HARMONIC MANIFOLDS ;
THE PROOF OF THE LICHNEROWICZ CONJECTURE
IN THE COMPACT CASE

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Introduction

The theory of harmonic manifolds has a relatively long history. It started with a work of H.S. Ruse in 1930, who made an attempt to find a solution for the equation $\Delta f = 0$ on a general Riemannian manifold which depends only on the geodesics distance $r(x, .)$. His main aim was to use these functions and develop harmonic analysis on Riemannian manifolds similar to the euclidean case.

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It turned out that such radial harmonic functions exist only in very special cases, namely, in the cases where the density function $\omega_p := \sqrt{|\det\ g_{ij}|}$ in the normal coordinate neighbourhood $\{x^1, \dots, x^n\}_p$ around each point p depends only on $r(p,.)$. From the well–known symmetry $\omega_p(q) = \omega_q(p)$ it can be easily seen that this is the case if and only if the function $\omega_p(q)$ is of the form

$$\omega_p(q) = \phi\,(r(p,q))\ ; \quad \phi : \mathbb{R}_+ \longrightarrow \mathbb{R}\ ;$$

A Riemannian manifold was defined to be harmonic precisely when its density function $\omega_p(q)$ satisfies this radial property.

For a precise formulation one can introduce the notions of global – local – and infinitesimal harmonicity [4]. Global and local harmonicity refer to the cases that the above radial property of the density function is global or local, respectively. For infinitesimal harmonicity we assume only that the derivatives $\overset{(k)}{\nabla}_{\xi_p \cdots \xi_p}\omega_p$ w.r.t. the unit vectors $\xi_p \in T_p(M^n)$ define constant functions on the manifold. These notions are obviously equivalent for analytic Riemannian manifolds [4].

The derivatives $\overset{(k)}{\nabla}_{\xi_p \cdots \xi_p}\omega_p$ can be expressed with the help of the curvature tensor and its covariant derivatives. For example, we have

$$\overset{(2)}{\nabla}_{\xi_p\xi_p}\omega_p = -\tfrac{1}{3}\,R(\xi_p,\xi_p)\,,$$

where $R(X,Y)$ is the Ricci curvature, so the harmonic manifolds of any type are Einstein manifolds. On the other hand, any Einstein metric is analytic in the harmonic and normal coordinates by Kazdan–De Turck theorem [48]. Thus we get:

The global, local and infinitesimal harmonicity are equivalent properties.

We mention that in another paper we shall prove that also those spaces which satisfy the Legendre curvature condition $R_{ij/k} + R_{jk/i} + R_{ki/j} = 0$ are real analytic. It follows that all the commutative spaces and D'Atri spaces are analytic.

An interesting equivalent formulation of harmonicity was found by Willmore [26]:

A Riemannian space is harmonic if and only if for any harmonic function u the classical mean–value theorem

$$u(p) = \frac{1}{\int_{S_{p;r}} dS_{p;r}(x)} \int_{S_{p;r}} u \, d\, S_{p;r}(x)$$

holds, where $dS_{p;r}(x)$ means the induced measure on the geodesic sphere $S_{p;r}$ with the centre p and radius r.

Any two–point homogeneous manifold is obviously harmonic. The main problem about the harmonic manifolds was to prove the Lichnevouricz conjecture [19] asserting the converse statement: Any harmonic manifold is two–point homogeneous.

The conjecture has been proved so far only for dimensions ≤ 4 [19], [25], [4]. All these solutions use the dimensionality very heavily, and did not give any hope for higher dimensions. In higher dimensions, only partial results were proved using an additional strong assumption. One such theorem is the following :

Any locally symmetric harmonic manifold is two–point homogeneous.

The harmonic spaces were investigated from a local point of view in most cases. Between the few global investigations we mention the Allamigeon theorem [2] and the "nice imbedding theorem" of Besse [4]. The first theorem asserts that any complete simply connected harmonic manifold is diffeomorphic either to $\mathbb{R}^d$ or to a Blaschke manifold, which has simple closed geodesics with the same length. In Besse's theorem an isometric imbedding $\phi : M^n \longrightarrow \mathbb{R}^d$ is constructed for compact simply connected harmonic spaces such that $\phi(M^n)$ is minimal in certain sphere and furthermore, all the geodesics are congruent screw lines in $\mathbb{R}^d$. Both theorems will be used in the present paper. Our aim is to prove the conjecture for simply connected compact harmonic spaces. Using the universal covering spaces, this proof gives a proof of the Lichnerowicz conjecture for the compact (infinitesimal, local or global) harmonic manifolds which have a finite fundamental group (and hence a compact universal covering space).

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1. The Basic Commutativity in harmonic spaces

For the sake of simplicity we investigate in this paper simply connected complete Riemannian manifolds (M^n,g). The metric g is assumed to be positive definite.

Let $(x_1, \dots ,x_n)_p$ be a normal coordinate neighbourhood around a point $p \in M^n$. The function

$$\omega_p := \omega\left[\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right]$$

stands for the volume density in the space. We introduce also the polar coordinate neighbourhood (r_p,φ) around p, where $r_p(q) = r(p,q)$ denotes the geodesic distance between p and q, and φ represents the points of the unit sphere in the tangent space $T_p(M^n)$. In this system the function ω_p can be written in the form $\omega_p(r,\varphi)$ and the density with respect to (r_p,φ) is

$$\Theta_p := r^{n-1}\omega_p .$$

A Riemannian manifold is said to be <u>harmonic</u> if the density Θ_p is a radial or spherical symmetric function around any point $p \in M^n$, i.e. it depends only on the variable r and thus it can be written as $\Theta_p(r)$.

It can be proved that the functions $\Theta_p(r)$, $p \in M^n$, are also independent of the points $p \in M^n$ in a harmonic space [4]. This statement follows from the symmetry $\Theta_p(r_p(q)) = \Theta_q(r_q(p))$ easily.

Now let $S_{p,R}$ be a geodesic sphere around p with the radius R whose Minkowskian mean curvature is denoted by $\sigma_p(R,\varphi)$. The formula

$$(1.1) \qquad \sigma_p(R,\varphi) = \Theta_p'(R,\varphi)/\Theta_p(R,\varphi) = \frac{n-1}{R} + \frac{\omega'(R,\varphi)}{\omega(R,\varphi)} = -(\Delta r_p)(R,\varphi)$$

is rather well known [4], where the comma means the derivation w.r.t. radial direction and $\Delta := -\nabla_i\nabla^i$ is the Laplace operator in the space.

One can prove from this formula that a Riemannian manifold is harmonic if the mean curvature function $\sigma_p(R,.)$ is a radial function of the form $\sigma_p\ (r,\varphi) = \sigma(r(p,.))$. The statement can be proved solving the equation

$$\sigma(r) - \frac{n-1}{r} = \frac{\omega'(r)}{\omega(r)}$$

with the initial condition $\omega(0) = 1$.

We also mention another connection between the Laplace operator Δ and the mean curvature function $\sigma_p(r,\varphi)$. Let $\tilde{\nabla}$ (resp. $\tilde{\Delta}$) be the covariant derivative (resp. the Laplace operator) of a geodesics sphere $S_{p,r}$ whose second fundamental form is denoted by $M_{p,r}(X,Y)$. Then the formula

$$\nabla^2 f(X,X) = X \cdot X(f) - (\nabla_X X) \cdot (f) = \tilde{\nabla}^2 f(X,X) + M(X,X)f'$$

holds for a function f in M^n and a vector field X tangent to $S_{p,r}$. So we get

(1.2) $$\Delta f = \tilde{\Delta} f - f'' - \sigma_p(r,\varphi) f' ,$$

and therefore the action of Δ on a radial function f (around p) is

(1.3) $$\Delta f := - f'' - \sigma_p(r,\varphi) f' .$$

We introduce also the so–called averaging operators A_p, $p \in M^n$, which play a very important role in the following discussions.

Let f be a smooth function of M^n. Then the averaged function $A_p(f)$ is defined as a radial function around p whose values are at the points of a geodesic sphere $S_{p,r}$ just the average of f on $S_{p,r}$, i.e.

(1.4) $$A_p(f)(r) = \frac{1}{\int \Theta_p(r,\varphi) d\varphi} \int f(r,\varphi) \Theta_p(r,\varphi) d\varphi .$$

The function $A_p(f)$ is defined only locally, namely for the small values of r which are less than the injectivite radius of M^n at $p \in M^n$. On the other hand, $A_p(f)$ is globally defined for any $p \in M^n$ if the space is a compact Blaschke manifold (i.e. for which the cut values are equal at any tangent space $T_p(M^n)$) or if it is a non–compact complete manifold with an infinite injectivity radius. In the last case M^n is diffeomorphic to $\mathbb{R}^n$ by the exponential map. We call these spaces as globally averageable spaces. The compact Blaschke manifolds have simple closed geodesics with a common length 2L such that the geodesics, starting from a point m, intersect the cut locus at the distance L orthogonally (see Corollary 5.42 and Proposition 7.9 in [4]). From this statement we get easily that the averaged function $A_p(f)$ of a function f of class C^k is a globally defined function of class C^k in any globally averageable space.

By the Allamigeon theorem any simply connected complete harmonic manifold is globally averageable space.

Lemma 1.1 (Basic Commutativity in harmonic spaces).

A Riemannian manifold (M^n,g) is harmonic if the Laplace operator Δ commutes with the local averaging operators A_p, $p \in M^n$, i.e.

$$A_p(\Delta f) = \Delta A_p(f) \tag{1.5}$$

yields for any smooth function f.

Proof. If M^n is harmonic then by (1.2), (1.3) and by the Stokes theorem we get

$$\begin{aligned} A_p(\Delta f) &= A_p(\tilde{\Delta} f) - A_p(f'') - A_p(\sigma_p(r)f') = \\ &= -(A_p(f))'' - \sigma_p(r)(A_p(f))' = \Delta A_p(f) \end{aligned} \tag{1.6}$$

which proves the commutativity in harmonic manifolds.

Conversely, if commutativity:

$$A_p(\Delta f) = \Delta A_p(f) = -(A_p(f))'' - \sigma_p(r,\varphi)(A_p(f))' \tag{1.7}$$

holds, then the mean curvature

(1.8) $$\sigma_p = [-(A_p(f))'' - A_p(\Delta f)] / (A_p f)'$$

is a radial function and the space is harmonic.

The above characterization of harmonic manifolds has serveral advantages. To make these perfectly clear we investigate here also the heat kernel on these manifolds.

At first we consider a compact Riemannian manifold M^n and the several investigations for the non–compact case will be given later.

The heat operator of M^n is defined by

(1.9) $$L := \Delta + \frac{\partial}{\partial t}$$

and a solution $u(x;t)$ of the heat equation $L(u) = 0$ is called a heat flow. The solutions of this equation can be determined by the heat kernel $H_t(x,y)$. This kernel function is defined on $M^n \times M^n \times \mathbb{R}_+$ and is characterized by the following properties:

1. It is of class C^1 w.r.t. the variable t and it is of class C^2 w.r.t. the other variables.

(1.10) 2. $L_y H_t(x,y) = 0$ for any fixed point $x \in M^n$.

3. Set $H_t^x(y)$ for the function $y \longrightarrow H_t(x,y)$, then

$$\lim_{t\to+0} H_t^x = \delta_x \quad \text{(Dirac } \delta-\text{function)}$$

is satisfied for any $x \in M^n$.

The <u>existence</u> of such a kernel is assured by well known constructions [30]. The usual simple proof of the <u>uniqueness</u> is as follows.

Let $\lambda_0 = 0 < \lambda_1 \leq \lambda_2 \leq \ldots$ be the (discrete!) spectrum of the Laplace operator Δ. Furthermore, let $(\varphi_0, \varphi_1, \varphi_2, \ldots)$ be the corresponding orthonormal set of eigenfunctions forming a basis in the L^2 function space of M^n. The series

$$H_t^x(y) = \sum f_i(x,t)\varphi_i(y)$$

stands for the L^2 expansion of $H_t^x(y)$ for the fixed points t and x, and thus

$$f_i(x,t) = \int H_t(x,y)\varphi_i(y)dy\,. \tag{1.11}$$

By the properties 1. and 2. we get

$$\frac{\partial f_i}{\partial t} = -\lambda_i f_i\,, \tag{1.12}$$

and therefore by 3.

$$f_i(x,t) = e^{-\lambda_i t}\varphi_i(x) ,$$

(1.13)

$$H_t(x,y) = \sum e^{-\lambda_i t}\varphi_i(x)\varphi_i(y)$$

follow, which proves the uniqueness of the kernel $H_t(x,y)$ on compact Riemannian manifolds.

The series (1.13) is absolutely convergent as it is nothing else than the Parseval formula for the integral

$$\int H_{t/2}(x,z)H_{t/2}(y,z)dz . \tag{1.14}$$

Also the series

$$\sum e^{-\lambda_i t} = \int H_t(x,x)dx , \quad t > 0 , \tag{1.15}$$

is convergent by the Beppo–Levi theorem.

The heat kernel $H_t(x,y)$ is used for the solution of the heat equation $L(u) = 0$ with the initial condition $u(x,0) = u_0(0)$ by the formula

$$u(x,t) = \int H_t(x,y)u_0(y)dy .$$

The situation is much more complicated about the heat kernel of a non–compact Riemannian space as the Laplace operator does not have a discrete spectrum in these cases and we cannot use an orthonormed basis of the eigenfunctions. On the other hand we can derive several heat kernels in such a spaces because of the determined boundary conditions. Very recent results refer to the existence and uniqueness for the heat kernel of a complete non–compact Riemannian manifold which vanishes at the infinity [35], [31], [34]. In the following we use this heat kernel in the case of complete non–compact manifolds. We mention that the assumption on the Rici curvature in Yau's theorem is trivially satisfied here, as now the manifold is Einsteinian with constante norm $||R||$ of the curvature tensor.

A complete Riemannian manifold without boundary is said to be strongly harmonic if the heat kernel $H_t(x,y)$ is a function of t and the distance $r(x,y)$ only, i.e. it is of the form $H_t(x,y) = H_t(r(x,y))$.

In this case the functions $H_t^x = H_t(x,.)$ are radial functions around x. Obviously this weaker property characterizes strong harmonicity, taking into account the symmetry $H_t(x,y) = H_t(y,x)$.

Any strongly harmonic manifold is harmonic (see in [4] p. 172), as can be seen from

$$\Delta_y H_t^x = -H_t^{x''} - \frac{\Theta_x'}{\Theta_x} H_t^{x'} = -\frac{\partial H}{\partial t}. \qquad (1.16)$$

From this equation we get that the mean curvature $\sigma_x = \Theta_x'/\Theta_x$ is also a radial function.

The converse statement is also true for simply connected and complete harmonic manifolds as was proved by D. Michel [32] using the technical method of Brownian motion for the proof. Since this theorem immediately follows from our Basic Commutativity (1.5), we describe the complete proof here.

Theorem 1.1. On the class of simply connected complete Riemannian manifolds, harmonicity and strong harmonicity are equivalent properties.

Proof. We have to prove only that harmonicity implies strong harmonicity.

The simply connected complete harmonic manifolds are globally averageable spaces by the Allamigeon theorem. Therefore the averaged kernel

$$\tilde{H}_t(x,y) := (A_x H_t^x(y)) \tag{1.17}$$

is a globally defined smooth function which obviously satisfies property 3. from (1.10). The equation $L_y \tilde{H}_t(x,y) = 0$ follows immediately from the Basic Commutativity (1.5), and $\tilde{H}_t(x,y) = H_t(x,y)$ follows by the uniqueness of the heat kernel. This proves the radial symmetry of the heat kernel which is just the statement of the Theorem.

2. The analysis of radial functions in harmonic spaces

Any function $f : \mathbb{R}_+ \longrightarrow \mathbb{R}$ generates a radial function f_x around a point $x \in M_n$ defined by $f_x(y) := f(r(x,y))$, where $r(x,y)$ means the geodesics distance between the points $x,y \in M_n$. This function is well defined only for the points y for which $r(x,y)$ is less than the injectivity radius at x. The <u>supporting radius</u> of the function f is defined by the infimum of the values $R \in \mathbb{R}_+$ for which $f([R,\infty)) = 0$ holds. If this radius is less then the injectivity radius at x, then the function f_x is globally defined on M^n.

Now let (M^n,g) be a harmonic manifold with the density function Θ_p. We consider also an eigenfunction φ of the Laplacian Δ with the eigenvalue $\lambda > 0$. From the Basic Commutativity $A_x \Delta\varphi = \Delta A_x\varphi$ we get that the radial function $(A_x\varphi)(r)$ is an eigenfunction with eigenvalue λ. Thus the function $z(r) := (A_x\varphi)(r)$ is the solution of the differential equation

$$z'' + \sigma(r)z' + \lambda z = 0 \tag{2.1}$$

with the initial conditions $z(0) = \varphi(x)$, $z'(0) = 0$. One of the difficulties about this equation is that $\sigma(r)$ has infinite value at $r = 0$; more precisely, it is of the form $\sigma(r) = \sigma^*(r)/r$ with $\sigma^*(0) = n - 1$. The following lemma plays an important role throughout the whole paper. (This statement can be found also in [3] with a different proof.)

<u>Lemma 2.1</u>. The differential equation

$$z'' + \sigma(r)z' + \lambda z = 0 ,$$

where $\lambda > 0$ and $\sigma(r) > 0$ near zero, has only one solution with the initial condition $z(0) = 1$, $z'(0) = 0$.

Proof. For this uniqueness it is enough to prove that the only solution of (2.1) with $z(0) = 0$, $z'(0) = 0$ is the zero function. Now let z be such a solution. So by multiplication with z' we get

$$z''z' + \sigma(z')^2 + \lambda zz' = 0\,;\ \frac{((z')^2)'}{2} + \sigma(z')^2 + \frac{(\lambda z^2)'}{2} = 0\,. \tag{2.2}$$

Introducing the function

$$v = \frac{1}{2}((z')^2 + \lambda z^2) \geq 0 \tag{2.3}$$

the second equation in (2.2) means

$$v' = -(z')^2\sigma \leq 0\,, \tag{2.4}$$

i.e. the function v is non–increasing in a neighbourhood of $r = 0$. Since $v \geq 0$ and $v(0) = 0$, so $v = 0$, $z' = 0$ and $z = 0$ in this neighbourhood. Thus $z = 0$ everywhere by the Picard–Lindelöff theorem. This proves the lemma completely.

If $\varphi(x) \neq 0$ at a point $x \in M^n$ for the eigenfunction φ with the eigenvalue λ then the function

$$\varphi^{\lambda}(r) := \frac{1}{\varphi(x)}(A_x\varphi)(r) \tag{2.5}$$

satisfies the equation (2.1) with the initial conditions $\varphi^{\lambda}(0) = 1$, $(\varphi^{\lambda})'(0) = 0$, so $\varphi^{\lambda}(r)$ is uniquely determined and it is independent from the choice of the point x. Furthermore also

$$(2.6) \qquad (A_x\varphi)(r) = \varphi(x)\varphi_x^{\lambda}(r),$$

holds. This formula can be considered as the generalization of the mean–value theorem for the eigenfunctions of the Laplacian in harmonic manifolds, as for harmonic functions φ we have $\varphi_x^0(r) = 1$.

Formula (2.6) says also that for a fixed point $x \in M^n$ the averaging operator A_x projects the space of eigenfunctions with a common eigenvalue λ onto a one–dimensional function space.

Also the formula

$$(2.7) \qquad A_y(A_x\varphi)(r) = (A_x\varphi)(y)\varphi_y^{\lambda}(r)$$

follows immediately from (2.6).

Next we give a new characterization of harmonicity.

If $H(x,y)$ and $G(x,y)$ are two kernel functions on a Riemannian manifold such that for any x the functions $H_x(.) := H(x,.)$, $G^x(.) := G(.,x)$ are L^2 – functions, then the convolution $H * G$ is defined as usual by

$$(2.8) \qquad H * G(x,y) := \int H(x,z)G(z,y)dz .$$

In the following Proposition we investigate the convolution of two radial kernel functions of the form $H(r(x,y))$; $G(r(x,y))$, where the functions H ; $G : \mathbb{R}_+ \longrightarrow \mathbb{R}$ are of compact support.

Proposition 2.1 A simply connected complete Riemannian manifold is harmonic if and only if the convolution of two radial kernel function is a radial kernel function again.

Proof First we prove that if in a space the convolution of two radial kernel functions is a radial kernel function, then the space is harmonic.

In fact, in this case for any $R > 0$ and for any smooth kernel function $H(r(x,y))$ the kernel functions

$$H^R(x,y) := \int_{S^{n-1}_{y;R}} H(x,p)dp \, ; \tag{2.9}$$

$$\frac{\partial H^R(x,y)}{\partial R}\Big/_{R=0} = \frac{(n+1)!}{2n}\left((\Delta_y H)(x,y) - \frac{1}{3}\rho(y)H(x,y)\right) \tag{2.10}$$

are radial, where Δ_y means the Laplacian acting on the second component and $\rho(y)$ is the Riemannian curvature scalar. This is possible, if ρ is constant and the Minkowskian mean curvature $\sigma_x(y) := \Theta_x'(y)/\Theta_x(y)$ of the geodesics spheres defines a radial function around x. From this harmonicity follows.

For the proof of the converse statement we consider first a simply connected compact harmonic manifold and two radial kernel functions $h(r(x,y)$; $g(r(x,y))$ on it. For a fixed point x the eigenfunctions $\varphi_x^{\lambda_i}$ form an orthogonal basis between the radial L^2 – functions around x, so h_x can be written in the L^2 – series from

$$(2.11) \qquad h_x = \sum_i \alpha_i \varphi_x^{\lambda_i} .$$

So from (2.7) we get

$$(2.12) \qquad \int h_x(z) g_y(z) dz = \sum_i \alpha_i \int \varphi_x^{\lambda_i}(z) g_y(z) dz =$$

$$= \sum_i \alpha_i \int_{S_y^{n-1}} \int_0^{R_g} \varphi_x^{\lambda_i}(r_y, \varphi) g(r) \Theta(r) dr d\varphi$$

$$= \Omega_{n-1} \sum_i \alpha_i \int_0^{R_g} (A_y \varphi_x^{\lambda_i})(r) g(r) \Theta(r) dr$$

$$= \Omega_{n-1} \sum_i \alpha_i \left(\int_0^{R_g} \varphi^{\lambda_i}(r) g(r) \Theta\ (r)\ dr\right) \varphi_x^{\lambda_i}(y) ,$$

where Ω_{n-1} denotes the hypersurface area of the euclidean unit sphere S^{n-1} and R_g is the supporting radius of the function $f : \mathbb{R}_+ \longrightarrow \mathbb{R}$.

As the functions $\varphi_x^{\lambda_i}$ are radial functions around x, so is the function $(h * g)_x(y) = \int h_x(z) g_y(z) dz$. This proves the converse statement in the compact case completely.

In the non–compact case the procedure is similar. In this case for any geodesic ball $B_{x;\delta}$ with the centre x and radius $R_h + R_g < \delta$, a system $\{\varphi_x^{\lambda_i}\}_{i=1}^{\infty}$ of radial eigenfunctions can be choosen in such a way that these span the radial L^2 function space defined on $B_{x;\delta}$ around x. Using these functions, the radiality of $(h * g)_x(y)$ follows with the same computation as before.

This proposition has serveral geometric corollaries supporting the Lichnerowicz conjecture.

Corollary 2.1. Let $B_{x_0 r_1}$ resp. $B_{y r_2}$ be geodesic balls in a harmonic manifold. Then the volume: $\mathrm{vol}(B_{x_0 r_1} \cap B_{y,r_2})$, the hypersurface area: Area $(B_{x_0 r_1} \cap S_{y,r_2})$ and in general the integral:

$$(2.13) \qquad \int_{B_{x_0 r_1} \cap S_{y r_2}} f(r_{x_0}(P))dS_{y,r_2}(P)\,; \quad f : R_+ \longrightarrow \mathbb{R}\,;$$

are constant as y move along the sphere with center x_0 and radius $R = r_{x_0}(y) = \text{constant}$.

The proof is straightforward using the characteristic functions of the balls in the above proposition.

Another simple but interesting corollary of the generalized mean value theorem (2.6) is the following.

For a radial kernel function $H(r(x,y))$ with $R_H < \infty$ we define the convolution $H * f$ on the $L^2(M^n)$ function space by

$$H * f(x) = \int H(r(x,y))f(y)dy \tag{2.14}$$

For a simply connected complete harmonic space we have.

Corollary 2.2. All the globally defined eigenfunctions φ of the Laplacian (with the eigenvalue λ) are the eigenfunctions of the operator $H *$ with the eigenvalue

$$\Omega_{n-1} \int_0^{R_h} \varphi^{\lambda}(r)H(r)\Theta(r)dr . \tag{2.15}$$

where Ω_{n-1} is the area of the euclidean unit sphere S^{n-1} .

The proof easily follows from (2.6) by

$$\int \varphi(z)H_y(z)dz = \Omega_{n-1} \int_0^{R_h} \varphi^{\lambda}(r)H(r)\Theta(r)dr\varphi(y) .$$

3. Besse's Nice Imbedding generalized

A.L. Besse states a beautiful theorem in [4], where he constructed isometric imbeddings of compact strongly harmonic manifolds into the euclidean spaces in such a way that the images of the geodesics are congruent screw lines in the euclidean space. He used for the proof the heat kernel of the manifolds considered.

Now we generalize this statement considerably as we construct similar imbeddings of an arbitrary harmonic manifold into the Hilbert space ℓ^2. Our method will be different from the method of Besse as the heat kernel cannot be used for such general cases. On the other hand, our method gives the Besse's result as a sperical case.

First of all we survey some facts about the screw lines in the Hilbert space ℓ^2. A coherent theory of these curves was given by J. von Neumann and I.J. Schoenberg in [33]. They defined these screw lines in ℓ^2 as the rectificable continuous curves $\underline{r}(s)$, parametrized by the arclength s, for which the distance $||\underline{r}(s_1) - \underline{r}(s_2)||$ in the ℓ^2 – space depends only on the arclength $s_1 - s_2$ for any two points $\underline{r}(s_1)$, $\underline{r}(s_2)$. They called the function

$$S(s) = ||\underline{r}(s_0 + s) - \underline{r}(s_0)||^2 \tag{3.1}$$

the screw function of the screw lines considered and they investigated these functions from the point of view of positive definite functions.

We mention that the above notion of the screw lines in the Hilbert space ℓ^2 can be traced back to the classical screw–line notion easily. In fact, let $\underline{r}(s) \subset \ell^2$ be a c^∞ curve in ℓ^2 which is a screw line in the above sense with the screw function $S(s)$. The Frenet frame $f_1(s) = \dot{\underline{r}}(s)$, $f_2(s) = \ddot{\underline{r}}(s)/|\ddot{\underline{r}}(s)|$, ... e.t.c. is defined as usual in the classical case together with the curvature $K_1 = |\dot{\underline{r}}| = 1$, $K_2 = |\ddot{\underline{r}}|$, ... ,$K_i$, ... e.t.c.

If we transfer the origin of the space ℓ^2 into $\underline{r}(0)$, then by the assumption the function $< \underline{r}(s) , \underline{r}(s) >$ is independent of the choice of the origin $s = 0$ on the curve $\underline{r}$. Therefore the derivatives of this function at $s = 0$ define constant functions along the curve. From this we shall see that curvatures K_i are constant.

We prove this statement by induction. By the Frenet formulas we get $< \underline{r}(s) , \underline{r}(s) >^{(4)}_{s=0} = -2K_2^2$, so K_2 is constant indeed. Assuming, that the curvatures $K_1, \dots ,K_{k-1}$ are constant we prove that K_k is constant.

In fact, by the formulas

(3.2) $$\underline{r}^{(1)} = \underline{f}_1 ,$$

$$\underline{r}^{(2)} = K_2 \underline{f}_2 ,$$

$$\underline{r}^{(3)} = K_2 K_3 \underline{f}_3 - K_2^2 \underline{f}_1 ,$$

$$\vdots$$

$$\underline{r}^{(k-1)} = K_2 K_3 \cdots K_{k-1} \underline{f}_{k-1} + T_{k-1}(K_1, \dots ,K_{k-2}, \underline{f}_{k-3}, \underline{f}_{k-5}, \dots),$$

we get

(3.3) $$\underline{r}^{(k)} = K_2K_3 \cdots K_k \underline{f}_k + T_k(K_1, \cdots, K_{k-1}, \underline{f}_{k-2}, \underline{f}_{k-4}, \cdots)$$

$$\underline{r}^{(k+1)} = K_k'K_2 \cdots K_{k-1} \underline{f}_k + K_2 \cdots K_k K_{k+1} \underline{f}_{k+1} - K_2 \cdots K_{k-1} K_k^2 \underline{f}_{k-1} + \\ + T_{k+1}(K_1, \cdots, K_{k-1}, \underline{f}_1, \cdots, \underline{f}_{k-1})$$

$$\vdots$$

$$\underline{r}^{(k+\ell)} = T^*_{k+\ell}(K_i^{(p)}, \dot{K}_i, \underline{f}_{k+\ell}, \cdots, \underline{f}_{k-\ell+1}) + \\ + (-1)^{\ell} K_2 \cdots K_{k-\ell} K_{k-\ell+1} \cdots K_k^2 \underline{f}_{k-\ell} + \\ + T_{k+\ell}(K_1, \cdots, K_{k-\ell}, \cdots, \underline{f}_{k-\ell}),$$

where the terms T_i, T_i^* are suitable functions (linear combinations) of the arguments. Thus for the derivative $< r,r >^{(2k)}_{s=0}$ we get

(3.4) $$< \underline{r},\underline{r} >^{(2k)}_{s=0} = \sum_{\ell=0}^{k} 2\binom{2k}{\ell} < r^{(\ell)}, r^{(2k-\ell)} >_{s=0} =$$

$$= \sum_{\ell=0}^{k} 2\binom{2k}{k-\ell} < r^{(k-\ell)}, r^{k+\ell} >_{s=0} =$$

$$= 2(-1)^{k+1} K_2^2 \cdots K_k^2 + \sum_{\ell=0}^{k} 2\binom{2k}{k-\ell} < r^{(k-\ell)}, T_{k+\ell} >_{s=0},$$

from which K_k = constant follows obviously.

So the smooth screw lines have constant curvatures. The converse is also obvious.

Returning to the investigations of von Neumann and Schoenberg, they constructed for any screw line $\underline{r}(s)$ a continuous one parametric family U_s of unitary transformations in the space ℓ^2 such that $\underline{r}(s)$ is the orbit of U_s of the form

$$\underline{r}(s) = U_s(\underline{r}(0)) .$$

It can be proved that for two screw lines $\underline{r}_1(s)$, $\underline{r}_2(s)$ with the same screw function $S_1(s) = S_2(s)$ an isometry $v : v_1 \longrightarrow v_2$ between the spaces v_i , spanned by $\{\underline{r}_i(s)\}_{s \in \mathbb{R}}$, exists which takes $\underline{r}_1$ onto $\underline{r}_2$.

After this introduction we construct an isometric immersion of a complete simply connected locally harmonic manifold M^n into the Hilbert space $L^2(M^n) \cong \ell^2$. We mention that this method gives also local imbeddings for a general harmonic manifold without any topological assumption.

For this construction we consider a function $h : \mathbb{R}_+ \longrightarrow R$ of class C^1 with $h'(0) = 0$ and with compact support whose supporting radius R_h is not greater than the radius i_p of injectivity at any point $p \in M^n$.

In the case $i_p = \infty$ we could consider also a function h for which $\int h^2 \Theta \, dt < \infty$; $\int (h')^2 \Theta \, dt < \infty$,i.e. $h , h' \in L^2_\Theta$.

With the help of h we define the map

$$\Phi_h : M^n \longrightarrow L^2(M^n) \tag{3.5}$$

by

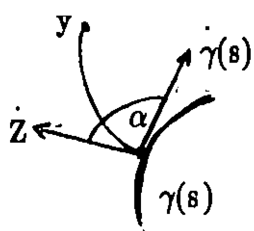
(3.6) $$\Phi_h : p(\in M^n) \longrightarrow h_p(\in L^2(M^n))$$

where $h_p(y)$ is defined by $h_p(y) := h(r(p,y))$ as in § 2.

If $\gamma(s)$ is a geodesic of M^n parametrized by the arc length s, then the tangent vectors of $\Phi_h(\gamma(s))$ are functions again. A simple calculation shows

(3.7) $$\frac{d\Phi_h(\gamma(s))}{ds}(y) = \lim_{t\to 0} \frac{h(r(\gamma(s+t),y)) - h(r(\gamma(s),y))}{t} =$$

$$= \lim_{t\to 0} \frac{h_y(\gamma(s+t)) - h_y(\gamma(s))}{t} = -h'(r(\gamma(s),y))\cos\alpha ,$$



where α is the angle between $\dot y(s)$ and the tangent vector $\dot z(\gamma(s))$ of the geodesic joining $\gamma(s)$ and y. So these tangent vectors have the constant norm:

(3.8) $$||\Phi_h'(\gamma(s))|| = \sqrt{\frac{\Omega_{n-1}}{n}\int_0^{R_h} (h')^2(r)\Theta(r)} = \frac{||h'||_\Theta}{\sqrt{n/\Omega_{n-1}}} .$$

Theorem 3.1 (Imbedding theorem of harmonic spaces)

1) For any radial kernel function $h(r(p,y))$ above the map

$$(3.9) \qquad \underline{r}_h = \phi_{qh} : M^n \longrightarrow L^2(M^n)\ ;\ \underline{r}_h : p \longrightarrow qh_p(y)\ ;$$

where $q := \sqrt{n/\Omega_{n-1}} / \|h'\|_\Theta$, is an isometric immersion of a harmonic space M^n into the sphere S_Q of $L^2(M^n)$ with the radius $Q = \sqrt{n}\ \|h\|_\Theta / \|h'\|_\Theta$.

2) The geodesics of $\underline{r}_h(M^n)$ are congruent screw lines in the space $L^2(M^n) \cong \ell^2$.

3) The submanifold $\underline{r}_h(M^n) \subset S_Q$ is minimal in the sphere S_Q iff the functions $h_p(y)$ are eigenfunctions of the Laplacian Δ. In this case the eigenvalues have the form $\lambda = n/Q^2$ automatically because $\int \nabla_i h \nabla^i h = \int h \Delta h$.

Proof Using (3.8), for any geodesic $\gamma(s)$ of M^n we have $\|r_h'(\gamma(s)\|_{L^2} = 1$, so $\underline{r}_h$ is an isometric immersion indeed. From $\int_{M^n} h_p^2 = \Omega_{n-1} \int h^2(r)\Theta(r)dr = \Omega_{n-1}\|h\|_\Theta^2$ we get

$$(3.10) \qquad \underline{r}_h(M^n) \subset S_Q$$

obviously.

Furthermore for any two points $\underline{r}_h(\gamma(s_1))$, $\underline{r}_h(\gamma(s_2)$ the inner product

$$(3.11)\qquad F_\gamma(s_1,s_2) := \langle \underline{r}_h(\gamma(s_1)), \underline{r}_h(\gamma(s_2)) \rangle =$$

$$= \frac{n/\Omega_{n-1}}{||h'||^2_\Theta} \int h_{\gamma(s_1)}(y) h_{\gamma(s_2)}(y)$$

depends only on the geodesics distance $|s_1 - s_2|$ by Proposition 2.1, so also the function

$$||r_h(\gamma(s_1)) - r_h(\gamma(s_2))||^2 = 2\,Q^2 - 2F_\gamma(s_1,s_2) = 2\,Q^2 - 2F(|s_1 - s_2|)$$

depends only on the geodesics distance $|s_1 - s_2|$. This means that the geodesics of $\underline{r}_h(M^n)$ are congruent screw lines in $L^2(M^n)$ with the common screw function $2(Q^2 - F(s)) = S(s)$.

For the proof of the last statement we consider also an orthonormal basis $\varphi_1, \varphi_2, \dots$ in the Hilbert space $L^2(M^n)$ with the coordinate functions

$$(3.12)\qquad x^i(p) = \langle \varphi_i, \underline{r}_h(p) \rangle = \int \varphi_i\, \underline{r}_h(p)\,.$$

By a well known theorem ([37], p. 342) the submanifold $\underline{r}_h(M_n) \subset S_Q$ is minimal in S_Q iff

$$(3.13)\qquad \Delta x^i = (n/Q^2)x^i\,, \quad i = 1,2, \dots ,$$

i.e. iff

$$(3.14)\qquad \int_{M_n} \varphi_i(x) \Delta_p h(r(p,x))dx = \frac{n}{Q^2} \int_{M_n} \varphi_i(x) h(p,x))dx$$

and consequently

$$(3.15) \qquad \Delta_p h(r(p,x) = (\Delta_x h_p)(x) = \frac{n}{Q^2} h_p(x)$$

satisfies for any $p \in M^n$. This proves the theorem completely.

In the cases of compact strongly harmonic manifolds the space is a Blaschke manifold with a simply closed geodesics with constant length $2L$ [4]. So for any eigenvalue $\lambda \in \text{spect}\,(\lambda_i)_{M^n}$ of the spectrum a uniquely determined radial eigenfunction φ_x^λ exists with $\varphi_x^\lambda(x) = 1$ and with the eigenvalue λ, as the space is globally averageable. Furthermore the functions $\varphi_x^\lambda(.) = \varphi^\lambda(r(x,.))$ span a finite dimensional subspace in $L^2(M^n)$, namely the eigensubspace V^λ. Thus the map

$$(3.16) \qquad \underline{r}_{\varphi^\lambda} : M^n \longrightarrow L^2(M^n)$$

maps the manifold M^n into the sphere S_Q of V^λ such that all the geodesics are congruent screw lines in V^λ. The minimality of $\underline{r}_{\varphi^\lambda}(M^n) \subset S_Q$ in S_Q follows from the fact that φ_x^λ are eigenfunctions for any x. Besse constructed exactly these maps for compact strongly harmonic manifolds and called them nice imbeddings of compact strongly harmonic manifolds.

We describe yet some more useful formulas. Let $\varphi_1, \dots, \varphi_\ell$ be the orthonormal basis in V^λ, so φ_x^λ is of the form

$$(3.17) \qquad \varphi_x^\lambda(y) = a_1\varphi_1(y) + \dots + a_\ell\varphi_\ell(y)\,,$$

with

$$(3.18)\qquad a_i(x) = \int_{M_n} \varphi_x^\lambda(y)\varphi_i(y)dy = \Omega_{n-1}\varphi_i(x)\int_0^L (\varphi^\lambda(r))^2\,\Theta(r)dr\,.$$

Thus for any strongly harmonic manifold, we have

$$(3.19)\qquad \varphi_x^\lambda(y) = \Omega_{n-1}\int_0^L (\varphi^\lambda(r))^2\,\Theta(r)dr\sum_{i=1}^{\ell}\varphi_i(x)\varphi_i(y) =$$

$$= A_\lambda \sum \varphi_i(x)\varphi_i(y)\,.$$

From these we get

$$(3.20)\qquad <\varphi_{x_1}^\lambda, \varphi_{x_2}^\lambda> = A_\lambda^2 \sum \varphi_i(x_1)\varphi_i(x_2) = A_\lambda \varphi_{x_1}^\lambda(x_2)\,,$$

which means that the restriction of the eigenfunction φ_x^λ onto a geodesic $\gamma(r)$ with $\gamma(0) = x$ is of the form

$$(3.21)\qquad \varphi^\lambda(r) = \varphi_x^\lambda(\gamma(r)) = B_\lambda < \underline{r}_{\varphi^\lambda}(x)\,, \underline{r}_{\varphi^\lambda}(\gamma(r)) >\,,$$

where B_λ depends only from λ obviously.

4. The proof of the Lichnerowicz conjecture for compact simply connected harmonic manifolds

We prove the conjecture for compact simply connected harmonic manifolds step by step using more lemmas. Note that then the conjecture is established for a compact harmonic manifold with finite fundamental group.

First of all we answer the following elementary question. Let $f_h(t) : f(t+h)$ stand for the parallel displacement of a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ w.r.t. a real number $h \in \mathbb{R}$. Our question is as follows: What are the continuous functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ for which the functions $\{f_h\}_{h \in \mathbb{R}}$ span a function space of finite dimension?

Although the following answer is classical, we will give a short proof here, for the sake of completeness.

Lemma 4.1. The functions $\{f_h\}_{h\in\mathbb{R}}$ span a function–space V of finite dimension iff f is of the form

$$f(x) = \sum_{i=1}^{k} P_i(x)\sin\alpha_i x + Q_i(x)\cos\beta_i x + R_i(x)e^{\gamma_i x}, \tag{4.1}$$

where $P_i(x)$, $Q_i(x)$, $R_i(x)$ are polynomials.

Proof. It is easy to show that for the functions of the form (4.1) the function–space V spanned by $\{f_n\}_{h\in\mathbb{R}}$ is of finite dimension indeed.

Conversely, if V is of finite dimension then let

(4.2) $$\Phi_h : V \longrightarrow V\,, \qquad \Phi_h : g(x) \longrightarrow g_h(x)$$

be the operator of the parallel displacement in V . Then $\{\Phi_h\}_{h\in\mathbb{R}}$ is a continuous one–parametric family of linear transformations in V , because

(4.3) $$\Phi_h(\alpha_1 g_1+\alpha_2 g_2) = \alpha_1\Phi_h(g_1)+\alpha_2\Phi_h(g_2); \;\; \alpha_1,\alpha_2 \in \mathbb{R},\;\; g_1,g_2 \in V$$
$$\Phi_0 = \mathrm{id}\,,\; \Phi_{h_1+h_2} = \Phi_{h_1} \circ \Phi_{h_2}$$

hold trivially. By the Cartan theorem (which is the finite dimensional version of the Stone theorem) Φ_h is of the form

(4.4) $$\Phi_h = \exp hX = \sum \frac{h}{k!} X^k$$

for a linear endomorphism $X : V \longrightarrow V$. So the function f(x) is not only continuous but of class C^∞ for which the i–th derivative is just the continuous function $X^i(f)$. More precisely, f is an analytic function, as the curve $c(h) : h \longrightarrow f_h = \Phi_h(f)$ in V is analytic with the convergent Taylor expansion

(4.5) $$f_h(x) = \sum \frac{h^k}{k!} X^k(f)/k \;\;,\; |h| < \epsilon < 0\;.$$

Therefore the Taylor expansion

(4.6) $$f(x) = \sum \frac{f^{(i)}(0)}{i!} x^i = \sum \frac{x^i}{i!} X^i(f)(0)$$

is convergent for $|x| < \epsilon$.

It is also plain that the derivatives $d^i f/dx^i := f^{(i)}$ belong to V and, as V is of finite dimension, $f^{(k)}$ is a linear combination of the functions $f^{(0)} = f, f^{(1)},\dots,f^{(k-1)}$ for some k . Therefore the function f is the solution of a differential equation of constant coefficients of the form

(4.7) $$\sum_{i=0}^{k} A_i f^{(i)} = 0 \ , \ A_i \in \mathbb{R} \ , \ A_k = 1 \ ,$$

so f is of the form (4.1) by a rather well known classical theorem.

Using Allamigeon's theorem, we assume that the space is normalized in such a way that the total length of a geodesic is 2π . So the generator function $\varphi^\lambda(r)$ of a radial eigenfunction with $\lambda \in \mathrm{Spect}\{\lambda_i\}_{M^n}$ is a function with period 2π .

<u>Lemma 4.2</u>. The functions $\varphi^\lambda(r)$, $\lambda \in \{\lambda_i\}/_{M^n}$ of a normalized harmonic manifold with the diameter π are of the form $\varphi^\lambda(r) = P_\lambda(\cos r)$, where the P_λ–s are polynomials.

<u>Proof</u>. The functions $\varphi^\lambda_x \in V^\lambda$ span the finite dimensional eigensubspace V^λ , so for any geodesic $\gamma(r)$ the functions $\varphi^\lambda_{\gamma(r)}$ span a finite dimensional space. The restrictions of the functions $\varphi^\lambda_{\gamma(r)}$ to γ form a parallel displaced family of functions in the above sense by

Lemma 4.1. As these span a finite dimensional function–space and these are even periodic functions, so the generator function $\varphi^\lambda(r)$ is of the form

$$\varphi^\lambda(r) = \sum_{i=1}^{k} A_i \cos \alpha_i r \ , \ A_i, \alpha_i \in \mathbb{R} \ .$$

We prove that the distinct (!) values α_i are uniquely determined natural numbers.

The distinct values α_i are uniquely determined for φ^λ. Supposing the contrary we have a non–trivial linear combination

$$\sum_{i=1}^{\ell} B_i \cos \alpha_i r = 0 \ . \tag{4.9}$$

By the derivation we have

$$\sum_{i=1}^{\ell} B_i \alpha_i^{2k} = 0 \ , \ k = 0,1,2,\ldots \ , \tag{4.10}$$

which is a contradiction, since the Vandermonde matrix $\{a_i^k\}_{i=1,\ldots,\ell}^{k=0,\ldots,\ell-1} := \{\alpha_i^{2k}\}$ has non–vanishing determinant.

From the periodicity $\varphi^\lambda(r+2\pi) = \varphi^\lambda(r)$ and from the above consideration we get $\cos\alpha_i\pi = 1$, $\sin 2\alpha_i\pi = 0$. So any value α_i in (4.8) is a natural number and therefore, by the Csebisev polynomials , $\varphi^\lambda(r)$ is a polynomial of $\cos r$.

At the next we prove a similar statement for the density function $\Theta^2(r)$.

Lemma 4.3. The function $\Theta^2(r)$ is also a trigonometric polynomial of the form $\Theta^2(r) = T(\cos r)$ for any compact normalized harmonic manifold.

Proof. Let

$$(4.11) \qquad \underline{\mathrm{I}}_\lambda : M^n \longrightarrow V^\lambda$$

be the Nice Imbedding of M^n into V^λ w.r.t. an eigenvalue $\lambda \in \{\lambda_i\}_{M^n}$. We consider a variation x_r^s, $-\epsilon < s < \epsilon$ of a geodesic $x_r = x_r^0$. Then the map

$$(4.12) \qquad \underline{\mathrm{I}}_\lambda(r,s) := \underline{\mathrm{I}}_\lambda(x_r^s) : \mathbb{R}\times(-\epsilon,\epsilon) \longrightarrow V^\lambda$$

has the property that for any values of s the curves $r \longrightarrow \underline{\mathrm{I}}_\lambda(r,s)$ are congruent screw lines in V^λ. So a differential operator

$$(4.13) \qquad L = \sum_{i=0}^{k} A_i \frac{d^i}{dr^i}, \quad A_i \in \mathbb{R}$$

of constant coefficients exists* such that

$$L(\underline{r}_\lambda) = 0 \tag{4.14}$$

holds for any point (r,s). So we get

$$0 = \frac{\partial}{\partial s} L(\underline{r}_\lambda) = L\left(\frac{\partial \underline{r}_\lambda}{\partial s}\right) , \tag{4.15}$$

which means that the Jacobian field

$$Y(r) := \frac{\partial \underline{r}_\lambda}{\partial s}_{s=0} \tag{4.16}$$

is also a solution of the differential equation

$$L(Y) = 0 \; . \tag{4.17}$$

Let $e_1,\ldots,e_p$ be an orthonormal basis in V^λ. As the differential equations $L(Y^i) = 0$ are satisfied for the functions $Y^i(r) = \langle Y(r),e_i\rangle$ we get (as in the previous lemma) that Y^i is a trigonometric polynomial of the form

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* This can be derived by the last Frenet formula using also the formulas (3.2) for the expression of the Frenet basis $\{\underline{f}_i\}$ with the help of $\underline{r}^{(k)}$'s.

$$Y^i(r) = \sum_{j=0}^{k_i} A_j \sin jx + B_j \cos jx \ . \tag{4.18}$$

Now let $Y_{(1)},\dots,Y_{(n-1)}$ be Jacobian fields along x_r with $Y_{(j)}(0) = 0$; furthermore the vectors $E_{(j)} := Y'_{(j)}(0)$ form an orthonormed basis in the hyperspace of $T_{x_0}(M^n)$ orthogonal to $\dot{x}_0$. So the norm of the (n–1)–form

$$Y_{(1)} \wedge Y_{(2)} \wedge \dots \wedge Y_{(n-1)}(r)$$

along x_r is just $\Theta(r)$.

On the other hand we have

$$Y_{(1)} \wedge \dots \wedge Y_{(n-1)}(r) = \sum_{1 \le j_1,\dots,j_{n-1} \le p} Y^{j_1}_{(1)}(r) \dots Y^{j_{n-1}}_{(n-1)} e_{j_1} \wedge \dots \wedge e_{j_{n-1}} =$$

$$= \sum_{1 \le i_1 < i_2 < \dots < i_{n-1} \le p} Q^{i_1 i_2 \dots i_{n-1}}(Y^\ell_{(k)}) e_{i_1} \wedge \dots \wedge e_{i_{n-1}} \ ,$$

where the functions $Q^{i_1,\dots,i_{n-1}}$ are suitable polynomials of the functions $Y^\ell_{(k)}$, i.e. these coefficients are trigonometric polynomials again. Thus the function

$$\Theta^2 = \sum (Q^{i_1 \dots i_{n-1}})^2$$

is a trigonometric polynomial. As $\Theta^2(r)$ is a periodic even function so it is of the form $\Theta^2(r) = T(\cos r)$ indeed, where $T(x)$ is a polynomial.

Now we examine the roots of the polynomials P_λ and T. Our aim is to prove that the polynomial T has only the roots $+1$ and -1. First of all we consider the polynomial P_λ.

Lemma 4.4. Neither $+1$ nor -1 is a root of P_λ; further more all the roots of P_λ have multiplicity one.

Proof. For $\varphi_\lambda(r) = P_\lambda(\cos r)$, $\varphi_\lambda(0) = P_\lambda(1) = 1$ holds so the value $+1$ is not a root of P_λ. Let us introduce also the function $z(r) := \varphi_\lambda(\pi - r)$, for which we have

$$z'' + \tilde{\sigma} z' = -\lambda z \quad , \tag{4.22}$$

where $\tilde{\sigma}(r) := -\sigma(\pi - r)$ is a positive function for small values of r, as the function σ is negative near π. (In fact, $\Theta(r)$ is a decreasing function near π.) As $z'(0) = -\varphi_\lambda'(\pi) = \sin(\pi)P'(\cos(\pi)) = 0$, we have $z(0) = \varphi_\lambda(\pi) = P_\lambda(\cos\pi) = P_\lambda(-1) \neq 0$ by virtue of Lemma 2.1. Thus the value -1 is not a root of P_λ.

Now we return to the second part of the lemma. The equation

$$\varphi_\lambda'' + \frac{\Theta'}{\Theta}\varphi_\lambda' = -\lambda\varphi_\lambda \tag{4.23}$$

can be written also in the form:

$$\frac{\theta'}{\theta} + \frac{\varphi''_\lambda}{\varphi'_\lambda} = \frac{1}{2}\left(\frac{(\theta^2)'}{\theta^2} + \frac{((\varphi'_\lambda)^2)'}{(\varphi'_\lambda)^2}\right) =$$

(4.24)

$$= \frac{1}{2}\frac{(\theta^2(\varphi'_\lambda)^2)'}{\theta^2(\varphi'_\lambda)^2} = -\lambda\,\frac{\varphi_\lambda}{\varphi'_\lambda} \ .$$

The function $\theta^2(\varphi'_\lambda)^2$ is a trigonometric polynomial of the form $\theta^2(r)(\varphi'_\lambda)^2(r) = Q(\cos r)$ by the above lemmas, therefore

(4.25) $$(\ell n\ Q(\cos r))' = 2\lambda\,\frac{P_\lambda(\cos r)}{\sin r\ P'(\cos r)}$$

(4.26) $$\ell n\ Q(\cos r) = -2\lambda \int \frac{P_\lambda(\cos r)}{-\sin r\ P'(\cos r)}\,dr \ .$$

Using the substitution $x = \cos r$ we get

(4.27) $$\ell n\ Q(x) = -2\lambda \int \frac{P_\lambda(x)}{(1-x^2)P'(x)}\,dx \ .$$

Let $K_1,\dots,K_r$ be the roots of P_λ with multiplicities $a_1,\dots,a_r$. Then the derived polynomial P'_λ has the values K_i as roots exactly with multiplicities (a_i-1). Furthermore for P'_λ we have additional new roots $\mu_1,\dots,\mu_\ell$ (different from the K_i) with multiplicities say $b_1,b_2,\dots,b_\ell$. So we have

(4.28) $$\ell n\ Q(x) = \int q\,\frac{(x-K_1)\dots(x-K_r)}{(1-x)(1+x)(x-\mu_1)^{b_1}(x-\mu_2)^{b_2}\dots(x-\mu_\ell)^{b_\ell}}\,dx \ ,$$

where $q = -2\lambda/(a_1 + \dots + a_r)$ is a constant.

Using the method of the partial fraction for the integration of the right side, we have that this integral is of the form $\ell n\, Q(x)$ for a polynomial $Q(x)$ iff $b_1 = b_2 = \dots = b_\ell = 1$ and $\mu_i \neq \pm 1$. Furthermore $Q(x)$ in this case is of the form

$$Q(x) = a(1-x)^A(1+x)^B(x-\mu_1)^{B_1} \dots (x-\mu_\ell)^{B_\ell} \tag{4.29}$$

with suitable constants $a, A, B, B_1, \dots, B_\ell$.

On the other side we have

$$\frac{(\theta^2)'}{\theta^2} = -\frac{((\varphi_\lambda')^2)'}{(\varphi_\lambda')^2} + \frac{(Q(\cos r))'}{Q(\cos r)}$$

(4.30)

$$T(\cos r) = (\varphi_\lambda')^{-2}(r)Q(\cos r) = (1-\cos^2 r)^{-1}(P'(\cos r))^{-2}Q(\cos r)$$

and so

$$T(x) = p(1-x)^{A-1}(1+x)^{B-1}(x-K_1)^{-2(a_1-1)} \dots$$

(4.31)

$$\dots (x-K_r)^{-2(a_r-1)}(x-\mu_1)^{B_1-2} \dots (x-\mu_\ell)^{B_\ell-2}$$

where $K_i \neq \mu_j$, $K_i \neq \pm 1$. So if some multiplicity a_i were greater than 1, then $-2(a_i-1) < 0$ and thus $T(x)$ would not be a polynomial. This proves the remaining statement

$$a_1 = a_2 = \ldots = a_r = 1$$

completely.

Now we turn to the examination of the roots of the polynomial $T(x)$. The values $+1$ and -1 are roots of T as the $\theta^2(r) = T(\cos r)$ vanishes at $r = 0$ and at $r = \pi$. The multiplicity of these roots are denoted by A resp. B.

Let $\gamma_1, \ldots \gamma_\ell$ be the other roots of $T(x)$ with the multiplicities $G_1, \ldots, G_\ell$. So $\theta^2(r)$ is of the form

$$\theta^2(r) = +c(1-\cos r)^A(\cos r+1)^B(\cos r-\gamma_1)^{G_1}\ldots(\cos r-\gamma_\ell)^{G_\ell} \,,$$

(4.32)

$$\theta^2(r) = c \sin^p r(1-\cos r)^q(\cos r-\gamma_1)^{G_1}\ldots(\cos r-\gamma_\ell)^{G_\ell}$$

with $p = 2A$, $q = B-A$.

Lemma 4.5. All the roots $\gamma_i \neq \pm 1$ of $T(x)$ are also the roots of the polynomial $P'_\lambda(x)$, $\lambda \in \{\lambda_i\}_{M^n}$.

Proof. From the equation

$$\frac{1}{2}\frac{(\theta^2)'}{\theta^2}\varphi'_\lambda = -\varphi''_\lambda - \lambda\varphi_\lambda \tag{4.33}$$

we have

$$\frac{1}{2}(1-x^2)\frac{T'(x)}{T(x)}P'(x) = -\lambda P(x)+xP'(x)-(1-x^2)P''(x) \tag{4.34}$$

and thus from (4.31) the function

$$\frac{1}{2}(1-x^2)P_\lambda'(x)\left(\frac{-A}{1-x}+\frac{B}{(1+x)}+\frac{G}{(x-\gamma_1)}+\dots+\frac{G_\ell}{(x-\gamma_\ell)}\right) \tag{4.35}$$

is a polynomial. This is possible iff the roots γ_i are also the roots of $P_\lambda'(x)$.

The following lemma is much more important in these considerations.

Lemma 4.6. All the roots $K_1,\dots,K_r$ of P_λ and all the roots $\mu_1,\dots,\mu_{r-1}$ of P_λ' are real numbers lying in the interval $(-1,1)$, i.e.

$$-1 < K_1 < \mu_1 < K_2 < \mu_2 < \dots < \mu_{r-1} < K_r < 1 \tag{4.36}$$

Proof. By the formula (4.24) we have

$$(\Theta^2(\varphi_\lambda')^2)' = -2\lambda\Theta^2\varphi_\lambda\varphi_\lambda' , \tag{4.37}$$

and so

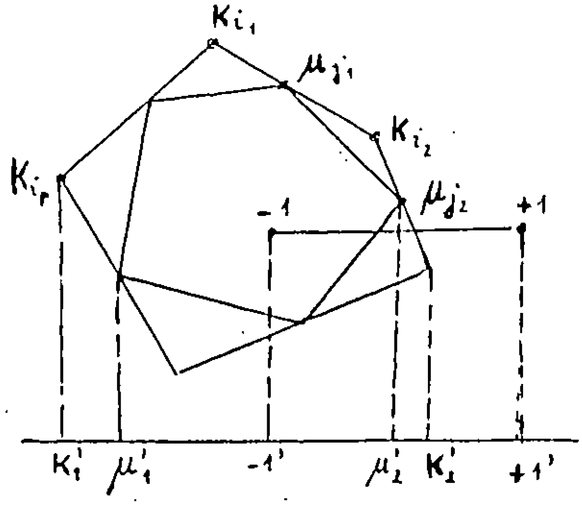
$$((1-x^2)T(x)P'(x)P'(x))' = -2\lambda T(x)P(x)P'(x) . \tag{4.38}$$

The roots of the polynomial $(1-x^2)T(x)P'(x)P'(x)$ are exactly the values $+1,-1,\mu_1,\dots,\mu_{r-1}$ and the roots of $T(x)P(x)P'(x)$ are exactly the values

$+1,-1,K_1,\dots,K_r$, $\mu_1,\dots,\mu_{r-1}$ by the above lemmas. As the roots of a derived polynomial lie in the convex hull of the roots of the original polynomial by Lucas' theorem, we have

$$(4.39) \qquad \{+1,-1,K_1,\dots,K_r,\mu_1,\dots,\mu_{r-1}\} \subset \operatorname{conv}\{+1,-1,\mu_1,\dots,\mu_{r-1}\} \ .$$

We show that this situation is possible only in the case where all the roots $K_1,\dots,K_r$ of P_λ (and consequently also all the roots $\mu_1,\dots,\mu_r$ of P'_λ) lie in the interval $(-1,+1)$. In fact, the convex hull of the roots $K_1,\dots,K_r$ of P_λ contains the roots $\mu_1,\dots,\mu_{r-1}$ of P'_λ . As the multiplicity of any root K_i is exactly one, the vertices $K_{i_1},\dots,K_{i_\ell}$ of $\operatorname{conv}\{K_1,\dots,K_r\}$ are different from the vertices $\mu_{j_\ell},\dots,\mu_{j_k}$ of $\operatorname{conv}\{\mu_1,\dots,\mu_{j_k}\}$.



So if g is such a line on the complex plane which is not orthogonal to any of the sides of $\operatorname{conv}\{K_1,\dots,K_r\}$, then the orthogonal projection of $\operatorname{conv}\{K_1,\dots,K_r\}$ onto g is an interval $[K'_1,K'_2]$ which <u>properly</u> contains the orthogonal projection $[\mu'_1,\mu'_2]$ of $\operatorname{conv}\{\mu_1,\dots,\mu_{r-1}\}$, i.e. $K'_1 < \mu'_1 < \mu'_2 < K'_2$ holds.

Now, if the roots $K_1,\dots,K_r$ did not lie in the interval $(-1,+1)$, then it would be possible to choose such a line g which has the additional property: The orthogonal projection $[-1',+1']$ of $[-1,+1]$ onto g does not contain the orthogonal projection $[K'_1K'_2]$ of $\operatorname{conv}\{K_1,\dots,K_r\}$. So at least one of the points K'_1,K'_2 (say K'_1) is not contained in $[-1',+1']$. In this case

$$K_1' \notin \operatorname{conv}\{\mu_1',\mu_2',-1',1'\} \ ,$$

(4.40)

$$K_{i_p} \notin \operatorname{conv}\{+1,-1,\mu_1,\dots,\mu_{r-1}\}$$

would hold, where K_{i_p} is the root of P_λ whose orthogonal projection onto g is just the point K_1'. This contradicts the property (4.39), so all the roots $K_1,\dots,K_r$ are contained in $(-1,+1)$ indeed. The arrangement (4.36) of the roots follows immediately from the fact that the multiplicity of any root K_i is one.

Now we return to the roots of the polynomial $T(x)$.

Lemma 4.7. The polynomial $T(x)$ has only the roots $+1,-1$, so the density function $\Theta(r)$ of a compact normalized harmonic manifold is of the form

$$\Theta(r) = \sin^p(r)(1-\cos r)^q \ . \tag{4.41}$$

Proof. If $T(x)$ had a root μ different from ± 1, then μ would be the root also of $P_\lambda'(x)$ by Lemma 4.5. Using Lemma 4.6, μ would be real with $-1 < \mu < 1$. So if $0 < r_0 < \pi$ were the value for which $\cos r_0 = \mu$ holds, then we would have
$\Theta^2(r_0) = T(\cos r_0) = T(\mu) = 0$, which is a contradiction as $\Theta^2(r)$ is strictly positive on the interval $0 < r < \pi$ and vanishes only at the endpoints 0 and π. So $T(x)$ has only the roots ± 1 and $\Theta(r)$ is of the form

$$\Theta(r) = (1-\cos r)^{A^*}(1+\cos r)^{B^*} = \sin^p r(1-\cos r)^q \ , \tag{4.42}$$

where $p = 2B^*$, $q = A^* - B^*$.

Lemma 4.8. Any normalized $(2L = 2\pi)$ compact strongly harmonic manifold has a Laplacian eigenfunction of the form

$$\varphi_\lambda = B\cos r + A\,,\quad A+B = 1\,, \tag{4.43}$$

whose eigenvalue λ is the least non-trivial eigenvalue of the Laplacian. The spectrum (without multiplicity!) is: $\{\lambda_n = n(n+p+q)\}_{n\in\mathbb{N}}$.

Proof. From $\Theta = \sin^p r(1-\cos r)^q$ we have

$$\frac{\Theta'}{\Theta} = \frac{p\cos r}{\sin r} + \frac{q\sin r}{1-\cos r} = \frac{(p+q)\cos r + q}{\sin r}\,, \tag{4.44}$$

so for the function $u = \cos r + q/p+q+1$ we get:

$$u'' + \frac{\Theta'}{\Theta}u' = -\cos r - ((p+q)\cos r + q) = -(p+q+1)u\,, \tag{4.45}$$

i.e. the function $u = \cos + q/p+q+1$ is an eigenfunction with the eigenvalue $\lambda = p+q+1$.

It can be seen easily that for any $n \in \mathbb{N}_+$ an eigenfunction of the form

$$\cos^n r + A_1\cos^{n-1} r + \dots + A_{n-1}\cos r + A_n\,,\quad A_i \in \mathbb{R} \tag{4.46}$$

exists whose eigenvalue is

$$\lambda_n = n(n+p+q)\,. \tag{4.47}$$

This proves the lemma completely.

Lemma 4.9. Let $\underline{r} : M^n \longrightarrow V^{\lambda_1}$ be the Nice Imbedding of a compact normalized harmonic manifold w.r.t. the first non–trivial eigenfunction $\cos r + A$. Then the geodesics of $\underline{r}(M^n)$ are circles of radius 1 in V^{λ_1} .

Proof. Let $\underline{r}(r) = \underline{r}(\gamma(r))$ be the image set of a geodesic $\gamma(r)$. Then by formula (3.21) and Lemma 4.8 the function $\langle \underline{r}(0), \underline{r}(r) \rangle$ is of the form $B\cos r + A \cdot B$, $A, B \in \mathbb{R}$, so for any r we get

$$\langle \underline{r}(0), \underline{r}''' + \underline{r}' \rangle = 0 \ . \tag{4.48}$$

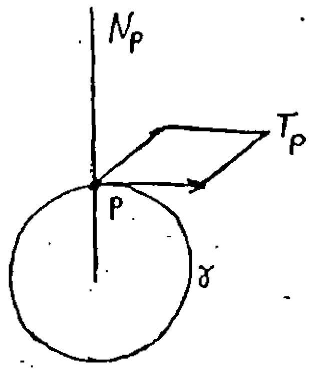
As $\underline{r}(0)$ is arbitrary on the geodesic $\underline{r}(\gamma)$ and as the vectors $\underline{r}''' + \underline{r}'$ lie in the subspace spanned by th vectors $\{\underline{r}(\gamma)\}$, we get

$$\underline{r}''' + \underline{r}' = 0 \ . \tag{4.49}$$

By the Frenet formulas we get that $\underline{r}(\gamma)$ is a plane curve of constant curvature $+1$, i.e. it is a circle.

The following lemma proves the conjecture for the compact harmonic manifolds with finite fundamental groups completely.

Lemma 4.10. Let $M^n \subseteq \mathbb{R}^{k+n}$ be a submanifold such that all the geodesics of M^n are circles in $\mathbb{R}^{k+n}$. Then M^n is a symmetric space; furthermore it is a two–point homogeneous space.



Proof. Let N_p be the orthogonal complement of the tangent space $T_p(M^n)$ at a point $p \in M^n$ in $\mathbb{R}^{n+k}$, and let

$$\tau_p : \mathbb{R}^{n+k} \longrightarrow \mathbb{R}^{n+k} \tag{4.50}$$

be the reflexion w.r.t. the subspace N_p. Thus τ_p is an isometry of the euclidean space $\mathbb{R}^{n+k}$.

As the curvature vectors $\underline{r}''_p$ of the geodesics through p lie in N_p, τ_p leaves these geodesics together with the whole submanifold M^n invariant. Thus τ_p induces an isometry on M^n which is the geodesics involution at p obviously. So M^n is symmetric space. Its rank need to be one, because all the other symmetric spaces have also non–closed geodesics on the maximal torus determined by the rank of the space.

This proves the Lemma and the conjecture completely.

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