AUTOMORPHISMS OF GALOIS COVERINGS OF GENERIC m-CANONICAL PROJECTIONS

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Abstract. The automorphism group of the Galois covering induced by a pluri-canonical generic covering of a projective space is investigated. It is shown that by means of such coverings one obtains, in dimensions one and two, serieses of specific actions of the symmetric groups S_d on curves and surfaces not deformable to an action of S_d which is not the full automorphism group. As an application, new DIF≠DEF examples for G-varieties in complex and real geometry are given.

> Perhaps the simplest combinatorial entity is the group of the $n!$ permutations of n things. This group has a different constitution for each individual number n. The question is whether there are nevertheless some asymptotic uniformities prevailing for large n or for some distinctive class of large n. Mathematics has still little to tell about such problems.
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> *H. Weyl, Philosophy of Mathematics and Natural Science, Appendix D, Princeton Univ. Press, 1949.*

Introduction.

0.1. **Terminology conventions.** By a *covering* we understand a branched covering, that is a finite morphism $f : X \to Y$ from a normal projective variety X onto a non-singular projective variety Y, all being defined over the field of complex numbers $\mathbb{C}$. To each covering f we associate the branch locus $B \subset Y$, the ramification locus $R \subset f^{-1}(B) \subset X$, and the unramified part

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$X \setminus f^{-1}(B) \to Y \setminus B$ (which is the maximal unramified subcovering). As is usual for unramified coverings, there is a homomorphism ψ which acts from the fundamental group $\pi_1(Y\setminus B)$ to the symmetric group S_d on d elements, where d is the degree of f. This homomorphism (called monodromy of f) is defined by f uniquely up to inner automorphisms of S_d; reciprocally, according to Grauert-Remmert-Riemann extension theorem the conjugacy class of ψ defines f up to an isomorphism. The image $G \subset S_d$ of ψ is a transitive subgroup of S_d.

As is well known, for each covering f there is a unique, up to isomorphism, minimal Galois covering $\widetilde{f} : \widetilde{X} \to Y$ which is factorized through f, $\widetilde{f} = f \circ h$, by means of a Galois covering $h : \widetilde{X} \to X$. The covering $\widetilde{f}$ is called the *Galois expansion of* f. The characteristic, minimality, property of the Galois expansion $\widetilde{f}$ is that any Galois covering which is factorized through f can be factorized through $\widetilde{f}$. The Galois expansion of f can be obtained by Grauert-Remmert-Riemann extension theorem from the non-diagonal component of the fibred product over Y of d copies of the unbranched part $X \setminus f^{-1}(B) \to Y \setminus B$ of f. In particular, the Galois group $Gal(\widetilde{X}/Y)$ of the covering $\widetilde{f} : \widetilde{X} \to Y$ is naturally identified with $G = \psi(\pi_1(Y \setminus B))$ as above.

In the present article, we study actions of finite groups on the Galois expansions of generic coverings of the projective spaces. To give a precise definition of a generic covering we need to introduce first some preliminary definitions concerning the actions of symmetric groups.

Let I be a finite set consisting of $|I| = d$ elements and let $I_1 \cup \cdots \cup I_k = I$ be a partition of I, $|I_i| = d_i \geq 1$, $\sum_{i=1}^k d_i = d$. Such a partition defines a unique, up to conjugation, imbedding of $S_{d_1} \times \cdots \times S_{d_k}$ in S_d which we call a *standard imbedding.* A representation $S_{d_i} \subset GL(V_i)$, where V_i is a vector space over $\mathbb{C}$, will be called a *standard representation of rank* $d_i - 1$ if there is a base $e_1, \dots, e_{r_i}$ of V_i, $r_i = \dim V_i \geq d_i - 1$, such that the action $\sigma_{(j,j+1)}$ of a transposition $(j, j + 1) \in S_{d_i}$ is given by

$$\sigma_{(j,j+1)}(e_l) = \begin{cases} e_l & \text{if } l \neq j, j+1, \\ e_{j+1} & \text{if } l = j, \end{cases}$$

for $j \neq d_i - 1$, and by

$$\sigma_{(d_i-1,d_i)}(e_l) = \begin{cases} e_l & \text{if } l \neq d_i - 1, \\ -\sum_{s=1}^{d_i-1} e_s & \text{if } l = d_i - 1, \end{cases}$$

in the remaining case. A collection of standard representations of symmetric groups $S_{d_i} \subset GL(V_i)$ of ranks $d_i - 1$, $i = 1, \dots, k$, defines a representation of $S_{d_1} \times \cdots \times S_{d_k} \subset GL(V)$ with $V = V_{i_1} \oplus \cdots \oplus V_{i_k}$ which we call a *standard representation of the product* $S_{d_1} \times \cdots \times S_{d_k}$ *of rank* $\sum d_i - k$. As is easy to see, if $S_{d_1} \times \cdots \times S_{d_k} \subset GL(V)$ is a standard representation of rank $\sum d_i - k$, then the codimension in V of the subspace consisting of the vectors fixed under the action of $S_{d_1} \times \cdots \times S_{d_k}$ is equal to $\sum d_i - k$.

Let the group S_d act on a smooth projective manifold Y. We say that the action of S_d on Y is *generic* if the stabilizer $St_a \subset S_d$ of each point $a \in Y$ is a standard imbedding of a product of symmetric groups and the action induced by St_a on the tangent space T_aY is a standard representation (both the product and the representation depending on a). According to this definition, if the action of S_d on Y is generic, then the factor-space Y/S_d is a smooth projective manifold.

A covering $f : X \to \mathbb{P}^{\dim X}$ of degree d is called *generic* if the Galois group $G = Gal(\widetilde{X}/\mathbb{P}^{\dim X})$ of the Galois expansion of f is the full symmetric group S_d, the varieties X and $\widetilde{X}$ are smooth, and the action of G on $\widetilde{X}$ is generic. From this definition it follows that, for any generic covering f of degree d, the group $Gal(\widetilde{X}/\mathbb{P}^{\dim X})$ is the full symmetric group S_d and the subgroup $Gal(\widetilde{X}/X)$ coincides with $S_{d-1} \subset S_d$.

If X is non-singular and $\dim X = 1$, then a covering $f : X \to \mathbb{P}^1$ branched over $B \subset \mathbb{P}^1$ is generic if and only if $|f^{-1}(b)| = \deg f - 1$ for any $b \in B$. Furthermore, in the case of generic covering at each point $\widetilde{b} \in \widetilde{f}^{-1}(b), b \in B$, the stabilizer group $St_{\widetilde{b}} \subset S_d = Gal(\widetilde{X}/\mathbb{P}^1)$ is generated by a transposition, $St_{\widetilde{b}} = S_2$.

If X is non-singular and $\dim X = 2$, then a covering $f : X \to \mathbb{P}^2$ is generic if and only if the following conditions are satisfied: f is branched over a cuspidal curve $B \subset \mathbb{P}^2$; $|f^{-1}(b)| = \deg f - 1$ for any nonsingular point b of B, and $|f^{-1}(b)| = \deg f - 2$ if b is a node or a cusp of B. In the case of generic covering at each point $\widetilde{b} \in \widetilde{f}^{-1}(b)$, $b \in B$, the stabilizer group $St_{\widetilde{b}} \subset S_d = Gal(\widetilde{X}/\mathbb{P}^2)$ is generated: by a transposition if b is a nonsingular point of B, and then $St_{\widetilde{b}} = S_2$; by two non-commuting transpositions if b is a cusp of B, and then $St_{\widetilde{b}} = S_3$; by two commuting transpositions if b is a node of B, and then $St_{\widetilde{b}} = S_2 \times S_2$ (for a detailed exposition see Subsection 3.1).

Whatever is the dimension of a generic covering, the automorphism group $Aut(\widetilde{X})$ of the manifold $\widetilde{X}$ contains the symmetric group S_d, but as the following examples show, one can not expect that $Aut(\widetilde{X})$ and S_d will necessarily coincide.

As a first example, let us pick a generic covering $f_1 : Y = \mathbb{P}^1 \to \mathbb{P}^1$ of degree $d+1$ and denote by $\widetilde{f} : \widetilde{Y} \to \mathbb{P}^1$ the Galois expansion of f_1, $\widetilde{f} = f_1 \circ h_1$. Then $Gal(\widetilde{Y}/\mathbb{P}^1) = S_{d+1}$, $Y = \widetilde{Y}/S_d$, and $h_1 : \widetilde{Y} \to Y = \mathbb{P}^1$ is a Galois covering with $Gal(\widetilde{Y}/Y) = S_d$. The covering $h_1 : \widetilde{Y} \to Y = \mathbb{P}^1$ can be considered as the Galois expansion $\widetilde{X} = \widetilde{Y} \to Y = \mathbb{P}^1$ (with Galois group $Gal(\widetilde{X}/\mathbb{P}^1) = S_d$) of a covering $f : X \to Y = \mathbb{P}^1$, $X = \widetilde{X}/S_{d-1}$. The latter is a generic covering of degree d, and, now, if we start from $f : X \to \mathbb{P}^1$, we obtain that its Galois group $Gal(\widetilde{X}/\mathbb{P}^1) = S_d$ does not coincide with $Aut(\widetilde{X})$, since $Aut(\widetilde{X}) = Aut(\widetilde{Y})$ contains at least the group S_{d+1}.

Another example can be obtained as follows. Let $f_1 : Y = \mathbb{P}^1 \to \mathbb{P}^1$ be a generic degree d covering branched over $B_1 \subset \mathbb{P}^1$. Let us choose two points $x, y \in \mathbb{P}^1$ not belonging to B_1, and let $f_2 : Z = \mathbb{P}^1 \to \mathbb{P}^1$ be a cyclic covering of degree p branched at x and y. Consider the fibred product $X = Y \times_{\mathbb{P}^1} Z$ and its projection $f : X \to Z = \mathbb{P}^1$ to the second factor. It is easy to see that f is a generic covering and $Aut(\widetilde{X})$ contains $Gal(\widetilde{X}/\mathbb{P}^1) \times \mathbb{Z}/p\mathbb{Z}$.

0.2. **Principal results.** The aim of our research is to give numerical conditions for a generic covering $f : X \to \mathbb{P}^{\dim X}$ which ensure that $Aut(\widetilde{X}) = Gal(\widetilde{X}/\mathbb{P}^{\dim X})$ and which are preserved under any deformation of the Galois expansion.

To state the results obtained we need to introduce one more auxiliary notion: a covering $f : X \to \mathbb{P}^{\dim X}$ is said to be (numerically) *m-canonical* if it is given by $\dim X + 1$ sections of a line bundle numerically equivalent to the m-th power $K_X^{\otimes m}$ of the canonical bundle K_X of X and these sections have no common zeros.

Certainly, m-canonical coverings exist only if the Kodaira dimension of X coincides with its dimension. If $\dim X = 2$, then, in addition, X should be minimal and it should not contain any (-2)-curve. If $\dim X = 1$, then its genus should be greater or equal to 2. Reciprocally, as is well known, any curve of genus $g \geqslant 2$ possesses a m-canonical covering for $m \geqslant 1$, and, as is shown in [13], any minimal surface of general type containing no (-2)-curves also possesses a m-canonical covering at least for $m \geqslant 10$.

Theorem 0.1. *Let X be a curve of genus $g \geqslant 2$ and $\widetilde{f} : \widetilde{X} \to \mathbb{P}^1$ be the Galois expansion of a m-canonical generic covering $f : X \to \mathbb{P}^1$. If $m(g-1) \geq 500$, the Galois group $Gal(\widetilde{X}/\mathbb{P}^1)$ is the full automorphism group of $\widetilde{X}$.*

Theorem 0.2. *Let X be a surface of general type, and assume that it possesses a m-canonical generic covering $f : X \to \mathbb{P}^2$, $m \geq 2$. If $m^2 K_X^2 \geq 2 \cdot 84^2$ and $\widetilde{f} : \widetilde{X} \to \mathbb{P}^2$ is the Galois expansion of f, the Galois group $Gal(\widetilde{X}/\mathbb{P}^2)$ is the full automorphism group of $\widetilde{X}$.*

As a consequence, the G-curves like in Theorem 0.1 and the G-surfaces like in Theorem 0.2 provide infinitely many examples of *saturated* connected components in the moduli space of G-varieties, $G = S_d$, where a component is called saturated if for any G-variety representing a point of this component G is the full automorphism group of V. (It may be worth pointing some easy series of saturated components with $G \neq S_d$, namely, the components given by curves and surfaces with the automorphism groups of maximal order, that is $84(g-1)$ for curves, and 42^2K^2 for surfaces. One can mention also deformation rigid varieties with nontrivial automorphism group.)

As another application of the above theorems, we give counter-examples to a Dif=Def problem for complex and real G-varieties. Namely, we construct pairs of complex (respectively, real) varieties V_1, V_2 such that the actions of $\operatorname{Aut} V_1$ and $\operatorname{Aut} V_2$ (respectively, $\operatorname{Kl} V_1$ and $\operatorname{Kl} V_2$; here, $\operatorname{Kl} V$ is the group formed by the regular isomorphisms $X \to X$ and $X \to \bar{X}$) are diffeomorphic but not deformation equivalent. Up to our knowledge, such examples, specially at the real setting, are new. (It may be worth noticing, that in [9] in our counter-examples to the Dif=Def problem for real structures the surfaces have diffeomorphic real structures, but the actions of the Klein group on these surfaces are not diffeomorphic.)

0.3. **Contents of the paper.** The proof of theorems 0.1 and 0.2 consists of two parts. In the beginning (Section 1), by methods of group theory, we investigate minimal expansions of the symmetric groups to restrict the number of possible cases, and then the possible cases are investigated by geometric methods (Section 2 for Theorem 0.1 and Section 3 for Theorem 0.2). Section 4 contains the applications mentioned above.

1. Minimal expansions of symmetric groups.

1.1. **Preliminary definitions.** To formulate group theoretic statements which we use in the proof of Theorems 0.1 and 0.2, we need to introduce few preliminary definitions. We say that a group G containing the symmetric group S_d satisfies the *minimality property* if there is no any proper subgroup G_1 of G which contains S_d and does not coincide with S_d, and call such a group G a *minimal expansion of* S_d.

An imbedding $\alpha : S_d \subset S_{d+2}$ is called *quasi-standard* if the image $\alpha(\sigma_{i,j})$ of each transposition $\sigma_{i,j} = (i,j) \in S_d$, $1 \leq i,j \leq d$, is the product $\alpha(\sigma_{i,j}) = (i,j)(d+1,d+2)$ of two transpositions (i,j) and $(d+1,d+2)$ in S_{d+2}. Note that for the quasi-standard imbedding the image of S_d is contained in the alternating subgroup A_{d+2} of S_{d+2}. This imbedding, $\alpha : S_d \hookrightarrow A_{d+2}$, is called *standard.*

Proposition 1.1. *Let G be a minimal expansion of the symmetric group S_d of index $k = (G : S_d)$. Assume that $k \leq cd^n$, where either* (i) $c = 63$ *and* $n = 1$, *or* (ii) $c = (4 \cdot 42)^2$ *and* $n = 2$.

If $d \geq \max(2c, 1000)$, then G is one of the following groups:

1. *$G = S_d \times \mathbb{Z}/p\mathbb{Z}$, p is a prime number, $p \leq cd^n$;*
2. *$G = A_d \rtimes D_r$, where $3 \leq r \leq cd^n$, r is odd, D_r is the dihedral group given by presentation*

$$D_r = \langle \sigma, \tau \mid \sigma^2 = \tau^2 = (\sigma\tau)^r = 1 \rangle,$$

and the action (*by conjugation*) *of σ and τ on A_d coincides with the action of the transposition $(1,2) \in S_d$ on $A_d \subset S_d$;*

(3) $G = S_{d+1}$ *is the symmetric group;*

(4) $G = A_{d+2}$ *is the alternating group, the imbedding of* S_d *in* A_{d+2} *is a standard one, and such expansions can appear only under assumption* (ii).

The rest of this section is devoted to the proof of Proposition 1.1.

Proof. A priori, one of the following two cases occurs.

Case I. *The group* G *contains a non-trivial normal subgroup.*

Case II. *The group* G *is simple.*

Since $d > 6$, the group S_d has the unique non-trivial normal subgroup, namely, the alternating group A_d and, consequently, Case I can be subdivided into the following subcases, where N denotes a nontrivial normal subgroup of G.

Case I_1. $S_d \subset N$.

Case I_2. $N \cap S_d = \{1\}$.

Case I_3. $N \cap S_d = A_d$.

In its turn, Case I_3 can be subdivided into two subcases.

Case I_{31}. A_d *is a normal subgroup of* G.

Case I_{32}. A_d *is not a normal subgroup of* G.

1.2. **Analysis of Case I_1.** It follows from the minimality property that $S_d = N$. Let g_1 be an arbitrary element of $G \setminus S_d$. The conjugation by g_1 induces an automorphism of S_d. Since $d \geq 7$, any automorphism of S_d is inner. Therefore, there is $g_2 \in S_d$ such that $g = g_1 g_2$ commutes with all elements of S_d. Hence, once more by the minimality property, the group G splits into the direct product of S_d and the cyclic group $\langle g \rangle$ generated by g. Moreover, the order of g is a prime number p.

1.3. **Analysis of Case I_{31}.** According to subsection 1.2 we can assume that S_d is not a normal subgroup of G. Therefore, there is $g \in G$ such that $S'_d = g^{-1} S_d g$ does not coincide with S_d (but, it is isomorphic to S_d).

Since the group A_d is a normal subgroup of G, we have $A_d \subset S'_d \cap S_d$. Furthermore, for any transposition $\sigma \in S_d$ the element $\tau = g^{-1} \sigma g$ (which we call a transposition in S'_d) does not belong to S_d. Thus, by the minimality property, it follows that the group G is generated by the elements of A_d and any two transpositions $\sigma \in S_d$ and $\tau \in S'_d$. Moreover, since conjugating by elements of S_d (respectively, S'_d) provides the full automorphism group $Aut(A_d)$ of A_d, we can choose the two generating transpositions $\sigma \in S_d$ and $\tau \in S'_d$ in a way that $\sigma\tau$ commutes with all elements of A_d. And above all, we can assume that the action (by conjugation) of σ and τ on A_d coincides with the action of the transposition $(1, 2) \in S_d$.

Denote by H a subgroup of G generated by σ and τ. Then H is isomorphic to a dihedral group

$$D_r = \langle \sigma, \tau \mid \sigma^2 = \tau^2 = (\sigma\tau)^r = 1\rangle$$

for some $r \in \mathbb{N}$.

As is known, any element $g \in D_r$ either belongs to the cyclic subgroup generated by $\sigma\tau$ or is conjugate to σ or τ. Therefore $A_d \cap H$ is a subgroup of $\langle \sigma\tau\rangle$, and since the element $\sigma\tau$ commutes with all elements of A_d and A_d has trivial center, we conclude that $A_d \cap H = \{1\}$. In addition, A_d is a normal subgroup of G and G satisfies the minimality property, which implies

$$G = A_d \rtimes H \simeq A_d \rtimes D_r.$$

Moreover, r is odd, since σ and τ are conjugate in G and, hence, in D_r.

1.4. **Analysis of Case I_2.** It follows from the minimality property that in this case the group G is isomorphic to a semi-direct product $N \rtimes S_d$.

If N is not a simple group, then we can find a minimal non-trivial normal subgroup N_1 of N. Note, first of all, that N_1 can not be a normal subgroup of G, since G satisfies the minimality property. Therefore, the set of subgroups of G conjugated to N_1 contains more than one element. Let $\{N_1, \dots, N_s\}$ be the set of subgroups conjugate to N_1 in G, $s \geqslant 2$. Each N_i, $1 \leqslant i \leqslant s$, is contained in N, since N is a normal subgroup of G. Moreover, each of them is a normal subgroup of N, since N_1 is normal in N and conjugation by any element of G induces an isomorphism of N. Besides, the action of S_d on the set $\{N_1, \dots, N_s\}$ is transitive and this set is an orbit of the action of S_d by conjugation on the whole set of subgroups of G.

Let us show that $N \simeq N_1 \times \dots \times N_{s_1}$ for some $s_1 \leq s$ (maybe after a renumbering the groups N_i). Note, first, that $N_i \cap N_j = \{1\}$ for $i \neq j$. Indeed, $N_1, \dots, N_s$ are minimal normal subgroups of N and the intersection $N_i \cap N_j$, as an intersection of any two normal subgroups, is a normal subgroup. Therefore, $[N_1, N_2] \subset N_1 \cap N_2 = \{1\}$ so that the subgroup N_1N_2 generated in N by the elements of the groups N_1 and N_2 is isomorphic to $N_1 \times N_2$, and this subgroup N_1N_2 is also a normal subgroup of N. By induction, assume that for some $i < s$ the subgroup $N_{1,i} = N_1 \dots N_i$ of N is normal and isomorphic to $N_1 \times \dots \times N_i$. Then, either $N_{1,i} \cap N_{i+1} = \{1\}$, or $N_{i+1} \subset N_{1,i}$ (here, once more, we use the observation that N_{i+1} is a minimal normal subgroup of N). If $N_{1,i} \cap N_{i+1} = 1$, then

$$N_{1,i+1} = N_1 \cdot \ldots \cdot N_{i+1} \simeq N_1 \times \dots \times N_{i+1},$$

since $[N_{1,i}, N_{i+1}] \subset N_{1,i} \cap N_{i+1} = 1$. This inductive procedure stops at some subgroup $N_{1,s_1} \subset N$ which, being normal in N and invariant under the action of S_d by conjugation, is a normal subgroup of G. Therefore, N_{1,s_1} coincides with N. As a consequence, $N \simeq N_1 \times \dots \times N_{s_1}$.

The groups $N_1, \dots, N_s$, which are isomorphic to each other, are simple. Indeed, if N_1 is not simple, there exists a non-trivial normal subgroup $\widetilde{N}_1$ of N_1, so that the group $\widetilde{N}_1 \times \{1\} \times \dots \times \{1\}$ is a normal subgroup of N, which is impossible, since, by assumption, N_1 is a minimal normal subgroup of N.

Next, let us show that $s_1 = s$ if N_1 is a non-abelian simple group. Here, we are reasoning by contradiction and suppose that $s_1 < s$. Then, $N_{s_1+1} \subset N_1 \times \dots \times N_{s_1}$, the projection of N_{s_1+1} to each of the factors is either an isomorphism or the trivial homomorphism, and at least two projections are isomorphisms. Without loss of generality, we can assume that the first two projections are isomorphisms. So, if an element $g \in N_{s_1+1}$ is written as a product $g = g_1 g_2 \dots g_{s_1}$ of elements $g_i \in N_i$, then g_1 is uniquely determined by g_2. On the other hand, N_{s_1+1} is a normal subgroup of N, which implies that for any $g = g_1 g_2 \dots g_{s_1} \in N_{s_1+1}$ and any $h \in N_1$ the products $h^{-1}gh = (h^{-1}g_1 h)g_2 \dots g_{s_1}$ belong to N_{s_1+1}. Contradiction.

If N_1 is a non-abelian simple group, then conjugation by the elements of S_d permutes $N_1, \dots, N_s$ transitively and defines a homomorphism $\psi : S_d \to S_s$. Since $d \geq 7$, there are only two possibilities: either $s \leq 2$ and $A_d \subset \ker \psi$, or ψ is an imbedding and, therefore, $s \geq d$.

Suppose that N_1 is a non-abelian simple group and $s = 2$. Then, conjugation by the elements of A_d defines a homomorphism $\psi : A_d \to Aut(N_1)$. Since the outer automorphism group $Out(N_1) = Aut(N_1)/Inn(N_1)$ is solvable for a simple group N_1 (see [5], Theorem 4.240) and the group A_d is simple, we see that $\psi(A_d) \subset Inn(N_1)$ and either $\psi(A_d) = 1$, or $\psi : A_d \to Inn(N_1)$ is an imbedding into the inner automorphism group. If $\psi(A_d) = 1$, any element of N_1 commutes with the elements of A_d, and therefore A_d is a normal subgroup of G, since G is generated by the elements of S_d and an element $h \in N_1$, $h \neq 1$, and since h commutes with the elements of A_d. But such a case is already considered above (Case I_{31}). If $\psi : A_d \to Inn(N_1)$ is an imbedding, then the order of N_1 is greater than $\frac{1}{2}d!$ and, consequently,

$$k = (G : S_d) \geq (\frac{1}{2}d!)^2,$$

which is impossible since, by assumption, $d \geq 1000$ and k is not greater than either $63d$ or $(4 \cdot 42)^2 d^2$.

Suppose, next, that N_1 is a non-abelian simple group and $s \geq d$. Then (as is well known, A_5 is the smallest non-abelian simple group)

$$k = (G : S_d) = |N|^s \geq |N|^d \geq 60^d$$

which is impossible by the same reason as above.

Now, let N_1 be an abelian simple group. Then $N \simeq N_1 \times \dots \times N_{s_1}$ and again conjugation by the elements of A_d defines a homomorphism $\psi : A_d \to Aut(N_1)$.

As above, if $\psi(A_d) = 1$, then A_d is a normal subgroup of G and this case is already considered.

If $\psi : A_d \to Aut(N_1)$ be an imbedding, then Lemma 1.3 (see below) implies that $s_1 \geq [\frac{d}{4}]$. Therefore,

$$k = (G : S_d) = |N_1|^{s_1} \geq 2^{[\frac{d}{4}]},$$

which contradicts the assumption that $d \geq 1000$ and k is not greater than either $63d$ or $(4 \cdot 42)^2 d^2$.

The rest of the subsection is devoted to a proof of Lemma 1.3, which is based, in its turn, on the following Lemma.

Lemma 1.2. *Let F be a finite field of characteristic p and let H be a subgroup of $PGL(F, n)$ isomorphic to the alternating group A_{4d_1}, $d_1 \in \mathbb{N}$. Then $n \geq d_1$.*

Proof. If $d_1 \leq 2$ the statement is obvious. Assume that it is true for $d_1 \leq k$ and put $d_1 = k + 1$.

Consider the natural epimorphism $\psi : GL(F, n) \to PGL(F, n)$. The kernel of ψ consists of scalar matrices $\lambda\mathrm{Id}$, $\lambda \in F^*$. Therefore the group $\psi^{-1}(A_{4(k+1)})$ is a central extension of the group $A_{4(k+1)}$.

Denote by Ξ the set of 4-tuples $\{i_1, i_2, i_3, i_4\}$ of pairwise distinct integers $1 \leq i_j \leq 4(k + 1)$ and for each $I \in \Xi$ denote by x_I the permutation $(i_1, i_2)(i_3, i_4) \in A_{4(k+1)}$. These permutations x_I, $I \in \Xi$, generate the group $A_{4(k+1)}$ and, since $4(k+1) > 8$, each two of them are conjugate in $A_{4(k+1)}$. For each pair $I_1, I_2 \in \Xi$ let us choose a word w_{I_1,I_2} in x_I such that $x_{I_1} = w^{-1}_{I_1,I_2} x_{I_2} w_{I_1,I_2}$ in $A_{4(k+1)}$, then pick elements $\hat{x}_I \in \psi^{-1}(x_I) \subset \psi^{-1}(A_{k+1})$ and put

$$\widetilde{x}_{I_0} = \hat{x}_{I_0} \text{ for } I_0 = \{4k + 1, 4k + 2, 4k + 3, 4k + 4\},$$
$$\widetilde{x}_I = \hat{w}^{-1}_{I,I_0} \widetilde{x}_{I_0} \hat{w}_{I,I_0} \text{ for } I \neq I_0,$$

where $\hat{w}_{I,I_0}$ is obtained from the word w_{I,I_0} by substitution the elements $\hat{x}_I$ instead of x_I.

For each $I \in \Xi$ we have $\widetilde{x}_I = \mu_I \hat{x}_I$, where $\mu_I \in \ker \psi$ are elements of the center of $GL(F, n)$. Therefore,

$$\widetilde{x}_I = \widetilde{w}^{-1}_{I,I_0} \widetilde{x}_{I_0} \widetilde{w}_{I,I_0},$$

where $\widetilde{w}_{I,I_0}$ is obtained from the word w_{I,I_0} by substitution the elements $\widetilde{x}_I$ instead of x_I. On the other hand, $x_I^2 = 1$ for each $I \in \Xi$ which implies that $\widetilde{x}_I^2 = \lambda_I \in \ker \psi$. Since $\widetilde{x}_I$ are conjugate to each other and $\widetilde{x}_I^2 = \lambda_I$ are the elements of the center, all λ_I should be equal to each other. We denote this element by λ.

Consider the group $GL(F, n)$ as a subgroup of $GL(\overline{F}, n)$, where $\overline{F}$ is the algebraic closure of the field F, and denote by $\widetilde{A}_{4(k+1)}$ the subgroup of $GL(\overline{F}, n)$ generated by $\widetilde{x}_I, I \in \Xi$, and the elements belonging to the center of $GL(\overline{F}, n)$. It is easy to see that $\widetilde{A}_{4(k+1)}$ is a central extension of the group $A_{4(k+1)}$.

In $\widetilde{A}_{4(k+1)}$ there is an element μ such that $\mu^2=\lambda^{-1}$. Put $y_I=\mu\widetilde{x}_I$. Then $y_I^2=1$ and all the y_I are conjugate to each other: $y_I=v_{I,I_0}^{-1}y_{I_0}v_{I,I_0}$, where v_{I,I_0} is obtained from the word w_{I,I_0} by substitution the elements y_I instead of x_I. Furthermore, for each $I=\{i_1,i_2,i_3,i_4\}\in\Xi$ with $1\leq i_j\leq 4k$, we have $x_Ix_{I_0}=x_{I_0}x_I$ which implies that $y_Iy_{I_0}=\mu_Iy_{I_0}y_I$ for some μ_I belonging to the center of $GL(\overline{F},n)$. Since $y_I^2=Id$, then $\mu_I=\pm Id$, and since all the y_I are conjugate to each other by words depending on the elements y_I, all μ_I should be equal to each other. As a result, all μ_I with $I\in\Xi$ are equal to either $\mu=Id$ or $\mu=-Id$.

Let us show that $\mu=Id$. Consider the elements $y_{1,2,3,4}$, $y_{1,2,5,6}$, and put $\widetilde{y}_{3,4,5,6}=y_{1,2,3,4}y_{1,2,5,6}$. We have $\widetilde{y}_{3,4,5,6}=\lambda y_{3,4,5,6}$, where λ is a central element, since $x_{3,4,5,6}=x_{1,2,3,4}x_{1,2,5,6}$. Also, we have $y_{I_0}y_{1,2,3,4}=\mu y_{1,2,3,4}y_{I_0}$ and $y_{I_0}y_{1,2,5,6}=\mu y_{1,2,5,6}y_{I_0}$. Hence, on the one hand,

$$y_{I_0}\widetilde{y}_{3,4,5,6}=y_{I_0}\lambda y_{3,4,5,6}=\lambda\mu y_{3,4,5,6}y_{I_0}=\mu\widetilde{y}_{3,4,5,6}y_{I_0}$$

and, on the other hand,

$$y_{I_0}\widetilde{y}_{3,4,5,6}=y_{I_0}y_{1,2,3,4}y_{1,2,5,6}=\mu^2y_{1,2,3,4}y_{1,2,5,6}y_{I_0}=\mu^2\widetilde{y}_{3,4,5,6}y_{I_0}.$$

Therefore, $\mu=Id$.

Denote by $\overline{A}_{4(k+1)}$ the subgroup of $GL(\overline{F},n)$ generated by $y_I, I\in\Xi$. Obviously, its image in $PGL(\overline{F},n)$ is $A_{4(k+1)}$. Consider a subgroup $\overline{A}_{4k}$ of $\overline{A}_{4(k+1)}$ generated by the elements y_I, $I=\{i_1,i_2,i_3,i_4\}\in\Xi$ with $1\leq i_j\leq 4k$. The elements of $\overline{A}_{4k}$ commute with y_{I_0}, and the image of $\overline{A}_{4k}$ in $PGL(\overline{F},n)$ is A_{4k}.

Assume, first, that the characteristic $p\neq 2$. Then the vector space $V=\overline{F}^n$ splits into a direct sum $E_+\oplus E_-$ of two eigen-spaces corresponding to the eigenvalues ±1 of y_{I_0}. Since y_{I_0} does not belong to the center, $\dim E_+\geq 1$ and $\dim E_-\geq 1$. Since the elements of $\overline{A}_{4k}$ and y_{I_0} commute, the both eigen-spaces $E_\pm$ are invariant under the action of A_{4k}. This action is non-trivial on at least one of these subspaces, say, on E_+. Furthermore, since A_{4k} is a simple group, this action induces an imbedding of A_{4k} into $PGL(E_+)$. Therefore, $\dim E_+\geq k$ and, as a result, $\dim V\geq k+1=d_1$.

Suppose now that $p=2$. Then, the subspace $E=\{v\in V\,|\,y_{I_0}(v)=v\}$ of V is invariant under the action of A_{4k} and it is of dimension $\dim E<\dim V$. If the action of A_{4k} on E is non-trivial, then $n=\dim V>\dim E\geq k$, that is, $n\geq k+1=d_1$.

To end the proof, let us show that if the action of A_{4k} on E is trivial, then the induced action of A_{4k} on V/E is non-trivial. Indeed, if the both actions are trivial, then we can choose a basis in V such that in this basis each $y\in A_{4k}$ can be represented by a matrix of shape

$$y=\begin{pmatrix} Id_a & A\\ 0 & Id_b\end{pmatrix},$$

where $a = \dim E$, $b = \dim V - a$, A is a $(a \times b)$-matrix, and 0 is the zero $(b \times a)$-matrix. But, it is impossible, since the such matrices form an abelian group, while the group A_{4k} is non-abelian. Thus, we conclude that the action of A_{4k} on V/E is non-trivial, and, hence, $n = \dim V > \dim V/E \geq k$, that is, $n \geq k + 1 = d_1$. □

Lemma 1.3. *Let F be a finite field of characteristic p and let H be a subgroup of $GL(F, n)$ isomorphic to the alternating group A_{4d_1}, $d_1 \in \mathbb{N}$. Then $n \geq d_1$.*

Proof. Since $PGL(F, n)$ is the quotient group of $GL(F, n)$ by its center and the alternating group has trivial center, Lemma 1.3 follows from Lemma 1.2. □

1.5. **Analysis of Case I_{32}.** Since N is a normal subgroup and $N \cap S_d = A_d$, the subgroup $\langle N, \sigma\rangle$ generated in G by the elements of N and a transposition $\sigma \in S_d$ is isomorphic to a semi-direct product $N \rtimes \langle\sigma\rangle$. The group S_d is contained in $\langle N, \sigma\rangle$, since $A_d \subset N$. Thus, the minimality property of G implies $G = N \rtimes \langle\sigma\rangle$.

Let us remind that, by assumption, A_d is not a normal subgroup of G.

Suppose first that the group N is not simple. Pick a minimal non-trivial normal subgroup N_1 of N. Then, either $N_1 \cap A_d = \{1\}$, or $N_1 \cap A_d = A_d$, since A_d is simple.

If $N_1 \cap A_d = \{1\}$, then the group $N_2 = \sigma^{-1} N_1 \sigma$ is a normal subgroup of N and $N_2 \cap A_d = \{1\}$. If $N_1 = N_2$, then N_1 is a normal subgroup of G and this case is already considered (Case I_2). If $N_1 \neq N_2$, then $[N_1, N_2] \subset N_1 \cap N_2 = \{1\}$ and the group $N_1 N_2 \simeq N_1 \times N_2$ is a normal subgroup of G. Again, the case when $N_1 N_2 \cap A_d = \{1\}$ is contained in Case I_2. Therefore, we can assume that $N = N_1 N_2$. Since, $N_i \cap A_d = \{1\}$ for $i = 1, 2$, the projections of A_d to the factors should be imbeddings. Therefore, $|N_i| \geq |A_d| = \frac{d!}{2}$. Hence,

$$k = (G : S_d) = (N : A_d) \geq \frac{d!}{2}$$

which is impossible since, by assumption, $d \geq 1000$ and k is not greater than either $63d$ or $(4 \cdot 42)^2 d^2$.

If $N_1 \cap A_d = A_d$, then $N_2 \cap A_d = A_d$, where $N_2 = \sigma^{-1} N_1 \sigma$. If $N_1 = N_2$, then N_1 is a normal subgroup of G and this case is contained in Case I_2. If $N_1 \neq N_2$, then $N_1 \cap N_2$ is a normal subgroup of N and $A_d \subset N_1 \cap N_2 \subset N_1$. Therefore, contrary to our initial assumptions, N_1 is not a minimal non-trivial normal subgroup of N.

Thus, it us remains to treat the case when N is a simple group and $G = N \rtimes \langle\,\sigma\rangle$. Obviously, N can not be a cyclic group.

If N is isomorphic to some alternating group A_{d_1}, then $d_1 - d = n_1 \geq 1$ and

$$k = (G : S_d) = (A_{d_1} : A_d) = (d + 1) \dots (d + n_1).$$

By the hypotheses, $d \geq \max(2c, 1000)$ and $k \leq cd^n$, where either (i) $c = 63$ and $n = 1$, or (ii) $c = (4 \cdot 42)^2$ and $n = 2$. Therefore $n_1 \leq 1$ under assumption (i) and $n_1 \leq 2$ under assumption (ii).

If $n_1 = 1$, then $G = S_{d+1}$ (and, moreover, the imbedding of S_d into $G = S_{d+1}$ is the standard one).

Let us show (before ending with an analysis of other simple groups) that it is impossible to have $n_1 = 2$ under assumption (ii).

Lemma 1.4. *An imbedding $\alpha : A_d \to A_{d+2}$ is conjugate to the standard one if $d \geq 9$.*

Proof. Consider the standard actions of $A_d \subset S_d$ and $A_{d+2} \subset S_{d+2}$ on the sets $I_d = \{1, 2, \dots, d\}$ and $I_{d+2} = \{1, 2, \dots, d+2\}$, respectively. If $\tau \in A_d$ is a cyclic permutation of length 3, then its image $\alpha(\tau)$ is a product $\tau_1 \dots \tau_s$ of pairwise disjoint cyclic permutations, and for each $i = 1, \dots, s$ it holds $\tau_i^3 = 1$. To prove Lemma 1.4, it suffices to show that $s = 1$ for any 3-cycle $\tau \in A_d$. Without loss of generality we may assume that $\tau = (d-2, d-1, d)$.

Under the action of $\alpha(\tau)$, the set I_{d+2} splits into a disjoint union of s orbits $O_{3,i}$, $i = 1, \dots, s$, of cardinality 3 and $d+2-3s$ orbits $O_{1,i}$, $i = 1, \dots, d+2-3s$, of cardinality 1. Consider the subgroup A_{d-3} of A_d which leaves fixed the elements $d-2, d-1, d \in I_d$. Each element of A_{d-3} commutes with τ. Hence, the group $\alpha(A_{d-3})$ acts on the set of orbits $O_{3,i}$ and on the set of orbits $O_{1,i}$. This action defines a homomorphism $\beta : A_{d-3} \to S_s \times S_{d+2-3s}$. But, $s < d-3$ if $d > 9$, and if $s > 1$, then $d + 2 - 3s < d - 3$. Therefore, β is the trivial homomorphism if $s > 1$, since A_{d-3} is a simple group and $|A_{d-3}| > |S_s|$, $|A_{d-3}| > |S_{d+2-3s}|$ if $d > 9$. Hence, the homomorphism α induces a homomorphism of A_{d-3} to the direct product of s copies of S_3, which again should be trivial. Finally, if $s > 1$, then α would not be an imbedding. $\square$

Due to Lemma 1.4, we can assume now that $G \simeq A_{d+2} \rtimes \langle \sigma \rangle$ and that the imbedding $A_d \subset A_{d+2}$ is standard. Let us show that in this case $G \simeq S_{d+2}$ and the imbedding $S_d \subset G \simeq S_{d+2}$ is standard too.

Indeed, let look at the natural homomorphism $i : Inn(G) \to Aut(A_{d+2}) \simeq S_{d+2}$. Obviously, $i(A_{d+2}) = A_{d+2} \subset S_{d+2}$, and to prove that $G \simeq S_{d+2}$, it suffices to show that $i(\sigma)$ is not an inner automorphism of A_{d+2} (recall that σ is a transposition as an element of S_d). If $i(\sigma) \in Inn(A_{d+2})$, then there is an element $\tau \in A_{d+2}$ such that $\gamma = \sigma\tau$ commutes with all the elements of A_{d+2}. In particular, it commutes with τ and therefore it commutes with σ. Since $\sigma \notin A_{d+2}$, we have $\gamma = \sigma\tau \neq 1$ and the group $\langle S_d, \gamma \rangle$ generated in G by γ and the elements of S_d is isomorphic to $S_d \times \langle \gamma \rangle$. But existence of such a subgroup in $G \simeq A_{d+2} \rtimes \langle \sigma \rangle$ contradicts the minimality property of G.

To show that the imbedding $j : S_d \subset G \simeq S_{d+2}$ is standard, note that $j(\sigma)$ is a product $\sigma_1 \dots \sigma_s$ of odd number of pairwise disjoint transpositions $\sigma_i \in S_{d+2}$. We must show that $s = 1$. Assume that $s \geq 3$. As in the proof of Lemma 1.4, consider the standard action of S_d and S_{d+2} on the sets $I_d = \{1, 2, \dots, d\}$ and $I_{d+2} = \{1, 2, \dots, d+2\}$, respectively. Let $\sigma \in S_d$ be the transposition $(d-1, d)$. Under the action of $j(\sigma)$, the set I_{d+2} splits into a disjoint union of s orbits $O_{2,l}$, $l = 1, \dots, s$, of cardinality 2 and $d+2-2s$ orbits $O_{1,l}$, $l = 1, \dots, d+2-2s$, of cardinality 1. Consider the subgroup S_{d-2} of S_d which leaves fixed the elements $d-1, d \in I_d$. Each element of S_{d-2} commutes with σ. Hence, the group $j(S_{d-2})$ acts on the set of orbits $O_{2,j}$ and on the set of orbits $O_{1,j}$. Thus action defines a homomorphism $\beta : S_{d-2} \to S_s \times S_{d+2-2s}$. But, $s < d-2$ (recall that $d \geq 1000$), and if $s \geq 3$, then $d+2-2s < d-3$. Therefore the composition of β with the projection to each factor has a non-trivial kernel if $s \geq 3$. This kernel is either A_{d-2} or the whole S_{d-2}. Therefore the image of each element of S_{d-2} under the imbedding j has the order not greater than 4, which is impossible if $d-2 \geq 5$.

The following Lemma forbids an appearance of other simple groups N in $G = N \rtimes \langle \sigma \rangle$ and thus completes the investigation of Case I_{23}.

Lemma 1.5. *Assume that a simple group G distinct from an alternating group contains a subgroup H_d isomorphic either to the symmetric group S_d or the alternating group A_d, $d \geq 1000$. Then $(G : H_d) > 168^2 d^2$.*

Proof. To prove Lemma, we use the classification of finite simple groups (see [5]).

The group G is non-abelian, since H_d is a non-abelian group.

The group G can not be a sporadic simple group, since the order of each sporadic simple group is not divisible by $\frac{d!}{2}$ if $d \geq 33$ (for the sporadic simple groups either the multiplicity of the prime number 11 in its order is not greater than 2, or the order is not divisible by 13), while $|G|$ is divisible by $|H_d|$, which is divisible by $\frac{d!}{2}$.

Let G be a group of Lie type. Then G is a subgroup of either $GL(F, n)$ or $PGL(F, n)$, where F is a finite field. Denote by q the number of elements of the field F. Since $H_d \subset G$, then, by Lemmas 1.2 and 1.3, we have the inequality $n \geq [\frac{d}{4}]$.

If G is one of the following groups: $A_n(q)$, $B_n(q)$, $C_n(q)$, $D_n(q)$, ${}^2A_n(q^2)$, ${}^2D_n(q^2)$, then

$$|G| \geq q^{r^2/2},$$

where $r = [\frac{n}{2}]$. Since $2^{\frac{1}{2}[\frac{d}{8}]^2 - 16} > (d+2)\log_2 d$ for $d \geq 1000$, we have that $(G : H_d) > 168^2 d^2$.

To complete the proof of Lemma, note that all the other simple groups of Lie type have a non-trivial irreducible linear representation of dimension less than 250 and therefore, by Lemma 1.3, they can not have a subgroup isomorphic to A_d if $d \geq 1000$. $\square$

1.6. **Analysis of Case** II**.** It follows from Lemma 1.5 that it remains to consider only the case $S_d \subset G = A_{d_1}$.

The imbedding $S_d \subset A_{d_1}$ induces an imbedding $S_d \subset S_{d_1}$. Since any imbedding $S_d \subset S_{d+1}$ is standard, we have $d_1 - d = n_1 \geq 2$. By the hypotheses, $d \geq \max(2c, 1000)$ and $(A_{d_1} : S_d) = \frac{1}{2}(d+1)\dots(d+n_1) \leq cd^n$, where either (i) $c = 63$ and $n = 1$, or (ii) $c = (4 \cdot 42)^2$ and $n = 2$. Therefore $n_1 \leq 2$.

Let us show that if $n_1 = 2$ then the imbedding $S_d \subset A_{d+2}$ is a standard one. Indeed, by Lemma 1.4, the imbedding $S_d \subset A_{d+2}$ induces a standard imbedding $A_d \subset A_{d+2}$. Moreover, the image in $A_{d+2} \subset S_{d+2}$ of a transposition $\sigma \in S_d$ is a product of an even number s of mutually commuting transpositions σ_i of S_{d+2}. To show that the imbedding $S_d \subset A_{d+2}$ is a standard one, it suffices to prove that $s = 2$. We omit this proof, since it almost word by word coincides with the proof of Lemma 1.4. □

2. Proof of Theorem 0.1.

2.1. **Minimal expansions of the Galois groups of generic coverings.** Denote by $\overline{g} = g - 1$ the arithmetic genus of X, $\overline{g} \geqslant 1$, and by B the branch locus of $f : X \to \mathbb{P}^1$. Since f is m-canonical,

$$d = \deg f = 2m\overline{g}.$$

By Hurwitz formula applied to f,

$$|B| = 2d + 2\overline{g} = 2(2m+1)\overline{g}.$$

The branch locus of $\widetilde{f}$ (the Galois expansion of f) coincides with B, and the ramification indices of the ramification points of $\widetilde{f}$ are all equal to 2. Therefore, by Hurwitz formula applied to $\widetilde{f}$,

$$2\widetilde{g} = -2d! + \frac{1}{2}d!|B| = d!(d + \overline{g} - 2), \tag{1}$$

where $\widetilde{g} = g(\widetilde{X}) - 1$ is the arithmetic genus of $\widetilde{X}$.

Assume that $Aut(\widetilde{X}) \neq Gal(\widetilde{X}/\mathbb{P}^1)$ and choose a subgroup G of $Aut(\widetilde{X})$ such that $S_d \subset G$ is a minimal expansion of S_d.

Denote by $k = (G : S_d)$ the index of S_d in G. The Hurwitz bound on the order of the automorphism groups of algebraic curves (see, for example, [4]) implies that $|G| \leq 84\widetilde{g}$. Therefore,

$$k \leq 42(d + \overline{g} - 2). \tag{2}$$

In particular, we have

$$k < 63d. \tag{3}$$

By Proposition 1.1, it follows that G is one of the following groups:

(1) $G = S_d \times \mathbb{Z}/p\mathbb{Z}$, $p \geqslant 2$, p is a prime number;

(2) $G = A_d \rtimes D_r$, where $r \geq 3$, r is odd, D_r is the dihedral group given by representation
$$D_r = \langle \sigma, \tau \mid \sigma^2 = \tau^2 = (\sigma\tau)^r = 1\rangle,$$
and the action (by conjugation) of σ and τ on A_d coincides with the action of the transposition $(1,2) \in S_d$ on $A_d \subset S_d$;

(3) $G = S_{d+1}$ is the symmetric group.

2.2. **Elimination of the remaining three cases.** Consider Case (1). Denote by γ a generator of $\mathbb{Z}/p\mathbb{Z}$.

Since the action of γ on $\widetilde{X}$ commutes with the action of any element of S_d, the action of the group $\langle\gamma\rangle$ on $\widetilde{X}$ descends to both X and $\mathbb{P}^1$. Denote by $\widetilde{X}_1 = \widetilde{X}/\langle\gamma\rangle$, $X_1 = X/\langle\gamma\rangle$ the corresponding factor-spaces and by $\widetilde{r} : \widetilde{X} \to \widetilde{X}_1$, $r : X \to X_1$, $h_1 : \widetilde{X}_1 \to X_1$, and $r_P : \mathbb{P}^1 \to \mathbb{P}^1/\langle\gamma\rangle \simeq \mathbb{P}^1$ the corresponding morphisms. We have the following commutative diagram:

$$\begin{tikzcd}
\widetilde{X} \arrow[r, "h"] \arrow[d, "\widetilde{r}"'] & X \arrow[r, "f"] \arrow[d, "r"] & \mathbb{P}^1 \arrow[d, "r_P"] \\
\widetilde{X}_1 \arrow[r, "h_1"'] & X_1 \arrow[r, "f_1"'] & \mathbb{P}^1.
\end{tikzcd}$$

The cyclic covering $r_P : \mathbb{P}^1 \to \mathbb{P}^1$ is of degree $p \geqslant 2$ and it is ramified at two points, say $x_1, x_2 \in \mathbb{P}^1$. Therefore, the cyclic covering r is ramified at least at $2(d-1)$ points lying in $f^{-1}(x_1) \cup f^{-1}(x_2)$ and their ramification index is equal to p. By Hurwitz formula,
$$2\overline{g} \geq 2p(g(X_1) - 1) + 2(d-1)(p-1)$$
which implies
$$2\overline{g} \geq -2p + 2(2m\overline{g} - 1)(p-1).$$
Finally, thus we get the inequality
$$p \leq \frac{(2m+1)\overline{g} + 1}{2m\overline{g} - 2} = 1 + \frac{\overline{g} + 3}{2m\overline{g} - 2} < 2$$
which shows that Case (1) is impossible.

Consider Case (2). For a suitable pair of generators σ, τ of D_r, we have $S_d = A_d \rtimes \langle\sigma\rangle \subset G$, while the group $S'_d = A_d \rtimes \langle\tau\rangle$ is conjugated to S_d and does not coincide with S_d (but, it is isomorphic to S_d). Besides, $A_d \subset S'_d \cap S_d$. Denote by $X_1 = \widetilde{X}/S'_{d-1}$, $\mathbb{P}^1 = \widetilde{X}/S'_d$, and $X_0 = \widetilde{X}/A_d$ the corresponding quotient spaces. They can be arranged in the following commutative diagram in which the morphisms f_{0i}, $i = 1, 2$, are of degree two and, since f is a generic covering, f_{01} is branched over all the points belonging to B.

$$\begin{tikzcd} & \widetilde{X} \arrow[dl, "h"] \arrow[d, "h_0"] \arrow[dr, "h_1"] & \\ X \arrow[d, "f"'] & X_0 \arrow[dl, "f_{01}"] \arrow[dr, "f_{02}"'] & X_1 \arrow[d, "f_1"] \\ \mathbb{P}^1 & & \mathbb{P}^1 \end{tikzcd}$$

The degree 2 morphisms f_{0i}, $i = 1, 2$, define an imbedding $i : X_0 \to \mathbb{P}^1 \times \mathbb{P}^1$ with $i(X_0)$ being a curve of bi-degree $(2, 2)$ in $\mathbb{P}^1 \times \mathbb{P}^1$. Therefore, $i(X_0)$ is an elliptic curve and the projection of $i(X_0)$ onto each factor is branched at four points. On the other hand, f_{01} is branched at every point of B and $|B| = 2d + 2\overline{g} > 4$. Therefore, Case (2) is impossible.

Consider Case (3). Note, first of all, that the imbedding of S_d into $G = S_{d+1}$ is the standard one.

Consider the quotient space $\widetilde{X}/G$ and the quotient map $\overline{f} : \widetilde{X} \to \widetilde{X}/G$. The latter factors through $\widetilde{f}$, so that $\widetilde{X}/G \simeq \mathbb{P}^1$ and $\overline{f}$ is the composition of the following morphisms

$$\widetilde{X} \xrightarrow{h} X \xrightarrow{f} \mathbb{P}^1 \xrightarrow{r} \mathbb{P}^1,$$

where r is a morphism of degree $d + 1$. Since S_d and S_{d+1} have no common normal subgroups, $\overline{f}$ is the Galois expansion of r.

Denote by $B_1 \subset \mathbb{P}^1$ the branch locus of r and compare the cardinality of B with the cardinality of $r(B) \subset B_1$.

The symmetric group S_{d+1} acts as the permutation group on the set $I = \{1, \dots, d{+}1\} \subset \mathbb{N}$. Denote by $H_i = \{\gamma \in S_{d+1} \mid \gamma(i) = i\}$, so that our $S_d = H_{d+1}$. All groups H_i are conjugated to each other. Therefore for each i the covering $\widetilde{f}_i : \widetilde{X} \to \widetilde{X}/H_i \simeq \mathbb{P}^1$ is the Galois expansion of a generic covering.

Let $a \in \widetilde{X}$ be a ramification point of $\widetilde{f}_{d+1} = \widetilde{f}$. The stabilizer group $St_a(\overline{f}) = \{g \in G \mid g(a) = a\}$ is a cyclic group; its order is equal to the ramification index of $\overline{f}$ at a. Let τ be a generator of $St_a(\overline{f})$. The intersection $St_a(\overline{f}) \cap S_d = St_a(\widetilde{f}_{d+1})$ is a group of order two generated by a transposition $\sigma \in H_{d+1}$, since $\widetilde{f}_{d+1}$ is the Galois expansion of a generic covering. Therefore, $\sigma = \tau^k$, where $\tau^{2k} = 1$,

Let us show, first, that k is odd. Indeed, let us write τ as the product of cyclic permutations: $\tau = (i_{1,1}, \dots, i_{1,k_1}) \dots (i_{s,1}, \dots, i_{s,k_s})$. We can assume that, up to renumbering, $\sigma = (i_{1,1}, \dots, i_{1,k_1})^k$ and $(i_{j,1}, \dots, i_{j,k_j})^k = 1$ for $j = 2, \dots, s$. Now, it easy to see that $k_1 = 2$, k is odd, and all k_j are divisors of k for $j = 2, \dots, s$.

Let us show that $k = 1$, so that $\tau = \sigma$. We have for each i the intersection $St_a(\overline{f}) \cap H_i = St_a(\widetilde{f}_i)$ is a group of order at most two and if its order equal to two, then it is also generated by a transposition $\sigma_i \in H_i$, since $\widetilde{f}_i$ is the Galois expansion conjugated to $\widetilde{f}_{d+1}$. On the other hand, the element

$$\sigma\tau = (i_{1,2}, \dots, i_{1,k_2}) \dots (i_{s,1}, \dots, i_{s,k_s}) \in St_a(\overline{f}) \cap H_{i_{1,1}} = St_a(\widetilde{f}_{i_{1,1}})$$

is of odd order. Therefore $\sigma = \tau$.

Now, consider the fibre $\overline{f}^{-1}(\overline{f}(a))$ containing the point a. The fibre $\overline{f}^{-1}(\overline{f}(a))$ can be identified with the set of right cosets $\{St_a(\overline{f})\gamma\}$ in S_{d+1}. The stabilizer group $St_{(a)\gamma}(\overline{f})$ of the point $(a)\gamma$ is generated by the transposition $\gamma^{-1}\sigma\gamma$.

The fibre $\overline{f}^{-1}(\overline{f}(a))$ splits into the disjoint union of orbits under the action of H_{d+1} each of which is a fibre of $\widetilde{f}_{d+1}$. Without loss of generality, we can assume that the group $St_a(\overline{f})$ is generated by $\sigma = (1, 2)$. Then it is easy to see that each of these orbits can be identified with one of $F_i = \{St_a(\overline{f})\gamma \mid \gamma \in \sigma_i H_{d+1}\}$, where $\sigma_i = (i, d+1)$ if $2 \leq i < d+1$, and $\sigma = (1, 2)$, if $i = d+1$. (The transpositions $\sigma_1 = (1, d+1)$ and $\sigma_2 = (2, d+1)$ give the same orbit under the action of H_{d+1}, since $(1, 2)(1, d+1)(1, 2) = Id \cdot (2, d+1)$.)

Now, the points $a_i = (a)\sigma_i$ have the same stabilizer group

$$St_{a_i}(\overline{f}) = \langle (1, 2) \rangle \subset H_{d+1}$$

if $i > 2$. Therefore for $i \geq 3$ the points belonging to F_i are the ramification points of $\widetilde{f}_{d+1}$ and hence $\widetilde{f}_{d+1}(F_i) \in B$. It is easy to see that the points belonging to F_2 are not the ramification points of $\widetilde{f}_{d+1}$. Therefore the point $\widetilde{f}_{d+1}(F_2)$ is a ramification point of r.

As a consequence, we obtain that if $\widetilde{b} \in r(B)$, then the fibre $r^{-1}(\widetilde{b})$ consists of $d-1$ points belonging to B and one point (a ramification point of r) which does not belong to B. Hence, $|B| = 2d + 2\overline{g} = 2(m+1)\overline{g}$ is divisible by $d - 1 = 2m\overline{g} - 1$. Then, $2\overline{g} + 1$ should also be divisible by $2m\overline{g} - 1$. But, it is possible only if $\overline{g} = 1$ and $m = 1$ or 2. □

3. Proof of Theorem 0.2.

3.1. **Local behavior of generic coverings and their Galois expansions.** In this subsection we specialize to surfaces the definitions related to generic actions of the symmetric group, compare our definitions with the traditional definition of generic coverings, deduce the local behavior of generic coverings from the local behavior of these actions, and fix the corresponding notation and notions.

Recall that the Galois expansion $\widetilde{f} : \widetilde{X} \to \mathbb{P}^2$ of a generic covering $f : X \to \mathbb{P}^2$ of degree d is factorized through f, $\widetilde{f} = f \circ h$, by means of a Galois covering $h : \widetilde{X} \to X$ with the Galois group $Gal(\widetilde{X}/X) = S_{d-1} \subset S_d = Gal(\widetilde{X}/\mathbb{P}^2)$. The branch locus $B \subset \mathbb{P}^2$ of f coincides with that of $\widetilde{f}$. We have called f generic if the action of S_d on $\widetilde{X}$ is generic. The latter means that for any point $a \in \widetilde{X}$ its stabilizer $St_a(S_d)$ is a standard imbedded in S_d product of symmetric groups (depending on a) and the actions induced by $St_a(S_d)$ on the tangent spaces $T_a\widetilde{X}$ are standard representations of rank ≤ 2 (see Introduction).

On the other hand, in dimension two more traditionally one understands under a generic covering $f : X \to \mathbb{P}^2$ of degree d a covering whose local behavior is as follows. The branch locus B of f is a cuspidal curve. Over a neighborhood U of a smooth point of B the preimage $f^{-1}(U)$ splits into a disjoint union of $d-1$ connected components, in one of them the covering is two-sheeted and isomorphic to the projection to x, y plane of the surface $x = z^2$ at a neighborhood of the origin, and in the other components it is a local isomorphism. Over a neighborhood of a cuspidal point of B the preimage splits into a disjoint union of $d-2$ neighborhoods, in one of them the covering is a pleat, that is a three-sheeted covering which is isomorphic to the projection to x, y plane of the surface $y = z^3 + xz$ at a neighborhood of the origin, and in the other components it is a local isomorphism. Over a neighborhood of a node of B the preimage splits into a disjoint union of $d-2$ neighborhoods, in two of them this covering is two-sheeted and isomorphic in the union of them to the projection to x, y plane of two surfaces $x = z^2$ and $y = z^2$ in a neighborhood of the origin and in the other components it is a local isomorphism. The nontrivial local Galois groups in the corresponding three cases are $\mathbb{Z}/2$, S_3, and $\mathbb{Z}/2 \times \mathbb{Z}/2$. But all their nontrivial representations in $GL(2, \mathbb{C})$ which produce a non-singular quotient are standard representations of rank ≤ 2, and therefore our definition of generic coverings coincides with the traditional one.

The local behavior of generic coverings is easily understandable from the above local models. In particular, one observes that: $f^*(B) = 2R + C$, where R (the ramification locus of f) is nonsingular, C is reduced and non-singular above the non-singular points of B; R and C intersect each other only above the nodes and cusps of B; they meet at two points above each node and intersect there transversally; and they meet at one point above each cusp and intersect there with simple tangency.

Let us observe the same local behavior from the point of view of the action of S_d on $\widetilde{X}$, which will help us in our further considerations in this section.

At a small neighborhood of any point $a \in \widetilde{X}$ the action of $St_a(S_d)$ can be linearized (Cartan's linearization procedure [2], which by a suitable change of coordinates identifies the local action with the action induced on the tangent space $T_a\widetilde{X}$, is reproduced below in the proof of Lemma 4.1). Let us treat case by case different possibilities for $St_a(S_d)$ and, by means of a linearization of the action and in accordance with the definition of standard representations, analyze the local behavior of f, $\widetilde{f}$, and h.

If $St_a(S_d) = S_2$ is generated by a transposition $\sigma \in S_d$, then in a neighborhood of a the ramification locus $\widetilde{R}$ of $\widetilde{f}$ coincides with the set of fixed points of σ, which we denote by $\widetilde{R}_\sigma$. The latter is smooth everywhere, and, in particular, $\widetilde{R}$ is smooth at a. The image $h(a)$ of a belongs to the ramification locus R of f (equivalently, a does not belong to the ramification locus of h) if and only if $\sigma \notin S_{d-1}$. Moreover, $h(\widetilde{R}_\sigma)$ coincides with R at a neighborhood of $h(a)$ if $\sigma \notin S_{d-1}$ (otherwise, it coincides with C introduced above). In both cases, $\sigma \in S_{d-1}$ and $\sigma \notin S_{d-1}$, the curve $h(\widetilde{R}_\sigma)$ is smooth at $h(a)$. Furthermore, in both cases, $\widetilde{f}(a)$ belongs to B and B is non-singular at $\widetilde{f}(a)$.

If $St_a(S_d) = S_2 \times S_2$ is generated by two commuting transpositions, $\sigma_1 \in S_d$ and $\sigma_2 \in S_d$, then the point a belongs to $\widetilde{R}_{\sigma_1} \cap \widetilde{R}_{\sigma_2}$, the curves $\widetilde{R}_{\sigma_1}$ and $\widetilde{R}_{\sigma_2}$ are nonsingular, and they meet transversally at a. Furthermore, $h(\widetilde{R}_{\sigma_1})$ and $h(\widetilde{R}_{\sigma_2})$ are nonsingular and meet transversally. If one of the transpositions, say σ_1, does not belong to S_{d-1} the curve $h(\widetilde{R}_{\sigma_1})$ is contained in R and, moreover, coincides with R in a neighborhood of $h(a)$. If $\sigma_1 \in S_{d-1}$ the curve $h(\widetilde{R}_{\sigma_1})$ is not contained in R (but contained in C). If both σ_1 and σ_2 belong to S_{d-1}, $h(a)$ is not a ramification point of f (and then it is a node of C with $C = h(\widetilde{R}_{\sigma_1}) \cup h(\widetilde{R}_{\sigma_2})$ in a neighborhood of $h(a)$). In any case, $\widetilde{f}(a)$ is a node of B.

If $St_a(S_d) = S_3$ is generated by two non-commuting transpositions, $\sigma_1 \in S_d$ and $\sigma_2 \in S_d$, then the point a belongs to $\widetilde{R}_{\sigma_1} \cap \widetilde{R}_{\sigma_2} \cap \widetilde{R}_{\sigma_3}$, where $\sigma_3 = \sigma_1\sigma_2\sigma_1$, the curves $\widetilde{R}_{\sigma_1}$, $\widetilde{R}_{\sigma_2}$, and $\widetilde{R}_{\sigma_3}$ are nonsingular and meet pairwise transversally at a. If all the three transpositions belong to S_{d-1}, the point $h(a)$ is not a ramification point of f. Otherwise, one and only one of the transpositions, say σ_3, belongs to S_{d-1}, and then: $h(\widetilde{R}_{\sigma_1}) = h(\widetilde{R}_{\sigma_2})$ and $h(\widetilde{R}_{\sigma_3})$ are nonsingular, they are tangent to each other, and $h(\widetilde{R}_{\sigma_1}) = h(\widetilde{R}_{\sigma_2})$ coincides with R (while $h(\widetilde{R}_{\sigma_3})$ coincides with C) in a neighborhood of $h(a)$. In any case, $\widetilde{f}(a)$ is a cusp of B.

3.2. **Invariants of m-canonical generic coverings.** Assume that $f : X \to \mathbb{P}^2$ is a generic m-canonical covering branched along a cuspidal curve $B \in \mathbb{P}^2$. Then X is a minimal surface of general type, it does not contain any (-2)-curve, and the degree of f is equal to

$$d = \deg f = m^2 K_X^2.$$

According to the formula for the canonical divisor of a finite covering, $K_X = f^*K_{\mathbb{P}^2} + [R]$. Hence, the divisor R is numerically equivalent to $(3m+1)K_X$. Since, in addition, the curve R is non-singular and X has no (-2)-curves (if f is a m-canonical generic covering, then K_X is ample), R is irreducible. Therefore, B as a curve birational to R is irreducible as well. Thus, we can apply the results from [11]. In particular, we get the following formulas for the degree $\deg B$ and the number c of cusps of B (see the proof of Theorem 2 in [11]):

$$\deg B = m(3m+1)K_X^2 \tag{4}$$

and

$$c=(12m^2+9m+3)K_X^2-12p_a, \tag{5}$$

where $p_a=p_g-q+1$ is the arithmetic genus of X. Note that if $\deg f\geq 3$ for a generic covering f, then its branch curve B should have cuspidal singular points, that is, $c>0$ (indeed, the image in S_d of the monodromy of f is a transitive subgroup of S_d and, for generic coverings, this image is generated by transpositions, hence coincides with S_d; therefore $\pi_i(\mathbb{P}^2\setminus B)$ is non-abelian if $d\geqslant 3$, while by Zariski's theorem $\pi_i(\mathbb{P}^2\setminus B)$ is an abelian group if B is a nodal curve).

Finally, applying the projection formula for the canonical divisor to $\widetilde{f}$ we obtain $K_{\widetilde{X}}=\widetilde{f}^*(K_{\mathbb{P}^2})+[\widetilde{R}]=\widetilde{f}^*(K_{\mathbb{P}^2}+\frac{1}{2}[B])$, and therefore

$$K_{\widetilde{X}}^2=\frac{1}{4}(\deg B-6)^2d!=d!(\frac{m(3m+1)}{2}K_X^2-3)^2. \tag{6}$$

3.3. **Minimal expansions of the Galois groups of generic coverings.** Assume that $Aut(\widetilde{X})\neq Gal(\widetilde{X}/\mathbb{P}^2)$ and choose a subgroup G of $Aut(\widetilde{X})$ such that $S_d\subset G$ is a minimal expansion of S_d. Denote by $k=(G:S_d)$ the index of S_d in G.

The Xiao bound (see [15]) on the order of the automorphism groups of surfaces of general type states that $|G|\leq 42^2K_{\widetilde{X}}^2$. It implies

$$k\leq 42^2(\frac{m(3m+1)}{2}K_X^2-3)^2. \tag{7}$$

Finally we get

$$k<(2\cdot 42)^2d^2. \tag{8}$$

By Proposition 1.1, it follows that the group G can be only one of the following groups:

(1) $G=S_d\times\mathbb{Z}/p\mathbb{Z}$, $p\geqslant 2$, p is a prime number;
(2) $G=A_d\rtimes D_r$, where $r\geq 3$, r is odd, D_r is the dihedral group given by representation

$$D_r=\langle\sigma,\tau\mid\sigma^2=\tau^2=(\sigma\tau)^r=1\rangle,$$

and the action (by conjugation) of σ and τ on A_d coincides with the action of the transposition $(1,2)\in S_d$ on $A_d\subset S_d$;
(3) $G=S_{d+1}$ is the symmetric group;
(4) $G=A_{d+2}$ is the alternating group, and the imbedding of S_d into $G=A_{d+2}$ is a standard one.

3.4. **Case** (1)**.** Denote by g a generator of $\mathbb{Z}/p\mathbb{Z}$. As in the proof of Theorem 0.1, since the action of g on $\widetilde{X}$ commutes with the action of any element of S_d, the action of the group $\langle g\rangle = \mathbb{Z}/p\mathbb{Z}$ on $\widetilde{X}$ descends to both X and $\mathbb{P}^2$. Denote by $\widetilde{X}_1 = \widetilde{X}/\langle g\rangle$, $X_1 = X/\langle g\rangle$ the corresponding factor-spaces and by $\widetilde{r} : \widetilde{X} \to \widetilde{X}_1$, $r : X \to X_1$, $h_1 : \widetilde{X}_1 \to X_1$, and $r_P : \mathbb{P}^2 \to \mathbb{P}^2/\langle g\rangle = Y$ the corresponding morphisms. We have the following commutative diagram:

$$\begin{tikzcd}
\widetilde{X} \arrow[r, "h"] \arrow[d, "\widetilde{r}"'] & X \arrow[r, "f"] \arrow[d, "r"] & \mathbb{P}^2 \arrow[d, "r_P"] \\
\widetilde{X}_1 \arrow[r, "h_1"'] & X_1 \arrow[r, "f_1"'] & Y.
\end{tikzcd}$$

The automorphism g on $\mathbb{P}^2$ is defined by a linear map $\mathbb{C}^3 \to \mathbb{C}^3$ of period p. Therefore, it has either three isolated fixed points, say x_1, $x_2, x_3 \in \mathbb{P}^2$, or an isolated fixed point, say $x \in \mathbb{P}^2$ and a fixed line $E \subset \mathbb{P}^2$. Respectively, the cyclic covering $r_P : \mathbb{P}^2 \to \mathbb{P}^2$ is of degree p and it is ramified either at three points x_1, x_2, x_3 or at the point x and along the line E. Consider this two cases separately.

If r_P is ramified at three isolated points, then r is ramified at at least $3(d-2)$ points lying in $F = f^{-1}(x_1) \cup f^{-1}(x_2) \cup f^{-1}(x_3)$ and the ramification index of each of them is equal to p. The points of F are the only fixed points of the automorphism g acting on X. Therefore, by Lefcshetz fixed point theorem we have

$$|F| = \sum_{i=0}^{4} (-1)^i tr_i, \tag{9}$$

where tr_i is the trace of the linear transformation g^* acting on $H^i(X, \mathbb{R})$. It follows from (9) that

$$3(d-2) \leqslant |F| \leq \sum_{i=0}^{4} |tr_i| \leq \sum_{i=0}^{4} b_i(X) = e(X) + 4b_1(X) \tag{10}$$

(where e stands for the topological Euler characteristic). On the other hand, Noether's formula $1 - q + p_g = \frac{K_X^2 + e(X)}{12}$ where $2q = b_1$ gives us

$$e(X) + 4b_1(X) = 12 - K_X^2 + 12p_g - 4q \leq 12 - K_X^2 + 12p_g \tag{11}$$

Therefore, combining (10) and (11) with Noether's inequality $2p_g \leqslant K_X^2 + 4$ we get

$$3(m^2K_X^2 - 2) \leq 12 - K_X^2 + 12p_g \leq 5K_X^2 + 36. \tag{12}$$

Hence,

$$(3m^2 - 5)K_X^2 \leq 42$$

which contradicts to $m^2K_X^2 \geq 2\cdot 84^2$ if $m\geq 2$, since $K_X^2\geq 1$ for any minimal surface of general type.

Now, let us assume that r_P is ramified at a point $x\in\mathbb{P}^2$ and along a line $E\subset\mathbb{P}^2$. In this case each line $L\subset\mathbb{P}^2$ passing through x is invariant under the action of g on $\mathbb{P}^2$. Therefore, each curve $C=f^{-1}(L)\subset X$ is invariant under the action of g on X. Pick a generic line L passing through x. By Hurwitz formula,

$$2(g(C)-1) = -2d+\deg B-2m^2K_X^2+m(3m+1)K_X^2 = m(m+1)K_X^2, \quad (13)$$

where $g(C)$ is the geometric genus of C.

Consider the restriction $r_{|C}: C\to C/\langle g\rangle = Z\subset X_1$ of r to C. The cyclic covering $r_{|C}$ has degree p and it is branched at at least $2d-3=2m^2K_X^2-3$ points belonging to $f^{-1}(x)\cup f^{-1}(L\cap E)$. Therefore,

$$2(g(C)-1)\geq 2p(g(Z)-1)+(2m^2K_X^2-3)(p-1). \quad (14)$$

It follows from (13) and (14) that

$$m(m+1)K_X^2\geq -2p+(2m^2K_X^2-3)(p-1)$$

which implies

$$m(3m+1)K_X^2\geq (2m^2K_X^2-5)p.$$

Finally, since $(m^2-m)K^2\geqslant 500>10$, we get the inequality

$$p\leq \frac{m(3m+1)K_X^2}{2m^2K_X^2-5}<2,$$

which is a contradiction.

3.5. **Case** (2)**. Group theoretic part.** Since r, $r\geqslant 3$, is odd, the conjugacy class of σ in D_r consists of r elements $\sigma_1=\sigma,\sigma_2=\tau,\dots,\sigma_r$. For each $i, 1\leqslant i\leqslant r$, the group $S_{d,i}$ generated in G by σ_i and the elements of A_d is isomorphic to $A_d\rtimes\langle\sigma_i\rangle\simeq S_d$. The groups $S_{d,i}$ are conjugate to each other in G. Besides, $\sigma_i\in S_{d,i}$ acts on $A_d\subset S_{d,i}$ as the transposition $(1,2)$. The element of $S_{d,i}$ which is conjugate to σ_i and acts on A_d as a transposition (i_1,i_2) will be denoted by $\sigma_{i,(i_1,i_2)}$. Given two disjoint subsets $J_1\neq\emptyset, J_2$ of $I=\{1,\dots,d\}$, denote by $S_{J_1\bigsqcup J_2,i}$ the subgroup of $G=A_d\rtimes D_r$ generated by the elements $\sigma_{i,(i_1,i_2)}$, $(i_1,i_2)\in(J_1\times J_1)\cup(J_2\times J_2)$.

Let $St_a\subset Aut\widetilde{X}$ be the stabilizer of a point $a\in\widetilde{X}$. For a subgroup H of $Aut\widetilde{X}$ put $St_a(H)=H\cap St_a$. For each point $a\in\widetilde{X}$ the action induced by $St_a(S_d)$ on the tangent space $T_a\widetilde{X}$ is a standard representation of rank ≤ 2, and the group $St_a(S_d)$ is trivial or can be expressed as $S_{J_1\bigsqcup J_2,1}$, where either $2\leqslant|J_1|\leqslant 3$ and $J_2=\emptyset$, or $|J_1|=|J_2|=2$. Since the groups $S_{d,i}$ are conjugate to each other, for each i and for each point $a\in\widetilde{X}$ the group $St_a(S_{d,i})$ has the same properties. Therefore, the intersection $St_a(S_{d,i})\cap A_d$ is generated by the cyclic permutation $(i_1,i_2,i_3)\in A_d$ if $J_1=\{i_1,i_2,i_3\}$ and $J_2=\emptyset$, and it is generated by the product of two transpositions if $J_1 = \{i_1, i_2\}$ and $J_2 = \{i_3, i_4\}$. In the remaining cases ($|J_1| = 2$ and $J_2 = \emptyset$, or $St_a(S_{d,i}) = \{1\}$) the group $St_a(S_{d,i}) \cap A_d$ is trivial. This implies that if $St_a(S_{d,1}) = S_{J_1 \bigsqcup J_2,1}$, where $|J_1| = 3$ and $J_2 = \emptyset$, then $St_a(S_{d,i}) = S_{J_1' \bigsqcup J_2',i}$, where $J_1' = J_1$ and $J_2' = \emptyset$, for each i. Similarly, if $St_a(S_{d,1}) = S_{J_1 \bigsqcup J_2,1}$, where $J_1 = \{i_1, i_2\}$ and $J_2 = \{i_3, i_4\}$, then $St_a(S_{d,i}) = S_{J_1' \bigsqcup J_2',i}$, where $J_1' = J_1$ and $J_2' = J_2$, for each i. If $St_a(S_{d,1}) = S_{J_1 \bigsqcup J_2,1}$, where $J_1 = \{i_1, i_2\}$ and $J_2 = \emptyset$, then for each i either $St_a(S_{d,i})$ is trivial or $St_a(S_{d,i}) = S_{J_1' \bigsqcup J_2',i}$, where $|J_1'| = 2$ and $J_2' = \emptyset$.

Let us examine more in details the case $St_a(S_{d,i}) = S_{J_1 \bigsqcup J_2,i}$ for each i, where $|J_1| = 3$ and $J_2 = \emptyset$. Denote by y the cyclic permutation $(i_1, i_2, i_3) \in A_d$, where $J_1 = \{i_1, i_2, i_3\}$. The group $St_a(G)$ contains a subgroup F_3 generated by three elements $x_1 = \sigma_{1,(i_1,i_2)}$ (conjugate to σ), $x_2 = \sigma_{2,(i_1,i_2)}$ (conjugate to τ), and y. It is easy to see that F_3 has the following presentation:

$$\begin{aligned} F_3 = \langle x_1, x_2, y \mid \ & x_1^2 = x_2^2 = (x_1x_2)^r = y^3 = [y, x_1x_2] = [y, x_2x_1] = 1, \\ & x_1^{-1}yx_1 = y^{-1},\ x_2^{-1}yx_2 = y^{-1} \rangle \end{aligned} \tag{15}$$

(recall that $r \geq 3$). The group F_3 is non-abelian. It contains a maximal normal subgroup N_3 generated by y and $z = x_1x_2$. This subgroup is isomorphic to the direct product $\langle y \rangle \times \langle z \rangle$ of two cyclic groups of orders 3 and r, respectively. On the other hand, according to well known properties of finite subgroups of $GL(2, \mathbb{C})$ (see, for example, [3]) the quotient of F_3 by its center should be either a cyclic group, or a dihedral group, or A_4, or A_5, or S_5. But, $F_3 = (\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/r\mathbb{Z}) \rtimes \mathbb{Z}/2\mathbb{Z}$ and thus it has the trivial center, since r is odd. All together, these arguments imply that r should not be divisible by 3 (recall that the branch curve does have at least one cuspidal point, see Subsection 3.2) and F_3 should be isomorphic to the dihedral group $D_{r'}$, $r' = 3r$.

In addition, once more due to the known classification of conjugacy classes of finite subgroups of $GL(2, \mathbb{C})$, the action of $F_3 \simeq D_{3r}$ near the point a is isomorphic to the unique 2-dimensional linear representation of D_{3r}. In particular, at a neighborhood of a the fixed points of $\sigma_{i,(i_1,i_2)}$, $i_1, i_2 \in J_1$, form a smooth curve, which we denote by $\widetilde{R}_{i,(i_1,i_2)}$, and any two of them, $\widetilde{R}_{i,(i_1,i_2)}$ and $\widetilde{R}_{i',(i_1',i_2')}$ with $(i, (i_1, i_2)) \neq (i', (i_1', i_2'))$, are distinct and meet each other at a transversally.

Next, let us examine the case $St_a(S_{d,i}) = S_{J_1 \bigsqcup J_2,i}$, where $J_1 = \{i_1, i_2\}$ and $J_2 = \{i_3, i_4\}$. The group $St_a(G)$ contains a subgroup $F_{2,2}$ generated by three elements $x_1 = \sigma_{1,(i_1,i_2)}$ (conjugate to σ), $x_2 = \sigma_{2,(i_1,i_2)}$ (conjugate to τ), and $y = (i_1, i_2)(i_3, i_4) \in A_d$. It is easy to see that $F_{2,2}$ has the following presentation:

$$F_{2,2} = \langle x_1, x_2, y \mid x_1^2 = x_2^2 = (x_1x_2)^r = y^2 = [y, x_1] = [y, x_2] = 1 \rangle \tag{16}$$

(recall that $r \geq 3$ and r is odd). The group $F_{2,2}$ contains a maximal normal subgroup $N_{2,2}$ generated by y and $z = x_1x_2$. This subgroup is isomorphic to the direct product $\langle y \rangle \times \langle z \rangle$ of two cyclic groups of orders 2 and r, respectively. Therefore, $F_{2,2}$ is isomorphic to the dihedral group D_{2r}. According to the classification of conjugacy classes of finite subgroups of $GL(2,\mathbb{C})$, the action of $F_{2,2}\simeq D_{2r}$ near the point a is isomorphic to the unique 2-dimensional linear representation of D_{2r}. In particular, similar to the previous case, for any $i_1,i_2,i_1',i_2'\in J_1\bigsqcup J_2$, the curves $\widetilde{R}_{i,(i_1,i_2)}$ and $\widetilde{R}_{i',(i_1',i_2')}$ with $(i,(i_1,i_2))\neq(i',(i_1',i_2'))$ are distinct and meet each other at a transversally.

Finally, consider the case $St_a(S_{d,1})=S_{J_1\bigsqcup J_2,1}$ and $St_a(S_{d,2})=S_{J_1'\bigsqcup J_2',2}$, where $J_1=\{j_1,j_2\}$, $J_2=\emptyset$, $J_1'=\{j_3,j_4\}$, $J_2'=\emptyset$. Let us show that either $J_1\cap J_1'=\emptyset$ and then $St_a(G)$ contains a subgroup isomorphic to D_r, or $J_1=J_1'$.

Indeed, if $|J_1\cap J_1'|=1$, then we can assume that $j_2=j_3$ so that $\sigma_{2,(j_3,j_4)}=\eta^{-1}\sigma_{2,(j_1,j_2)}\eta$, where $\eta=(j_4,j_2,j_1)\in A_d$. We have $\eta^3=1$ and

$$\begin{aligned}(\sigma_{1,(j_1,j_2)}\sigma_{2,(j_3,j_4)})^r = \ & (\sigma_{1,(j_1,j_2)}\eta\sigma_{2,(j_1,j_2)}\eta^{-1})^r=(\sigma_{1,(j_1,j_2)}\sigma_{2,(j_1,j_2)}\eta)^r=\\ & (\sigma_{1,(j_1,j_2)}\sigma_{2,(j_1,j_2)})^r\eta^r=\eta^r=\eta^{\pm1},\end{aligned}$$

since r is not divisible by 3. Therefore, $\eta\in St_a(G)$ which contradicts the assumption that $St_a(S_{d,1})=S_{J_1\bigsqcup J_2,1}$.

If $J_1\cap J_1'=\emptyset$, then $\sigma_{2,(j_3,j_4)}=\eta^{-1}\sigma_{2,(j_1,j_2)}\eta$, where $\eta=(i_1,i_3)(i_2,i_4)\in A_d$. We have $\eta^2=1$ and

$$\begin{aligned}&(\sigma_{1,(j_1,j_2)}\sigma_{2,(j_3,j_4)})^r=(\sigma_{1,(j_1,j_2)}\eta\sigma_{2,(j_1,j_2)}\eta^{-1})^r=\\ &(\sigma_{1,(j_1,j_2)}\sigma_{2,(j_1,j_2)}\eta^2)^r=(\sigma_{1,(j_1,j_2)}\sigma_{2,(j_1,j_2)})^r=1.\end{aligned}$$

Therefore, the subgroup H_2 generated in G by $\sigma_{1,(j_1,j_2)}$ and $\sigma_{2,(j_3,j_4)}$ is isomorphic to D_r. Note in addition that for each i the group $St_a(S_{d,i})=G\cap S_{d,i}$ is non-trivial and it is contains either $\sigma_{i,(j_1,j_2)}$ or $\sigma_{i,(j_3,j_4)}$.

3.6. **Case** (2)**. Geometrical part.** We have $A_d=S_{d,i}\cap S_{d,j}$ for $i\neq j$. Denote by $X_i=\widetilde{X}/S_{d-1,i}$, $\mathbb{P}^2=\widetilde{X}/S_{d,i}$, and $X_0=\widetilde{X}/A_d$ the corresponding quotient spaces. They can be arranged in the following commutative diagram (a fragment of which is drown below) in which the morphisms f_{0i}, $i=1,\dots,r$, are of degree two.

$$\begin{tikzcd}
& \widetilde{X} \arrow[dl, "h_i"'] \arrow[d, "h_0"] \arrow[dr, "h_j"] & \\
X_i \arrow[d, "f_i"'] & X_0 \arrow[dl, "f_{0i}"] \arrow[dr, "f_{0j}"'] & X_j \arrow[d, "f_j"] \\
\mathbb{P}^2 & & \mathbb{P}^2
\end{tikzcd}$$

Since each f_{0i} is conjugate to the covering $f_{01}:X_0\to\mathbb{P}^2$ branched along $B_1=B$, the covering f_{0i} is branched over the points of a cuspidal curve $B_i\subset\mathbb{P}^2$ having the same degree and the same number of nodes and cusps as B has. Denote by $R_{i,0}\subset X_0$ the ramification curve of the covering f_{0i}.

The group D_r acts on X_0. The image of $\sigma_{i,(j_1,j_2)} \in S_{d,i}$ (see subsection 3.5) under the natural epimorphism of G to D_r coincides with σ_i. Therefore, the fixed point set of σ_i coincides with $R_{i,0}$.

The surface X_0 is a normal projective variety. The set of its singular points coincides with $f_{01}^{-1}(Sing B_1)$: over each cusp we have a singular point of type A_2, and over each node we have a singular point of type A_1. Therefore, for any i, $1 \leqslant i \leqslant r$, all the points of $f_{01}^{-1}(Sing B_1)$ belong to $R_{i,0}$. In addition, as it follows from an observation made at the end of subsection 3.5, if two curves, say $R_{1,0}$ and $R_{2,0}$, meet at a nonsingular point $b \in X_0$, then the point b is common for all the curves $R_{i,0}$ and at this point each pair of curves, $R_{i,0}$ and $R_{j,0}$, meet transversally. Denote by e the number of nonsingular points $b \in X_0$ common to all the curves $R_{i,0}$.

Let $\nu : Z \to X_0$ be the minimal resolution of singularities. The exceptional divisor of this resolution look as follows:

$$\overline{E} = \nu^{-1}(Sing X_0) = \bigcup_{k=1}^{c} (\overline{E}_{1,s_k} \cup \overline{E}_{2,s_k}) \cup \bigcup_{l=2c+1}^{2c+n} \overline{E}_{s_l},$$

where $\overline{E}_{1,s_k}$, $\overline{E}_{2,s_k}$, $(1 \leqslant k \leqslant c)$, are the irreducible components of $\overline{E}$ contracted to the cusp s_k of X_0, and $\overline{E}_{s_l}$ $(l = 2c+1, \dots, 2c+n)$ is the irreducible component of $\overline{E}$ contracted to the node s_l of X_0.

Since f_{0i} is a double covering branched along a cuspidal curve B_{i0}, the above minimal resolution of singularities fits into the following commutative diagram

$$\begin{array}{ccc} Z & \xrightarrow{\nu} & X_0 \\ {\scriptstyle \overline{f}_{0i}}\downarrow & & \downarrow{\scriptstyle f_{0i}} \\ \overline{\mathbb{P}}^2 & \xrightarrow[\nu_i]{} & \mathbb{P}^2, \end{array}$$

where ν_i blows up once each of the singular points of B_i, and $\overline{f}_{0i}$ is a two-sheeted covering of $\overline{\mathbb{P}}^2$ branched over the strict transform $\overline{B}_i \subset \overline{\mathbb{P}}^2$ of B_i.

Here is a more explicit description convenient for counting intersection numbers.

Let s be a cusp of B_i. Denote by $E \subset \overline{\mathbb{P}}^2$ the exceptional curve $\nu_i^{-1}(s) = \mathbb{P}^1$ of ν_i lying over s. Then $\overline{B}_i \subset \overline{\mathbb{P}}^2$ meets E at one point, it is non-singular at this point and has there a simple tangency to E. The lift $\overline{R}_{i,0} = f_{0i}^{-1}(\overline{B}_i)$ is the ramification curve of f_{0i}, it is non-singular and coincides with the proper transform of $R_{i,0}$. While $f_{0i}^{-1}(E)$ splits into $\overline{E}_{1,s} \cup \overline{E}_{2,s} \subset Z$, a union of two smooth curves intersecting transversally, so that

$$(\overline{E}_{1,s}^2)_Z = (\overline{E}_{2,s}^2)_Z = -2$$

and

$$(\overline{E}_{1,s}, \overline{E}_{2,s})_Z = (\overline{E}_{1,s}, \overline{R}_{i,0})_Z = (\overline{E}_{2,s}, \overline{R}_{i,0})_Z = 1.$$

Let s be a node of B_i. Then $\overline{B}_i \subset \overline{\mathbb{P}}^2$ meets the exceptional curve $E = \nu_i^{-1}(s)$ at two points, $\overline{B}_i$ is nonsingular at these points, and intersects E transversally. The lift $f_{0i}^{-1}(E) = \overline{E}_s \subset Z$ is the exceptional curve of ν, it meets $\overline{R}_{i,0} = f_{0i}^{-1}(\overline{B}_i)$, which is non-singular, transversally, so that $(\overline{E}_s^2)_Z = -2$ and $(\overline{E}_s, \overline{R}_{i,0})_Z = 2$.

Now, let us show that $(\overline{R}_{i,0}, \overline{R}_{j,0})_Z$ does not depend on i and j if $i \neq j$. Consider the commutative diagram

$$\begin{array}{ccc} \overline{X} & \xrightarrow{\overline{\mu}} & \widetilde{X} \\ {\scriptstyle \overline{h}_0}\downarrow & & \downarrow{\scriptstyle h_0} \\ \overline{Z} & \xrightarrow[\mu]{} & X_0, \end{array}$$

where $\overline{X} = \widetilde{X} \times_{X_0} \overline{Z}$ is the fibre product of $\widetilde{X}$ and $\overline{Z}$ over X_0, and $\mu : \overline{Z} \to X_0$ is the composition of ν and the blow ups of the all intersection points of (-2)-curves $\overline{E}_{1,s_k}$ and $\overline{E}_{2,s_k}$ $(k = 1, \dots, c)$ lying in Z (these curves are those components of divisor $\overline{E}$ which are constructed by ν to the cusps of X_0). Denote by $\overline{E}_{1,2,s_k} \subset \overline{Z}$ the exceptional curve lying over the point $\overline{E}_{1,s_k} \cap \overline{E}_{2,s_k}$ and, to simplify the notation, denote by the same symbols, $\overline{R}_{i,0}$, $\overline{E}_{1,s_k}$ and $\overline{E}_{2,s_k}$, and $\overline{E}_{s_l}$, the strict transforms in $\overline{Z}$ of the curves $\overline{R}_{i,0}$, $\overline{E}_{1,s_k}$ and $\overline{E}_{2,s_k}$ $(k = 1, \dots, c)$, and $\overline{E}_{s_l}$ $(l = 2c+1, \dots, 2c+n)$ in Z.

It is easy to see that $\overline{h}_0$ is a Galois covering branched along the curves $\overline{E}_{\cdot,s_k}$ and $\overline{E}_{s_l}$. The ramification indices over the curves $\overline{E}_{\cdot,s_k}$ are equal to 3 (cf. local calculations in [13], §2) and the ramification indices over the curves $\overline{E}_{s_l}$ are equal to 2. The morphism $\overline{\mu}$ blows up once each of the points lying over the nodes of X_0 and performs three blow ups at each of the points lying over the cusps of X_0. Therefore, the strict transforms $\overline{\mu}^{-1}(\widetilde{R}_{i,(j_1,j_2)})$, $1 \leq i \leq r$, $1 \leq j_1, j_2 \leq d$, pairwise do not meet. But, $\bigcup_{j_1,j_2} \overline{\mu}^{-1}(\widetilde{R}_{i,(j_1,j_2)}) = \overline{h}_0^{-1}(\overline{R}_{i,0})$. Hence, after the blow down of all curves $\overline{E}_{1,2,s_k}$, we get $(\overline{R}_{i,0}, \overline{R}_{j,0})_Z = c + e$ for $i \neq j$ and these intersection numbers do not depend on i and j. Note also that the intersection numbers of the curves $\overline{R}_{j,0}$ and an irreducible component of $\overline{E}$ do not depend on j.

The action of D_r on X_0 lifts to an action on Z. The curve $\overline{R}_{i,0} \subset Z$ (respectively, $R_{i,0} \subset X_0$) is the set of fixed points of $\sigma_i \in D_r$. Since $\sigma_i^{-1}\sigma_j\sigma_i \neq \sigma_j$ for $j \neq i$, the curve $\sigma_i(\overline{R}_{j,0}) \neq \overline{R}_{j,0}$ (respectively, $\sigma_i(R_{j,0}) \neq R_{j,0}$) for $j \neq i$. In particular, $R_{3,0} = \sigma_1(R_{2,0}) \neq R_{2,0}$ and hence $R_{2,0} + R_{3,0} = f_{01}^{-1}(D)$ for some curve $D \subset \mathbb{P}^2$.

Since D_r acts transitively on the set of curves $\overline{R}_{i,0}$ (respectively, on the set of curves $R_{i,0}$), we have $(\overline{R}_{1,0}^2)_Z = (\overline{R}_{2,0}^2)_Z = (\overline{R}_{3,0}^2)_Z$ and it was shown that

$$(\overline{R}_{1,0}, \overline{R}_{2,0})_Z = (\overline{R}_{1,0}, \overline{R}_{3,0})_Z = (\overline{R}_{2,0}, \overline{R}_{3,0})_Z.$$

Denote by L the subspace of the Neron-Severi group $NS(Z)\otimes\mathbb{Q}$ orthogonal (via the intersection form) to the subspace V_E generated by $\overline{E}_{1,s_k}$, $\overline{E}_{2,s_k}$, $k=1,\dots,c$, and $\overline{E}_{s_l}$, $l=2c+1,\dots,2c+n$. The intersection form is negative definite on V_E. Therefore, by Hodge index theorem, the intersection form on L has signature $(1,\dim L-1)$.

In what follows we make, first, certain calculations of intersection numbers of some divisors in L. We project the Neron-Severi classes of the divisors $\overline{R}_{i,0}$ to L, denote the projections by $(\overline{R}_{i,0})_L$ and their intersections in L by $(\overline{R}_{i,0}\cdot\overline{R}_{j,0})_L$ (the latter numbers, indeed, are equal to the corresponding $\mathbb{Q}$-intersection numbers on the $\mathbb{Q}$-variety X_0).

Observe, first, that $f_{i,0}^*B_i=2R_{i,0}$ and $\nu^*R_{i,0}=2\overline{R}_{i,0} \mod L$. Thus,

$$(\overline{R}_{i,0}^2)_L=(\overline{R}_{j,0}^2)_L=\frac{1}{2}(\deg B)^2>0$$

for all i,j. Let $(\overline{R}_{2,0})_L=\lambda(\overline{R}_{1,0})_L+T$, where $T\in L$ is orthogonal to $(\overline{R}_{1,0})_L$. We have

$$(\overline{R}_{2,0}\cdot\overline{R}_{1,0})_L=\lambda(\overline{R}_{1,0}^2)_L=(\overline{R}_{3,0}\cdot\overline{R}_{1,0})_L=(\overline{R}_{2,0}\cdot\overline{R}_{3,0})_L,$$

since the intersection numbers of the curves $\overline{R}_{j,0}$ and an irreducible component of $\overline{E}$ do not depend on j and the intersection numbers $(\overline{R}_{i,0},\overline{R}_{j,0})_Z$ also do not depend on i and j for $i\neq j$.

Next, $\overline{R}_{2,0}+\overline{R}_{3,0}$ coincides with $\nu^*(f_{01}^*(D))$. Therefore, $(\overline{R}_{2,0}+\overline{R}_{3,0})_L$ is proportional to $(\overline{R}_{1,0})_L=\frac{1}{2}\nu^*(f_{01}^*(B_1))$. Hence $(\overline{R}_{3,0})_L=\lambda(\overline{R}_{1,0})_L-T$ and $\lambda>0$. We have $(\overline{R}_{2,0}^2)_L=\lambda^2(\overline{R}_{1,0}^2)_L+T^2=(\overline{R}_{1,0}^2)_L$, therefore

$$T^2=(1-\lambda^2)(\overline{R}_{1,0}^2)_L\leq 0.$$

Hence $\lambda\geq 1$ and $\lambda=1$ if and only if $T^2=0$, that is, if and only if $T=0\in L$.

Since $(\overline{R}_{2,0}\cdot\overline{R}_{3,0})_L=(\overline{R}_{2,0}\cdot\overline{R}_{1,0})_L$, we have

$$\lambda^2(\overline{R}_{1,0}^2)_L-T^2=\lambda(\overline{R}_{1,0}^2)_L$$

and

$$T^2=(\lambda^2-\lambda)(\overline{R}_{1,0}^2)_L\leq 0,$$

which yields $\lambda\leq 1$. Combining this with the previous observations, we get $\lambda=1$ and $T=0$, which implies that $(\overline{R}_{i,0})_L=(\overline{R}_{j,0})_L$ for any i,j. It allows us to conclude that $\deg D=\deg B$, since $2\deg D=((R_{2,0}+R_{3,0})^2)_L=4(R_{2,0}^2)_L=2\deg B$.

Therefore, we have

$$\begin{aligned}(\overline{R}_{1,0}^2)_Z= {}& (\tfrac{1}{2}\nu_1^*(B)-\textstyle\sum(E_{1,s_k}+E_{2,s_k})-\sum E_{s_l})^2=\\ & \tfrac{1}{2}(\deg B)^2-2c-2n=\tfrac{1}{2}(\overline{R}_{1,0},\overline{R}_{2,0}+\overline{R}_{3,0})_Z\geq c.\end{aligned}$$

Hence
$$(\deg B)^2 - 4n \geq 6c.$$
On the other hand, as is known (see the proof of Lemma 3 in [11]),
$$(\deg B)^2 - 2n \leq 6c$$
for any generic covering of degree $d \geqslant 3$, while due to [12] $n > 0$ if $d > 6$. The contradiction between these bounds eleminates Case (2).

3.7. **Case** (3)**.** The symmetric group $G = S_{d+1}$ acts as the permutation group on the set $I = \{1, \dots, d+1\} \subset \mathbb{N}$. Denote by $H_i = \{\gamma \in S_{d+1} \,|\, \gamma(i) = i\}$, so that our $S_d = H_{d+1}$.

As in the proof of Theorem 0.1, consider the quotient space $\widetilde{X}/G = Y$ and the quotient map $\overline{f} : \widetilde{X} \to Y$. The surface Y is a normal projective variety. The morphism $\overline{f}$ factors through $\widetilde{f}_i$, so that $\overline{f}$ is the composition of the following morphisms
$$\widetilde{X} \xrightarrow{h} X \xrightarrow{f} \mathbb{P}^2 \xrightarrow{r} Y,$$
where r is a finite morphism of degree $d+1$. Since S_d and S_{d+1} have no common normal subgroups, $\overline{f}$ is the Galois expansion of r.

Let $\overline{B} \subset Y$ be the branch locus of r. We have $r(B) = B_1 \subset \overline{B}$. The preimage $r^{-1}(B_1)$ is the union of B and some curve $B' \subset \mathbb{P}^2$.

Since Y is a normal projective surface, we can find a non-singular projective curve $L \subset Y \setminus Sing(Y)$ which intersects $\overline{B}$ transversally. Let $E = r^{-1}(L)$, $F = f^{-1}(E)$, and $\widetilde{F} = \widetilde{f}^{-1}(E)$. Then $f_{|F} : F \to E$ is a generic covering branched over $B \cap E$, $\widetilde{f}_{|\widetilde{F}} : \widetilde{F} \to E$ is the Galois expansion of the generic covering $f_{|F} : F \to E$ with $Gal(\widetilde{F}/E) = S_d$, and $\overline{f}_{|\widetilde{F}} : \widetilde{F} \to L$ is the Galois expansion of the covering $r_{|E} : E \to L$ with $Gal(\widetilde{F}/L) = S_{d+1}$.

Consider the image $b_1 = r(b)$ of a point $b \in B \cap E$. As in the proof of Theorem 0.1, it is easy to see that $r^{-1}(b_1)$ consists of $d-1$ points belonging to B and one point point belonging to B' (the ramification point of $r_{|E}$). In other words, the covering $r_{|B'} : B' \to B_1$ is of degree 1, the covering $r_{|B} : B \to B_1$ is of degree $d-1$, and $r^*(B_1) = B + 2B'$. In particular,
$$\begin{aligned} \deg B' \cdot \deg E = &\quad (B', E)_{\mathbb{P}^2} = (B_1, L)_Y, \\ \deg B \cdot \deg E = &\quad (B, E)_{\mathbb{P}^2} = (d-1)(B_1, L)_Y. \end{aligned}$$
Therefore
$$\deg B = (d-1) \deg B'.$$
It follows, since $d = m^2 K_X^2$ and $\deg B = m(3m+1)K_X^2$, that $mK_X^2 + 3$ is divisible by $m^2 K_X^2 - 1$, which contradicts the assumption $m^2 K_X^2 \geq 2 \cdot 84^2$.

3.8. **Case** (4)**.** Denote the standard imbedding $S_d \to A_{d+2}$ by α. For each transposition $\sigma \in S_d$, the set $\widetilde{X}_\sigma = \widetilde{X}_{\alpha(\sigma)} \subset \widetilde{X}$ of fixed points of σ is a nonsingular curve. Hence, for each $\tau \in A_{d+2}$ which is conjugate to $\alpha(\sigma)$ the set $\widetilde{X}_\tau$ of fixed points of τ is also a nonsingular curve.

By [12], if $d > 6$, then the branch curve $B \subset \mathbb{P}^2$ has at least one node. Therefore for each product $\eta = \sigma_1\sigma_2$ of two commuting transpositions $\sigma_1, \sigma_2 \in S_d$, the set $\widetilde{X}_\eta$ of fixed points of η is finite and non-empty. It implies that for any η', conjugated to $\alpha(\eta)$ in A_{d+2}, the set $\widetilde{X}_{\eta'}$ is also finite and non-empty. On the other hand, if $\sigma_1 = (j_1, j_2)$ and $\sigma_2 = (j_3, j_4)$, then $\alpha(\sigma_1) = (j_1, j_2)((d+1), (d+2))$, $\alpha(\sigma_2) = (j_3, j_4)((d+1), (d+2))$, and $\alpha(\eta) = (j_1, j_2)(j_3, j_4)$ are conjugate to each other in A_{d+2}. Contradiction.

4. Few applications

4.1. **Deformation stability.** The aim of this subsection is to prove a certain deformation stability of examples given by Theorems 0.1 and 0.2. To state the corresponding results we need to fix few notions. Namely, by a *G-manifold* we will mean a non-singular projective manifold equipped with a regular action of the group G, and by a smooth *G-family*, or *G-deformation*, of G-manifolds we will mean a proper smooth morphism (*i.e.*, a proper submersion) $p : \mathcal{X} \to B$, where $\mathcal{X}$ and B are smooth quasi-projective varieties, and $\mathcal{X}$ is equipped with a regular action of G preserving each fiber of p (preservation of fibers means $p \circ G = p$).

Proposition 4.1. *If one of the fibers of a smooth G-family is the Galois expansion of a generic covering of* $\mathbb{P}^n$, *then the whole family is constituted from the Galois expansions of generic coverings of* $\mathbb{P}^n$.

To prove Proposition 4.1 we need the following Lemma.

Lemma 4.2. *Let* $p : \mathcal{X} \to B$ *be a smooth G-deformation of G-manifolds, G being a finite group. Then for each element* $g \in G$ *we have:*

- (i) *the set* $\mathcal{X}^g = \{x \in X \mid g(x) = x\}$ *of fixed points of* g *is a smooth closed submanifold of* $\mathcal{X}$*;*
- (ii) *the restriction of* p *to* $\mathcal{X}^g$ *is a smooth proper surjective morphism;*
- (iii) *the intersection of* $\mathcal{X}^g$ *and each fibre* $X_t, t \in B$ *of* p *is transversal.*

Proof. As is known, at any point $x \in \mathcal{X}$ fixed by a subgroup H of G, the action of H can be linearized, which means an existence of local analytic coordinates with respect to which the action of H is linear.

Let us resume Cartan's linearization procedure, see [2]. Start from any system of local coordinates $z_1, \dots, z_n$ taking value 0 at a chosen point x fixed by H. For any $h \in H$ denote by h' the linear part of the Taylor expansion (with respect to $z_1, \dots, z_n$) of h at x. Then the change of coordinates defined by the map

$\sigma=\frac{1}{|H|}\sum_{g\in H}(g')^{-1}g$ makes linear the action of H. Namely, it conjugates h and h' for any $h\in H$, since $\sigma\circ h=h'\circ\sigma$ (indeed, $\sigma\circ h=\frac{1}{|H|}\sum_{g\in H}(g')^{-1}g\circ h=\frac{1}{|H|}\sum_{g\in H}h'(g'\circ h')^{-1}g\circ h=\frac{1}{|H|}h'\sum_{e\in H}(e')^{-1}e=h'\circ\sigma$).

This change of coordinates is tangent to identity and it acts as identity on each linear, with respect to $z_1,\dots,z_n$, subspace on which H acts already linear. Therefore, to prove (i) it is sufficient to linearize the action of g (then, in new coordinates the set $\mathcal{X}^g$ becomes linear), and to prove (ii) and (iii) it is sufficient to pick any system of local coordinates at $t\in B$ and include their lift into a system of local coordinates $z_1,\dots,z_n$ (thus one gets for granted the surjectivity of the projection at the level of tangent spaces, $T_x(\mathcal{X}^g)\to T_{p(x)}B$; the properness and surjectivity of $p:\mathcal{X}^g\to B$ then follow from properness of $p:\mathcal{X}\to B$ and closeness of $\mathcal{X}^g$ in $\mathcal{X}$). □

Proof of Proposition 4.1. We will give the proof Proposition 4.1 only in the case $n\le 2$, since the proof in the general case is similar.

Let $X_o, o\in B$, be a fiber of p which is the Galois expansion of a generic covering $X_o\to\mathbb{P}^n$. The ramification locus of this covering is a union of smooth codimension one manifolds $R_{o(i,j)}$, $1\leqslant i<j\leqslant d$; these latter manifolds are the fixed point sets of the transpositions $(i,j)\in S_d$. Due to Lemma 4.2, the fixed point sets $\mathcal{X}^{(i,j)}\subset\mathcal{X}$ of the same transpositions acting in $\mathcal{X}$ are also smooth codimension one manifolds, and for any $t\in B$ the intersection $R_{t(i,j)}=\mathcal{X}^{(i,j)}\cap X_t$ is transversal for each $(i,j)\in S_d$. Besides, if $\mathcal{X}^g\neq\emptyset$ for some $g\in S_d$, $g\neq 1$, then, by Lemma 4.2, $\mathcal{X}^g\cap X_o\neq\emptyset$, and since the action of S_d on X_o is generic, it implies that g is a transposition in case $n=1$ and g is either a transposition, or a product of two disjoint transpositions, or a cyclic permutation of length three in case $n=2$.

Let $n=2$ and g be a cyclic permutation (j_1,j_2,j_3) (the other cases can be treated in a similar way). Let us show that the action of $S_{\{j_1,j_2,j_3\}}$ on $\mathcal{X}$, as well as on each of the fibers X_t, $t\in B$, is generic. Indeed, without loss of generality we can assume that $\dim B=1$. Then, by Lemma 4.2, $\mathcal{X}^g$ is a smooth curve, $\mathcal{X}^{(j_1,j_2)}$ and $\mathcal{X}^{(j_1,j_3)}$ are smooth surfaces, and they all meet the fibers transversally. Since $\mathcal{X}^g\cap X_o=R_{o(j_1,j_2)}\cap R_{o(j_1,j_3)}$, by applying once more Lemma 4.2, we obtain that $\mathcal{X}^{(j_1,j_2)}\cap\mathcal{X}^{(j_1,j_3)}=\mathcal{X}^g$ and $X_t^g=R_{t(j_1,j_2)}\cap R_{t(j_1,j_3)}$ for any $t\in B$. Therefore $\mathcal{X}^g$ (respectively, X_t^g) coincides with the fixed point set under the action of $S_{\{j_1,j_2,j_3\}}$ in $\mathcal{X}$ (respectively, in X_t).

As a result, the action of S_d on $\mathcal{X}$ and in each of $X_t, t\in B$, is generic. Hence, the factor-space $\mathcal{X}/S_d$ is a smooth manifold and the induced morphism $p_1:\mathcal{X}/S_d\to B$ is smooth and proper. Thus, it remains to notice that $X_t/S_d=X_o/S_d=\mathbb{P}^n$ for any $t\in B$, since due to a projective manifold M is isomorphic to $\mathbb{P}^n$ as soon as there exists a C^∞-diffeomorphism $M\to\mathbb{P}^n$ which maps the canonical class to the canonical class (for $n=1$ it is known till Riemann; for $n=2$ one can use the Enriques-Kodaira classification, see [1]; it may be worth mentioning also that, in fact, due to Siu [14] in any dimension every compact complex manifold deformation equivalent to $\mathbb{P}^n$ is isomorphic to $\mathbb{P}^n$). $\square$

Corollary 4.3. *G-varieties like in Theorems* 0.1 *and* 0.2 *form connected components in the moduli space of, respectively, G-curves and G-surfaces of general type. These components are saturated* (see the definition in Introduction). *In dimension* 1*, G-varieties like in Theorem* 0.1 *also form proper subvarieties in the moduli space of curves of general type.*

Proof. The first statement follows from Proposition 4.1. The second statement follows from the first one and Theorems 0.1 and 0.2. The third statement follows from the first one and an observation that any birational transformation of a one-parameter deformation family of genus $g \geqslant 2$ curves which preserves each fiber and regular at the all points of all the fibers, except a finite collection of fibers, extends indeed to a transformation regular everywhere. $\square$

4.2. **Examples of Dif$\neq$Def complex G-manifolds.** Here we consider regular actions of finite groups on complex surfaces and construct diffeomorphic actions which are not deformation equivalent. The idea is to pick diffeomorphic, but not deformation equivalent, surfaces and apply to them Theorem 0.2.

Let X be a rigid non real minimal surface of general type, that is a minimal surface of general type which is stable under deformations and not isomorphic to its own conjugate, $\bar{X}$. Such surfaces are found in [10]. Denote by $Y_1 = \widetilde{X}$ the Galois expansion of a generic m-canonical covering $X \to \mathbb{P}^2$, and by Y_2 its conjugate, $Y_2 = \bar{Y}_1$.

Proposition 4.4. *Let Y_1 and Y_2 be as above and let m be like in Theorem* 0.2*. Then the actions of $S_d = \operatorname{Aut} Y_1 = \operatorname{Aut} Y_2$ on Y_1 and Y_2 are diffeomorphic, but Y_1 and Y_2 are not S_d-deformation equivalent.*

Proof. According to Theorem 0.2, $\operatorname{Aut} Y_1 = \operatorname{Aut} Y_2 = S_d$ where d is the degree of the m-canonical covering $X \to \mathbb{P}^2$. The action of S_d in Y_1 is tautologically diffeomorphic to that in Y_2, since $Y_2 = \bar{Y}_1$.

Assume that $Y_1 = \widetilde{X}$ and $Y_2 = \bar{Y}_1$ are S_d-deformation equivalent. Let $p : \mathcal{X} \to B$ be a smooth S_d-deformation connecting them (the treatment of a chain of deformation families is literally the same). By Proposition 4.1, for any $t \in B$ the covering $X_t \to \mathbb{P}^n$ is generic. Hence, $\mathcal{X}/S_{d-1} \to B$ is a deformation family connecting $X = Y_1/S_{d-1}$ with $\bar{X} = Y_2/S_{d-1}$, which is a contradiction. $\square$

4.3. **Examples of Dif$\neq$Def real G-manifolds.** Here we extend the category of G-manifolds, namely, we consider finite subgroups of the Klein extension of the automorphism group. Let us recall that the Klein group $\mathrm{K}\,l(X)$ of a complex variety X is, by definition, the group consisting of biregular isomorphisms $X \to X$ and $X \to \bar{X}$ (some people call it the group of dianalytic automorphisms). If X is a real manifold and c its real structure, then there is an exact sequence

$$1 \to \langle c \rangle = \mathbb{Z}/2 \to \mathrm{Kl}(X) \to \mathrm{Aut}\, X \to 1.$$

Pick two real Campedelli surfaces (X_1, c_1), (X_2, c_2) constructed in [9], Section 2. As is shown in [9], these particular surfaces are not real deformation equivalent, but their real structures, $c_1 : X_1 \to X_1$ and $c_2 : X_2 \to X_2$, are diffeomorphic.

The Campedelli surfaces are minimal surfaces of general type. Thus, we can consider m-canonical generic coverings $X_1 \to \mathbb{P}^2$ and $X_2 \to \mathbb{P}^2$. Moreover, we can choose these coverings to be real, that is to be equivariant with respect to the usual, complex conjugation, real structure on $\mathbb{P}^2$ and the real structures c_1, c_2 on X_1, X_2. Denote by $\widetilde{X}_1 \to \mathbb{P}^2$ and $\widetilde{X}_2 \to \mathbb{P}^2$ the Galois expansions. The surfaces $\widetilde{X}_1$ and $\widetilde{X}_2$ are real with real structures lifted from $\mathbb{P}^2$.

Proposition 4.5. *The Klein groups* $\mathrm{Kl}(\widetilde{X}_1)$ *and* $\mathrm{Kl}(\widetilde{X}_2)$ *are isomorphic, their actions are diffeomorphic, while there exists no equivariant deformation connecting* $(\widetilde{X}_1, \mathrm{Kl}(\widetilde{X}_1))$ *with* $(\widetilde{X}_2, \mathrm{Kl}(\widetilde{X}_2))$.

Proof. As in the proof of Proposition 4.4, the non existence of an equivariant deformation is a straightforward consequence of Proposition 4.1.

Due to Theorem 0.2 to prove the other two statements, it is sufficient to construct a real diffeomorphism $\widetilde{X}_1 \to \widetilde{X}_2$ respecting the Galois action. In its turn, to reach this task, it is sufficient to construct real diffeomorphisms $X_1 \to X_2$, $\mathbb{P}^2 \to \mathbb{P}^2$ commuting with the initial m-canonical generic coverings, $X_1 \to \mathbb{P}^2$ and $X_2 \to \mathbb{P}^2$.

Now, we need to recall some details of the construction of surfaces X_1, X_2 from [9]. The construction starts from a real one-parameter family of Campedelli line arrangements $\mathcal{L}(t)$, $t \in T = \{|t| \leqslant 1, t \in \mathbb{C}\}$, consisting of seven lines $L_1(t), \dots L_7(t)$ labeled by non-zero elements $\alpha \in (\mathbb{Z}/2\mathbb{Z})^3$. The lines are real for real values of t, and the family performs a triangular transformation at $t = 0$ (see the definition of triangular transformation in [9]). Consider the Galois covering $Y \to \mathbb{P}^2 \times T$ with Galois group $(\mathbb{Z}/2\mathbb{Z})^3$ branched in $\sum_1^7 \mathcal{L}_i, \mathcal{L}_i = \{(p,t) \in \mathbb{P}^2 \times T : p \in L_i(t)\}$ and defined by the chosen labelling of the lines. The fibers Y_t under the projection of Y to T are nonsingular Campedelli surfaces for generic t, in particular, for any $t \neq 0$ close to 0. The surfaces X_1, X_2 we are interested in are given by Y_t with, respectively, positive and negative t close to 0. The fiber Y_0 has two singular points; each of these points is a so-called $T(-4)$-singularity; these points are not real but complex conjugate to each other. For each nonsingular Campedelli surface Y_t, $t \neq 0$, and each $1 \leqslant i \leqslant 7$, the pull-back $L_i^*(t) \subset Y_t$ of $L_i(t) \subset \mathbb{P}^2$ represents the bi-canonical class, $[L_i^*(t)] = 2K_{Y_t}$.

Since $Y \to \mathbb{P}^2 \times T$ is a finite morphism, the divisors $E_m = m[\mathcal{L}_i^*]$ are relatively very ample for any m sufficiently large (and any i), see for example [6]. Pick such an integer m and consider the real, relative to T, imbedding of Y into $\mathbb{P}^N \times T$ defined by the linear system $|E_m|$ (to show existence of relative to T global sections, one can twist E_m by a pull-back of a very ample divisor on T). Note, that according to $[L_i^*(t)] = 2K_{Y_t}$ it defines $2m$-canonical imbedding of Campedelli surfaces Y_t to $\mathbb{P}^N$. As it follows from Theorem 0.1 in [13], if $m \geq 5$, then the projection $Y_t \to \mathbb{P}^2$ from a generic $\mathbb{P}^{N-3}$ is a generic covering for any but finite number of $t \in T$, in particular, for any $t \neq 0$ close to 0. For a real $\mathbb{P}^{N-3}$ the projection $Y \to \mathbb{P}^2 \times T$ is real and, for a sufficiently generic real $\mathbb{P}^{N-3}$, the two singular points of Y_0 project to two distinct complex conjugate points. Denote the singular points by y, $\bar{y}$, and their projections by b, $\bar{b}$. One can show, repeating word-by-word the arguments from [13], that the projection $Y_0 \to \mathbb{P}^2$ is generic everywhere (generic at singular point means that the fibre of projection passing through the singular point $y \in Y_0$ (resp. $\bar{y} \in Y_0$) is in generic position with respect to the tangent cone $C_b Y_0$ (resp. $C_{\bar{b}} Y_0$)).

Restrict, now, our attention to small values of t. The coverings $Y_t \to \mathbb{P}^2$ are generic for $t \neq 0$, and the branching curves B_t are cuspidal. For $t = 0$, the branching curve B_0 is cuspidal everywhere, except two distinct, complex conjugate, points, b and $\bar{b}$. Cut out small, Milnor, complex conjugate balls $V(b), V(\bar{b})$ around these two points. Use a family of Morse-Lefschetz diffeomorphisms to complete the isotopy $B_{te^{i\phi}} \setminus (V(b) \cup V(\bar{b}))$ by an isotopy inside $V(b)$, and then complete it by a complex conjugate isotopy inside $V(\bar{b})$. This isotopy provides an equivariant diffeomorphism between Galois coverings branched in B_t and, respectively, B_{-t}. The Morse-Lefschetz diffeomorphisms can be seen as transformations acting identically on the complement of $V(b) \cup V(\bar{b})$. Therefore, the epimorphism $\pi_1(\mathbb{P}^2 \setminus B_t) \to S_d$ defining the Galois coverings is not changing, so that the constructed diffeomorphism between the Galois coverings acts from $\widetilde{X}_1$ to $\widetilde{X}_2$ and it is equivariant with respect to the Galois action. It is also equivariant with respect to the real structure. Thus, it remains to notice that due to Theorem 0.2 the full automorphism groups $\mathrm{Aut}(\widetilde{X}_1)$ and $\mathrm{Aut}(\widetilde{X}_2)$ coincide with the Galois group. $\square$

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