A COMPACTNESS THEOREM FOR SURFACES WITH L_p-BOUNDED SECOND FUNDAMENTAL FORM

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1. Introduction

A number of well-known results in differential geometry (see, e.g., [2], [4], [8]) admit the following general description: a set Ω whose elements are Riemannian manifolds satisfying certain bounds on geometric quantities such as curvature, volume, injectivity radius, etc., is compact with respect to some metric topology on Ω.

For analogous results in submanifold theory, one might assume bounds on the second fundamental form and seek a correspondingly extrinsic type of convergence, e.g., in a more standard function space sense. Our main result can be viewed as a compactness theorem in this spirit for the classical situation of surfaces in R^3.

In the following theorem we denote by dV the induced area element, II denotes the second fundamental form, and k_1, k_2 the principal curvatures:

Compactness Theorem Given constants A, E, and $p>2$, let Ω be the set of immersed surfaces $\psi: M \to R^3$ satifying $\text{Area}(\psi) = \int_M dV < A$, $E_p(\psi) = \int_M |II|^p\, dV = \int_M (k_1^2+k_2^2)^{p/2}\, dV < E$, and $\int_M \psi\, dV = 0$ (ψ has "center of gravity " at the origin). Then for any sequence $\{\psi^n\}$ in Ω, there exists a sequence of surface diffeomorphisms $\{\phi^n\}$ such that a subsequence of $\{\psi^n \circ \phi^n\}$ converges in the C^1 topology to an immersion ψ in Ω.

Here M is assumed only to be complete (and without boundary), but as shown in [5], the above bounds imply M is in fact compact, so we might as well assume compactness of M from now on. Also, though the topological type of M may be allowed to vary over Ω, the finiteness of topological types represented by Ω happens not to be at issue here, as the Gauss-Bonnet theorem obviously implies a bound on the Euler characteristic of such M ; thus, we might as well fix M for the purpose of further discussion. Finally we specify the smoothness of the above maps: Ω is a subset of the Sobolev space $L_{2,p}(M,R^3)$ of maps of M into R^3 whose derivatives up to order 2 are p-integrable, and the diffeomorphisms ϕ_i have the same smoothness.

In the course of proving the above theorem, convergence is actually obtained in the weak - $L_{2,p}$ sense. Moreover, the functional E_p is shown below to be weakly lower-semicontinuous on $Imm_{2,p}(M,R^3) = \{ \psi \text{ in } L_{2,p}(M,R^3) : \psi \text{ is an immersion} \}$. Thus we arrive at the following

Existence Theorem For $p > 2$, the functional E_p achieves its infimum within each component of $Imm_{2,p} \cap Area^{-1}(1)$, i.e., in each regular homotopy class of unit area $L_{2,p}$-immersions of M into R^3.

As the case $p = 2$ is "borderline" for the Sobolev inequality ($L_{2,p}(M,R^3)$ is included in $C^1(M,R^3)$ only for $p > 2$), the above compactness theorem is apparently unimprovable in the sense that a bound on E_2 does not prevent immersed surfaces from degenerating (an interesting example of this behavior is described at the end of Section 2).

On the other hand, this leaves open the possibility that the solutions given by the above existence theorem persist in the limit as p approaches 2. This point is of particular interest since E_2 is variationally equivalent to the total squared mean curvature functional, $F(\psi) = \int_M H^2\, dV$; to be precise, if $\chi(M)$ is the Euler characteristic of M, the Gauss-Bonnet theorem gives $E_2(\psi) = 4F(\psi) - 4\pi\chi(M)$. While some beautiful results about F have been proved, it is not even known, e.g., if there exist any minima for F (among immersed surfaces in R^3) other than the standard sphere.

It appears extremely likely that the "Clifford torus", $T_{\sqrt{2}}$, obtained by revolving the circle $C = \{(x,o,z) \text{ in } R^3 : (x - \sqrt{2})^2 + z^2 = 1\}$ about the z-axis, yields a minimum for F among immersed tori. However, the Willmore inequality $\int_{T^2} H^2\, dV \geq 2\pi^2 = F(T_{\sqrt{2}})$ remains a conjecture except in certain cases the (result has been established recently for canal surfaces by U. Pinkall, and for a large class of conformal types of tori by Li and Yau [6]).One strategy for proving the general conjecture is to prove the above existence theorem for the case $M = T^2$, $p = 2$, and show that all critical points of F (among tori) satisfy the inequality.

All known critical points of F arise from two sources. First, a result of Weiner [9] states that minimal submanifolds of S^3 are taken to critical points of F under stereographic projection of S^3 onto R^3. This result is a consequence of the fact that a generalization of F is invariant under conformal changes of metric in the ambient space. Second, a very recent paper of Bryant [1] classifies all critical points of F for $M = S^2$, after proving the beautiful result that to such a critical point one can associate a holomorphic quartic differential on M.

2. Non-Parametric Estimates

A basic difficulty to be confronted is that the functional E_p does not depend on parametrization and therefore cannot control derivatives of ψ. Our strategy for overcoming this problem begins with the fact that any immersed surface $\psi: M \longrightarrow R^3$ can be considered locally as the graph of a function $h : D_r \longrightarrow R$, where $D_r \subset R^2$ is the open disc of sufficiently small radius r. In this section we obtain estimates which refer to the functions h which arise in this way, rather than to the global parametrization ψ.

First we fix some notation. We will denote the norm on $C^0(D_r)$ by $\| \ \|_0$, the norm on $L_p(D_r)$ by $\| \ \|_p$, and the norm on $L_{k,p}(D_r)$ by $\| \ \|_{k,p}$. We will use the multi-index notation; for instance, if $\gamma = (i,j)$ for positive integers i,j, then $[\gamma] = i+j$, and

$$D^\gamma = \frac{\partial^{[\gamma]}}{\partial x_1^i \, \partial x_2^j}, \quad \text{thus} \quad \| h\|^p_{k,p} = \sum_{[\gamma]\le k} \| D^\gamma h \|_p^p .$$

The following lemma is essentially a statement of the fact that in the "non-parametric", i.e. graph case, the functional E_p is non-uniformly elliptic.

<u>Lemma 2.1</u> Let $\psi \in L_{2,p}(D_r,R^3)$ be of the special form $\psi(x,y) = (x,y,h(x,y))$, where $h:D_r \to \mathbb{R}$ is a smooth function. Then for $p \ge 2$,

$$\Big(\sum_{[\gamma]=2} \| D^\gamma h \|_p\Big)^p \le 4^p (1+\|\nabla h\|_0)^{3p-1} \, E_p(\psi)$$

<u>Proof</u>: We set $s = h_x$, $t = h_y$, $U = h_{xx}$, $V = h_{yy}$, $W = h_{xy}$, and $A = 1 + s^2 + t^2$. Then the graph of h has mean and Gaussian curvatures $H = \frac{1}{2} A^{-\frac{3}{2}}[(1+s^2)V + (1+t^2)U - 2stW]$, $K = A^{-2}(UV - W^2) = A^{-3}(1 + s^2 + t^2)(UV - W^2)$. So we get the following inequality for the integrand of $E_p(\psi)$: $(4H^2-2K)^{\frac{p}{2}}A^{\frac{1}{2}} =$

$$A^{\frac{1-3p}{2}}\Big[(1+s^2)^2V^2 + (1+t^2)^2U^2 + 2(1+s^2+t^2+2s^2t^2)W^2 + 2s^2t^2UV - 4(1+s^2)stVW - 4(1+t^2)stUW\Big]^{p/2}$$

$$= A^{\frac{1-3p}{2}}\Big[(s^2V+t^2U-2stW)^2 + 2(sV-tW)^2 + 2(tU-sW)^2 + (U^2+V^2+2W^2)\Big]^{\frac{p}{2}}$$

$$\geq A^{\frac{1-3p}{2}}\Big[U^2 + V^2 + 2W^2\Big]^{\frac{p}{2}} \geq A^{\frac{1-3p}{2}}\Big[|U|^p + |V|^p + 2|W|^p\Big].$$

Since $A = 1 + |\nabla h|^2 \leq (1 + |\nabla h|)^2$, we have $A^{\frac{1-3p}{2}}(x,y) \geq (1 + \|\nabla h\|_0)^{1-3p}$, $\forall (x,y) \in M$. Therefore, integrating over M gives $E_p(\psi) \geq (1+\|\nabla h\|_0)^{1-3p}\int_M |U|^p+|V|^p+2|W|^p dydx$ $= (1+\|\nabla h\|_0)^{1-3p} \sum_{[\gamma]=2} \|D^\gamma h\|_p^p \geq 4^{-p}(1+\|\nabla h\|_0)^{1-3p}(\sum_{[\gamma]=2} \|D^\gamma h\|_p)^p$. □

In order to deal with the non-uniformity, i.e., to control ∇h, we need the following Sobolev-type inequality due to Morrey (a proof is given in [3], p. 23):

<u>Lemma 2.2</u> Let $p>2$, and $r>0$. Then for all h in $L_{1,p}(D_r)$ and for all z in D_r,

$$|h(z)-h(0)| \leq 16\, r^{1-\frac{2}{p}} \|\nabla h\|_p .$$

Applying Lemma 2.2 to $\frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}$ and summing, we obtain:

<u>Lemma 2.3</u> Suppose $p>2$, $r>0$, and h is in $L_{2,p}(D_r)$ and satisfies $\nabla h(0)=0$. Then $\|\nabla h\|_0 \le 64\, r^{1-\frac{2}{p}} \sum_{[\gamma]=2} \|D^\gamma h\|_p$.

To summarize the above in words: a uniform bound on ∇h implie an L_p-bound on second derivatives of h (given the bound on E_p), while an L_p-bound on second derivatives gives rise to a uniform bound on ∇h. Of course the situation is not so circular as this statement makes it sound, for the radius of the domain D_r enters into the latter estimate.

In order to make use of this information we first introduce some notation. Given $q \in M$, let $A: R^3 \to R^3$ be the Euclidean isometry which takes the origin to $\psi(q)$ and whose differential takes $e_3=(0,0,1)$ to the inward unit normal to ψ at q (to simplify notation we are assuming M to be orientable). Let π be the standard projection of R^3 onto the x,y-plane,

and define $U_{r,q}$ to be the q-component of $(\pi \circ A^{-1} \circ \psi)^{-1}(D_r)$. For $r, \alpha > 0$, let us call an immersion $\psi : M \longrightarrow R^3$ an <u>(r,α)-immersion</u> if, for each point $q \in M$, $A^{-1} \circ \psi(U_{r,q})$ is the graph of a C^1 function $h: D_r \longrightarrow R$ satisfying $\|\nabla h\|_0 \le \alpha$.

<u>Theorem 2.4</u> Let $p > 2$, let $0 < \alpha \le 1$, and suppose $\psi \in \mathrm{Imm}_{2,p}(M, R^3)$. Then ψ is an (r,α)-immersion for all r satisfyi

$$2^{-11p}\, \frac{\alpha^p}{E_p(\psi)} .$$

<u>Proof</u> Pick $q \in M$, and suppose $s > o$ is small enough so that $d(\pi_q \circ \psi)$ is non-singular on $U_{s,q}$. Then since M is compact the local homeomorphism $\pi_q \circ \psi : U_{s,q} \longrightarrow D_{s,q}$ is actually a covering projection. But $D_{s,q}$ is simply connected, so in fact $\pi_q \circ \psi : U_{s,q} \longrightarrow D_{s,q}$ is a homeomorphism. It follows that $\psi(U_{s,q})$ is the graph of a function over $D_{s,q}$.

Now let S be the largest number such that $\psi(U_{S,q})$ is the graph of a function $h : D_{S,q} \longrightarrow R$. By the previous paragraph, it follows that $\|\nabla h\|_0 = \infty$. Therefore, since $\nabla h(o) = o$, there must exist r_q, $0 < r_q < S$ such that $\|\nabla h\|_0 = \alpha$.

It remains to estimate r_q. Using Lemmas 2.1 and 2.3, we get

$$\alpha = \|\nabla h_q\|_0 \le 2^6 r_q^{1-\frac{2}{p}} \sum_{[\gamma]=2} \|D^\gamma h_q\|_p \le 2^6 r_q^{1-\frac{2}{p}} 4(1+\alpha)^{\frac{3p-1}{p}} (E_p(\psi))^{\frac{1}{p}} .$$

Solving for r_q^{p-2} and using the fact that $1+\alpha \le 2$ one obtains $r_q > 2^{-11p} \dfrac{\alpha^p}{E_p(\psi)}$. □

Thus, given a bound on E_p, ψ cannot degenerate to a non-immersion. It should be noted, however, that the (r,α)-bound does <u>not</u> imply a lower bound on the injectivity radius. In fact, we can easily describe a function $h : R^2 \longrightarrow R$ (and satisfying $\|\nabla h\|_0 < \alpha$) but having a graph with arbitrarity small injectivity radius. To do so, we begin by choosing a positive integer n, setting $\varepsilon = \frac{\alpha^2}{8n}$, and defining a continuous function $\overline{h}$ by

$$\overline{h}(x,y) = \begin{cases} \alpha(\varepsilon - |(x,y)-(2m\varepsilon,0)|), & \text{if } |(x,y)-(2m\varepsilon,0)| < \varepsilon, \text{ m an integer} \\ 0 & \text{if } (x,y) \text{ not in one of the above discs} . \end{cases}$$

We then smooth $\bar{h}$ to get h such that $\|\nabla h\|_0 \leq \|\nabla \bar{h}\|_0 = \alpha$, h is C^1 close to $\bar{h}$, and h preserves the symmetry of $\bar{h}$ with respect to the x-axis. Then the origin is joined to the point (x,o) by a geodesic of length approximately $\frac{\sqrt{1+\alpha^2}}{n}$ (the geodesic lying directly over the x-axis), while a path making a "detour around the cones" can join the same two points with length approximately $\frac{1}{n} + 2\varepsilon < \frac{\sqrt{1+\alpha^2}}{n}$.

Finally, we show by an explicit example how Theorem 2.4 (hence also the compactness theorem itself) can fail in the case p=2. Our example depends on the fact that E_2 is invariant under conformal transformations of $R^3 - \left\{ \text{point not in } \psi(M) \right\}$ (essentially a special case of the conformal invariance property mentioned in the introduction).

Let $X_n = -(\sqrt{2} + \frac{1}{n}, o, o)$ and let ψ_n be obtained by inverting $T_{\sqrt{2}}$ (see introduction) about X_n and normalizing area by the appropriate dilation. Then $\{\psi_n\}$ is a sequence of critical points of E_2 (in fact minima, if the Willmore conjecture is true) satisfying $E_2(\psi_n) = 2\pi^2$, Area $(\psi_n) = 1$, for all n. Yet the geodesic C of the original torus $T_{\sqrt{2}}$ is taken to a geodesic whose length is approaching zero; thus, no reparametrization of ψ_n can approach an immersion. Curiously, though, the <u>image</u> of ψ_n is approaching the standard sphere of unit area.

3. Convergence

The previous section suggests that it may be convenient, for the present purpose, to consider an immersed surface as a system of graphs and to work with a notion of convergence adapted to this point of view. We begin this section by making this idea precise and showing that, given a sequence of immersions in the set Ω defined in the introduction, one can extract a subsequence $\{\psi^n\}$ which converges in such a manner.

This enables us to reparametrize the immersions by a kind of "averaged normal projection" of ψ^n onto ψ^{n+k} (for some fixed large integer n) in such a way that the reparametrized sequence converges weakly in $L_{2,p}(M,R^3)$ to an immersion. The reparametrization step involves a bit more work, and we relegate to the appendix some detailed verifications of "intuitively obvious" properties of the projection construction.

In the following lemma we record some useful facts which refer to the notation introduced before Theorem 2.4:

<u>Lemma 3.1</u> Let $\psi : M \longrightarrow R^3$ be an (r,α)-surface and let $p,q \in M$.

a) If $p \in U_{r,q}$ then $|\psi(q) - \psi(p)| < (1+\alpha^2)r$

b) If $\alpha^2 < \frac{1}{3}$ and $U_{\frac{r}{4},p} \cap U_{\frac{r}{4},p} \neq \emptyset$ then $U_{\frac{r}{4},p} \subset U_{r,q}$.

<u>Proof:</u> Part a) is obvious. To prove b) , let

$$x \in U_{\frac{r}{4},p} \text{ , } y \in U_{\frac{r}{4},q} \cap U_{\frac{r}{4},p} \text{ . Then}$$

$|\varphi_q(x)| \leq |\psi(x)-\psi(q)| \leq |\psi(x)-\psi(p)| + |\psi(p)-\psi(y)| + |\psi(y)-\psi(q)|$ $\leq 3(1+\alpha^2)\frac{r}{4} < r$. So $U_{\frac{r}{4},p} \cap U_{\frac{r}{4},q} \subseteq \varphi_q^{-1}(D_r)$. But $U_{\frac{r}{4},p} \cup U_{\frac{r}{4},q}$

is a connected set containing q, so in fact, it must be contained in the q-component of $\varphi_q^{-1}(D_r)$, i.e., $U_{r,q}$. So

$U_{\frac{r}{4},p} \subseteq U_{r,q}$. □

Let $Q = \{q_1,\dots,q_m\}$ be a finite set of points in M, and let $0<\delta<r$. We call Q a <u>δ-net for ψ</u> if $M = \bigcup_{i=1}^{m} U_{\delta,q_i}$.

<u>Lemma 3.2</u> If ψ is an (r,α)-surface, $\alpha^2<\frac{1}{3}$, and $0<\delta<r$, then there exists a δ-net for ψ which contains

fewer than $\frac{6}{\delta^2}\cdot \text{Area}(\psi)$ points.

<u>Proof</u>: Pick q_1 arbitrarily, and pick $q_2 \notin U_{\delta,q_1}$. By Lemma 3.1,

$$U_{\frac{\delta}{4},q_1} \cap U_{\frac{\delta}{4},q_2} = \phi.$$ If $\{U_{\delta,q_1}, U_{\delta,q_2}\}$ is not a cover for M, pick $q_3 \notin U_{\delta,q_1} \cup U_{\delta,q_2}$. Lemma3.1 implies

$U_{\frac{\delta}{4},q_3} \cap U_{\frac{\delta}{4},q_i} = \phi$, $i = 1,2$. We continue in this way. Since

$\text{Area}(\psi) \geq \sum_{i=1}^{m} \text{Area}(U_{\frac{\delta}{4},q_i}) \geq \sum_{i=1}^{m} \pi(\frac{\delta}{4})^2 \geq m\,\frac{\delta^2}{6}$, we see that this procedure must yield a cover after at most $\frac{6}{\delta^2}$ Area(ψ) steps. □

Now suppose $\psi: M \longrightarrow R^3$ is an (r,α)-immersion and $Q = \{q_1,\dots,q_m\}$ is a δ-net for ψ. To each $q_i \in Q$ we can associate a Euclidean isometry A_i, a neighborhood $U_i = U_{r,q_i}$ of q_i and a C^1-function $h_i : D_r \longrightarrow R$, as described earlier. We can also assign to each q_i a set $Z_i \subset \{1,2,\dots,m\}$ as follows: $j \in Z_i$ if and only if U_{δ,q_i} has non-empty intersection with U_{δ,q_j}.

Thus we associate to ψ, Q, r, δ, a <u>graph system</u>, $\Gamma = \left\{ A_i, h_i, Z_i \right\}_{i=1}^{m}$ (of course A_i, h_i are not quite uniquely determined, but in any case $0 = h(0) = \nabla h(0)$).

Next we consider a sequence of immersions $\{\psi^n\}$ in Ω and show how to extract a subsequence which converges in the sense of graph systems. We begin by choosing $0<\alpha^2<\frac{1}{3}$, and $0<r<1$ such that Theorem 2.4 implies all the ψ^n are (r,α)-immersions. Next we take $\delta<r$ and use Lemma 3.2 to choose δ-nets Q^n containing at most $\frac{6}{\delta^2}A$ points. Since we can always pass to a subsequence if necessary, we might as well assume the nets Q^n all have m elements and that $Z_i^n = Z_i$ for some fixed sets Z_i.

By Lemma 3.1 a), we know that the diameter of ψ^n is bounded by $2m(1+\alpha)r$. But the ψ^n all have center of gravity zero, so in fact, they are all contained in some fixed ball. It follows that we can assume all the Euclidean isometries A_i^n converge to some fixed isometries A_i in the sense that $\| A_i^n - A_i \| = \sup_{|v|=1} |A_i^n(v) - A_i(v)|$ tends to zero.

Finally, using Lemma 2.1, the (r,α)-bound, the bound on E_p, and the fact that $0=h_i^n(0)$, we obtain a bound on $\| h_i^n \|_{2,p}$. Since $L_{2,p}(D_r)$ is compactly embedded in $C^1(D_r)$ (for $p>2$), we can therefore assume that the h_i^n converge in C^1 to functions h_i.

Defining the distance between two graph systems $\Gamma = \left\{ A_i, h_i, Z_i \right\}_{i=1}^{m}$ and $\bar{\Gamma} = \left\{ \bar{A}_i, \bar{h}_i, Z_i \right\}_{i=1}^{m}$ by $\| \Gamma - \bar{\Gamma} \| = \sum_{i=1}^{m} \| A_i - \bar{A}_i \| + \| h_i - \bar{h}_i \|_{C^1}$, we can summarize the above by

<u>Theorem 3.3</u> Any sequence of immersions in Ω has a subsequence $\{\psi^n\}$ which admits a Cauchy sequence of graph systems.

In order to obtain a stronger type of convergence we first take a bit more care in our choice of constants: let $\alpha=\frac{1}{100}$, then choose $r\le 1$ according to Theorem 2.4, and set $\delta=\frac{r}{10}$. Now let $\psi^o,\psi^1,\psi^2,\dots$ be the sequence of (r,α)- immersions provided by the above theorem, with corresponding Cauchy sequence of graph systems $\Gamma^o,\Gamma^1,\Gamma^2,\dots,$ and δ-nets $Q^n = \left\{ q_i^n=A_i^n(o) \right\}_{i=1}^m$, $n = o,1,2,\dots$ Since we can always pass to a subsequence if necessary, we might as well assume that $\|\Gamma^n-\Gamma^o\| < \varepsilon = \alpha r$ for all n. Moreover, we might as well assume, for convenience, that ψ^o itself is C^∞ since we could make a C^1-perturbation to smooth ψ^o (it isn't quite obvious that such a perturbation need not do slight damage to the (r,α)-bound or the estimate on $\|\Gamma^n-\Gamma^o\|$, but we can always start with slightly smaller α,ε).

The object now is to "project" ψ^o onto the subsequent immersions $\psi^1,\psi^2,\psi^3,\dots,$ thus inducing diffeomorphisms $\phi^n : M \longrightarrow M$ such that the reparametrizations $\psi^n \circ \phi^n$ converge to an immersion, as claimed in the compactness theorem. In discussing the projection construction we will simplify notation by dropping the superscript o and replacing the superscript n by a bar above the letter; for example, we will write

ψ, h_i, U_{r,q_i}, $\bar\psi$, $\bar h_i$, $\bar U_{r,\bar q_i}$ in place of ψ^o, h_i^o, U^o_{r,q_i^o}, ψ^n, h_i^n, U^n_{r,q_i^n}, respectively.

The first step is to define an approximate unit normal along ψ, $X:M \longrightarrow S^2$, which satisfies much better bounds than the actual (inward) unit normal along ψ, $N:M \longrightarrow S^2$ (as before, we are assuming M to be orientable to keep the notation simple).

Set $N_i = N(q_i)$, and for $q \in M$, let $Z(q)$ be the set of integers i such that $q \in U_{r,q_i}$. We will make use of an auxilliary function $g : R^+ \longrightarrow R$ which is C^∞ and which satisfies: $g(t)=1$ for $t<\frac{1}{4}$, $g(t)=o$ for $t>1$, and $-2<g'(t)<o$ for all t. We now define a vector $S(q)$ by

$$S(q) = \sum_{i \in Z(q)} g\left(\frac{|\psi(q)-\psi(q_i)|}{r}\right) N_i \,. \tag{3.1}$$

Finally, our approximate unit normal is the vector $X(q) = \frac{S(q)}{|S(q)|}$.

Now let ℓ_q be the line parametrized by $\ell_q(t) = \psi(q) + tX(q)$. It is shown in the appendix that if q lies in U_{δ,q_i} then there exists a unique number, $T(q)$, such that $\ell_q(T(q))$ lies in $\bar{\psi}(\bar{U}_{r,\bar{q}_i})$. So let us define a map $\Psi : U_{\delta,q_i} \longrightarrow R^3$ by

$$\Psi(q) = \ell_q(T(q)) = \psi(q) + T(q)X(q) \tag{3.2}$$

and a map $\phi : U_{\delta,q_i} \longrightarrow M$ by letting $\phi(q)$ be the unique point $\bar{q}$ in $\bar{U}_{r,\bar{q}_i}$ such that $\bar{\psi}(\bar{q}) = \Psi(q)$. It is verified in the appendix that if q also happens to lie in U_{δ,q_j} then the corresponding construction leads to the same $T(q)$, $\Psi(q)$, $\phi(q)$. Since M is covered by the sets U_{δ,q_i} we thus obtain globally defined maps $T : M \longrightarrow R$, $\Psi : M \longrightarrow R^3$, and $\phi : M \longrightarrow M$. Of course, $\Psi = \bar{\psi} \circ \phi$.

It remains to show that Ψ is an immersion which is C^1 close to ψ and which satisfies an $L_{2,p}$-bound; returning to the sequence $\{\psi^n\}$ we will then be able to extract from the corresponding sequence $\{\Psi^n = \psi^n \circ \phi^n\}$ a subsequence which converges to an immersion in C^1 and converges weakly in $L_{2,p}$ (by the Eberlein - Shmulyan Theorem). The fact that the ϕ^n are diffeo-orphisms is automatic. For ϕ^n is an immersion if Ψ^n is, hence by compactness of M, $\phi^n : M \longrightarrow M$ is actually a covering projection. Since ϕ^n is obviously not a multiple covering, ϕ^n must be a diffeomorphism.

To obtain the necessary local information about derivatives of $\Psi = \psi + TX$ we use the atlas on M induced by the graph system Γ to pull back the maps $\psi|_{U_{r,q_i}}$, $X|_{U_{r,q_i}}$, $T|_{U_{r,q_i}}$ to the disc D_r, and we use the atlas induced by $\bar{\Gamma}$ to pull back the maps $\bar{\psi}_{\bar{U}_{r,\bar{q}_i}}$ to D_r. The same letters will be used to denote these pulled-back maps; thus, e.g., we will write

$\psi(w) = A_i\,(w, h_i(w))$, $\bar{\psi}(w) = \bar{A}_i\,(w, \bar{h}_i(w))$, for $w \in D_r$.

Furthermore, we can assume $\bar{A}_i$ is the identity on R^3.

Let π^3, π denote the orthogonal projections of R^3 onto the z-axis and x,y-plane, respectively. Define $G : D_r \times R \longrightarrow R$ by

$$(3.3) \qquad G(w,t) = \bar{h}_i(\pi\,\psi(w) + t\pi\,X(w)) - \pi^3\psi(w) - t\pi^3 X(w).$$

Applying π^3 to equation (3.2) and comparing with (3.3) yields

$$(3.4) \qquad 0 = G(w, T(w)),$$

for all $w \in D_\delta$.

In the appendix, the implicit function theorem is applied to (3.4) to show that T is differentiable. Also, by refining our choice of ε, we obtain sufficiently good estimates for $|T|$ and $\|DT\|$ (the double bars denote the operator norm), to conclude that at each $w \in D_\delta$,
$\|D\Psi - D\psi\| \le \|DT\| + |T|\,\|DX\| < \frac{1}{2}$. It follows that Ψ is an immersion with a differential which satisfies $|D\Psi(w)u| > \frac{1}{2}$ for all $w \in D_r$ and unit vectors $u \in R^2$.

To get the second derivative bound on $\Psi = \psi + TX$ recall that ψ, X are fixed smooth maps, so it suffices to bound T in $L_{2,p}(M,R)$. We can define our norm on $L_{2,p}$ via the fixed atlas used above (that induced by Γ). Thus, we are clearly done once we bound D^2T in $L_p(D_r)$. As shown in the appendix, this bound follows readily from the second derivative of equation (3.4) together with the estimate on $\|\bar{h}_i\|_{2,p}$ provided by Lemma 2.1.

4. Lower Semi-continuity

In the previous section we obtained a sequence $\{\Psi^n\}$ in Ω which converges weakly in $L_{2,p}$ to an immersion $\Psi \in L_{2,p}$. It remains to show that $E_p(\Psi) \leq \liminf_{n \to \infty} E_p(\Psi^n)$, hence $\Psi \in \Omega$, thus completing the proof of the compactness theorem. Also, by choosi as our original sequence a minimizing sequence for E_p in a given regular homotopy class of unit area immersions, we will simutane ly obtain the existence theorem.

We begin by stating a straightforward generalization of a theorem of Morrey (see [7], p. 22) to higher order integrals. Fo Λ a bounded domain in R^d and $\psi \in L_{k,p}(\Lambda, R^m)$, we will let ν stand for the derivatives of ψ of order 0 up to k-1, and ζ will stand for the derivatives of ψ of order k. An element of Λ will be denoted by ξ.

Theorem 4.1 Suppose $F=F(\xi,\nu,\zeta)$ is C^2, $F \geq 0$, and F is convex in the set of variables ζ, for each fixed ξ,ν. Then the functional $I(\psi) = \int_\Lambda F(\xi,\zeta,\nu)\, d\xi^1 \ldots d\xi^d$ is lower semicontinuous with respect to weak $L_{k,1}(\Lambda, R^m)$-convergence.

At this point one could easily apply Theorem 4.1 to E_p in the graph case $\psi(x,y)=(x,y,h(x,y))$, and then make use of the graph system convergence to obtain the desired inequality. However, we choose to work with arbitrary global parametrization and obtain a result of independent interest; in particular, for the lower semi-continuity result itself there is no reason to exclude the case p=2.

For $\Lambda \subset \mathbb{R}^2$, and $\psi : \Lambda \to \mathbb{R}^3$ an $L_{2,p}(\Lambda,\mathbb{R}^3)$-immersion, let $s = \frac{\partial\psi}{\partial x}$, $t = \frac{\partial\psi}{\partial y}$, $U = \frac{\partial^2\psi}{\partial x^2}$, $V = \frac{\partial^2\psi}{\partial y^2}$, $W = \frac{\partial^2\psi}{\partial x\partial y}$, $E = \langle s,s\rangle$, $F = \langle s,t\rangle$, $G = \langle t,t\rangle$, $A = EG - F^2$, $\ell = A^{-\frac{1}{2}}\det\begin{pmatrix}U\\s\\t\end{pmatrix}$, $m = A^{-\frac{1}{2}}\det\begin{pmatrix}W\\s\\t\end{pmatrix}$, $n = A^{-\frac{1}{2}}\det\begin{pmatrix}V\\s\\t\end{pmatrix}$. Then the surface ψ has mean and Gaussian curvatures $H = \frac{1}{2}A^{-1}(En - 2Fm + G\ell)$, $K = A^{-1}(\ell n - m^2)$. We set

$$F_p(x,y,s,t,U,V,W) = \left[A(s,t)\right]^{\frac{1}{2}}\left[4H^2(s,t,U,V,W) - 2K(s,t,U,V,W)\right]^{\frac{p}{2}},$$

and $E_p(\psi) = \int_\Lambda F_p(x,y,s,t,U,V,W)\,dy\,dx$.

<u>Lemma 4.2</u> For $p \geq 2$, F_p is convex in the variables U^i, V^i, W^i $i = 1,2,3$ (the components of U, V, W).

<u>Proof</u>: Let β_i be in $\mathbb{R}^3$, $i=1,2,3$, and let $\beta=(0,\ldots 0,\beta_1,\beta_2,\beta_3)$. Observe that F_2 is a homogeneous second degree polynomial in U^i, V^i, W^i, $i=1,2,3$. Thus, $D^2F_2(x,y,s,t,U,V,W)(\beta,\beta)=2F_2(x,y,s,t,\beta_1,\beta_2,\beta_3)\geq 0$.

By the chain rule, the case $p>2$ now follows from the case $p=2$. □

<u>Theorem 4.3</u> Let $p\geq 2$, and suppose a sequence of immersions $\{\psi^n\}$ converges weakly in $L_{2,p}(M,\mathbb{R}^3)$ to an immersion ψ. Then $E_p(\psi) \leq \liminf_{n\to\infty} E_p(\psi^n)$.

<u>Proof</u>: By Lemma 4.2 we see that F_p satisfies the hypothesis of Theorem 4.1 (where it has been defined, i.e., for (ξ,ζ,ν) coming from an immersion). Of course, our maps ψ are defined on a manifold M rather than on a domain Λ in $\mathbb{R}^2$. However, we can think of M as a polygon with identifications on the boundary; thus, omitting a set of measure zero, we can pull our maps back by a fixed diffeomorphism to a polygonal domain in $\mathbb{R}^2$ and then apply Theorem 4.1. □

5. Appendix

Here we discuss details of the projection construction of Section 3. Thus we are considering two (r,α)-immersions $\psi, \bar{\psi}$, and corresponding graph systems $\Gamma, \bar{\Gamma}$, satisfying $\| \Gamma - \bar{\Gamma} \| < \varepsilon = \alpha r$. We also have the δ-nets $Q = \left\{ q_i = A_i(o) \right\}_{i=1}^{m}$ and $\bar{Q} = \left\{ \bar{q}_i = \bar{A}_i(o) \right\}_{i=1}^{m}$, on which the graph systems $\Gamma, \bar{\Gamma}$ are based. The numbers α and δ (and hence also ε) are assumed to be small, and it will be seen below how precisely to determine these values such that the projection construction works.

<u>Lemma 5.1</u> For any $q \in M$, $|X(q) - N(q)| \le \sqrt{2}\,\alpha$. Also, if $i \in Z(q)$, i.e., if $q \in U_{r,q_i}$, then $|N(q) - N(q_i)| \le \sqrt{2}\,\alpha$.

<u>Proof:</u> We might as well assume that $\psi(q_i)$ is the origin and $N(q_i) = (o,o,1)$, hence $\psi(U_{r,q_i})$ is the graph of $h : D_r \longrightarrow R$ which satisfies $\| \nabla h \|_o < \alpha$.

Now $N(q)$ is a unit normal to the graph of h, so it can be written $N(q) = (1+|\nabla h|^2)^{-\frac{1}{2}} \left(-\frac{\partial h}{\partial x}, -\frac{\partial h}{\partial y}, 1\right)\Big|_{(x_o,y_o)}$.

Setting $N(q) = (a,b,c)$, it follows that $\frac{a^2+b^2}{c^2} \le \alpha^2$.

Since $a^2+b^2+c^2=1$, we conclude that $c^2 \ge \frac{1}{1+\alpha^2}$, which implies $|N(q) - N(q_i)| = 2(1-c) \le 2\alpha^2$.

The first statement now follows from the observation that $X(q)$ is a normalized (weighted) sum of the vectors $X_i = N(q_i)$, $i \in Z(q)$, all of which lie in the hemisphere whose "pole" is $N(q)$.

Lemma 5.2 Assume $\bar{\delta} = \delta + 10\varepsilon < r < 1$, and suppose $q \in U_{\delta,q_i}$ Then the line ℓ_q (defined in section 3) has non-empty intersection with $\bar{\psi}\,(\bar{U}_{\bar{\delta},\bar{q}_i})$.

Proof: Let C be the boundary of $\bar{\psi}\,(\bar{U}_{\bar{\delta},\bar{q}_i})$ (so C is an embedded circle in R^3). The idea of the proof is to show that C is not null-homotopic in $R^3-\{\ell_q\}$.

We might as well assume $X(q) = (o,o,1)$ (so ℓ_q is parallel to the z-axis) and $\bar{\psi}(\bar{q}_i)$ is the origin. We note that the distance between $(o,o,1)$ and the unit normal to $\bar{\psi}$ at $\bar{q}_i$ can be estimated by $|\bar{N}(\bar{q}_i)-X(q)| \le |\bar{N}(\bar{q}_i)-N(q_i)| + |N(q_i)-N(q)| + |N(q)-X(q)| < \varepsilon + 2\sqrt{2}\,\alpha < 4\,\alpha$ (here we used Lemma 5.1 and the fact that $\|\Gamma-\bar{\Gamma}\| < \varepsilon$).

It follows easily from this estimate, together with the (r,α)-bound on $\bar{\psi}$ that C can be parametrized as a curve $C(\theta)$ which stays very close to the circle $\gamma(\theta) = \bar{\delta}(\cos\theta, \sin\theta, 0)$; to be precise, $\|C-\gamma\|_o < 4\,\alpha\,(1+\alpha^2)\,\bar{\delta} < 8\,\varepsilon$. Of course, the projection of C onto the x,y-plane is at least as close to γ.

On the other hand, the projection of ℓ_q is the point obtained by projecting $\psi(q)$. But $|\psi(q)| = |\psi(q)-\bar{\psi}(\bar{q}_i)| \le |\psi(q)-\psi(q_i)| + |\psi(q_i)-\bar{\psi}(\bar{q}_i)| \le (1+\alpha^2)\delta+\varepsilon < \delta+2\varepsilon$.
Since $\delta+2\varepsilon < \bar{\delta}-8\varepsilon$, the result follows. □

Lemma 5.3 In addition to the hypothesis of Lemma 5.2 assume $\alpha \le \frac{1}{10}$. Then ℓ_q intersects $\bar{\psi}(\bar{U}_{r,\bar{q}_i})$ in a unique point.

Proof: This time it is convenient to assume $\bar{N}(\bar{q}_i) = (o,o,1)$. Then from the estimate $|\bar{N}(\bar{q}_i) - X(q)| < 4\alpha$ of the previous proof we conclude that ℓ_q has "slope" greater than $\frac{1}{2}$. Now if ℓ_q intersects $\bar{\psi}(\bar{U}_{r,\bar{q}_i})$ in at least two points the mean value theorem implies that the curve obtained by projecting ℓ_q vertically onto $\bar{\psi}(\bar{U}_{r,\bar{q}_i})$ also has "slope" somewhere greater than $\frac{1}{2}$. But this contradicts the (r,α)-bound on $\bar{\psi}$.

Lemma 5.4 In addition to the hypothesis of Lemma 5.3 assume $\bar{\delta} \le \frac{r}{4}$. Suppose also $q \in U_{\delta,q_j}$ for some j. Then $\bar{\psi}^{-1}(\ell_q)$ intersects $\bar{U}_{\bar{\delta},\bar{q}_i}$ and $\bar{U}_{\bar{\delta},\bar{q}_j}$ in the same point.

Proof: The assumption $q \in U_{\delta,q_i}$ implies $j \in Z_i$ (see Section 3 for the definition of Z_i). Thus, $\bar{U}_{\bar{\delta},\bar{q}_i}$ intersects $\bar{U}_{\bar{\delta},\bar{q}_j}$ non-trivially. Lemma 3.1 now implies that $\bar{U}_{\bar{\delta},\bar{q}_j}$ is contained in $\bar{U}_{r,\bar{q}_i}$, so the statement follows from Lemmas 5.2 and 5.3. □

On inspection of the above lemmas, we see that the projection construction makes sense if we choose our constants as follows: $r < 1$, $\alpha = \frac{1}{100}$ $\left(\text{so } \varepsilon = \frac{r}{100}\right)$, and $\delta = \frac{r}{10}$.

However, to ensure that the resulting map $\Psi = \psi + TX$ is an immersion satisfying the required bounds, we will make one further refinement of the above choice of constants, namely, we set $\varepsilon = \frac{(\alpha r)^3}{\max(A,}$

This will enable us to prove the following

<u>Lemma 5.5</u> Let $T:D_r \to R$, $X:D_r \to S^2$ be the pulled-back maps defined before equation (3.3). Then at each $w \in D_\delta$,

$|T| < 15\varepsilon$ and $||DX|| < \frac{30\,A}{\delta^2 r}$, hence $|T|\,||DX|| < \frac{1}{10}$.

<u>Proof</u>: From $||\Gamma-\bar{\Gamma}||<\varepsilon$ one easily checks that the two maps ψ, $\bar{\psi}:D_r \to R$ satisfy $||\psi-\bar{\psi}||_0<4\varepsilon$. This enables one to make a homotopy argument very similar to that of Lemma 5.2 to show that for $w \in D_\delta$, $\ell_w(t)=\psi(w)+tX(w)$ intersects $\bar{\psi}(\bar{U}_{10\varepsilon,w})$. It follows that $|T(w)| \leq 4\varepsilon+(1+\alpha^2)10\,\varepsilon < 15\varepsilon$.

Differentiating equation (3.1) and using the fact that $|Z(w)| \leq |Q| = m \leq \frac{6A}{\delta^2}$, we obtain $||DS(w)|| \leq \sum_{i\in Z(w)} \frac{2}{r}(1+\alpha^2) \leq \frac{15A}{\delta^2 r}$. Since $S(w)$ is a weighted sum of unit vectors N_i which lie within 3α of each other, and since at least one of the "weights", $g(\cdot)$, is equal to one, it follows that $|S(w)| \geq 1$. Therefore,

$$||DX|| \leq 2\frac{||DS||}{||S||} \leq \frac{30A}{\delta^2 r}.$$ □

<u>Lemma 5.6</u> $T:D_\delta \to R$ is differentiable and satisfies $||DT||<\frac{1}{4}$.

<u>Proof</u>: Let us denote the derivative of $G:D_\delta \times R \to R$ with respect to the first factor by D_1G and the derivative with respect to the second factor by $\frac{\partial G}{\partial t}$. By Lemma 5.1, $|X(w)-(0,0,1)| <5\alpha$, so differentiation of equation (3.3) gives $|\frac{\partial}{\partial t}G| = |D\bar{h}_i \pi X - \pi^3 X| \geq |\pi^3 X| - |D\bar{h}_i \pi X| > \frac{1}{2}$. The implicit function theorem and equation (3.3) now imply T is differentiable.

Furthermore, since $D\psi$ is easily seen to satisfy $||D\psi||<1+\alpha$ and $||\pi^3 D\psi||<2\alpha$, we obtain

$$||D_1G|| = ||D\bar{h}_i\, \pi(D\psi+TDX) - \pi^3(D\psi+TDX)|| \leq \alpha\left((1+\alpha)+\frac{1}{10}\right)+2\alpha+\frac{1}{10} \leq \frac{1}{8}.$$ Thus we have

$$||DT|| = \frac{||D_1G||}{|\frac{\partial G}{\partial t}|} \leq \frac{1}{4} .$$ □

Lemma 5.7 $T:D_\delta \to R$ is in $L_{2,p}(D_\delta)$, and $\|D^2 T\|_p$ is bounded by a constant which does not depend on $\bar{\psi}$.

Proof: Set $w=(w^1,w^2)$. Taking $\frac{\partial^2}{\partial w^i \partial w^j}$ of equation (3.4) (using equation (3.3)) yields an equation of the form

$$0 = \left(\sum_{\beta,\gamma=1}^{2} C_{\beta,\gamma} \frac{\partial^2 \bar{h}_i}{\partial w^\beta \partial w^\gamma} \right) + \frac{\partial G}{\partial t} \frac{\partial^2 T}{\partial w^k \partial w^l} + C ,$$

where $C_{\beta,\gamma}$, C depend on derivatives of ψ and X up to order two and derivatives of $\bar{h}_i$ and T up to order one, all of which quantities we have uniform bounds for.

By Lemma 2.1 the second derivatives of $\bar{h}_i$ are bounded in terms of α and E, and by Lemma 5.6, $\frac{\partial G}{\partial t}$ is bounded away from 0. Thus, the result follows by solving for the second derivative of T in the above equation.

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