Small Elliptic Quantum Group $e_{\tau,\gamma}(\mathfrak{sl}_N)$

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Abstract. The small elliptic quantum group $e_{\tau,\gamma}(\mathfrak{sl}_N)$, introduced in the paper, is an elliptic dynamical analogue of the universal enveloping algebra $U(\mathfrak{sl}_N)$. We define highest weight modules, Verma modules and contragradient modules over $e_{\tau,\gamma}(\mathfrak{sl}_N)$, the dynamical Shapovalov form for $e_{\tau,\gamma}(\mathfrak{sl}_N)$ and the contravariant form for highest weight $e_{\tau,\gamma}(\mathfrak{sl}_N)$-modules. We show that any finite-dimensional $\mathfrak{sl}_N$-module and any Verma module over $\mathfrak{sl}_N$ can be lifted to the corresponding $e_{\tau,\gamma}(\mathfrak{sl}_N)$-module on the same vector space. For the elliptic quantum group $E_{\tau,\gamma}(\mathfrak{sl}_N)$ we construct the evaluation morphism $E_{\tau,\gamma}(\mathfrak{sl}_N) \to e_{\tau,\gamma}(\mathfrak{sl}_N)$, thus making any $e_{\tau,\gamma}(\mathfrak{sl}_N)$-module into an evaluation $E_{\tau,\gamma}(\mathfrak{sl}_N)$-module.

Introduction

The main purpose of this paper is to define a dynamical quantum group $e_{\tau,\gamma}(\mathfrak{sl}_N)$ which is an elliptic dynamical analogue of the universal enveloping algebra $U(\mathfrak{sl}_N)$. We call $e_{\tau,\gamma}(\mathfrak{sl}_N)$ the small elliptic quantum group, comparing it with the elliptic quantum group $E_{\tau,\gamma}(\mathfrak{sl}_N)$ introduced in [F]. Our initial motivation to study this object arises from the wish to understand the structure of evaluation modules over $E_{\tau,\gamma}(\mathfrak{sl}_N)$, which should be analogous to evaluation modules over the Yangian $Y(\mathfrak{sl}_N)$.

Evaluation modules over $E_{\tau,\gamma}(\mathfrak{sl}_2)$ have been defined in [FV1]. They appear naturally in the descripion of transition matrices for the trigonometric *qKZ* difference equation [TV1]. They also serve for the definiton of the *qKZB* difference equations and occur in the description of its monodromies, see [FTV]. One should expect evaluation modules over $E_{\tau,\gamma}(\mathfrak{sl}_N)$ for $N > 2$ to play a similar role. Symmetric and exterior powers of the vector representation of $E_{\tau,\gamma}(\mathfrak{sl}_N)$, developed in [FV2], are examples of evaluation modules over $E_{\tau,\gamma}(\mathfrak{sl}_N)$. In general, evaluation modules over $E_{\tau,\gamma}(\mathfrak{sl}_N)$ arise from $e_{\tau,\gamma}(\mathfrak{sl}_N)$-modules via the evaluation morphism $E_{\tau,\gamma}(\mathfrak{sl}_N) \to e_{\tau,\gamma}(\mathfrak{sl}_N)$, see Corollary 3.4, analogous to the evaluation homomorphism $Y(\mathfrak{sl}_N) \to U(\mathfrak{sl}_N)$.

In this paper we prove the PBW type theorem for the small elliptic quantum group $e_{\tau,\gamma}(\mathfrak{sl}_N)$. We define highest weight modules, Verma modules and contragradient modules over $e_{\tau,\gamma}(\mathfrak{sl}_N)$. We show that for any finite-dimensional $\mathfrak{sl}_N$-module and any Verma module over $\mathfrak{sl}_N$ one can define the corresponding $e_{\tau,\gamma}(\mathfrak{sl}_N)$-module on the same vector space. Pulling them back through the evaluation morphism we get finite-dimensional evaluation modules and evaluation Verma modules over $E_{\tau,\gamma}(\mathfrak{sl}_N)$. Conjecturally, the same picture takes place for any highest weight $\mathfrak{sl}_N$-module.

We introduce the dynamical Shapovalov form for $e_{\tau,\gamma}(\mathfrak{sl}_N)$, the dynamical Shapovalov pairing and the contravariant form for the highest weight modules over $e_{\tau,\gamma}(\mathfrak{sl}_N)$. They play an important role in the construction of finite-dimensional highest weight $e_{\tau,\gamma}(\mathfrak{sl}_N)$-modules. From another point of view the contravariant form for $e_{\tau,\gamma}(\mathfrak{sl}_2)$-modules appeared in the disguised form in [TV, Appendix C].

The small elliptic quantum group $e_{\tau,\gamma}(\mathfrak{sl}_N)$ admits the trigonometric and rational degenerations. They are closely related to the exchange quantum groups $F_q\bigl(SL(N)\bigr)$ and $F\bigl(SL(N)\bigr)$ introduced in [EV2]. In this paper we consider only the rational dynamical quantum group $e_{rat}(\mathfrak{sl}_N)$ and its relation to the exchange quantum group $F\bigl(SL(N)\bigr)$. We contstruct a functor from a certain category of $\mathfrak{sl}_N$-modules to a category of $e_{rat}(\mathfrak{sl}_N)$-modules, see Theorems 7.5, 7.6, and 9.9. We also establish an equivalence of certain tensor categories of $e_{rat}(\mathfrak{sl}_N)$-modules and rational dynamical representations of $F\bigl(SL(N)\bigr)$, see Theorem 10.4. In particular, this gives a new contruction of highest weight representations for the exchange quantum group $F\bigl(SL(N)\bigr)$.

Notice that while $e_{rat}(\mathfrak{sl}_N)$ and exchange quantum groups have a coproduct structure, no coproduct structure is known for the elliptic quantum group $e_{\tau,\gamma}(\mathfrak{sl}_N)$. This makes the small elliptic group similar to the Sklyanin algebra [S], [HW].

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The paper is organized as follows. After introducing basic notation we recall the definiton of the elliptic quantum group $E_{\tau,\gamma}(\mathfrak{sl}_N)$. The small elliptic quantum group $e_{\tau,\gamma}(\mathfrak{sl}_N)$ is defined in Section 3. In Section 4 we introduce highest weight modules and Verma modules over $e_{\tau,\gamma}(\mathfrak{sl}_N)$. The dynamical Shapovalov form for $e_{\tau,\gamma}(\mathfrak{sl}_N)$ is defined in Section 5. Contragradient modules and the contravariant form for highest weight $e_{\tau,\gamma}(\mathfrak{sl}_N)$-modules are defined in Section 6. In Section 7 we study the rational dynamical quantum group $e_{rat}(\mathfrak{sl}_N)$. We construct irreducible finite-dimensional $e_{\tau,\gamma}(\mathfrak{sl}_N)$-modules in Section 8. A functor from a certain category of $\mathfrak{sl}_N$-modules to a category of $e_{rat}(\mathfrak{sl}_N)$-modules is defined in Section 9. Relations between $e_{rat}(\mathfrak{sl}_N)$ and $F\big(SL(N)\big)$ are studied in Section 10. There are six Appendices in the paper; they contain useful technichal information, the example of $e_{\tau,\gamma}(\mathfrak{sl}_2)$, and some proofs.

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1. Basic notation

Let τ be a complex number such that $\operatorname{Im}\tau > 0$. Let $\theta(u;\tau)$ be the Jacobi theta function:

$$\theta(u;\tau) \,=\, -\sum_{m=-\infty}^{\infty} \exp\big(\pi i\tau(m+1/2)^2 + 2\pi i(m+1/2)(u+1/2)\big)\,.$$

There is a product formula

$$\theta(u;\tau) \,=\, 2e^{\pi i\tau/4}\sin(\pi u)\prod_{s=1}^{\infty}(1-e^{2\pi i s\tau})(1-e^{2\pi i(s\tau+u)})(1-e^{2\pi i(s\tau-u)})\,.$$

The function $\theta(u;\tau)$ has multipliers -1 and $-\exp(-2\pi i u-\pi i\tau)$ as $u\to u+1$ and $u\to u+\tau$, respectively. $\theta(u;\tau)$ is an entire function with only simple zeros lying on the lattice $\mathbb{Z}+\tau\mathbb{Z}$. Usually, we do not indicate the second argument of the theta function explicitly, writing $\theta(u)$ instead of $\theta(u;\tau)$.

Let $\mathfrak{h}$ be a finite-dimensional commutative Lie algebra and let $\mathfrak{h}^*$ be the dual space. An $\mathfrak{h}$-module V is called *diagonalizable* if it admits a weight decomposition

$$V \,=\, \bigoplus_{\mu\,\in\,\mathfrak{h}^*} V[\mu]\,,$$

all weight subspaces $V[\mu]$ are finite-dimensional and the set $\{\mu \mid V[\mu]\neq 0\}$ is at most countable.

Let $V_1,\dots,V_k$ be $\mathfrak{h}$-modules. For any function $f:\mathfrak{h}^*\to\operatorname{End}(V_1\otimes\ldots\otimes V_k)$ and any $i=1,\dots,k$ we define an operator $f(h^{(i)})\in\operatorname{End}(V_1\otimes\ldots\otimes V_k)$ by the rule:

$$f(h^{(i)})\,v_1\otimes\ldots\otimes v_N \,=\, f(\mu)\,v_1\otimes\ldots\otimes v_N \qquad \text{for any}\quad v\in V_1\otimes\ldots\otimes V_i[\mu]\otimes\ldots\otimes V_k\,. \tag{1.1}$$

For a finite-dimensional vector space V over $\mathbb{C}$ denote by $Fun(V)$ the space of V-valued meromorphic functions on $\mathfrak{h}^*$. If V is a diagonalizable $\mathfrak{h}$-module, set

$$Fun(V) \,=\, \bigoplus_{\mu\,\in\,\mathfrak{h}^*} Fun\big(V[\mu]\big)\,.$$

$Fun(V)$ is a vector space over $Fun(\mathbb{C})$. The space V is naturally embedded in $Fun(V)$ as the subspace of constant functions. If V is an $\mathfrak{h}$-module, then $Fun(V)$ is an $\mathfrak{h}$-module with the natural pointwise action of $\mathfrak{h}$ and $Fun(V)[\mu]=Fun\big(V[\mu]\big)$.

Let U be a diagonalizable $\mathfrak{h}$-module and a vector space over $Fun(\mathbb{C})$. Suppose that the action of $\mathfrak{h}$ commutes with multiplication by functions. Then each weight subspace $U[\mu]$ is a vector space over $Fun(\mathbb{C})$. Assume that all the weight subspaces are finite-dimensional over $Fun(\mathbb{C})$. Then one can define a diagonalizable $\mathfrak{h}$-module V, such that $U=Fun(V)$ as $\mathfrak{h}$-modules, in the following way. For any μ such that $U[\mu]\neq 0$ pick up a basis $f_1,\dots,f_k$ of $U[\mu]$ over $Fun(\mathbb{C})$ and set $V[\mu]=\bigoplus_{i=1}^{k}\mathbb{C}f_i$, otherwise, set $V[\mu]=0$. Then define $V=\bigoplus_{\mu\,\in\,\mathfrak{h}^*}V[\mu]$ to be the diagonalizable $\mathfrak{h}$-module such that $V[\mu]$ is a weight subspace of weight μ.

Let V, W be diagonalizable $\mathfrak{h}$-modules. The space $\mathrm{Hom}(V,W)$ has the natural $\mathfrak{h}$-module structure, but the weight subspaces are infinite-dimensional in general. We set

$$Fun\big(\mathrm{Hom}(V,W)\big) = \mathrm{Hom}\big(V, Fun(W)\big).$$

A function $\varphi \in Fun\big(\mathrm{Hom}(V,W)\big)$ induces a linear map $Fun(V) \to Fun(W)$, acting pointwise: $f(\lambda) \mapsto \varphi(\lambda) f(\lambda)$. This map is usually denoted by the same letter.

Denote by $D(V)$ the space of difference operators acting in $Fun(V)$. It is spanned over $\mathbb{C}$ by operators of the form $f(\lambda) \mapsto \varphi(\lambda) f(\lambda+\mu)$ where $\varphi \in Fun\big(\mathrm{End}(V)\big)$ and $\mu \in \mathfrak{h}^*$.

As a rule we do not distinguish a function $\varphi(\lambda) \in Fun(\mathbb{C})$ and a function $\varphi(\lambda)\cdot\mathrm{id} \in Fun\big(\mathrm{End}(V)\big)$.

In this paper we take $\mathfrak{h}$ to be the Cartan subalgebra of the Lie algebra $\mathfrak{sl}_N$. Fix a basis $h_1, \dots, h_{N-1}$ of $\mathfrak{h}$. Let $\omega_1, \dots, \omega_{N-1}$ be the fundamental weights: $\langle \omega_a, h_b\rangle = \delta_{ab}$. Let $\mathbb{P} \subset \mathfrak{h}^*$ be the weight lattice: $\mathbb{P} = \bigoplus_{a=1}^{N-1} \mathbb{Z}\,\omega_a$. For any $a = 1, \dots, N$ set $\varepsilon_a = \omega_a - \omega_{a-1}$, where by convention $\omega_0 = \omega_N = 0$. Let $\alpha_1, \dots, \alpha_{N-1}$ be the simple roots: $\alpha_a = \varepsilon_a - \varepsilon_{a+1}$. For $\lambda, \mu \in \mathfrak{h}^*$ say that $\lambda \geqslant \mu$ if $\lambda - \mu \in \bigoplus_{a=1}^{N-1} \mathbb{Z}_{\geqslant 0}\,\alpha_a$.

Define a bilinear form $(\,,)$ on $\mathfrak{h}^*$ by the rule $(\alpha_a, \omega_b) = \delta_{ab}$ for any $a, b = 1, \dots, N-1$. For any $\lambda \in \mathfrak{h}^*$ set $\lambda_a = (\lambda, \varepsilon_a)$. It is easy to see that $\lambda = \sum_{a=1}^{N} \lambda_a \varepsilon_a$, $\sum_{a=1}^{N} \lambda_a = 0$ and $(\lambda, \mu) = \sum_{a=1}^{N} \lambda_a \mu_a$. The Weyl group W acts on $\mathfrak{h}^*$ as the symmetric group $\mathbf{S}_N$ permuting the coordinates $\lambda_1, \dots, \lambda_N$.

Let $\rho = \sum_{a=1}^{N-1} \omega_a = -\sum_{a=1}^{N} a\varepsilon_a$ be the half-sum of positive roots. For any $w \in W$ and $\lambda \in \mathfrak{h}^*$ set $w \cdot \lambda = w(\lambda+\rho) - \rho$. Notice that $(\lambda, \rho) = -\lambda_1 - 2\lambda_2 - \ldots - N\lambda_N$.

Let $E_{ab} \in \mathrm{End}(\mathbb{C}^N)$ be the matrix with the only nonzero entry equal to 1 at the intersection of the a-th row and b-th column. The assignment $h_a \mapsto E_{aa} - E_{a+1,a+1}$, $a = 1, \dots, N-1$, makes $\mathbb{C}^N$ into an $\mathfrak{h}$-module, called the *vector representation* of $\mathfrak{h}$. Henceforth, we always consider $\mathbb{C}^N$ as the vector representation of $\mathfrak{h}$.

Let γ be a nonzero complex number. Introduce functions $\alpha(u,\xi)$ and $\beta(u,\xi)$ as follows:

$$\alpha(u,\xi) = \frac{\theta(u)\theta(\xi+\gamma)}{\theta(u-\gamma)\theta(\xi)}, \qquad \beta(u,\xi) = -\frac{\theta(u+\xi)\theta(\gamma)}{\theta(u-\gamma)\theta(\xi)}. \tag{1.2}$$

Let $R(u,\lambda)$ be the *elliptic dynamical R-matrix* [F]:

$$R(u,\lambda) = \sum_{a=1}^{N} E_{aa} \otimes E_{aa} + \sum_{\substack{a,b=1\\ a\neq b}}^{N} \big(\alpha(u,\lambda_{ab}) E_{aa} \otimes E_{bb} + \beta(u,\lambda_{ab}) E_{ab} \otimes E_{ba}\big), \tag{1.3}$$

where $\lambda \in \mathfrak{h}^*$ and $\lambda_{ab} = \lambda_a - \lambda_b$. The dynamical R-matrix has zero weight:

$$\big[R(u,\lambda), h^{(1)} + h^{(2)}\big] = 0, \tag{1.4}$$

satisfies the inversion relation:

$$R(u,\lambda) R^{(21)}(-u,\lambda) = 1, \tag{1.5}$$

and the dynamical Yang-Baxter equation:

$$\begin{aligned} R^{(12)}(u-v, \lambda - \gamma h^{(3)}) R^{(13)}(u,\lambda) R^{(23)}(v, \lambda - \gamma h^{(1)}) &= \\ = R^{(23)}(v,\lambda) R^{(13)}(u, \lambda - \gamma h^{(2)}) R^{(12)}(u-v,\lambda)&. \end{aligned} \tag{1.6}$$

The last equality holds in $\mathrm{End}(\mathbb{C}^N \otimes \mathbb{C}^N \otimes \mathbb{C}^N)$. By standard convention, we assume that $R^{(ij)}(u,\lambda)$ acts as $R(u,\lambda)$ in the i-th and j-th tensor factors and as the identity operator in the remaining factors. For instance, in formula (1.6) we have $R^{(12)} = R \otimes \mathrm{id}$ and $R^{(23)} = \mathrm{id} \otimes R$. Notice that for the R-matrix (1.3) we in addition have

$$R\big(u, \lambda - \gamma(h^{(1)} + h^{(2)})\big) = R(u,\lambda).$$

2. Elliptic quantum group $E_{\tau,\gamma}(\mathfrak{sl}_N)$

A *module over the elliptic quantum group* $E_{\tau,\gamma}(\mathfrak{sl}_N)$ is a diagonalizable $\mathfrak{h}$-module V together with $D(V)$-valued meromorphic functions $T_{ab}(u)$, $a,b=1,\dots,N$, of a complex variable u, subject to relations (2.1) – (2.3). We combine the functions $T_{ab}(u)$ into a matrix $T(u)$ with noncommuting entries:

$$T(u)\,=\,\sum_{a,b}E_{ab}\otimes T_{ab}(u)\,.$$

The defining relations are:

$$T_{ab}(u)\,\varphi(\lambda)\,=\,\varphi(\lambda-\gamma\varepsilon_b)\,T_{ab}(u) \tag{2.1}$$

for any $\varphi\in \mathit{Fun}(\mathbb{C})$,

$$\bigl[T(u)\,,h^{(1)}+h^{(2)}\bigr]\,=\,0\,, \tag{2.2}$$

$$R^{(12)}(u-v,\lambda-\gamma h^{(3)})T^{(13)}(u)T^{(23)}(v)\,=\,T^{(23)}(v)T^{(13)}(u)\,R^{(12)}(u-v,\lambda)\,. \tag{2.3}$$

The last equality holds in $\mathrm{End}\bigl(\mathbb{C}^N\otimes\mathbb{C}^N\otimes \mathit{Fun}(V)\bigr)$. Here $T^{(13)}(u)\,=\,\sum\limits_{a,b}E_{ab}\otimes\mathrm{id}\otimes T_{ab}(u)$ and $T^{(23)}(u)=\sum\limits_{a,b}\mathrm{id}\otimes E_{ab}\otimes T_{ab}(u)$.

Relations (2.1) can be written as follows: $T(u)\,\varphi(\lambda+\gamma h^{(1)})\,=\,\varphi(\lambda)\,T(u)$ for any $\varphi\in \mathit{Fun}(\mathbb{C})$. Formula (2.2) means that for any $\mu\in\mathfrak{h}^*$

$$T_{ab}(u)\ \mathit{Fun}\bigl(V[\mu]\bigr)\,\subset\,\mathit{Fun}\bigl(V[\mu-\varepsilon_a+\varepsilon_b]\bigr)\,. \tag{2.4}$$

Introduce the *quantum determinant* $\mathrm{Det}\,T(u)$, cf. [FV1], [FV2], by the rule

$$\mathrm{Det}\,T(u)\,=\,\frac{\Theta(\lambda)}{\Theta(\lambda-\gamma h)}\sum_{\boldsymbol{i}\in\mathbf{S}_N}\mathrm{sign}(\boldsymbol{i})\ T_{i_N,N}\bigl(u+(N-1)\gamma\bigr)\,\dots\,T_{i_2,2}(u+\gamma)\,T_{i_1,1}(u) \tag{2.5}$$

where $\Theta(\lambda)\,=\prod\limits_{1\leqslant a<b\leqslant N}\theta(\lambda_a-\lambda_b)$, the sum is taken over all permutations $\boldsymbol{i}=(i_1,\dots,i_N)$ and $\mathrm{sign}(\boldsymbol{i})$ is the sign of the permutation. It is clear that $\mathrm{Det}\,T(u)$ commutes with multiplication by any function $\varphi(\lambda)\in \mathit{Fun}(\mathbb{C})$ and with the action of $\mathfrak{h}$. Hence, $\mathrm{Det}\,T(u)$ acts in $\mathit{Fun}(V)$ as multiplication by some $\mathrm{End}\,(V)$-valued meromorphic function of u and λ. We denote this function by $\mathrm{Det}\,L(u,\lambda)$, cf. (2.6).

Proposition 2.1. $\bigl[\mathrm{Det}\,T(u),T_{ab}(v)\bigr]\,=\,0$ *for any* $a,b=1,\dots,N$.

The proposition is proved in Appendix B.

According to (2.1), $T_{ab}(u)$ is a difference operator; for any $v\in \mathit{Fun}(V)$ we have

$$\bigl(T_{ab}(u)\,v\bigr)(\lambda)\,=\,L_{ab}(u,\lambda)\,v(\lambda-\gamma\varepsilon_b) \tag{2.6}$$

where $L_{ab}(u,\lambda)\in \mathit{Fun}\bigl(\mathrm{End}\,(V)\bigr)$. Set $L(u,\lambda)\,=\sum\limits_{a,b}E_{ab}\otimes L_{ab}(u,\lambda)$.

Example. For any $x\in\mathbb{C}$ the assignment $L(u,\lambda)\mapsto R(u-x,\lambda)$ makes $\mathbb{C}^N$ into an $E_{\tau,\gamma}(\mathfrak{sl}_N)$-module. The module is called the *vector representation* of $E_{\tau,\gamma}(\mathfrak{sl}_N)$ with the *evaluation point* x. The quantum determinant in this $E_{\tau,\gamma}(\mathfrak{sl}_N)$-module is

$$\mathrm{Det}\,L(u,\lambda)\,=\,\frac{\theta\bigl(u-x+(N-1)\gamma\bigr)}{\theta(u-x)}\,.$$

Let V,W be $E_{\tau,\gamma}(\mathfrak{sl}_N)$-modules. An element $\varphi\in \mathit{Fun}\bigl(\mathrm{Hom}_{\mathfrak{h}}(V,W)\bigr)$ is called a *morphism* of $E_{\tau,\gamma}(\mathfrak{sl}_N)$-modules if the induced map $\mathit{Fun}(V)\to \mathit{Fun}(W)$ satisfies

$$\varphi(\lambda)\,T_{ab}(u)\big|_{\mathit{Fun}(V)}\,=\,T_{ab}(u)\big|_{\mathit{Fun}(W)}\,\varphi(\lambda)$$

for any $a, b = 1, \ldots, N$. Denote by $\operatorname{Mor}(V, W)$ the space of all morphisms from V to W. A morphism φ is called an *isomorphism* if the map $\varphi(\lambda)$ is bijective for generic λ.

An $E_{\tau,\gamma}(\mathfrak{sl}_N)$-module V is called *irreducible* if for any nontrivial morphism $\varphi \in \operatorname{Mor}(W, V)$ the map map $\varphi(\lambda)$ is surjective for generic λ, and *reducible*, otherwise.

Let V, W be $E_{\tau,\gamma}(\mathfrak{sl}_N)$-modules. Then the $\mathfrak{h}$-module $V \otimes W$ is made into an $E_{\tau,\gamma}(\mathfrak{sl}_N)$-module by the rule

$$L_{ab}(u, \lambda)\big|_{V\otimes W} = \sum_{c=1}^{N} L_{ac}(u, \lambda - \gamma h^{(2)}) \otimes L_{cb}(u, \lambda)\,, \tag{2.7}$$

and $T_{ab}(u)$ acts in $\mathit{Fun}(V \otimes W)$ according to (2.6). Triple tensor products $(U \otimes V) \otimes W$ and $U \otimes (V \otimes W)$ are canonically isomorphic as $E_{\tau,\gamma}(\mathfrak{sl}_N)$-modules. The quantum determinant is group-like, it acts in the $E_{\tau,\gamma}(\mathfrak{sl}_N)$-module $V \otimes W$ by

$$\operatorname{Det} L(u, \lambda)\big|_{V\otimes W} = \operatorname{Det} L(u, \lambda - \gamma h^{(2)}) \otimes \operatorname{Det} L(u, \lambda)\,.$$

If φ_1, φ_2 are morphisms of $E_{\tau,\gamma}(\mathfrak{sl}_N)$-modules: $\varphi_1 \in \operatorname{Mor}(V_1, W_1)$, $\varphi_2 \in \operatorname{Mor}(V_2, W_2)$, then

$$\varphi_1(\lambda - \gamma h^{(2)}) \otimes \varphi_2(\lambda)$$

is a morphism of $E_{\tau,\gamma}(\mathfrak{sl}_N)$-modules $V_1 \otimes V_2$ and $W_1 \otimes W_2$.

Remark. Notice that the given definition of modules over the elliptic quantum group $E_{\tau,\gamma}(\mathfrak{sl}_N)$ is slightly different from the corresponding definition in [FV1]. The definition of morphisms of $E_{\tau,\gamma}(\mathfrak{sl}_N)$-modules is also suitably modified. We choose the present version in order to simplify the exposition.

3. Small elliptic quantum group $e_{\tau,\gamma}(\mathfrak{sl}_N)$

Let $\mathit{Fun}^{\otimes 2}(\mathbb{C})$ be the ring of meromorphic functions $f(\lambda^{\{1\}}, \lambda^{\{2\}})$ on $\mathfrak{h}^* \oplus \mathfrak{h}^*$ such that location of singularities of $f(\lambda^{\{1\}}, \lambda^{\{2\}})$ in $\lambda^{\{1\}}$ does not depend on $\lambda^{\{2\}}$ and vice versa. For brevity, we write $f(\lambda^{\{1\}})$ or $f(\lambda^{\{2\}})$ instead of $f(\lambda^{\{1\}}, \lambda^{\{2\}})$ if the function does not depend on the other variable.

Given a diagonalizable $\mathfrak{h}$-module V we define an action of $\mathit{Fun}^{\otimes 2}(\mathbb{C})$ in $\mathit{Fun}(V)$: for any $f \in \mathit{Fun}^{\otimes 2}(\mathbb{C})$ set $f : v(\lambda) \mapsto f(\lambda, \lambda - \gamma h)\, v(\lambda)$. For instance, for any $\varphi \in \mathit{Fun}(\mathbb{C})$ we have

$$\varphi(\lambda^{\{1\}}) : v(\lambda) \mapsto \varphi(\lambda)\, v(\lambda)\,, \qquad \varphi(\lambda^{\{2\}}) : v(\lambda) \mapsto \varphi(\lambda - \gamma h)\, v(\lambda)\,. \tag{3.1}$$

We always assume that $\mathit{Fun}^{\otimes 2}(\mathbb{C})$ acts in $\mathit{Fun}(V)$ in this way.

The *operator algebra* $e^{\mathcal{O}}_{\tau,\gamma}(\mathfrak{sl}_N)$ is a unital associative algebra over $\mathbb{C}$ generated by elements t_{ab}, $a, b = 1, \ldots, N$, and functions $f \in \mathit{Fun}^{\otimes 2}(\mathbb{C})$ subject to relations

$$t_{ab}\, f(\lambda^{\{1\}}, \lambda^{\{2\}}) = f(\lambda^{\{1\}} - \gamma\varepsilon_a, \lambda^{\{2\}} - \gamma\varepsilon_b)\, t_{ab} \tag{3.2}$$

for any $f \in \mathit{Fun}^{\otimes 2}(\mathbb{C})$,

$$t_{ab}\, t_{ac} = t_{ac}\, t_{ab}\,, \tag{3.3}$$

$$t_{ac}\, t_{bc} = \frac{\theta(\lambda^{\{1\}}_{ab} + \gamma)}{\theta(\lambda^{\{1\}}_{ab} - \gamma)}\, t_{bc}\, t_{ac}\,, \qquad \text{for} \quad a \neq b\,, \tag{3.4}$$

$$\frac{\theta(\lambda^{\{2\}}_{bd} + \gamma)}{\theta(\lambda^{\{2\}}_{bd})}\, t_{ab}\, t_{cd} - \frac{\theta(\lambda^{\{1\}}_{ac} + \gamma)}{\theta(\lambda^{\{1\}}_{ac})}\, t_{cd}\, t_{ab} = \frac{\theta(\lambda^{\{1\}}_{ac} + \lambda^{\{2\}}_{bd})\, \theta(\gamma)}{\theta(\lambda^{\{1\}}_{ac})\, \theta(\lambda^{\{2\}}_{bd})}\, t_{ad}\, t_{cb}\,, \tag{3.5}$$

for $a \neq c$ and $b \neq d$. Here $\lambda^{\{i\}}_{ab} = \lambda^{\{i\}}_a - \lambda^{\{i\}}_b$.

The ring $\mathit{Fun}^{\otimes 2}(\mathbb{C})$ is embedded into $e^{\mathcal{O}}_{\tau,\gamma}(\mathfrak{sl}_N)$ as a commutative subalgebra. It acts on $e^{\mathcal{O}}_{\tau,\gamma}(\mathfrak{sl}_N)$ by left multiplication. In this paper we consider $e^{\mathcal{O}}_{\tau,\gamma}(\mathfrak{sl}_N)$ as the corresponding $\mathit{Fun}^{\otimes 2}(\mathbb{C})$-module.

The operator algebras $e^{\mathcal{O}}_{\tau,\gamma}(\mathfrak{sl}_N)$, $e^{\mathcal{O}}_{\tau+1,\gamma}(\mathfrak{sl}_N)$ and $e^{\mathcal{O}}_{-1/\tau,-\gamma/\tau}(\mathfrak{sl}_N)$ are isomorphic. The isomorphism $e^{\mathcal{O}}_{\tau,\gamma}(\mathfrak{sl}_N) \to e^{\mathcal{O}}_{\tau+1,\gamma}(\mathfrak{sl}_N)$ corresponds to the property $\theta(u;\tau) = e^{\pi i/4}\,\theta(u;\tau+1)$ of the theta function and is tautological. The isomorphism $e^{\mathcal{O}}_{\tau,\gamma}(\mathfrak{sl}_N) \to e^{\mathcal{O}}_{-1/\tau,-\gamma/\tau}(\mathfrak{sl}_N)$ corresponds to the equality

$$\theta(u;-1/\tau) \,=\, (i\tau)^{1/2}\exp(\pi i\tau u^2)\,\theta(-\tau u;\tau)\,, \qquad \operatorname{Im}(i\tau)^{1/2} > 0\,,$$

and is given by the following formulae

$$f(\lambda^{\{1\}},\lambda^{\{2\}}) \;\mapsto\; f(-\tau\lambda^{\{1\}},-\tau\lambda^{\{2\}})\,,$$

$$t_{ab} \;\mapsto\; \exp\Bigl(\frac{\pi i\tau}{2N}\,\bigl((N+1)\sum_{c=1}^{N}(\lambda^{\{1\}}_{ca})^2 - (N-1)\sum_{c=1}^{N}(\lambda^{\{2\}}_{cb})^2 - 4N\lambda^{\{1\}}_a\lambda^{\{2\}}_b\bigr)\Bigr)\,t_{ab}\,.$$

Introduce a matrix $\mathcal{T}(u)$ with noncommuting entries:

$$\mathcal{T}(u) \,=\, \sum_{a,b} E_{ab}\otimes\mathcal{T}_{ab}(u)\,, \qquad \mathcal{T}_{ab}(u) \,=\, \theta(u-\lambda^{\{1\}}_b+\lambda^{\{2\}}_a)\,t_{ba}\,. \tag{3.6}$$

Theorem 3.1. *The commutation relations* (3.3) – (3.5) *are equivalent to*

$$R^{(12)}(u-v,\lambda^{\{2\}})\,\mathcal{T}^{(13)}(u)\,\mathcal{T}^{(23)}(v) \,=\, \mathcal{T}^{(23)}(v)\,\mathcal{T}^{(13)}(u)\,R^{(12)}(u-v,\lambda^{\{1\}})\,, \tag{3.7}$$

The proof is straightforward and is based on summation formulae for the theta function. Notice that formula (3.7) is similar to (2.3).

Introduce the *quantum determinant* $\operatorname{Det}\mathcal{T}(u)$ like in (2.5):

$$\operatorname{Det}\mathcal{T}(u) \,=\, \frac{\Theta(\lambda^{\{1\}})}{\Theta(\lambda^{\{2\}})}\sum_{\boldsymbol{i}\in\mathbf{S}_N}\operatorname{sign}(\boldsymbol{i})\;\mathcal{T}_{i_N,N}\bigl(u+(N-1)\gamma\bigr)\,\ldots\,\mathcal{T}_{i_2,2}(u+\gamma)\,\mathcal{T}_{i_1,1}(u) \tag{3.8}$$

where $\Theta(\lambda) \,=\, \prod_{1\leqslant a<b\leqslant N}\theta(\lambda_a-\lambda_b)$. By (3.6) and (3.2) we obtain

$$\operatorname{Det}\mathcal{T}(u) \,=\, \frac{\Theta(\lambda^{\{1\}})}{\Theta(\lambda^{\{2\}})}\sum_{\boldsymbol{i}\in\mathbf{S}_N}\operatorname{sign}(\boldsymbol{i})\prod_{a=1}^{N}\theta\bigl(u+(a-1)\gamma-\lambda^{\{1\}}_a+\lambda^{\{2\}}_{i_a}\bigr)\;t_{N,i_N}\,\ldots\,t_{1,i_1}\,. \tag{3.9}$$

It is clear that $\operatorname{Det}\mathcal{T}(u)\,f(\lambda^{\{1\}},\lambda^{\{2\}}) = f(\lambda^{\{1\}},\lambda^{\{2\}})\,\operatorname{Det}\mathcal{T}(u)$ for any $f\in \mathit{Fun}^{\otimes 2}(\mathbb{C})$.

Proposition 3.2. $\bigl[\operatorname{Det}\mathcal{T}(u)\,,t_{ab}\bigr] \,=\, 0$ *for any* $a,b=1,\ldots,N$.

The proof is similar to the proof of Proposition 2.1.

Let $\bar f = (f_1,\ldots,f_N)$ be a multiplicative cocycle with coefficients in $\mathit{Fun}(\mathbb{C})$, that is, the functions $f_1,\ldots,f_N\in\mathit{Fun}(\mathbb{C})$ satisfy the condition

$$f_a(\lambda)\,f_b(\lambda-\gamma\varepsilon_a) \,=\, f_a(\lambda-\gamma\varepsilon_b)\,f_b(\lambda)$$

for any $a,b=1,\ldots,N$, cf. Appendix C. Then the assignments

$$t_{ab}\mapsto f_a(\lambda^{\{1\}})\,t_{ab} \qquad\text{and}\qquad t_{ab}\mapsto f_b(\lambda^{\{2\}})\,t_{ab} \tag{3.10}$$

define two endomorphisms of the operator algebra $e^{\mathcal{O}}_{\tau,\gamma}(\mathfrak{sl}_N)$. The endomorphisms are automorphisms if $\bar f$ is nondegenerate, that is, if $f_a\neq 0$ for any $a=1,\ldots,N$. The automorphisms are inner if $\bar f$ is a multiplicative coboundary: $f_a(\lambda) = \varphi(\lambda-\gamma\varepsilon_a)/\varphi(\lambda)$ for a certain function $\varphi\in\mathit{Fun}(\mathbb{C})$, see (3.2).

Remark. In this paper the inequality $f\neq 0$ for a meromorphic function f means that the function f is not identically zero.

Proposition 3.3. *The assignment*

$$f(\lambda^{\{1\}},\lambda^{\{2\}}) \mapsto f(-\lambda^{\{2\}},-\lambda^{\{1\}}), \tag{3.11}$$

$$t_{ab} \mapsto \prod_{\substack{c=1\\c\neq b}}^{N} \theta(\lambda^{\{1\}}_{cb}) \prod_{\substack{c=1\\c\neq a}}^{N} \bigl(\theta(\lambda^{\{2\}}_{ca}+\gamma)\bigr)^{-1} t_{ba}$$

defines an involutive antiautomorphism of the operator algebra $e^{\mathcal{O}}_{\tau,\gamma}(\mathfrak{sl}_N)$.

A *module over the small elliptic quantum group* $e_{\tau,\gamma}(\mathfrak{sl}_N)$ is a diagonalizable $\mathfrak{h}$-module V endowed with an action of the operator algebra $e^{\mathcal{O}}_{\tau,\gamma}(\mathfrak{sl}_N)$ in the space $Fun(V)$. Relations (3.1) and (3.2) mean that t_{ab} acts in $Fun(V)$ as a difference operator:

$$(t_{ab}\,v)(\lambda) = \ell_{ab}(\lambda)\,v(\lambda-\gamma\varepsilon_a) \qquad \text{for any} \quad v\in Fun(V) \tag{3.12}$$

where $\ell_{ab}(\lambda)\in Fun\bigl(\mathrm{End}\,(V)\bigr)$ is a suitable function. In $D(V)$ relations (3.1), (3.2) are equivalent to

$$t_{ab}\,\varphi(\lambda) = \varphi(\lambda-\gamma\varepsilon_a)\,t_{ab}$$

for any $\varphi\in Fun(\mathbb{C})$, and

$$t_{ab}\;Fun\bigl(V[\mu]\bigr) \subset Fun\bigl(V[\mu+\varepsilon_a-\varepsilon_b]\bigr)$$

for any $\mu\in\mathfrak{h}^*$, which are similar to (2.1) and (2.4). For the quantum determinant we have that

$$\bigl(\mathrm{Det}\,\mathcal{T}(u)\,v\bigr)(\lambda) = \mathcal{D}(u,\lambda)\,v(\lambda)$$

for any $v\in Fun(V)$ where $\mathcal{D}(u,\lambda)\in Fun\bigl(\mathrm{End}\,(V)\bigr)$ is a suitable function.

Example. The assignment

$$\ell_{aa}(\lambda) \mapsto E_{aa} + \sum_{\substack{b=1\\b\neq a}}^{N} \frac{\theta(\lambda_{ab}+\gamma)}{\theta(\lambda_{ab})}\,E_{bb}\,, \tag{3.13}$$

$$\ell_{ab}(\lambda) \mapsto \frac{\theta(\gamma)}{\theta(\lambda_{ab})}\,E_{ab}\,, \qquad a\neq b\,,$$

$a,b=1,\dots,N$, makes $\mathbb{C}^N$ into an $e_{\tau,\gamma}(\mathfrak{sl}_N)$-module. The module is called the *vector representation* of $e_{\tau,\gamma}(\mathfrak{sl}_N)$. In the vector representation

$$\mathcal{D}(u,\lambda) = \theta(u-\gamma)\,\theta(u+\gamma)\,\theta(u+2\gamma)\,\dots\,\theta\bigl(u+(N-1)\gamma\bigr)\,.$$

Corollary 3.4. *For any* $e_{\tau,\gamma}(\mathfrak{sl}_N)$*-module* V *and any* $x\in\mathbb{C}$ *the rule* $T_{ab}(u) = \mathcal{T}_{ab}(u-x)\bigr|_V$ *makes* V *into an* $E_{\tau,\gamma}(\mathfrak{sl}_N)$*-module called the evaluation module* $V(x)$ *over* $E_{\tau,\gamma}(\mathfrak{sl}_N)$.

By abuse of notation we call the assignment $T(u)\mapsto\mathcal{T}(u)$ the *evaluation morphism* $E_{\tau,\gamma}(\mathfrak{sl}_N)\to e_{\tau,\gamma}(\mathfrak{sl}_N)$. It is analogous to the evaluation homomorphism from the Yangian $Y(\mathfrak{sl}_N)$ to $U(\mathfrak{sl}_N)$.

Remark. Corollary 3.4 was our main motivation to discover and study the small elliptic quantum group $e_{\tau,\gamma}(\mathfrak{sl}_N)$.

Let V,W be $e_{\tau,\gamma}(\mathfrak{sl}_N)$-modules. An element $\varphi\in Fun\bigl(\mathrm{Hom}_{\mathfrak{h}}(V,W)\bigr)$ is a *morphism* of $e_{\tau,\gamma}(\mathfrak{sl}_N)$ modules if the induced map intertwines the corresponding actions of $e^{\mathcal{O}}_{\tau,\gamma}(\mathfrak{sl}_N)$:

$$\varphi(\lambda)\,t_{ab}\bigr|_{Fun(V)} = t_{ab}\bigr|_{Fun(W)}\,\varphi(\lambda)$$

for any $a,b=1,\dots,N$. Denote by $\mathrm{Mor}\,(V,W)$ the space of all morphisms from V to W. A morphism φ is called an *isomorphism* if the map $\varphi(\lambda)$ is bijective for generic λ.

An $e_{\tau,\gamma}(\mathfrak{sl}_N)$-module V is called *irreducible* if for any nontrivial morphism $\varphi \in \mathrm{Mor}\,(W,V)$ the map $\varphi(\lambda)$ is surjective for generic λ, and *reducible*, otherwise.

Say that an $e_{\tau,\gamma}(\mathfrak{sl}_N)$-module W is a *submodule* of V if there is a morphism $\varphi \in \mathrm{Mor}\,(W,V)$ such that the map $\varphi(\lambda)$ is injective for generic λ. The morphism φ is called an *embedding*. The submodule W is called *proper* if φ is not an isomorphism. Any $e_{\tau,\gamma}(\mathfrak{sl}_N)$-module V has at least two submodules: V itself and the *trivial submodule* $\{0\}$ with obvious embeddings.

Let W be a submodule of V. Then one can define the *quotient* $e_{\tau,\gamma}(\mathfrak{sl}_N)$-module V/W as follows. Fix an embedding φ. The subspace $\varphi\bigl(\mathit{Fun}(W)\bigr) \subset \mathit{Fun}(V)$ is invariant with respect to the action of $e^{\mathcal{O}}_{\tau,\gamma}(\mathfrak{sl}_N)$, hence $e^{\mathcal{O}}_{\tau,\gamma}(\mathfrak{sl}_N)$ acts on $\mathit{Fun}(V)/\varphi\bigl(\mathit{Fun}(W)\bigr)$. Let $\lambda_0 \in \mathfrak{h}^*$ be such that the map $\varphi(\lambda_0)$ is injective. Take a complement U of $\varphi(\lambda_0)W$ in V, that is, $V = U \oplus \varphi(\lambda_0)W$ as a vector space. Notice that $V = U \oplus \varphi(\lambda)W$ for generic λ as well. Then $\mathit{Fun}(V) = \mathit{Fun}(U) \oplus \varphi\bigl(\mathit{Fun}(W)\bigr)$ and, therefore, $\mathit{Fun}(U) = \mathit{Fun}(V)/\varphi\bigl(\mathit{Fun}(W)\bigr)$, which induces an action of $e^{\mathcal{O}}_{\tau,\gamma}(\mathfrak{sl}_N)$ on $\mathit{Fun}(U)$ and makes U into an $e_{\tau,\gamma}(\mathfrak{sl}_N)$-module. The constructed $e_{\tau,\gamma}(\mathfrak{sl}_N)$-module does not depend on a choice φ, λ_0 and U up to an isomorphism of $e_{\tau,\gamma}(\mathfrak{sl}_N)$-modules and is denoted by V/W.

Lemma 3.5. *An $e_{\tau,\gamma}(\mathfrak{sl}_N)$-module V is reducible if and only if it has a nontrivial proper submodule.*

Lemma 3.6. *An $e_{\tau,\gamma}(\mathfrak{sl}_N)$-module V is irreducible if and only if for any nontrivial morphism $\varphi \in \mathrm{Mor}\,(V,W)$ the map $\varphi(\lambda)$ is injective for generic λ.*

4. Highest weight modules over $e_{\tau,\gamma}(\mathfrak{sl}_N)$

For any monomial $t_{a_1b_1}\dots t_{a_kb_k}$ set $\deg\,(t_{a_1b_1}\dots t_{a_kb_k}) = k$. For $k=0$ we assume that the monomial equals 1. The operator algebra $e^{\mathcal{O}}_{\tau,\gamma}(\mathfrak{sl}_N)$ is generated by all monomials $t_{a_1b_1}\dots t_{a_kb_k}$, $k = 0,1,\dots$, as a $\mathit{Fun}^{\otimes 2}(\mathbb{C})$-module.

For any function $\varphi \in \mathit{Fun}^{\otimes 2}(\mathbb{C})$ set $\deg\,(\varphi) = 0$. Since relations (3.2) – (3.5) are homogeneous, the algebra $e^{\mathcal{O}}_{\tau,\gamma}(\mathfrak{sl}_N)$ is $\mathbb{Z}_{\geqslant 0}$-graded by $\deg$. Let $\mathfrak{e}_k = \{x \in e^{\mathcal{O}}_{\tau,\gamma}(\mathfrak{sl}_N) \mid \deg\,(x) = k\}$ be the homogeneous subspace of degree k. Each subspace $\mathfrak{e}_k$ is finitely generated over $\mathit{Fun}^{\otimes 2}(\mathbb{C})$.

Consider the *normal ordering* of generators: $t_{ab} < t_{cd}$ if $a-b < c-d$, or $a-b = c-d$ and $a<c$. Say that the monomial $t_{a_1b_1}\dots t_{a_kb_k}$ is *normally ordered* if $t_{a_ib_i} < t_{a_jb_j}$ for any $i<j$, or $k=0$.

Theorem 4.1. *For any $k \in \mathbb{Z}_{\geqslant 0}$ the normally ordered monomials of degree k form a basis of $\mathfrak{e}_k$ over $\mathit{Fun}^{\otimes 2}(\mathbb{C})$.*

Proof. For $k=0$ and $k=1$ the claim is immediate. Let $k>1$. Here we prove that the normally ordered monomials of degree k span $\mathfrak{e}_k$ over $\mathit{Fun}^{\otimes 2}(\mathbb{C})$. The linear independence of the normally ordered monomials is proved in Appendix D.

It is clear from relations (3.3) – (3.5) that any product $t_{ab}t_{cd}$ can be written as a linear combination of normally ordered products. Given a monomial $t_{a_1b_1}\dots t_{a_kb_k}$ we take any disordered product of adjacent factors and replace it by a suitable sum of normally ordered products, then do the same for each of the obtained monomials. To see that the procedure terminates and, hence, produces a linear combination of normally ordered monomials, introduce auxilary gradings on monomials by the rule

$$r(t_{a_1b_1}\dots t_{a_kb_k}) = \sum_{i=1}^{k} i\,(a_i-b_i)\,, \qquad r'(t_{a_1b_1}\dots t_{a_kb_k}) = \sum_{i=1}^{k} i\,b_i\,,$$

and observe that at each nontrivial step of the procedure we replace a monomial by a sum of monomials of either less degree r, or the same degree r and less degree r'. □

Introduce modified generators of the algebra $e^{\mathcal{O}}_{\tau,\gamma}(\mathfrak{sl}_N)$. For any $a,b = 1,\dots,N$ set

$$\hat{t}_{ab} = \prod_{1\leqslant c<a} \theta(\lambda^{\{1\}}_{ca}) \prod_{1\leqslant c<b} \bigl(\theta(\lambda^{\{2\}}_{cb})\bigr)^{-1}\, t_{ab}\,. \tag{4.1}$$

Let V be an $e_{\tau,\gamma}(\mathfrak{sl}_N)$-module. A nonzero function $v \in \mathit{Fun}(V)$ is called a *singular vector* if $t_{ab}\,v = 0$ for any $1 \leqslant a < b \leqslant N$. Say that v is a *regular singular vector* if in addition v is a weight vector with respect to the action of $\mathfrak{h}$ and

$$(\hat{t}_{aa}v)(\lambda) = Q_a(\lambda)\,v(\lambda)\,, \qquad a = 1,\dots,N\,, \tag{4.2}$$

for certain functions $Q_1, \ldots, Q_N \in \mathit{Fun}(\mathbb{C})$. We call $\widehat{Q} = (Q_1, \ldots, Q_N)$ the *dynamical weight* of v. Relation (A.2) implies that $\widehat{Q}$ is a multiplicative cocycle:

$$Q_a(\lambda)\, Q_b(\lambda - \gamma\varepsilon_a) \,=\, Q_a(\lambda - \gamma\varepsilon_b)\, Q_b(\lambda)$$

for any $a, b = 1, \ldots, N$. If $f \in \mathit{Fun}(\mathbb{C})$, then the function $\tilde{v}(\lambda) = f(\lambda)\, v(\lambda)$ is a regular singular vector of dynamical weight $(\widetilde{Q}_1, \ldots, \widetilde{Q}_N)$ where

$$\widetilde{Q}_a(\lambda) \,=\, Q_a(\lambda)\, \frac{f(\lambda - \gamma\varepsilon_a)}{f(\lambda)}\,, \qquad a = 1, \ldots, N.$$

Hence, the subspace $\mathit{Fun}(\mathbb{C})\, v$ determines the dynamical weight up to a multiplicative coboundary.

Say that $\widehat{Q}$ and v are *nondegenerate* if $Q_a \neq 0$ for any $a = 1, \ldots, N$. Say that $\widehat{Q}$ and v are *standard* if $Q_a = 1$ for any $a = 1, \ldots, N$.

By formula (3.9) the quantum determinant acts on a regular singular vector v of weight μ and dynamical weight $\widehat{Q}$ as follows:

$$\text{(4.3)} \qquad \bigl(\operatorname{Det} \mathcal{T}(u)\, v\bigr)(\lambda) \,=\, \prod_{a=1}^{N} \theta\bigl(u - \gamma(\mu_a - a + 1)\bigr) \prod_{a=1}^{N} Q_a\bigl(\lambda - \sum_{a<b\leqslant N} \gamma\varepsilon_b\bigr)\, v(\lambda).$$

An $e_{\tau,\gamma}(\mathfrak{sl}_N)$-module V is called a *highest weight module* with *highest weight* μ, *dynamical highest weight* $\widehat{Q}$ and *highest weight vector* v if v is a regular singular vector of weight μ and dynamical weight $\widehat{Q}$ generating $\mathit{Fun}(V)$ over $e^{\mathcal{O}}_{\tau,\gamma}(\mathfrak{sl}_N)$. If $\widehat{Q}$ is standard (nondegenerate), then V is called a *standard* (*nondegenerate*) $e_{\tau,\gamma}(\mathfrak{sl}_N)$-module of highest weight μ. For example, the vector representation is a standard $e_{\tau,\gamma}(\mathfrak{sl}_N)$-module of highest weight ω_1.

It is clear that any nondegenerate highest weight module is isomorphic to a pullback of a standard highest weight module of the same highest weight through a suitable automorphism of the form (3.10).

Proposition 4.2. *Let V be a highest weight module with highest weight μ. Then*
a) $V = \bigoplus_{\nu \leqslant \mu} V[\nu]$ *and* $\dim_{\mathbb{C}} V[\mu] = 1$;
b) *V is reducible if and only if it has a singular vector of weight $\nu < \mu$.*

Lemma 4.3. *Let V be a highest weight module of highest weight μ and dynamical highest weight $(Q_1, \ldots, Q_N)$. Then for any $v \in \mathit{Fun}(V)$*

$$\text{(4.4)} \qquad \bigl(\operatorname{Det} \mathcal{T}(u)\, v\bigr)(\lambda) \,=\, \prod_{a=1}^{N} \theta\bigl(u - \gamma(\mu_a - a + 1)\bigr) \prod_{a=1}^{N} Q_a\bigl(\lambda - \sum_{a<b\leqslant N} \gamma\varepsilon_b\bigr)\, v(\lambda).$$

The statement follows from Proposition 3.2 and formula (4.3).

Corollary 4.4. *Let $\gamma \notin \mathbb{Q} + \tau\mathbb{Q}$. Then a nondegenerate highest weight module V of highest weight μ is irreducible unless $V[w \cdot \mu] \neq 0$ for some $w \in W$ such that $w \cdot \mu < \mu$. Here W is the Weil group.*

Proof. If V is reducible, then it has a singular vector v of weight $\nu < \mu$. Comparing formulae (4.3) and (4.4) for the action of $\operatorname{Det} \mathcal{T}(u)$ on v we obtain that $(\nu_1 - 1, \ldots, \nu_N - N) = (\mu_{i_1} - i_1, \ldots, \mu_{i_N} - i_N)$ for some permutation $(i_1, \ldots, i_N)$ since $\gamma \notin \mathbb{Q} + \tau\mathbb{Q}$. That is, $\nu = w(\mu + \rho) - \rho = w \cdot \mu$ for some $w \in W$ because $\rho = -\varepsilon_1 - 2\varepsilon_2 - \ldots - N\varepsilon_N$. □

Let $e(\mathfrak{b}_+)$ and $e(\mathfrak{n}_+)$ be the left ideals in $e^{\mathcal{O}}_{\tau,\gamma}(\mathfrak{sl}_N)$ generated by the elements t_{ab} with $a \leqslant b$ and $a < b$, respectively. Let $\mathcal{B}$, $\mathcal{B}'$, $\mathcal{N}$, $\mathcal{N}'$ be the following sets of normally ordered monomials

$$\text{(4.5)} \qquad \begin{aligned} \mathcal{B} &= \{\hat{t}_{a_1 b_1} \ldots \hat{t}_{a_k b_k} \mid k \geqslant 0\,,\ \ a_i \geqslant b_i\,,\ \ i = 1, \ldots, k\}\,, \\ \mathcal{B}' &= \{\hat{t}_{a_1 b_1} \ldots \hat{t}_{a_k b_k} \mid k > 0\,,\ \ a_i \leqslant b_i\,,\ \ i = 1, \ldots, k\}\,, \\ \mathcal{N} &= \{\hat{t}_{a_1 b_1} \ldots \hat{t}_{a_k b_k} \mid k \geqslant 0\,,\ \ a_i > b_i\,,\ \ i = 1, \ldots, k\}\,, \\ \mathcal{N}' &= \{\hat{t}_{a_1 b_1} \ldots \hat{t}_{a_k b_k} \mid k > 0\,,\ \ a_i < b_i\,,\ \ i = 1, \ldots, k\}\,. \end{aligned}$$

Lemma 4.5. *The normally ordered monomials of the form* $m\,m'$ *where* $m \in \mathcal{N}$ *and* $m' \in \mathcal{B}'$ *form a basis of* $e(\mathfrak{b}_+)$ *over* $Fun^{\otimes 2}(\mathbb{C})$.

Lemma 4.6. *The normally ordered monomials of the form* $m\,m'$ *where* $m \in \mathcal{B}$ *and* $m' \in \mathcal{N}'$ *form a basis of* $e(\mathfrak{n}_+)$ *over* $Fun^{\otimes 2}(\mathbb{C})$.

The statements easily follows from relations (3.3) – (3.5) and Theorem 4.1.

For any monomial $t_{a_1 b_1} \ldots t_{a_k b_k}$ set $wt(t_{a_1 b_1} \ldots t_{a_k b_k}) = \sum_{i=1}^{k} (\varepsilon_{a_i} - \varepsilon_{b_i})$ and for any function $\varphi \in Fun^{\otimes 2}(\mathbb{C})$ set $wt(\varphi) = 0$. Since relations (3.2) – (3.5) are homogeneous, the algebra $e^{\mathcal{O}}_{\tau,\gamma}(\mathfrak{sl}_N)$ is $\mathbb{P}$ graded by wt.

Let $\mu \in \mathfrak{h}^*$ and $\widehat{Q}$ be a multiplicative cocycle. Below we define a *Verma module* $M_{\mu,\widehat{Q}}$ of highest weight μ and dynamical highest weight $\widehat{Q}$ over $e_{\tau,\gamma}(\mathfrak{sl}_N)$.

Let $\mathcal{N}_{\mathbb{C}} = \bigoplus_{m \in \mathcal{N}} \mathbb{C}m$ be a diagonalizable $\mathfrak{h}$-module such that a monomial m has weight $wt(m)$, and let $\mathbb{C}v_{\mu,\widehat{Q}}$ be an $\mathfrak{h}$-module such that $v_{\mu,\widehat{Q}}$ has weight μ. Then $M_{\mu,\widehat{Q}} = \mathcal{N}_{\mathbb{C}} \otimes \mathbb{C}v_{\mu,\widehat{Q}}$ as an $\mathfrak{h}$-module. We define an action of $e^{\mathcal{O}}_{\tau,\gamma}(\mathfrak{sl}_N)$ in $Fun(M_{\mu,\widehat{Q}})$ by the rule: $1 \otimes v_{\mu,\widehat{Q}}$ is a regular singular vector of weight μ and dynamical weight $\widehat{Q}$ and

$$m\,(1 \otimes v_{\mu,\widehat{Q}}) \,=\, m \otimes v_{\mu,\widehat{Q}}$$

for any $m \in \mathcal{N}$. This determines an action on $v_{\mu,\widehat{Q}}$ by any normally ordered monomial and, hence, by any element of $e^{\mathcal{O}}_{\tau,\gamma}(\mathfrak{sl}_N)$, cf. Theorem 4.1. Finally, for any $x \in e^{\mathcal{O}}_{\tau,\gamma}(\mathfrak{sl}_N)$ set

$$x\,(m \otimes v_{\mu,\widehat{Q}}) \,=\, (xm)\,(1 \otimes v_{\mu,\widehat{Q}})$$

where the product xm should be represented as a linear combination of normally ordered monomials.

Proposition 4.7. $M_{\mu,\widehat{Q}}$ *is a well-defined* $e_{\tau,\gamma}(\mathfrak{sl}_N)$*-module with highest weight* μ, *dynamical highest weight* $\widehat{Q}$ *and highest weight vector* $1 \otimes v_{\mu,\widehat{Q}}$.

The statement follows from Lemmas 4.5, 4.6 and Theorem 4.1.

From now on we suppress the symbol of tensor product in the definition of the Verma module $M_{\mu,\widehat{Q}}$, for instance, we write $v_{\mu,\widehat{Q}}$ instead of $1 \otimes v_{\mu,\widehat{Q}}$.

If $\widehat{Q}$ is standard, then the $e_{\tau,\gamma}(\mathfrak{sl}_N)$-module $M_\mu = M_{\mu,\widehat{Q}}$ is called the *standard* Verma module with highest weight μ and the highest weight vector $v_\mu = v_{\mu,\widehat{Q}}$.

Lemma 4.8. *Let* V *be an* $e_{\tau,\gamma}(\mathfrak{sl}_N)$*-module. For any* $v \in Fun(V)$ *which is a regular singular vector of weight* μ *and dynamical weight* $\widehat{Q}$, *there is a unique morphism* $\varphi \in \mathrm{Mor}\,(M_{\mu,\widehat{Q}}\,, V)$ *which sends* $v_{\mu,\widehat{Q}}$ *to* v.

Proposition 4.9. *Any highest weight* $e_{\tau,\gamma}(\mathfrak{sl}_N)$*-module is isomorphic to a suitable quotient of the Verma module over* $e_{\tau,\gamma}(\mathfrak{sl}_N)$ *of the same highest weight and dynamical highest weight.*

Proposition 4.10. *For any* $\mu \in \mathfrak{h}^*$ *and a dynamical weight* $\widehat{Q}$ *there exists a unique up to an isomorphism irreducible highest weight* $e_{\tau,\gamma}(\mathfrak{sl}_N)$*-module with highest weight* μ *and dynamical highest weight* $\widehat{Q}$.

Proposition 4.11. *Let* $\gamma \notin \mathbb{Q} + \tau\mathbb{Q}$. *Then a highest weight* $e_{\tau,\gamma}(\mathfrak{sl}_N)$*-module with highest weight* μ *and a nondegenerate dynamical highest weight* $\widehat{Q}$ *is isomorphic to the Verma module* $M_{\mu,\widehat{Q}}$ *unless* $w \cdot \mu < \mu$ *for some* $w \in W$.

5. Dynamical Shapovalov form

Let $e(\mathfrak{n}_-)$ be the right ideal in $e^{\mathcal{O}}_{\tau,\gamma}(\mathfrak{sl}_N)$ generated by the elements t_{ab} with $a > b$, let $\mathfrak{d}$ be the $Fun^{\otimes 2}(\mathbb{C})$-submodule generated by normally ordered monomials of the form $t_{a_1 a_1} \ldots t_{a_k a_k}$, and let

$$\mathfrak{e}[0] \,=\, \{x \in e^{\mathcal{O}}_{\tau,\gamma}(\mathfrak{sl}_N) \mid wt(x) = 0\}$$

be the subalgebra of zero weight elements in $e^{\mathcal{O}}_{\tau,\gamma}(\mathfrak{sl}_N)$. Consider the quotient

$$e(\mathfrak{h}) = e^{\mathcal{O}}_{\tau,\gamma}(\mathfrak{sl}_N)/\big(e(\mathfrak{n}_+)+e(\mathfrak{n}_-)\big).$$

Let $\eta : e^{\mathcal{O}}_{\tau,\gamma}(\mathfrak{sl}_N) \to e(\mathfrak{h})$ be the natural projection. By Theorem 4.1 the restriction of η to $\mathfrak{d}$ is a bijection. Denote by $\bar\eta : e(\mathfrak{h}) \to \mathfrak{d}$ the inverse map. For any $x, y \in e(\mathfrak{h})$ define their product by the rule $xy = \eta\big(\bar\eta(x)\bar\eta(y)\big)$. It is easy to see that this defines an algebra structure on $e(\mathfrak{h})$.

Lemma 5.1. *The restriction of η to $\mathfrak{e}[0]$ is a homomorphism.*

Set $q_a = \eta(\hat t_{aa})$, $a = 1,\dots,N$. It follows from (A.2) that $e(\mathfrak{h})$ is generated by functions $f \in Fun^{\otimes 2}(\mathbb{C})$ and the pairwise commuting elements $q_1,\dots,q_N$ subject to relations

$$q_a\, f(\lambda^{\{1\}},\lambda^{\{2\}}) = f(\lambda^{\{1\}}-\gamma\varepsilon_a\,,\lambda^{\{2\}}-\gamma\varepsilon_a)\,q_a\,. \tag{5.1}$$

The assignment
$$\varpi : f(\lambda^{\{1\}},\lambda^{\{2\}}) \mapsto f(-\lambda^{\{2\}},-\lambda^{\{1\}}),$$

$$\varpi : t_{ab} \mapsto \prod_{1\leqslant c<b}\theta(\lambda^{\{1\}}_{cb})\,\theta(\lambda^{\{1\}}_{bc}-\gamma)\prod_{1\leqslant c<a}\big(\theta(\lambda^{\{2\}}_{ca})\,\theta(\lambda^{\{2\}}_{ac}-\gamma)\big)^{-1}\,t_{ba}$$

defines an involutive antiautomorphism of the operator algebra $e^{\mathcal{O}}_{\tau,\gamma}(\mathfrak{sl}_N)$, which differs from the antiautomorphism (3.11) by a suitable automorphism of the form (3.10). We have

$$\varpi(\hat t_{ab}) = \hat t_{ba}\,. \tag{5.2}$$

For any $x \in e(\mathfrak{h})$ set $\varpi(x) = \eta\big(\varpi(\bar\eta(x))\big)$.

Lemma 5.2. *For any $m \in e^{\mathcal{O}}_{\tau,\gamma}(\mathfrak{sl}_N)$ we have $\eta\big(\varpi(m)\big) = \varpi\big(\eta(m)\big)$.*

For any $m_1, m_2 \in e^{\mathcal{O}}_{\tau,\gamma}(\mathfrak{sl}_N)$ set

$$S(m_1\,,m_2) = \eta\big(\varpi(m_1)m_2\big) \in e(\mathfrak{h})\,. \tag{5.3}$$

Since ϖ is involutive, $S(m_1\,,m_2) = \varpi\big(S(m_2\,,m_1)\big)$. Moreover, $S(m_1m_2\,,m_3) = S\big(m_2\,,\varpi(m_1)m_3\big)$ for any $m_1,m_2,m_3 \in e^{\mathcal{O}}_{\tau,\gamma}(\mathfrak{sl}_N)$. S is called the *dynamical Shapovalov form* on $e^{\mathcal{O}}_{\tau,\gamma}(\mathfrak{sl}_N)$.

Example. Let $a<b$. Then $S(\hat t_{ba}\,,\hat t_{ba}) = -\dfrac{\theta(\lambda^{\{1\}}_{ab}-\lambda^{\{2\}}_{ab})\,\theta(\gamma)}{\theta(\lambda^{\{1\}}_{ab})\,\theta(\lambda^{\{2\}}_{ab})}\,q_a\,q_b\,.$

Let $\mu \in \mathfrak{h}^*$ and let $\widehat Q = (Q_1,\dots,Q_N)$ be a multiplicative cocycle. Then there is an algebra homomorphism $\chi_{\mu,\widehat Q} : e(\mathfrak{h}) \to D(\mathbb{C})$:

$$\chi_{\mu,\widehat Q}\big(f(\lambda^{\{1\}},\lambda^{\{2\}})\big) : \varphi(\lambda) \mapsto f(\lambda,\lambda-\gamma\mu)\,\varphi(\lambda)\,,$$

$$\chi_{\mu,\widehat Q}(q_a) : \varphi(\lambda) \mapsto Q_a(\lambda)\,\varphi(\lambda-\gamma\varepsilon_a)\,.$$

Consider the Verma module $M_{\mu,\widehat Q}$. For any $m \in e^{\mathcal{O}}_{\tau,\gamma}(\mathfrak{sl}_N)$ and $v \in Fun(M_{\mu,\widehat Q})$ set

$$S_{\mu,\widehat Q}(m\,,v) = \chi_{\mu,\widehat Q}\big(S(m\,,m')\big)\cdot 1 \in Fun(\mathbb{C})\,. \tag{5.4}$$

Here $m' \in e^{\mathcal{O}}_{\tau,\gamma}(\mathfrak{sl}_N)$ is determined by $v = m'v_{\mu,\widehat Q}$ and $1 \in Fun(\mathbb{C})$ is the constant function. It is easy to see that $S_{\mu,\widehat Q}(m\,,v)$ does not depend on a choice of m'. We call $S_{\mu,\widehat Q}$ the *dynamical Shapovalov pairing* for $M_{\mu,\widehat Q}$.

Lemma 5.3. *For any $m_1,m_2 \in e^{\mathcal{O}}_{\tau,\gamma}(\mathfrak{sl}_N)$ and $v \in Fun(M_{\mu,\widehat Q})$ we have*

$$S_{\mu,\widehat Q}(m_1\,,m_2v) = S_{\mu,\widehat Q}\big(\varpi(m_2)m_1\,,v\big) = S_{\mu,\widehat Q}\big(1\,,\varpi(m_1)m_2v\big)\,.$$

Here $1 \in e^{\mathcal{O}}_{\tau,\gamma}(\mathfrak{sl}_N)$ is the identity element.

Lemma 5.4. *Let $m \in e^{\mathcal{O}}_{\tau,\gamma}(\mathfrak{sl}_N)$ be a wt-homogeneous element, and $v \in M_{\mu,\widehat{Q}}[\nu]$. If $wt(m) \neq \mu - \nu$, then $S_{\mu,\widehat{Q}}(m, v) = 0$. Otherwise, $\varpi(m)\, v = S_{\mu,\widehat{Q}}(m, v)\, v_{\mu,\widehat{Q}}$.*

Example. $S_{\mu,\widehat{Q}}(1, v_{\mu,\widehat{Q}}) = 1$. If $a < b$, then

$$S_{\mu,\widehat{Q}}(\hat{t}_{ba}, \hat{t}_{ba} v_{\mu,\widehat{Q}}) \,=\, -\,\frac{\theta(\gamma\mu_{ab})\,\theta(\gamma)}{\theta(\lambda_{ab})\,\theta(\lambda_{ab} - \gamma\mu_{ab})}\; Q_a(\lambda)\, Q_b(\lambda - \gamma\varepsilon_a)\,.$$

Set
$$\operatorname{Ker} S_{\mu,\widehat{Q}} \,=\, \{ v \in Fun(M_{\mu,\widehat{Q}}) \mid S_{\mu,\widehat{Q}}(m, v) = 0 \quad \text{for any} \quad m \in e^{\mathcal{O}}_{\tau,\gamma}(\mathfrak{sl}_N) \}\,.$$

The subspace $\operatorname{Ker} S_{\mu,\widehat{Q}}$ is invariant under the action of $e^{\mathcal{O}}_{\tau,\gamma}(\mathfrak{sl}_N)$ and defines a proper submodule $N_{\mu,\widehat{Q}}$ of $M_{\mu,\widehat{Q}}$.

Proposition 5.5. *$N_{\mu,\widehat{Q}}$ is the maximal proper submodule of $M_{\mu,\widehat{Q}}$, that is, for any proper submodule U of $M_{\mu,\widehat{Q}}$ the embedded image of $Fun(U)$ is contained in $Fun(N_{\mu,\widehat{Q}}) = \operatorname{Ker} S_{\mu,\widehat{Q}}$.*

Proof. Let $\varphi \in \operatorname{Mor}(U, M_{\mu,\widehat{Q}})$ be the embedding. $\varphi\bigl(Fun(U)\bigr)$ is a direct sum of its weight components and $\varphi\bigl(Fun(U)\bigr)[\mu] = 0$ since U is a proper submodule. Therefore, if $v \in \varphi\bigl(Fun(U)\bigr)[\nu]$ and $wt(m) = \mu - \nu$, then $\varpi(m)v = 0$. Hence, by Lemma 5.4 $S_{\mu,\widehat{Q}}(m, v) = 0$ for any $m \in e^{\mathcal{O}}_{\tau,\gamma}(\mathfrak{sl}_N)$ and $v \in \varphi\bigl(Fun(U)\bigr)$. □

Corollary 5.6. *The quotient module $V_{\mu,\widehat{Q}} = M_{\mu,\widehat{Q}} / N_{\mu,\widehat{Q}}$ is the irreducible highest weight $e_{\tau,\gamma}(\mathfrak{sl}_N)$ module with highest weight μ and dynamical highest weight $\widehat{Q}$.*

For any highest weight module V with highest weight μ, dynamical highest weight $\widehat{Q}$ and highest weight vector v one can define the Shapovalov pairing similarly to (5.4): for any $m \in e^{\mathcal{O}}_{\tau,\gamma}(\mathfrak{sl}_N)$ and $v' \in Fun(V)$ set
$$S^V_{\mu,\widehat{Q}}(m, v') \,=\, \chi_{\mu,\widehat{Q}}\bigl(S(m, m')\bigr) \cdot 1 \in Fun(\mathbb{C})$$

where $m' \in e^{\mathcal{O}}_{\tau,\gamma}(\mathfrak{sl}_N)$ is determined by $v' = m'v$. Propositions 4.9 and 5.5 imply that $S^V_{\mu,\widehat{Q}}(m, v)$ does not depend on a choice of m'.

Proposition 5.7. *The module V is irreducible if and only if $\operatorname{Ker} S^V_{\mu,\widehat{Q}}$ is trivial. Otherwise, $\operatorname{Ker} S^V_{\mu,\widehat{Q}}$ defines the maximal proper submodule of V.*

Let $V^{\mathfrak{sl}}_\mu$ be the irreducible $\mathfrak{sl}_N$-module of highest weight μ. Set $d_\mu[\nu] = \dim_{\mathbb{C}} V^{\mathfrak{sl}}_\mu[\nu]$.

Theorem 5.8. *Let $\mu \in \mathfrak{h}^*$ be a dominant integral weight. Then $\dim_{\mathbb{C}} V_{\mu,\widehat{Q}}[\nu] \leqslant d_\mu[\nu]$ for any $\nu \in \mathfrak{h}^*$. In particular, the $e_{\tau,\gamma}(\mathfrak{sl}_N)$-module $V_{\mu,\widehat{Q}}$ is finite-dimensional.*

Theorem 5.9. *Let $\gamma \notin \mathbb{Q} + \tau\mathbb{Q}$. The $e_{\tau,\gamma}(\mathfrak{sl}_N)$-module $V_{\mu,\widehat{Q}}$ for a nondegenerate dynamical weight $\widehat{Q}$ is finite-dimensional if and only if μ is a dominant integral weight. Moreover, $\dim_{\mathbb{C}} V_{\mu,\widehat{Q}}[\nu] = d_\mu[\nu]$ for any $\nu \in \mathfrak{h}^*$.*

Theorems 5.8 and 5.9 are proved in Section 8.

If $\widehat{Q}$ is standard, we set $\chi_\mu = \chi_{\mu,\widehat{Q}}$, $S_\mu = S_{\mu,\widehat{Q}}$, $N_\mu = N_{\mu,\widehat{Q}}$ and $V_\mu = V_{\mu,\widehat{Q}}$.

6. Contragradient modules over $e_{\tau,\gamma}(\mathfrak{sl}_N)$ and contravariant form

For any diagonalizable $\mathfrak{h}$-module V define an involutive linear map $\psi : Fun(V) \to Fun(V)$ by the rule: if $f \in Fun\bigl(V[\mu]\bigr)$, then $(\psi f)(\lambda) = f(-\lambda + \gamma\mu)$.

Let V be a diagonalizable $\mathfrak{h}$-module, and let $V^* = \bigoplus_{\mu \in \mathfrak{h}^*} \bigl(V[\mu]\bigr)^*$ be its restricted dual space. We consider V^* as a diagonalizable $\mathfrak{h}$-module such that $\bigl(V[\mu]\bigr)^*$ is a weight subspace of weight μ. For any $B \in \operatorname{End}(V)$ we denote by $B^* \in \operatorname{End}(V^*)$ the dual map. For a difference operator $A \in D(V)$ we define the dual operator $A' \in D(V^*)$ by the rule:

$$\text{if} \quad (Av)(\lambda) = B(\lambda)\, v(\lambda + \mu)\,, \quad \text{then} \quad (A'\varphi)(\lambda) \,=\, \bigl(B(\lambda - \mu)\bigr)^* \varphi(\lambda - \mu)\,,$$

and the operator $A^\dagger \in D(V^*)$: $A^\dagger = \psi A' \psi$. The assignment $A \mapsto A^\dagger$ is an antihomomorphism $D(V) \to D(V^*)$.

Example. Let V be an $e_{\tau,\gamma}(\mathfrak{sl}_N)$-module. Then

$$\big((t_{ab})^\dagger\varphi\big)(\lambda) \,=\, \big(\ell_{ab}\big(-\lambda+(\mu+\varepsilon_b)\gamma\big)\big)^*\varphi(\lambda-\gamma\varepsilon_b)$$

for any $\varphi\in \mathit{Fun}\big(V^*[\mu]\big)$, cf. (3.12).

Given an $e_{\tau,\gamma}(\mathfrak{sl}_N)$-module V we make the $\mathfrak{h}$-module V^* into an $e_{\tau,\gamma}(\mathfrak{sl}_N)$-module as follows: the action of an element $m\in e^{\mathcal{O}}_{\tau,\gamma}(\mathfrak{sl}_N)$ in $\mathit{Fun}(V^*)$ is given by $\big(\varpi(m)\big)^\dagger$ where $\varpi(m)$ is understood as a difference operator acting in $\mathit{Fun}(V)$. It is easy to check that the definition is consistent, that is, the ring $\mathit{Fun}^{\otimes 2}(\mathbb{C})$ acts in $\mathit{Fun}(V^*)$ in the prescribed way, cf. (3.1). The obtained $e_{\tau,\gamma}(\mathfrak{sl}_N)$-module V^* is called the *contragradient module* to the $e_{\tau,\gamma}(\mathfrak{sl}_N)$-module V.

Consider the Verma module $M_{\mu,\widehat{Q}}$. Recall that $M_{\mu,\widehat{Q}}[\mu]=\mathbb{C}v_{\mu,\widehat{Q}}$. Fix $v^*_{\mu,\widehat{Q}}\in M^*_{\mu,\widehat{Q}}[\mu]$ by the rule $\langle v^*_{\mu,\widehat{Q}}\,,v_{\mu,\widehat{Q}}\rangle=1$. For any $a=1,\dots,N$ set

$$\widetilde{Q}_a(\lambda) \,=\, Q_a\big(-\lambda+(\mu+\varepsilon_a)\gamma\big)\,.$$

Notice that $Q_a(\lambda)=\widetilde{Q}_a\big(-\lambda+(\mu+\varepsilon_a)\gamma\big)$ as well.

Proposition 6.1. *For the contragradient Verma module $M^*_{\mu,\widehat{Q}}$ the constant function $v^*_{\mu,\widehat{Q}}$ is a regular singular vector of weight μ and dynamical weight $\widetilde{Q}=(\widetilde{Q}_1,\dots,\widetilde{Q}_N)$.*

Corollary 6.2. *There is a unique morphism $\pi_{\mu,\widehat{Q}}\in\mathrm{Mor}\,(M_{\mu,\widehat{Q}}\,,M^*_{\mu,\widetilde{Q}})$ which sends $v_{\mu,\widehat{Q}}$ to $v^*_{\mu,\widetilde{Q}}$.*

Theorem 6.3. $\mathrm{Ker}\,\pi_{\mu,\widehat{Q}}\,=\,\mathrm{Ker}\,S_{\mu,\widehat{Q}}$.

The morphism $\pi_{\mu,\widehat{Q}}$ induces a $\mathit{Fun}(\mathbb{C})$-bilinear map $B_{\mu,\widehat{Q}}:\mathit{Fun}(M_{\mu,\widehat{Q}})\otimes_{\mathbb{C}}\mathit{Fun}(M_{\mu,\widetilde{Q}})\to\mathit{Fun}(\mathbb{C})$:

$$B_{\mu,\widehat{Q}}(v\,,\tilde v) \,=\, \langle\pi_{\mu,\widehat{Q}}v\,,\tilde v\rangle\,.$$

Define a bilinear map $C_{\mu,\widehat{Q}}:\mathit{Fun}(M_{\mu,\widehat{Q}})\otimes_{\mathbb{C}}\mathit{Fun}(M_{\mu,\widetilde{Q}})\to\mathit{Fun}(\mathbb{C})$ by the rule

$$C_{\mu,\widehat{Q}}(v\,,\tilde v) \,=\, B_{\mu,\widehat{Q}}(v\,,\psi\tilde v)\,. \tag{6.1}$$

The map $C_{\mu,\widehat{Q}}$ is called the *contravariant form.*

Theorem 6.4. *Let $v\in\mathit{Fun}\big(M_{\mu,\widehat{Q}}[\nu]\big)$ and $\tilde v\in\mathit{Fun}\big(M_{\mu,\widetilde{Q}}[\tilde\nu]\big)$. Then $C_{\mu,\widehat{Q}}(v\,,\tilde v)=0$ unless $\nu=\tilde\nu$. Moreover,*

$$C_{\mu,\widehat{Q}}(v\,,\tilde v)(\lambda) \,=\, C_{\mu,\widetilde{Q}}(\tilde v\,,v)(-\lambda+\gamma\nu)\,.$$

Theorem 6.5.
a) $C_{\mu,\widehat{Q}}(v\,,\tilde v)=0$ *for any* $\tilde v\in\mathit{Fun}(M_{\mu,\widetilde{Q}})$ *if and only if* $v\in\mathrm{Ker}\,S_{\mu,\widehat{Q}}$.
b) $C_{\mu,\widehat{Q}}(v\,,\tilde v)=0$ *for any* $v\in\mathit{Fun}(M_{\mu,\widehat{Q}})$ *if and only if* $\tilde v\in\mathrm{Ker}\,S_{\mu,\widetilde{Q}}$.

Theorems 6.3 – 6.5 are proved at the end of the section.

Let V and $\widetilde{V}$ be highest weight $e_{\tau,\gamma}(\mathfrak{sl}_N)$-modules of the same highest weight μ and dynamical highest weights $\widehat{Q}$ and $\widetilde{Q}$, respectively. By the last corollary the form $C_{\mu,\widehat{Q}}$ descends to a form $\mathit{Fun}(V)\otimes_{\mathbb{C}}\mathit{Fun}(\widetilde{V})\to\mathit{Fun}(\mathbb{C})$, which is denoted by the same letter. After obvious modification Theorem 6.5 remains true in this case. In particular, for the irreducible highest weight $e_{\tau,\gamma}(\mathfrak{sl}_N)$-modules $V_{\mu,\widehat{Q}}$ and $V_{\mu,\widetilde{Q}}$ the corresponding function-valued bilinear form $\mathit{Fun}(V_{\mu,\widehat{Q}})\otimes_{\mathbb{C}}\mathit{Fun}(V_{\mu,\widetilde{Q}})\to\mathit{Fun}(\mathbb{C})$ is nondegenerate.

Example. $C_{\mu,\widehat{Q}}(v_{\mu,\widehat{Q}}\,,v_{\mu,\widetilde{Q}})=1$. If $a<b$, then

$$C_{\mu,\widehat{Q}}(\hat t_{ba}v_{\mu,\widehat{Q}}\,,\hat t_{ba}v_{\mu,\widetilde{Q}}) \,=\, -\,\frac{\theta(\gamma\mu_{ab})\,\theta(\gamma)}{\theta(\lambda_{ab}+\gamma)\,\theta(\lambda_{ab}-\gamma\mu_{ab}+\gamma)}\;Q_a(\lambda+\gamma\varepsilon_a)\,Q_b(\lambda)\,.$$

For a monomial $m=\hat t_{a_1b_1}\dots\hat t_{a_kb_k}$ set $\zeta'(m)=\sum\limits_{i=1}^k\varepsilon_{a_i}$ and $\zeta''(m)=\sum\limits_{i=1}^k\varepsilon_{b_i}$. Notice that $wt(m)=\zeta'(m)-\zeta''(m)$.

Lemma 6.6. *For any* $m_1, m_2 \in e^{\mathcal{O}}_{\tau,\gamma}(\mathfrak{sl}_N)$ *we have*

$$S_{\mu,\widehat{Q}}(m_1\,, m_2 v_{\mu,\widehat{Q}})(\lambda) \;=\; S_{\mu,\widetilde{Q}}(m_2\,, m_1 v_{\mu,\widetilde{Q}})\bigl(-\lambda + \gamma\mu + \gamma\zeta'(m_1) + \zeta''(m_2)\bigr)\,.$$

Proof. Without loss of generality we can assume that $\zeta'(m_1) + \zeta''(m_2) = \zeta''(m_1) + \zeta'(m_2)$ because expressions in both sides of the formula otherwise equal zero by Lemma 5.4.

Let $m_1 = \hat{t}_{a_1 b_1} \ldots \hat{t}_{a_k b_k}$ and $m_2 = \hat{t}_{c_1 d_1} \ldots \hat{t}_{c_l d_l}$. Set $(s_1, \ldots, s_{k+l}) = (a_1, \ldots, a_k, b_1, \ldots, b_l)$. By the definition of the Shapovalov pairing, cf. (5.4), for any multiplicative cocycle $\widehat{Q}$ we have

$$S_{\mu,\widehat{Q}}(m_1\,, m_2 v_{\mu,\widehat{Q}})(\lambda) \;=\; S_\mu(m_1\,, m_2 v_\mu)(\lambda) \prod_{i=1}^{k+l} Q_{s_i}\bigl(\lambda - \sum_{1\leqslant j<i} \gamma\varepsilon_{s_j}\bigr)\,. \tag{6.2}$$

Here S_μ and v_μ correspond to the so-called standard case, cf. Section 5. Since $\widehat{Q}$ is a multiplicative cocycle, the product can be written also as $\prod\limits_i Q_{s_i}\bigl(\lambda - \sum\limits_{j>i}\gamma\varepsilon_{s_j}\bigr)$. By formula (6.2) and the last remark it suffices to verify that

$$S_\mu(m_1\,, m_2 v_\mu)(\lambda) \;=\; S_\mu(m_2\,, m_1 v_{\mu,\widetilde{Q}})\bigl(-\lambda + \gamma\mu + \gamma\zeta'(m_1) + \zeta''(m_2)\bigr)$$

which follows from the property $S(m_1\,, m_2) \,=\, \varpi\bigl(S(m_2\,, m_1)\bigr)$, commutation relations (5.1) and formula (5.4). □

Proposition 6.7. *For any* $m_1, m_2 \in \mathcal{N}$ *we have*

$$C_{\mu,\widehat{Q}}(m_1 v_{\mu,\widehat{Q}}\,, m_2 v_{\mu,\widetilde{Q}})(\lambda) \;=\; S_{\mu,\widehat{Q}}(m_2\,, m_1 v_{\mu,\widehat{Q}})\bigl(\lambda + \gamma\zeta''(m_2)\bigr)\,. \tag{6.3}$$

Proof. Recall that by the definition of Verma modules $m_1 v_{\mu,\widehat{Q}}$ and $m_2 v_{\mu,\widetilde{Q}}$ are constant functions since $m_1, m_2 \in \mathcal{N}$. If $wt(m_1) \neq wt(m_2)$, then expressions in both sides of formula (6.3) vanish. For $wt(m_1) = wt(m_2)$ the straightforward application of the definition of the contravariant form together with Lemma 5.4 gives

$$C_{\mu,\widehat{Q}}(m_1 v_{\mu,\widehat{Q}}\,, m_2 v_{\mu,\widetilde{Q}})(\lambda) \;=\; S_{\mu,\widetilde{Q}}(m_2\,, m_1 v_{\mu,\widetilde{Q}})\bigl(-\lambda + \gamma\mu + \gamma\zeta'(m_1)\bigr)\,, \tag{6.4}$$

and using Lemma 6.6 we complete the proof. □

Proof of Theorem 6.4. The first part of the theorem is an easy consequence of the definition of the contravariant form. The second part follows from Proposition 6.7, Lemma 6.6, and the property

$$C_{\mu,\widehat{Q}}\bigl(f(\lambda)v\,, g(\lambda)\tilde{v}\bigr) \;=\; f(\lambda)\,g(-\lambda+\gamma\nu)\,C_{\mu,\widehat{Q}}(v\,,\tilde{v}) \tag{6.5}$$

for any $f, g \in \mathit{Fun}(\mathbb{C})$, $v \in \mathit{Fun}\bigl(M_{\mu,\widehat{Q}}[\nu]\bigr)$ and $\tilde{v} \in \mathit{Fun}\bigl(M_{\mu,\widetilde{Q}}[\nu]\bigr)$, cf. (6.1). □

Proof of Theorems 6.3 and 6.5. It is clear that $\operatorname{Ker}\pi_{\mu,\widehat{Q}} = \bigl\{v \in \mathit{Fun}(M_{\mu,\widehat{Q}}) \mid C_{\mu,\widehat{Q}}(v\,,\tilde{v}) = 0$ for any $\tilde{v} \in \mathit{Fun}(M_{\mu,\widetilde{Q}})\bigr\}$. So Theorem 6.3 is equivalent to claim a) of Theorem 6.5. Claim b) of the latter follows from claim a) and Theorem 6.4.

Since $\pi_{\mu,\widehat{Q}} v_{\mu,\widehat{Q}} \,=\, v^*_{\mu,\widetilde{Q}} \neq 0$, the subspace $\operatorname{Ker}\pi_{\mu,\widehat{Q}} \subset \mathit{Fun}(M_{\mu,\widehat{Q}})$ defines a proper submodule of $M_{\mu,\widehat{Q}}$, and by Proposition 5.5 we have that $\operatorname{Ker}\pi_{\mu,\widehat{Q}} \subset \operatorname{Ker} S_{\mu,\widehat{Q}}$.

Let $v \in \operatorname{Ker} S_{\mu,\widehat{Q}}$. We write it as a linear combination of basis vectors: $v \,=\, \sum\limits_{m\in\mathcal{N}} \varphi_m(\lambda)\, m v_{\mu,\widehat{Q}}$. Then by Proposition 6.7 for any $\widetilde{m} \in \mathcal{N}$ we have

$$\begin{aligned} C_{\mu,\widehat{Q}}(v\,, \widetilde{m} v_{\mu,\widetilde{Q}})(\lambda) \;&=\; \sum_{m\in\mathcal{N}} \varphi_m(\lambda)\, S_{\mu,\widehat{Q}}(\widetilde{m} v_{\mu,\widetilde{Q}}\,, m v_{\mu,\widehat{Q}})\bigl(\lambda + \gamma\zeta''(\widetilde{m})\bigr) \;= \\ &=\; S_{\mu,\widehat{Q}}(\widetilde{m} v_{\mu,\widetilde{Q}}\,, v)\bigl(\lambda + \gamma\zeta''(\widetilde{m})\bigr) \;=\; 0\,. \end{aligned}$$

Therefore, $C_{\mu,\widehat{Q}}(v\,,\tilde{v}) = 0$ for any $\tilde{v} \in \mathit{Fun}(M_{\mu,\widetilde{Q}})$ by property (6.5). □

7. Rational dynamical quantum group $e_{rat}(\mathfrak{sl}_N)$

Introduce the spaces $Rat(\mathbb{C})$, $Rat(V)$ and $Rat\bigl(\mathrm{Hom}\,(V,W)\bigr)$ similar to the spaces $Fun(\mathbb{C})$, $Fun(V)$ and $Fun\bigl(\mathrm{Hom}\,(V,W)\bigr)$, replacing meromorphic function by rational functions in the definitions. Let $Rat^{\otimes 2}(\mathbb{C}) = Rat(\mathbb{C}) \otimes_{\mathbb{C}} Rat(\mathbb{C})$.

We define the *operator algebra* $e^{\mathcal{O}}_{rat}(\mathfrak{sl}_N)$ and *modules over the rational dynamical quantum group* $e_{rat}(\mathfrak{sl}_N)$ similar to the elliptic case with the following modification: we replace the spaces of meromorphic functions by the respective spaces of rational functions, substitute the theta function $\theta(u)$ by the linear function $u \mapsto u$, and set $\gamma = 1$. For instance, formulae (3.2), (3.4), (4.1) and (3.12) take the form

$$t_{ab}\, f(\lambda^{\{1\}}, \lambda^{\{2\}}) \,=\, f(\lambda^{\{1\}} - \varepsilon_a\,, \lambda^{\{2\}} - \varepsilon_b)\, t_{ab}$$

$$t_{ac}\, t_{bc} \,=\, \frac{\lambda^{\{1\}}_{ab} + 1}{\lambda^{\{1\}}_{ab} - 1}\; t_{bc}\, t_{ac}\,, \qquad \text{for} \quad a \neq b\,,$$

$$\hat{t}_{ab} \,=\, \prod_{1 \leqslant c < a} \lambda^{\{1\}}_{ca} \prod_{1 \leqslant c < b} (\lambda^{\{2\}}_{cb})^{-1}\, t_{ab}\,, \tag{7.1}$$

$$(t_{ab}\, v)(\lambda) \,=\, \ell_{ab}(\lambda)\, v(\lambda - \varepsilon_a) \qquad \text{for any} \quad v \in Rat(V)\,. \tag{7.2}$$

In the last formula V is an $e_{rat}(\mathfrak{sl}_N)$-module. The definitions of highest weight modules, dynamical weights, etc. can be obviously transfered to the rational case.

The rational case can be considered as a degeneration of the elliptic case obtained by rescaling variables: $u \to \gamma u$, $\lambda \to \gamma\lambda$ and taking the limit $\gamma \to 0$.

Consider the limit $\bar{R}(\lambda)$ of the rational version of the R-matrix (1.3) as $u \to \infty$:

$$\bar{R}(\lambda) \,=\, \sum_{a,b=1}^{N} E_{aa} \otimes E_{bb} \,+\, \sum_{\substack{a,b=1 \\ a \neq b}}^{N} \frac{E_{aa} \otimes E_{bb} - E_{ab} \otimes E_{ba}}{\lambda_{ab}}\,. \tag{7.3}$$

It is a constant solution of the dynamical Yang-Baxter equation:

$$\bar{R}^{(12)}(\lambda - h^{(3)})\, \bar{R}^{(13)}(\lambda)\, \bar{R}^{(23)}(\lambda - h^{(1)}) \,=\, \bar{R}^{(23)}(\lambda)\, \bar{R}^{(13)}(\lambda - h^{(2)})\, \bar{R}^{(12)}(\lambda)\,.$$

$\bar{R}(\lambda)$ is the simplest example of the Hecke type dynamical R-matrix, see [EV1].

Let $\bar{\mathcal{T}} \,=\, \sum_{a,b} E_{ba} \otimes t_{ab}$, where t_{ab} are the generators of $e^{\mathcal{O}}_{rat}(\mathfrak{sl}_N)$ obeying the rational version of commutation relations (3.2) – (3.5). Then one can write relations (3.3) – (3.5) in the R-matrix form:

$$\bar{R}^{(12)}(\lambda^{\{2\}})\, \bar{\mathcal{T}}^{(13)}\, \bar{\mathcal{T}}^{(23)} \,=\, \bar{\mathcal{T}}^{(23)}\, \bar{\mathcal{T}}^{(13)}\, \bar{R}^{(12)}(\lambda^{\{1\}})\,. \tag{7.4}$$

Let V, W be $e_{rat}(\mathfrak{sl}_N)$-modules. Then the $\mathfrak{h}$-module $V \otimes W$ is made into an $e_{rat}(\mathfrak{sl}_N)$-module by the rule

$$\ell_{ab}(u, \lambda)\big|_{V \otimes W} \,=\, \sum_{c=1}^{N} \ell_{cb}(u, \lambda - h^{(2)}) \otimes \ell_{ac}(u, \lambda)\,, \tag{7.5}$$

and t_{ab} acts in $Rat(V \otimes W)$ according to (7.2).

Consider the following element in $e^{\mathcal{O}}_{rat}(\mathfrak{sl}_N)$:

$$t^{\wedge N} \,=\, \sum_{\boldsymbol{i} \in \mathbf{S}_N} \mathrm{sign}(\boldsymbol{i})\; t_{N, i_N} \, \ldots \, t_{1, i_1}\,. \tag{7.6}$$

The product $D \,=\, \prod_{1 \leqslant a < b \leqslant N} \bigl(\lambda^{\{1\}}_{ab} / \lambda^{\{2\}}_{ab}\bigr)\; t^{\wedge N}$ coincides with the top coefficient of the rational version of the quantum determinant $\mathrm{Det}\, \mathcal{T}(u)$, cf. (3.8). Hence, D is a central element in $e^{\mathcal{O}}_{rat}(\mathfrak{sl}_N)$.

Let V be an $e_{rat}(\mathfrak{sl}_N)$-module. The action of D in $Rat(V)$ commutes with multiplication by any function $\varphi(\lambda) \in Rat(\mathbb{C})$ and, therefore, is given by multiplication by a certain function $D(\lambda) \in Rat\bigl(\mathrm{End}\,(V)\bigr)$. The module V is called *nondegenerate* if $D(\lambda)$ is invertible for generic λ, and *semi-standard* if $D(\lambda) = 1$. For instance, any nondegenerate (standard) highest weight $e_{rat}(\mathfrak{sl}_N)$-module is nondegenerate (semistandard) in the new sense.

One can check that the element D is group-like, it acts in the $E_{\tau,\gamma}(\mathfrak{sl}_N)$-module $V \otimes W$ by

$$D(\lambda)\big|_{V\otimes W} = D(\lambda - h^{(2)}) \otimes D(\lambda)\,.$$

Therefore, a tensor product of nondegenerate (semistandard) $e_{\tau,\gamma}(\mathfrak{sl}_N)$-modules is nondegenerate (semi-standard).

Let e_{ab}, $a, b = 1, \dots, N$, be the standard generators of the Lie algebra $\mathfrak{gl}_N$:

$$[e_{ab}\,, e_{cd}] = \delta_{bc}\, e_{ad} - \delta_{ad}\, e_{cb}\,. \tag{7.7}$$

We identify the Lie algebra $\mathfrak{sl}_N$ with the subalgebra of traceless elements in $\mathfrak{gl}_N$:

$$\mathfrak{sl}_N = \Bigl\{ \sum_{a,b} x_{ab}\, e_{ab} \;\Big|\; \sum_a x_{aa} = 0 \Bigr\} \tag{7.8}$$

and $\mathfrak{h}$ with the subalgebra of diagonal elements in $\mathfrak{sl}_N$: $\mathfrak{h} = \bigl\{ \sum_a x_{aa}\, e_{aa} \;\big|\; \sum_a x_{aa} = 0 \bigr\}$. The standard basis of $\mathfrak{h}$ is $h_a = e_{aa} - e_{a+1,a+1}$, $a = 1, \dots, N-1$. The assignment $e_{ab} \mapsto E_{ab}$, $a, b = 1, \dots, N$, makes $\mathbb{C}^N$ into the *vector representation* of $\mathfrak{gl}_N$ and $\mathfrak{sl}_N$.

Let V be an $e_{rat}(\mathfrak{sl}_N)$-module. The elements $\hat t_{ab}$ act in $Rat(V)$ as difference operators:

$$(\hat t_{ab}\, v)(\lambda) = \hat\ell_{ab}(\lambda)\, v(\lambda - \varepsilon_a)\,, \tag{7.9}$$

with coefficients $\hat\ell_{ab}(\lambda) \in Rat\bigl(\mathrm{End}\,(V)\bigr)$. The module V is called *perturbative* if these coefficients have the following behaviour as λ goes to infinity in a generic direction:

a) for any $a = 1, \dots, N$ the function $\hat\ell_{aa}(\lambda)$ has a limit $\tilde\ell_{aa}$ which is an invertible operator;
b) for any $a, b = 1, \dots, N$, $a \neq b$, the function $\lambda_{ab}\hat\ell_{ab}(\lambda)$ has a limit $\tilde\ell_{ab}$.

The operators $\tilde\ell_{ab}$, $a, b = 1, \dots, N$ satisfy the following commutation relations:

$$[x\,, \tilde\ell_{bc}] = (x\,, \varepsilon_b - \varepsilon_c)\, \tilde\ell_{bc}\,, \qquad [\tilde\ell_{aa}\,, \tilde\ell_{bc}] = 0\,,$$

for any $x \in \mathfrak{h}$ and $a, b, c = 1, \dots, N$,

$$[\tilde\ell_{ab}\,, \tilde\ell_{ba}] = (e_{aa} - e_{bb})\, \tilde\ell_{aa}\, \tilde\ell_{bb}\,,$$

for $a \neq b$, and

$$[\tilde\ell_{ab}\,, \tilde\ell_{bc}] = \tilde\ell_{ac}\, \tilde\ell_{bb}\,,$$

for pairwise distinct a, b, c. Hence, the assignment $e_{ab} \mapsto \tilde\ell_{aa}^{-1}\, \tilde\ell_{ab}$ for $a \neq b$, supplemented by the action of $\mathfrak{h}$, makes V into an $\mathfrak{sl}_N$-module which we denote by $\mathcal{C}(V)$. We say that V is a *perturbation* of $\mathcal{C}(V)$. It is clear that V coincide with $\mathcal{C}(V)$ as a vector space.

Lemma 7.1. *Let V be a perturbative $e_{\tau,\gamma}(\mathfrak{sl}_N)$-module. Then V is nondegenerate.*

Proof. The operator $D(\lambda)$ is invertible for generic λ because it has an invertible limit $\tilde\ell_{11} \dots \tilde\ell_{NN}$ as λ goes to infinity in a generic direction. □

Lemma 7.2. *A tensor product of perturbative $e_{\tau,\gamma}(\mathfrak{sl}_N)$-modules is perturbative.*

Example. The assignment

$$\hat\ell_{aa}(\lambda) \mapsto 1 - \sum_{a<b\leqslant N} \frac{E_{bb}}{\lambda_{ab}^2}\,,$$

$$\hat\ell_{ab}(\lambda) \mapsto \frac{E_{ab}}{\lambda_{ab}}\,, \qquad a \neq b\,,$$

$a,b=1,\ldots,N$, makes $\mathbb{C}^N$ into an $e_{rat}(\mathfrak{sl}_N)$-module V, which is a perturbation of the vector representation of $\mathfrak{sl}_N$. The module V is isomorphic to the vector representation U of $e_{rat}(\mathfrak{sl}_N)$, cf. (3.13), by the following isomorphism:

$$\varphi(\lambda) = \sum_{a=1}^{N} \prod_{1\leqslant b<a} \lambda_{ba} E_{aa} \in \operatorname{Mor}(V,U).$$

Lemma 7.3. *Let V be a perturbative $e_{rat}(\mathfrak{sl}_N)$-module. If V has a weight singular vector, then $\mathcal{C}(V)$ has a singular vector of the same weight;*

Proof. Let $v\in Rat\big(V[\mu]\big)$ be a singular vector. This means that for generic λ the value $v(\lambda)$ belongs to the subspace $K_\lambda = \bigcap\limits_{a,b} \operatorname{Ker} \hat{\ell}_{ab}(\lambda+\varepsilon_a)\big|_{V[\mu]} \subset V$. It is clear that $\dim K_\lambda$ does not depend on λ for generic λ. Moreover, K_λ has a limit K_∞ as λ goes to infinity in a certain generic direction, and $\dim K_\infty = \dim K_\lambda \geqslant 1$. To complete the proof we observe that the subspace of singular vectors in $\mathcal{C}(V)[\mu]$ contains K_∞. □

Lemma 7.4. *Let V be a perturbative $e_{rat}(\mathfrak{sl}_N)$-module. Then*
a) *if $\mathcal{C}(V)$ is irreducible, then V is irreducible;*
b) *if $\mathcal{C}(V)$ is a highest weight module, then V is a highest weight module of the same highest weight;*
c) *if $\mathcal{C}(V)$ is a Verma module, then V is isomorphic to a Verma module of the same highest weight.*

Proof. If V is reducible, then $Fun(V)$ has a nontrivial proper invariant $Fun(\mathbb{C})$-linear subspace U, which is a direct sum of its weight components. For each weight component $U[\mu]$ consider the subspace $U_\lambda[\mu]\subset V$ spanned by values of functions from $U[\mu]$ regular at λ. It is clear that $\dim_{\mathbb{C}} U_\lambda[\mu] = \dim_{Fun(\mathbb{C})} U[\mu]$ for generic λ. Moreover, $U_\lambda[\mu]$ has a limit $U_\infty[\mu]$ as λ goes to infinity in a certain generic direction, $\dim U_\infty[\mu] = \dim U_\lambda[\mu]$, and the direction can be taken the same for all μ. Then $U_\infty = \bigoplus\limits_{\mu} U_\infty[\mu]$ is a nontrivial proper invariant subspace in $\mathcal{C}(V)$ since $\dim U_\infty[\mu] < \dim V[\mu]$ at least for some μ. Claim a) is proved.

Let v be the highest weight vector of $\mathcal{C}(V)$. Then the constant function $v\in Rat(V)$ is a regular singular vector by the weight reason. Any weight subspace $V[\mu]$ has a basis given by vectors of the form $e_{a_1b_1}\ldots e_{a_kb_k}v$. Then the corresponding functions $\hat{t}_{a_1b_1}\ldots\hat{t}_{a_kb_k}v$ span $Rat\big(V[\mu]\big)$, which proves claim b). Claim c) follows from claim b) and comparison of dimensions of weight subspaces. □

Say that an $\mathfrak{sl}_N$-module V is *admissible* if V is a diagonalizable $\mathfrak{h}$-module. Notice, that any highest weight $\mathfrak{sl}_N$-module is admissible. In Section 9 we define a functor $\mathcal{E}$ from the category of admissible $\mathfrak{sl}_N$-modules to the category of semistandard $e_{rat}(\mathfrak{sl}_N)$-modules, cf. Theorem 9.9. We summarize the properties of this functor in the next two theorems.

Theorem 7.5. *Let V be an admissible $\mathfrak{sl}_N$-module. Then*
a) *$\mathcal{E}(V)$ coincides with V as an $\mathfrak{h}$-module;*
b) *$\mathcal{E}(V)$ is a perturbation of V, moreover, $\tilde{\ell}_{aa}=1$ for any $a=1,\ldots,N$;*
c) *$\mathcal{E}(V)$ is a semistandard $e_{rat}(\mathfrak{sl}_N)$-module;*
d) *if $v\in V$ is a weight singular vector, then $v\in Fun(V)$ considered as a constant function, is a standard singular vector of the same weight for $\mathcal{E}(V)$;*
e) *if V is a highest weight module, then $\mathcal{E}(V)$ is a standard highest weight module with the same highest weight and highest weight vector;*
f) *if V is a Verma module, then $\mathcal{E}(V)$ is isomorpic to the standard Verma module with the same highest weight and highest weight vector.*

Theorem 7.6. *Let U,V be highest weight $\mathfrak{sl}_N$-modules. Then*
a) *any element of $\operatorname{Mor}\big(\mathcal{E}(U),\mathcal{E}(V)\big)$ is a constant function;*
b) *the map $\operatorname{Hom}_{\mathfrak{sl}_N}(U,V)\to \operatorname{Mor}\big(\mathcal{E}(U),\mathcal{E}(V)\big)$ defined by $\mathcal{E}$ is an isomorphism;*
c) *the above isomorphism coincides with the restriction of the natural embedding of $\operatorname{Hom}(U,V)$ into $Fun\big(\operatorname{Hom}(U,V)\big)$.*

The theorems are proved in Section 9.

Let V be a highest weight $\mathfrak{sl}_N$-module with highest weight μ and highest weight vector v. Let S be the $\mathfrak{sl}_N$ Shapovalov form on V, and let S_μ be the dynamical Shapovalov pairing for $\mathcal{E}(V)$.

Proposition 7.7. Ker $S_\mu = Fun(\mathrm{Ker}\, S)$.

Proof. Since Ker S is an $\mathfrak{sl}_N$ submodule of V, then $\mathcal{E}(\mathrm{Ker}\, S)$ is an $e_{rat}(\mathfrak{sl}_N)$ submodule of $\mathcal{E}(V)$, and $Fun(\mathrm{Ker}\, S) \subset \mathrm{Ker}\, S_\mu$ by Proposition 5.7. On the other hand, $V/\mathrm{Ker}\, S$ is an irreducible $\mathfrak{sl}_N$-module, therefore, $\mathcal{E}(V)/\mathcal{E}(\mathrm{Ker}\, S) = \mathcal{E}\big(V/\mathrm{Ker}\, S\big)$ is an irreducible $e_{rat}(\mathfrak{sl}_N)$-module by Lemma 7.4. Hence, $\mathrm{Ker}\, S_\mu \subset Fun(\mathrm{Ker}\, S)$ by Proposition 5.7. $\square$

Lemma 7.8. *Let $a_i \neq b_i$ for any $i = 1, \ldots, k$ and let $c_j \neq d_j$ for any $j = 1, \ldots, l$. Then*

$$(-1)^k \prod_{i=1}^{k} \lambda_{a_i b_i} \prod_{j=1}^{l} \lambda_{c_j d_j}\ S_\mu(\hat t_{a_1 b_1} \ldots \hat t_{a_k b_k}, \hat t_{c_1 d_1} \ldots \hat t_{c_l d_l} v_\mu) \to S(e_{a_1 b_1} \ldots e_{a_k b_k} v_\mu, e_{c_1 d_1} \ldots e_{c_l d_l} v_\mu)$$

as λ goes to infinity in a generic direction.

8. Finite-dimensional highest weight modules over $e_{\tau,\gamma}(\mathfrak{sl}_N)$

In this section we assume that $\gamma \notin \mathbb{Q} + \tau\mathbb{Q}$ and consider only nondegenerate dynamical weights. We do not mention this assumption explicitly. To save writing we usually formulate the results only for standard highest weight $e_{\tau,\gamma}(\mathfrak{sl}_N)$-modules if they can be generalized to arbitrary highest weight modules by pulling back through automorphisms (3.10).

Let v be a standard singular vector of weight μ, and k be a nonnegative integer.

Lemma 8.1. *Let $a < b$. Then* $\hat t_{ab} \hat t^{\,k}_{ba} v = -\dfrac{\theta\big((\mu_{ab} - k + 1)\gamma\big)\, \theta(k\gamma)}{\theta(\lambda_{ab})\, \theta\big(\lambda_{ab} - (\mu_{ab} - 2k + 2)\gamma\big)}\, \hat t^{\,k-1}_{ba} v$.

Proof. Take formula (A.11) for $a = b$, $c = d$, and replace c by b. Since $\hat t_{aa} v = \hat t_{bb} v = v$, we have

$$\hat t_{ab} \hat t^{\,k}_{ba} v = -\frac{\theta\big(\lambda^{\{1\}}_{ab} - \lambda^{\{2\}}_{ab} + (k-1)\gamma\big)\, \theta(k\gamma)}{\theta(\lambda^{\{1\}}_{ab})\, \theta(\lambda^{\{2\}}_{ab})}\, \hat t^{\,k-1}_{ba} v .$$

Now apply convention (3.1) and observe that the vector $\hat t^{\,k-1}_{ba} v$ has weight $\mu - (k-1)(\varepsilon_a - \varepsilon_b)$. Since $\nu_{ab} = (\nu, \varepsilon_a - \varepsilon_b)$ for any $\nu \in \mathfrak{h}^*$ and $(\varepsilon_a - \varepsilon_b, \varepsilon_a - \varepsilon_b) = 2$, the lemma is proved. $\square$

Corollary 8.2. *Let $a < b$ and $\mu_{ab} \notin \{0, 1, \ldots, k-1\}$. Then $\hat t^{\,k}_{ba} v \neq 0$.*

Proof. By Lemma 8.1

$$\hat t^{\,k}_{ab} \hat t^{\,k}_{ba} v = (-1)^k \prod_{j=0}^{k-1} \frac{\theta\big((\mu_{ab} - j)\gamma\big)\, \theta\big((j+1)\gamma\big)}{\theta(\lambda_{ab} - j\gamma)\, \theta\big(\lambda_{ab} - (\mu_{ab} - j)\gamma\big)}\, v \neq 0 .$$

$\square$

Lemma 8.3. $\hat t_{cd} \hat t^{\,k}_{a+1,a} v = 0$ *for any $c < d$, $(c,d) \neq (a, a+1)$.* $\hat t_{cc} \hat t^{\,k}_{a+1,a} v = v$ *for any $c \neq a, a+1$.*

Lemma 8.4. *If $(\mu, \alpha_a) = k - 1$ and $\hat t^{\,k}_{a+1,a} v \neq 0$, then $\prod_{j=1}^{k} \theta(\lambda_{a,a+1} + j\gamma)\, \hat t^{\,k}_{a+1,a} v$ is a standard singular vector of weight $\mu - k\alpha_a$.*

Proof. By Lemmas 8.1 and 8.3 the function $\hat t^{\,k}_{a+1,a} v$ is a singular vector and $\hat t_{cc} \hat t^{\,k}_{a+1,a} v = v$ for any $c \neq a, a+1$. On the other hand, it follows from formulae (A.3) and (A.4) that

$$\hat t_{aa} \hat t^{\,k}_{a+1,a} v = \frac{\theta(\lambda_{a,a+1} + k\gamma)}{\theta(\lambda_{a,a+1})}\, \hat t^{\,k}_{a+1,a} v , \tag{8.1}$$

$$\hat t_{a+1,a+1} \hat t^{\,k}_{a+1,a} v = \frac{\theta(\lambda_{a,a+1} - \gamma\mu_{a,a+1} + k\gamma)}{\theta(\lambda_{a,a+1} - \gamma\mu_{a,a+1} + 2k\gamma)}\, \hat t^{\,k}_{a+1,a} v = \frac{\theta(\lambda_{a,a+1} + \gamma)}{\theta\big(\lambda_{a,a+1} + (k+1)\gamma\big)}\, \hat t^{\,k}_{a+1,a} v \tag{8.2}$$

because $\mu_{a,a+1} = (\mu, \alpha_a) = k - 1$. Hence, multiplying $\hat t^{\,k}_{a+1,a} v$ by the product $\prod_{j=1}^{k} \theta(\lambda_{a,a+1} + j\gamma)$ one gets a standard singular vector. $\square$

Proposition 8.5. *An irreducible standard highest weight $e_{\tau,\gamma}(\mathfrak{sl}_N)$-module with highest weight μ is infinite-dimensional if μ is not a dominant integral weight.*

Proof. Let v be the highest weight vector. Assume that μ is not a dominant integral weight, and let a be such that $(\mu,\alpha_a)\notin\mathbb{Z}_{\geqslant 0}$. Then the functions v, $\hat t_{a+1,a}v$, $\hat t^{2}_{a+1,a}v$, ... are linearly independent over $Fun(\mathbb{C})$ because all of them are nonzero by Corollary 8.2 and they have distinct weights with respect to the action of $\mathfrak{h}$. □

Proposition 8.6. *Let $(\mu,\alpha_a)=k-1$. Then $\hat t^{k}_{a+1,a}v_\mu$ generates a submodule of the Verma module M_μ isomorphic to the Verma module $M_{\mu-k\alpha_a}$.*

Proof. By Lemma 8.4 there is a nontrivial morphism $\varphi\in\operatorname{Mor}(M_{\mu-k\alpha_a},M_\mu)$ which sends the highest weight vector $v_{\mu-k\alpha_a}\in M_{\mu-k\alpha_a}$ to $\prod_{j=1}^{k}\theta(\lambda_{a,a+1}+j\gamma)\;\hat t^{k}_{a+1,a}v_\mu$, and it remains to show that φ is an embedding. In other words, one has to prove that the induced map $Fun(M_{\mu-k\alpha_a})\to Fun(M_\mu)$ is injective.

For any Verma module M_ν set $Fun_j(M_\nu)=\{\,xv_\nu\mid x\in e^{\mathcal{O}}_{\tau,\gamma}(\mathfrak{sl}_N)\,,\ \deg(x)\leqslant j\,\}$ and

$$Fun_\bullet(M_\nu)\,=\,\bigoplus_{j=0}^{\infty}Fun_j(M_\nu)\big/Fun_{j-1}(M_\nu)\,.$$

Let $\mathcal{N}$ be given by (4.5). It is clear that the set $\{\,xv_\nu\mid x\in\mathcal{N}\,,\ \deg(x)\leqslant j\,\}$ is a basis of $Fun_j(M_\nu)$ over $Fun(\mathbb{C})$, and the set $\{\,xv_\nu\mid x\in\mathcal{N}\,,\ \deg(x)=j\,\}$ induces a basis of $Fun_j(M_\nu)\big/Fun_{j-1}(M_\nu)$. We identify $Fun_\bullet(M_\nu)$ with the space of polynomials in the variables $u_{21},u_{31},\dots,u_{N,N-1}$ with coefficients in $Fun(\mathbb{C})$: for any monomial $\hat t_{b_1c_1}\dots\hat t_{b_jc_j}\in\mathcal{N}$ a class of the function $\hat t_{b_1c_1}\dots\hat t_{b_jc_j}v_\nu$ in the quotient $Fun_j(M_\nu)\big/Fun_{j-1}(M_\nu)$ is mapped to $u_{b_1c_1}\dots u_{b_jc_j}$.

The map $\varphi: Fun(M_{\mu-k\alpha_a})\to Fun(M_\mu)$ induces the map $\varphi_\bullet: Fun_\bullet(M_{\mu-k\alpha_a})\to Fun_\bullet(M_\mu)$. Consider $\varphi_\bullet$ as a map from $Fun(\mathbb{C})[u_{21},\dots,u_{N,N-1}]$ to itself. By formula (A.10) we find that

$$\varphi_\bullet(u_{b_1c_1}\dots u_{b_jc_j})\,=\,f\bigl(\lambda-\gamma\sum_{i=1}^{j}\varepsilon_{b_i}\bigr)\,u_{b_1c_1}\dots u_{b_jc_j}\,u^{k}_{a+1,a}$$

where $f(\lambda)=\prod_{j=1}^{k}\theta(\lambda_{a,a+1}+j\gamma)$. Hence, $\varphi_\bullet$ is injective, and so does φ. □

From now on till the end of the section fix a dominant integral weight μ and set $k_a=(\mu,\alpha_a)+1$, $a=1,\dots,N-1$. Notice that $k_a\in\mathbb{Z}_{>0}$ for any a. Denote by Z_μ the subspace in $Fun(M_\mu)$ generated over $e^{\mathcal{O}}_{\tau,\gamma}(\mathfrak{sl}_N)$ by the functions $\hat t^{k_a}_{a+1,a}v_\mu$, $a=1,\dots,N-1$. Notice that the functions $\hat t^{k_a}_{a+1,a}v_\mu$ are regular singular vectors, cf. Lemma 8.4.

Let S_μ be the Shapovalov pairing for M_μ, cf. (5.3).

Proposition 8.7. $Z_\mu\subset\operatorname{Ker}S_\mu$.

Proof. Lemma 8.4 implies that Z_μ is an invariant $Fun(\mathbb{C})$-linear subspace in $Fun(M_\mu)$, therefore it defines a submodule of M_μ. Hence, the statement follows from Proposition 5.5. □

Theorem 8.8. $\operatorname{Ker}S_\mu=Z_\mu$ *for generic* γ.

Proof. Since both $\operatorname{Ker}S_\mu$ and Z_μ are direct sums of their weight components, we have to prove that $(\operatorname{Ker}S_\mu)[\nu]=Z_\mu[\nu]$ for any $\nu\leqslant\mu$. Notice that $(\operatorname{Ker}S_\mu)[\mu]=Fun(\mathbb{C})\,v_\mu=Z_\mu[\mu]$.

By Proposition 8.7 it suffices to prove that $(\operatorname{Ker}S_\mu)[\nu]$ and $Z_\mu[\nu]$ have the same dimensions. For doing this we employ the deformation arguments.

Consider the Verma module $M^{\mathfrak{sl}}_\mu$ of highest weight μ over $\mathfrak{sl}_N$ and the Shapovalov form $S^{\mathfrak{sl}}_\mu$ on it. Let $M^{rat}_\mu=\mathcal{E}(M^{\mathfrak{sl}}_\mu)$, see Theorem 7.5, and let S^{rat}_μ be the corresponding dynamical Shapovalov pairing. Recall that, M^{rat}_μ is isomorphic to the Verma module of highest weight μ over $e_{rat}(\mathfrak{sl}_N)$. By abuse of notation we denote generators of $e^{\mathcal{O}}_{\tau,\gamma}(\mathfrak{sl}_N)$ and $e^{\mathcal{O}}_{rat}(\mathfrak{sl}_N)$ by the same letters and write v_μ for the highest weight vector of each of the modules M_μ, $M^{\mathfrak{sl}}_\mu$ and M^{rat}_μ.

Let $Z^{\mathfrak{sl}}_\mu$ be the $\mathfrak{sl}_N$ submodule in $M^{\mathfrak{sl}}_\mu$ generated by singular vectors $e^{k_a}_{a+1,a}v_\mu$, $a=1,\dots,N-1$, and let Z^{rat}_μ be the subspace in $Rat(M^{rat}_\mu)$ generated over $e^{\mathcal{O}}_{rat}(\mathfrak{sl}_N)$ by functions $\hat t^{k_a}_{a+1,a}v_\mu$, $a=1,\dots,N-1$. It is known that $\operatorname{Ker}S^{\mathfrak{sl}}_\mu=Z^{\mathfrak{sl}}_\mu$. By Theorem 7.5 and Proposition 7.7 we have that $\operatorname{Ker}S^{rat}_\mu=Rat(S^{\mathfrak{sl}}_\mu)$.

Lemma 8.9. $\prod_{j=1}^{k_a}(\lambda_{a,a+1}+j)\,\hat t^{k_a}_{a+1,a}v_\mu \,=\, (-1)^{k_a}\,e^{k_a}_{a+1,a}v_\mu$ *for any* $a=1,\dots,N-1$.

Proof. Both $\prod_{j=1}^{k_a}(\lambda_{a,a+1}+j)\,\hat t^{k_a}_{a+1,a}v_\mu$ and $e^{k_a}_{a+1,a}v_\mu$ have weight $\mu-k_a\alpha_a$. So they are proportional over $Rat(\mathbb{C})$ because $\dim_{\mathbb{C}} M^{\mathfrak{sl}}_\mu[\mu-k_a\alpha_a]=1$. The proportionality coefficient is a constant function since both of them are standard singular vectors by Theorem 7.5 and the rational version of Lemma 8.4. The constant equals 1 because $\lambda^{k_a}_{a,a+1}\hat t^{k_a}_{a+1,a}v_\mu \to (-1)^{k_a}\,e^{k_a}_{a+1,a}v_\mu$ as λ goes to infinity in a generic direction. □

Corollary 8.10. $\operatorname{Ker} S^{rat}_\mu = Z^{rat}_\mu = Rat(Z^{\mathfrak{sl}}_\mu)$.

Since the elliptic case is a deformation of the rational one we obtain that for generic γ

$$(8.3)\qquad \begin{aligned}\dim_{Fun(\mathbb{C})}(\operatorname{Ker} S_\mu)[\nu] &\leqslant \dim_{Rat(\mathbb{C})}(\operatorname{Ker} S^{rat}_\mu)[\nu] \,=\, \dim_{\mathbb{C}}(\operatorname{Ker} S^{\mathfrak{sl}}_\mu)[\nu] \,=\\ &= \dim_{\mathbb{C}} Z^{\mathfrak{sl}}_\mu[\nu] \,=\, \dim_{Rat(\mathbb{C})} Z^{rat}_\mu[\nu] \leqslant \dim_{Fun(\mathbb{C})} Z_\mu[\nu] \leqslant \dim_{Fun(\mathbb{C})}(\operatorname{Ker} S_\mu)[\nu]\,.\end{aligned}$$

Here the last inequality is due to Proposition 8.7. Therefore, all the dimensions in (8.3) are the same, which proves the theorem. The rest of the proof is a justification of this informal reasoning.

For a pair of sequences $\boldsymbol{a}=(a_1,\dots,a_j)$ and $\boldsymbol{b}=(b_1,\dots,b_j)$ let $\hat t_{\boldsymbol{ab}}=\hat t_{a_1b_1}\dots\hat t_{a_jb_j}$. Set

$$wt(\boldsymbol{a},\boldsymbol{b}) \,=\, wt(\hat t_{\boldsymbol{ab}}) \,=\, \sum_{i=1}^{j}(\varepsilon_{a_i}-\varepsilon_{b_i})\,, \qquad\qquad \zeta(\boldsymbol{a},\boldsymbol{b}) \,=\, \sum_{i=1}^{j}\varepsilon_{b_i}\,.$$

Let $\mathcal{M}=\{(\boldsymbol{a},\boldsymbol{b})\mid \hat t_{\boldsymbol{ab}}\in\mathcal{N}\}$ where the set $\mathcal{N}$ is defined in (4.5). Set

$$\mathcal{M}[\beta] \,=\, \{m\in\mathcal{M}\ \mid\ wt(m)=\beta\}\,, \qquad\qquad \mathcal{M}_\mu[\beta] \,=\, \bigcup_{a=1}^{N-1}\mathcal{M}[\beta+k_a\alpha_a]\,.$$

Let $\beta=\nu-\mu$. Introduce a matrix $A^\gamma(\lambda)$ with entires labeled by pairs of elements of $\mathcal{M}[\beta]$:

$$(8.4)\qquad A^\gamma_{m,m'}(\lambda) \,=\, S_\mu(\hat t_m\,,\hat t_{m'}v_\mu)\bigl(\lambda+\gamma\zeta(m)\bigr)\,,$$

and a matrix $B^\gamma(\lambda)$ with entries labeled by pairs $(m,m')\in\mathcal{M}[\beta]\times\mathcal{M}_\mu[\beta]$ and given by the rule:

$$m'\,\hat t^{k_a}_{a+1,a}v_\mu \,=\, \sum_{m\in\mathcal{M}[\beta]} B^\gamma_{m,m'}(\lambda)\,\hat t_m v_\mu\,, \qquad\qquad m'\in\mathcal{M}[\beta+k_a\alpha_a]\,.$$

Both $A^\gamma(\lambda)$ and $B^\gamma(\lambda)$ are meromorphic functions of γ,λ. Moreover, if $\gamma\to 0$, then

$$(8.5)\qquad A^\gamma(\gamma\lambda)\to A^{rat}(\lambda)\,, \qquad\qquad B^\gamma(\gamma\lambda)\to B^{rat}(\lambda)\,,$$

where $A^{rat}(\lambda)$ and $B^{rat}(\lambda)$ are defined in the same way for the rational case (in formula (8.4) the argument in the right hand side is $\lambda+\zeta(m)$ for the rational case).

Each matrix naturally defines a linear map: $A^\gamma(\lambda)$ and $A^{rat}(\lambda)$ act in $\mathcal{M}_{\mathbb{C}}[\beta]=\bigoplus_{m\in\mathcal{M}[\beta]}\mathbb{C}m$, whilst $B^\gamma(\lambda)$ and $B^{rat}(\lambda)$ map $\bigoplus_{m\in\mathcal{M}_\mu[\beta]}\mathbb{C}m$ to $\mathcal{M}_{\mathbb{C}}[\beta]$.

Given functions $\varphi_m(\lambda)\in Fun(\mathbb{C})$, $m\in\mathcal{M}[\beta]$, consider a vector $\check\varphi(\lambda)=\sum_{m\in\mathcal{M}[\beta]}\varphi_m(\lambda)\,m\in\mathcal{M}_{\mathbb{C}}[\beta]$. Set $\vec\varphi(\lambda)=\sum_{m\in\mathcal{M}[\beta]}\varphi_m(\lambda)\,\hat t_m v_\mu\in Fun(M_\mu)$.

Lemma 8.11.

a) $\vec\varphi\in(\operatorname{Ker} S_\mu)[\nu]$ *iff* $\check\varphi(\lambda)\in\operatorname{Ker} A^\gamma(\lambda)$ *for generic* λ;

b) $\vec\varphi\in(\operatorname{Ker} S^{rat}_\mu)[\nu]$ *iff* $\check\varphi(\lambda)\in\operatorname{Ker} A^{rat}(\lambda)$ *for generic* λ;

c) $\vec\varphi\in Z_\mu[\nu]$ *iff* $\check\varphi(\lambda)\in\operatorname{Im} B^\gamma(\lambda)$ *for generic* λ;

d) $\vec\varphi\in Z^{rat}_\mu[\nu]$ *iff* $\check\varphi(\lambda)\in\operatorname{Im} B^{rat}(\lambda)$ *for generic* λ.

Proof. Claims a) and b) follow from formulae (5.2) – (5.4) and their rational versions, respectively. Claims c) and d) are straightforward. □

Corollary 8.10 gives that $\mathrm{Ker}\, A^{rat}(\lambda) = \mathrm{Im}\, B^{rat}(\lambda)$ for generic λ. The standard deformation reasoning, cf. (8.5), shows that

$$\dim_{\mathbb{C}} \mathrm{Ker}\, A^{\gamma}(\lambda) \leqslant \dim_{\mathbb{C}} \mathrm{Ker}\, A^{rat}(\lambda) = \dim_{\mathbb{C}} \mathrm{Im}\, B^{rat}(\lambda) = \dim_{\mathbb{C}} \mathrm{Im}\, B^{\gamma}(\lambda) \leqslant \dim_{\mathbb{C}} \mathrm{Ker}\, A^{\gamma}(\lambda) \tag{8.6}$$

for generic λ, if γ is generic. Here the last inequality is due to Proposition 8.7 which implies that $\mathrm{Im}\, B^{\gamma}(\lambda) \subset \mathrm{Ker}\, A^{\gamma}(\lambda)$ for generic λ. Therefore, all the dimensions in (8.6) are the same. Hence $\mathrm{Ker}\, A^{\gamma}(\lambda) = \mathrm{Im}\, B^{\gamma}(\lambda)$ for generic λ, if γ is generic. By Lemma 8.11 we get that $(\mathrm{Ker}\, S_\mu)[\nu] = Z_\mu[\nu]$ for generic γ. Theorem 8.8 is proved. □

Let V_μ be the irreducible standard highest weight $e_{\tau,\gamma}(\mathfrak{sl}_N)$-module of highest weight μ. Let N_μ be the $e_{rat}(\mathfrak{sl}_N)$-submodule of M_μ such that $Fun(N_\mu) = \mathrm{Ker}\, S_\mu$. We have $V_\mu = M_\mu / N_\mu$, cf. Corollary 5.6. Let $V_\mu^{\mathfrak{sl}}$ be the irreducible $\mathfrak{sl}_N$-module of highest weight μ. Set $d_\mu[\nu] = \dim_{\mathbb{C}} V_\mu^{\mathfrak{sl}}[\nu]$.

Proof of Theorem 5.8. It follows from the proof of Theorem 8.8 that for generic γ

$$\begin{aligned} \dim_{\mathbb{C}} V_\mu[\nu] &= \dim_{\mathbb{C}} M_\mu[\nu] - \dim_{Fun(\mathbb{C})} (\mathrm{Ker}\, S_\mu)[\nu] = \\ &= \dim_{\mathbb{C}} M_\mu^{\mathfrak{sl}}[\nu] - \dim_{\mathbb{C}} (\mathrm{Ker}\, S_\mu^{\mathfrak{sl}})[\nu] = \dim_{\mathbb{C}} V_\mu^{\mathfrak{sl}}[\nu] = d_\mu[\nu]. \end{aligned} \tag{8.7}$$

Since $\dim_{Fun(\mathbb{C})} (\mathrm{Ker}\, S_\mu)[\nu]$ can only jump up at specific γ, we have that $\dim_{\mathbb{C}} V_\mu[\nu] \leqslant d_\mu[\nu]$ for arbitrary γ. □

Proof of Theorem 5.9. We have proved already that $\dim_{\mathbb{C}} V_\mu[\nu] = d_\mu[\nu]$ for any ν, if γ is generic, cf. (8.7). Consider an $e_{\tau,\gamma}(\mathfrak{sl}_N)$-module U_μ defined in the following way. Let $U_\mu = V_\mu$ if γ is generic, that is, if $\dim_{\mathbb{C}} V_\mu[\nu] = d_\mu[\nu]$. Otherwise, define U_μ by analytic continuation from generic γ. It is not difficult to justify the given definition of U_μ, and to see that U_μ is a highest weight $e_{\tau,\gamma}(\mathfrak{sl}_N)$-module with highest weight μ and $\dim_{\mathbb{C}} U_\mu[\nu] = d_\mu[\nu]$ for any ν.

If μ is a dominant integral weight, then $w \cdot \mu < \mu$ and $d_\mu[w \cdot \mu] = 0$ for any nontrivial element $w \in W$. Hence, U_μ is irreducible by Corollary 4.4 if $\gamma \notin \mathbb{Q} + \tau\mathbb{Q}$, that is, $U_\mu = V_\mu$. The theorem is proved. □

Remark. There is another approach to constructing finite-dimensional irreducible representation of $e_{\tau,\gamma}(\mathfrak{sl}_N)$. One can start from the vector representation of $e_{\tau,\gamma}(\mathfrak{sl}_N)$ and apply the fusion procedure technique developed in the nondynamical case, cf. [C], [N]. If $\gamma \notin \mathbb{Q} + \tau\mathbb{Q}$, then any irreducible finite-dimensional standard highest weight $e_{\tau,\gamma}(\mathfrak{sl}_N)$-module can be obtained in this way. The symmetric and exterior powers of the vector representation of $e_{\tau,\gamma}(\mathfrak{sl}_N)$ has been constructed by this technique in [FV2]. We will address this approach elsewhere.

9. Definition of the functor $\mathcal{E}$

In this section we construct a functor from the category of admissible $\mathfrak{sl}_N$-modules to the category of semistandard $e_{rat}(\mathfrak{sl}_N)$-modules. The construction is similar to the construction of the functor from the category of finite-dimensional $\mathfrak{sl}_N$-modules to the category of rational representations of the exchange quantum group $F\bigl(SL(N)\bigr)$, developed in [EV2]. We discuss in detail the relation of two constructions in Section 10.

Let $\mathfrak{n}_+ = \bigl\{ \sum\limits_{a<b} x_{ab}\, e_{ab} \bigr\}$ and $\mathfrak{n}_- = \bigl\{ \sum\limits_{a>b} x_{ab}\, e_{ab} \bigr\}$ be the standard nilpotent subalgebras in $\mathfrak{sl}_N$, and let $\mathfrak{b}_\pm = \mathfrak{h} \oplus \mathfrak{n}_\pm$. Set

$$\Xi = \frac{1}{2} \sum_{a=1}^{N} e_{aa}^2 - \frac{1}{2N} \bigl(\sum_{a=1}^{N} e_{aa} \bigr)^2 \in U(\mathfrak{h}).$$

Proposition 9.1. *There exists a unique power series $\mathcal{J}(\lambda; z)$ in z with coefficients in $U(\mathfrak{sl}_N) \otimes U(\mathfrak{sl}_N)$ valued functions of λ with the properties:*

a) *$\mathcal{J}(\lambda; z)$ satisfies the rational ABRR equation*

$$\mathcal{J}(\lambda; z) \bigl(1 \otimes (\lambda - z\,\Xi)\bigr) = \bigl(1 \otimes (\lambda - z\,\Xi) + z \sum_{1 \leqslant a<b \leqslant N} e_{ab} \otimes e_{ba}\bigr)\, \mathcal{J}(\lambda; z); \tag{9.1}$$

b) *the coefficients of the series $\bigl(\mathcal{J}(\lambda; z) - 1\bigr)$ are $U(\mathfrak{b}_+)\mathfrak{n}_+ \otimes \mathfrak{n}_- U(\mathfrak{b}_-)$-valued functions of λ.*

Proof. Let $\mathcal{J}(\lambda;z) = \sum\limits_{k=0}^{\infty} \mathcal{J}_k(\lambda) z^k$. Equation (9.1) is equivalent to certain recurrence relations for coefficients $\mathcal{J}_k(\lambda)$ with the initial condition $\mathcal{J}_0(\lambda) = 1$. It is straightforward to verify that at each step the recurrence relations uniquely determine $\mathcal{J}_k$ from $\mathcal{J}_0, \dots, \mathcal{J}_{k-1}$. □

It follows from the proof of the last proposition that the coefficients of the series $\mathcal{J}(\lambda;z)$ are rational functions of λ, and for any $x \in \mathfrak{h}$

$$[\mathcal{J}(\lambda;z), x\otimes 1 + 1\otimes x] = 0. \tag{9.2}$$

Denote by $\Delta : U(\mathfrak{sl}_N) \to U(\mathfrak{sl}_N) \otimes U(\mathfrak{sl}_N)$ the coproduct for $U(\mathfrak{sl}_N)$.

Theorem 9.2. *The series $\mathcal{J}(\lambda;z)$ satisfies the equation*

$$\mathcal{J}^{((12)3)}(\lambda;z)\,\mathcal{J}^{(12)}(\lambda - zh^{(3)};z) = \mathcal{J}^{(1(23))}(\lambda;z)\,\mathcal{J}^{(23)}(\lambda;z). \tag{9.3}$$

Here $\mathcal{J}^{((12)3)} = (\Delta\otimes \mathrm{id})(\mathcal{J})$, $\mathcal{J}^{(1(23))} = (\mathrm{id}\otimes\Delta)(\mathcal{J})$, $\mathcal{J}^{(12)} = \mathcal{J}\otimes 1$, $\mathcal{J}^{(23)} = 1\otimes\mathcal{J}$, and the meaning of $\mathcal{J}^{(12)}(\lambda - zh^{(3)};z)$ is explained below, cf. (9.4).

Proof. The statement is a degeneration of Theorem 3.1 in [ESS], and can be proved in the same way. □

Remark. Let $x_1, \dots, x_{N-1}$ be a basis of $\mathfrak{h}^*$, and let $x^1, \dots, x^{N-1}$ be the dual basis of $\mathfrak{h}$. Write $\lambda = \lambda^1 x_1 + \ldots + \lambda^{N-1} x_{N-1}$. For a rational function $f(\lambda)$ we define a series $f(\lambda - zh^{(3)})$ by the Taylor series expansion:

$$f(\lambda - zh^{(3)}) = f(\lambda;z) - z\sum_{a=1}^{N-1} \frac{\partial f(\lambda)}{\partial\lambda^a}\,(x^a)^{(3)} + \ldots, \tag{9.4}$$

and extend the definition to series in z with coefiicients in rational functions of λ in the natural way.

Remark. The equation (9.3) is usually called the *dynamical* 2*-cocycle condition.*

Define a *rational exchange matrix* $\mathcal{R}(\lambda;z)$ by the rule:

$$\mathcal{R}(\lambda;z) = \bigl(\mathcal{J}(\lambda;z)\bigr)^{-1}\mathcal{J}^{(21)}(\lambda;z). \tag{9.5}$$

Theorem 9.3. *$\mathcal{R}(\lambda)$ satisfies the dynamical Yang-Baxter equation:*

$$\mathcal{R}^{(12)}(\lambda - zh^{(3)};z)\,\mathcal{R}^{(13)}(\lambda;z)\,\mathcal{R}^{(23)}(\lambda - zh^{(1)};z) = \mathcal{R}^{(23)}(\lambda;z)\,\mathcal{R}^{(13)}(\lambda - zh^{(2)};z)\,\mathcal{R}^{(12)}(\lambda;z).$$

The statement follows from Theorem 9.2 and cocommutativity of the coproduct Δ.

Say that an n-tuple $V_1, \dots, V_n$ of $\mathfrak{sl}_N$-modules is admissible if for any pairwise distinct $i_1, \dots, i_k$ the tensor product $V_{i_1}\otimes\ldots\otimes V_{i_k}$ is a diagonalizable $\mathfrak{h}$-module; that is, $V_1, \dots, V_n$ are diagonalizable $\mathfrak{h}$-modules and all weight subspaces of any tensor product $V_{i_1}\otimes\ldots\otimes V_{i_k}$ are finite-dimensional.

Let V, W be an admissible pair of $\mathfrak{sl}_N$-modules. Denote by $J_{VW}(\lambda;z) \in \mathrm{End}(V\otimes W)$ the action of $\mathcal{J}(\lambda;z)$ in the tensor product $V\otimes W$. It follows from the explicit form of recurrence relations in the proof of Proposition 9.1 that there is a unique function $J_{VW}(\lambda) \in Rat\bigl(\mathrm{End}(V\otimes W)\bigr)$ such that the series $J_{VW}(\lambda;z)$ coincides with the expansion of $J_{VW}(\lambda/z)$ at $z=0$. The function $J_{VW}(\lambda)$ admits the following description.

Lemma 9.4. *$J_{VW}(\lambda)$ is the unique solution of the equation*

$$J_{VW}(\lambda)\,\bigl(1\otimes(\lambda - \Xi)\bigr)\big|_{V\otimes W} = \Bigl(1\otimes(\lambda-\Xi) + \sum_{1\leqslant a<b\leqslant N} e_{ab}\otimes e_{ba}\Bigr)\Big|_{V\otimes W} J_{VW}(\lambda),$$

such that $\bigl(J_{VW}(\lambda) - 1\bigr) \in \bigl(U(\mathfrak{b}_+)\mathfrak{n}_+\otimes\mathfrak{n}_- U(\mathfrak{b}_-)\bigr)\big|_{V\otimes W}$. Moreover, $J_{VW}(\lambda)$ commutes with the action of $\mathfrak{h}$ in $V\otimes W$, cf. (9.2).

Proposition 9.5. *For any admissible triple of $\mathfrak{sl}_N$-modules U, V, W we have*

$$J_{U\otimes V,W}(\lambda)\bigl(J_{UV}(\lambda - h^{(3)})\otimes 1\bigr) = J_{U,V\otimes W}(\lambda)\bigl(1\otimes J_{VW}(\lambda)\bigr). \tag{9.6}$$

Proof. The function $\left(J_{UV}(\lambda-h^{(3)})\otimes 1\right)$ is defined by the rule (1.1). To get relation (9.6) from formula (9.3) one needs to verify that the series obtained by expansion of $\left(J_{UV}(z^{-1}\lambda-h^{(3)})\otimes 1\right)$ at $z=0$ coincides with the action of $\mathcal{J}^{(12)}(\lambda-zh^{(3)};z)$, defined by (9.4), in $U\otimes V\otimes W$, which is simple. □

For any $A\in\mathrm{End}(W\otimes V)$ let $A^{(21)}=PAP^{-1}\in\mathrm{End}(V\otimes W)$ where $P:W\otimes V\to V\otimes W$ is the permutation map: $P(x\otimes y)=y\otimes x$. Set

$$R_{VW}(\lambda)\,=\,\left(J_{VW}(\lambda)\right)^{-1}\left(J_{WV}(\lambda)\right)^{(21)}. \tag{9.7}$$

It is clear that the action of the series $\mathcal{R}(\lambda;z)$ in $V\otimes W$ coincides with the expansion of $R_{VW}(\lambda/z)$ at $z=0$. Like in the proof of Proposition 9.5 we get the following assertion from Theorem 9.3.

Proposition 9.6. *For any admissible triple of $\mathfrak{sl}_N$-modules U,V,W we have*

$$R^{(12)}_{UV}(\lambda-h^{(3)})\,R^{(13)}_{UW}(\lambda)\,R^{(23)}_{VW}(\lambda-h^{(1)})\;=\;R^{(23)}_{VW}(\lambda)\,R^{(13)}_{UW}(\lambda-h^{(2)})\,R^{(12)}_{UV}(\lambda)\,.$$

Let $\widetilde{J}_{VW}(\lambda)$ be the fusion matrix for $U(\mathfrak{sl}_N)$ defined in [EV2], and let

$$\widetilde{R}_{VW}(\lambda)\,=\,\left(\widetilde{J}_{VW}(\lambda)\right)^{-1}\left(\widetilde{J}_{WV}(\lambda)\right)^{(21)} \tag{9.8}$$

be the corresponding dynamical R-matrix. Let X be the matrix $\sum_{a=1}^{N}(-1)^{a-1}E_{a,N-a+1}$ considered as an element of the group $SL(N)$, and let w_X be the longest element of the Weil group. We have $\mathrm{Ad}_X(e_{ab})=e_{N-a+1,N-b+1}$ and $w_X(\varepsilon_a)=\varepsilon_{N-a+1}$ for any $a,b=1,\dots,N$.

Lemma 9.7. *For any finite-dimensional $\mathfrak{sl}_N$-modules V,W we have*

$$\widetilde{J}_{VW}(\lambda)\;=\;(X\otimes X)\,J_{VW}\big(w_X(\lambda+\rho)\big)\,(X\otimes X)^{-1}. \tag{9.9}$$

Proof. It is shown in [EV2, Section 9] that $\widetilde{J}_{VW}(\lambda)$ is the unique solution of the equation

$$\widetilde{J}_{VW}(\lambda)\,\big(1\otimes(\lambda+\rho-\Xi)\big)\big|_{V\otimes W}\;=\;\Big(1\otimes(\lambda+\rho-\Xi)+\sum_{1\leqslant a<b\leqslant N}e_{ba}\otimes e_{ab}\Big)\Big|_{V\otimes W}\,\widetilde{J}_{VW}(\lambda)\,,$$

such that $\big(\widetilde{J}_{VW}(\lambda)-1\big)\in\big(\mathfrak{n}_-U(\mathfrak{b}_-)\otimes U(\mathfrak{b}_+)\mathfrak{n}_+\big)\big|_{V\otimes W}$. By Lemma 9.4 the right hand side of formula (9.9) has the same properties, which proves the claim. □

Corollary 9.8. $\widetilde{R}_{VW}(\lambda)\;=\;(X\otimes X)\,R_{VW}\big(w_X(\lambda+\rho)\big)\,(X\otimes X)^{-1}$.

Henceforward, let U be the vector representation of $\mathfrak{sl}_N$. By formula (36) in [EV2] we have

$$\widetilde{R}_{UU}(\lambda)\;=\;\sum_{a,b=1}^{N}E_{aa}\otimes E_{bb}\;-\sum_{1\leqslant a<b\leqslant N}\frac{E_{bb}\otimes E_{aa}}{(\lambda_{ab}-a+b)^2}\;-\sum_{\substack{a,b=1\\ a\neq b}}^{N}\frac{E_{ab}\otimes E_{ba}}{\lambda_{ab}-a+b}\,, \tag{9.10}$$

Therefore,

$$R_{UU}(\lambda)\;=\;\sum_{a,b=1}^{N}E_{aa}\otimes E_{bb}\;-\sum_{1\leqslant a<b\leqslant N}\frac{E_{aa}\otimes E_{bb}}{\lambda_{ab}^2}\;-\sum_{\substack{a,b=1\\ a\neq b}}^{N}\frac{E_{ab}\otimes E_{ba}}{\lambda_{ab}}\,. \tag{9.11}$$

Let V be an admissible $\mathfrak{sl}_N$-module. Introduce functions $\hat{\ell}^{V}_{ab}\in Rat\big(\mathrm{End}(V)\big)$, $a,b=1,\dots,N$, by the equality

$$R_{UV}(\lambda)\;=\;\sum_{a,b=1}^{N}E_{ba}\otimes\hat{\ell}^{V}_{ab}(\lambda)\,.$$

Theorem 9.9. *Let V be an admissible $\mathfrak{sl}_N$-module. Then the rule $(\hat{t}_{ab}\,v)(\lambda)=\hat{\ell}^{V}_{ab}(\lambda)v(\lambda-\varepsilon_a)$, for any $a,b=1,\dots,N$ and any $v\in Rat(V)$, endows V with an $e_{rat}(\mathfrak{sl}_N)$-module structure. The constructed $e_{rat}(\mathfrak{sl}_N)$-module is denoted by $\mathcal{E}(V)$.*

Proof. The statement follows from Proposition 9.6, and formulae (7.1), (7.4), (7.9) and (9.11). □

We define the functor $\mathcal{E}$ from the category of admissible $\mathfrak{sl}_N$-modules to the category of semistandard $e_{rat}(\mathfrak{sl}_N)$-modules sending an object V to $\mathcal{E}(V)$ and a morphism $\varphi \in \mathrm{Hom}\,(V,W)$ to the corresponding constant function $\varphi \in \mathrm{Mor}\,(V,W)$.

Proof of Theorem 7.5. Claim a) of the theorem is immediate. Say that $f(\lambda) = O\bigl(|\lambda|^k\bigr)$ if $f(s\lambda) = O(s^k)$ as $s \to +\infty$ for generic λ. From Lemma 9.4 and formula (9.7) we have that

$$J_{UV}(\lambda) \,=\, 1 - \sum_{1\leqslant a<b\leqslant N} \frac{E_{ba}\otimes e_{ab}}{\lambda_{ab}} + O\bigl(|\lambda|^{-2}\bigr)\,,$$

$$R_{UV}(\lambda) \,=\, 1 + \sum_{\substack{a,b=1\\ a\neq b}}^{N} \frac{E_{ba}\otimes e_{ab}}{\lambda_{ab}} + O\bigl(|\lambda|^{-2}\bigr)\,,$$

which proves claim b). Claim c) follows from Lemma 7.1. Claims e) and f) follow from claims b) and d), and Lemma 7.4.

To prove claim d) one should show that for any singular vector $v \in V$ we have $\hat\ell^{\,V}_{aa}(\lambda)\,v = v$ for any a, and $\hat\ell^{\,V}_{ab}(\lambda)\,v = 0$ for any $a<b$. By Lemma 9.4 and formula (9.7) we see that $\bigl(\hat\ell^{\,V}_{ab}(\lambda)-1\bigr) \in \bigl(\mathfrak{n}_-U(\mathfrak{sl}_N)\,\mathfrak{n}_+\bigr)\big|_V$ for any a, and $\hat\ell^{\,V}_{ab}(\lambda) \in \bigl(U(\mathfrak{sl}_N)\,\mathfrak{n}_+\bigr)\big|_V$ for any $a<b$, which implies claim d).

It remains to prove claim c. The element D acts in $Rat(V)$ as multiplication by $D(\lambda)$. It is clear from the definition of $\mathcal{E}(V)$ that there exists some independent of V element in a certain completion of $U(\mathfrak{sl}_N)$ such that its action in V coincides with $D(\lambda)$. Since an element of $U(\mathfrak{sl}_N)$ is uniquely determined by its action in highest weight modules, it suffices to prove claim c) under the assumption that V is a highest weight module. In the last case claim c) follows from claim d). Theorem 7.5 is proved. □

The proof of Theorem 7.6 is similar to the proof of Theorem 45 in [EV2].

10. Exchange quantum group $F\bigl(SL(N)\bigr)$

For any $a = 1,\dots,N$ let $\boldsymbol{i}^a = (1,\dots,a-1,a+1,\dots,N)$. Set

$$\bar t_{ab} \,=\, (-1)^{a+b}\sum_{\boldsymbol{j}\in\mathbf{S}_{N-1}} \mathrm{sign}(\boldsymbol{j})\; t_{i^a_{N-1},\,i^b_{j_{N-1}}}\,\dots\,t_{i^a_1,\,i^b_{j_1}}\,.$$

Lemma 10.1. $\sum\limits_{c=1}^{N} \bar t_{ac}\,t_{bc} \,=\, \delta_{ab}\,t^{\wedge N}$, *where* $t^{\wedge N}$ *is defined by* (7.6).

Proof. The formula coincides with the equality of the top coefficients in the rational version of formula (B.6) for $k=1$ and $T(u) = \mathcal{T}(u)$, cf. (3.6). □

Consider the exchange quantum group $F\bigl(SL(N)\bigr)$ defined in [EV2]. It admits the following description, see [EV2, Section 5.3]. $F\bigl(SL(N)\bigr)$ is a unital associative algebra over $\mathbb{C}$ generated by functions $f \in Rat^{\otimes 2}(\mathbb{C})$ and elements T^+_{ab}, T^-_{ab}, $a,b = 1,\dots,N$, subject to relations (10.1)–(10.3) and (10.5).

Let $\widetilde R(\lambda) = \widetilde R_{UU}(\lambda)$, cf. (9.10). Set $T^\pm = \sum\limits_{a,b=1}^{N} E_{ab}\otimes T^\pm_{ab}$. The defining relations for $F\bigl(SL(N)\bigr)$ are

$$T^+T^- \,=\, T^-T^+ \,=\, \mathrm{id}\otimes 1\,, \tag{10.1}$$

$$T^+_{ab}\,f(\lambda^{\{1\}},\lambda^{\{2\}}) \,=\, f(\lambda^{\{1\}}-\varepsilon_b\,,\lambda^{\{2\}}-\varepsilon_a)\,T^+_{ab} \tag{10.2}$$

for any $f \in Rat^{\otimes 2}(\mathbb{C})$,

$$\widetilde R^{(12)}(\lambda^{\{2\}})\,T^{(13)}T^{(23)} \,=\, T^{(23)}T^{(13)}\,\widetilde R^{(12)}(\lambda^{\{1\}}) \tag{10.3}$$

where $T^{(13)} = \sum\limits_{a,b} E_{ab}\otimes\mathrm{id}\otimes T^+_{ab}$ and $T^{(23)} = \sum\limits_{a,b} \mathrm{id}\otimes E_{ab}\otimes T^+_{ab}$, and relation (10.5).

Remark. In this paper the variables $\lambda^{\{1\}},\lambda^{\{2\}}$ and the generators T^+_{ab}, T^-_{ab} correspond to the variables λ^2,λ^1 and the generators L_{ab}, L^{-1}_{ab} in [EV2].

For any permutation $\boldsymbol{i} \in \mathbf{S}_N$ let $\Lambda_{\boldsymbol{i}}(\lambda) = \prod_{\substack{1\leqslant a<b\leqslant N \\ i_a<i_b}} (1+\lambda_{ab}^{-1})$. Set

$$\mathrm{Det}\, T^+ \,=\, \frac{1}{\Lambda_{\mathrm{id}}(\lambda^{\{1\}})} \sum_{\boldsymbol{i}\in\mathbf{S}_N} \mathrm{sign}(\boldsymbol{i})\, \Lambda_{\boldsymbol{i}}(\lambda^{\{2\}})\, T^+_{i_N,N} \ldots T^+_{i_1,1} \tag{10.4}$$

where $\mathrm{id} = (1,\ldots,N)$. The last defining relation for $F\bigl(SL(N)\bigr)$ is

$$\mathrm{Det}\, T^+ \,=\, 1\,. \tag{10.5}$$

Remark. The element $\mathrm{Det}\, T^+$ corresponds to the element D in [EV2]. Formula (10.4) can be derived from the definition of D therein. The complete proof of formula (10.4) will appear elsewhere.

Recall that given a diagonalizable $\mathfrak{h}$-module V we assume the following action of $\mathit{Rat}^{\otimes 2}(\mathbb{C})$ on V-valued functions:

$$f\,:\, v(\lambda) \,\mapsto\, f(\lambda, \lambda-\mu)\, v(\lambda)$$

for any $f \in \mathit{Rat}^{\otimes 2}(\mathbb{C})$ and any function $v(\lambda)$ with values in $V[\mu]$.

A rational dynamical representation of $F\bigl(SL(N)\bigr)$ is a diagonalizable $\mathfrak{h}$-module V endowed with an action of $F\bigl(SL(N)\bigr)$ on V-valued meromorphic functions by difference operators with rational coefficients:

$$(T^+_{ab}\, v)(\lambda) \,=\, L_{ab}(\lambda) v(\lambda - \varepsilon_b)\,, \qquad a,b = 1,\ldots,N\,, \tag{10.6}$$

where $L_{ab}(\lambda) \in \mathit{Rat}\bigl(\mathrm{End}\,(V)\bigr)$ are suitable functions.

Proposition 10.2. *Let V be a rational dynamical representation of $F\bigl(SL(N)\bigr)$. Then the rule*

$$(t_{ab}\, v)(\lambda) \,=\, \prod_{1\leqslant c<a} (\lambda_{ca}+1)^{-1} \prod_{1\leqslant c<b} (\lambda_{cb} + e_{bb} - e_{cc} + 1)\, L_{ba}(\lambda-\rho)\, v(\lambda-\varepsilon_a) \tag{10.7}$$

for any $a,b = 1,\ldots,N$ and any $v \in \mathit{Rat}(V)$, endows V with a structure of a semistandard $e_{rat}(\mathfrak{sl}_N)$ module.

The proof is straightforward.

Proposition 10.3. *Let V be a semistandard $e_{rat}(\mathfrak{sl}_N)$-module. Then formulae* (10.6), (10.7) *make V into a rational dynamical representation of $F\bigl(SL(N)\bigr)$.*

Proof. It is straightforward to verify relations (10.2), (10.3) and (10.5). To complete the proof it remains to find the action of elements T^-_{ab} to obey relations (10.1). This can be done using Proposition 10.1, since the $e_{rat}(\mathfrak{sl}_N)$-module V is nondegenerate. ☐

The last two propositions define a functor $\mathcal{F}$ from the category of rational dynamical representations of $F\bigl(SL(N)\bigr)$ to the category of semistandard $e_{rat}(\mathfrak{sl}_N)$-modules: an object V goes to itself and a morphism $\varphi(\lambda) \in \mathit{Rat}\bigl(\mathrm{Hom}\,(V,W)\bigr)$ goes to $\varphi(\lambda-\rho) \in \mathrm{Mor}\,(V,W)$. Furthermore, the propositions imply the following assertion.

Theorem 10.4. *The functor $\mathcal{F}$ is an equivalence of the categories.*

For both categories involved in the last theorem the subcategories of finite-dimensional objects are tensor categories, the tensor product of rational dynamical representations of $F\bigl(SL(N)\bigr)$ being defined in [EV2]. One can show that the restriction of the functor $\mathcal{F}$ to these subcategories is an equivalence of tensor categories.

Let $\mathcal{G}$ be the functor from the category of finite-dimensional $\mathfrak{sl}_N$-modules to the category of finite-dimensional dynamical representations of $F\bigl(SL(N)\bigr)$ defined in [EV2]: an $\mathfrak{sl}_N$-module V goes to the representation $\mathcal{G}(V)$ of $F\bigl(SL(N)\bigr)$ given by the rule

$$\widetilde{R}_{UV}(\lambda) \,=\, \sum_{a,b=1}^{N} E_{ab} \otimes L_{ab}(\lambda)\,, \tag{10.8}$$

and a morphism $\varphi \in \mathrm{Hom}\,(V,W)$ goes to the corresponding constant function $\varphi \in \mathit{Rat}\bigl(\mathrm{Hom}\,(V,W)\bigr)$. The composition $\widetilde{\mathcal{E}} = \mathcal{F}\circ\mathcal{G}$ is a functor from the category of finite-dimensional $\mathfrak{sl}_N$-modules to the category of semistandard $e_{rat}(\mathfrak{sl}_N)$-modules.

Theorem 10.5. *The functor $\widetilde{\mathcal{E}}$ is isomorphic to the restriction of the functor $\mathcal{E}$ to the category of finite-dimensional $\mathfrak{sl}_N$-modules.*

The theorem is proved in Appendix F.

Remark. Let V be an irreducible finite-dimensional $\mathfrak{sl}_N$-module. Then the $e_{rat}(\mathfrak{sl}_N)$-module $\mathcal{E}(V)$ is an irreducible standard highest weight module over the dynamical quantum group $e_{rat}(\mathfrak{sl}_N)$. Such modules have been described in Section 8. Applying the functor inverse to $\mathcal{F}$ we get a new description of the dynamical representations of $F\bigl(SL(N)\bigr)$ induced from irreducible finite-dimensional $\mathfrak{sl}_N$-modules. This new description is a new highest weight module theory for the exchange dynamical quantum group $F\bigl(SL(N)\bigr)$.

Appendix A. Commutation relations in $e^{\mathcal{O}}_{\tau,\gamma}(\mathfrak{sl}_N)$

In this section we collect useful commutation relations which hold in the operator algebra $e^{\mathcal{O}}_{\tau,\gamma}(\mathfrak{sl}_N)$.

In the definition of $e^{\mathcal{O}}_{\tau,\gamma}(\mathfrak{sl}_N)$ one can replace relations (3.5) by the following relations:

$$\frac{\theta(\lambda^{\{1\}}_{ac}-\gamma)}{\theta(\lambda^{\{1\}}_{ac})}\,t_{ab}\,t_{cd}\;-\;\frac{\theta(\lambda^{\{2\}}_{bd}-\gamma)}{\theta(\lambda^{\{2\}}_{bd})}\,t_{cd}\,t_{ab}\;=\;\frac{\theta(\lambda^{\{1\}}_{ac}+\lambda^{\{2\}}_{bd})\,\theta(\gamma)}{\theta(\lambda^{\{1\}}_{ac})\,\theta(\lambda^{\{2\}}_{bd})}\,t_{cb}\,t_{ad}\,, \tag{A.1}$$

for $a\neq c$ and $b\neq d$. The last formula implies that

$$\hat t_{aa}\,\hat t_{bb}\;-\;\hat t_{bb}\,\hat t_{aa}\;=\;\frac{\theta(\lambda^{\{1\}}_{ab}+\lambda^{\{2\}}_{ab})\,\theta(\gamma)}{\theta(\lambda^{\{1\}}_{ab})\,\theta(\lambda^{\{2\}}_{ab})}\,\hat t_{ba}\,\hat t_{ab} \tag{A.2}$$

for $a<b$. Under the same assumption we have

$$\frac{\theta(\lambda^{\{2\}}_{ab}-\gamma)\,\theta(\lambda^{\{2\}}_{ab}+\gamma)}{\theta(\lambda^{\{2\}}_{ab})^2}\,\hat t_{aa}\,\hat t_{bb}\;-\;\frac{\theta(\lambda^{\{1\}}_{ab}-\gamma)\,\theta(\lambda^{\{1\}}_{ab}+\gamma)}{\theta(\lambda^{\{1\}}_{ab})^2}\,\hat t_{bb}\,\hat t_{aa}\;=\;\frac{\theta(\lambda^{\{1\}}_{ab}+\lambda^{\{2\}}_{ab})\,\theta(\gamma)}{\theta(\lambda^{\{1\}}_{ab})\,\theta(\lambda^{\{2\}}_{ab})}\,\hat t_{ab}\,\hat t_{ba}\,,$$

$$\hat t_{ab}\,\hat t_{ba}\;-\;\frac{\theta(\lambda^{\{1\}}_{ab}-\gamma)\,\theta(\lambda^{\{1\}}_{ab}+\gamma)}{\theta(\lambda^{\{1\}}_{ab})^2}\,\hat t_{ba}\,\hat t_{ab}\;=\;-\,\frac{\theta(\lambda^{\{1\}}_{ab}-\lambda^{\{2\}}_{ab})\,\theta(\gamma)}{\theta(\lambda^{\{1\}}_{ab})\,\theta(\lambda^{\{2\}}_{ab})}\,\hat t_{aa}\,\hat t_{bb}\,,$$

$$\hat t_{ab}\,\hat t_{ba}\;-\;\frac{\theta(\lambda^{\{2\}}_{ab}-\gamma)\,\theta(\lambda^{\{2\}}_{ab}+\gamma)}{\theta(\lambda^{\{2\}}_{ab})^2}\,\hat t_{ba}\,\hat t_{ab}\;=\;-\,\frac{\theta(\lambda^{\{1\}}_{ab}-\lambda^{\{2\}}_{ab})\,\theta(\gamma)}{\theta(\lambda^{\{1\}}_{ab})\,\theta(\lambda^{\{2\}}_{ab})}\,\hat t_{bb}\,\hat t_{aa}\,.$$

More general relations are listed below. We assume that $a<c$ and $b<d$ therein.

$$\hat t_{ab}\,\hat t^{\,k}_{cb}\;=\;\frac{\theta(\lambda^{\{1\}}_{ac}+k\gamma)}{\theta(\lambda^{\{1\}}_{ac})}\,\hat t^{\,k}_{cb}\,\hat t_{ab}\,,\qquad\qquad \hat t^{\,k}_{ab}\,\hat t_{cb}\;=\;\frac{\theta(\lambda^{\{1\}}_{ac}+\gamma)}{\theta\bigl(\lambda^{\{1\}}_{ac}-(k-1)\gamma\bigr)}\,\hat t_{cb}\,\hat t^{\,k}_{ab}\,, \tag{A.3}$$

$$\hat t_{ad}\,\hat t^{\,k}_{ab}\;=\;\frac{\theta(\lambda^{\{2\}}_{bd}-k\gamma)}{\theta(\lambda^{\{2\}}_{bd})}\,\hat t^{\,k}_{ab}\,\hat t_{ad}\,,\qquad\qquad \hat t^{\,k}_{ad}\,\hat t_{ab}\;=\;\frac{\theta(\lambda^{\{2\}}_{bd}-\gamma)}{\theta\bigl(\lambda^{\{2\}}_{bd}+(k-1)\gamma\bigr)}\,\hat t_{ab}\,\hat t^{\,k}_{ad}\,, \tag{A.4}$$

$$\hat t_{ab}\,\hat t_{cd}\;-\;\hat t_{cd}\,\hat t_{ab}\;=\;\frac{\theta(\lambda^{\{1\}}_{ac}+\lambda^{\{2\}}_{bd})\,\theta(\gamma)}{\theta(\lambda^{\{1\}}_{ac})\,\theta(\lambda^{\{2\}}_{bd})}\,\hat t_{cb}\,\hat t_{ad}\,, \tag{A.5}$$

$$\frac{\theta(\lambda^{\{2\}}_{bd}+\gamma)\,\theta(\lambda^{\{2\}}_{bd}-\gamma)}{\bigl(\theta(\lambda^{\{2\}}_{bd})\bigr)^2}\,\hat t_{ab}\,\hat t_{cd}\;-\;\frac{\theta(\lambda^{\{1\}}_{ac}+\gamma)\,\theta(\lambda^{\{1\}}_{ac}-\gamma)}{\bigl(\theta(\lambda^{\{1\}}_{ac})\bigr)^2}\,\hat t_{cd}\,\hat t_{ab}\;=\;\frac{\theta(\lambda^{\{1\}}_{ac}+\lambda^{\{2\}}_{bd})\,\theta(\gamma)}{\theta(\lambda^{\{1\}}_{ac})\,\theta(\lambda^{\{2\}}_{bd})}\,\hat t_{ad}\,\hat t_{cb}\,, \tag{A.6}$$

$$\hat t_{ad}\,\hat t_{cb}\;-\;\frac{\theta(\lambda^{\{1\}}_{ac}+\gamma)\,\theta(\lambda^{\{1\}}_{ac}-\gamma)}{\bigl(\theta(\lambda^{\{1\}}_{ac})\bigr)^2}\,\hat t_{cb}\,\hat t_{ad}\;=\;-\,\frac{\theta\bigl(\lambda^{\{1\}}_{ac}-\lambda^{\{2\}}_{bd}\bigr)\,\theta(\gamma)}{\theta(\lambda^{\{1\}}_{ac})\,\theta(\lambda^{\{2\}}_{bd})}\,\hat t_{ab}\,\hat t_{cd}\,, \tag{A.7}$$

$$\hat t_{ad}\,\hat t_{cb}\;-\;\frac{\theta(\lambda^{\{2\}}_{bd}+\gamma)\,\theta(\lambda^{\{2\}}_{bd}-\gamma)}{\bigl(\theta(\lambda^{\{2\}}_{bd})\bigr)^2}\,\hat t_{cb}\,\hat t_{ad}\;=\;-\,\frac{\theta\bigl(\lambda^{\{1\}}_{ac}-\lambda^{\{2\}}_{bd}\bigr)\,\theta(\gamma)}{\theta(\lambda^{\{1\}}_{ac})\,\theta(\lambda^{\{2\}}_{bd})}\,\hat t_{cd}\,\hat t_{ab}\,.$$

All these formulae follow from (3.2) – (3.5) and (A.1), and the summation formulae for the theta function. Combining formula (A.5) with formulae (A.3) and (A.4) for $k = 1$ we can obtain the following Serre-type relations:

$$\begin{aligned}
&\theta(\lambda_{ac}^{\{1\}} + \lambda_{bd}^{\{2\}})\,\theta(\lambda_{ac}^{\{1\}} - \gamma)\,\theta(\lambda_{bd}^{\{2\}} - 2\gamma)\,\theta(\gamma)\ \hat{t}_{ab}^{2}\,\hat{t}_{cd}\ - \\
&-\ \theta(\lambda_{ac}^{\{1\}} + \lambda_{bd}^{\{2\}} - \gamma)\,\theta(\lambda_{ac}^{\{1\}})\,\theta(\lambda_{bd}^{\{2\}} - \gamma)\,\theta(2\gamma)\ \hat{t}_{ab}\,\hat{t}_{cd}\,\hat{t}_{ab}\ + \\
&+\ \theta(\lambda_{ac}^{\{1\}} + \lambda_{bd}^{\{2\}} - 2\gamma)\,\theta(\lambda_{ac}^{\{1\}} + \gamma)\,\theta(\lambda_{bd}^{\{2\}})\,\theta(\gamma)\ \hat{t}_{cd}\,\hat{t}_{ab}^{2}\ =\ 0\,,
\end{aligned} \tag{A.8}$$

$$\begin{aligned}
&\theta(\lambda_{ac}^{\{1\}} + \lambda_{bd}^{\{2\}} + 2\gamma)\,\theta(\lambda_{ac}^{\{1\}})\,\theta(\lambda_{bd}^{\{2\}} - \gamma)\,\theta(\gamma)\ \hat{t}_{ab}\,\hat{t}_{cd}^{2}\ - \\
&-\ \theta(\lambda_{ac}^{\{1\}} + \lambda_{bd}^{\{2\}} + \gamma)\,\theta(\lambda_{ac}^{\{1\}} + \gamma)\,\theta(\lambda_{bd}^{\{2\}})\,\theta(2\gamma)\ \hat{t}_{cd}\,\hat{t}_{ab}\,\hat{t}_{cd}\ + \\
&+\ \theta(\lambda_{ac}^{\{1\}} + \lambda_{bd}^{\{2\}})\,\theta(\lambda_{ac}^{\{1\}} + 2\gamma)\,\theta(\lambda_{bd}^{\{2\}} + \gamma)\,\theta(\gamma)\ \hat{t}_{cd}^{2}\,\hat{t}_{ab}\ =\ 0\,.
\end{aligned} \tag{A.9}$$

Lemma A.1. *Let $a < c$ and $b < d$. Then the following relations hold:*

$$\hat{t}_{ab}\,\hat{t}_{cd}^{k} - \hat{t}_{cd}^{k}\,\hat{t}_{ab} = \frac{\theta\big(\lambda_{ac}^{\{1\}} + \lambda_{bd}^{\{2\}} + (k-1)\gamma\big)\,\theta(k\gamma)}{\theta(\lambda_{ac}^{\{1\}})\,\theta\big(\lambda_{bd}^{\{2\}} + (k-1)\gamma\big)}\ \hat{t}_{cb}\,\hat{t}_{cd}^{k-1}\,\hat{t}_{ad}\,,$$

$$\hat{t}_{ab}^{k}\,\hat{t}_{cd} - \hat{t}_{cd}\,\hat{t}_{ab}^{k} = \frac{\theta\big(\lambda_{ac}^{\{1\}} + \lambda_{bd}^{\{2\}} - (k-1)\gamma\big)\,\theta(k\gamma)}{\theta\big(\lambda_{ac}^{\{1\}} - (k-1)\gamma\big)\,\theta(\lambda_{bd}^{\{2\}})}\ \hat{t}_{cb}\,\hat{t}_{ab}^{k-1}\,\hat{t}_{ad}\,, \tag{A.10}$$

$$\begin{aligned}
&\hat{t}_{ad}\,\hat{t}_{cb}^{k} - \frac{\theta(\lambda_{ac}^{\{1\}} + k\gamma)\,\theta(\lambda_{ac}^{\{1\}} - \gamma)\,\theta\big(\lambda_{bd}^{\{2\}} - (k-1)\gamma\big)}{\big(\theta(\lambda_{ac}^{\{1\}})\big)^{2}\,\theta(\lambda_{bd}^{\{2\}})}\ \hat{t}_{cb}^{k}\,\hat{t}_{ad} = \\
&\qquad = -\,\frac{\theta\big(\lambda_{ac}^{\{1\}} - \lambda_{bd}^{\{2\}} + (k-1)\gamma\big)\,\theta(k\gamma)}{\theta(\lambda_{ac}^{\{1\}})\,\theta(\lambda_{bd}^{\{2\}})}\ \hat{t}_{cb}^{k-1}\,\hat{t}_{ab}\,\hat{t}_{cd}\,.
\end{aligned} \tag{A.11}$$

Proof. For $k = 1$ these formulae coincide with formulae (A.5) and (A.7), respectively. All the proofs for $k > 1$ are similar to each other. So we will prove only formula (A.11).

Multiply formula (A.6) by $\hat{t}_{cb}^{k-1}$ from the right, and push the factor $\hat{t}_{cb}^{k-1}$ in the left hand side through all the products from right to left using relations (A.3). Taking into account formula (A.5) we get

$$\begin{aligned}
&\hat{t}_{ad}\,\hat{t}_{cb}^{k} - \frac{\theta(\lambda_{ac}^{\{1\}} + k\gamma)\,\theta(\lambda_{ac}^{\{1\}} - \gamma)\,\theta\big(\lambda_{bd}^{\{2\}} - (k-1)\gamma\big)}{\big(\theta(\lambda_{ac}^{\{1\}})\big)^{2}\,\theta(\lambda_{bd}^{\{2\}})}\ \hat{t}_{cb}^{k}\,\hat{t}_{ad} = \\
&\qquad = \big(F(\lambda_{ac}^{\{1\}}, \lambda_{bd}^{\{2\}}) + F(-\lambda_{bd}^{\{2\}}, -\lambda_{ac}^{\{1\}})\big)\ \hat{t}_{cb}^{k-1}\,\hat{t}_{ab}\,\hat{t}_{cd}
\end{aligned}$$

where

$$F(u,v) = \frac{\theta\big(u + (k-1)\gamma\big)\,\theta(v+\gamma)\,\theta(v - k\gamma)}{\theta(u+v)\,\theta(v)\,\theta(\gamma)}\,.$$

$F(u,v) + F(-v,-u)$ is a quasiperiodic function of u with periods 1 and τ and has only simple poles. So, it is completely determined by its multipliers and residues. Hence,

$$F(u,v) + F(-v,-u) = -\,\frac{\theta\big(u - v + (k-1)\gamma\big)\,\theta(k\gamma)}{\theta(u)\,\theta(v)}$$

□

Appendix B. Quantum determinant

The construction of $\mathrm{Det}\, T(u)$ and the proof of Proposition 2.1 can be obtained by extending the standard fusion procedure technique to the dynamical case.

For any $k = 2, \dots, N$ let A_k be the complete antisymmetrizer in $(\mathbb{C}^N)^{\otimes k}$:

$$A_k(x_1 \otimes \ldots \otimes x_k) \;=\; \frac{1}{k!} \sum_{\boldsymbol{i} \in \mathbf{S}_k} \mathrm{sign}(\boldsymbol{i})\, (x_{i_1} \otimes \ldots \otimes x_{i_k})\,.$$

Set $A = A_2$ and let $S = 1 - A$ be the corresponding symmetrizer. The R-matrix $R(u, \lambda)$ has a simple pole at $u = \gamma$. Denote by $Q(\lambda)$ the resique of $R(u, \lambda)$ at $u = \gamma$.

Lemma B.1. $\mathrm{Ker}\, Q(\lambda) = \mathrm{Ker}\, A = \mathrm{Im}\, S$.

Due to the inversion relation (1.5) we can write relation (2.3) in the following form:

$$T^{(13)}(u) T^{(23)}(v)\, R^{(21)}(u - v, \lambda) \;=\; R^{(21)}(u - v, \lambda - \gamma h^{(3)}) T^{(23)}(v) T^{(13)}(u)\,. \tag{B.1}$$

Then by Lemma B.1 we have that $A^{(12)}\, T^{(23)}(u + \gamma) T^{(13)}(u)\, S^{(12)} \;=\; 0$, which is equivalent to each of the following relations:

$$\begin{aligned} T^{(23)}(u + \gamma) T^{(13)}(u)\, S^{(12)} \;&=\; S^{(12)}\, T^{(23)}(u + \gamma) T^{(13)}(u)\, S^{(12)}\,, \\ A^{(12)}\, T^{(23)}(u + \gamma) T^{(13)}(u) \;&=\; A^{(12)}\, T^{(23)}(u + \gamma) T^{(13)}(u)\, A^{(12)}\,. \end{aligned} \tag{B.2}$$

Formula (B.2) shows that for any $i = 1, \dots, k-1$ we have

$$\begin{aligned} & T^{(k,k+1)}\bigl(u + (k-1)\gamma\bigr) \ldots T^{(1,k+1)}(u)\, S^{(i,i+1)} \;= \\ & \;=\; S^{(i,i+1)}\, T^{(k,k+1)}\bigl(u + (k-1)\gamma\bigr) \ldots T^{(1,k+1)}(u)\, S^{(i,i+1)}\,, \end{aligned}$$

which implies that

$$\begin{aligned} & A_k^{(1,\dots,k)}\, T^{(k,k+1)}\bigl(u + (k-1)\gamma\bigr) \ldots T^{(1,k+1)}(u) \;= \\ & \;=\; A_k^{(1,\dots,k)}\, T^{(k,k+1)}\bigl(u + (k-1)\gamma\bigr) \ldots T^{(1,k+1)}(u)\, A_k^{(1,\dots,k)} \end{aligned} \tag{B.3}$$

because $\mathrm{Ker}\, A_k \;=\; \sum_{i=1}^{k-1} \mathrm{Im}\, S^{(i,i+1)}$ and $(\mathbb{C}^N)^{\otimes k} = \mathrm{Im}\, A_k \oplus \mathrm{Ker}\, A_k$. We denote the restriction of

$$A_k^{(1,\dots,k)}\, T^{(k,k+1)}\bigl(u + (k-1)\gamma\bigr) \ldots T^{(1,k+1)}(u)$$

to $\mathrm{Im}\, A_k \otimes \mathit{Fun}(V)$ by $T^{\wedge k}(u)$ and call it the *k-th exterior power* of $T(u)$. Since $\mathrm{Im}\, A_N$ is one-dimensional, the top exterior power $T^{\wedge N}(u)$ can be considered as an element of $\mathrm{End}\bigl(\mathit{Fun}(V)\bigr)$.

Lemma B.2. *For any permutation $\boldsymbol{j} \in \mathbf{S}_N$ we have*

$$T^{\wedge N}(u) \;=\; \mathrm{sign}(\boldsymbol{j}) \sum_{\boldsymbol{i} \in \mathbf{S}_N} \mathrm{sign}(\boldsymbol{i})\, T_{i_N, j_N}\bigl(u + (N-1)\gamma\bigr) \ldots T_{i_2, j_2}(u + \gamma)\, T_{i_1, j_1}(u)\,. \tag{B.4}$$

Proof. Let $v_1, \dots, v_N$ be the standard basis of $\mathbb{C}^N$. Then for any $\boldsymbol{j} \in \mathbf{S}_N$ we have

$$A_N\, (v_{j_1} \otimes \ldots \otimes v_{j_N}) \;=\; \mathrm{sign}(\boldsymbol{j})\, A_N\, (v_1 \otimes \ldots \otimes v_N)\,. \tag{B.5}$$

Let $v \in \mathit{Fun}(V)$. By the definition of $T^{\wedge N}(u)$ and using relation (B.3) we get

$$\begin{aligned} & A_N\, (v_{j_1} \otimes \ldots \otimes v_{j_N}) \otimes T^{\wedge N}(u)\, v \;= \\ & \;=\; \sum_{\boldsymbol{i} \in \mathbf{S}_N} A_N\, (v_{i_1} \otimes \ldots \otimes v_{i_N}) \otimes T_{i_N, j_N}\bigl(u + (N-1)\gamma\bigr) \ldots T_{i_2, j_2}(u + \gamma)\, T_{i_1, j_1}(u)\, v\,. \end{aligned}$$

By formula (B.5) the expression in the right hand side equals

$$\mathrm{sign}(\boldsymbol{j})\, A_N\, (v_{j_1} \otimes \ldots \otimes v_{j_N}) \;\otimes \sum_{\boldsymbol{i} \in \mathbf{S}_N} \mathrm{sign}(\boldsymbol{i})\, T_{i_N, j_N}\bigl(u + (N-1)\gamma\bigr) \ldots T_{i_2, j_2}(u + \gamma)\, T_{i_1, j_1}(u)\, v\,,$$

which proves the lemma. □

Like in the ordinary linear algebra there is a connection between the complementary exterior powers $T^{\wedge k}(u)$ and $T^{\wedge(N-k)}(u)$ and the top exterior power $T^{\wedge N}(u)$, cf. Theorem B.4. For any two sequences $\boldsymbol{a}=(a_1,\dots,a_k)$ and $\boldsymbol{b}=(b_1,\dots,b_k)$ set

$$T^{\wedge k}_{\boldsymbol{ab}}(u) \;=\; \sum_{\boldsymbol{i}\in\mathbf{S}_k} \operatorname{sign}(\boldsymbol{i})\, T_{a_{i_k}b_k}\bigl(u+(k-1)\gamma\bigr)\,\dots\,T_{a_{i_2}b_2}(u+\gamma)\,T_{a_{i_1}b_1}(u)\,.$$

Lemma B.3. *Let $\boldsymbol{i},\boldsymbol{j}\in\mathbf{S}_k$, and let $\boldsymbol{a}'=(a_{i_1},\dots,a_{i_k})$, $\boldsymbol{b}'=(a_{j_1},\dots,a_{j_k})$. Then*

$$T^{\wedge k}_{\boldsymbol{a}'\boldsymbol{b}'}(u) \;=\; \operatorname{sign}(\boldsymbol{i})\operatorname{sign}(\boldsymbol{j})\,T^{\wedge k}_{\boldsymbol{ab}}(u)\,.$$

The proof is similar to the proof of Lemma B.4.

Denote by Y_k the set of increasing k-tuples of integers from $\{1,\dots,N\}$. For any $\boldsymbol{a}\in Y_k$ define $\bar{\boldsymbol{a}}\in Y_{N-k}$ to be the complement of $\boldsymbol{a}$, that is, $\{a_1,\dots,a_k,\bar a_1,\dots,\bar a_{N-k}\}=\{1,\dots,N\}$. Denote by $\boldsymbol{a}\bar{\boldsymbol{a}}$ the permutation $(a_1,\dots,a_k,\bar a_1,\dots,\bar a_{N-k})\in\mathbf{S}_N$.

Theorem B.4.

$$\operatorname{sign}(\boldsymbol{a}\bar{\boldsymbol{a}})\sum_{\boldsymbol{c}\in Y_k}\operatorname{sign}(\boldsymbol{c}\bar{\boldsymbol{c}})\,T^{\wedge(N-k)}_{\bar{\boldsymbol{c}}\bar{\boldsymbol{a}}}\bigl(u-(k-1)\gamma\bigr)\,T^{\wedge k}_{\boldsymbol{cb}}(u) \;=\; \delta_{\boldsymbol{ab}}\,T^{\wedge N}(u)\,. \tag{B.6}$$

The proof is similar to the proof of the analogous formlula in the ordinary linear algebra.

It is clear from relation (2.1) and formula (B.4) that the difference operator $T^{\wedge N}(u)$ commutes with multiplication by scalar functions. So, there exists a function $L^{\wedge N}(u,\lambda)\in \mathit{Fun}\bigl(\mathrm{End}(V)\bigr)$ such that

$$\bigl(T^{\wedge N}(u)\,v\bigr)(\lambda) \;=\; L^{\wedge N}(u,\lambda)\,v(\lambda)$$

for any $v\in \mathit{Fun}(V)$. Denote by $R^{\wedge N}(u,\lambda)$ the function $L^{\wedge N}(u,\lambda)$ for the vector representation of $E_{\tau,\gamma}(\mathfrak{sl}_N)$ with the evaluation point $x=0$.

Lemma B.5.

$$R^{\wedge N}(u,\lambda) \;=\; \frac{\theta\bigl(u+(N-1)\gamma\bigr)}{\theta(u)}\,\sum_{a=1}^N\,\prod_{\substack{b=1\\ b\neq a}}^N \frac{\theta(\lambda_{ab}-\gamma)}{\theta(\lambda_{ab})}\,E_{aa}\,.$$

Proof. Recall that in the vector representation

$$\bigl(T_{aa}(u)\,v\bigr)(\lambda) \;=\; E_{aa}v(\lambda-\gamma\varepsilon_a)\,+\sum_{\substack{a,b=1\\ a\neq b}}^N \alpha(u,\lambda_{ab})\,E_{bb}\,v(\lambda-\gamma\varepsilon_b)$$

and $\bigl(T_{ab}(u)\,v\bigr)(\lambda) \;=\; \beta(u,\lambda_{ab})\,E_{ba}v(\lambda-\gamma\varepsilon_b)$, where $\alpha(u,\xi)$ and $\beta(u,\xi)$ are given by (1.2). By Lemma (B.2) this implies that $R^{\wedge N}(u,\lambda)$ is a linear combination of the matrices $E_{11},\dots,E_{NN}$ with some functional coefficients. Moreover, taking formula (B.4) for the permutation $\boldsymbol{j}$ such that $j_1=a$ we observe that only the term in the sum with $\boldsymbol{i}=\boldsymbol{j}$ contribute to the coefficient of E_{aa}, which, therefore, can be found easily. $\square$

Proof of Proposition 2.1. From relations (2.3) and (2.1) following the definition of $T^{\wedge N}(u)$ we find that

$$(R^{\wedge N})^{(1)}(u-v,\lambda-\gamma h^{(2)})\,(T^{\wedge N})^{(2)}(u)\,T^{(12)}(v) \;=\; T^{(12)}(v)\,(T^{\wedge N})^{(2)}(u)\,(R^{\wedge N})^{(1)}(u-v,\lambda+\gamma h^{(1)})\,.$$

By Lemma B.5 this is equivalent to

$$\prod_{\substack{c=1\\ c\neq a}}^N \frac{\theta(\lambda_{ac}-\gamma h_{ac}-\gamma)}{\theta(\lambda_{ac}-\gamma h_{ac})}\,T^{\wedge N}(u)\,T_{ab}(v) \;=\; \prod_{\substack{c=1\\ c\neq b}}^N \frac{\theta(\lambda_{bc}-\gamma)}{\theta(\lambda_{bc})}\,T_{ab}(v)\,T^{\wedge N}(u)\,.$$

for any $a,b=1,\dots,N$. Since $\operatorname{Det}T(u) = \dfrac{\Theta(\lambda)}{\Theta(\lambda-\gamma h)}\,T^{\wedge N}(u)$ where $\Theta(\lambda)=\prod_{1\leqslant a<b\leqslant N}\theta(\lambda_{ab})$, the proposition is proved. $\square$

Appendix C. Multiplicative forms

In this section we essentially follow [EV1, Section 1.4]. Notice that all over the paper the words *cocycle* and *coboundary* mean 1-*cocycle* and 1-*coboundary*, respectively.

Let I_k be the set of k-tuples of pairwise distinct integers from $\{1,\ldots,N\}$. A *multiplicative k-form* $\bar f$ is a map $I_k \to \mathit{Fun}(\mathbb{C})$, $\bar f : \boldsymbol{a} \mapsto f_{\boldsymbol{a}}$, such that for any $\boldsymbol{a} \in I_k$ and any $i = 1,\ldots,k$ we have

$$f_{a_1,\ldots,a_k}(\lambda)\, f_{a_1,\ldots,a_{i-1},a_{i+1},a_i,a_{i+2},\ldots,a_k}(\lambda) \;=\; 1$$

Let Ω^k be the set of all multiplicative k-forms. If $\bar f$ and $\bar g$ are multiplicative k-forms, then $\bar f\bar g : \boldsymbol{a} \mapsto f_{\boldsymbol{a}} g_{\boldsymbol{a}}$ and $\bar f/\bar g : \boldsymbol{a} \mapsto f_{\boldsymbol{a}}/g_{\boldsymbol{a}}$ are multiplicative k-forms, which defines an abelian group structure on Ω^k. The neutral element is the form $\bar 1$: $1_{\boldsymbol{a}}(\lambda) = 1$ for any $\boldsymbol{a} \in I_k$.

For any nonzero function $f(\lambda)$ and any $a = 1,\ldots,N$ set $(\delta_a f)(\lambda) = f(\lambda - \gamma\varepsilon_a)/f(\lambda)$, and for any $\bar f \in \Omega^k$ define a multiplicative form $d\bar f \in \Omega^{k+1}$ by

$$(d\bar f)_{a_1,\ldots,a_{k+1}}(\lambda) \;=\; \prod_{i=1}^{k+1} \bigl((\delta_{a_i} f_{a_1,\ldots,a_{i-1},a_{i+1},\ldots,a_{k+1}})(\lambda)\bigr)^{(-1)^{i-1}}.$$

For any multiplicative form $\bar f$ we have $d^2\bar f = \bar 1$. The multiplicative form $\bar f$ is called a *multiplicative cocycle* if $d\bar f = \bar 1$, and a *multiplicative coboundary* if $\bar f = d\bar g$ for a suitable multiplicative form $\bar g$.

For any meromorphic function $f(\xi)$ in one variable the multiplicative 1-form $\bar F = (F_1,\ldots,F_N)$:

$$F_a(\lambda) \;=\; \prod_{1\leqslant c<a} f(\lambda_{ca}) \prod_{a<c\leqslant N} \bigl(f(\lambda_{ac}-\gamma)\bigr)^{-1},$$

is a multiplicative 1-cocycle. If $f(\xi) = g(\xi+\gamma)/g(\xi)$, then $\bar F$ is a multiplicative 1-coboundary: $\bar F = dG$, where $G(\lambda) = \prod_{1\leqslant b<c\leqslant N} g(\lambda_{bc})$.

If $(F_1,\ldots,F_N)$ is a multiplicative 1-cocycle, then so does $(G_1,\ldots,G_N)$: $G_a(\lambda) = F_a(-\lambda+\gamma\varepsilon_a)$.

Appendix D. Proof of Theorem 4.1

In this section we introduce another ordering on generators of $e^{\mathcal{O}}_{\tau,\gamma}(\mathfrak{sl}_N)$, called ordering by rows, and prove the analogue of Theorem 4.1 for the monomials ordered by rows, cf. Theorem D.1. Since the number or ordered by rows monomials of degree k equals the number of normally ordered monomials of the same degree, and each monomial can be transformed to a linear combination of normally ordered monomials, Theorem D.1 implies Theorem 4.1. We introduce ordered by rows monomials for technical reason because this allows us to reduce the number of cases to be examined at some stage of the proof.

Consider the *ordering by rows* of generators: $t_{ab} \prec t_{cd}$ if $a<c$, or $a=c$ and $b<d$. Say that the monomial $t_{a_1b_1}\ldots t_{a_kb_k}$ is *ordered by rows* if $t_{a_ib_i} \prec t_{a_jb_j}$ for any $i<j$, or $k=0$.

Theorem D.1. *For any $k \in \mathbb{Z}_{\geqslant 0}$ the ordered by rows monomials of degree k form a basis of $\mathfrak{e}_k$ over $\mathit{Fun}^{\otimes 2}(\mathbb{C})$.*

Proof. To save words from now on we write *ordered* instead of *ordered by rows*. We call an equality

$$\textit{disordered monomial} \;=\; \textit{linear combination of ordered monomials}$$

an *ordering rule* for the monomial in the left hand side. The commutation relations (3.3) – (3.5) have the important property:

A. Any relation is a linear combinations of ordering rules, and the complete list of linear independent ordering rules contains precisely one rule for each disordered product of generators.

For $k=0$ and $k=1$ the claim of Theorem D.1 is immediate. Let $k>1$. First we prove that the ordered monomials of degree k span $\mathfrak{e}_k$ over $\mathit{Fun}^{\otimes 2}(\mathbb{C})$. Indeed, one can transform any monomial $t_{a_1b_1}\ldots t_{a_kb_k}$ to a linear combination of ordered monomials by the following procedure. Pick up any disordered product of adjacent factors and replace it by a sum of ordered products using the ordering rule, then do the same for each of the obtained monomials. To see that the procedure terminates and, hence, produces a linear combination of ordered monomials, introduce auxilary gradings on monomials by the rule

$$r(t_{a_1b_1}\dots t_{a_kb_k}) = \sum_{i=1}^{k} i\,a_i\,, \qquad\qquad r'(t_{a_1b_1}\dots t_{a_kb_k}) = \sum_{i=1}^{k} i\,b_i\,,$$

and observe that at each nontrivial step of the procedure we replace a monomial by a sum of monomials of either less degree r, or the same degree r and less degree r'. We call the described procedure a *regular transformation* of the monomial $t_{a_1b_1}\dots t_{a_kb_k}$ to a linear combination of ordered monomials.

If an ordering rule is applied to a product $t_{a_ib_i}t_{a_{i+1}b_{i+1}}$ in the monomial $t_{a_1b_1}\dots t_{a_kb_k}$, then we say that the ordering rule is applied *at the i-th place*.

By the standard reasoning the property A of the commutation relations (3.3) – (3.5) implies that Theorem D.1 follows from Proposition D.2. □

Proposition D.2. *Any regular transformation of the monomial $t_{a_1b_1}\dots t_{a_kb_k}$ produces the same linear combination of the ordered monomials.*

Proof. For $k=2$ the claim follows from the property A. For $k=3$ the claim can be verified in a straightforward way. We discuss more details of $k=3$ case at the end of the proof.

Let $k>3$. For the proof we use induction with respect to the lexicographical ordering on monomials defined by pair of degrees (r,r'). The claim of the proposition is obvious for the ordered monomials, which provides the base of induction.

Let I and II be two regular transformations of the monomial $t_{a_1b_1}\dots t_{a_kb_k}$ to linear combinations of ordered monomials. If for I and II the first steps coincide, then they produce the same results by the induction assumption. Otherwise, let us construct two additional regular transformations III and IV such that the first steps of I and III coincide, the same holds for II and IV, and III and IV produce the same results. By the previous remark, this proves the proposition.

Assume that for the transformation I an ordering rule at the first step is applied at the i-th place, and for the transformation II at the j-th place. If $|i-j|>1$ then we define the transformation III as follows: first apply an ordering rule at the i-th place, next apply an ordering rule at the j-th place for all monomials obtained at the first step, then continue in any possible way. The transformation IV is defined similarly with i and j interchanged. By the induction assumption the transformations III and IV produce the same results because after the first two steps of both III and IV one has identical linear combinations of monomials, each of them being lexicographically smaller than the initial monomial $t_{a_1b_1}\dots t_{a_kb_k}$.

The cases $j=i\pm 1$ are similar. Assume for example that $j=i+1$. Define the transformation III as follows: apply a regular transformation of the product $t_{a_ib_i}t_{a_{i+1}b_{i+1}}t_{a_{i+2}b_{i+2}}$ to a linear combination of ordered triple products, making the first step at the i-th place, then continue in any possible way. Define the transformation IV similarly, but making the first step at the $(i+1)$-th place. Then the claim of the proposition for $k=3$ shows that after the first stages of both III and IV one has identical linear combinations of monomials, each of them being lexicographically smaller than the initial monomial $t_{a_1b_1}\dots t_{a_kb_k}$. Therefore, by the induction assumption the transformations III and IV produce the same results.

It remains to complete the proof for $k=3$. For any monomial $t_{a_1b_1}t_{a_2b_2}t_{a_3b_3}$ there are at most two regular transformations, and the regular transformation is unique unless $t_{a_3b_3}\prec t_{a_2b_2}\prec t_{a_1b_1}$. The rest of the proof is given by the straightforward calculations. The simplest cases appear when $a_1=a_2=a_3$ or $b_1=b_2=b_3$. We present below the most bulky example when $a_1>a_2>a_3$ and b_1,b_2,b_3 are pairwise distinct. □

Example. Regular transformations of the monomial $t_{34}t_{25}t_{16}$. Let functions $\alpha(u,\xi)$ and $\beta(u,\xi)$ be given by (1.2). Making the first step at the first place we have

$$\begin{aligned}
t_{34}t_{25}t_{16} &= \alpha(\lambda^{\{1\}}_{32},\lambda^{\{2\}}_{54})\,t_{25}t_{34}t_{16} - \beta(\lambda^{\{1\}}_{32},\lambda^{\{2\}}_{45})\,t_{24}t_{35}t_{16} = \\
&= \alpha(\lambda^{\{1\}}_{32},\lambda^{\{2\}}_{54})\bigl(\alpha(\lambda^{\{1\}}_{31},\lambda^{\{2\}}_{64})\,t_{25}t_{16}t_{34} - \beta(\lambda^{\{1\}}_{31},\lambda^{\{2\}}_{46})\,t_{25}t_{14}t_{36}\bigr) + \\
&- \beta(\lambda^{\{1\}}_{32},\lambda^{\{2\}}_{45})\bigl(\alpha(\lambda^{\{1\}}_{31},\lambda^{\{2\}}_{65})\,t_{24}t_{16}t_{35} - \beta(\lambda^{\{1\}}_{31},\lambda^{\{2\}}_{56})\,t_{24}t_{15}t_{36}\bigr) =
\end{aligned}$$

$$= \alpha(\lambda_{32}^{\{1\}},\lambda_{54}^{\{2\}})\,\alpha(\lambda_{31}^{\{1\}},\lambda_{64}^{\{2\}})\bigl(\alpha(\lambda_{21}^{\{1\}},\lambda_{65}^{\{2\}})\,t_{16}t_{25}t_{34} - \beta(\lambda_{21}^{\{1\}},\lambda_{56}^{\{2\}})\,t_{15}t_{26}t_{34}\bigr) +$$
$$- \alpha(\lambda_{32}^{\{1\}},\lambda_{54}^{\{2\}})\,\beta(\lambda_{31}^{\{1\}},\lambda_{46}^{\{2\}})\bigl(\alpha(\lambda_{21}^{\{1\}},\lambda_{45}^{\{2\}})\,t_{14}t_{25}t_{36} - \beta(\lambda_{21}^{\{1\}},\lambda_{54}^{\{2\}})\,t_{15}t_{24}t_{36}\bigr) +$$
$$- \beta(\lambda_{32}^{\{1\}},\lambda_{45}^{\{2\}})\,\alpha(\lambda_{31}^{\{1\}},\lambda_{65}^{\{2\}})\bigl(\alpha(\lambda_{21}^{\{1\}},\lambda_{64}^{\{2\}})\,t_{16}t_{24}t_{35} - \beta(\lambda_{21}^{\{1\}},\lambda_{46}^{\{2\}})\,t_{14}t_{26}t_{35}\bigr) +$$
$$+ \beta(\lambda_{32}^{\{1\}},\lambda_{45}^{\{2\}})\,\beta(\lambda_{31}^{\{1\}},\lambda_{56}^{\{2\}})\bigl(\alpha(\lambda_{21}^{\{1\}},\lambda_{54}^{\{2\}})\,t_{15}t_{24}t_{36} - \beta(\lambda_{21}^{\{1\}},\lambda_{45}^{\{2\}})\,t_{14}t_{25}t_{36}\bigr),$$

while making the first step at the second place we have

$$t_{34}t_{25}t_{16} = \alpha(\lambda_{21}^{\{1\}},\lambda_{65}^{\{2\}})\,t_{34}t_{16}t_{25} - \beta(\lambda_{21}^{\{1\}},\lambda_{56}^{\{2\}})\,t_{34}t_{15}t_{26} =$$
$$= \alpha(\lambda_{21}^{\{1\}},\lambda_{65}^{\{2\}})\bigl(\alpha(\lambda_{31}^{\{1\}},\lambda_{64}^{\{2\}})\,t_{16}t_{34}t_{25} - \beta(\lambda_{31}^{\{1\}},\lambda_{46}^{\{2\}})\,t_{14}t_{36}t_{25}\bigr) +$$
$$- \beta(\lambda_{21}^{\{1\}},\lambda_{56}^{\{2\}})\bigl(\alpha(\lambda_{31}^{\{1\}},\lambda_{54}^{\{2\}})\,t_{15}t_{34}t_{26} - \beta(\lambda_{31}^{\{1\}},\lambda_{45}^{\{2\}})\,t_{14}t_{35}t_{26}\bigr) =$$
$$= \alpha(\lambda_{21}^{\{1\}},\lambda_{65}^{\{2\}})\,\alpha(\lambda_{31}^{\{1\}},\lambda_{64}^{\{2\}})\bigl(\alpha(\lambda_{32}^{\{1\}},\lambda_{54}^{\{2\}})\,t_{16}t_{25}t_{34} - \beta(\lambda_{32}^{\{1\}},\lambda_{45}^{\{2\}})\,t_{16}t_{24}t_{35}\bigr) +$$
$$- \alpha(\lambda_{21}^{\{1\}},\lambda_{65}^{\{2\}})\,\beta(\lambda_{31}^{\{1\}},\lambda_{46}^{\{2\}})\bigl(\alpha(\lambda_{32}^{\{1\}},\lambda_{56}^{\{2\}})\,t_{14}t_{25}t_{36} - \beta(\lambda_{32}^{\{1\}},\lambda_{65}^{\{2\}})\,t_{14}t_{26}t_{35}\bigr) +$$
$$- \beta(\lambda_{21}^{\{1\}},\lambda_{56}^{\{2\}})\,\alpha(\lambda_{31}^{\{1\}},\lambda_{54}^{\{2\}})\bigl(\alpha(\lambda_{32}^{\{1\}},\lambda_{64}^{\{2\}})\,t_{15}t_{26}t_{34} - \beta(\lambda_{32}^{\{1\}},\lambda_{46}^{\{2\}})\,t_{15}t_{24}t_{36}\bigr) +$$
$$+ \beta(\lambda_{21}^{\{1\}},\lambda_{56}^{\{2\}})\,\beta(\lambda_{31}^{\{1\}},\lambda_{45}^{\{2\}})\bigl(\alpha(\lambda_{32}^{\{1\}},\lambda_{65}^{\{2\}})\,t_{14}t_{26}t_{35} - \beta(\lambda_{32}^{\{1\}},\lambda_{56}^{\{2\}})\,t_{14}t_{25}t_{36}\bigr).$$

The coefficients of the monomials $t_{16}t_{25}t_{34}$, $t_{15}t_{26}t_{34}$ and $t_{16}t_{24}t_{35}$ coincide identically. To check that the coefficients for other monomials are the same one should take into account the explicit form of $\alpha(u,\xi)$ and $\beta(u,\xi)$ and use summation formulae for the theta function.

Appendix E. Elliptic quantum group $e_{\tau,\gamma}(\mathfrak{sl}_2)$

This section is an illustration of general constructions in the simplest case $N=2$. In this case $\omega_1 = \varepsilon_1 = -\varepsilon_2 = \rho = \alpha_1/2$ and $\mathfrak{h}^* = \mathbb{C}\omega_1$. For any $\lambda \in \mathfrak{h}^*$ we have $\lambda_1 = -\lambda_2 = \lambda_{12}/2$ and $\lambda = \lambda_{12}\omega_1$. We identify functions on $\mathfrak{h}^*$ with functions of one variable λ_{12}.

The operator algebra $e^{\mathcal{O}}_{\tau,\gamma}(\mathfrak{sl}_2)$ is generated over $\mathbb{C}$ by elements $t_{11}, t_{12}, t_{21}, t_{22}$ and functions $f \in \mathit{Fun}^{\otimes 2}(\mathbb{C})$. According to Theorem 4.1 monomials $t_{21}^{k_{21}}t_{11}^{k_{11}}t_{22}^{k_{22}}t_{12}^{k_{12}}$, $k_{11},k_{12},k_{21},k_{22} \in \mathbb{Z}_{\geqslant 0}$, form a basis of $e^{\mathcal{O}}_{\tau,\gamma}(\mathfrak{sl}_2)$ as a $\mathit{Fun}^{\otimes 2}(\mathbb{C})$-module.

Let $\widehat{Q} = (Q_1,Q_2)$ be a multiplicative cocycle, which means in this case that

$$Q_1(\lambda_{12})\,Q_2(\lambda_{12}-\gamma) = Q_1(\lambda_{12}+\gamma)\,Q_2(\lambda_{12}).$$

A Verma module $M_{\mu,\widehat{Q}}$ of highest weight $\mu \in \mathfrak{h}^*$ and dynamical highest weight $\widehat{Q}$ over $e_{\tau,\gamma}(\mathfrak{sl}_2)$ is constructed as follows. As an $\mathfrak{h}$-module $M_{\mu,\widehat{Q}} = \bigoplus_{k=0}^{\infty}\mathbb{C}v_{\mu,\widehat{Q}}[k]$, the vector $v_{\mu,\widehat{Q}}[k]$ having weight $\mu - k\alpha_1$. Notice that $v_{\mu,\widehat{Q}} = v_{\mu,\widehat{Q}}[0]$. The generators $\hat{t}_{11},\hat{t}_{12},\hat{t}_{21},\hat{t}_{22}$ act in $\mathit{Fun}(M_{\mu,\widehat{Q}})$ by the rule

$$\hat{t}_{21}\,v_{\mu,\widehat{Q}}[k] = v_{\mu,\widehat{Q}}[k+1],$$

$$\hat{t}_{11}\,v_{\mu,\widehat{Q}}[k] = Q_1(\lambda_{12}+k\gamma)\,\frac{\theta(\lambda_{12}+k\gamma)}{\theta(\lambda_{12})}\,v_{\mu,\widehat{Q}}[k],$$

$$\hat{t}_{22}\,v_{\mu,\widehat{Q}}[k] = Q_2(\lambda_{12}+k\gamma)\,\frac{\theta\bigl(\lambda_{12}-(\mu_{12}-k)\gamma\bigr)}{\theta\bigl(\lambda_{12}-(\mu_{12}-2k)\gamma\bigr)}\,v_{\mu,\widehat{Q}}[k],$$

$$\hat{t}_{12}\,v_{\mu,\widehat{Q}}[k] = Q_1(\lambda_{12}+k\gamma)\,Q_2\bigl(\lambda_{12}+(k-1)\gamma\bigr)\,\frac{\theta\bigl((\mu_{12}-k+1)\gamma\bigr)\,\theta(k\gamma)}{\theta(\lambda_{12})\,\theta\bigl(\lambda_{12}-(\mu_{12}-2k+2)\gamma\bigr)}\,v_{\mu,\widehat{Q}}[k-1],$$

see (8.1), (8.2) and Lemma 8.1. Taking into account relations (4.1) and (3.6), and Corollary 3.4, one reproduces from these formulae the construction, given in [FV1], of the evaluation Verma module over $E_{\tau,\gamma}(\mathfrak{sl}_2)$ tensored with a one-dimensional representation of $E_{\tau,\gamma}(\mathfrak{sl}_2)$.

In general, to compute the dynamical Shapovalov form on $e^{\mathcal{O}}_{\tau,\gamma}(\mathfrak{sl}_2)$ one has to find

$$S\big(f(\lambda_{12}^{\{1\}},\lambda_{12}^{\{2\}}))\,\hat t_{21}^{k_{21}}\,\hat t_{11}^{k_{11}}\,\hat t_{22}^{k_{22}}\,\hat t_{12}^{k_{12}},\; g(\lambda_{12}^{\{1\}},\lambda_{12}^{\{2\}})\,\hat t_{21}^{m_{21}}\,\hat t_{11}^{m_{11}}\,\hat t_{22}^{m_{22}}\,\hat t_{12}^{m_{12}}\big) \tag{E.1}$$

for any $f,g\in Fun^{\otimes 2}(\mathbb{C})$ and any monomials $\hat t_{21}^{k_{21}}\hat t_{11}^{k_{11}}\hat t_{22}^{k_{22}}\hat t_{12}^{k_{12}}$, $\hat t_{21}^{m_{21}}\hat t_{11}^{m_{11}}\hat t_{22}^{m_{22}}\hat t_{12}^{m_{12}}$. The answer is zero unless $k_{12}=m_{12}=0$ and $k_{21}=m_{21}$. Moreover, contributions of the functions f,g and factors t_{aa} are easy to calculate, and we find that in the nonzero case expression (E.1) equals

$$\begin{aligned} &f\big(-\lambda_{12}^{\{2\}}+(k_{11}-k_{21}-k_{22})\gamma,\,-\lambda_{12}^{\{1\}}+(k_{11}+k_{21}-k_{22})\gamma\big)\;\times\\ &\times\; g\big(\lambda_{12}^{\{1\}}-(k_{11}+k_{21}-k_{22})\gamma,\,\lambda_{12}^{\{2\}}-(k_{11}-k_{21}-k_{22})\gamma\big)\,q_1^{k_{11}}\,q_2^{k_{22}}\,S(\hat t_{21}^{k_{21}},\hat t_{21}^{k_{21}})\,q_1^{m_{11}}\,q_2^{m_{22}}\end{aligned}$$

where

$$S(\hat t_{21}^{k},\hat t_{21}^{k}) \;=\; (-1)^k\prod_{j=0}^{k-1}\frac{\theta(\lambda_{12}^{\{1\}}-\lambda_{12}^{\{2\}}-j\gamma))\,\theta\big((j+1)\gamma\big)}{\theta(\lambda_{12}^{\{1\}}-j\gamma)\,\theta(\lambda_{12}^{\{2\}}+j\gamma)}\;q_1^k\,q_2^k\,,\qquad k\in\mathbb{Z}_{\geqslant 0}\,.$$

For the corresponding part of the dynamical Shapovalov pairing for $M_{\mu,\widehat Q}$ we have

$$S_{\mu,\widehat Q}\big(\hat t_{21}^{k},v_{\mu,\widehat Q}[k]\big) \;=\; (-1)^k\prod_{j=0}^{k-1}\Big(Q_1\big(\lambda_{12}+(j+1)\gamma\big)\,Q_2(\lambda_{12}+j\gamma)\;\frac{\theta\big((\mu_{12}-j)\gamma\big)\,\theta\big((j+1)\gamma\big)}{\theta(\lambda_{12}-j\gamma)\,\theta\big(\lambda_{12}-(\mu_{12}-j)\gamma\big)}\Big)\,.$$

The contravariant form $C_{\mu,\widehat Q}: Fun(M_{\mu,\widehat Q})\otimes_{\mathbb{C}} Fun(M_{\mu,\widetilde Q})\to Fun(\mathbb{C})$ is given by

$$\begin{aligned} C_{\mu,\widehat Q}\big(v_{\mu,\widehat Q}[k],v_{\mu,\widetilde Q}[l]\big) \;=\; \delta_{kl}\,(-1)^k\prod_{j=0}^{k-1}\Big(&Q_1\big(\lambda_{12}+(j+1)\gamma\big)\,Q_2(\lambda_{12}+j\gamma)\;\times\\ &\times\;\frac{\theta\big((\mu_{12}-j)\gamma\big)\,\theta\big((j+1)\gamma\big)}{\theta\big(\lambda_{12}+(j+1)\gamma\big)\,\theta\big(\lambda_{12}-(\mu_{12}-j-k)\gamma\big)}\Big)\,.\end{aligned}$$

Let $\gamma\notin\mathbb{Q}+\tau\mathbb{Q}$ and assume that $\widehat Q$ is nondegenerate. Then the module $M_{\mu,\widehat Q}$ is irreducible provided that $\mu_{12}\notin\mathbb{Z}_{\geqslant 0}$. If $\mu_{12}\in\mathbb{Z}_{\geqslant 0}$, then $v_{\mu,\widehat Q}[\mu_{12}+1]$ is a regular singular vector generating an irreducible submodule $N_{\mu,\widehat Q}$ isomorphic to $M_{-\mu-2\rho,\widetilde Q}$ where $\widetilde Q(\lambda_{12})=\widehat Q(\lambda_{12}-\mu_{12}-2)$. The quotient module $V_{\mu,\widehat Q}=M_{\mu,\widehat Q}/N_{\mu,\widehat Q}$ is the irreducible highest weight $e_{\tau,\gamma}(\mathfrak{sl}_2)$-module with highest weight μ and dynamical highest weight $\widehat Q$, and it has dimension $\mu_{12}+1$, the same as the irreducible $\mathfrak{sl}_2$-module of highest weight μ.

Appendix F. Proof of Theorem 10.5

In [TV2, Section 2.6] for any semisimple Lie algebra $\mathfrak g$ we defined rational functions $B_w(\lambda)$ of λ labeled by elements of the Weil group. The functions takes values in a certain completion of $U(\mathfrak g)$. Here we need the function $B_w(\lambda-\rho)$ for the particular case of $\mathfrak g=\mathfrak{sl}_N$ and $w=w_X$, the longest element of the Weil group. For brevity we denote this function by $B_X(\lambda)$. We list the required properties of $B_X(\lambda)$ below using the notation of the present paper. All of them easily follow from the properties of $B_w(\lambda)$ given in [TV2].

Consider the following series

$$G_{ab}(\lambda) \;=\; \sum_{k=0}^{\infty} e_{ba}^k\,e_{ab}^k\prod_{j=1}^{k}\frac{1}{j\,(\lambda_{ab}-e_{aa}+e_{bb}-j)}\,.$$

Then $B_X(\lambda)$ equals the ordered product $\prod\limits_{1\leqslant a<b\leqslant N} G_{ab}(\lambda)$ where the factor G_{ab} is to the left from the factor G_{cd} if $a<c$, or $a=c$ and $b>d$. For instance, if $N=3$, then $B_X(\lambda)=G_{12}(\lambda)G_{13}(\lambda)G_{23}(\lambda)$.

The series $B_X(\lambda)$ acts in any finite-dimensional $\mathfrak{sl}_N$-module V, commuting with the $\mathfrak{h}$-action. The action of $B_X(\lambda)$ gives an element of $Rat\bigl(\mathrm{End}(V)\bigr)$, which tends to 1 as λ goes to infinity in a generic direction and, therefore, is invertible for generic λ.

Proposition F.1. *For any finite-dimensional $\mathfrak{sl}_N$-modules V, W we have*

$$B_X(\lambda)\big|_{V\otimes W}(X\otimes X)\,J_{VW}\bigl(w_X(\lambda)\bigr)(X\otimes X)^{-1} \,=\, J_{VW}(\lambda)\bigl(B_X(\lambda-h^{(2)})\otimes B_X(\lambda)\bigr)\,.$$

Corollary F.2. $\widetilde{R}_{VW}(\lambda-\rho) \,=\, \bigl(B_X(\lambda-h^{(2)})\otimes B_X(\lambda)\bigr)^{-1}R_{VW}(\lambda)\bigl(B_X(\lambda)\otimes B_X(\lambda-h^{(1)})\bigr)\,.$

The statement follows from Corollary 9.8 and cocommutativity of the coproduct Δ.

Lemma F.3. *Let U be the vector representation of $\mathfrak{sl}_N$. Then* $B_X(\lambda)\big|_U \,=\, \sum\limits_{a=1}^{N} E_{aa}\prod\limits_{1\leqslant b<a}(1+\lambda_{ba}^{-1})\,.$

Proof of Theorem 10.5. By the definition of isomorpic functors the assertion of the theorem means that for any finite-dimensional $\mathfrak{sl}_N$-module V there exists an isomorphism $\psi_V\in\mathrm{Mor}\bigl(\widetilde{\mathcal{E}}(V),\mathcal{E}(V)\bigr)$, and for any morphism $\varphi: V\to W$ of $\mathfrak{sl}_N$-modules one has $\mathcal{E}(\varphi)\circ\psi_V=\psi_W\circ\widetilde{\mathcal{E}}(\varphi)$.

It follows from formula (10.8), Proposition 10.2, Corollary F.2 and Lemma F.3 that $B_X(\lambda)\big|_V$ is a morphism from $\widetilde{\mathcal{E}}(V)$ to $\mathcal{E}(V)$. It is an isomorphism since $B_X(\lambda)\big|_V$ is invertible for generic λ. Moreover, for any morphism $\varphi: V\to W$ of $\mathfrak{sl}_N$-modules we have $\varphi B_X(\lambda)\big|_V = B_X(\lambda)\big|_W\varphi$. Since both $\mathcal{E}$ and $\widetilde{\mathcal{E}}$ send φ to the constant function $\varphi\in Rat\bigl(\mathrm{Hom}(V,W)\bigr)$, the theorem is proved. □

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