THE ASYMPTOTIC DIMENSION OF A CURVE GRAPH IS FINITE

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Abstract. We find an upper bound for the asymptotic dimension of a hyperbolic metric space with a set of geodesics satisfying a certain boundedness condition studied by Bowditch. The primary example is a collection of tight geodesics on a curve graph. We use this to conclude that a curve graph has finite asymptotic dimension. It follows then that a curve graph has property A_1. We also show that the asymptotic dimension of mapping class groups of surfaces with genus ≤ 2 is finite.

1. Introduction

The asymptotic dimension of a metric space was defined by Gromov [17, page 29] as the large-scale analog of Lebesgue covering dimension. Gromov originally denoted asymptotic dimension asdim_+, but the notation asdim has become more standard. There are several equivalent definitions of asymptotic dimension of a metric space (see [2, 31]), but we will only use the following one in terms of *r-multiplicity* of a cover. In a metric space X, let $N(x;r)$ denote the r-ball centered at x in X. Define $\operatorname{asdim} X \leq n$ if for every (large) $r \in \mathbb{N}$ there exists a cover $\{U_i\}_i$ of X by uniformly bounded subsets of X so that $\sharp\{i \mid N(x;r) \cap U_i \neq \emptyset\} \leq n+1$ for all $x \in X$. Asymptotic dimension is a quasi-isometry invariant; moreover it is a coarse invariant (see below). For more information on asymptotic dimension see the survey article [2] or Roe's book [31], where asymptotic dimension is defined for coarse spaces.

A metric space is said to be *proper* if closed, bounded sets are compact. Most of the interest in asymptotic dimension arose following a result of Yu [39] saying that the coarse Baum-Connes conjecture holds for proper metric spaces with finite asymptotic dimension. Applying the so-called descent principle, he concludes that the Novikov conjecture holds for finitely generated groups with finite classifying space and finite asymptotic dimension. This result has since been improved upon by Yu [40] and Kasparov-Yu [26].

The notion of a δ-hyperbolic space is due to Gromov [16] and has been studied extensively. A geodesic triangle in a metric space X is said to be *δ-thin* if each of its sides is contained in the δ-neighborhood of the union of the other two sides. A geodesic space X will be called *δ-hyperbolic* if every geodesic triangle in X is δ-thin. By a *hyperbolic metric space* we mean a geodesic metric space which is δ-hyperbolic for some $\delta \geq 0$. We will also say that a finitely generated group is *hyperbolic* if its Cayley graph is a δ-hyperbolic metric space for some $\delta \geq 0$. Observe that this property is a quasi-isometry invariant – although the value of δ is not – so being hyperbolic is invariant of the choice of finite generating set.

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Gromov remarked [17, page 31] that finitely generated hyperbolic groups have finite asymptotic dimension. Recently Roe [32] improved on this result, showing that a hyperbolic geodesic metric space with bounded growth (see below) has finite asymptotic dimension. This also follows from a result of Bonk and Schramm [6].

A subset A of a metric space X is *r-discrete* if $d(a, a') \geq r$ for all $a \neq a'$ in A. The *r-capacity* of a set $Y \subset X$ is the maximal cardinality of an r-discrete set in Y. A metric space X has *bounded geometry* if there is an $r > 0$ and a function $c : [0, \infty) \to [0, \infty)$ so that for any $x \in X$ the r-capacity of $N(x; R)$ does not exceed $c(R)$. For a discrete metric space, this simply means that the cardinality of any R-ball is bounded by $c(R)$.

A metric space X has *bounded growth at some scale* if there are constants r and R with $R > r > 0$, and $N \in \mathbb{N}$ such that every open ball of radius R in X can be covered by N open balls of radius r. A metric space with bounded geometry has bounded growth: take r from the definition of bounded geometry, take $R > r$ and put $N = c(R)$. Clearly the Cayley graph of a finitely generated hyperbolic group has bounded geometry, and so has bounded growth.

Below we consider the asymptotic dimension of hyperbolic geodesic graphs which are not locally finite but satisfy some boundedness condition in terms of geodesics. Although the property is stated locally, it is indeed a global condition. For example, an infinite-valence tree does not satisfy the bounded growth condition, but does satisfy our boundedness condition.

We mentioned that asymptotic dimension is a quasi-isometry invariant, but we will need a stronger statement. It is also a coarse invariant [31]. Let X and Y be metric spaces. A map $f : X \to Y$ is *bornologous* if for every $R > 0$ there is an $S > 0$ such that $d(f(x), f(x')) < S$ whenever $d(x, x') < R$. The map f is *metrically proper* if $f^{-1}(B)$ has bounded diameter in X for each bounded subset $B \subset Y$. A *coarse map* is a map $f : X \to Y$ that is bornologous and metrically proper. A coarse map $f : X \to Y$ is a *coarse equivalence* between X and Y if there is a coarse map $g : Y \to X$ and a constant $K \geq 0$ so that $d(fg(y), y) \leq K$ and $d(gf(x), x) \leq K$ for all $x \in X$ and $y \in Y$.

As mentioned above, one of the main applications of finite asymptotic dimension is to the coarse Baum-Connes and Novikov conjectures. The coarse Baum-Connes conjecture is a conjecture about proper metric spaces. Many of the examples we consider in this paper are not proper and moreover, they are not even coarsely equivalent to proper metric spaces, so the coarse Baum-Connes conjecture does not apply. In fact, Yu showed in [39] that the requirement of bounded geometry in the coarse Baum-Connes conjecture could not be dropped, by exhibiting a counter-example to the conjecture that admits a uniform embedding into Hilbert space, but does not have bounded geometry. In [40], Yu defines a property of discrete metric spaces called "property A" guaranteeing the existence of a large-scale (or uniform) embedding into Hilbert space. The existence of this embedding implies the coarse Baum-Connes conjecture for discrete spaces with bounded geometry. As a consequence, finitely generated groups with this property satisfy the Novikov conjecture. Yu's property A is a coarse invariant [40, 37].

In the fourth section of this paper we show that spaces with our boundedness condition have property A in the sense of Tu [37], which we call property A_1. This property reduces to Yu's property A when the spaces are discrete with bounded geometry. Property A_1 for spaces with our boundedness condition follows from the fact that they have finite asymptotic dimension and the proof of a theorem of Higson-Roe [20, Lemma 4.3].

This paper is organized in the following way. In the second section we state our boundedness criterion and find an upper bound for the asymptotic dimension of hyperbolic metric spaces with a set of geodesics satisfying this condition. In the third section we apply this result to the curve complex of a compact orientable surface and a Farey graph. In the fourth section we show that hyperbolic spaces with a set of geodesics satisfying our boundedness condition have property A_1. We change tack in the fifth section and derive upper bounds for the asymptotic dimension of certain mapping class groups (assuming finite dimensionality of others). In particular, we are able to show that the mapping class group of a surface with genus at most 2 is finite; moreover combining our upper bounds with a recent result of Dranishnikov's [14], we obtain exact formulas for the asdim of mapping class groups of surfaces with genus 0 or 1. We conclude the paper with a list of open questions about curve complexes and mapping class groups.

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2. Boundedness condition

Let Γ be a δ-hyperbolic graph. Suppose that $\mathcal{L}$ is a set of geodesics in Γ such that for any $a, b \in V(\Gamma)$, there exists $\gamma \in \mathcal{L}$ connecting a, b.

For $a, b \in V(\Gamma)$, let $\mathcal{L}(a, b)$ be the set of all geodesics in $\mathcal{L}$ connecting a, b. We write $G(a, b) = \cup\mathcal{L}(a, b) \subset \Gamma$. Given $A, B \subset V(\Gamma)$, let $\mathcal{L}(A, B) = \cup\{\mathcal{L}(a, b) \mid a \in A, b \in B\}$ and $G(A, B) = \cup\mathcal{L}(A, B)$. We write $G(a, b; r)$ to mean the set $G(N(a; r), N(b; r))$.

In [8], Bowditch considered the following boundedness property, which we call *property B*: there are constants ℓ, k, D such that if $a, b \in V(\Gamma), r \in \mathbb{N}$ and $c \in G(a, b)$ with $d(c, \{a, b\}) \geq r + \ell$, then $G(a, b; r) \cap N(c; k)$ has at most D elements. We also assume that any geodesic connecting $N(a; r)$ to $N(b; r)$ must intersect $N(c; k)$.

We remark that if this property is satisfied, then we can assume that $k = 2\delta$ because Γ is δ-hyperbolic. Also, it should be noted that this property is not a quasi-isometry invariant, as the following example shows.

Example. We shall define a graph X, which is quasi-isometric to $\mathbb{R}$, but does not have property B. For the set of vertices of X take the set $\{(i, j) \mid i \in \mathbb{Z}, j \in \mathbb{Z}\} \cup \{c_i\}_{i \in \mathbb{Z}}$. There is an edge connecting every pair of vertices of the form (i, j) and $(i + 1, j)$, for all i and j and for each i there is an edge between each vertex of the form (i, j) and c_i, for all $j \in \mathbb{Z}$. Now, the set $\{(i, 0) \mid i \in \mathbb{Z}\}$ is an embedded copy of the graph $\mathbb{R}$ with integer vertices, which is quasi-isometric to X. On the other hand, let $\mathcal{L}$ be a set of geodesics so that any two points of X are connected by an $\mathcal{L}$-geodesic. We show $\mathcal{L}$ does not have property B. For each $i > 0$ and each j, there is a unique geodesic in X connecting $(-i, j)$ to (i, j), which has to be in $\mathcal{L}$. The geodesic contains the point $(0, j)$. Note that for each i, all points of the form (i, j), $j \in \mathbb{Z}$ are contained in $N((i, 0); 2)$, and also all points of the form $(-i, j), j \in \mathbb{Z}$ are contained in $N((-i,0);2)$. Therefore, $G((-i,0),(i,0);2)\cap N((0,0);2)$ contains the infinite set $\{(0,j)\mid j\in\mathbb{Z}\}$. Letting $i\to\infty$, we find that $\mathcal{L}$ does not have property B.

Theorem 1. *Let Γ be a δ-hyperbolic graph. Suppose $\mathcal{L}$ is a set of geodesics in Γ such that any two vertices of Γ are joined by a geodesic in $\mathcal{L}$. If $\mathcal{L}$ satisfies property B with constants $\ell, k=2\delta$ and D, then the asymptotic dimension of Γ is at most $2D-1$.*

Proof. Suppose $r\in\mathbb{N}$ is given. We may assume that $\ell\in\mathbb{N}$, and $\ell\geq 10\delta$. Fix $x_0\in\Gamma$. For each $n\in\mathbb{N}$, define

$$A_n=\{x\in\Gamma\mid 10(n-1)(r+\ell)\leq d(x,x_0)\leq 10n(r+\ell)\}.$$

Clearly $\cup_n A_n=\Gamma$. Let

$$S_n=\{x\in\Gamma\mid d(x,x_0)=10n(r+\ell)\}.$$

For each $n\geq 3$, we define subsets $\{U_i^n\}_i$ of A_n such that $\cup_i U_i^n=A_n$ as follows. Write out the elements of S_{n-2} as $\{s_i\}$. Define

$$U_i^n=\{x\in A_n\mid^{\exists}[x,x_0]\in\mathcal{L}\,\text{s.t.}\, s_i\in[x,x_0]\},$$

where $[x,x_0]$ is a geodesic from x to x_0. If $n=1$ or 2, let $U_1^n=A_n$. Clearly, $\cup_i U_i^n=A_n$ for each n, so the collection $\{U_i^n\}_{i,n}$ covers Γ.
Claim: (1) for any i and for each n, $\operatorname{diam} U_i^n\leq 40(r+\ell)$; and
(2) for an $r/2$-ball V in Γ and for each n, there are at most D subsets U_i^n such that $U_i^n\cap V\neq\emptyset$.

To see (1), suppose $x,y\in U_i^n$. If $n=1$ or 2 this is clear, so suppose $n\geq 3$. Then, $d(x,s_i)\leq 20(r+\ell)$, and $d(y,s_i)\leq 20(r+\ell)$, so that $d(x,y)\leq 40(r+\ell)$.

For (2), we observe that this is clear if $n=1$ or 2, so suppose $n\geq 3$. Suppose $U_i^n\cap V\neq\emptyset$ and $U_j^n\cap V\neq\emptyset$. Choose $y_i\in U_i^n\cap V, y_j\in U_j^n\cap V$. Then, $d(y_i,y_j)\leq r$, so that $y_j\in N(y_i,r)$. Therefore, $s_j\in G(y_i,x_0;r)$, because $s_j\in[x_0,y_j]\in\mathcal{L}$. By the δ-hyperbolicity of Γ, $s_j\in N(s_i;2\delta)$, since $d(y_i,s_i)\geq r+\ell$ and $\ell\geq 10\delta$. We find $s_j\in G(y_i,x_0;r)\cap N(s_i;2\delta)$. Fixing i and allowing j to vary we see, by property B (letting $x_0=a, y_i=b, s_i=c$), that there are at most D such s_j satisfying this property since $d(x_0,s_i)\geq r+\ell$. So we have the claim.

It follows that the collection $\{U_i^n\}_{i,n}$ is a uniformly bounded cover with $r/2$-multiplicity $\leq 2D$. The last item is clear from claim (2) because for any $x\in\Gamma$ there can be at most two sets of the form A_n with $A_n\cap N(x;r/2)\neq\emptyset$. Thus, $\operatorname{asdim}\Gamma\leq 2D-1$ as required. □

Examples.
(1) Let T be a tree, which is 0-hyperbolic. Let $\mathcal{L}$ be the set of all geodesics in T. $\mathcal{L}$ satisfies the property B with $D=1,k=0,\ell=0$, so that the asymptotic dimension of T is at most 1. We should note that this was known to Gromov [17, **1.E.**, Example (b)] and an explicit proof appears in Roe's book [31].
(2) Let Γ be the Cayley graph of a word-hyperbolic group G with respect to a finite generating set, S, with $S^{-1}=S$. Let $|S|=s$. Let $\mathcal{L}$ be the set of all geodesics in Γ. Any two points are joined by a geodesic. Suppose that Γ is δ-hyperbolic. For a point $a\in V(\Gamma)$, let $D=|N(a;2\delta)|\leq s^{2\delta}$. Note that D does not depend on a since G acts transitively on Γ by isometries. Then, $\mathcal{L}$ satisfies property B with $k=2\delta,\ell=10\delta$ and D. Thus, we conclude that $\operatorname{asdim}\Gamma\leq 2s^{2\delta}-1$.

3. Curve graph

Let $S = S_{g,p}$ be a compact orientable surface of genus g with p punctures. The *curve complex* of S was defined by Harvey [19] and has been successfully used in the study of mapping class groups, for example in [5, 18, 22]. For our purposes, we will restrict attention to the 1-skeleton of the curve complex, called the *curve graph*. The curve graph X of S is a graph whose vertices are isotopy classes of essential, nonparallel, nonperipheral, simple closed curves in S and two distinct vertices are joined by an edge if the corresponding curves can be realized simultaneously by pairwise disjoint curves. In certain sporadic cases X as defined above is 0-dimensional or empty. (This happens when there are no curve systems consisting of two curves, i.e. $3g-3+p \leq 1$.) In the theorem below these cases are excluded. (One could rectify the situation by declaring that in those cases two vertices are joined by an edge if the corresponding curves can be realized with only one intersection point.) The mapping class group $Mod(S)$ of S (see section 5) acts on X by $f \cdot a = f(a)$. H. Masur and Y. Minsky proved the following remarkable result.

Theorem 2 ([28]). *Let S be a compact orientable surface of genus g with p punctures. Suppose $3g-3+p > 1$. Then the curve graph $X(S)$ is δ-hyperbolic.*

It is known (see [28]) that X is connected, locally infinite, and its diameter is infinite. In general, for $a, b \in X$ there are infinitely many geodesics connecting them, however Masur-Minsky found a set, $\mathcal{L}$, of geodesics, called *tight geodesics* such that $\mathcal{L}(a,b)$ is not empty and finite for any $a, b \in X$.

Bowditch showed the following.

Theorem 3 (Theorem 1.2, [8]). *Let X denote the curve graph of a surface $S_{g,p}$ with $3g-3+p > 1$. Let $\mathcal{L}$ be the set of tight geodesics on X. Then $\mathcal{L}$ satisfies property B for some k, ℓ and D.*

It follows from Theorem 1 that the asymptotic dimension of the curve graph X is at most $2D-1$ so in particular it is finite.

Corollary 4. *Let $S_{g,p}$ be an orientable surface of genus g with p punctures such that $3g-3+p > 1$. Then $\operatorname{asdim} X(S_{g,p}) < \infty$.*

Remark. As we said, the curve graph of S is the 1-skelton of the curve complex of S. They are quasi-isometric, so they have same asymptotic dimension.

There is a modified definition of a curve graph for an exceptional surface. For example, if S satisfies $g = 1, p = 1$ (a once-punctured torus) we put an edge between curves if their intersection number is 1. Then we obtain a Farey graph, X. It is the 1-skelton of a planar 2-complex whose dual tree is a binary tree. The following result was explained to us by Bowditch.

Proposition 5 ([7]). *A Farey graph is quasi-isometric to a regular infinite-valence tree.*

Applying Example 1 from Section 2, we immediately obtain the following result (see the remarks before Question 3 in Section 6 for the lower bound).

Corollary 6. *The asymptotic dimension of a Farey graph is* 1.

From Proposition 5 it is easy to see that a Farey graph is not quasi-isometric to a finite valence tree. (See Proposition 11 for a stronger statement.) It seems that there is no proof of Proposition 5 in the literature. We follow an argument given by Bowditch in lectures at Osaka University [9].

Proof of Proposition 5. Let G be a Farey graph, with $V(G)$ and $E(G)$ denoting the sets of vertices and edges, respectively. Observe that G is a planar graph. For a vertex $v \in V(G)$, label the edges at v as e_i, $i \in \mathbb{Z}$ in such a way that the index i increases monotonically clockwise around v. For each i, the edges e_i and e_{i+1} are contained in a unique triangle (face) at v, which we denote by f_i. For a given i, we produce a connected, planar graph, G', by "cutting" at (v, f_i) as follows. We call (v, f_i) a *cut*. $V(G')$ is obtained from $V(G)$ by "separating" v into two new vertices, v_1 and v_2. Namely, $V(G') = (V(G)\backslash v) \cup \{v_1, v_2\}$. For the set of edges, we have $E(G') = E(G)$ and form G' as follows:

- $e \in E(G)$ is connected to $a \in (V(G)\backslash v)$ if and only if $e \in E(G')$ is connected to $a \in (V(G')\backslash\{v_1, v_2\})$.
- $e \in E(G)$ is connected to v if and only if $e \in E(G')$ is connected to v_1 or v_2 such that if $e = e_j$ $(j \leq i)$ then e is connected to v_1, and if $e = e_j$ $(i+1 \leq j)$ then e is connected to v_2.

Our plan is to cut G at all but two vertices (see below) to obtain a tree which is quasi-isometric to G. Let $\{f_i\}, i \in \mathbb{Z}$ be the set of faces of G. Suppose that for each i, there is a vertex $v_i \in V(G)$ which is contained in f_i such that if $i \neq j$ then $v_i \neq v_j$. We call such collection a *system of cuts*.

First, we show that such a system of cuts must exist for a Farey graph. Let T be the dual tree of a Farey graph G. The set of vertices of T is $\{f_i\}$. Take f_0 as a base vertex. For each $i \neq 0$, the vertex $f_i \in V(T)$ is joined to f_0 in T by a unique geodesic, γ_i, which is an edge path in T. The dual of an edge of T is an edge of G, and γ_i contains only one edge, $e_i \in E(G)$ (as a dual), which is contained in the face f_i. Let $v_i \in V(G)$ be the vertex that is in the face f_i but not in the edge e_i. For f_0, choose $v_0 \in V(G)$ to be any vertex in the face f_0. By the way we chose v_i using the based dual tree T, if $i \neq j$, then $v_i \neq v_j$, so that $\{(v_i, f_i)\}$ is a system of cuts for G. We use this system in what follows. We remark that $\{v_i\} = V(G)\backslash\{u, w\}$, where u, w are the vertices of f_0 which are not v_0.

For every i, we cut G at (v_i, f_i) and obtain a planar graph Γ. Each vertex of Γ has countably many edges. We claim that Γ is a tree and quasi-isometric to G. To see that Γ is quasi-isometric to G, we construct a surjective map $p : \Gamma \to G$ as follows. Since $E(\Gamma) = E(G)$, define $p(e) = e$ for each edge e. There are two types of vertices in Γ: those belonging to $V(G)$ and those produced by separating vertices in $V(G)$. For the first kind, we define $p(v) = v$. For the second kind, if $v \in V(G)$ is separated into $v_1, v_2 \in V(\Gamma)$, then define $p(v_i) = v$. We claim that p is a quasi-isometry. It is clear that for any $x, y \in V(\Gamma)$, $d(p(x), p(y)) \leq d(x, y)$ because $E(G) = E(\Gamma)$. On the other hand, each vertex $v \in V(G)$ admits at most one cut (i.e., one of the form (f, v)). So, if a geodesic (i.e., a sequence of edges) in G is given, then we obtain a (connected edge) path in Γ by replacing a vertex $v \in V(G)$ in the geodesic by the three edges of f whenever the system of cuts contains a cut of the form (f, v). It is clear then that the length of the path is at most four times of the length of the geodesic. So, we have shown for any $x, y \in V(\Gamma)$,

$$d(p(x), p(y)) \leq d(x, y) \leq 4d(p(x), p(y)) + 3.$$

So G and Γ are quasi-isometric via the quasi-isometry p. We have also shown that Γ is connected.

Lastly, the connected planar graph $\Gamma \subset \mathbb{R}^2$ is a retract of $\mathbb{R}^2$, which shows that Γ is a tree. Indeed, since every face of G (i.e., every bounded component of $\mathbb{R}^2 \backslash G$) contains a cut in our system, it becomes unbounded after we cut G at there. Therefore there exists no bounded component of $\mathbb{R}^2 \backslash \Gamma$. Since $\Gamma \subset \mathbb{R}^2$ is connected, it is a retract. □

4. Property A_1

G. Yu introduced property A for discrete metric spaces as a means to guarantee the existence of a uniform embedding into Hilbert space [40]. The existence of this embedding implies the coarse Baum-Connes conjecture for discrete metric spaces with bounded geometry. (As mentioned in the introduction many of the spaces we consider are not examples of such spaces.) Yu's definition was the following:

Definition. Let $\mathrm{fin}(Z)$ denote the collection of finite, nonempty subsets of Z. The discrete metric space Z has *property A* if there are maps $A_n : Z \to \mathrm{fin}(Z \times \mathbb{N})$ ($n = 1, 2, \ldots$) such that

(1) for each n there is some $R > 0$ so that
$$A_n(z) \subset \{(z', j) \in Z \times \mathbb{N} \mid d(z, z') < R\}$$
for every $z \in Z$ and

(2) for every $K > 0$
$$\lim_{n\to\infty} \sup_{d(z,w)<K} \frac{|A_n(z) \Delta A_n(w)|}{|A_n(z) \cap A_n(w)|} = 0.$$

Following Tu [37] and Dranishnikov [13], define (for $p \in \mathbb{R} \cup +\infty$) a metric space X to have *property A_p* if there exist maps $a^n : X \to \ell^p(X)$ such that $\|a^n_x\|_p = 1$, $a^n_x(y) \geq 0$ for all $x, y \in X$, $n \in \mathbb{N}$, and

(1) there is a function $R = R(n)$ so that for all $x \in X$, $\mathrm{supp}(a^n_x) \subset N(x; R)$, and

(2) for every $K > 0$,
$$\lim_{n\to\infty} \sup_{d(x,y)<K} \|a^n_x - a^n_y\|_p = 0.$$

This definition is similar to a characterization of property A for discrete metric spaces with bounded geometry given by Higson and Roe in [20]. There they show that for discrete metric spaces with bounded geometry property A_1 is equivalent to Yu's Property A. Dranishnikov showed in [13] that discrete metric spaces with property A_p admit a coarse embedding in ℓ^p. In [37, Proposition 3.2], Tu shows that property A_1 and A_2 are equivalent. The proposition is stated for discrete metric spaces, but the remarks following the proof state that the equivalence holds in general. Also in remarks following this proposition, Tu states that Property A (in the sense of Yu) implies property A_1. On the other hand, it is unknown whether A_1 implies property A without the requirement of bounded geometry. In [13, Proposition 3.2], Dranishnikov shows that properties A_p and A_q are equivalent for any $1 \leq p, q < \infty$.

The relation between finite asymptotic dimension and these properties is the following:

Lemma 7 (Lemma 4.3, [20])**.** *Let X be a discrete metric space with bounded geometry and finite asymptotic dimension. Then X has property A.*

In fact, they show X has property A_1, which is equivalent to property A for such spaces. Their argument along with Theorem 1 can be used to show the following result:

Theorem 8. *Let Γ be a δ-hyperbolic graph. Suppose $\mathcal{L}$ is a set of geodesics in Γ such that any two vertices of Γ are joined by a geodesic in $\mathcal{L}$. If $\mathcal{L}$ satisfies property B with constants $\ell, k = 2\delta$ and D, then Γ has property A_1.*

We reproduce the argument from [20] for the reader's convenience. We also fill in some details using our definition of asymptotic dimension in terms of r-multiplicity of the cover.

Proof. By Theorem 1, $\operatorname{asdim}\Gamma \leq 2D-1$. By the definition of asdim, we know that for any $r \in \mathbb{N}$ we can find a uniformly bounded cover $\mathcal{U}$ of Γ with $5r$-multiplicity $\leq 2D$. Define a cover $\mathcal{V}$ of Γ by $\mathcal{V} = \{N(U; 2r) \mid U \in \mathcal{U}\}$, where $N(U; 2r)$ denotes the $2r$-neighborhood of U in Γ. Clearly $\mathcal{V}$ covers Γ and $\operatorname{diam}(V) \leq 4r + \operatorname{diam}(U)$, so $\mathcal{V}$ consists of sets with uniformly bounded diameters. Next, if $x \in N(U; 2r)$ then $N(x; 5r)$ meets U. For any x there are at most $2D$ such U, so the multiplicity (or order) of $\mathcal{V}$ is $\leq 2D$. Finally, let $A \subset \Gamma$ with $\operatorname{diam}(A) < r$. If $x \in A$, then $A \subset N(U; 2r)$ where $U \in \mathcal{U}$ is any set containing x. Thus, any set with diameter $< r$ is entirely contained within some element of the cover $\mathcal{V}$. (This says exactly that r is a *Lebesgue number* for $\mathcal{V}$.)

For each $V \in \mathcal{V}$, define a map $\phi_V^r : \Gamma \to [0,1]$ by

$$\phi_V^r(x) = \frac{d(x, \bar{V})}{\sum_{W \in \mathcal{V}} d(x, \bar{W})},$$

where the bar denotes the complement in Γ. Clearly, each $\phi_V^r \not\equiv 0$. For each $V \in \mathcal{V}$ let x_V denote some point with $\phi_V^r(x_V) \neq 0$. Now, define $a^r : \Gamma \to \ell^1(\Gamma)$ by

$$a_x^r(z) = \sum_{V \in \mathcal{V}} \phi_V^r(x) \delta_{x_V}(z),$$

where δ_{x_V} is the Dirac-δ function. Clearly a_x^r assumes a nonzero value at no more than $2D$ points in Γ and so $a_x^r \in \ell^1(\Gamma)$.

Claim:

(1) $a_x^r(z) \geq 0$ for all $z \in \Gamma$;
(2) $\|a_x^r\|_1 = 1$;
(3) there is some $R = R(r)$ so that $\operatorname{supp}(a_x^r) \subset N(x; R)$ for all $x \in \Gamma$; and
(4) for every $K > 0$

$$\lim_{r \to \infty} \sup_{d(z,w) < K} \|a_z^r - a_w^r\|_1 = 0.$$

Item (1) is clear. For item (2), we compute

$$\begin{aligned} \|a_x^r\|_1 &= \textstyle\sum_{z \in \Gamma} \left| \sum_{V \in \mathcal{V}} \phi_V^r(x) \delta_{x_V}(z) \right| \\ &= \textstyle\sum_{V \in \mathcal{V}} \phi_V^r(x) = 1. \end{aligned}$$

To see item (3), we take $R(r) = 4r + \operatorname{diam}(\mathcal{U})$ where $\operatorname{diam}(\mathcal{U})$ is an upper bound for the diameters of the sets in $\mathcal{U}$. Now, $a_x^r(z) > 0$ means that $z = x_V$ for some V containing x. Thus, $z \in V$, so $d(z, x) \leq R$.

Finally, for item (4), let $K > 0$ and $\epsilon > 0$ be given. Take r so large that $(4D+1)^2K/r < \epsilon$. The triangle inequality says if $z, w \in \Gamma$ and $V \in \mathcal{V}$, then $|d(z,\bar{V}) - d(w,\bar{V})| \leq d(z,w)$. Also, observe that for any $w \in \Gamma$, $\sum_{U\in\mathcal{V}} d(w,\bar{U}) \geq r$ since r is a Lebesgue number for the cover $\mathcal{V}$. Thus, we have

$$\begin{aligned}|\phi_V^r(z) - \phi_V^r(w)| &= \left|\frac{d(z,\bar{V})}{\sum_{U\in\mathcal{V}} d(z,\bar{U})} - \frac{d(w,\bar{V})}{\sum_{U\in\mathcal{V}} d(w,\bar{U})}\right| \\ &\leq \frac{\left|d(z,\bar{V}) - d(w,\bar{V})\right|}{\sum_{U\in\mathcal{V}} d(z,\bar{U})} + \left|\frac{d(w,\bar{V})}{\sum_{U\in\mathcal{V}} d(z,\bar{U})} - \frac{d(w,\bar{V})}{\sum_{U\in\mathcal{V}} d(w,\bar{U})}\right| \\ &\leq \frac{d(z,w)}{\sum_{U\in\mathcal{V}} d(z,\bar{U})} + \frac{d(w,\bar{V})}{\sum_{U\in\mathcal{V}} d(z,\bar{U}) \sum_{U\in\mathcal{V}} d(w,\bar{U})} \sum_{U\in\mathcal{V}} \left|d(w,\bar{U}) - d(z,\bar{U})\right| \\ &\leq \frac{1}{r} d(z,w) + \frac{1}{r}\left(\sum_{U\in\mathcal{V}} \left|d(z,\bar{U}) - d(w,\bar{U})\right|\right) \\ &\leq \frac{1}{r} d(z,w) + \frac{4Dd(z,w)}{r} = \frac{(4D+1)}{r} d(z,w).\end{aligned}$$

Then,

$$\begin{aligned}\|a_z^r - a_w^r\|_1 &= \textstyle\sum_{x\in\Gamma} |a_z^r(x) - a_w^r(x)| \\ &= \textstyle\sum_{x\in\Gamma} \left|\sum_V \phi_V^r(z)\delta_{x_V}(x) - \sum_V \phi_V^r(w)\delta_{x_V}(x)\right| \\ &= \left|\textstyle\sum_V \left(\phi_V^r(z) - \phi_V^r(w)\right)\delta_{x_V}\right| \\ &\leq |\phi_V^r(z) - \phi_V^r(w)|\, 4D \\ &\leq 4D\tfrac{(4D+1)}{r} d(z,w) \leq \tfrac{(4D+1)^2K}{r} < \epsilon.\end{aligned}$$

So item (4) is proved. Thus, Γ has property A_1. $\square$

As mentioned in Theorem 3, Masur-Minsky's tight geodesics $\mathcal{L}$ on a curve graph satisfy property B. Thus we have the following result.

Corollary 9. *Let S be an orientable surface of genus g with p punctures such that $3g - 3 + p > 1$. Then the curve graph of S has property A_1.*

Tu [37, Proposition 8.1] also proved that a discrete hyperbolic metric space X with bounded geometry has property A_1 by fixing $a \in \partial X$ and taking for the functions collections of geodesic rays marching off to a. As it stands, this argument cannot be applied directly to discrete hyperbolic spaces with a collection of geodesics satisfying property B as the following example shows.

Example. Let T be the following tree. Let x_0 be a vertex with infinite valence. Issuing from x_0 take edges of length $1, 2, 3, \ldots$. Clearly, T is hyperbolic and the collection of all geodesics in T satisfies property B, but $\partial T = \emptyset$, so Tu's argument does not apply.

On the other hand, suppose Γ were a discrete hyperbolic space with a collection of geodesics $\mathcal{L}$ with property B such that any two points in Γ are connected by a geodesic in $\mathcal{L}$. If Γ has the additional properties that $\partial\Gamma \neq \emptyset$ and that $\gamma \in \mathcal{L}$ implies any subgeodesic of γ is also in $\mathcal{L}$, then Tu's argument can be applied with very few changes to show that Γ has property A_1. The point of this remark is that when we follow his argument we use only geodesics in the set $\mathcal{L}$ to deal with the problem that Γ may not have bounded geometry. We note that since the set of tight geodesics of a curve graph satisfies these additional properties, this approach gives another argument to show that a curve graph has property A_1. We are informed that Y. Kida [27] has shown that a curve graph has not only property A_1 but also property A.

A natural question at this point is whether the coarse Baum-Connes conjecture is true for a curve complex. We settle this question by showing that a Farey graph is not coarsely equivalent to a proper metric space, so the coarse Baum-Connes conjecture cannot be formulated for it. The following lemma is well known.

Lemma 10. *Let $f : X \to Y$ be a coarse equivalence with coarse inverse $g : Y \to X$. Then, for every R there is an S so that $d(x, x') \geq S$ implies $d(f(x), f(x')) \geq R$.*

Proof. Since X and Y are coarsely equivalent, there is some $K > 0$ so that $d(fg, 1_Y) \leq K$ and $d(gf, 1_X) \leq K$. Let R be given. Since g is a coarse map, there is a $S_g > 0$ so that $d(y, y') \leq R$ implies that $d(g(y), g(y')) \leq S_g$. Put $S > 2S_g + 2K$. Then, if $d(x, x') \geq S$, we have $d(gf(x), gf(x')) \geq 2S_g + 2K - 2K$. So, if $d(f(x), f(x')) < R$, then we have $d(gf(x), gf(x')) \leq S_g$, which is a contradiction. □

Proposition 11. *Let G be a Farey graph. Then G is not coarsely equivalent to a proper metric space.*

Proof. Let T be a tree with infinite valence. We saw that G and T are quasi-isometric. Since a coarse isometry between G and a proper metric space X would yield a coarse isometry between X and T, it suffices to show that T and X are not coarsely isometric.

To this end, suppose that there were coarse maps $f : T \to X$ and $g : X \to T$ whose compositions were K-close to the identity. Let $R = 1$ and take S so that $d(t, t') \geq S$ implies that $d(f(t), f(t')) \geq 1$. Put $L = S + K$. Fix some vertex $t_0 \in T$ and define $S(t_0; L) = \{t \in T \mid d(t_0, t) = L\}$. Now, $S(t_0; L)$ is a bounded, $2L$-discrete, $2L$-bounded subset of T. Since $L > K$ we know that $gf(S(t_0; L))$ fixes $S(t_0; L)$ pointwise. Since $f(S(t_0; L))$ is an infinite bounded subset in the proper metric space X it contains a convergent subsequence. But, $d(f(t), f(t')) \geq 1$ for all $t,\ t' \in T$, a contradiction. □

We remark that the above argument shows that there is no map $f : G \to X$ such that there exist $C > 0$ and $\epsilon \geq 0$ such that for any $x, y \in G$, $\frac{1}{C} d(x, y) - \epsilon \leq d(f(x), f(y)) \leq C d(x, y) + \epsilon$.

5. Mapping class groups

In this section we turn our attention to mapping class groups. The reader is referred to Ivanov's paper [21] for an thorough introduction to mapping class groups. We reproduce many of the definitions and results contained therein for the reader's convenience. Let $S_{g,p}$ denote the compact orientable surface of genus g with p punctures. The *mapping class group of* $S_{g,p}$, $\mathrm{Mod}(S_{g,p})$, is the group of isotopy classes of orientation-preserving diffeomorphisms $S_{g,p} \to S_{g,p}$. They are also known by the name *modular groups*, which explains the notation.

Let Γ be a finitely generated group. Fixing a (finite) set of generators $S = S^{-1}$ endows Γ with a left-invariant word metric defined by $d_S(g, h) = \|g^{-1}h\|_S$, where $\|g^{-1}h\|_S$ is the length of the shortest S-word presenting the element $g^{-1}h$. Notice that two finite generating sets give rise to quasi-isometric metric spaces, so we define $\operatorname{asdim} \Gamma$ to be the asymptotic dimension of (Γ, d_S) where $S = S^{-1}$ is any finite generating set. We will have occasion to consider countable groups that are not finitely generated. J. Smith [34] showed that every countable group could be endowed with a left-invariant proper metric and that all left-invariant proper metrics are coarsely equivalent. Thus, when speaking of countable groups that may not be finitely generated, we will always assume that the group has a proper left-invariant metric on it. As a consequence, we see that $\operatorname{asdim} H \leq \operatorname{asdim} G$ for any subgroup H of a finitely generated group G.

The mapping class group, $\operatorname{Mod}(S_{g,p})$ is finitely generated, (in fact it is finitely presented, see [21]). Recently, Storm [35] combined a theorem of T. Kato and a result of Hamenstädt to show the Novikov conjecture for mapping class groups. A natural question is whether these groups have finite asymptotic dimension.

It is easy to obtain an obvious lower bound on $\operatorname{asdim} \operatorname{Mod}(S_{g,p})$, namely

$$\operatorname{asdim} \operatorname{Mod}(S_{g,p}) \geq 3g + p - 3.$$

Indeed, there are $k = 3g + p - 3$ simple closed curves on $S_{g,p}$ and Dehn twists by these curves commute because they are disjoint. Thus, there is a copy of $\mathbb{Z}^k$ inside $\operatorname{Mod}(S_{g,p})$. Dranishnikov, Keesling and Uspenskij showed that $\operatorname{asdim} \mathbb{Z}^k = k$ in [15]. Finally, if $Y \subset X$ in a metric space X, then any uniformly bounded cover of X by sets with r-multiplicity $\leq n + 1$ will restrict to a uniformly bounded cover of Y with r-multiplicity $\leq n + 1$, so $\operatorname{asdim} Y \leq \operatorname{asdim} X$. Thus, we conclude that $\operatorname{asdim} \operatorname{Mod}(S_{g,p}) \geq k$.

This lower bound is unlikely to give the exact asymptotic dimension as we now demonstrate.

Observe that the Euler characteristic of $S_{g,p}$ is $2-2g-p$. The second part of [21, Theorem 2.8.C] states that when $2 - 2g - p < 0$, the following sequence is exact:

$$1 \to \pi_1(S_{g,p}) \to \operatorname{PMod}(S_{g,p+1}) \to \operatorname{PMod}(S_{g,p}) \to 1,$$

where $\operatorname{PMod}(S)$ is the *pure mapping class group* of S, defined as the group of all isotopy classes of all orientation-preserving diffeomorphisms preserving the boundary components of S set-wise. Since $\operatorname{PMod}(S)$ is the kernel of the action of $\operatorname{Mod}(S)$ on the set of punctures, $\operatorname{PMod}(S) \subset \operatorname{Mod}(S)$ with finite index. Thus, $\operatorname{asdim} \operatorname{PMod}(S) = \operatorname{asdim} \operatorname{Mod}(S)$.

A. Dranishnikov has informed us of the following inequality for virtually finitely presented groups [14]:

$$\operatorname{vcd}\Gamma \leq \operatorname{asdim} \Gamma. \tag{1}$$

Ivanov computes the virtual cohomological dimension of mapping class groups in [21], Theorems 6.4.A, 6.4.B and 6.4.C:

Theorem 12 ([21]). *The following equalities hold for $S_{g,p}$, the surface of genus g and p boundary components:*

$$\operatorname{vcd} \operatorname{Mod}(S_{0,p}) = \begin{cases} 0, & \text{if } p \leq 3; \\ p - 3, & \text{if } p \geq 3; \end{cases}$$

$$\operatorname{vcd} \operatorname{Mod}(S_{1,p}) = \begin{cases} 1, & \text{if } p = 0; \\ p, & \text{if } p \geq 1; \end{cases}$$

$$\operatorname{vcd} \operatorname{Mod}(S_{g,p}) = \begin{cases} 4g - 5, & \text{if } g \geq 2,\ p = 0; \\ 4g - 4 + p, & \text{if } g \geq 2,\ p \geq 1. \end{cases}$$

Combining Theorem 12 with Dranishnikov's result (1) gives an improved lower bound on $\operatorname{asdim}\operatorname{Mod}(S_{g,p})$ in certain cases.

Next, we turn our attention to an upper bound for $\operatorname{asdim}\operatorname{Mod}(S_{g,p})$. Let G be a finitely generated group and let $1 \to K \to G \to H \to 1$ be exact. Being the surjective image of G, H is finitely generated. On the other hand, K is countable but not necessarily finitely generated, so we consider any proper left-invariant metric on K. For example, we could give K the metric it inherits as a subset of G.

Suppose $1 \to K \to G \to H \to 1$ is exact and G is finitely generated. When $\operatorname{asdim} K$ and $\operatorname{asdim} H$ are both finite, Bell and Dranishnikov prove that $\operatorname{asdim} G \leq \operatorname{asdim} H + \operatorname{asdim} K$ in [3]. If we abuse notation slightly, allowing the terms of our inequalities to be infinity, we get a formula

$$\operatorname{asdim}\operatorname{PMod}(S_{g,p+1}) \leq \operatorname{asdim}\operatorname{PMod}(S_{g,p}) + \operatorname{asdim}\pi_1(S_{g,p}),$$

when $2-2g-p<0$. Since $\operatorname{asdim}\operatorname{PMod}(S) = \operatorname{asdim}\operatorname{Mod}(S)$, we get

$$\operatorname{asdim}\operatorname{Mod}(S_{g,p+1}) \leq \operatorname{asdim}\operatorname{Mod}(S_{g,p}) + \operatorname{asdim}\pi_1(S_{g,p}),$$

when $2-2g-p<0$. Thus, we can apply an inductive argument on the number of punctures of S. We begin with an easy computation.

Lemma 13. *Let $S_{g,p}$ denote the compact surface with genus g and p punctures. Then*

$$\operatorname{asdim}\pi_1(S_{g,p}) = \begin{cases} 1, & \text{if } p>0; \\ 2, & \text{if } p=0. \end{cases}$$

Proof. Observe first that $\pi_1(S_{g,p})$ is a free group if $p>0$, so in this case $\operatorname{asdim}\pi_1(S_{g,p}) = 1$. If $p=0$ and $g>1$, then $\pi_1(S_{p,0})$ is quasi-isometric to $\mathbb{H}^2$, so $\operatorname{asdim}\pi_1(S_{p,0}) = 2$. Finally, $\pi_1(S_{1,0}) = \mathbb{Z}\oplus\mathbb{Z}$ so $\operatorname{asdim}\pi_1(S_{1,0}) = 2$. □

Theorem 14. *Let $p \geq 0$ and $2-2g-p<0$. If $\operatorname{asdim}\operatorname{Mod}(S_{g,0}) < \infty$, then*

$$\operatorname{asdim}\operatorname{Mod}(S_{g,p}) \leq \operatorname{asdim}\operatorname{Mod}(S_{g,0}) + p + 1.$$

In particular, if $\operatorname{asdim}\operatorname{Mod}(S_{g,0}) < \infty$ then $\operatorname{asdim}\operatorname{Mod}(S_{g,p}) < \infty$ for $p \geq 0$.

Proof. This follows easily from the lemma. Indeed,

$$\begin{aligned} \operatorname{asdim}\operatorname{Mod}(S_{g,p}) &\leq \operatorname{asdim}\operatorname{Mod}(S_{g,p-1}) + \operatorname{asdim}\pi_1(S_{g,p-1}) \\ &\ \ \vdots \\ &\leq \operatorname{asdim}\operatorname{Mod}(S_{g,0}) + \operatorname{asdim}\pi_1(S_{g,p-1}) + \cdots + \operatorname{asdim}\pi_1(S_{g,0}) \\ &= \operatorname{asdim}\operatorname{Mod}(S_{g,0}) + p + 1 \end{aligned}$$

□

We can say more when $g \leq 1$. Ivanov explains in [21, 9.2] that $\operatorname{Mod}(S_{0,4})$ is commensurable with $\operatorname{PSL}(2,\mathbb{Z})$. Since $\operatorname{PSL}(2,\mathbb{Z}) \cong \mathbb{Z}_2 * \mathbb{Z}_3$, $\operatorname{asdim}\operatorname{Mod}(S_{0,4}) = 1$ in these cases by a formula of Bell-Dranishnikov-Keesling [4]. Ivanov also explains that $\operatorname{Mod}(S_{1,1}) \cong SL(2,\mathbb{Z}) \cong \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$, and so $\operatorname{asdim}\operatorname{Mod}(S_{1,1}) = 1$. Thus we obtain upper bounds for $\operatorname{asdim}\operatorname{Mod}(S_{g,p})$ for $g=0$ or 1. The lower bounds follow from Theorem 12 and Dranishnikov's estimate (1); combining these bounds gives the following equalities.

Corollary 15. *If $p \geq 4$ then*

$$\operatorname{asdim}\operatorname{Mod}(S_{0,p}) = p-3$$

and for $p \geq 1$,

$$\operatorname{asdim}\operatorname{Mod}(S_{1,p}) = p$$

Finally, we can give a rough upper bound for the case $g = 2$. The *Torelli group* $\mathcal{I}_g$ is the subgroup of $\mathrm{Mod}(S_{g,0})$ acting trivially on $H_1(S_{g,0}, \mathbb{Z})$, i.e. the group $\mathcal{I}_g$ arising in the following exact sequence:

$$1 \to \mathcal{I}_g \to \mathrm{Mod}(S_{g,0}) \to Sp(2g, \mathbb{Z}) \to 1.$$

Johnson showed in [24] that $\mathcal{I}_g$ is finitely generated if $g \geq 3$, but $\mathcal{I}_2$ is non-finitely generated and free, [29]. To make sense of the asymptotic dimension of $\mathcal{I}_2$, we endow it with a proper left-invariant metric. One such metric on $\mathcal{I}_2$ is the one it inherits as a subset of $\mathrm{Mod}(S_{2,0})$. Since $\mathcal{I}_2$ is isomorphic to a subset of F_2 the free group on two letters, $\operatorname{asdim} \mathcal{I}_2 \leq 1$. Since it contains a bi-infinite geodesic we can conclude that $\operatorname{asdim} \mathcal{I}_2 = 1$. This computation allows us to get the following bounds on asdim:

Corollary 16. *When $p = 0$, we have*

$$3 \leq \operatorname{asdim} \mathrm{Mod}(S_{2,0}) \leq 10.$$

If $p \geq 1$, then

$$4 + p \leq \operatorname{asdim} \mathrm{Mod}(S_{2,p}) \leq 11 + p.$$

Proof. The lower bounds are by Theorem 12 and (1). First, since $\mathrm{Sp}(4, \mathbb{Z}) \subset \mathrm{GL}(4, \mathbb{Z})$, we have $\operatorname{asdim} \mathrm{Sp}(4, \mathbb{Z}) \leq \operatorname{asdim} \mathrm{GL}(4, \mathbb{Z}) \leq 9$ by [12], where 9 is the dimension of the symmetric space associated to $\mathrm{SL}(4, \mathbb{R})$. Thus, $\mathrm{Mod}(S_{2,0}) \leq 10$. Next, since $g = 2$, $\chi(S_{g,p}) < 0$ for all $p \geq 0$, so we can apply Theorem 14 to get the desired upper bound. □

We should remark that this result can be improved upon by finding the exact value of $\operatorname{asdim} \mathrm{Sp}(4, \mathbb{Z})$.

It is interesting to observe that putting our lower bound for $\operatorname{asdim} \mathrm{Mod}(S_{g,p})$ together with Theorem 14 we get

$$4g - 4 + p \leq \operatorname{asdim} \mathrm{Mod}(S_{g,p}) \leq \operatorname{asdim} \mathrm{Mod}(S_{g,0}) + 1 + p,$$

(when $2 - 2g - p < 0$) so that if $\operatorname{asdim} \mathrm{Mod}(S_{g,0})$ is finite, then $\operatorname{asdim} \mathrm{Mod}(S_{g,p})$ increases like the number of punctures.

Lemma 17. *Suppose $1 \to K \xrightarrow{\iota} G \xrightarrow{\phi} H \to 1$ is an exact sequence of groups with G finitely generated. Suppose* $\operatorname{asdim} H < \infty$ *and K has property A with respect to some (hence any) proper, left-invariant metric. Then G has property A.*

Proof. Take a finite, symmetric generating set, $S = S^{-1}$, for G and consider the word metric d_S on G. Taking the word metric $d_{\phi(S)}$ corresponding to the generating set $\phi(S)$ on H implies that ϕ is 1-Lipshcitz. Let G act on H by the rule $g.h = \phi(g)h$. In [1], the first author showed that if a finitely generated group G acts on a group H with finite asymptotic dimension so that for every R, the set $W_R = \{g \in G \mid d_{\phi(S)}(g.e, e) \leq R\}$ has property A, then G has property A.

We will prove that $W_R = N(\iota(K); R)$, where the R-neighborhood is taken in G, with respect to the word metric in G. Thus, W_R is quasi-isometric to $\iota(K)$. Thus W_R has property A if $\iota(K)$ does. Since K is countable, the metric space $\iota(K)$ and the group K with its given proper metric are coarsely equivalent. Since property A is an invariant of coarse equivalence, this will say that W_R has property A for every R, and hence, that G has property A.

First assume $g \in W_R$. Then $\|\phi(g)\|_{\phi(S)} \leq R$. So, there exist $s_1, \ldots, s_k \in S$, with $k \leq R$, so that $\phi(g) = \phi(s_1) \cdots \phi(s_k)$. Take $g' = s_k^{-1} \cdots s_1^{-1} \in G$. Then, $gg' \in K$, and $d(gg', g) \leq R$. On the other hand, if $g \in N(K; R)$, then there is a $k \in K$ with $d(g, k) \leq R$. Since ϕ is 1-Lipschitz, we have $d(\phi(g), e) \leq R$, as required. $\square$

Corollary 18. *Endow the Torelli group $\mathcal{I}_g$ with a proper (left-invariant) metric. Then $\mathcal{I}_g$ has finite asymptotic dimension if and only if* $\mathrm{Mod}(S_{g,0})$ *does. It has property A if and only if* $\mathrm{Mod}(S_{g,0})$ *does.*

Proof. We showed in Corollary 15 that $\operatorname{asdim} \mathrm{Mod}(S_{g,0}) < \infty$ when $g < 2$ and so these groups (being finitely generated) also have property A. For $g < 2$ the Torelli groups are trivial so they, too, have both properties.

If $g = 2$ we only have to show that $\mathcal{I}_2$ has property A, but this follows from the corresponding fact for the free group on two generators, which contains it as a subgroup, and the fact that property A is a coarse invariant.

Let $g \geq 3$. In both cases these properties pass to subsets. The exact sequence

$$1 \to \mathcal{I}_g \to \mathrm{Mod}(S_{g,0}) \to Sp(2g, \mathbb{Z}) \to 1$$

and the fact that asdim is finite for arithmetic groups by [25] implies the result for finite asymptotic dimension. Lemma 17 implies the result for property A. $\square$

We briefly discuss Teichmüller spaces. Let $T(S)$ be the Teichmüller space of the surface S with negative Euler characteristic.

See [23] or Chapter 5 in [21] for definitions and information. The group $\mathrm{Mod}(S)$ acts on $T(S)$ naturally. We set $d(S_{g,p}) = 3g - 3 + p$. It is known that $T(S_{g,p})$ is homeomorphic to $\mathbb{R}^m$, where $m = 6g - 6 + 2p = 2d(S)$. There is a natural metric on $T(S)$, called the *Teichmüller metric*, which is a Finsler metric. With respect to this metric, $T(S)$ is proper, and the action by $\mathrm{Mod}(S)$ is proper and by isometries. Therefore, we obtain

Proposition 19. *Suppose S has negative Euler characteristic and endow $T(S)$ with the Teichmüller metric. Then* $\operatorname{asdim} \mathrm{Mod}(S) \leq \operatorname{asdim} T(S)$*, where we allow the possibility that the terms of this inequality are infinite.*

Proof. The group $\mathrm{Mod}(S)$ acts on $T(S)$ (with the Teichmüller metric) properly by isometries [23, Chapter 6.3], so Proposition 2.3 from [25] applies to this setting. We outline an argument. We fix a point $x \in T(S)$ and identify each element $g \in \mathrm{Mod}(S)$ with $g.x \in T(S)$. We wish to consider the metric on $\mathrm{Mod}(S)$ induced from being a subset in $T(S)$, but this is a pseudo-metric as we could have non-trivial g with $g.x = x$. Since the action is proper, such a g must be torsion, so we may (if necessary) pass to a torsion-free subgroup of $\mathrm{Mod}(S)$ of finite index, [21]. This subgroup has the same asdim as $\mathrm{Mod}(S)$. The point is that the metric this group inherits as a subset of $T(S)$ is a coarsely equivalent to a word metric on $\mathrm{Mod}(S)$. Thus, the asymptotic dimension of $\mathrm{Mod}(S)$ as a finitely generated group is the same as the asymptotic dimension of $\mathrm{Mod}(S)$ as a subset of $T(S)$. It follows that $\operatorname{asdim} \mathrm{Mod}(S) \leq \operatorname{asdim} T(S)$. $\square$

6. Some open problems

Our results lead to some natural questions. Again, we let $S_{g,p}$ be the compact orientable surface of genus g with p punctures and let $X(S_{g,p})$ denote its curve graph.

Question 1. What is the asymptotic dimension of $X(S_{g,p})$?

We remark that it is known that $\dim X(S_{g,p}) = 3g+p-4$ (in the non-exceptional cases where this number is positive). Generally, there is no relation between dim and asdim, indeed there are countable metric spaces with arbitrary (even infinite) asdim whereas dim of these spaces is necessarily 0. On the other hand, any compact metric space has asdim $= 0$.

In Theorem 14, we showed that $\operatorname{asdim} \operatorname{Mod}(S_{g,p})$ (if finite) depends on g and p. It would be very interesting if $\operatorname{asdim} X(S_{g,p})$ were independent of g and p. This leads to the following questions.

Question 2. What is an upper bound for $\operatorname{asdim} X(S_{g,p})$? Does it depend on g and p or is there some $C < \infty$ so that $\operatorname{asdim} X(S_{g,p}) \leq C$ for all g and p?

Perhaps a promising approach to these problems would be to examine the boundary $\partial X(S_{g,p})$. For finitely generated hyperbolic groups Γ, one can combine results of [11] and [36] to conclude that $\dim \partial\Gamma + 1 = \operatorname{asdim} \Gamma$. Unfortunately, for more general hyperbolic spaces, X one can only conclude [30] that $\dim \partial X \leq \operatorname{asdim} X$. The drawback to this approach is that the boundary of $X(S_{g,p})$ is not well understood.

In the non-exceptional cases $X(S_{g,p})$ contains a bi-infinite geodesic, (see [8]) so $\operatorname{asdim} X(S_{g,p}) \geq 1$, but it appears that we are not even able to answer the following question.

Question 3. Do we have $\operatorname{asdim} X(S_{g,p}) \leq 1$ for all g, p ?

Although a Farey graph is quasi-isometric to a tree, so that its asymptotic dimension is one, it appears unlikely that a curve graph is quasi-isometric or even coarsely equivalent to a tree. Indeed, S. Schleimer [33] applied results of Behrstock and Leininger to show the following property for $X = X(S_{g,1})$ with $g \geq 2$: for any $x \in X$ and $r \in \mathbb{N}$, $X \backslash N(x; r)$ is connected. This easily implies X is not coarsely equivalent to a tree. As Schleimer points out, it is unclear how to extend his result to the case of more (or fewer) punctures.

Although we had some finiteness results for $\operatorname{asdim} \operatorname{Mod}(S_{g,p})$ the question remains open when $g \geq 3$.

Question 4. Is the asymptotic dimension of $\operatorname{Mod}(S_{g,p})$ finite when $g \geq 3$?

Indeed, to our knowledge, the following question remains unanswered.

Question 5. Does $\operatorname{Mod}(S_{g,p})$ have property A when $g \geq 3$?

Notice that an affirmative answer to question 4 would immediately imply an affirmative answer to this question. As we mentioned, Storm [35] showed that the Novikov conjecture holds for mapping class groups, so it would be very surprising if they did not have property A.

A naive approach to questions 4 and 5 would be to attempt to use the Hurewicz-type theorem of Bell and Dranishnikov [1, 3]. The group $\operatorname{Mod}(S_{g,p})$ acts on $X(S_{g,p})$ by isometries. Since $X(S_{g,p})$ has finite asymptotic dimension, we would be able to conclude that $\operatorname{Mod}(S_{g,p})$ has finite asymptotic dimension (or property A) provided we could show that the set $\{g \in \operatorname{Mod}(S_{g,p}) \mid d_X(g.x_0, x_0) \leq r\}$ has finite asymptotic dimension (respectively, property A) for all $r \in \mathbb{N}$. Here, x_0 is any point of X. We remark that if we regard x_0 as a curve on S the set we need to analyze contains the stabilizer subgroup of x_0, which is $\operatorname{Mod}(S \backslash x_0)$. A problem is that those two sets are not necessarily coarsely equivalent (cf. proof of Lemma 17).

L. Ji proved, in [25], that arithmetic groups have finite asymptotic dimension. Ivanov draws a comparison between mapping class groups and arithmetic groups in Chapter 9 of [21]. Although the groups are different (cf. [5]), many things that are proved true for arithmetic groups are later proved to be true for mapping class groups. So, an affirmative answer to question 4 does not seem unlikely. Ji gave an upper bound of the asymptotic dimension of an arithmetic group, Γ, as follows (cf. proof of Proposition 19). Let X be the symmetric space associated to Γ. Then $\operatorname{asdim}\Gamma \leq \operatorname{asdim} X \dim X$ So, in connection with question 4, we ask the following (see Proposition 19).

Question 6. Let $S = S_{g,p}$ with negative Euler characteristic and endow $T(S)$ with the Teichmüller metric. What is $\operatorname{asdim} T(S)$? Is it finite? Is $\operatorname{asdim} T(S)$ strictly bigger than $\operatorname{asdim} \operatorname{Mod}(S)$? If not (i.e., if they are same), then is $\operatorname{Mod}(S)$ coarsely equivalent to $T(S)$?

We can ask similar questions on the asymptotic dimensions of a Teichmüller space with the Weil-Petersson metric (cf.[23]), g_{WP}, but we do not even know if a statement similar to Proposition 19 holds because the space is not proper. Let n be the greatest integer less than or equal to $(d(S)+1)/2$. Brock and Farb (Theorem 1.4 [10]) proved that there is a quasi-isometric embedding $\mathbb{R}^n \to (T(S_{g,p}), g_{\mathrm{WP}})$. It follows that $\operatorname{asdim}(T(S), g_{\mathrm{WP}}) \geq n$.

Let S, S' be compact orientable surfaces. If there exists a system of disjoint curves, C, on S so that one of the connected components of $S\backslash C$ is homeomorphic to the interior of S', we write $S' < S$. Then we have $\operatorname{asdim}(T(S'), g_{\mathrm{WP}}) \leq \operatorname{asdim}(T(S), g_{\mathrm{WP}})$. This follows from the fact that $(T(S), g_{\mathrm{WP}})$ contains a subset which is quasi-isometric to $(T(S'), g_{\mathrm{WP}})$. Indeed (cf. [38]), although the metric space $(T(S), g_{\mathrm{WP}})$ is not complete, its metric completion $\overline{(T(S), g_{\mathrm{WP}})}$ contains $\overline{(T(S'), g_{\mathrm{WP}})}$ as a totally geodesic subspace in such a way that $(T(S'), g_{\mathrm{WP}}) \subset \overline{(T(S), g_{\mathrm{WP}})} \backslash (T(S), g_{\mathrm{WP}})$. For $\epsilon > 0$, let N be the ϵ-neighborhood of $T(S')$ in $\overline{T(S)}$. Then $(N \backslash T(S')) \subset T(S)$ is quasi-isometric to $T(S')$.

References

[1] G. Bell, *Property A for groups acting on metric spaces*, *Topology Appl.* **130** (2003) 239–251.
[2] G. Bell and A. Dranishnikov, *Asymptotic dimension in Będlewo*, preprint, arXiv:math.GR/0507570 (2005).
[3] G. Bell and A. Dranishnikov, *A Hurewicz-type theorem for asymptotic dimension and applications to geometric group theory*, Preprint arXiv:math.GR/0407431 (2004).
[4] G. Bell, A. Dranishnikov, and J. Keesling, *On a formula for the asymptotic dimension of free products*, Fund. Math. **183** (2004), no. 1, 39–45.
[5] M. Bestvina and K. Fujiwara, *Bounded cohomology of subgroups of the mapping class groups*, Geom. & Topol. **6** (2002) 69–89.
[6] M. Bonk and O. Schramm, *Embeddings of Gromov hyperbolic spaces*, Geom. Funct. Anal. **10** (2000), no. 2, 266–306.
[7] Brian Bowditch, *Intersection numbers and the hyperbolicity of the curve complex*, preprint, www.maths.soton.ac.uk/staff/Bowditch/ (2003).
[8] Brian Bowditch, *Tight geodesics in the curve complex*, preprint, www.maths.soton.ac.uk/staff/Bowditch/ (2003).
[9] Brian Bowditch, Lectures at Osaka University, 2005.
[10] J. Brock, B. Farb, Curvature and rank of Teichmüller space. To appear, American Journal of Mathematics .
[11] S. Buyalo and V. Schroeder, *A product of trees as universal space for hyperbolic groups*, on-line at http://arXiv.org/abs/math.GR/0509355 (2005).

[12] G. Carlsson and B. Goldfarb, *On homological coherence of discrete groups*, J. Algebra **276** (2004), 502–514.

[13] A. Dranishnikov, *Groups with a polynomial dimension growth*, Preprint, available on-line at http://arXiv.org/abs/math.MG/0405239 (2004).

[14] A. Dranishnikov, *Cohomological approach to asymptotic dimension*, in preparation.

[15] A. Dranishnikov, J. Keesling, and V. Uspenskij, *On the Higson corona of uniformly contractible spaces*, Topology **37** (1998), no. 4, 791–803.

[16] M. Gromov, *Hyperbolic groups*, Essays in group theory, Math. Sci. Res. Inst. Publ., vol. 8, Springer, 1987, pp. 75–263.

[17] M. Gromov, *Asymptotic invariants of infinite groups*, Geometric Group Theory, London Math. Soc. Lecture Note Ser. (G. Niblo and M. Roller, eds.), no. 182, 1993.

[18] J. L. Harer, *The virtual cohomological dimension of the mapping class groups of orientable surfaces*, Invent. Math. **84** (1986), 157–176.

[19] W. J. Harvey. Boundary structure of the modular group. In *Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, N.Y., 1978)*, pages 245–251, Princeton, N.J., 1981. Princeton Univ. Press.

[20] N. Higson and J. Roe, *Amenable group actions and the Novikov conjecture*, J. Reine Angew. Math. **519** (2000), 143–153.

[21] N. V. Ivanov, *Mapping class groups*, in Handbook of Geometric Topology, (R. Daverman and R. Sher, eds.), 523–633 North-Holland, Amsterdam 2002.

[22] N. V. Ivanov, *Automorphisms of complexes of curves and of Teichmüller spaces*, Internat. Math. Res. Notices (1997) 651–666.

[23] Y. Imayoshi, M. Taniguchi, An introduction to Teichmüller spaces. Springer, 1992.

[24] D. Johnson, *The structure of the Torelli group I: a finite set of generators for $\mathcal{T}$*, Annals of Math. **118** (1983) 423–442.

[25] L. Ji, *Asymptotic dimension and the integral K-theoretic Novikov conjecture for arithmetic groups,* J. Differential Geom. **68** (2004), no. 3, 535–544

[26] G. Kasparov and G. Yu, *The coarse geometric Novikov conjecture and uniform convexity*, to appear in Advances in Mathematics (2005).

[27] Y. Kida, Master thesis, Kyoto University, 2005.

[28] Howard A. Masur and Yair N. Minsky. Geometry of the complex of curves. I. Hyperbolicity. *Invent. Math.*, **138** (1999) 103–149.

[29] G. Mess, *The Torelli groups for genus* 2 *and* 3 *surfaces*, Topology **31** (1992), no. 4, 775–790.

[30] P. Papasoglu and T. Gentimis, *Asymptotic dimension and boundaries of hyperbolic spaces*, Preprint (2005).

[31] J. Roe, *Lectures on coarse geometry*, University Lecture Series, vol. 31, AMS, 2003.

[32] J. Roe, *Hyperbolic groups have finite asymptotic dimension*, Proc. Amer. Math. Soc. **133** (2005), no. 9, 2489–2490.

[33] S. Schleimer, *Notes on the complex of curves*, available on-line at http://www.math.rutgers.edu/∼saulsch/ Maths/notes.pdf (2005).

[34] J. Smith, *On asymptotic dimension of countable abelian groups*, Preprint: arXiv math.GR/0504447 (2005).

[35] P. Storm, *The Novikov conjecture for mapping class groups as a corollary of Hamenstadt's theorem* Preprint, available on-line at http://arxiv.org/abs/math.GT/0504248 (2005).

[36] J. Świątkowski, *On the asymptotic homological dimension of hyperbolic groups*, Bull. London Math. Soc. **27** (1995), no. 3, 209–221.

[37] J.-L. Tu, *Remarks on Yu's "property A" for discrete metric spaces and groups*, Bull. Soc. Math. France **129** (2001), no. 1, 115–139.

[38] S. Wolpert *Geometry of Weil-Petersson completion of Teichmüller space*, in Surveys in Differential Geometry, VIII: Papers in Honor of Calabi, Lawson, Siu and Uhlenbeck, (S. T. Yau, ed.) International Press, 2003.

[39] G. Yu, *The Novikov conjecture for groups with finite asymptotic dimension*, Annals of Mathematics **147** (1998), no. 2, 325–355.

[40] G. Yu, *The coarse Baum-Connes conjecture for spaces which admit a uniform embedding into a Hilbert space*, Invent. Math. **139** (2000), 201–240.

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