Birational geometry of Fano direct products

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Abstract

Combining the connectedness principle of Shokurov and Kollár with the technique of hypertangent divisors, we develop a new technique of studying birational maps of natural Fano fiber spaces. We prove that the only structures of a rationally connected fibration on direct products of typical Fano varieties are projections onto the factors. In particular, the groups of biregular and birational self-maps of Fano direct products coincide.

CONTENTS

Introduction

The aim of this paper is to prove that Fano direct products

$$F_1 \times \ldots \times F_K,$$

where F_i belong to a large class of primitive Fano varieties (which includes general Fano hypersurfaces $V_M \subset \mathbb{P}^M$, $M \geq 6$, and Fano double spaces of index 1 and dimension ≥ 3) admit no other structures of a rationally connected fibration than those given by the projections onto direct factors $F_{i_1} \times \ldots \times F_{i_k}$.

The proof is given in full detail. However, I do not explain the details of one of the basic facts underlying the proof, that is, the connectedness principle of Shokurov and Kollár and its immediate follow up, the inversion of adjunction ([K]), which are in their turn based on the Kawamata-Viehweg vanishing theorem [Kw,V]. Our method is based on the connectedness principle combined with the (modified) technique of hypertangent divisors (see, for instance, [P3]).

For the Fano hypersurfaces of low dimension, that is, the famous three-dimensional quartic [IM] and four-dimensional quintic [P1], the proof is not completed yet. There is, however, no doubt that general varieties of these two types are divisorially canonical (see Definition 4 below).

The main result of this paper remains true in slightly weaker assumptions: one should assume divisorial *log*-canonicity of the factors together with the fact that *moving* divisors are canonical. No modification of the argument is required.

The result was obtained in October 2003 during my stay at Max-Planck-Institut für Mathematik in Bonn. However, it was not until Easter holidays (April 2004) that I could start writing up. I am very grateful to the Institute for the excellent conditions of work.

1 Effective divisors on rationally connected fibrations

1.1 Effective divisors and essentially non-effective divisors

Let X be an (irreducible) algebraic variety, non-singular in codimension one, that is, $\operatorname{codim} \operatorname{Sing} \geq 2$.

Definition 1. An algebraic family of irreducible curves on a variety X

$$\mathcal{C} = (C_t, t \in T),$$

parameterized by an irreducible quasi-projective variety T, is said to be *free*, if for any closed subset $Y \subset X$ of codimension 2 the set

$$T_Y = \{t \in T \,|\, C_t \cap Y \neq \emptyset\}$$

is a proper closed subset in T.

Definition 2. A divisor D on the variety X is said to be *essentially non-effective*, if on X there is a free algebraic family of irreducible curves $\mathcal{C} = (C_t, t \in T)$, the general curve of which satisfies the inequality

$$(D \cdot C_t) < 0.$$

Remark 1. Since the variety X by assumption is non-singular in codimension 1, whereas the family of curves $\mathcal{C}$ is free, for a general $t \in T$ the curve C_t lies entirely in the smooth part of the variety X, so that the intersection index $(D \cdot C_t)$ is well defined.

Definition 3. We say that a smooth rationally connected variety X (respectively, a rationally connected fiber space $\pi\colon X \to S$ with X, S smooth) satisfies the necessary condition of effectivity of divisors or briefly the *condition* (E) (respectively, the relative necessary condition of effectivity of divisors or briefly the *condition* (R)), if

- for any irreducible divisor D,

$$lD \in |-nK_X + Z|,$$

  where Z is an effective divisor on X (respectively, $Z = \pi^* Z_S$ is the pull back of an effective divisor Z_S on the base S) and $n \in \mathbb{Z}_+$, $l \geq 1$ a positive integer (in the relative case $l = 1$),

- for any non-trivial sequence of blow ups

$$\varphi\colon \widetilde{X} \to X$$

  the divisor

$$Z^\sharp = l\widetilde{D} + nK_{\widetilde{X}}$$

is not essentially non-effective, where $\widetilde{D} \subset \widetilde{X}$ is the strict transform of the irreducible divisor D on the variety $\widetilde{X}$.

Remark 2. Let $\mathcal{E}$ be the set of exceptional divisors of the birational morphism φ. Let $a_E \in \mathbb{Z}_+$ be the discrepancy of the divisor $E \in \mathcal{E}$. There is a set of non-negative integers $d_E \in \mathbb{Z}_+$, $E \in \mathcal{E}$, such that

$$\widetilde{D} \sim \varphi^* D - \sum_{E \in \mathcal{E}} d_E E.$$

Therefore

$$Z^\sharp = \varphi^* Z + \sum_{E \in \mathcal{E}} (na_E - ld_E) E.$$

If the condition (E) or (R) holds, then for any free family of irreducible curves $\mathcal{C} = (C_t, t \in T)$ on $\widetilde{X}$ the inequality

$$(Z^\sharp \cdot C_t) \geq 0$$

holds. On the other hand, if the condition (E)(or (R)) does not hold, then for some family $\mathcal{C}$ we have $(Z^\sharp\cdot C_t)<0$. Since the divisor Z is effective, in this case there is an exceptional divisor $E\in\mathcal{E}$ satisfying the Noether-Fano inequality

$$ld_E>n\cdot a_E.$$

1.2 Divisorially canonical varieties

Consider a smooth primitive Fano variety V of dimension $\dim V\geq 3$, $\operatorname{Pic}V=\mathbb{Z}K_V$.

Definition 4. We say that the variety V is *divisorially canonical* or, briefly speaking, satisfies the condition (C), if for any effective divisor

$$D\in|-nK_V|,$$

$n\in\mathbb{Z}_+$, the pair $(V,\frac{1}{n}D)$ has canonical singularities.

In other words, V satisfies the condition (C), if for any effective divisor $D\in|-nK_V|$, any non-trivial sequence of blow ups

$$\varphi\colon\widetilde{V}\to V$$

and any exceptional divisor $E\subset\widetilde{V}$ the inequality

$$\nu_E(D)\leq na(E),$$

holds, where $\nu_E(D)=\operatorname{ord}_E\varphi^*D$ is the multiplicity of the divisor D with respect to the discrete valuation $\nu_E(\cdot)$, $a(E)$ is the discrepancy of the divisor E.

Theorem 1. *If the primitive Fano variety V satisfies the condition (C), then it satisfies the condition (E), too.*

Proof. Assume the converse: the condition (E) does not hold. Let $H=-K_V$ be the positive generator of the Picard group. In accordance with Definition 3, there is an effective divisor D,

$$lD\in|(n+m)H|,$$

where n, $m\in\mathbb{Z}_+$, $l\geq 1$ and a non-trivial sequence of blow ups $\varphi\colon\widetilde{V}\to V$ such that the divisor

$$Z^\sharp=l\widetilde{D}+nK_{\widetilde{V}}$$

is essentially non-effective on $\widetilde{V}$. Let $\mathcal{C}=(C_t,t\in T)$ be a free family of irreducible curves on $\widetilde{V}$, on which $Z^\sharp$ is negative. Let $\mathcal{E}$ be the set of exceptional divisors of the birational morphism φ. Then

$$l\widetilde{D}=(n+m)H-\sum_{E\in\mathcal{E}}d_EE,\quad K_{\widetilde{V}}=-H+\sum_{E\in\mathcal{E}}a_EE,$$

where a_E is the discrepancy and we write H instead of φ^*H. If the number $\gamma=-(K_{\widetilde{V}}\cdot C_t)\leq 0$, then the inequality $(Z^\sharp\cdot C_t)<0$ implies that

$$(\widetilde{D}\cdot C_t)<0,$$

which is impossible, since $\widetilde{D}$ is effective and $\mathcal{C}$ is a dense family of irreducible curves. Therefore $\gamma > 0$. Besides, $n \geq 1$.

Now compare the inequalities

$$(Z^\sharp \cdot C_t) = m(H \cdot C_t) - \sum_{E \in \mathcal{E}} (ld_E - na_E)(E \cdot C_t) < 0 \tag{1}$$

and

$$-(K_{\widetilde{V}} \cdot C_t) = (H \cdot C_t) - \sum_{E \in \mathcal{E}} a_E (E \cdot C_t) > 0 \tag{2}$$

Obviously, $(H \cdot C_t) > 0$ and $(E \cdot C_t) \geq 0$ for all $E \in \mathcal{E}$, whereas for some $E \in \mathcal{E}$ we have $(E \cdot C_t) > 0$, otherwise the inequality (1) would not be possible. Set

$$\mathcal{E}^+ = \{E \in \mathcal{E} | (E \cdot C_t) > 0\}.$$

Lemma 1. *There exists an exceptional divisor $E \in \mathcal{E}^+$* such that

$$ld_E > (n+m)a_E \tag{3}$$

Proof. Rewrite the inequality (1) in the following way

$$(Z^\sharp \cdot C_t) = m[(H \cdot C_t) - \sum_{E \in \mathcal{E}^+} a_E (E \cdot C_t)] + \sum_{E \in \mathcal{E}^+} (-ld_E + (n+m)a_E)(E \cdot C_t) < 0$$

Taking into consideration the inequality (2), we obtain the claim of the lemma.

The inequality (3) is the Noether-Fano inequality for the divisor lD. Thus the pair $(V, \frac{l}{n+m}D)$ is not canonical and the condition (C) is not satisfied. A contradiction.

Q.E.D. for the theorem.

1.3 Direct products

Theorem 2. *Let S be a smooth rationally connected variety, F a smooth primitive Fano variety. Assume that:*

(1) the variety S satisfies the condition (E),

(2) the variety F satisfies the condition (C).

Then the Fano fiber space

$$\pi : F \times S \to S$$

(π is the projection onto the second direct factor) satisfies the condition (R).

Proof. Assume the converse: the condition (R) is not satisfied by the Fano fiber space $\pi : V = F \times S \to S$. Therefore there exists a prime divisor D,

$$D \in |-nK_V + Z|$$

(recall that in the relative case $l = 1$), where $Z = \pi^* Z_S = F \times Z_S$ is an effective divisor on the base S and a non-trivial sequence of blow ups

$$\varphi : \widetilde{V} \to V,$$

such that the class $Z^\sharp = \widetilde{D} + nK_{\widetilde{V}}$ is negative on the free family of irreducible curves $\mathcal{C} = (C_t, t \in T)$.

First of all, let us see to what extent this situation depends on the choice of the sequence of blow ups φ. Let

$$\psi \colon V^+ \to V$$

be another model of the variety V, that is, ψ is a birational morphism. Assume that the birational map

$$\gamma = \psi^{-1} \circ \varphi \colon \widetilde{V} - - \to V^+$$

is an isomorphism in codimension one (in other words, γ is biregular at the generic point of each exceptional divisor $E \subset \widetilde{V}$ of the morphism φ). Then there exist open sets

$$U \subset \widetilde{V}, \quad U^+ \subset V^+$$

such that $\gamma(U) = U^+$, $\gamma \colon U \to U^+$ is an isomorphism and

$$\operatorname{codim}_{\widetilde{V}}(\widetilde{V} \setminus U) \geq 2.$$

Since the family of curves $\mathcal{C}$ is free, for a general $t \in T$ we get

$$C_t \subset U.$$

Set $T^+ = \{t \in T \mid C_t \subset U\}$. Then the family of irreducible curves

$$\mathcal{C}^+ = (C_t^+ = \gamma(C_t), t \in T^+)$$

is a free family of irreducible curves on V^+. Set D^+ to be the strict transform of the divisor D on V^+, $K^+ = K_{V^+}$. Obviously,

$$((D^+ + nK^+) \cdot C_t^+) = ((\widetilde{D} + nK_{\widetilde{V}}) \cdot C_t)$$

for $t \in T^+$, so that the class $Z^+ = D^+ + nK^+$ is negative on the curves C_t^+ and thus essentially non-effective.

From what was said it follows that the sequence of blow ups φ can be replaced by another sequence of blow ups $\psi \colon V^+ \to V$ provided that the map $\psi^{-1} \circ \varphi$ is biregular at the generic point of each exceptional divisor $E \subset \widetilde{V}$ of the morphism φ. Note also that for the condition of being essential non-effective only those exceptional divisors $E \subset \widetilde{V}$ matter, for which $(E \cdot C_t) \neq 0$.

By what was said, we may assume from the start that the sequence of blow ups φ has the following structure:

$$\varphi \colon \widetilde{V} \xrightarrow{\gamma} V^+ \xrightarrow{\mu} V,$$

$\varphi = \mu \circ \gamma$, where the properties of birational morphisms γ and μ are described below.

The morphism μ is a part of the commutative diagram

$$\begin{array}{ccccc} & V^+ & \xrightarrow{\mu} & V & \\ \pi & \downarrow & & \downarrow & \pi \\ & S^+ & \xrightarrow{\mu_S} & S, & \end{array}$$

where $\mu_S\colon S^+ \to S$ is a sequence of blow ups on the base S, $\mu = (id_F, \mu_S)$, $V^+ = F \times S^+$, the symbol π in both cases stands for the projection onto the second factor. Let $\mathcal{E}_S$ be the set of exceptional divisors of the birational morphism μ_S. Then the exceptional divisors of the morphism μ are of the form $F \times E = \pi^* E$ for $E \in \mathcal{E}_S$.

The morphism $\gamma\colon \widetilde{V} \to V^+$ is a sequence of blow ups with the set of exceptional divisors $\mathcal{E}^+$. We denote the composition $\pi \circ \gamma\colon \widetilde{V} \to S^+$ by the same symbol π. For each exceptional divisor $E \in \mathcal{E}^+$ we have $\pi^{-1}(\pi(E)) \neq E$, that is, it is not a pull back of a divisor on the base. For any exceptional divisor $E \in \mathcal{E}^+$ satisfying the condition $(E \cdot C_t) \neq 0$, one of the following two cases realize:

1) either $\pi(E) = S^+$,

2) or $\pi(E) \subset S^+$ is a prime divisor on S^+.

Existence of this modification of the morphism φ follows from the remark above. Indeed, let $\varphi_1\colon V_1 \to V$ be an arbitrary sequence of blow ups, for which the condition (E) is violated. Let $\mathcal{E}_1$ be the set of exceptional divisors. Let us break the set $\mathcal{E}_1$ into two subsets:

$$\mathcal{E}_1 = \mathcal{E}_1^\sharp \cup \mathcal{E}_1^*,$$

where $\varphi_1(E)$ covers the base S for $E \in \mathcal{E}_1^\sharp$ and $\pi \circ \varphi_1(E) \subset S$ is a proper closed subset for $E \in \mathcal{E}_1^*$. Now we construct an arbitrary sequence of blow ups

$$\mu_S\colon S^+ \to S,$$

such that for any discrete valuation ν_E, $E \in \mathcal{E}_1^*$, its center

$$B_E = \mathrm{centre}(\nu_E, V^+)$$

on $V^+ = F \times S^+$ covers a divisor on the base:

$$\pi(B_E) \subset S^+$$

is a prime divisor. Now let $\gamma\colon \widetilde{V} \to V$ be an arbitrary sequence of blow ups, the centers of which are not pulled back from the base S^+, such that all the discrete valuations

$$\nu_E, \quad E \in \mathcal{E}_1 = \mathcal{E}_1^\sharp \cup \mathcal{E}_1^*,$$

are realized by prime divisors on $\widetilde{V}$. It is easy to see that $\varphi = \mu \circ \gamma$, $\mu = (id_F, \mu_S)$ satisfies the properties described above.

Let D^+ be the strict transform of the prime divisor D on V^+, $K^+ = K_{V^+}$ the canonical class.

Lemma 2. *The algebraic family of curves*

$$\mathcal{C}_S = \pi_* \gamma_* \mathcal{C} = (\pi\gamma(C_t), t \in T)$$

on S^+ is free unless all the curves C_t, $t \in T$, lie in the fibers of the morphism π.

Proof. Let $Y \subset S_+$ be an irreducible closed subset of codimension ≥ 2. We have

$$\pi^{-1}(Y) = \bigcup_{i=1}^{N} Y_i \subset \widetilde{V}.$$

It is clear that the curve $\pi(C_t)$ intersects Y if and only if the curve C_t intersects at least one of the sets Y_i, $i = 1, ..., N$. If $\operatorname{codim} Y_i \geq 2$, then $C_t \cap Y_i = \emptyset$, since the family $\mathcal{C}$ is free. If $Y_i \subset \widetilde{V}$ is a prime divisor, then $Y_i = E \in \mathcal{E}^+$ is an exceptional divisor of the morphism γ. However, by the properties of the morphism γ

$$(C_t \cdot E) = 0,$$

since $\operatorname{codim} \pi(E) \geq \operatorname{codim} Y \geq 2$. Once again we obtain that for a general $t \in T$ the curve C_t does not intersect $E = Y_i$. Consequently, for a general $t \in T$

$$\pi(C_t) \cap Y = \emptyset$$

Q.E.D. for the lemma.

Corollary 1. *The class*

$$Z^+ = D^+ + nK^+$$

is non-negative on the dense (although generally speaking, not free) family of curves

$$\mathcal{C}^+ = \gamma_* \mathcal{C} = (\gamma(C_t), t \in T).$$

Proof. Assume the converse:

$$(Z^+ \cdot C_t^+) < 0 \tag{4}$$

$C^+ = \gamma(C_t)$. Clearly, in this case the curves C_t do not lie in the fibers of the projection π. Explicitly, we get

$$D^+ = \mu^* D - \sum_{E \in \mathcal{E}_S} d_E \pi^* E,$$

$$K^+ = \mu^* K_V + \sum_{E \in \mathcal{E}_S} a_E \pi^* E,$$

where a_E is the discrepancy of the exceptional divisor $E \subset S^+$ with respect to S. Taking into account that

$$D + nK_V = \pi^* Z_S$$

for some effective class Z_S on S, we obtain that the class $Z^+ = \pi^* Z_S^+$, where

$$Z_S^+ = Z_S - \sum_{E \in \mathcal{E}_S} (d_E - na_E) E,$$

is negative on the dense family of curves $\mathcal{C}^+$ and thus the class Z_S^+ is negative on the free family of curves $\mathcal{C}_S$.

Since $V = F \times S$, we get

$$\operatorname{Pic} V = \pi_F^* \operatorname{Pic} F \oplus \pi^* \operatorname{Pic} S = \mathbb{Z} K_V \oplus \pi^* \operatorname{Pic} S,$$

where $K_V = \pi_F^* K_F + \pi^* K_S$. In particular, for the divisor D we obtain the presentation

$$D = -n\pi_F^* K_F + \pi^* D_S,$$

where $D_S = -nK_S + Z_S$. Let $z \in F$ be a point of general position. Set

$$D_z = D \cap \{z\} \times S.$$

Obviously, D_z is an effective reduced divisor on S, and moreover

$$D_z \in |D_S|,$$

and for a sufficiently general point $z \in F$ we have

$$D_z^+ \in |\mu_S^* D_S - \sum_{E \in \mathcal{E}_S} d_E E|,$$

where D_z^+ is the strict transform of the divisor D_z on S^+. Setting

$$K_S^+ = K_{S^+} = \mu_S^* K_S + \sum_{E \in \mathcal{E}} a_E E$$

to be the canonical class of the variety S^+, we obtain the presentation

$$Z_S^+ = D_S^+ + nK_S^+.$$

Note that in the situation under consideration the sequence of blow ups μ_S cannot be trivial, since the class Z_S is effective.

Thus assuming that the inequality (4) is true, we obtain a non-trivial sequence of blow ups $\mu_S \colon S^+ \to S$ such that the class $(D_S^+ + nK_S^+)$ is essentially non-effective, whereas the divisor $D_z \sim D_S = -nK_S + Z_S$ is effective and Z_S is an effective class. The only problem which prevents us from applying the condition (E) is that the divisor D_z can be reducible. Let us consider this problem.

For a general point $z \in F$ the divisor D_z can be reducible although the divisor D is irreducible.

Consider the set of divisors

$$D = \{D_z \,|\, z \in F, \{z\} \times S \not\subset D\}.$$

These divisors have no common irreducible components: otherwise the divisor D would have been reducible (or pulled back from the base S, which is impossible). Therefore the linear system $|D_S|$ has no fixed components. In a similar way, the linear system of divisors

$$|D_S^+|$$

is free from fixed components. Let

$$\varepsilon \colon S - - \to \mathbb{P}^N$$

be the rational map defined by this linear system. Two cases are possible: $\dim \varepsilon(S) = 1$ and $\dim \varepsilon(S) \geq 2$.

In the second case the general divisor $\Delta^+ \in |D_S^+|$ is irreducible. Set

$$\Delta = \mu_S(\Delta^+) \subset S.$$

Obviously, Δ^+ is the strict transform of the divisor Δ on S^+, $\Delta \sim D_z$, $\Delta^+ \sim D_z^+$. Now we come to a contradiction with the condition (E) and conclude that the inequality (4) cannot be true.

In the first case note that the image $\varepsilon(S)$ is a rational curve (recall that the variety S is rationally connected). Therefore for a general point $z \in F$

$$D_z = D_1(z) + D_2(z) + \ldots + D_k(z),$$

$$D_z^+ = D_1^+(z) + D_z^+(z) + \ldots + D_k^+(z).$$

The number of components $k \geq 2$ does not depend on z (for a general z) and

$$D_1(z) \sim D_2(z) \sim \ldots \sim D_k(z),$$

and similarly

$$D_1^+(z) \sim D_2^+(z) \sim \ldots \sim D_k^+(z),$$

that is, $D_i(z) \in \Pi$, $D_i^+(z) \in \Pi^+$, where Π is a pencil of divisors on S, and Π^+ its strict transform on S^+. Thus

$$D_z \sim kD_1(z), \quad D_z^+ \sim kD_1^+(z)$$

and we obtain an irreducible divisor $D_1(z)$ such that

$$kD_1(z) \sim -nK_S + Z_S$$

and the class

$$kD_1^+(z) + nK_S^+$$

is essentially non-effective. Once again we come to a contradiction with the condition (E).

Q.E.D. for Corollary 1.

Remark 3. It is clear from the last part of the proof of Corollary 1, why do we need to introduce this additional parameter $l \geq 1$. It can be avoided if one works with $\mathbb{Q}$-divisors, that is, in the presentation

$$D \sim -nK_V + Z$$

one allows that $n \in \mathbb{Q}_+$ and Z is a $\mathbb{Q}$-divisor (however, the divisor D remains to be integral).

Remark 4. We require that the divisor D is irreducible for the only reason: one cannot allow that D had components pulled back from the base.

Let us continue the proof of Theorem 2.

For the divisor $Z^\sharp$ on $\widetilde{V}$ we have the presentation

$$Z^\sharp = \gamma^* Z^+ + \sum_{E\in\mathcal{E}^+} (-d^+_E + na^+_E)E,$$

where a^+_E is the discrepancy of the exceptional divisor $E \subset \widetilde{V}$ with respect to V^+, d^+_E is the multiplicity of the divisor D^+ with respect to the discrete valuation $\mathrm{ord}_E(\cdot)$. By assumption, $(Z^\sharp \cdot C_t) < 0$. On the other hand,

$$(\gamma^* Z^+ \cdot C_t) = (Z^+ \cdot \gamma(C_t)) \geq 0.$$

Therefore there exists an exceptional divisor $E \in \mathcal{E}^+$ such that

$$d^+_E > na^+_E \tag{5}$$

and

$$(E \cdot C_t) > 0.$$

By the properties of the morphism γ we get that the closed set $\pi\gamma(E) \subset S^+$ is either the whole variety S^+ or an exceptional divisor on S^+. In any case for a point of general position $s \in \pi\gamma(E)$ we get

$$F_s = F \times \{s\} = \pi^{-1}(s) \not\subset D^+,$$

since the divisor D^+ is irreducible. On the other hand, $\gamma(E)$ intersects F_s, because $s \in \pi\gamma(E)$. The Noether-Fano inequality (5) means that the pair

$$(V^+, \frac{1}{n}D^+)$$

is not canonical along $\gamma(E) \cap F_s$. By the inversion of adjunction this implies that the pair

$$(F_s, \frac{1}{n}D^+_s),$$

where $D^+_s = D^+|_{F_s}$, is not log-canonical and the more so not canonical. However, $F_s \cong F$ and $D^+_s \in |nK_F|$ by construction, since $K^+_V|_{F_s} = K_{F_s}$ and the divisor Z is pulled back from the base.

This gives a contradiction with the condition (C).

Q.E.D. for the theorem.

2 Fano direct products

2.1 The relative and absolute cases

Let $F_1, \ldots, F_K$, $K \geq 2$, be primitive Fano varieties, $\mathrm{Pic}\, F_i = \mathbb{Z}K_{F_i}$, $i = 1, \ldots, K$. Let us denote by the symbol H_i the positive generator of the Picard group, $H_i = -K_{F_i}$. Set

$$V = F_1 \times \ldots \times F_K = \prod_{i=1}^{K} F_i,$$

$$S_j = \prod_{\substack{i=1\\ i\neq j}}^{K} F_i, \quad j=1,\dots,K,$$

so that $V = F_j \times S_j$, $j=1,\dots,K$. The projections of the variety V onto F_j and S_j we denote by λ_j and π_j, respectively.

Theorem 3. *Assume that each of the Fano fiber spaces $\pi_j\colon V \to S_j$, $j=1,\dots,K$ satisfies the condition (R). Then the variety V satisfies the condition (E).*

Proof. Obviously,

$$\operatorname{Pic} V = \lambda_j^* \operatorname{Pic} F_j \oplus \pi_j^* \operatorname{Pic} Sj = \bigoplus_{i=1}^{K} \lambda_i^* \operatorname{Pic} F_i = \bigoplus_{i=1}^{K} \mathbb{Z}(\lambda_i^* H_i).$$

Besides,

$$K_V = \lambda_j^* K_{F_j} + \pi_j^* K_{S_j} = -\sum_{i=1}^{K} \lambda_i^* H_i.$$

Let D,

$$lD \sim -nK_V + Z,$$

be an irreducible divisor, where Z is an effective divisor on V. To simplify the notations, we write H_i instead of $\lambda_i^* H_i$. We get

$$Z = \sum_{i=1}^{K} m_i H_i$$

with the coefficients $m_i \in \mathbb{Z}_+$. Setting

$$m = m_e = \min_{i \in \{1,\dots,K\}} m_i,$$

we write Z in the form

$$Z = -mK_V + \pi_e^* Y, \tag{6}$$

where

$$Y = \sum_{\substack{i=1\\ i\neq e}}^{K} (m_i - m) H_i$$

is an effective divisor on S_e.

The remaining part of the arguments is similar to the proof of Theorem 1. Fix a non-trivial sequence of blow ups

$$\varphi\colon \widetilde{V} \to V$$

with the set of exceptional divisors $\mathcal{E}$. Let $\widetilde{D} \subset \widetilde{V}$ be the strict transform of the divisor D. We get

$$l\widetilde{D} \sim -nK_{\widetilde{V}} + Z^\sharp,$$

where

$$Z^\sharp = \varphi^* Z - \sum_{E\in\mathcal{E}} (ld_E - na_E)E.$$

Taking into account the presentation (6), rewrite the latter formula in the form

$$Z^\sharp = -mK_{\widetilde{V}} + \varphi^*\pi_e^* Y - \sum_{E\in\mathcal{E}} (ld_E - (n+m)a_E)E.$$

Assume now that the class $Z^\sharp$ is negative on the free family of irreducible curves $\mathcal{C} = (C_t, t\in T)$ on $\widetilde{V}$. First of all, we get

$$-(K_{\widetilde{V}}\cdot C_t) > 0,$$

otherwise $(\widetilde{D}\cdot C_t) < 0$, which is impossible. It follows that the class

$$Y^\sharp = \varphi^*\pi_e^* Y - \sum_{E\in\mathcal{E}} (ld_E - (n+m)a_E)E$$

is negative on the curves of the family $\mathcal{C}$. Since

$$l\widetilde{D} \sim -(n+m)K_{\widetilde{V}} + Y^\sharp,$$

$$lD \sim -(n+m)K_V + \pi_e^* Y,$$

we obtain a contradiction with the condition (R) for the Fano fiber space $\pi_e\colon V\to S_e$.

Therefore the class $Z^\sharp$ cannot be negative on the curves of the family $\mathcal{C}$.

This means that the variety V satisfies the condition (E). Q.E.D. for the theorem.

2.2 Geometry of Fano direct products

Theorem 4. *Assume that each of the varieties F_j, $i = 1,\dots,K$, satisfies the condition (C) of divisorial canonicity. Then their direct product $V = F_1\times\dots\times F_K$ satisfies the condition (E).*

Proof. Let us argue by induction on $K\geq 1$. For $K=1$ the condition (E) is proved in Theorem 1. Assume that the claim of the theorem is proved for the smaller values of K. Then all the varieties S_j, $j=1,\dots,K$ satisfy the condition (E). Since the varieties F_j, $j=1,\dots,K$ are divisorially canonical, Theorem 2 implies that each of the Fano fiber spaces

$$\pi_j\colon V\to S_j$$

satisfies the relative condition (R). Now we apply Theorem 3 and conclude that V satisfies the absolute condition (E). Q.E.D. for the theorem.

Theorem 5. *In the assumptions of Theorem 4 let*

$$\chi\colon V - - \to V^\sharp$$

be a birational map, where $\beta\colon V^\sharp \to S^\sharp$ is a rationally connected fibration. Then there exists an index $e \in \{1,\dots,K\}$ such that the following diagram of maps is commutative:

$$\begin{array}{ccccc} & V & \stackrel{\chi}{- - \to} & V^\sharp & \\ \pi_e & \downarrow & & \downarrow & \beta \\ & S_e & \stackrel{\alpha}{- - \to} & S^\sharp & \end{array}$$

for some rational dominant map $\alpha\colon S_e - - \to S^\sharp$, that is, the map χ is fiber-wise with respect to the structures of rationally connected fibrations V/S_e and $V^\sharp/S^\sharp$.

Corollary 2. (i) *Let $\rho\colon V - - \to Y$ be a rational dominant map, the general fiber of which is a rationally connected variety. Then there is a set of indices*

$$I = \{i_1,\dots,i_k\} \subset \{1,\dots,K\},$$

such that Y is birational to

$$\prod_{i\in I} F_i$$

and the map ρ is equivalent to the projection with respect to the direct decomposition

$$V \cong \prod_{i\in I} F_i \times \prod_{i\notin I} F_i.$$

(ii) *The groups of birational and biregular self-maps of the variety V coincide:*

$$\operatorname{Bir} V = \operatorname{Aut} V.$$

(iii) *The variety V is non-rational.*

Proof of the corollary. Let us prove (i). The other two claims are obvious. Let $\rho\colon V - - \to Y$ be a rational dominant map, the general fiber of which is rationally connected. Theorem 5 implies that ρ factors through the map $\pi_e\colon V \to S_e$ for some $e \in \{1,\dots,K\}$, that is, $\rho = \rho_1 \circ \pi_e$ for some rational dominant map $\rho_1\colon S_e - - \to Y$. Obviously, for a general fiber of the map ρ_1 we get

$$\rho^{-1}(y) = F_e \times \rho_1^{-1}(y),$$

so that the general fiber of the map ρ_1 is rationally connected. Iterating this procedure, we obtain the claim (i). Q.E.D. for the corollary.

Remark 5. Let us break the set of varieties $F_1,\dots,F_K$ into subsets of pair-wise isomorphic varieties:

$$I = \{1,\dots,K\} = \bigvee_{j=1}^{l} I_j,$$

where $F_i \cong F_j$ if and only if $\{i,j\} \subset I_k$ for some $k \in \{1,\dots,l\}$. It is easy to see that

$$\operatorname{Aut} V = \prod_{j=1}^{l} \operatorname{Aut}(\prod_{i\in I_j} F_i).$$

In particular, if the varieties $F_1,\dots,F_K$ are pair-wise non-isomorphic, then

$$\operatorname{Aut} V=\prod_{i=1}^{K}\operatorname{Aut} F_i.$$

Now consider the opposite case when

$$F_1\cong F_2\cong\ldots\cong F_K\cong F.$$

It is easy to see that in this case we get the exact sequence

$$1\to(\operatorname{Aut} F)^{\times K}\to\operatorname{Aut} V\to S_K\to 1,$$

where S_K is the symmetric group of permutations of K elements.

2.3 Proof of Theorem 5

Let $\chi\colon V--\to V^\sharp$ be a birational map, where $\beta\colon V^\sharp\to S^\sharp$ is a rationally connected fibration. Choose a very ample class $H^\sharp$ on $S^\sharp$ and a general divisor

$$Y\in|H^\sharp|$$

in the linear system $|H^\sharp|$. The divisor Y and its pull back β^*Y on $V^\sharp$ are irreducible. Let D be the strict transform of the divisor β^*Y with respect to χ.

If there is an index $e\in\{1,\dots,K\}$ such that

$$D=\pi_e^*\Delta$$

for some (irreducible) divisor Δ on S_e, then the claim of the theorem is true: χ factors through the projection π_e.

Assume the converse: D is not pulled back from S_e for some $e\in\{1,\dots,K\}$. It follows that

$$D\sim\sum_{i=1}^{K}n_iH_i,$$

where all the integers $n_i\geq 1$. Therefore for

$$n=\min_{i\in\{1,\dots,K\}}n_i\geq 1$$

we get

$$D\sim -nK+Z,$$

where Z is an effective divisor on V. (In fact, $Z=\pi_e^*X$ is pulled back from the base S_e for e such that $n_e=n$, but we do not need that.) Let

$$\varphi\colon\widetilde{V}\to V$$

be a resolution of singularities of the map χ. Consider a free system of irreducible rational curves

$$\mathcal{C}^\sharp = (C^\sharp_t, t \in T^\sharp)$$

on the variety $V^\sharp$, lying in the fibers of the morphism β. Obviously,

$$(C^\sharp_t \cdot K_{V^\sharp}) < 0.$$

Let $\mathcal{C} = (C_t, t \in T)$ be the strict transform of the family $\mathcal{C}^\sharp$ on the variety $\widetilde{V}$. We get

$$(C_t \cdot K_{\widetilde{V}}) = (C^\sharp_t \cdot K_{V^\sharp}) < 0.$$

Finally, let $\widetilde{D} \subset \widetilde{V}$ be the strict transform of the divisor D. We get

$$(\widetilde{D} \cdot C_t) = (\beta^* Y \cdot C^\sharp_T) = 0.$$

Therefore we get

$$(Z^\sharp \cdot C_t) = ((\widetilde{D} + nK_{\widetilde{V}}) \cdot C_t) < 0,$$

where

$$\widetilde{D} \sim -nK_{\widetilde{V}} + Z^\sharp.$$

The family of curves $\mathcal{C}$ is free. We obtain a contradiction with the condition (E). Q.E.D. for the theorem.

3 Divisorially canonical Fano varieties

3.1 $2n$-inequality

Let $x \in S$ be a germ of a smooth surface, $C \subset S$ a germ of an effective (possibly reducible) curve, $x \in C$. Consider a sequence of blow ups

$$\varphi_{i,i-1} \colon S_i \to S_{i-1},$$

$S_0 = S$, $i = 1, \dots, N$, $\varphi_{i,i-1}$ blows up a point $x_{i-1} \in S_{i-1}$, $E_i = \varphi^{-1}_{i,i-1}(x_{i-1}) \subset S_i$ is the exceptional line. For $i > j$ set

$$\varphi_{i,j} = \varphi_{i,i+1,j} \circ \ldots \circ \varphi_{i,i-1} \colon S_i \to S_j,$$

$\varphi = \varphi_{N,0}$, $\widetilde{S} = S_N$. We assume that the points x_i lie one over another, that is, $x_i \in E_i$, and that $x_0 = x$, so that all the points x_i, $i \geq 1$, lie over x:

$$x_{i,0}(x_i) = x \in S.$$

Let Γ be the graph with the vertices $1, \dots, N$ and oriented edges (arrows)

$$i \to j,$$

that connect i and j if and only if $i > j$ and

$$x_{i-1} \in E_j^{i-1},$$

where for a curve $Y \subset S_j$ its strict transform on S_a, $a \geq j$, is denoted by the symbol Y^a. Assume that the point x is an isolated non-log-canonical singularity of the pair

$$(S, \frac{1}{n}C)$$

for some $n \geq 1$. This means that for some exceptional divisor $E \subset \widetilde{S}$ the inequality

$$\nu_E(C) = \mathrm{ord}_E\, \varphi^* C > n(a_E + 1), \tag{7}$$

holds, where a_E is the discrepancy of E. Without loss of generality we may assume that $E = E_N$ is the last exceptional divisor. The inequality (7) is called *the log Noether-Fano inequality.*

For $i > j$ let the symbol p_{ij} denote the number of paths in the graph Γ from the vertex i to the vertex j. For $i < j$ set $p_{ij} = 0$. Set also $p_{ii} = 1$. In terms of the numbers p_{ij} the log Noether-Fano inequality (7) takes the traditional form

$$\sum_{i=1}^{N} p_{Ni}\mu_i > n(\sum_{i=1}^{N} p_{Ni} + 1), \tag{8}$$

where

$$\mu_i = \mathrm{mult}_{x_{i-1}} C^{i-1}.$$

Proposition 1. *Either $\mu_1 > 2n$ (that is, the first exceptional divisor $E_1 \subset S_1$ already gives a non-log-canonical singularity of the pair $(S, (1/n)C)$), or $N \geq 2$ and the following inequality holds:*

$$\mu_1 + \mu_2 > 2n.$$

Proof. If $N = 1$, then $\mu_1 > 2n$ by means of log Noether-Fano inequality. Assume that $\mu_1 \leq 2n$, then $N \geq 2$. Obviously, $\mu_1 > n$. If $\mu_2 \geq n$, then $\mu_1 + \mu_2 > 2n$, as we claim. So assume that $\mu_2 < n$. Then for each $i \in \{1, \dots, N\}$, $i \geq 2$, we have $\mu_i \leq \mu_2 < n$ (since the point x_{i-1} lies over x_1). Therefore from the inequality (8) we get

$$p_{N1}(\mu_1 - n) + \sum_{i=2}^{N} p_{Ni}(\mu_2 - n) > n.$$

However,

$$p_{N1} = \sum_{j \to 1} p_{Nj} \leq \sum_{i=2}^{N} p_{Ni},$$

so that the more so

$$\sum_{i=2}^{N} p_{Ni}(\mu_1 + \mu_2 - 2n) > n.$$

Therefore $\mu_1 + \mu_2 > 2n$. Q.E.D. for the proposition.

3.2 The regularity conditions

Let $V = V_M \subset \mathbb{P} = \mathbb{P}^M$ be a smooth Fano hypersurface. For a point $x \in V$ fix a system of affine coordinates $z_1, \ldots, z_M$ with the origin at x and let

$$f = q_1 + q_2 + \ldots + q_M$$

be the equation of the hypersurface V, $q_i = q_i(z_*)$ are homogeneous polynomials of degree $\deg q_i = i$. In order to prove that this variety is divisorially canonical, we need stronger regularity conditions (that is, the conditions of general position) for f than those that were used for proving birational rigidity in [P3].

($G1$) The sequence

$$q_1, q_2, \ldots, q_{M-1}$$

is regular in $\mathcal{O}_{x,\mathbb{P}}$, that is, the system of equations

$$q_1 = q_2 = \ldots = q_{M-1} = 0$$

defines a one-dimensional subset, a finite set of lines in $\mathbb{P}$, passing through the point x. This is the standard regularity condition, which was used in the previous papers (see [P3]).

($G2$) The linear span of any irreducible component of the closed algebraic set

$$q_1 = q_2 = q_3 = 0$$

in $\mathbb{C}^M$ is the hyperplane $q_1 = 0$ (that is, the tangent hyperplane $T_x V$).

($G3$) The closed algebraic set

$$\overline{\{f_1 = f_2 = 0\} \cap V} = \overline{\{q_1 = q_2 = 0\} \cap V} \subset \mathbb{P}$$

(the bar $^-$ means the closure in $\mathbb{P}$) is irreducible and any section of this set by a hyperplane $P \ni x$ is

- either also irreducible and reduced,
- or breaks into two irreducible components $B_1 + B_2$, where $B_i = V \cap S_i$ is the section of V by a plane $S_i \subset \mathbb{P}$ of codimension 3, and moreover $\mathrm{mult}_x B_i = 3$,
- or is non-reduced and is of the form $2B$, where $B = V \cap S$ is the section of V by a plane S of codimension 3, and moreover $\mathrm{mult}_x B = 3$.

Proposition 2. *For $M \geq 6$ a general (in the sense of Zariski topology) hypersurface V satisfies the conditions (G1), (G2) and (G3) at every point $x \in V$.*

Proof. For the condition ($G1$) see [P3]. A general hypersurface satisfies the condition ($G1$) at each point; in particular, through each point $x \in V$ there are at most finitely many lines lying on V.

The conditions ($G2$) and ($G3$) are checked by straightforward computations. We omit these easy considerations.

Note that for $M \geq 8$ a general hypersurface V satisfies the condition ($G3$) with the first option only.

3.3 Fano hypersurfaces are divisorially canonical

Theorem 6. *Assume that the Fano hypersurface $V = V_M \subset \mathbb{P} = \mathbb{P}^M$ satisfies the conditions (G1),(G2) and (G3) at every point $x \in V$. Then the primitive Fano variety V is divisorially canonical.*

Proof. Let $\Delta \in |nH|$ be a prime divisor, $n \geq 1$, where $H \in \operatorname{Pic} V$ is the class of a hyperplane section, $K_V = -H$. We must prove that the pair $(V, \frac{1}{n}\Delta)$ has canonical singularities.

Assume the converse. Taking into account that for any curve $C \subset V$ the following inequality holds,

$$\operatorname{mult}_C \Delta \leq n,$$

we obtain, by Proposition 1, that there is a point $x \in V$ and a hyperplane $B \subset E$, where $E \subset \widetilde{V}$ is the exceptional divisor of the blow up

$$\varphi\colon \widetilde{V} \to V$$

of the point x, $E \cong \mathbb{P}^{M-2}$, such that the inequality

$$\operatorname{mult}_x \Delta + \operatorname{mult}_B \widetilde{\Delta} > 2n, \tag{9}$$

holds, where $\widetilde{\Delta} \subset \widetilde{V}$ is the strict transform of the divisor Δ.

Let $\mathbb{T} = \overline{T_x V} \subset \mathbb{P}$ be the tangent hyperplane at the point x. The divisor E can be naturally identified with the projectivization

$$\mathbb{P}(T_x\mathbb{T}) = \mathbb{P}(T_x V).$$

There is a unique hyperplane $\mathbb{B} \subset \mathbb{T}$, $x \in \mathbb{B}$, such that

$$B = \mathbb{P}(T_x\mathbb{B})$$

with respect to the above-mentioned identification. Let $\Lambda_{\mathbb{B}}$ be the pencil of hyperplanes in $\mathbb{P}$, containing $\mathbb{B}$, and $\Lambda_B = \Lambda_{\mathbb{B}}|_V \subset |H|$ its restriction onto V. Consider a general divisor $R \in \Lambda_B$. It is a hypersurface of degree M in $\mathbb{P}^{M-1}$, smooth at the point x. Let $\widetilde{R} \subset \widetilde{V}$ be the strict transform of the divisor R. Obviously,

$$\widetilde{R} \cap E = B.$$

Set $\Delta_R = \Delta|_R = \Delta \cap R$. It is an effective divisor on the hypersurface R.

Lemma 3. *The following estimate holds:*

$$\operatorname{mult}_x \Delta_R > 2n. \tag{10}$$

Proof. We have

$$(\widetilde{\Delta} \circ \widetilde{R}) = \widetilde{\Delta}_R + Z,$$

where Z is an effective divisor on E. According to the elementary rules of the intersection theory [F],

$$\operatorname{mult}_x \Delta_R = \operatorname{mult}_x \Delta + \deg Z,$$

since $\mathrm{mult}_x R = 1$. However, Z contains B with multiplicity at least $\mathrm{mult}_B \widetilde{\Delta}$. Therefore, the inequality (9) implies the estimate (10). Q.E.D. for the lemma.

Lemma 4. *The divisor*

$$T_R = T_x R \cap R$$

on the hypersurface R is irreducible and has multiplicity exactly 2 at the point x.

Proof. The irreducibility is obvious (for instance, for $M \geq 6$ one can apply the Lefschetz theorem). By the condition ($G2$) the quadric

$$\{q_2|_E = 0\}$$

does not contain a hyperplane in E as a component, in particular, it does not contain the hyperplane $B \subset E$. Thus the quadratic component of the equation of the divisor T_R, that is, the polynomial

$$q_2|_B,$$

is non-zero. Q.E.D. for the lemma.

Let us continue our proof of Theorem 6. By Lemmas 3 and 4 we can write

$$\Delta_R = aT_R + \Delta_R^\sharp,$$

where $a \in \mathbb{Z}_+$ and the effective divisor $\Delta_R^\sharp \in |n^\sharp H_R|$ on the hypersurface R satisfies the estimate

$$\mathrm{mult}_x \Delta_R^\sharp > 2n^\sharp.$$

Moreover, $\Delta_R^\sharp$ does not contain the divisor T_R as a component. Without loss of generality we can assume the divisor $\Delta_R^\sharp$ to be irreducible and reduced.

Now consider the second hypertangent system

$$\Lambda_2^R = |s_0 f_2 + s_1 f_1|_R,$$

where s_i are homogeneous polynomials of degree i in the linear coordinates z_*. Its base set

$$S_R = \{q_1|_R = q_2|_R = 0\}$$

is by condition ($G3$) of codimension 2 in R and either irreducible and of multiplicity 6 at the point x, or breaks into two plane sections of R, each of multiplicity 3 at the point x. In any case, for a general divisor $D \in \Lambda_2^R$ we get $\Delta_R^\sharp \not\subset \mathrm{Supp}\, D$, so that the following effective cycle of codimension two on R,

$$\Delta_D = (D \circ \Delta_R^\sharp),$$

is well defined. Since $\mathrm{mult}_x D = 3$ and $\Lambda_2^R \subset |2H_R|$, the cycle Δ_D satisfies the estimate

$$\frac{\mathrm{mult}_x}{\deg} \Delta_D > \frac{3}{M}. \tag{11}$$

We can replace the cycle Δ_D by its suitable irreducible component and thus assume it to be an irreducible subvariety of codimension 2 in R. Comparing the estimate (11) with the description of the set S_R given above, we see that

$$\Delta_D \not\subset S_R.$$

This implies that $\Delta_D \not\subset T_R$. Indeed, if this were not true, we would have got

$$f_1|_{\Delta_D} \equiv q_1|_{\Delta_D} \equiv 0. \tag{12}$$

However, $\Delta_D \subset D$, so that for some $s_0 \neq 0$, $s_1 \neq 0$ (the divisor D is chosen to be general) we have

$$(s_0 f_2 + s_1 f_1)|_{\Delta_D} \equiv 0.$$

By (12) this implies that

$$f_2|_{\Delta_D} \equiv (q_1 + q_2)|_{\Delta_D} \equiv 0$$

(since $s_0 \neq 0$ is just a constant), so that

$$\Delta_D \subset S_R.$$

A contradiction.

Thus $\Delta_D \not\subset T_R$. Therefore the effective cycle

$$\Delta^+ = (\Delta_D \circ T_R)$$

is well defined. It satisfies the estimate

$$\frac{\mathrm{mult}_x}{\deg} \Delta^+ > \frac{6}{M}. \tag{13}$$

The effective cycle Δ^+ as a cycle on V is of codimension 4. In [P3] it was shown that the inequality (13) for an effective cycle of codimension 4 is impossible provided that the regularity condition $(G1)$ holds.

Q.E.D. for the theorem.

3.4 Fano double spaces

Theorem 6 together with Proposition 2 give examples of divisorially canonical Fano varieties of any dimension starting with 5. Let us give an example embracing all dimensions starting with 3. Let $\sigma: V \to \mathbb{P} = \mathbb{P}^M$ be a double space of index 1, that is, σ is branched over a smooth hypersurface $W = W_{2M} \subset \mathbb{P}$ of degree $2M$, $M \geq 3$, see [P2]. The variety V is obviously a primitive Fano variety of index 1.

Theorem 7. *A general variety V (corresponding to a general hypersurface $W \subset \mathbb{P}$) is divisorially canonical.*

Proof. Let $D \in |nH|$ be an irreducible divisor, where $H = -K_V = \sigma^* H_{\mathbb{P}}$, $H_{\mathbb{P}}$ is the class of a hyperplane in $\mathbb{P}$. In [P4] it was proved that

$$\mathrm{mult}_C D \leq n$$

for any curve $C \subset V$. Assume now that the singularities of the pair $(V, \frac{1}{n}D)$ are not canonical. Then there is a point $x \in V$ and a hyperplane $B \subset E$ in the exceptional divisor $E \subset \widetilde{V}$ of the blow up $\varphi: \widetilde{V} \to V$ of the point x, such that

$$\mathrm{mult}_x D + \mathrm{mult}_B \widetilde{D} > 2n,$$

where $\widetilde{D} \subset \widetilde{V}$ is the strict transform of D.

Case 1. The point $z = \sigma(x) \notin W$. In this case let $P \subset \mathbb{P}$ be the hyperplane such that $P \ni z$ and

$$\widetilde{V}_P \cap E = B,$$

where $V_P = V \cap P$. Such hyperplane is unique. The divisor V_P is obviously irreducible and satisfies the equality

$$\operatorname{mult}_x V_P + \operatorname{mult}_B \widetilde{V}_P = 2.$$

Since $V_P \in |H|$, this implies that $V_P \neq D$. Therefore,

$$Z = (V_P \circ D)$$

is a well defined effective cycle of codimension two on V. We get

$$\deg Z = 2n, \quad \operatorname{mult}_x Z \geq \operatorname{mult}_x D + \operatorname{mult}_B \widetilde{D} > 2n,$$

which is impossible. Thus the case $z \notin W$ does not realize.

Case 2. The point $z = \sigma(x) \in W$. In this case let

$$P = T_z W \subset \mathbb{P}$$

be the tangent hyperplane. Set again $V_P = V \cap P$. We get $V_P \in |H|$, $\operatorname{mult}_x V_P \geq 2$. The divisor V_P is irreducible, so that if $V_P \neq D$, then for the cycle $Z = (V_P \circ D)$ we get again

$$\deg Z = 2n, \quad \operatorname{mult}_x Z \geq 2 \operatorname{mult}_x D > 2n, \tag{14}$$

since $\operatorname{mult}_x D \geq \operatorname{mult}_B \widetilde{D}$. Since there is no cycle Z, satisfying the condition(14), we conclude that

$$V_P = D.$$

Up to this moment we have never used the arguments of general position.

Lemma 5. *For a general $W \subset \mathbb{P}$ the pair*

$$(V, V_P)$$

is canonical at every point $z \in W$.

Proof is straightforward and we omit it.

Now the proof of Theorem 7 is complete: we have come to a contradiction.

Remark 5. It is easy to see that the claim of Theorem 7 remains true if one admits simple singularities, for instance, non-degenerate factorial double points in dimension three.

The methods developed above make it possible to obtain the following result.

Theorem 8. *A general Fano double hypersurface of index 1 and a general Fano complete intersection*

$$V_{d_1 \cdot \ldots \cdot d_k} \subset \mathbb{P}^{M+k}$$

of index 1, where $M \geq 3k+3$, $k \geq 2$, is divisorially canonical.

Conjecture. *A general Fano complete intersection $V \subset \mathbb{P}$ in a weighted projective space is divisorially canonical in any dimension ≥ 3.*

3.5 Further results

The technique developed above makes it possible to prove the following fact.

Theorem 9. *Let $F_1, \ldots, F_K$ be general Fano hypersurfaces of index 1 and dimension ≥ 5. Let*

$$\chi: V = F_1 \times \ldots \times F_K - - \to V^\sharp$$

be a birational map, where $\pi^\sharp: V^\sharp \to S^\sharp$ is a K-trivial fibration. Then there exist:

(i) *a partition $\{1, \ldots, K\} = I \vee I^+$ into two disjoint subsets;*

(ii) *for each $i \in I$ a pencil of hyperplane sections Λ_i on F_i, defining a rational map*

$$\lambda_i: F_i - - \to \mathbb{P}^1,$$

the fiber of which is K-trivial;

(iii) *a birational map*

$$\alpha: \prod_{i \in I} \mathbb{P}^1 \times \prod_{i \in I^+} F_i - - \to S^\sharp$$

such that

$$\pi^\sharp \circ \chi = \alpha \circ \left(\prod_{i \in I} \lambda_i \times \prod_{i \in I^+} \mathrm{id}_{F_i} \right),$$

that is, all structures of a K-trivial fibration on V are given by rational maps

$$\prod_{i \in I} \lambda_i \times \prod_{i \in I^+} \mathrm{id}_{F_i}: V - - \to \prod_{i \in I} \mathbb{P}^1 \times \prod_{i \in I^+} F_i.$$

A modification of the proof of Theorem combined with the arguments that prove Sarkisov's theorem [S] make it possible to prove the following fact.

Theorem 10. *Let F be a divisorially canonical primitive Fano variety of dimension ≥ 3. Then all structures of a rationally connected fibration on the variety*

$$V = F \times \mathbb{P}^1$$

are given by the projections

$$\pi_l: V \to F \quad \textit{and} \quad \pi_r: V \to \mathbb{P}^1.$$

A more sophisticated argument makes it possible to prove the following fact.

Theorem 11. *Let F be a divisorially canonical primitive Fano variety of dimension ≥ 3. Then all structures of a rationally connected fibration on the variety*

$$V = F \times \mathbb{P}^2$$

are either given by the projections

$$\pi_l: V \to F \quad \textit{and} \quad \pi_r: V \to \mathbb{P}^2,$$

or are of the form

$$\begin{array}{cccc} \lambda\colon & F\times\mathbb{P}^2 & --\to & F\times\mathbb{P}^1 \\ & \downarrow & & \downarrow \\ & F & \longleftrightarrow & F, \end{array}$$

where for a general $x\in F$ the fibers of the corresponding map

$$\lambda_x\colon\mathbb{P}^2 - - \to\mathbb{P}^1$$

are rational curves, or, finally, are obtained by taking the composition of a map λ just described with the projection of $F\times\mathbb{P}^1$ onto $\mathbb{P}^1$.

In particular, the variety V is non-rational. Theorems 10 and 11 can be considered as the first step in solving the problem of stable rationality for Fano varieties.

As we mentioned above, for a smooth point $x\in F=F_M\subset\mathbb{P}$ on a Fano hypersurface of index 1 failure to satisfy the regularity conditions ($G1$)-($G3$) gives a high codimension $\geq M+1$ for the hypersurface F as a point in the space

$$\mathbb{P}(H^0(\mathbb{P},\mathcal{O}_{\mathbb{P}}(M))).$$

Therefore for a general Fano fiber space $\pi\colon V\to\mathbb{P}^1$ with the fiber $F=F_M\subset\mathbb{P}$, $M\geq 6$, each fiber F satisfies these conditions at every smooth point $x\in F$. Therefore the methods developed above give the following fact.

Theorem 12. *Let $\pi\colon V\to\mathbb{P}^1$ be a general Fano fiber space with the fiber $F=F_M\subset\mathbb{P}=\mathbb{P}^M$, $M\geq 6$, and moreover*

$$\operatorname{Pic}=\mathbb{Z}K_V\oplus\pi^*\operatorname{Pic}\mathbb{P}^1=\mathbb{Z}K_V\oplus\mathbb{Z}F.$$

Let $D\in|-nK_V+lF|$ be an irreducible divisor, where $l\in\mathbb{Z}_+$ and $n\geq 1$. Then the pair

$$(V,\frac{1}{n}D)$$

has canonical singularities. In particular, if

$$\chi\colon V - - \to V^\sharp$$

is a birational map, where $\pi^\sharp\colon V^\sharp\to S^\sharp$ is a rationally connected fibration, $\Sigma^\sharp$ is a movable linear system on $V^\sharp$, pulled back from the base $S^\sharp$, and Σ its strict transform on V, then either $\Sigma=|F|$ is a pencil of fibers or $\Sigma\subset|-nK_V+lF|$ with $l\leq -1$.

Similar results are true for fiber spaces $V/\mathbb{P}^1$, the fiber of which is a Fano double hypersurface and Fano complete intersection from the class described above.

The detailed proofs will be given in another paper.

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