varepsilon_\rho^2 = 1$$

which means that one has a representation of the symmetric group.

The representation of the braid group is evaluated by using a left inverse ϕ of ρ, i.e. a linear mapping from $\mathcal{A}$ to $\mathcal{A}$ such that $\rho\phi$ is a conditional expectation from $\mathcal{A}$ to $\rho(\mathcal{A})$. $\phi(\varepsilon_\rho)$ commutes with $\rho(\mathcal{A})$, hence for irreducible ρ we have

$$\phi(\varepsilon_\rho) = \lambda_\rho 1$$

where $\lambda_\rho \in \mathbb{C}$ is the so–called statistics parameter of ρ. By iterating ϕ one obtains a Markov trace on the braid group representation with Markov parameter λ_ρ, and by rescaling a link invariant.

The Markov trace fixes the braid group representation up to equivalence. If ρ^2 is irreducible, ε_ρ is a multiple of the identity, thus the induced braid group representation is one dimensional. In this case ρ turns out to be an automorphism, thus $\phi = \rho^{-1}$. If ρ^2

is a direct sum of 2 irreducible subrepresentations one obtains the Jones–Ocneanu–Wenzl representations of the braid group and the corresponding link invariants. In the general case ρ^2 is a direct sum of at most $|\lambda_\rho|^{-2}$ irreducible subrepresentations provided $\lambda_\rho \neq 0$. Thus the corresponding braid group representation, restricted to the braid group with n strands, is a multiple of a finite dimensional representation which, however, has not been determined in general up to now.

The discrete and locally finite nature of the superselection rules in the case $\lambda_\rho \neq 0$ leads to a direct definition of fusion rules and R matrices which satisfy all relations (pentagon–, hexagon relations etc.) usually attributed to conformal field theory. The only exception so far I can see are the implications of modular invariance found by Verlinde, namely the symmetry of the orthogonal matrix which diagonalizes the fusion rules. This relation describes a sort of self duality of the superselection rules and has (at least in the moment) no counterpart in the general theory.

There is also a direct connection to the classification of subfactors of von Neumann algebras due to Jones et Ocneanu. The number $|\lambda_\rho|^{-2}$ for instance turns out to be an index in the sense of Jones. Actually by a generalization of the notion of index for properly infinite algebras Longo [6] has shown that the index of the inclusion $\rho(\mathcal{A}(\mathcal{O})) \subset \mathcal{A}(\mathcal{O})$ for sufficiently large $\mathcal{O}$ is $|\lambda_\rho|^{-2}$.

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5) D. Buchholz, K. Fredenhagen, Comm. Math. Phys. 84, 1 (1982)

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RATIONAL CONFORMAL FIELD THEORY AND GROUP THEORY

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The seminar described work done in [1–5]. Further references to the literature may be found in these references.

The goal of the seminar was to explain the reasoning behind the conjecture [4] that all RCFT's may be obtained from some Chern–Simons gauge theory for a compact gauge group, along the lines described by E. Witten [6].

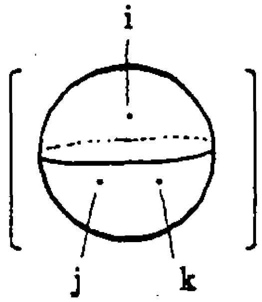
I. Duality Equations and the Tannaka–Krein viewpoint.

As is well–known from many points of view in RCFT one associates to a Riemann surface Σ with punctures P_i and corresponding representations j_i (of some chiral algebra) a corresponding vector space –

$$(\Sigma,P_i,j_i) \longrightarrow \mathscr{H}(\Sigma,P_i,j_i)\ .$$

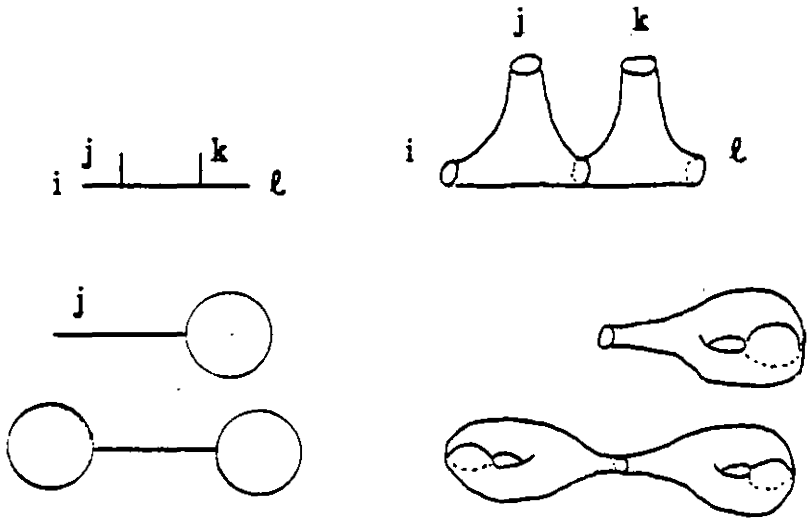
In RCFT this vector space is finite–dimensional and is known as the space of conformal blocks. Varying Σ we obtain a projectively flat vector bundle over the moduli space of curves. This is the basic datum in Friedan–Shenker "modular geometry" [7].

The vector spaces $\mathscr{H}$ may be characterized as the space of intertwiners for the chiral algebra defined by Σ (see, e.g. [3] and/or papers on the "operator formalism"), and, as in group theory the intertwining spaces can always be written in terms of those corresponding to trivalent couplings. In RCFT we therefore define the space of 3–point couplings to be:

$V_{ijk} = \mathscr{H}$ 

We may obtain such decompositions as follows. From a trivalent graph (duality diagram) we obtain an asymptotic region of Teichmüller space by 1) thickening the graph to obtain a surface, 2) using the graph to define a pants decomposition – hence a Fenchel—Nielsen coordinate system for Teichmüller space, 3) choosing the length parameters ℓ_i to be small. (The twist parameters range: $-\infty < \theta_i < \infty$. We should divide the line into 2π intervals, choose a region from each interval and regard these as distinct asymptotic regions).

A few examples of the corresponding decomposition ("physical factorization") might be helpful:



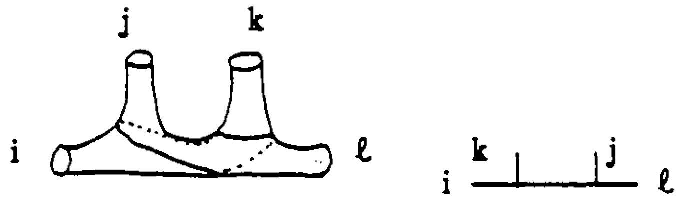
$$\mathscr{H}(\Sigma) \cong \bigoplus_p V_{ijp} \otimes V_{pk\ell}$$

$$\bigoplus_p V_{jpp}$$

$$\bigoplus_{p,j,\ell} V_{jjp} \otimes V_{p\ell\ell}$$

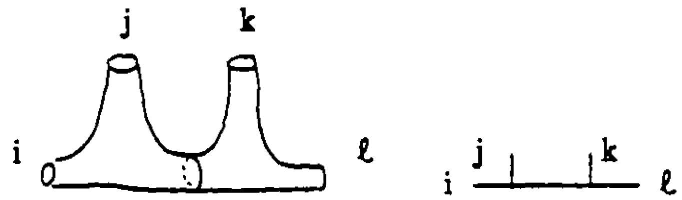
There are many asymptotic regions, so the decomposition is not unique.

Nevertheless we merely describe the same vectorspace in different ways so we deduce the existence of isomorphisms B and F :



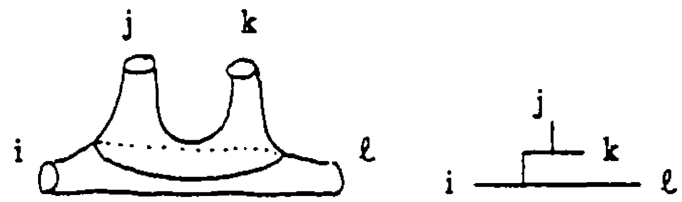
$$\bigoplus_p V_{ikp} \otimes V_{pj\ell}$$

$$\| \wr \; B\begin{bmatrix} j & k \\ i & \ell \end{bmatrix}$$



$$\bigoplus_p V_{ijp} \otimes V_{pk\ell}$$

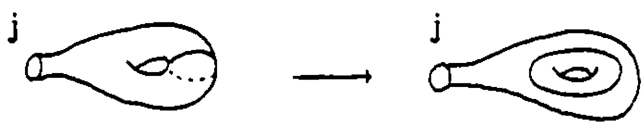
$$\| \wr \; F\begin{bmatrix} j & k \\ i & \ell \end{bmatrix}$$



$$\bigoplus_p V_{jkp} \otimes V_{pi\ell}$$

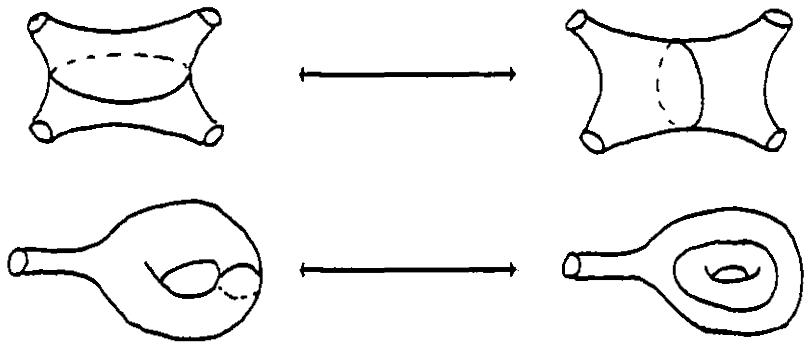
B and F are known as braiding and fusing isomorphisms, respectively.

Similarly, comparing asymptotic regions of $g = 1$ Teichmüller space related by $\tau \longrightarrow -1/\tau$ (τ = modular parameter) we deduce the duality tranformation $S_{(j)}$:



$$S_{(j)} : \bigoplus_p V_{jpp} \longrightarrow \bigoplus_p V_{jpp}$$

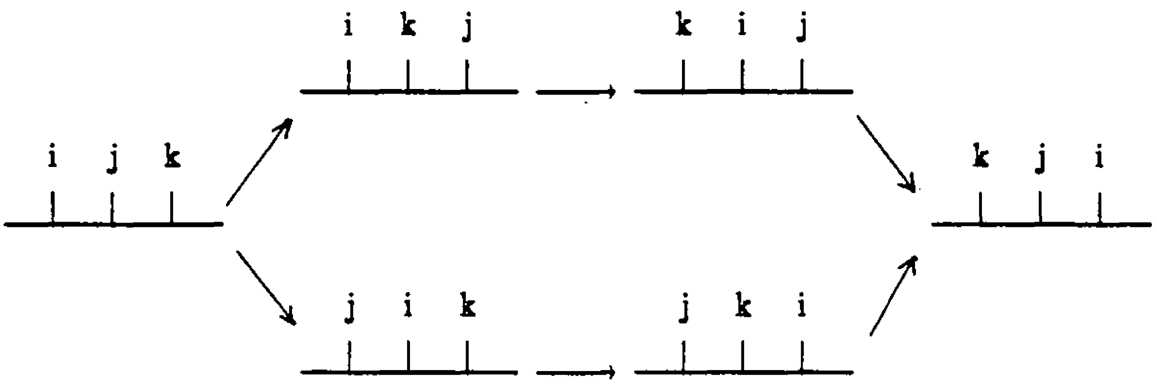
The first main point is that with the additional data of $e^{2\pi i c/24}$ (c = central charge of the Virasoro algebra) we can express *all* duality transformations for all surfaces in terms of the data B,F,S . The basic reason for this is that if one forms a complex whose vertices correspond to asymptotic regions (hence sewings) with 1–simplices generated by the simple–moves



then the resulting complex is connected.

We would like to try to characterize RCFT's in terms of the data B,F,S and to that end we should study the relations on these transformations. Such relations arise from closed loops on the 1–complex defined above. That is, the same duality transformation can be expressed in terms of different paths of simple moves. A famous example of this is given by the hexagon:

(*)



From this one learns that $B_{pq}\begin{bmatrix} j & k \\ i & \ell \end{bmatrix} : V_{ijp} \otimes V_{pk\ell} \longrightarrow V_{ikq} \otimes V_{qj\ell}$ satisfies the Yang–Baxter equations in IRF–form. While the relation (*) is very well–known it is only part of a much larger story. There are an infinite number of such duality relations following from all the closed loops on all possible moduli spaces.

The second main point is that, of all the relations described above, there are only a finite number of independent relations [1,3]. This is known as the completeness theorem.

To write these relations we first recall that there is a special representation $\mathscr{H}_0$ of the chiral algebra generated by the unit operator. The special properties of the unit operator imply properties:

$$V_{oij} \cong V_{ioj} \cong \cdots \cong \delta_{ij}\mathbb{C} \, .$$

(For simplicity we assume all representations are self–conjugate. More precise statements can be found in the references.) Thus if we put an external representation $\ell = o$ the B–matrix becomes a transformation:

$$\Omega_{ijk} = B\begin{bmatrix} j & k \\ i & o \end{bmatrix} \quad : \quad V_{ijk} \longrightarrow V_{ikj}$$

$$: \quad i \overset{j}{\longrightarrow} k \longrightarrow i \overset{k}{\longrightarrow} j \, .$$

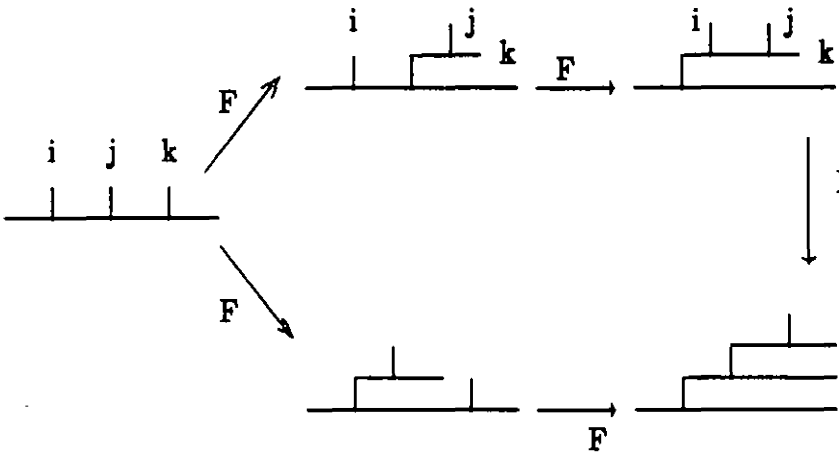
Here we have introduced a graphical notation for vector spaces.

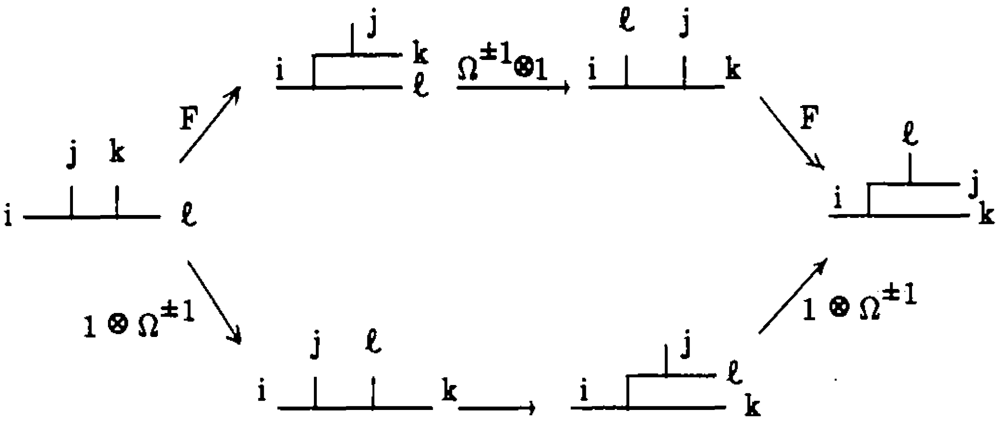
In conformal field theory Ω^2 is a scalar operator, but is not 1. Rather

$$\Omega^2_{ijk} = e^{2\pi i(\Delta_j + \Delta_k - \Delta_i)} 1$$

is the mutual locality factor.

The basic relations are

1) 

2,3) 

4) $$S^2_{(j)} = 1 \times \text{(phase)}$$

5) $$(ST)^3 = 1 \times \text{(phase)}$$

together with a relation expressing S in terms of B,F . For $j = o$ this relation is

(6) $$\frac{S_{ij}}{S_{oo}} = \frac{\left[B\begin{bmatrix} i & j \\ i & j \end{bmatrix} B\begin{bmatrix} j & i \\ i & j \end{bmatrix} \right]_{oo}}{F_i F_j}$$

where F_i is the "gauge–invariant" ([1]) fusion matrix element

$$F_i = F_{oo}\begin{bmatrix} i & i \\ i & i \end{bmatrix} .$$

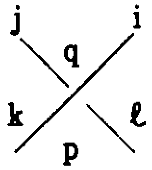
There is a formula for $S_{(j)}$ similar to (6) for the case $j \neq o$.

Further details and a description of the phases in 4,5 may be found in [3].

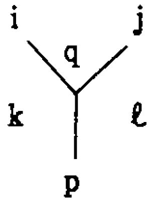
There is a strong analogy between the above relations and the pentagon/hexagon relations satisfied by commutativity and associativity constraints in tensor categories [8]. In fact, when $\Omega^2 = 1$ conditions 2,3 become identical and 1,2 are just the pentagon/hexagon relations of category theory.

Regarding the above equations as axioms on the data V_{ijk},B,F,S there is a close relationship with the axioms of a Tannakian category. The relation can be made precise by considering the classical limit of a conformal field theory. It often happens that conformal field theories naturally lie in a sequence of theories whose fusion rules stabilize, such that the duality matrices have a well–defined limit. For example in level k SU(2) current algebra we have $SU(2)_k \longrightarrow SU(2)_{k+1} \longrightarrow \ldots$

In such a limit $\Omega^2 \longrightarrow 1$ and we obtain precisely the axioms of a rigid abelian tensor category. With one more axiom it follows from the work of P. Deligne that we in fact obtain a Tannakian category, the category of representations of some compact group. (Indeed, in the large level limit of WZW models B,F become 6j symbols and V_{ijk} become the spaces of interwiners $\mathrm{Hom}_G(R_i, R_j \otimes R_k)$.) We may express Deligne's extra magic condition in terms of "classical knot invariants". It is well–known that one can compute invariants of knotted graphs via the rules [10, 11].

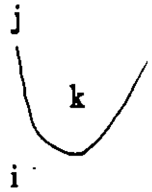


$$\sim B_{pq}\begin{bmatrix} j & k \\ i & \ell \end{bmatrix}$$

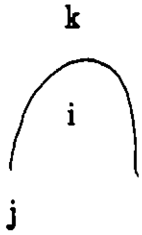


$$\sim F_{pq}\begin{bmatrix} i & j \\ k & \ell \end{bmatrix} .$$

Instead of introducing a Markov trace we may also introduce creation/annihilation amplitudes:



$$= \alpha_j F_{ok}\begin{bmatrix} i & j \\ i & j \end{bmatrix}$$



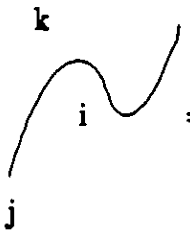
$$= \beta_j F_{io}\begin{bmatrix} k & k \\ j & j \end{bmatrix} .$$

This form for the amplitude can be deduced from consistency conditions such as

=

etc.

The coefficients α_j, β_j are determined from



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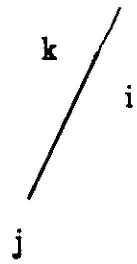
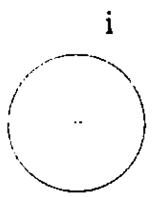


.

We learn $\alpha_j \beta_j = F_j^{-1}$. Thus the value of an unknotted, unlinked circle is



$$= \frac{1}{F_i} .$$

All of the above works in the classical "group theory" case. The knot invariants are not very interesting – but they *are* normalized. One may calculate classically that F_i^{-1} is the dimension of the representation R_i . Deligne's condition is that F_i^{-1} should be a nonnegative integer.

In RCFT F_i^{-1} is no longer an integer but, as shown in [12]

$$\frac{1}{F_i} = \frac{S_{io}}{S_{oo}} = \lim_{q\to 1} \frac{\mathrm{tr}_i q^{L_o - c/24}}{\mathrm{tr}_o q^{L_o - c/24}} = \frac{\text{"dim } \mathscr{H}_i\text{"}}{\dim \mathscr{H}_o} .$$

The first equality follows from (6) above.

Thus, it is not unreasonable to hope that by adding an appropriate axiom on the quantity F_i^{-1} we may find a quantum version of the reconstruction theorem.

Of course, reconstruction is much easier given a knowledge of what it is one wants to reconstruct. It is here that Witten's 3–dimensional viewpoint proves quite helpful.

II. Three–Dimensional Perspective

In [6] Witten showed that $\mathscr{H}(\Sigma)$ for level k G–current algebra is the same as the space of physical states in a corresponding Chern–Simons gauge theory when we canonically quantize on the 3–manifold $\Sigma \times \mathbb{R}$. One obvious question left open in [6] is whether other RCFT's can be described in a similar fashion. In the physics literature one finds a veritable zoo of RCFT's but these always seem to be one of the following 3 types: 1) Extensions of affine algebras, 2) coset models, 3) orbifolds of the above. One describes these as follows:

1) To form extended algebras, one uses the "spectral flow" transformation associated to automorphisms of extended Dynkin diagrams. Thus, if we wish to extend G_k current algebra we begin with $\theta \in \mathrm{Center}(G)$ and write $\theta = e^{2\pi\mu}$ for some weight vector μ . (For simplicity we take $G = SU(n)$, the discussion can be generalized.) The integrable level k representations are given by the points in the Weyl alcove:

$$\Lambda_{wt} / W \ltimes k \Lambda_{rt} \cdot$$

The transformation $\lambda \longrightarrow \lambda + k\mu$ is equivalent, via the affine Weyl group to a transformation $\lambda \longrightarrow \mu(\lambda)$ of highest weight representations. For example, for $SU(2)_k$ thetransformation is $j \longrightarrow \frac{k}{2} - j$. For any subgroup $Z \subset \mathrm{Center}(G)$ we can "mod out" by this action thus obtaining the extended chiral algebra

$$\mathfrak{A} = \bigoplus_{\mu \in Z} \mathscr{H}_{\mu(0)} \cdot$$

(The algebra is only consistent for appropriate values of k .) In CSGT this simply corresponds to modifying the gauge group $G \longrightarrow G/Z$.

Such a change in the gauge group has important consequences for the observables of the theory. For example, for $SO(3) = SU(2)/\mathbb{Z}_2$ the observables are the Wilson lines:

$$W_j(\mathscr{C}) = \mathrm{Tr}_j(P \exp \oint_{\mathscr{C}} A)$$

but we have

(a) $j \in \mathbb{Z}$, only odd–dimensional representations exist. Moreover, $k = 0 \bmod 4$, to avoid global anomalies.

(b) $W_j(\mathscr{C}) \cong W_{\frac{k}{2}-j}(\mathscr{C})$ in the sense that they have the same correlation functions

(c) $W_{j=\frac{k}{4}}(\mathscr{C}) = \mathcal{O}^+ + \mathcal{O}^-$ where the operators $\mathcal{O}^{\pm}$ cannot be conveniently expressed in terms of Wilson lines.

The rules a,b,c apply quite generally in the representation theory of extended algebras.

2) GKO Coset models. We can define a coset model $\hat{\mathfrak{g}}_{k_G}/\hat{\mathfrak{h}}_{k_H}$ when H is a subgroup of G , $H \hookrightarrow G$ such that if the embedding index is ℓ then $k_H = \ell k_G$. Decomposing the $\widetilde{LG}$ representations $\mathscr{H}_\Lambda$ (thus Λ is an integrable level k_G representation of $\widetilde{LG}$)

in terms of $\widetilde{LH}$ representations $\mathcal{H}_\lambda$ we have

$$\mathcal{H}_\Lambda \cong \bigoplus_\lambda \mathcal{H}_{\Lambda,\lambda} \otimes \mathcal{H}_\lambda \, .$$

As Goddard–Kent–Olive showed the spaces $\mathcal{H}_{\Lambda,\lambda} = (\mathcal{H}_\Lambda \otimes \mathcal{H}^*_\lambda)^{LH}$ are Virasoro algebra modules with central charge $C_G - C_H$ and can be used to define a rational conformal field theory with chiral algebra:

$$\mathfrak{A}^{G/H} = \mathcal{H}_{\Lambda=0,\lambda=0} \, .$$

To obtain a CFT we must have a unique vacuum, which excludes conformal embeddings.

To reproduce these theories from CSGT we introduce G,H gauge fields A,B , respectively, choose the action

$$\frac{k_G}{4\pi} \int \mathrm{Tr}(AdA + \tfrac{2}{3} A^3) - \frac{k_H}{4\pi} \int \mathrm{Tr}(BdB + \tfrac{2}{3} B^3)$$

and take the gauge group $\frac{G \times H}{Z}$ where Z is the common center. (This description of the gauge group must be used with care if there are U(1) factors.) To see–at least heuristically – why this prescription works let us return to the quantization of CSGT for the group G at level k on the 3–fold $D \times \mathbb{R}$ where D = disk . We rephrase the argument of [6] as a change of variables in the functional integral

$$\int \frac{DA}{\mathrm{vol}\, \mathfrak{G}} e^{\frac{ik}{4\pi} \int \mathrm{Tr}(AdA + \frac{2}{3} A^3)} \, . \tag{7}$$

To do the path–integral we must specify boundary conditions on A , which can be determined by requiring that there are no boundary contributions to the equations of motion. Thus, the variation of the action is

$$\delta S = \frac{k}{4\pi} \int_{\partial D \times \mathbb{R}} \mathrm{Tr}(\delta A A) + \underbrace{\ldots}_{\text{volume term}}$$

so we may choose $A_t\Big|_{\partial D\times\mathbb{R}} = 0$, where the $\mathbb{R}$–direction is identified with time t . The gauge group $\mathfrak{G}$ must preserve boundary conditions, hence g is time–independent on $\partial D \times \mathbb{R}$.

Seperating out global symmetries which are not 1 at $t = \pm\infty$ we have the gauge group:

$$\mathfrak{G} = \{g : D \times \mathbb{R} \longrightarrow G \quad | \quad g|_{\partial D \times \mathbb{R}} = 1\} .$$

We may now split the gauge field and exterior derivative into time and space components

$$A = A_t + \tilde{A}$$

$$d = dt \frac{\partial}{\partial t} + \tilde{d}$$

and rewrite the action

$$S = \frac{k}{4\pi} \int \mathrm{Tr}(\tilde{A} \frac{\partial}{\partial t} \tilde{A})dt + \frac{k}{2\pi} \int \mathrm{Tr}\, A_t \tilde{F} .$$

Now integrate over A_t to produce the δ–function $\prod_{x,a} \delta(\tilde{F}^a(x))$. The constraint is solved by $\tilde{A} = -\tilde{d}U\, U^{-1}$ for $U : D \times \mathbb{R} \longrightarrow G$. One may show that there is no Jacobian for transforming $D\tilde{A}\, \delta(\tilde{F}) = DU$, the second measure being defined by the Haar measure. Hence the path–integral (7) becomes

$$\int_{U:D\times\mathbb{R}\to G} \frac{DU}{\mathrm{vol}\ \mathfrak{G}}\, e^{\frac{ik}{4\pi} \int_{\partial D\times\mathbb{R}} \mathrm{Tr}(U^{-1}\partial_\varphi U U^{-1}\partial_t U) + ik\Gamma_{WZ}(U)}$$

where φ is the angle on ∂D and Γ_{WZ} is the Wess–Zumino term. Since the action only depends on the boundary values of U we can trivially factor out the gauge volume. The resulting action is first order in time, from which one deduces the phase space LG/G together with the correct symplectic structure to give the basic representation of LG upon quantization. A similar exercise shows that if a Wilson line in representation λ pierces the disk in the time direction then quantization produces the space of states $\mathscr{H}_\lambda$.

For the coset models we must evaluate a similar path integral

$$\int \frac{DA\,DB}{\mathrm{vol}\ \mathfrak{G}}\, e^{ik_G CS(A) - ik_H CS(B)} \tag{8}$$

where CS is the Chern–Simons functional. Proceeding as before we vary the action

$$\delta S = \frac{k_G}{4\pi}\int \mathrm{Tr}\,\delta A A - \frac{k_H}{4\pi}\int \mathrm{Tr}\,\delta B B + \ldots$$

When $H \hookrightarrow G$ and $\ell k_G = k_H$ we may choose a special boundary condition. Let $\pi : \mathfrak{g} \longrightarrow \mathfrak{h}$ be the orthogonal projection with the Killing form and set:

$$\pi(A) = B \quad \text{on} \quad \partial D \times \mathbb{R}$$

$$\pi(A_t) = A_t \quad \text{on} \quad \partial D \times \mathbb{R}\,.$$

By arguments parallel to those above we find the gauge group should be:

$$\mathfrak{G} = \{(g,h) : D \times \mathbb{R} \longrightarrow \frac{G \times H}{Z} \;\Big|\; g = h|_{\partial D \times \mathbb{R}}\}\,.$$

We may carry out the change of variables in the same way so that:

$$A = -\tilde{d}U\,U^{-1} \qquad U : D \times \mathbb{R} \longrightarrow G$$

$$B = -\tilde{d}V\,V^{-1} \qquad V : D \times \mathbb{R} \longrightarrow H\,.$$

Taking account of gauge–fixing we obtain from (8)

$$\int DU\, DV\, D\lambda\, e^{ik_G S_{WZW}(U) - ik_H S_{WZW}(V) + \int \mathrm{Tr}\, \lambda(U^{-1}\partial_\varphi U - V^{-1}\partial_\varphi V)}$$

where S_{WZW} is the WZW functional and $\lambda : D \times \mathbb{R} \longrightarrow \mathfrak{h}$ is a Lagrange multiplier. Taking account of the first order constraint we see that quantization gives (with insertion of a Wilson Line) the space of states $\mathscr{H}_{\Lambda,\lambda} = (\mathscr{H}_\Lambda \otimes \mathscr{H}^*_\lambda)^{LH}$ of the coset model.

3) Orbifolds. To obtain a CSGT description of orbifolds of the above theories we use an automorphism group P of the group G to form a new gauge group $P \ltimes G$. The above arguments show that the chiral algebra $\mathfrak{A}$ of the G–theory is reduced to the chiral algebra $\mathfrak{A}/P$.

Thus, the entire zoo of RCFT's is nicely organized – from the $2+1$ dimensional perspective – by a choice of gauge group and levels for simple factors (including U(1)'s). This leads to a natural conjecture that all RCFT's may be obtained, along the lines indicated above, from some CSGT with a compact gauge group G. The proof might well proceed by a version of quantum reconstruction. For reasons indicated in the following section one might try to use the axioms sketched in section I to reproduce the category of representations of a quantized universal enveloping algebra for suitable values of the parameters q (i.e. suitable roots of unity).

The above conjecture may be interpreted as saying that there are no new RCFT's beyond the ones we know, and as such would be something of a disappointment for string—theory model builders. The idea that the coset construction essentially exhausts nontrivial rational models was probably first stated by E. Martinec in [13] and has been proposed by P. Goddard, V. Bazhanov, N. Reshetikhin, the authors of [14], and perhaps many others.

The point of [4] was that a) the *axioms* naturally lead to such a conjecture b) the simplicity of the CSGT description naturally leads to a precise version of the conjecture and c) we have a framework for trying to prove the conjecture.

III. Some Remarks on Quantum Groups

We noted above that for level k G–current–algebra, in the classical limit ($k \longrightarrow \infty$) the F,B matrices become 6j–symbols. In fact, as noted by many people for finite k,F and B are precisely related to the 6j–symbols of the quantized universal enveloping algebra $U_q(G)$ for an appropriate value of q. For example for SU(2) level k RCFT we consider $U_q(s\ell(2))$, generated by $J^{\pm}, H$ satisfying

$$[H,J^{\pm}] = \pm 2J^{\pm}$$

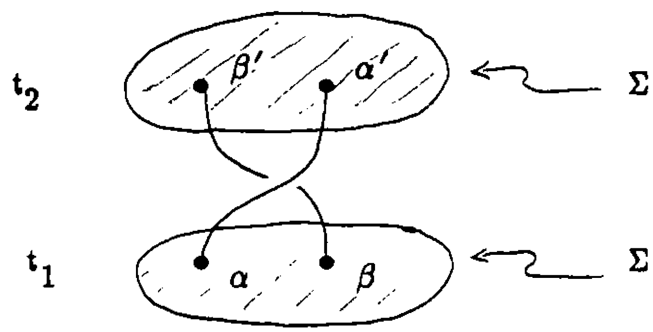
(9)

$$[J^+,J^-] = \frac{q^{H/2} - q^{-H/2}}{q^{1/2} - q^{-1/2}}$$

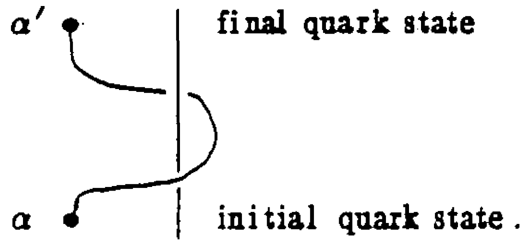
for $q = e^{2\pi i/k+2}$. For generic q the representation theory of (9) is similar to that of $s\ell(2)$, in particular, representations are parametrized by a 1/2–integer spin j :

$$\tilde{W}_j = \text{Span}\ \{\ |\ j,\alpha > : -j \le \alpha\ \le j\}\ .$$

This continues to be true for the special vlaues of q in the case of "good representations" $(j \le K/2)$. In [5] and in the contribution of Fröhlich to this conference a class of operators in RCFT was defined which helps one to understand the coincidence of F and B matrices. Rather than repeat those formulae here we give an intuitive discussion with in the framework of CSGT. In the Chern–Simons theory the new operators act like quarks with a gauge theory index and a quantum group charge α corresponding to the state $|\ j,\alpha > \in \tilde{W}_j$. Quarks terminate Wilson lines and therefore we may consider transition amplitudes like:



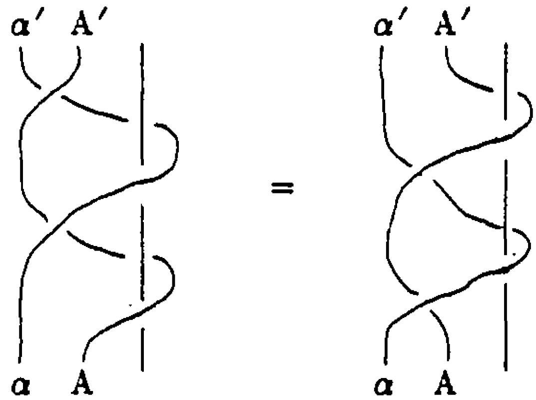
with amplitude $R_{\alpha\ \beta}^{\beta'\alpha'} = \rho \otimes \rho\,(\mathfrak{R})$, where $\mathfrak{R}$ is the universal $\mathfrak{R}$–matrix and ρ is the spin–j representation. With this picture we can interpret one Wilson line wrapping around the other as an operator on quark states:



In formulas we have defined an operator:

$$L_\alpha{}^{\alpha'} : |\beta> \longrightarrow \sum_{\beta'} (R_{12}R_{21})_{\alpha}{}^{\alpha'}{}_{\beta}{}^{\beta'} \, |\beta'>$$

We now ask what relations the operators $L_\alpha{}^{\alpha'}$ satisfy.. Consider two representations $|\alpha> \in \tilde{W}_{j_1}$, $|A> \in \tilde{W}_{j_2}$ and corresponding matrices of operators $L_\alpha{}^{\alpha'}$ and $L_A{}^{A'}$. Furthermore, consider the Yang–Baxter matrix of c–numbers $R_{12} = \rho_{j_1} \otimes \rho_{j_2}(R)$. The diagram



implies the relations

(10) $$R_{12}(1 \otimes L)R_{12}(L \otimes 1) = (L \otimes 1)R_{21}(1 \otimes L)R_{12}$$

where: $$1 \otimes L = \delta_\alpha{}^{\alpha'} \, L_A{}^{A'}$$

$$L \otimes 1 = L_\alpha{}^{\alpha'} \, \delta_A{}^{A'} .$$

One may check that (10) is indeed a presentation of the defining relations of a quantum group. For example, choosing $j_1 = j_2$ to be the spin–1/2 representation we may parametrize

$$L = \begin{bmatrix} q^{1/2H} & (1-q^{-1})q^{1/4}q^{1/4H}J^- \\ (1-q^{-1})q^{1/4}J^+q^{1/4H} & q^{-1/2H}+(1-q^{-1})^2q^{1/2}J^+J^- \end{bmatrix}$$

from (10) one recovers (9). The above gives a description of the quantum group action in CSGT. Unfortunately, the definition of the quark operators is, at present, somewhat contrived and their existence awaits a more natural explanation.

It would be useful to understand the quantum group connection better since, if the above conjecture is true then to every RCFT/CSGT one may associate a quantum group for special values of q . This strongly suggests that the proof of the conjecture will proceed by showing that the defining axioms lead to the representation theory of a quantum group for which the deformation parameters are roots of unity.

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MONODROMY OF BRAID GROUPS AND QUANTUM GROUPS

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By investigating the holonomy of the Knizhnik–Zamolodchikov connection, we show that the monodromy representations of the braid groups appearing in the conformal field theory on the Riemann sphere with gauge symmetry can be described by means of the quantized universal enveloping algebras in the sense of Drinfel'd and Jimbo. For any simple Lie algebra $\mathfrak{g}$ and its irreducible representations $P_i : \mathfrak{g} \longrightarrow \mathrm{End}(V_i)$, $1 \le i \le n$, the Knizhnik–Zamolodchikov connection is defined to be the 1–form $\omega = \sum_{i<j} \lambda\, \Omega_{ij}\, d\log(z_i - z_j)$, $\lambda \in \mathbb{C}$, where $\Omega_{ij} \in \mathrm{End}(V_1 \otimes \dots \otimes V_n)$ is given by $\Omega_{ij} = \sum_{\mu} P_i(I_\mu) P_j(I_\mu)$ by using the Casimir element $\Sigma\, I_\mu \cdot I_\mu$. The quadratic relations $[\Omega_{ik}, \Omega_{ij} + \Omega_{jk}] = 0$ (i,j,k distinct) and $[\Omega_{ij}, \Omega_{k\ell}] = 0$ (i,j,k,ℓ distinct) provide the integrability of the connection ω and they are considered to be the infinitesimal version of the relations for the pure braid group. In fact it can be shown by means of K.T. Chen's iterated integrals that the completion of the group ring of the pure braid group over $\mathbb{C}$ is isomorphic to the algebra $\mathbb{C} \ll X_{ij} \gg / J$, where $\mathbb{C} \ll X_{ij} \gg$ is the non–commutative formal power series with indeterminates X_{ij}, $1 \le i < j \le n$, and J is the ideal generated by the above infinitesimal pure braid relations for X_{ij}. Based on this method, we describe the holonomy of the Knizhnik–Zamolodchikov connection by means of the R–matrix with $q = \exp \pi\sqrt{-1}\,\lambda$.

Now we consider the situation of the rational conformal field theory on the Riemann sphere with gauge symmetry of the affine Lie algebra $\hat{\mathfrak{g}}$ at level k. For the chiral vertex operators $\phi_i(u_i,z_i)$, $u_i \in V_{\pi_i}$, $z_i \in \mathbb{C} \setminus \{0\}$, sending $\mathscr{H}_{\Lambda_{i-1}}$ to $\mathscr{H}_{\Lambda_i}$, the n–point function $\phi(z) = \langle 0 | \phi_n(u_n,z_n) \cdots \phi_1(u_1,z_1) | 0 \rangle$ is a horizontal section of the connection ω with $\lambda = \frac{1}{k+h}$, where h is the dual Coxeter number.

On the other hand, associated with the fusion path $p = (\Lambda_0,\Lambda_1, \dots ,\Lambda_n)$, we consider the composition of the q–Clebsch–Gordan coefficients

$C_{\Lambda_i}^{\pi_i \Lambda_{i-1}} : V_{\pi_i} \otimes V_{\Lambda_{i-1}} \longrightarrow V_{\Lambda_i}$, which are the intertwiners as $U_q(\mathfrak{g})$–module. The action of the braid group on these fusion paths via the R–matrix was described by Reshetikhin. We show that under a suitable normalization of the n–point functions the action of the braid group is given by the above action on q–Clebsch–Gordan coefficients with $q = \exp \frac{\pi \sqrt{-1}}{k+h}$. This braiding matrix was explicitly obtained by Tsuchiya and Kanie in the case $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$ and each P_i is the vector representation. It turns out that the associated monodromy representation of the braid group factors through the Jones algebra with index $4 \cos^2 \frac{\pi}{k+2}$. We claim in a more general situation that the monodromy of the braid group factors through semi–simple algebras with a positive Markov trace.

ALGEBRA COCHAINS AND CYCLIC COHOMOLOGY

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In this talk we describe a formalism of algebra cochains which was developed to understand certain calculations in cyclic cohomology.

Let A be a non unital algebra over $\mathbb{C}$ and let B(A) be the bar construction of the augmented algebra $\mathbb{C} \oplus A$. The bar construction is naturally a DG coalgebra, whose cocommutator subspace $B(A)^{\#}$ (this is the coalgebra analogue of $R/[R,R]$ for an algebra R) can be identified with the cyclic complex $C^{\lambda}(A)$ by means of the norm map $N : C^{\lambda}(A) \longrightarrow B(A)^{\#}$.

If R is an algebra, then $\mathrm{Hom}_{\mathbb{C}}(B(A),R)$ is naturally a DG algebra whose elements of degree p are multilinear functions $f(a_1, \dots ,a_p)$ on A called cochains with values in R. If τ is a trace defined on an ideal J in R, then $\tau^{\#}(f) = \tau f N$ is a trace on the ideal Hom(B(A),J) with values in cyclic cochains.

As an application consider the following situation studied by Connes. Let $\rho : A \longrightarrow R$ be a linear map which is an algebra homomorphism modulo an ideal I in R, and let τ be a trace on I^m. We view ρ as a 1–cochain analogous to a "connection" form in Chern–Weil theory. The "curvature" $\omega = d\rho + \rho^2$ is the 2–cochain $\omega(a_1,a_2) = \rho(a_1,a_2) - \rho(a_1)\rho(a_2)$ having values in I. Standard arguments using the Bianchi identity show that the cyclic cochains $\tau^{\#}(\omega^n)$ for $n \geq m$ are cyclic cocycles, and these turn out to be the odd degree cyclic cocycles produced by Connes in this situation.

In order to prove S–relations among cyclic cohomology classes, as well as to understand better entire cyclic cohomology, we need to produce Hochschild cochains, that is, cochains where the differential is $\pm b$ instead of $\pm b'$. There is an operation which associates to cochains f , g as above and a trace τ a Hochschild cochain $\tau^{\#}(\partial f\, g)$ in such a way as to be compatible with differentials and such that

$$\tau^{\#}(\partial(f_1 f_2)g) = \tau^{\#}(\partial f_1(f_2 g)) \pm \tau^{\#}(\partial f_2(g f_1))$$

Using this operation the entire cyclic cocycle of Jaffe–Lesniewski–Osterwalder can be interpreted as the analogue of a superconnection character form in this cochain formalism.

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CONNECTION ON THE SPACE OF CONFORMAL BLOCKS VIA GEOMETRIC QUANTIZATION

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An outstanding problem in geometric quantization is to construct a natural identification, $h_{J'J} : \mathscr{H}_J \xrightarrow{\cong} \mathscr{H}_{J'}$, between the Hilbert spaces obtained by Kähler quantizing a symplectic manifold, $\mathfrak{M}$, using complex structures J and J' . It should be reasonable in the sense that all quantum operators on $\mathscr{H}_J$ are taken to those on $\mathscr{H}_{J'}$ and $h_{J''J'}h_{J'J} = h_{J''J} \times$ (projective factor) . In this way the dependence of the quantum Hilbert space and operators on the auxiliary choice of J is trivial.

In our work we apply the known result for an affine symplectic manifold and geometric invariance theory to find the desired identification for $\mathfrak{M}$ = a symplectic quotient of an affine manifold, $\mathscr{A}$, by a subgroup, $\mathscr{G}$, of the affine symplectic group.

This is naturally accomplished as follows. Let $\mathscr{H} \longrightarrow \{J\}$ be the bundle over the space of $\mathscr{G}$ invariant complex structures, J , with fibers equal to the Hilbert space obtained from Kähler quantizing $\mathscr{A}$.

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* Joint work with Ed Witten and Steve Della Pietra in progress

Let $\tilde{\mathscr{H}} \longrightarrow \{J\}$ be the bundle with fiber $\tilde{\mathscr{H}}_J = \mathscr{H}_J^{\mathscr{G}}$ which by geometric invariance theory is the Hilbert space obtained from quantizing $\mathfrak{M}$. The old results for quantizing an affine space give us the desired identification in that case as the parallel transport of a projectively flat connection on $\mathscr{H}$. This connection is compatible with the $\mathscr{G}$ action on the fibers of $\mathscr{H}$ and so restricts to a projectively flat connection on $\tilde{\mathscr{H}}$, which gives the desired identification for the quotient case.

We need to do a version of regularization to make this connection well defined in the case where $\mathscr{A}$ is the infinite dimensional space of connections for a G–bundle over a surface Σ, and $\mathfrak{M}$ is the symplectic quotient of $\mathscr{A}$ by the group of gauge transformations. The complex structures on $\mathscr{A}$ are obtained as those induced by picking a complex structure, also called J, on Σ. The quantization of the space $\mathfrak{M}$ is part of Witten's solution of the Chern–Simon's gauge theory which gives a conceptual explanation of invariants of links in 3–manifolds. Witten uses the fact that the bundle $\tilde{\mathscr{H}}$ is the same as the bundle of conformal blocks for the well understood Wess–Zumino–Witten model for the group G.

Presumably all the results he uses about the conformal field theory can be derived from the three dimensional point of view. In this case, the projectively flat connection which we find from this point of view is simply the connection on the space of conformal blocks originally produced by conformal field theory. Here we show that the connection has the same curvature as that calculated in conformal field theory.

The approach to regularization and calculation of the curvature is as follows. First one rewrites the connection in the finite dimensional case in a way which is still well defined in the Chern–Simon's gauge theory case where $\mathscr{A}$ is infinite dimensional. This involves picking a metric on the group $\mathscr{G}$ and evaluating the determinant of $T^{+}T : (\mathrm{Ker}\ T)^{\perp} \longrightarrow (\mathrm{Ker}\ T)^{\perp}$ where T is the generator of the group action. To get this data in the gauge theory case one picks a metric on Σ and a good regularization of the determinant of the laplacian on 0–forms . The explicit finite dimensional calculation that the connection in this form is well defined now goes through except for one anomaly. This is compensated for by inserting a factor similar to that in the Sugawara construction. The calculation of the curvature in the finite dimensional case goes through to the gauge theory case with no new anomalies. Happily, it yields the answer in the form given by conformal field theory.

HOLOMORPHIC LINKING AND NON–HAUSDORFF RIEMANN SURFACES

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In this note I present two twistor–motivated ideas, neither of which has been developed very far as yet and, indeed, for which the basic concepts still remain to be formulated precisely. Informal accounts of these two ideas have appeared in Penrose (1988) and Penrose (1989), respectively.

A brief reminder of some of the fundamentals of twistor theory is in order. (For detailed accounts of this theory, see Penrose and Rindler 1986, Huggett and Tod 1985, Ward and Wells 1989.) The basic twistor correspondence arises from regarding the complexification $\mathbb{C}\mathbb{M}$ of (conformally compactified) Minkowski 4–space $\mathbb{M}$ as the Grassmannian of lines in a $\mathbb{C}\mathbb{P}^3$ (projective twistor space $\mathbb{P}\mathbb{T}$) , i.e. of linear subspaces of dimension 2 in $\mathbb{C}^4$ (non–projective twistor space $\mathbb{T}$) . In twistor theory it is *global holomorphic structure* that codes physical information, which appears as *local* field/curvature information in space–time. Linear massless fields in $\mathbb{M}$, or $\mathbb{C}\mathbb{M}$, are coded as cohomology elements in (regions of) $\mathbb{P}\mathbb{T}$; (anti–) self–dual Yang–Mills fields are coded as holomorphic vector bundles over (parts of) $\mathbb{P}\mathbb{T}$; (anti–) self–dual vacuum Einstein fields are coded as holomorphic deformations of regions of $\mathbb{P}\mathbb{T}$. In each case the twistor information disappears when the construction is restricted to a small enough region of the twistor space.

Holomorphic Linking

The idea of holomorphic linking is partially motivated by the fact that in a topological quantum field theory (TQFT) one also has a structure in which all information disappears in small enough regions. This, of course has importance for applications in mathematics (invariants for links, knots, braids, etc.) where one may be essentially interested only in topological matters. But the relevance of TQFT to *physics* is obscure, if one is thinking of the background space of the theory as being space–time, since physically one expects to have *local* space–time field quantities. The idea, then, is to use (projective) *twistor* space, instead, as the background in which we could imagine the TQFT to be taking place.

I shall be concerned here with the question of linking only. (I shall not attempt to address more complicated issues raised by knots and braids.) The immediate problem that arises, if one is to consider the linking of two curves in (projective) twistor space, is that complex curves are two–dimensional (as real manifolds) and they lie in a six–dimensional space (the real dimensionality of $\mathbb{P}\mathbb{T}$). Thus they cannot be considered as linking in the ordinary sense of topology. However, the Gauss formula

$$L(X,Y) = \frac{1}{4\pi}\oint \frac{(\vec{x}-\vec{y})\cdot d\vec{y}\wedge d\vec{x}}{|\vec{x}-\vec{y}|^3},$$

for the linking number of two closed curves X and Y in $\mathbb{R}^3$ (where $\vec{x}$ and $\vec{y}$ are position vectors of variable points on X and Y , respectively) can be generalized directly to holomorphic curves $\mathbb{C}X$ and $\mathbb{C}Y$ in $\mathbb{C}^3$. For this, we must interpret the distance

$$|\vec{x} - \vec{y}| = \{(\vec{x} - \vec{y}) \cdot (\vec{x} - \vec{y})\}^{1/2}$$

holomorphically, and the integral is over a two–dimensional contour ($\cong S^1 \times S^1$) in the four–dimensional product space $\mathbb{C}X \times \mathbb{C}Y$ of two (not necessarily compact) holomorphic curves. We may regard $\mathbb{C}X$ and $\mathbb{C}Y$ to be the complexifications of real analytic curves X and Y, respectively, and the contour to be $X \times Y$, and then the formula is the same as it was before; but the expression is now more general than that. The formula makes sense provided that the contour (taken to be compact) avoids the zero–distance locus

$$Z \subset \mathbb{C}X \times \mathbb{C}Y$$

defined by the holomorphic equation

$$|\vec{x} - \vec{y}|^2 = 0 .$$

The value L of the integral clearly depends only on the homology class of the contour within the space

$$\mathbb{C}X \times \mathbb{C}Y - Z ,$$

but, more than this, it is independent of continuous variation of

(a) the two curves $\mathbb{C}X$ and $\mathbb{C}Y$

and of

(b) the particular choice of (complexified) Euclidean metric

where the variations are made in such a way that the condition $|\vec{x}-\vec{y}|^2 \neq 0$ can be maintained over the entire contour. In (b), the "metrics" can be taken to be those complex Euclidean metrics that are compatible with the projective structure of $\mathbb{P}\mathbb{T}$ (where we regard our $\mathbb{C}^3$ as being completed to a $\mathbb{C}\mathbb{P}^3$ (namely $\mathbb{P}\mathbb{T}$) by the addition of a plane at infinity). These are given by complex quadratic forms

$$Q(W) = Q^{\alpha\beta} W_\alpha W_\beta = Q \cdot WW$$

in the *dual* variables W_α, where the matrix of coefficients of Q has rank 3.

The integral for L becomes a simple numerical multiple of

$$\oint \frac{\overline{\mathrm{X}\overline{\mathrm{QQQ}}}\mathrm{Y} \quad \overline{\mathrm{XY}\,d\mathrm{X} \wedge d\mathrm{Y}}}{\{\overline{\mathrm{XYQQ}}\,\overline{\mathrm{XY}}\}^{3/2}}$$

when written in terms of twistor variables X^α, Y^α ($\mathbb{C}^4$ coordinates for $\mathbb{T}$) and where the bars denote contractions of sets of four upper indices with Levi–Civita symbols $\epsilon_{\alpha\beta\gamma\delta}$. It now turns out that the rank of $Q^{\alpha\beta}$ can be 4 (complexified sphere metric) or 3 (Euclidean). This integral can also be re–expressed as a four–dimensional (non–projective) contour integral, where the form $\overline{\mathrm{XY}d\mathrm{X} \wedge d\mathrm{Y}}$ is replaced by

$$-\frac{1}{4\pi^2}\,\overline{d\mathrm{X} \wedge d\mathrm{X} \wedge d\mathrm{Y} \wedge d\mathrm{Y}} = \frac{1}{4\pi^2}\,\overline{d^2\mathrm{X} \wedge d^2\mathrm{Y}}\,.$$

By use of certain known expressions from twistor diagram theory (and assistance from Andrew Hodges is greatly appreciated here), one can convert the integral to numerical multiple of the eight–dimensional contour integral expression

$$\oint \frac{\log(Q \cdot WW / R \cdot WW) \; \overline{d^4W} \wedge \overline{d^2X \wedge d^2Y}}{(W \cdot X)^2 \, (W \cdot Y)^2}$$

on, more simply, to

$$\oint \frac{\overline{d^4W} \wedge \overline{d^2X \wedge d^2Y}}{(W \cdot X)^2 (W \cdot Y)^2}$$

where the contour is now not closed but is allowed to have boundary on the regions $Q \cdot WW = 0$, $R \cdot WW = 0$. Here R is another (arbitrary) quantity of the same nature as Q . By converting the integral to this expression, we see that it depends as little upon Q as it does upon R , in accordance with (b).

Using a (somewhat formal) identity from twistor diagram theory, we can reduce the expression for L to a sum (and difference) of terms of the form

$$\frac{1}{2} \frac{1}{(2\pi i)^4} \frac{df}{f} \wedge \frac{dg}{g} \wedge \frac{dh}{h} \wedge \frac{dj}{j}$$

where $\mathbb{C}X$ is given locally by $f = g = 0$ and $\mathbb{C}Y$, locally by $h = j = 0$. For algebraic curves order p and q , respectively, there appears to b a "canonical" answer for this sum, which is

$$\frac{1}{2} pq ,$$

but other answers are also possible, depending on the original choice of contour (and some uncertainties depending upon the formal nature of the derivation). More rigorous (and cohomological) derivations and concepts seem to be required.

I am grateful to Michael Atiyah for helpful discussions and for pointing out a relation to Green's functions. For information on twistor diagrams, see Hodges (1982), (1985), and I am grateful to him also for discussions.

Non–Hausdorff Riemann Surfaces

Complex manifolds with a (rather mild) form of non–Hausdorffness have played a significant role in twistor theory from time–to–time (Penrose and Sparling 1979, Bailey 1985). Most recently, there is the work of Woodhouse and Mason (1988), who use a construction due to Ward (1983) for the stationary axi–symmetric solutions of the Einstein vacuum equations. The symmetry gives a dimensional reduction on the twistor space by two complex dimensions. The original Ward construction is given in terms of holomorphic (rank 2) vector bundles over regions of $\mathbb{P}\mathbf{T}$, and by the Woodhouse–Mason procedure, this is reduced to bundles over a (compact) complex manifold of one dimension. This is no ordinary Riemann surface, however, but it has some essential non–Hausdorff features. (In their main example, the surface is constructed by taking an ordinary Riemann sphere, but then detaching a closed neighbourhood of the equator and reattaching it wrapped around twice, so that two points of the boundary are attached to the north and south caps where there was one point before.)

There are other reasons that one might be interested in non–Hausdorff Riemann surfaces in twistor theory, and one of these has to do with the fact that some of them have a close relationship to certain "chaotic" structures, namely Julia sets and the Mandelbrot set. (If twistor theory is ever to be able to describe *general* solutions of the Yang–Mills or Einstein equations, then it must be able to incorporate the "chaotic" behaviour that is characteristic of non–integrable systems.)

Let us first recall how an ordinary Riemann surface is described. For genus 1, we may think of a parallelogram cut from the Argand plane, with vertices at points 0 , 1 , λ , $1+\lambda$, where opposite edges are to be identified. The universal covering space of this torus provides a tiling of the plane with this parallelogram, where there are rigid matching rules as to which edges may be placed against one another. In the same way, for genus >1 , we get a tiling of the hyperbolic plane, again by some polygonal shape with rigid matching rules. Now one can construct a polygonal shape which will tile the hyperbolic plane only in a *non*–periodic way, if we interpret "periodic" in the restrictive sense that there are two independent motions of the entire tiling to itself (see Fig. 1 and Penrose 1979).

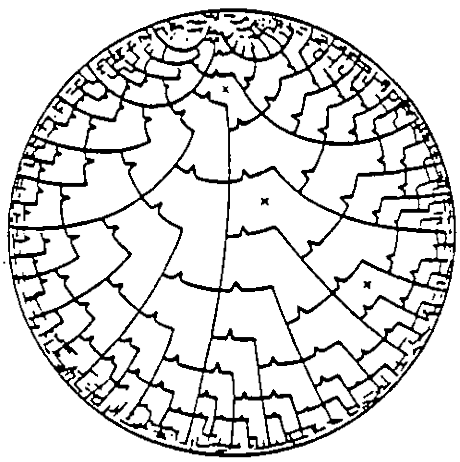

Figure 1: A forced "non–periodic" tiling
of the hyperbolic plane

If we try to fold this tile into a Riemann surface, by identifying edges which are to be matched in the tiling, we get a non–Hausdorff Riemann surface, because two edges must be matched to one (in two different places).

Let us consider another way of constructing the ordinary torus. Instead of using a parallelogram, we can use the annular region between the circles $|z| = R$ and $|z| = |\mu| R$ in the Argand plane. The inner circle is now to be identified with the outer one according to the correspondence

$$z \longmapsto \mu z$$

(where I am taking $|\mu| > 1$) .

Now let us imagine replacing this linear map by a *quadratic* one:

$$z \longmapsto z^2 + c .$$

We consider the annular region between the inner circle $|z| = R$ and an outer one of radius R^2 and centre c (where R is chosen large enough). We wish to identify the inner and outer circles, as before, but now we get a non–Hausdorffness, because z and $-z$ must both be identified with the same point $z^2 + c$. (To get a proper non–Hausdorff manifold, we must consider that our annular region contains its boundary at the inner circle, but not at the outer circle.)

In a certain sense the surface we have constructed contains the information of the map $z \longmapsto z^2 + c$. But, as things stand, it contains too much information because the circle $|z| = R$ is singled out. To remove this feature, we must adopt a certain point of view with regard to *equivalence* of non–Hausdorff manifolds. Unfortunately it is not yet clear what is the best way to formulate this equivalence precisely. Roughly speaking, one should be allowed to move the edge along which the non–Hausdorffness lies, either by "splitting" the surface apart or by "reglueing" it. This splitting or reglueing proceeds only locally and must not change the topological structure of the space. A rule of this kind (more precisely formulated) would also be in line with other requirements suggested by twistor theory.

Adopting such a viewpoint, we can separate off the region between $|z| = R$ and $|z| = R + \varepsilon$, identify opposite points on this region, stretch it out conformally, and then attach it again at the outside. This removes the special nature of the particular circle $|z| = R$. A procedure of this kind can also be applied in reverse, and again an "equivalent" non–Hausdorff manifold is obtained. If this reverse kind of procedure is iterated indefinitely (and maximally), the Julia set for the map $z \longmapsto z^2 + c$ is converged upon. The values of c for which the Julia set is connected and those for which it is a Cantor set lead to drastically different structures. The first case arises when c lies in the Mandelbrot set and the second, when c lies outside the Mandelbrot set. Thus, we see that the "chaotic" Mandelbrot set forms an important part of the modulus space (the analogue of a Teichmüller space) for this particular class of "non–Hausdorff Riemann surfaces".

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QUANTUM GROUPS AND RATIONAL CONFORMAL FIELD THEORY

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There are several approaches to the classification of Rational Conformal Field Theories. One can use three dimensional topological theories as suggested by E. Witten [1] in his treatment of knot and link invariants, or one can exploit the duality properties of conformal theories as summarized by the polynomial equations written by Moore and Seiberg [2]. Both approaches capture purely topological information about conformed theories. In recent work in collaboration with C. Gomez and G. Sierra, we have approached the second point of view using the theory of Quasi–triangular Yang–Baxter algebras or quantum groups [3]. For quantum groups with deformation parameters equal to a root of unity some special features appear. First the number of representations truncates to a finite number, and their behavior under tensor products reproduces the fusion rules of the corresponding Wess–Zumino–Witten theories [4] whose chiral algebra is a Kac–Moody algebra. The defining properties of a Quasi–triangular Yang–Baxter algebra can easily be understood in simple terms. They essentially imply that the representation of the conformal blocks in this language furnish representations of the braid group. They contain the condition that fusing and braiding are compatible operations and they also relate the duality properties of blocks with some fields in their external lines with those of the conjugate fields. The modular transformations can be understood in terms of the remnants of the Virasoro algebra that survive the reduction of the features of a conformal field theory to its duality properties. Any of the quantum groups contains an invertible element which can be interpreted as $\exp 2\pi i L_0$, where L_0 is the energy operator on the Virasoro algebra. The modular transformation properties of the characters, and in particular the matrix S which diagonalizes the fusion rules can be expressed in terms of the co–multiplication of this element. Furthermore, the structural information contained in quantum groups naturally leads to knot and link invariants [5]. It is reasonable to expect that the series of Rational Theories which admit a classical limit in the sense defined in [2] can be obtained from the quantization (as quantum groups) of the group describing this limit.

What is not yet clear is how to classify possible theories (if any) which do not admit a classical limit, and in particular we do not know which of the approaches will be more powerful.

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CONFORMAL FIELD THEORY ON UNIVERSAL FAMILY OF STABLE CURVES

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§ 0 Introduction

In this talk, I'll talk about a conformal field theory under the gauge symmetry associated with integrable representations of an affine Lie algebra over modular family of stable curves.

This is joint work with K. Ueno and Y. Yamada.

We realize this conformal field theory by constructing a holonomic system with regular singularity over moduli space satisfied by so called conformal blocks. This D–module is constructed by using Ward–Takahashi identity for Energy–Momentum tensor.

Our main theorems say that these D–module constitute finite dimensional vector bundle over moduli space and the factorization principle at the discriminant locus hold. And the dimensions of these vector bundles can be calculated combinatorially from fusion rules.

One essential point that finite dimensionality of vector bundles hold comes from integrability of the representations of affine Lie algebra.

§ 1. Integral representation of affine Lie agebra.

At first we briefly sketch the theory of integrable representation of affine Lie algebras.

Let $\mathbb{C}((\xi))$ be the field of Laurent series that is $\mathbb{C}((\xi)) = \{f(\xi) = = a_n \xi^n + a_{n+1}\xi^{n+1} + \ldots\}$ and set $\mathbb{C}[[\xi]] = \{f(\xi) = a_0 + a_1 \xi + \ldots\}$.

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ then associated affine Lie algebra $\hat{\mathfrak{g}}$ is defined by

$$\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}((\xi)) \oplus \mathbb{C}c$$

with usual commutation relations. We fix the Gauss decomposition

$$\hat{\mathfrak{g}} = \hat{\mathfrak{g}}_- \oplus \mathfrak{g} \oplus \mathbb{C}c \oplus \hat{\mathfrak{g}}_+$$

where $\hat{\mathfrak{g}}_+ = \mathfrak{g} \otimes \mathbb{C}[[\xi]]\xi$, $\hat{\mathfrak{g}}_- = \mathfrak{g} \otimes \mathbb{C}[\xi^{-1}]\xi^{-1}$. We fix the Killing form (,) on $\mathfrak{g}$ normalized by $(\theta,\theta) = 2$ where θ is the highest root.

Fix a positive integer ℓ called level and put

$$P_\ell = \{\lambda \in P_+ : 0 \leq (\theta,\lambda) \leq \ell\}$$

where P_+ is the set of dominant integral weights of $\mathfrak{g}$.

For each $\lambda \in P_+$, $\mathscr{H}_\lambda$ denote the integrable representation of $\hat{\mathfrak{g}}$ with level ℓ and classical highest weight λ. We set $\mathscr{H}_\lambda^+ = \mathrm{Hom}_{\mathbb{C}}(\mathscr{H}_\lambda, \mathbb{C})$, then $\hat{\mathfrak{g}}$ acts from right and $\langle \mid \rangle : \mathscr{H}_\lambda^+ \times \mathscr{H}_\lambda \longrightarrow \mathbb{C}$ denote the canonical complete pairing.

Let $T(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$ be the energy momentum tensor in Sugawara form. Element of $\hat{\mathfrak{g}}$ is written by

$$X \otimes f = X[f] \quad \text{for } f \in \mathbb{C}((\xi)), \ X \in \mathfrak{g}.$$

For $\ell = \ell(\xi) \frac{d}{d\xi} \in \mathbb{C}((\xi)) \frac{d}{d\xi}$ we set

$$T[\ell] = \frac{1}{2\pi\sqrt{-1}} \oint_0 d\xi \, T[\xi] \, \ell(\xi).$$

Then $X[f]$ and $T[\ell]$ operate on $\mathscr{H}_\lambda$ and satisfy

$$[T[\ell], X[f]] = -[\ell(f)].$$

§ 2. The space of conformal blocks.

Fix integer $g \geq 0$, and $N \geq 0$.

Consider N–pointed stable curve of genus g , $\mathfrak{X} = (C : Q_1 \dots Q_N)$ that is

C : complete algebraic curve of genus g with only ordinary double points singularities.

$Q_1, \dots, Q_N$: distinct non–singular points on C and some mild stability conditions,

and assume at first the following condition (Q)

(Q) : on each irreducible components C_j of C there is at least one point Q_i .

Furthermore at each point Q_j we fix formal local coordinate
$t_j : \hat{\mathcal{O}}_{Q_j} \xrightarrow{\sim} \mathbb{C}[[\xi_j]]$.

Consider the data $\mathfrak{X} = (C : Q_1 \dots Q_N, t_1 \dots t_N)$. For each $\vec{\lambda} = (\lambda_1 \dots \lambda_N) \in P_\ell^N$ we consider space $\mathcal{H}_{\vec{\lambda}} = \mathcal{H}_{\lambda_1} \otimes \dots \otimes \mathcal{H}_{\lambda_N}$, and $\mathcal{H}^\dagger_{\vec{\lambda}} = \mathrm{Hom}_{\mathbb{C}}(\mathcal{H}_{\vec{\lambda}}, \mathbb{C}) = \mathcal{H}^\dagger_{\lambda_1} \hat{\otimes} \dots \hat{\otimes} \mathcal{H}^\dagger_{\lambda_N}$. The space of the conformal block associated to the data $(\mathfrak{X}, \vec{\lambda})$ is defined by

$$V^{+}_{\vec{\lambda}}(\mathfrak{X}) \equiv \left\{ \langle \Psi | \in \mathscr{H}^{+}_{\vec{\lambda}} : \sum_{j=1}^{N} \langle \Psi | \varphi_j(X \otimes f(\xi_j)) = 0 \quad \text{for any} \right.$$

$$\left. f \in H^0(C : \mathcal{O}_C(\star Q_1 + \dots + \star Q_N)) \right\} .$$

The dual space of $V^{+}(\mathfrak{X})$ is

$$V_{\vec{\lambda}}(\mathfrak{X}) = \mathscr{H}_{\vec{\lambda}}/\mathfrak{g} \otimes H^0(C, \mathcal{O}_C(\star Q_1 + \dots + \star Q_N))\mathscr{H}_{\vec{\lambda}}$$

and there is a complete pairing

$$\langle \, | \, \rangle : V^{+}_{\vec{\lambda}}(\mathfrak{X}) \times V_{\vec{\lambda}}(\mathfrak{X}) \longrightarrow \mathbb{C} .$$

Main problems are the following:

1) Is $\dim_{\mathbb{C}} V^{+}_{\vec{\lambda}}(\mathfrak{X})$ finite?

2) Is $\dim_{\mathbb{C}} V^{+}_{\vec{\lambda}}(\mathfrak{X})$ independent on $\mathfrak{X}$?

3) How does $V^{+}_{\vec{\lambda}}(\mathfrak{X})$ depend on $\mathfrak{X}$?

We give a complete answer for these problems.

§ 3 Local universal family of pointed stable curves.

Fix nonnegative integers g and N with $2g - 2 + N > 0$.

Let $(\pi : C \longrightarrow S : s_1, \dots, s_n)$ the local universal family of N–pointed stable curves of genus g , that is:

1) C and S are complex manifolds of dimension $3g - 2 + N$ and $3g - 3 + N$ respectively.

2) $\pi : C \longrightarrow S$: proper flat holomorphic map

3) $s_i : S \longrightarrow C$: cross–section

4) For each $m \in S$, $(\pi^{-1}(m) = C_m, Q_j = s_j(m))$ is an N–pointed stable curve of genus g .

5) Local universality condition is satisfied.

Now set

$$\Sigma = \{P \in C : d\pi_p : T_pC \longrightarrow T\pi_{(p)}S : \text{not surjective}\} \text{ critical locus of } \pi$$

$$D = \pi(\Sigma) : \text{discriminant locus.}$$

Then Σ is codimension 2 closed submanifold of C and $\pi|_\Sigma : \Sigma \longrightarrow S$ is smooth map and $D \subset S$ is normal crossing divisor. Each element m of S consist of an N–pointed

stable curve $m = (C : Q_1, \dots, Q_N)$, we associate formal coordinate t_j at Q_j and consider the set $S^{(\infty)} = \{\mathfrak{X} = (C : Q_1, \dots, Q_N, t_1, \dots, t_N)\}$, then $S^{(\infty)}$ is an infinite dimensional complex manifold and the canonical projection $\pi : S^{(\infty)} \longrightarrow S$ is a principal bundle with structure group D^N where
$D = \{\varphi(\xi) = a_0\xi + a_1\xi^2 + \dots, \ a_0 \neq 0\}$ acts on $S^{(\infty)}$ as coordinate change. Also we consider the set

$$S^{(1)} = \{\mathfrak{X}^{(1)} = (C : Q_1, \dots, Q_N, t_1^{(1)}, \dots, t_N^{(1)})$$

$$\text{where } t_1^{(1)} : \text{order 1 constant element of } Q_i\} .$$

Then we have the canonical map

$$S^{(\infty)} \longrightarrow S^{(1)} \longrightarrow S$$

and $S^{(1)} \longrightarrow S$ is a principal $(\mathbb{C}^*)^N$–bundle and $S^{(\infty)} \longrightarrow S^{(1)}$ is principal $D^{(1)N}$–bundle.

§ 4 Sheafication

At first fix $\vec{\lambda} = (\lambda_1, \dots, \lambda_N) \in P_\ell^N$ and consider the family of vector spaces over $S^{(\infty)}$

$$\bigcup_{\mathfrak{X}} V_{\vec{\lambda}}(\mathfrak{X}) \longrightarrow S^{(\infty)} .$$

By using more precise construction, we can construct an $\mathcal{O}_{S^{(\infty)}}$–module $\tilde{V}_{\vec{\lambda}}$ on $S^{(\infty)}$ which is sheaf version of the above family of vector spaces, where $\mathcal{O}_{S^{(\infty)}}$ is the sheaf of holomorphic functions on $S^{(\infty)}$.

<u>Proposition 1</u> The actions of $D^{(1)N}$ on $S^{(\infty)}$ can be lifted of the action on $\tilde{V}_{\vec{\lambda}}$ and we can construct $\mathcal{O}_{S^{(1)}}$ module $\tilde{V}^{(1)}_{\vec{\lambda}}$ on $S^{(1)}$ as invariant part of $\tilde{V}_{\vec{\lambda}}$.

<u>Theorem A</u> The $\mathcal{O}_{S^{(1)}}$ module $\tilde{V}^{(1)}_{\vec{\lambda}}$ is a coherent sheaf.

<u>Corollary</u> For each $\mathfrak{X} \in S^{(\infty)}$ the spaces $V_{\vec{\lambda}}(\mathfrak{X})$ and $V^{+}_{\vec{\lambda}}(\mathfrak{X})$ are finite dimensional.

§ 5 Connection on $\tilde{V}^{(1)}_{\vec{\lambda}}$

Now consider the following situation

$$\begin{array}{ccc} S^{(1)} & \xrightarrow{\ \pi\ } & S \\ \uparrow & & \uparrow \\ D^{(1)} & \longrightarrow & D \end{array} \qquad \text{where} \qquad D^{(1)} = \pi^{-1}(D)\,.$$

Consider the following Lie algebra sheaf on $S^{(1)}$.

$$\Theta_{S^{(1)}}(-\log D^{(1)}) = \{v : \text{local holomorphic vector field on } S^{(1)} \text{ such that } v(I_{D^{(1)}}) \subset I_{D^{(1)}}\}$$

where $I_{D^{(1)}}$ is defining ideal of $D^{(1)}$.

If locally $(\tau_1, \ldots, \tau_M)$ is local coordinate set $\tau_1 \cdots \tau_k = 0$ is defining equation of $D^{(1)}$, then $\Theta_{S^{(1)}}(-\log D^{(1)})$ is free $\mathcal{O}_{S^{(1)}}$ module generated by

$$\tau_1\, \partial/\partial\tau_1, \ldots, \tau_k\, \partial/\partial\tau_k,\quad \partial/\partial\tau_{k+1},\quad \partial/\partial\tau_M .$$

Due to the conformal anomaly, the sheaf $\Theta_{S^{(1)}}(-\log D^{(1)})$ can not act on $\tilde{V}^{(1)}_{\vec{\lambda}}$. But we can construct extension of Lie algebra sheaf for each $c_v \in \mathbb{C}$,

$$0 \longrightarrow \mathcal{O}_{S^{(1)}} \longrightarrow D^1_{S^{(1)}}(-\log D^{(1)}, c_v) \longrightarrow \Theta_{S^{(1)}}(-\log D^{(1)}) \longrightarrow 0 .$$

Locally the Lie algebra $D^1_{S^{(1)}}(-\log D^{(1)}, c_v)$ is isomorphic to

$$\Theta_{S^{(1)}}(-\log D^{(1)}) \oplus \mathcal{O}_{S^{(1)}}$$

but this isomorphism is not canonical.

<u>Theorem B</u> For $c_v = \dfrac{\ell \dim \mathfrak{g}}{g^* + \ell}$ where g^* is the dual Coxter number of $\mathfrak{g}$, the Lie

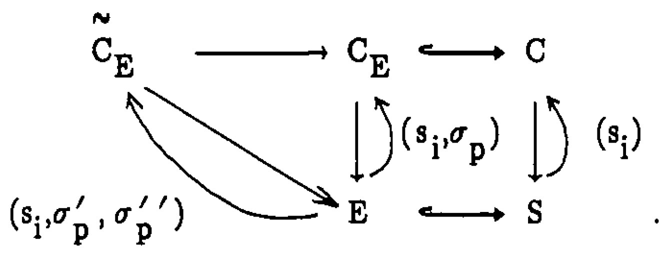
algebra sheaf $D^1_{S^{(1)}}(-\log D^{(1)}, c_v)$ acts on $\tilde{V}^{(1)}_{\vec{\lambda}}$ as twisted first order differential operators.

<u>Corollary</u> On $S^{(1)} - D^{(1)}$, the $\mathcal{O}_{S^{(1)}}$ module $\tilde{V}^{(1)}_{\vec{\lambda}}$ is locally free, so constitutes a vector bundle.

§ 6 Locally freeness and factorization.

Now we study the behavior of $\tilde{V}^{(1)}_{\vec{\lambda}}$ near the discriminant locus.

We take S small and fix a local coordinate $\tau_1, \dots, \tau_M$ of S such that $\tau_1 \cdots \tau_k = 0$ is a defining equation of D. Let $D = D_1 \cup \dots \cup D_k$ and set $E = D_1 \cap \dots \cap D_k$ and $E^{(1)} = D_1^{(1)} \cap \dots \cap D_k^{(1)}$.



Now let $C_E \longrightarrow E$: the restriction of C on E and $\tilde{C}_E \longrightarrow C_E$: the simultaneous normalization of $\pi_E : C_E \longrightarrow E$ and $\sigma'_p, \sigma''_p : E \longrightarrow \tilde{C}_E$ are the cross–sections corresponding to normalization points.

<u>Proposition 2</u> $(\tilde{\pi}_E : \tilde{C}_E \longrightarrow E\ ,\ s_j\ ,\ \sigma'_p\ ,\sigma''_p)$ is local universal family of $(N+2k)$ pointed stable curves and each fiber is a non–singular curve.

Now fix a trivialization

$$D^1_{S^{(1)}}(-\log D^{(1)} : C_v) \simeq \Theta_{S^{(1)}}(-\log D^{(1)}) \oplus \mathcal{O}_{S^{(1)}} .$$

We introduce the following V–filtration on each $\mathcal{O}_{S^{(1)}}$ module $\mathcal{F}$ such as

$$\Theta_{S^{(1)}}(-\log D^{(1)})\ ,\ \mathcal{O}_{S^{(1)}}\ ,\ \tilde{V}^{(1)}_{\vec{\lambda}} .$$

For each $\mathbb{P} = (p_1, \dots ,p_k) \in \mathbb{Z}^k$

$$V_{\mathbb{P}}\mathcal{F} = I_1^{-p_1} \cdots I_k^{-p_k}\mathcal{F}$$

where I_j is defining ideal of $D^{(1)}_j$. Take associated graded modules

$$Gr^V_* \mathcal{F} = \sum_{\mathbb{P}\in\mathbb{Z}^k} Gr^V_{\mathbb{P}}(\mathcal{F})$$

$$Gr^V_{\mathbb{P}}(\mathcal{F}) = V_{\mathbb{P}}\mathcal{F} / \sum_{j=1}^{k} V_{\mathbb{P}-\varepsilon_j}\mathcal{F}$$

where $\varepsilon_j = (0 \dots 0 \dots 1, \dots 0)$.

Then we have

$$\mathrm{Gr}_*^{\mathrm{V}} \mathcal{O}_{S^{(1)}} = \mathcal{O}_{E^{(1)}}[\tau_1, \dots, \tau_k]$$

$$\mathrm{Gr}_*^{\mathrm{V}} \Theta_{S^{(1)}}(-\log D^{(1)}) = \Big(\sum_{j=1}^{k} \mathcal{O}_{E^{(1)}}\tau_j \partial/\partial\tau_j \oplus \sum_{j=k+1}^{M} \mathcal{O}_{E^{(1)}}\partial/\partial\tau_j\Big) \otimes \mathbb{C}[\tau_1, \dots, \tau_k]$$

where degree $\tau_j = -\varepsilon_j$.

<u>Theorem C</u> As $\mathrm{Gr}_*^{\mathrm{V}} \Theta_{S^{(1)}}(-\log D^{(1)}) \oplus \mathrm{Gr}_*^{\mathrm{V}} \mathcal{O}_{S^{(1)}}$ module we have

$$\mathrm{Gr}_0^{\mathrm{V}} \tilde{V}^{(1)}_{\vec{\lambda}} \otimes \mathbb{C}[\tau_1, \dots, \tau_k] \xrightarrow{\sim} \mathrm{Gr}_*^{\mathrm{V}} \tilde{V}^{(1)}_{\vec{\lambda}} .$$

<u>Theorem D</u> factorization

There exist the following isomorphism as $\mathcal{O}_{E^{(1)}}$ module

$$\mathrm{Gr}_0^{\mathrm{V}} \tilde{V}^{(1)}_{\lambda} \underset{\mathcal{O}_{E^{(1)}}}{\otimes} \mathcal{O}_{\tilde{E}^{(1)}} \xrightarrow{\sim} \sum_{\mu \in P_\ell^k} \tilde{V}^{(1)}_{(\mu, \mu^+, \lambda)}$$

here $\tilde{E}^{(1)} \longrightarrow E$ is the 1–st order structure of the family $(\tilde{\pi}_E : \tilde{C}_E \longrightarrow E : (s, \sigma', \sigma''))$ so $\tilde{E}^{(1)} \longrightarrow E^{(1)}$ is $(C^*)^{2k}$–principal bundle.

<u>Theorem E</u> $\tilde{V}^{(1)}_{\vec{\lambda}}$ is locally free as $\mathcal{O}_{S^{(1)}}$ module.

Corollary The rank of $\tilde{V}^{(1)}_{\vec{\lambda}}$ can be calculated combinatorially from fusion rule. So we calculate the dimension of $\tilde{V}^{(1)}_{\vec{\lambda}}$ in the case of $g = 0$ and $N = 3$. In this case $S = \{\text{points}\}$. For each $\lambda,\mu,\nu \in P_\ell$ we set $N_{\lambda,\mu,\nu} = \text{rank } V^{(1)}_{\lambda,\mu,\nu}$.

Proposition $\dim N_{\lambda,\mu,\nu} = \dim_{\mathbb{C}} W_{\lambda,\mu,\nu}$ where

$$W_{\lambda,\mu,\nu} = \{\varphi \in \text{Hom}_{\mathfrak{g}}(V_\lambda \otimes V_\mu \otimes V_\nu : \mathbb{C}) \text{ which satisfy } (\star)\} ;$$

now we state the condition $(\star)$.

Set $k_\theta = \mathbb{C}X_\theta \oplus \mathbb{C}X_{-\theta} \oplus \mathbb{C}H_\theta$ be the three dimensional sub–algebra of $\mathfrak{g}$ corresponding to highest root θ.

Set $V_\lambda = \sum_{j=0}^{\ell/2} W_{\lambda,j}$: the decomposition of $\mathfrak{g}$–module V_λ as k_θ module. Then the condition $(\star)$ is

$$\varphi|_{W_{\lambda,i} \otimes W_{\mu,j} \otimes W_{\nu,k}} = 0$$

for each $i + j + k > \ell$.

On the space of conformal blocks in rational conformal field theory

summary of talk given by Erik Verlinde

Given a semi-simple compact Lie group G and a positive integer k one can associate to a closed Riemann surface Σ a finite dimensional vector space $\mathcal{H}_\Sigma^{G,k}$, called the space of conformal blocks. We will give three different descriptions of this space, which can be considered to be 2, 1 and 3 dimensional respectively. After that we will present a formula for the dimension of $\mathcal{H}_\Sigma^{G,k}$.

Three descriptions of $\mathcal{H}_\Sigma^{G,k}$

i) Conformal blocks arise in rational conformal field theory as the 'chiral' building blocks of the partition function, i.e. which depend holomorphically on the complex strucure of Σ. In the case of the Wess-Zumino-Witten model one can alternatively consider the partition function

$$Z_\Sigma^{G,k}[A,\overline{A}] = \int [dg] e^{ikS_\Sigma(g,A,\overline{A})}$$

as a function of the background gauge field $(A,\overline{A})$ (A and $\overline{A}$ denote the holomorphic and anti-holomorphic component respectively). The partition function turns out to have the following form:

$$Z_\Sigma^{G,k}[A,\overline{A}] = \sum_{I=1}^{\dim \mathcal{H}_\Sigma^{G,k}} \Psi_I[A]\,\overline{\Psi}_I[\overline{A}]\,e^{-ik\int_\Sigma \mathrm{tr} A\overline{A}}$$

The holomorphic functionals $\Psi_I[A]$ are the conformal blocks and span the vector space $\mathcal{H}_\Sigma^{G,k}$.

ii) We consider the basic representation $\mathcal{H}_0^{G,k}$ of the Kac-Moody algebra:

$$[J(\epsilon_1), J(\epsilon_2)] = J([\epsilon_1,\epsilon_2]) + k\int_{S^1} \mathrm{tr}(\epsilon_1\epsilon_2')$$

where $\epsilon_{1,2}$ are liealgebra-valued functions on S^1. We want to construct $\mathcal{H}_\Sigma^{G,k}$ as a subspace of $\mathcal{H}_0^{G,k}$. To this end we choose a point P on Σ and a circle around P which we identify with S^1. Next we introduce the set $\mathcal{E}_\Sigma$ of local Laurent expansions $\epsilon(z)$ at P which extend to liealgebra-valued holomorphic functions on Σ. Then we define $\mathcal{H}_\Sigma^{G,k} \subset \mathcal{H}_0^{G,k}$ as

$$\mathcal{H}_\Sigma^{G,k} = \{ \, |\Psi\rangle \in \mathcal{H}_0^{G,k}; \;\; J(\epsilon)|\Psi\rangle = 0 \text{ for } \epsilon \in \mathcal{E}_\Sigma \}$$

By considering the action of the loop group on these states one can relate this definition to *i*) and also to the third description which we will now discuss.

iii) A beautiful characterization of $\mathcal{H}_\Sigma^{G,k}$, which is due to Witten, is that it represents the Hilbert space of the 2+1 dimensional Chern-Simons theory on $\Sigma \times \mathbf{R}$. This theory is described by the action

$$S[A] = k \int_{\Sigma \times \mathbf{R}} \mathrm{tr}\Big(AdA + \tfrac{2}{3}A^3\Big)$$

and has as its classical phase space the moduli space of flat connections on Σ. By making use of a complex structure on Σ one can quantize this phase space using Kähler quantization. The components A and $\overline{A}$ are canonical conjugate operators and the Hilbert space is given by gauge-invariant functionals of A:

$$\mathcal{H}_\Sigma^{G,k} = \{\Psi[A]; \;\; F(A,\overline{A})\Psi[A] = 0 \,\}$$

where we used that the curvature $F(A,\overline{A}) = \partial\overline{A} - \overline{\partial}A + [A,\overline{A}]$ is the generator of gauge transformations.

A dimension formula for $\mathcal{H}_\Sigma^{G,k}$

It is possible to extend the definition of the space of conformal blocks to surfaces with punctures. In this case one has to assign to each puncture a representation of G, where only those representations are allowed which lead to integrable representations of the Kac-Moody algebra. These are labeled by positive weights $\lambda \in P_k$ where:

$$P_k = \{\lambda \in P_+ \,; \;\; \psi \cdot \lambda \leq k\}$$

where ψ denotes the longest root.

When Σ degenerates into a surface with nodes one finds that the conformal blocks on Σ factorize into those on the punctured surfaces. Therefore, $\mathcal{H}_\Sigma^{G,k}$ is isomorphic to a direct sum of tensor products of the spaces associated with these punctured surfaces. This means that in order to be able to compute $\dim \mathcal{H}_\Sigma^{G,k}$ one only has to know the dimensions $N_{\lambda\mu\nu}$ $(\lambda, \mu, \nu \in P_k)$ for the case of the three-punctured sphere. This information is contained in the fusion rules, which can be described as follows.

We consider the Weyl characters $ch_\lambda(\theta) = \mathrm{tr}_\lambda(e^{\theta\cdot H})$ for $\lambda \in P_k$. It turn out that these characters form a closed algebra when we restrict θ to the following discrete set of points on the Cartan torus:

$$\theta \in \left\{\theta_\nu = 2\pi\left(\tfrac{\nu+\rho}{k+h}\right) \; ; \;\; \nu \in P_k\right\}$$

where $\rho = \sum_{\alpha\in\Delta_+} \frac{1}{2}\alpha$ and h is the dual Coxeter number. These points correspond to the set of zeroes of the characters of the first 'bad' representations with $\psi\cdot\lambda > k$. The multiplication rule of the Weyl characters is precisely given by the fusion rules:

$$ch_\lambda(\theta_\nu)ch_{\lambda'}(\theta_\nu) = \sum_{\mu\in P_k} N_{\lambda\lambda'}{}^{\mu}\, ch_\mu(\theta_\nu)$$

and this determines the multiplicities $N_{\lambda\mu\nu}$. Then using Weyl's character formula and some combinatorics it becomes straightforward to compute $\dim \mathcal{H}_\Sigma^{G,k}$. The result is:

$$\dim \mathcal{H}_\Sigma^{G,k} = \Big((k+h)C\Big)^{-\frac{1}{2}\chi_\Sigma} \sum_{\lambda\in P_k} \prod_{\alpha\in\Delta} \left(1 - e^{\alpha(\theta_\lambda)}\right)^{\frac{1}{2}\chi_\Sigma}$$

where χ_Σ is the Euler characteristic of Σ. The constant C, which is independent of k and Σ, is uniquely determined by the requirement that $\dim \mathcal{H}_{S^2}^{G,k} = 1$. The form of this result actually suggests that it should be possible to derive $\dim \mathcal{H}_\Sigma^{G,k}$ from a fixed-point formula.

As a final remark we mention that also for discrete groups one can construct a space of conformal blocks, given by the L^2-functions on the moduli space of G-bundles. The fusion rules in this case turn out to be related to work of Lusztig on equivariant K-theory. One might expect that also in the continuous case there is such a relation.

COMPLEX IMMERSIONS AND CHARACTERISTIC CLASSES

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Orsay

The purpose of this talk is to review some recent results on refined characteristic classes associated with immersions of complex manifolds and resolutions of vector bundles, in relation with Arakelov theory.

Let $i : Y \longrightarrow X$ be an immersion of complex manifolds, let $\xi \longrightarrow i_*\eta \longrightarrow 0$ be a resolution of a vector bundle η on Y by a complex ξ on X. Assume that all the considered vector bundles are equipped with metrics.

For $u \geq 0$, one can construct smooth Chern character forms ω_u on X by using Quillen's superconnections. Then a result I obtained recently [1] states that under compatibility assumptions on the metrics of ξ with metrics on η and on the normal bundle N to Y, then as $u \longrightarrow +\infty$

$$\omega_u \longrightarrow \frac{ch(\eta)}{Td(N)} \delta_{\{Y\}} .$$

Using the analytic torsion formalism, in joint work with Gillet and Soulé [3,4], we construct currents $T(\xi)$ on X which solve the current equation

$$\frac{\bar{\partial}\partial}{2i\pi} T(\xi) = -\mathrm{ch}(\xi) + \frac{\mathrm{ch}(\eta)}{\mathrm{Td}(N)}\, \delta_{\{Y\}} .$$

With Gillet and Soulé [3,4], we proved natural functorial properties of the current $T(\xi)$. In particular, to a commutative diagram of immersions

$$\begin{array}{ccc} Y \cap Y' & \longrightarrow & Y \\ \downarrow & \searrow & \downarrow \\ Y' & \longrightarrow & X \end{array}$$

we associate a corresponding commutative diagram of currents.

The second part of the talk is to explain how to interpret the Todd form of a vector bundle as a generalized Chern character form in the superconnection formalism [2].

The third part of the talk is devoted to the construction of a complicate characteristic class associated with an exact sequence of holomorphic Hermitian vector bundles $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ [2]. The idea is to consider a double complex in M which describes the resolution of the sheaf $\mathcal{O}(L)$ by the Koszul complex of N, and to calculate the corresponding generalized Chern character of the family. In this way, one calculates a form $\mathbb{B}(L,M,g^M)$ (where g^M is the metric of M) solving the equation

$$\frac{\bar{\partial}\partial}{2i\pi} \mathbb{B}(L,M,g^M) = \frac{\mathrm{Td}(M)}{\mathrm{Td}(N)} - \mathrm{Td}(L) .$$

The evaluation of $\mathbb{B}(L,M,g^M)$ in terms of the Bott–Chern class $\widetilde{Td}(L,M,g^M)$ shows that modulo irrelevant coboundaries

$$\mathbb{B}(L,M,g^M) = -\,Td^{-1}(N)\widetilde{Td}(L,M,g^M) + Td(L)D(N)$$

where D is the additive class associated with the derivative at 0 of the Mellin transform of $\dfrac{-\frac{\partial\varphi}{\partial x}(u,x)}{\varphi(u,x)}$, where

$$\varphi(u,x) = \frac{4}{u}\sinh\left[\frac{-x+\sqrt{x^2+4u}}{4}\right]\sinh\left[\frac{x+\sqrt{x^2+4u}}{4}\right].$$

In a joint calculation with Soulé (appendix of [2]), we obtain the formula

$$D(x) = \sum_{n\ \mathrm{odd}}\left[\sum_{1}^{n}\frac{1}{j} + \frac{2\zeta'(-n)}{\zeta(-n)} + \Gamma'(1)\right]\frac{\zeta(-n)x^n}{n!}.$$

The series $D(x)$ is closely related with a series $R(x)$ introduced by Gillet, Soulé [5]

$$R(x) = \sum_{n\ \mathrm{odd}}\left[\sum_{1}^{n}\frac{1}{j} + \frac{2\zeta'(-n)}{\zeta(-n)}\right]\frac{\zeta(-n)x^n}{n!}$$

which they conjectured to appear in a refined arithmetic Todd genus.

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THE DIRECT OPERATOR APPROACH IN QUANTUM STRING THEORY

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Let's consider a nonsingular Riemann surface Γ of a genus g with punctures P_i and with fixed real numbers p_i such that

$$\sum p_i = 0 \ , \ i = 1, \dots, m .$$

The set of data (Γ, P_i, p_i) will be called a "multistring diagram". For any such diagram there exists a unique differential dk which satisfies the following properties:

a) it is holomorphic on Γ outside the punctures P_i

b) at every point P_i it has a simple pole with the residue equal to p_i

c) the periods of dk over an arbitrary closed cycle on $\Gamma \setminus (\cup P_i)$ are pure imaginary

$$\mathrm{Re} \oint_\gamma dk = 0 .$$

The real part of the multivalued function $k(z)$ is therefore single-valued. This function

$$\tau(z) = \mathrm{Re}\, k(z)$$

is called the "time" (euclidean). We denote the curves $\tau(z) = \mathrm{const} = \tau$ by C_τ and the domains $\tau_1 < \tau(z) < \tau_2$ by $C_{\tau_1\tau_2}$. In the case of "one–string" diagrams we have $m = 2$, $p_1 < 0$, $p_2 > 0$. Using a transformation $\tau \longrightarrow a\tau$, $a > 0$, we may always assume that $p_+ = p_2 = 1$, $p_- = p_1 = -1$, $P_2 = P_+$, $P_1 = P_-$.

I. Fourier–Laurent–type bases on the Riemann surfaces.

For any integer λ and one–string diagram $(\Gamma, P_\pm)$ in general position, for "almost" any number $n \in \mathbb{Z} + S(g,\lambda)$ there exists a unique up to a constant factor tensor $f_n^\lambda(z)$ of weight λ such that: a) it is holomorphic on Γ outside the points $P_\pm$; b) it has the form

$$f_n^\lambda = \varphi_{n,\lambda}^{\pm} z_\pm^{\pm n - S}(1 + O(z_\pm))(dz_\pm)^\lambda , \quad S = g/2 - \lambda(g-1)$$

near the points $P_\pm$. Here $z_\pm$ are local coordinates $z_\pm(P_\pm) = 0$, "almost" means except a finite number of n's for $\lambda = 0,1$ or $g = 1$. An analogous statement is true also for spinors $\lambda = 1/2$. Corrections for exceptional values of n, g and λ see in the papers [1], [2]. The bases f_n^λ and $f_m^{1-\lambda}$ are dual:

$$\frac{1}{2\pi i} \oint_{C_\tau} f_n^\lambda f_m^{1-\lambda} = \delta_{n,-m} .$$

Theorem 1. Let C_τ be nonsingular. Then for any smooth tensor $f^\lambda(\sigma)$ of weight λ on C_τ the (Fourier type) expansion is valid:

$$f^\lambda(\sigma) = \sum_n f_n^\lambda(\sigma)\left[\frac{1}{2\pi i}\oint_{C_\tau} f^\lambda(\sigma')f_{-n}^{1-\lambda}(\sigma')\right].$$

The same expansion is valid for tensors holomorphic in the domain $C_{\tau_1\tau_2}$ (Laurent–type). Theorem 1 is valid also for $\lambda = 1/2$.

We shall use special notations for $\lambda = -1, 0, 1/2, 1, 2$:

$$e_n = f_n^{-1} \text{ (vector fields)}, \quad A_n = f_n^0 \text{ (scalars)}$$

$$f_n^{1/2} = \phi_n \text{ (spinors)}, \quad f_n^1(\text{forms}) = d\omega_{-n},$$

$$f_n^2 = d^2\Omega_{-n} \text{ (quadratic differentials)}.$$

Multiplication of our elements f_n^λ has the important "almost graded" property:

$$f_n^\lambda f_m^\mu = \sum_{|k|\le g/2} Q_{n,m}^{\lambda,\mu,k} f_{m+n-k}^{\lambda+\mu}$$

$$[e_n, f_m^\lambda] = \sum_{|k|\le g_0} R_{n,m}^{\lambda,k} f_{n+m-k}^\lambda, \quad g_0 = 3g/2.$$

For $\lambda = 0$ we have the commutative algebra A, and for $\lambda = -1$ the Lie algebra L.

The Riemann analogue of the Heisenberg algebra is generated by the elements $\alpha_n \longleftrightarrow A_n$ and t with the following relations ($A_{-g/2} = \text{const}$) :

$$[\alpha_n, t] = 0 \,, \quad [\alpha_n, \alpha_m] = \gamma_{mn} \cdot t$$

$$\gamma_{mn} = -\frac{1}{2\pi i} \oint_{C_\tau} A_n d\, A_m \,, \quad \gamma_{mn} = 0 \,, \quad |n| > g/2 \,, \quad |n| > g/2 \,, \quad |m+n| > g$$

and $\gamma_{mn} = 0$ for all (m,n) if $|m+n| > 2g$.

The analogue of Virasoro algebra is generated by the elements e_n , t with commuting relations

$$[e_n, e_m] = \sum_{|k| \le g_0} C^k_{mn}\, e_{m+n-k} + \chi_{mn} \cdot t$$

$$C^k_{mn} = R^{-1,k}_{m,n} \,, \quad \chi_{mn} = \chi(e_n, e_m)$$

$$\chi(f,g) = \frac{i}{48\pi i} \oint_{C_\tau} [(f'''g - g'''f) - 2(f'g - g'f)R]\, dz$$

$$f = f(z)\partial/\partial z \,, \quad g = g(z)\partial/\partial z \,,$$

$$R(w) = R(z)w'^2 + (w'''/w' - \tfrac{3}{2}(w''/w')^2) \,, \quad w' = dw/dz \,.$$

<u>Conjecture</u>. $H^2(L,R) = H_1(\Gamma \setminus (P_+ \cup P_-), R)$.

Theorem 2. The central extension of algebra L , which is almost graded, is unique.

II. Riemann analogues of Heisenberg and Virasoro algebras in string theory.

Let X^μ and P^μ be the quantized coordinates and momenta ($\mu = 1, \dots ,d$) with standard commuting relations, $J^\mu(\sigma) = \partial_\sigma X^\mu + \pi P^\mu = \sum_n \alpha_n^\mu d\omega_n(\sigma)$.

Lemma. $[\alpha_n^\mu, \alpha_m^\nu] = \eta^{\mu\nu}\gamma_{mn}$, $\eta^{\mu\nu} = \mathrm{diag}(\pm 1,1, \dots ,1)$. The holomorphic parts on "vacuum–sectors" of "in" and "out" Fock spaces are defined by relations:

$$\mathcal{H}^+ : \alpha_n | 0 > = 0 \, , \ n > g/2 \, , \ n = - g/2 \quad (\text{"in"})$$

$$\mathcal{H}^- : < 0 | \alpha_n = 0 \, , \ n \le - g/2 \quad (\text{"out"}) \, .$$

For energy–momentum tensor we have

$$T = \frac{1}{2} : J(z)J(z) :$$

Normal product is not unique. It is such that

$$: \alpha_n \alpha_m := \alpha_n \alpha_m \, , \quad (m,n) \in \Sigma^+$$

$$: \alpha_n \alpha_m := \alpha_m \alpha_n \, , \quad (m,n) \in \Sigma^-$$

$$\Sigma^- \cup \Sigma^+ = \mathbb{Z}^2$$

and Σ^+ differs from the integer half–plane $m \le n$ only in the strip $|m+n| \le g-2$.

Theorem 3. $T(Q) = \Sigma L_k d\,\Omega_k$ and L_k generates the Riemann analogue of Virasoro algebra with $t = 1$ and some R :

$$L_k = \frac{1}{2} \sum_{(m,n)} \ell^k_{m,n} : \alpha_n \alpha_m : , \quad \ell^k_{mn} = \frac{1}{2\pi i} \oint_{C_\tau} e_k d\omega_m d\omega_n .$$

Theorem 4. Let $\tau(z) > \tau(w)$ and $z \longrightarrow w$. We have the expansion

$$J(z)J(w) = d\,\frac{dz dz}{(z-w)^2} + 2\tilde{T}(z) + O\,(z-w)$$

$$\tilde{T}(z)\tilde{T}(w) = \frac{d}{2}\,\frac{1}{(z-w)^4} + \frac{2\tilde{T}(z)}{(z-w)^2} + \frac{\tilde{T}_z(z)}{z-w} + O\,(1)\,.$$

Here $\tilde{T} = T + R \cdot 1$ (pseudotensor).

The holomorphic operator fields are therefore constructed. For the calculation of physical quantities we need the "pairing" between right and left Fock spaces corresponding to "in" and "out" states. It was done in the papers [1 – 3] on the base of fermionization.

All these results were obtained by the author and I.M. Krichever.

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ON THE EULER NUMBER OF AN ORBIFOLD

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Let G be a finite group acting on a compact differentiable manifold X. Topological invariants like Betti numbers of the quotient space X/G are well-known:

$$b_i(X/G) = \dim H^i(X,\mathbf{R})^G = \frac{1}{|G|}\sum_{g\in G} tr(\, g^* \mid H^i(X,\mathbf{R}))$$

The topological Euler characteristic is determined by the Euler characteristic of the fixed point sets X^g:

$$e(X/G) = \frac{1}{|G|}\sum_{g\in G} e(X^g)$$

Physicists' Formula: Viewed as an orbifold, X/G still carries some information on the group action. In [DHVW$_{1,2}$], [Va] one finds the following string-theoretic definition of the 'orbifold Euler characteristic':

$$\boxed{e(X,G) = \frac{1}{|G|}\sum_{gh=hg} e(X^{(g,h)})}$$

Here summation runs over all pairs of commuting elements in $G\times G$, and $X^{(g,h)}$ denotes the common fixed point set of g and h. The physicists are mainly interested in the case where X is a complex threefold with trivial canonical bundle and G is a finite subgroup of $SU(3)$. They point out that in some situations where X/G has a resolution of singularities $\widetilde{X/G} \xrightarrow{f} X/G$ with trivial canonical bundle $e(X,G)$ is just the Euler characteristic of this resolution ([DHVW$_2$], [Str-Wi]).

In this paper we consider some well-known examples from algebraic geometry and check to what extent the formula

$$e(X,G) = e(\widetilde{X/G})$$

holds. We will also do this in the local situation of a matrix group $G \subset U(n)$ acting on $\mathbf{C}^n$, since in this non-compact case all the invariants considered here are meaningful as well.

Some elementary calculations: For a fixed $g \in G$ the elements commuting with g form the centralizer $C(g)$. The conjugacy class $[g]$ is a system of representatives for $G/C(g)$, so we have

$$\#C(g) \cdot \#([g]) = |G|.$$

Since simultaneous conjugation of g and h by some element of G leaves $e(X^{\langle g,h\rangle})$ fixed, using the classical formula for $e(X/G)$ we can write $e(X,G)$ as a sum over the conjugacy classes of G:

$$\begin{aligned} e(X,G) &= \frac{1}{|G|}\sum_{[g]}\#([g]) \sum_{h\in C(g)} e(X^{\langle g,h\rangle}) \\ &= \frac{1}{|G|}\sum_{[g]}\#([g]) \cdot \#C(g) \cdot e(X^g/C(g)) \end{aligned}$$

So we get an equivalent definition which sometimes is more useful than the original one:

$$\boxed{e(X,G) = \sum_{[g]} e(X^g/C(g))}$$

For a free action we immediately get $e(X,G) = e(X/G)$, and we also see that some assumption is neccessary: For a cyclic group of order n acting on $\mathbf{P}^1(\mathbf{C})$ with two fixed points, the quotient is $\mathbf{P}^1(\mathbf{C})$ again, whereas $e(\mathbf{P}^1, G) = e(\mathbf{P}^1) + (n-1)\cdot 2 = 2n$.

Loop spaces: For $g \in G$ we consider the space of paths

$$\mathcal{L}(X,g) := \{\alpha : \mathbf{R} \to X \mid \alpha(t+1) = g\,\alpha(t)\}.$$

G acts on the disjoint union of these spaces by $(h\alpha)(t) := h \cdot \alpha(t)$. Obviously h transforms $\mathcal{L}(X,g)$ into $\mathcal{L}(X, hgh^{-1})$. We form the quotient

$$\mathcal{L}(X,G) := \left(\dot{\bigcup_g} \mathcal{L}(X,g)\right) / G = \dot{\bigcup_{[g]}} (\mathcal{L}(X,g)/C(g)).$$

The real numbers act on the $\mathcal{L}(X,g)$ and on $\mathcal{L}(X,G)$ by transforming $\alpha(t)$ to $\alpha(t+c)$. The fixed point set of this action is

$$\dot{\bigcup_{[g]}} (X^g/C(g)) \subset \mathcal{L}(X,G)$$

where X^g is embedded in $\mathcal{L}(X,g)$ as the set of constant paths. This corresponds to the inclusion of X in the ordinary loop space $\mathcal{L}(X)$ as the fixed point set of the obvious S^1-action. On each component $\mathcal{L}(X,g)$ our $\mathbf{R}$-action is in fact an action of S^1 as well because $\alpha(t + ord(g)) = \alpha(t)$. So we can take the Euler characteristic with respect to this action, i.e. the Euler characteristic of the fixed point set, and get the orbifold invariant $e(X,G)$.

Quotient singularities: If G is a finite subgroup of $U(n)$ acting on $\mathbf{C}^n$, then every fixed point set is contractible. Thus $e(\mathbf{C}^n, G)$ equals the number of conjugacy classes, i.e. the number of isomorphism classes of irreducible representations of G.

If in particular $G \subset SU(2)$, then the corresponding 2-dimensional quotient singularity has a minimal resolution $\widetilde{\mathbf{C}^2/G}$ by a configuration of rational (-2)-curves. This is equivalent to $\widetilde{\mathbf{C}^2/G}$ having trivial canonical bundle. If the number of exceptional curves is k, then $e(\widetilde{\mathbf{C}^2/G}) = k+1$. Now the McKay correspondence states that the number of non-trivial irreducible representations of G equals this number k of exceptional curves, hence $e(\widetilde{\mathbf{C}^2/G}) = k+1 = e(\mathbf{C}^2, G)$.

For resolution configurations containing other than (-2)-curves and therefore having non-trivial canonical divisor the result is false: If G is a cyclic subgroup of $U(2)$ generated by

$$\begin{pmatrix} \exp(2\pi i \frac{p}{n}) & 0 \\ 0 & \exp(2\pi i \frac{q}{n}) \end{pmatrix},$$

p, q relatively prime to n, we have $e(\mathbf{C}^2, G) = n$. But now the resolution graph consists of rational curves with self-intersections $-a_i$ determined by the continued fraction $\frac{n}{r} = a_1 - \frac{1}{a_2 - \frac{1}{\cdots}}$, where $r \equiv p/q \mod n$, $0 < r < n$. In the case $G \subset SU(2)$ considered above we have $r = n-1$, the continued fraction has length $n-1$ with entries $a_i = 2$, and the result is true. But for $p = q$ there is just one ($-n$)-curve, so $e(\widetilde{\mathbf{C}^2/G}) = 2$ equals $e(\mathbf{C}^2, G) = n$ only if $n = 2$, i.e. $G \subset SU(2)$.

In higher dimensions the same phenomenon occurs: If $G \subset SU(n)$ is generated by a diagonal matrix $diag(\zeta, \ldots, \zeta)$ for ζ a primitive n-th root of unity, then a resolution of $(\mathbf{C}^n/G)$ consists of a single $\mathbf{P}^{n-1}$ with normal bundle $\mathcal{O}(-n)$ and we have $e(\widetilde{\mathbf{C}^n/G}) = n = e(\mathbf{C}^n, G)$.

Kummer surfaces: The quotient of an abelian surface (two-dimensional complex torus) X by the involution $\tau : x \to -x$ has 16 singularities corresponding to the 16 fixed points of τ. Each singularity can be resolved by a single (-2)-curve. This minimal resolution $\widetilde{X/\langle\tau\rangle}$ is called the Kummer surface of X. It is a K3-surface with Euler characteristic 24. On the other hand $e(X, \langle\tau\rangle) = \frac{1}{2}(e(X) + 3 \cdot e(X^\tau)) = \frac{1}{2}(0 + 3 \cdot 16) = 24$.

A Calabi-Yau manifold: This is a corresponding example in dimension three. If C is the elliptic curve with complex multiplication of order 3, the cyclic group $G = \langle\rho\rangle$ of order 3 operates also on $X = C \times C \times C$ with 27 fixed points. As described above, each of the corresponding singularities is resolved by a $\mathbf{P}^2$, and we get

$$\begin{aligned} e(X, G) &= \frac{1}{3}(e(X) + 8 \cdot e(X^\rho)) = \frac{1}{3}(0 + 8 \cdot 27) = 72 \\ e(\widetilde{X/G}) &= e(X/G) - 27 + 27 \cdot e(\mathbf{P}^2) = \frac{1}{3}(e(X) + 2 \cdot e(X^\rho)) + 54 = 72. \end{aligned}$$

These global results are not too surprising if one has the local results for quotient singularities, since $e(X,G) = e(X_1,G) + e(X_2,G)$ for reasonable disjoint unions $X = X_1 \cup X_2$ of G-invariant subsets for which the Euler characteristic is defined.

Göttsche's Formula [Gö$_{1,2}$]: One important class of examples consists in the symmetric powers $S^{(n)}$ of a smooth (complex-)algebraic surface S. The symmetric power is a quotient of the cartesian power S^n by the obvious action of the symmetric group $\mathcal{S}_n$. Algebraic Geometry provides a canonical resolution

$$Hilb^n(S) =: S^{[n]} \xrightarrow{f} S^{(n)}$$

by the Hilbert scheme of finite subschemes of length n. The action leaves the canonical divisor of S^n invariant, so it descends to a canonical divisor on $S^{(n)}$. This divisor is not affected by the resolution, i.e. $f^*\mathcal{K}_{S^{(n)}} = \mathcal{K}_{S^{[n]}}$. If in particular S has trivial canonical divisor then so does $S^{[n]}$, but we will see that $e(S^{[n]}) = e(S^n, \mathcal{S}_n)$ holds in general.

In his Diplom thesis Lothar Göttsche computed the Betti numbers of $S^{[n]}$ for an algebraic surface S. His main result is

$$\sum_{n=0}^{\infty} \tilde{P}(S^{[n]}, z)\cdot t^n \quad = \quad \exp\left(\sum_{k=1}^{\infty} \frac{t^k}{k}\cdot\frac{\tilde{P}(S,z^k)}{1-z^{2k}t^k}\right)$$

where $\tilde{P}(X,z)$ denotes the modified Poincaré polynomial $\tilde{P}(X,z) = P(X,-z) = \sum(-1)^i b_i(X) z^i$. For the Euler characteristic $e(X) = \tilde{P}(X,1)$ this simplifies to

$$\begin{aligned}
\sum_{n=0}^{\infty} e(S^{[n]})\cdot t^n &= \exp\left(e(S)\sum_{i=1}^{\infty}\frac{1}{i}\frac{t^i}{1-t^i}\right) \\
&= \exp\left(e(S)\sum_{i=1}^{\infty}\frac{1}{i}\sum_{k=1}^{\infty} t^{ik}\right) \\
&= \exp\left(e(S)\sum_{k=1}^{\infty} -\log(1-t^k)\right) \\
&= \prod_{k=1}^{\infty}(1-t^k)^{-e(S)}.
\end{aligned}$$

Compare these formulae to those obtained for symmetric powers by I.M. Macdonald ([Ma], [Za]), for example:

$$\sum_{n=0}^{\infty} e(S^{(n)})\cdot t^n \quad = \quad (1-t)^{-e(S)}$$

Verification of $e(S^{[n]}) = e(S^n, \mathcal{S}_n)$ for symmetric powers of algebraic surfaces: Let $\mathcal{M}(n)$ denote the set of all series $(\alpha) = (\alpha_1, \alpha_2, \ldots)$ of nonnegative integers with $\sum_i i\alpha_i = n$, and $\mathcal{M} := \bigcup \mathcal{M}(n)$. The conjugacy class of a permutation $\sigma \in \mathcal{S}_n$ is determined by its type $(\alpha) = (\alpha_1, \alpha_2, \ldots) \in \mathcal{M}(n)$ where α_i denotes the number of i-cycles in σ. Its fixed point set in S^n consists of all n-tuples $(x_1, \ldots, x_n)$ with $x_{\nu_1} = \ldots = x_{\nu_i}$ for any i-cycle $(\nu_1 \ldots \nu_i)$ in σ and is therefore isomorphic to $\prod_i S^{\alpha_i}$. Any element τ in the centralizer $C(\sigma)$ permutes the cycles of σ respecting their length, i.e. it induces permutations π_i of α_i elements. Thus $C(\sigma)$ maps onto $\prod_i \mathcal{S}_{\alpha_i}$, the kernel acting trivially on $\prod_i S^{\alpha_i}$. Therefore $(S^n)^\sigma / C(\sigma) = \prod_i S^{(\alpha_i)}$ is a product of symmetric powers. We can compute $e(S^n, \mathcal{S}_n)$ using the formulae of Macdonald and Göttsche:

$$\begin{aligned}
\sum_{n=0}^{\infty} e(S^n, \mathcal{S}_n) \cdot t^n &= \sum_{n=0}^{\infty} \sum_{[\sigma] \subset \mathcal{S}_n} e((S^n)^\sigma / C(\sigma)) \cdot t^n \\
&= \sum_{n=0}^{\infty} \sum_{(\alpha) \in \mathcal{M}(n)} \left(\prod_i e(S^{(\alpha_i)}) \right) \cdot t^n \\
&= \sum_{(\alpha) \in \mathcal{M}} \prod_{i \geq 1} \left(e(S^{(\alpha_i)}) \cdot t^{i\alpha_i} \right) \\
&= \prod_{i=1}^{\infty} \sum_{\alpha_i = 0}^{\infty} \left(e(S^{(\alpha_i)}) \cdot t^{i\alpha_i} \right) \\
&= \prod_{i=1}^{\infty} \frac{1}{(1 - t^i)^{e(S)}} \\
&= \sum_{n=0}^{\infty} e(S^{[n]}) \cdot t^n
\end{aligned}$$

Graeme Segal's interpretation (Equivariant K-theory): Equivariant K-theory of (X,G) and ordinary K-theory of the fixed point sets are related by an isomorphism of complex vector spaces [Se]

$$K_G(X) \otimes \mathbf{C} \xrightarrow{\sim} \bigoplus_{[g]} K(X^g / C(g)) \otimes \mathbf{C}.$$

The image of an equivariant vector bundle E on X is defined as follows: On $E|_{X^g}$ the element g still acts, leaving the base points fixed. Thus $E|_{X^g}$ splits into a direct sum of vector bundles consisting of the eigenspaces of g in every fibre. We put the corresponding eigenvalue in the second factor and get an element in $K(X^g) \otimes \mathbf{C}$. Now as $C(g)$ still acts on X^g, we can take the invariants and get something in $K(X^g)^{C(g)} \otimes \mathbf{C} = K(X^g / C(g)) \otimes \mathbf{C}$. The same also holds for $K_G^1(X)$, and by the standard fact that the Euler characteristic of the complex $K^*(X)$ equals the topological Euler characteristic we can deduce

$$\begin{aligned}
e(K_G^*(X) \otimes \mathbf{C}) &= \dim_{\mathbf{C}} K_G^0(X) \otimes \mathbf{C} - \dim_{\mathbf{C}} K_G^1(X) \otimes \mathbf{C} \\
&= \sum_{[g]} e(X^g / C(g)) \\
&= e(X, G).
\end{aligned}$$

However, since the isomorphism does not commute with Adams operations, we cannot say anything about the single Betti numbers.

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