Quasifinite representations of classical Lie subalgebras of $\mathcal{W}_{1+\infty}$

Victor G. Kac[1], Weiqiang Wang[2*] and Catherine H. Yan[3]

[1]Department of Mathematics
MIT, Cambridge, MA 02139, USA
E-mail: kac@math.mit.edu

[2]Max-Planck Institut für Mathematematik
Gottfried-Claren Str. 26, 53225 Bonn, Germany
E-mail: wqwang@mpim-bonn.mpg.de

[3]Courant Institute of Mathematical Sciences
251 Mercer Street, New York, NY 10012, USA
E-mail: yanhuaf@math1.cims.nyu.edu

Abstract

We show that there are precisely two, up to conjugation, anti-involutions $\sigma_\pm$ of the algebra of differential operators on the circle preserving the principal gradation. We classify the irreducible quasifinite highest weight representations of the central extension $\widehat{\mathcal{D}}^\pm$ of the Lie subalgebra of this algebra fixed by $-\sigma_\pm$, and find the unitary ones. We realize them in terms of highest weight representations of the central extension of the Lie algebra of infinite matrices with finitely many non-zero diagonals over the algebra $\mathbb{C}[u]/(u^{m+1})$ and its classical Lie subalgebras of B, C and D types. Character formulas for *positive primitive* representations of $\widehat{\mathcal{D}}^\pm$ (including all the unitary ones) are obtained. We also realize a class of primitive representations of $\widehat{\mathcal{D}}^\pm$ in terms of free fields and establish a number of duality results between these primitive representations and finite-dimensional irreducible representations of finite-dimensional Lie groups and supergroups. We show that the vacuum module V_c of $\widehat{\mathcal{D}}^+$ carries a vertex algebra structure and establish a relationship between V_c for $c \in \frac{1}{2}\mathbb{Z}$ and $\mathcal{W}$-algebras.

*On leave from Department of Mathematics, Yale University, USA

Contents

0 Introduction

The systematic study of quasifinite highest weight modules of the universal central extension $\widehat{\mathcal{D}}$ of the Lie algebra of differential operators over the circle (described first in [KP]) was initiated by Kac and Radul in [KR1] and further studied in [Ma, FKRW, AFMO, KR2, W1] and many others. The Lie algebra $\widehat{\mathcal{D}}$ is also known in the literature as $\mathcal{W}_{1+\infty}$ [PRS] and as one of the universal $\mathcal{W}$-algebras (cf. [BS] [BEH^3] and references therein), and it has various connections with conformal field theory, the quantum Hall effect [CTZ] etc.

The difficulty in understanding representation theory of a Lie algebra of this sort is that although $\widehat{\mathcal{D}}$ admits a natural "principal" $\mathbb{Z}$-gradation and thus the associated triangular decomposition, each of the graded subspaces of $\widehat{\mathcal{D}}$ is still infinite-dimensional in contrast to the more familiar cases such as the Virasoro algebra and Kac-Moody algebras. The study of the highest weight modules of $\widehat{\mathcal{D}}$ with the finiteness requirement on the dimensions of their graded subspaces (which we will refer to as quasifiniteness condition throughout our paper) thus becomes a highly non-trivial problem. By analyzing for which parabolic subalgebras of $\widehat{\mathcal{D}}$ the corresponding generalized Verma modules are quasifinite, Kac and Radul [KR1] were able to give an elegant characterization of quasifinite highest weight $\widehat{\mathcal{D}}$-modules in terms of a certain generating function of highest weights. They constructed all these $\widehat{\mathcal{D}}$-modules in terms of representations of the Lie algebra $\widehat{\mathfrak{gl}}^{[m]}$ which is the central extension of the Lie algebra $\mathfrak{gl}^{[m]}$ of infinite matrices with finitely many non-zero diagonals taking values in the truncated polynomial algebra $R_m = \mathbb{C}[u]/(u^{m+1})$. They also classified all such $\widehat{\mathcal{D}}$-modules which are unitary.

It is well known that the Lie algebra $\widehat{\mathfrak{gl}}$ [KP, DJKM], a special case of $\widehat{\mathfrak{gl}}^{[m]}$ with $m = 0$, has Lie subalgebras of B, C, D types as in the finite dimensional case [K]. So a natural question which arises here is: what are the Lie subalgebras of $\widehat{\mathcal{D}}$ which correspond to the classical Lie subalgebras of $\widehat{\mathfrak{gl}}$ of B, C, D types. It is the goal of this paper to give a complete answer to this question and to present the representation theory of these Lie subalgebras of $\widehat{\mathcal{D}}$.

We show that there are two, up to conjugation, anti-involutions of $\widehat{\mathcal{D}}$, denoted by $\sigma_{\pm}$, which preserve the principal $\mathbb{Z}$-gradation. We thus obtain two different Lie subalgebras, denoted by $\widehat{\mathcal{D}}^{\pm}$, fixed by $-\sigma_{\pm}$. The graded subspaces of $\widehat{\mathcal{D}}^{\pm}$ are still infinite-dimensional. We regard $\widehat{\mathcal{D}}^{+}$ to be more fundamental than $\widehat{\mathcal{D}}^{-}$ as we shall show that the vacuum module of $\widehat{\mathcal{D}}^{+}$ carries a canonical vertex algebra structure.

We first give a description of parabolic subalgebras of $\widehat{\mathcal{D}}^{\pm}$ in terms of the so-called characteristic polynomials $\{b_k(w)\}_{k\in\mathbb{N}}$. It turns out that the quasifiniteness of a generalized Verma module is completely determined by the non-vanishing of the first characteristic polynomial $b_1(w)$. The corresponding condition in defining the generalized Verma module induced from a parabolic subalgebra $\mathcal{P}$ leads to a characterization of highest weights for which the corresponding $\widehat{\mathcal{D}}^{\pm}$-module is quasifinite in terms of a generating function $\Delta(x)$. This is the content of our

Theorem 4.1.

Next for each $s \in \mathbb{C}$ we construct a Lie algebra homomorphism $\widehat{\phi}_s^{[m]}$ from $\widehat{\mathcal{D}}^{\pm}$ to $\widehat{\mathfrak{gl}}^{[m]}$. While $\widehat{\mathcal{D}}^{\pm}$ is the central extension of a Lie algebra of differential operators with polynomial coefficients, it is important to consider a Lie algebra $\widehat{\mathcal{D}}^{\mathcal{O},\pm}$ which is an analytic completion of $\widehat{\mathcal{D}}^{\pm}$ and to extend the homomorphisms $\widehat{\phi}_s^{[m]}$ to $\widehat{\mathcal{D}}^{\mathcal{O},\pm}$. It turns out that for $s \notin \mathbb{Z}/2$ the homomorphism $\widehat{\phi}_s^{[m]}$ from $\widehat{\mathcal{D}}^{\mathcal{O},\pm}$ to $\widehat{\mathfrak{gl}}^{[m]}$ is surjective, but for $s \in \mathbb{Z}/2$, this is no longer true. The image for $s \in \mathbb{Z}/2$ turns out to be various classical Lie subalgebras of $\widehat{\mathfrak{gl}}^{[m]}$.

More generally we define a family of Lie algebra homomorphisms $\widehat{\phi}_{\vec{s}}^{[\vec{m}]}$ from $\widehat{\mathcal{D}}^{\pm}$ to $\mathfrak{g}^{[\vec{m}]}$ for a vector $\vec{m} = (m_1, \ldots, m_N) \in \mathbb{Z}_+$ and a vector $\vec{s} = (s_1, \ldots, s_N) \in \mathbb{C}^N$ satisfying a certain condition $(\star\pm)$, where $\mathfrak{g}^{[\vec{m}]}$ is a direct sum of Lie algebras $b_\infty^{[m_i]}$, $\tilde{b}_\infty^{[m_i]}$, $c_\infty^{[m_i]}$, $d_\infty^{[m_i]}$, and $\widehat{\mathfrak{gl}}^{[m_i]}$ satisfying a certain consistency condition. These homomorphisms $\widehat{\phi}_{\vec{s}}^{[\vec{m}]}$ again extend to $\widehat{\mathcal{D}}^{\mathcal{O},\pm}$ and then become surjective. The principal $\mathbb{Z}$-gradations on $b_\infty^{[m_i]}$, $c_\infty^{[m_i]}$ and $\widehat{\mathfrak{gl}}^{[m_i]}$, and the specialized $\mathbb{Z}$-gradation on $d_\infty^{[m_i]}$ of type $(2,1,1,\cdots)$ induce one on $\mathfrak{g}^{[\vec{m}]}$. The homomorphism $\widehat{\phi}_{\vec{s}}^{[\vec{m}]}$ matches the induced $\mathbb{Z}$-gradation on $\mathfrak{g}^{[\vec{m}]}$ with the principal $\mathbb{Z}$-gradation on $\widehat{\mathcal{D}}^{\pm}$.

Quasifinite highest weight modules (QHWM's) of $\widehat{\mathfrak{gl}}^{[m]}$ and its classical subalgebras are better understood (particularly when $m = 0$). Irreducible QHWM's over $\mathfrak{g}^{[\vec{m}]}$ can be regarded as modules over $\widehat{\mathcal{D}}^{\pm}$ via the homomorphisms $\widehat{\phi}_{\vec{s}}^{[\vec{m}]}$. Our Theorem 6.1 asserts that they are irreducible over $\widehat{\mathcal{D}}^{\pm}$ as well. Our Theorem 6.2 asserts that all irreducible quasifinite highest weight modules over $\widehat{\mathcal{D}}^{\pm}$ can be realized in this way.

There is a natural anti-involution ω on $\widehat{\mathcal{D}}^{\pm}$ which is compatible with the standard Cartan involution on $\widehat{\mathfrak{gl}}$ via the Lie algebra homomorphism $\widehat{\phi}_s$. We show that unitary irreducible quasifinite highest weight modules over $\widehat{\mathcal{D}}^{\pm}$ with respect to ω are the pullbacks of those over $\widehat{\mathfrak{gl}}$ via the homomorphisms $\widehat{\phi}_s$ for real vectors $\vec{s}$. We give a simple characterization for these unitary $\widehat{\mathcal{D}}^{\pm}$-modules in terms of the generating function $\Delta(x)$ for the highest weights with respect to $\widehat{\mathcal{D}}^{\pm}$ and the central charge c. This is our Theorem 7.1. The proofs of Theorems 4.1, 6.1, 6.2 and 7.1 are similar to those in [KR1] for $\widehat{\mathcal{D}}$.

The q-character formulas in accordance with the $\mathbb{Z}$-gradation for unitary quasifinite highest weight modules of b_∞, c_∞ and d_∞ are worked out explicitly. This leads to the q-character formulas for the so-called *positive primitive* quasifinite $\widehat{\mathcal{D}}^{\pm}$-modules, which include all unitary quasifinite $\widehat{\mathcal{D}}^{\pm}$-modules.

In the second half of this paper we study primitive quasifinite representations of $\widehat{\mathcal{D}}^{\pm}$ in terms of free bosonic and fermionic fields. Recall that representations of b_∞ or c_∞ (resp. $\tilde{b}_\infty$ and d_∞) were realized in various Fock spaces [DJKM1, W2]. A number of duality results in various Fock spaces between finite dimensional irreducible representations of a finite dimensional Lie group or superalgebra and quasifinite representations of b_∞ or c_∞ (resp. $\tilde{b}_\infty$ or d_∞) were established in [W2]. Lie groups and Lie superalgebra appearing in these duality results are

$Sp(2l), Pin(2l)$, $Spin(2l+1)$, $O(2l)$, $O(2l+1)$ and $\mathfrak{osp}(1,2l)$ respectively. Combining with our homomorphisms $\widehat{\phi}_0$ or $\widehat{\phi}_{1/2}$ (resp. $\widehat{\phi}_{-1/2}$), we obtain free field realizations of $\widehat{\mathcal{D}}^-$ (resp. $\widehat{\mathcal{D}}^+$) and corresponding dual pairs involving $\widehat{\mathcal{D}}^-$ (resp. $\widehat{\mathcal{D}}^+$). We present explicit descriptions of the Fock space decompositions into isotypic subspaces with respect to the joint action of the corresponding dual pairs and calculate the corresponding *exponents* and their *multiplicities* (which characterize the highest weights for $\widehat{\mathcal{D}}^\pm$) in each isotypic subspace. The explicit formulas for the highest weight vectors in isotypic subspaces were already given in [W2]. Free field realizations of $\widehat{\mathcal{D}}$ were studied earlier in [FKRW, KR2] and a duality between the general linear Lie group and $\widehat{\mathcal{D}}$ was established in these papers.

It is a remarkable new feature of dual pairs in our infinite dimensional setting that b_∞ and c_∞ (resp. $\tilde{b}_\infty$ and d_∞) involved in different dual pairs are uniformly replaced now by a single Lie algebra $\widehat{\mathcal{D}}^-$ (resp. $\widehat{\mathcal{D}}^+$). The ranks of these finite dimensional Lie groups, while varying for different Fock spaces, correspond neatly to the central charges of the primitive $\widehat{\mathcal{D}}^\pm$-modules appearing in these Fock spaces. Note that [W2] the Lie algebras of these Lie groups turn out to be the horizontal subalgebras of the twisted or untwisted affine algebras acting on the corresponding Fock spaces with level ± 1 [F1] [KP] [FF].

We further show that the vacuum module M_c of $\widehat{\mathcal{D}}^+$ with central charge c and its irreducible quotient V_c admit a natural structure of a vertex algebra [B] [FLM] [DL] [K2]. The module M_c is irreducible if and only if $c \notin \frac{1}{2}\mathbb{Z}$. For the central charge $c = l, l+1/2, -l$ or $-l+1/2$, V_c is shown to be isomorphic to the vertex algebra of invariants of the various Fock spaces $\mathcal{F}^{\bigotimes l}, \mathcal{F}^{\bigotimes l+\frac{1}{2}}, \mathcal{F}^{\bigotimes -l}$ or $\mathcal{F}^{\bigotimes -l+\frac{1}{2}}$ (see the text for notations) with respect to the action of $O(2l)$, $O(2l+1)$, $Sp(2l)$ or $\mathfrak{osp}(1,2l)$ respectively. Those quasifinite representations of $\widehat{\mathcal{D}}^+$ appearing from the Fock space decompositions are representations of the vertex algebra V_c for the corresponding $c \in \frac{1}{2}\mathbb{Z}$.

It is well known that to any complex simple Lie algebra $\mathfrak{g}$ and $c \in \mathbb{C}$ one canonically associates a vertex algebra with central charge c, called $\mathcal{W}$-algebra and denoted by $\mathcal{W}\mathfrak{g}$, cf. [BS, FeF] and references therein. It is known [BS, F2] that the associated $\mathcal{W}$-algebra $\mathcal{WD}_l$ with central charge l is isomorphic to the vertex algebra of the $SO(2l)$-invariants in the basic representation of the affine algebra $\widehat{\mathfrak{so}}(2l)$. We prove that $\mathcal{WD}_l$ with central charge l is a sum of the vertex algebra V_l and an irreducible representation of V_l. This provides us a new way of computing the q-character formula of $\mathcal{WD}_l$.

In a similar fashion, we identify the vertex algebra of $SO(2l+1)$-invariants in the Fock space $\mathcal{F}^{\bigotimes l+\frac{1}{2}}$ as a $\mathcal{W}$-superalgebra $\mathcal{WB}(0,l)$ [Ito] with l bosonic generating fields of conformal weight $2, 4, \ldots, 2l$ and a fermionic generating field of conformal weight $l+1/2$. The even part of $\mathcal{WB}(0,l)$ is isomorphic to $V_{l+1/2}$ while the odd part of $\mathcal{WB}(0,l)$ is an irreducible representation of $V_{l+1/2}$.

The approach of this paper can be modified to study the representation theory of another interesting Lie subalgebra of $\widehat{\mathcal{D}}$ considered by Bloch [Bl]. This will be treated in a separate publication [W3].

The paper is organized as follows. In Section 1 we describe the infinite rank Lie algebras $\widehat{\mathfrak{gl}}^{[m]}, b_{\infty}^{[m]}, \tilde{b}_{\infty}^{[m]}, c_{\infty}^{[m]}, d_{\infty}^{[m]}$ and present the q-character formulas of unitary highest weight modules over them in the case $m = 0$ (see also Appendix). In Section 2 we review the Lie algebra $\widehat{\mathcal{D}}$ and classify the anti-involutions of $\widehat{\mathcal{D}}$ preserving the principal $\mathbb{Z}$-gradation of $\widehat{\mathcal{D}}$. In Section 3 we study the structure of parabolic subalgebras of $\widehat{\mathcal{D}}^{\pm}$. In Section 4 we give a characterization for a $\widehat{\mathcal{D}}^{\pm}$-module to be quasifinite. In Section 5 we study the connection between $\widehat{\mathcal{D}}^{\pm}$ and the infinite rank Lie algebras. In Section 6 we realize quasifinite $\widehat{\mathcal{D}}^{\pm}$-modules in terms of modules over these infinite rank Lie algebras. In Section 7 we classify all unitary quasifinite $\widehat{\mathcal{D}}^{\pm}$-modules. In Sections 8, 9 and 10, we realize a class of primitive $\widehat{\mathcal{D}}^{-}$-modules with a half-integral central charge in some Fock spaces and establish some duality between this class of primitive $\widehat{\mathcal{D}}^{-}$-modules with central charge $c = -l$ (resp. $-l - 1/2,\ l,\ l + 1/2$) and finite dimensional irreducible modules of $O(2l)$ (resp. $O(2l + 1)$, $Sp(2l)$ and $Pin(2l)$, $\mathfrak{osp}(1, 2l)$, $Spin(2l + 1)$). In Sections 11, 12 and 13, we realize in a similar way a class of primitive $\widehat{\mathcal{D}}^{+}$-modules and establish various duality results for $\widehat{\mathcal{D}}^{+}$. In Section 14, we study the representations of $\widehat{\mathcal{D}}^{\pm}$ from the viewpoint of vertex algebras.

Acknowledgement This paper is based on two preprints [KWY]. W. W. wishes to thank Max-Planck-Institut für Mathematik for its hospitality.

1 Lie algebra $\widehat{\mathfrak{gl}}^{[m]}$ and its classical subalgebras

1.1 Lie algebra $\widehat{\mathfrak{gl}}^{[m]}$

Denote by R_m the quotient algebra $\mathbb{C}[u]/(u^{m+1})$ of the polynomial algebra $\mathbb{C}[u]$ by the ideal generated by u^{m+1} ($m \in \mathbb{Z}_+$). Denote by $\mathbf{1}$ the identity element of R_m. Denote by $\mathfrak{gl}_f^{[m]}$ the complex Lie algebra of all infinite matrices $(a_{ij})_{i,j \in \mathbb{Z}}$ with finitely many non-zero entries in R_m. Denote by $\mathfrak{gl}^{[m]}$ the Lie algebra of all matrices $(a_{ij})_{i,j \in \mathbb{Z}}$ with only finitely many nonzero diagonals with entries in R_m. Denote by E_{ij} the infinite matrix with 1 at (i, j) place and 0 elsewhere. Obviously $\mathfrak{gl}_f^{[m]}$ is a Lie subalgebra of $\mathfrak{gl}^{[m]}$. There is a natural automorphism ν of $\widehat{\mathfrak{gl}}^{[m]}$ given by

$$\nu(E_{i,j}) = E_{i+1,j+1}. \tag{1.1}$$

Let the weight of E_{ij} be $j - i$. This defines the *principal* $\mathbb{Z}$-gradation $\mathfrak{gl}^{[m]} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{gl}_j^{[m]}$. Denote by $\widehat{\mathfrak{gl}}^{[m]} = \mathfrak{gl}^{[m]} \bigoplus R_m$ the central extension of $\widehat{\mathfrak{gl}}^{[m]}$ given by the following 2–cocycle with values in R_m [KP, DJKM]:

$$C(A, B) = \mathrm{Tr}\ ([J, A]B) \tag{1.2}$$

where $J = \sum_{j\leq 0} E_{ii}$. The $\mathbb{Z}$–gradation of Lie algebra $\mathfrak{gl}^{[m]}$ extends to $\widehat{\mathfrak{gl}}^{[m]}$ by putting the weight R_m to be 0. In particular, we have the *triangular decomposition*

$$\widehat{\mathfrak{gl}}^{[m]} = \widehat{\mathfrak{gl}}^{[m]}_{+} \bigoplus \widehat{\mathfrak{gl}}^{[m]}_{0} \bigoplus \widehat{\mathfrak{gl}}^{[m]}_{-},$$

where

$$\widehat{gl}_{\pm} = \bigoplus_{j\in\mathbb{N}} \widehat{\mathfrak{gl}}^{[m]}_{\pm j}, \quad \widehat{\mathfrak{gl}}^{[m]}_{0} = \mathfrak{gl}^{[m]}_{0} \bigoplus R_m.$$

Given $\Lambda \in \left(\widehat{\mathfrak{gl}}^{[m]}_{0}\right)^*$, we let

$$\begin{aligned} c_j &= \Lambda(u^j), \\ {}^a\lambda_i^{(j)} &= \Lambda(u^j E_{ii}), \\ {}^aH_i^{(j)} &= u^j E_{ii} - u^j E_{i+1,i+1} + \delta_{i,0} u^j, \\ {}^ah_i^{(j)} &= \Lambda({}^aH_i^{(j)}) = {}^a\lambda_i^{(j)} - {}^a\lambda_{i+1}^{(j)} + \delta_{i,0}\, c_j, \end{aligned}$$

where $i \in \mathbb{Z}$, $j = 0, \ldots, m$. The superscript a here denotes $\widehat{\mathfrak{gl}}^{[m]}$ which is of A type. Denote by $L(\widehat{\mathfrak{gl}}^{[m]}; \Lambda)$ the highest weight $\widehat{\mathfrak{gl}}^{[m]}$–module with highest weight Λ. The c_j are called the *central charges* and ${}^a\lambda_i^{(j)}$ are called the *labels* of $L(\widehat{\mathfrak{gl}}^{[m]}; \Lambda)$.

In particular putting $m = 0$ we recover the well-known Lie algebras $\mathfrak{gl} = \mathfrak{gl}^{[0]}$, $\widehat{\mathfrak{gl}} = \widehat{\mathfrak{gl}}^{[0]}$. In this case, we drop the superscript $[0]$. Define ${}^a\Lambda_j \in \mathfrak{gl}_0^*$ $(j \in \mathbb{Z})$ as follows:

$${}^a\Lambda_j(E_{ii}) = \begin{cases} 1, & \text{for } \ 0 < i \leq j \\ -1, & \text{for } \ j < i \leq 0 \\ 0, & \text{otherwise.} \end{cases} \tag{1.3}$$

Define ${}^a\widehat{\Lambda}_0 \in \widehat{\mathfrak{gl}}_0^*$ by

$${}^a\widehat{\Lambda}_0(C) = 1, \quad {}^a\widehat{\Lambda}_0(E_{ii}) = 0 \text{ for all } i \in \mathbb{Z}$$

and extend Λ_j from $\mathfrak{gl}_0^*$ to $\widehat{\mathfrak{gl}}_0^*$ by letting $\Lambda_j(C) = 0$. Then

$${}^a\widehat{\Lambda}_j = {}^a\Lambda_j + {}^a\widehat{\Lambda}_0 \quad (j \in \mathbb{Z})$$

are the *fundamental weights*, i.e. ${}^a\widehat{\Lambda}_j({}^aH_i) = \delta_{ij}$.

Recall that the q-character (i.e. principally specialized character) formula for an integrable highest weight module $L(\lambda)$ of a Kac-Moody algebra $\mathfrak{g}$ ([K], chap. 10) with respect to the principal gradation of $\mathfrak{g}$ is given by

$$\mathrm{ch}_q L(\lambda) = \prod_{\alpha^\vee \in \Delta_+^\vee} \left(\frac{1 - q^{\langle \lambda+\rho, \alpha^\vee \rangle}}{1 - q^{\langle \rho, \alpha^\vee \rangle}} \right)^{mult\ \alpha^\vee} \tag{1.4}$$

where $\Delta_+^\vee$ is the positive coroots of the Lie algebra $\mathfrak{g}$, and ρ satisfies $\langle \rho, \alpha_i^\vee \rangle = 1$ for all simple coroots $\alpha_i^\vee$.

Since $\widehat{\mathfrak{gl}}$ is a completed infinite rank affine algebra of Kac-Moody type, the above formula (1.4) applies. This gives the following explicit formula (cf. [FKRW])

$$\mathrm{ch}_q L(\widehat{\mathfrak{gl}}, \Lambda) = \frac{\prod_{1 \leq i < j \leq c} (1 - q^{n_i - n_j + j - i})}{\varphi(q)^c} \tag{1.5}$$

where $\Lambda = \widehat{\Lambda}_{n_1} + \widehat{\Lambda}_{n_2} + \ldots + \widehat{\Lambda}_{n_c}$, $n_1 \geq n_2 \geq \ldots \geq n_c$, $c \in \mathbb{N}$, and $\varphi(q) = \prod_{j=1}^{\infty}(1 - q^j)$ is the Euler product.

1.2 Lie algebra $b_{\infty}^{[m]}$

Now consider the vector space $R_m[t, t^{-1}]$ and take its basis $v_i = t^i$ $(i \in \mathbb{Z})$ over R_m. The Lie algebra $\mathfrak{gl}^{[m]}$ acts on this vector space via the usual formula

$$E_{ij} v_k = \delta_{j,k} v_i. \tag{1.6}$$

Let us consider the following $\mathbb{C}$-bilinear forms on this space:

$$B^{\pm}(u^m v_i, u^n v_j) \quad = \quad u^m (-u)^n (\pm 1)^i \delta_{i,-j}.$$

Denote by $\overline{b}_{\infty}^{-[m]}$ (resp. $\overline{b}_{\infty}^{+[m]}$) the Lie subalgebra of $\mathfrak{gl}^{[m]}$ which preserves the bilinear form B^- (resp. B^+), namely we have

$$\begin{aligned}
\overline{b}_{\infty}^{-[m]} &= \{g \in \mathfrak{gl}^{[m]} \mid B(a(u), v) + B(u, a(v)) = 0\} \\
&= \{(a_{ij}(u))_{i,j \in \mathbb{Z}} \in \mathfrak{gl}^{[m]} \mid a_{ij}(u) = (-1)^{i+j+1} a_{-j,-i}(-u)\}, \\
\overline{b}_{\infty}^{+[m]} &= \{g \in \mathfrak{gl}^{[m]} \mid B(a(u), v) + B(u, a(v)) = 0\} \\
&= \{(a_{ij}(u))_{i,j \in \mathbb{Z}} \in \mathfrak{gl}^{[m]} \mid a_{ij}(u) = -a_{-j,-i}(-u)\}.
\end{aligned}$$

Denote by $b_{\infty}^{[m]} = \overline{b}_{\infty}^{-[m]} \bigoplus R_m$ (resp. $\tilde{b}_{\infty}^{[m]} = \overline{b}_{\infty}^{+[m]} \bigoplus R_m$) the central extension of $\overline{b}_{\infty}^{-[m]}$ (resp. $\overline{b}_{\infty}^{+[m]}$) given by the 2-cocycle (1.2) restricted to $\overline{b}_{\infty}^{-[m]}$ (resp. $\overline{b}_{\infty}^{+[m]}$). Then $b_{\infty}^{[m]}$ (resp. $\tilde{b}_{\infty}^{[m]}$) inherits from $\widehat{\mathfrak{gl}}^{[m]}$ the principal $\mathbb{Z}$-gradation and the triangular decomposition:

$$\begin{aligned}
b_{\infty}^{[m]} = \bigoplus_{j \in \mathbb{Z}} b_{\infty,j}^{[m]}, \quad b_{\infty}^{[m]} = b_{\infty,+}^{[m]} \oplus b_{\infty,0}^{[m]} \oplus b_{\infty,-}^{[m]}, \\
\tilde{b}_{\infty}^{[m]} = \bigoplus_{j \in \mathbb{Z}} \tilde{b}_{\infty,j}^{[m]}, \quad \tilde{b}_{\infty}^{[m]} = \tilde{b}_{\infty,+}^{[m]} \oplus \tilde{b}_{\infty,0}^{[m]} \oplus \tilde{b}_{\infty,-}^{[m]}
\end{aligned}$$

where

$$\begin{aligned}
b_{\infty,j}^{[m]} = b_{\infty}^{[m]} \cap \widehat{\mathfrak{gl}}_j^{[m]}, \quad b_{\infty \pm}^{[m]} = b_{\infty}^{[m]} \cap \mathfrak{gl}_{\pm}^{[m]}, \quad b_{\infty,0}^{[m]} = b_{\infty}^{[m]} \cap \mathfrak{gl}_0^{[m]}, \\
\tilde{b}_{\infty,j}^{[m]} = \tilde{b}_{\infty}^{[m]} \cap \widehat{\mathfrak{gl}}_j^{[m]}, \quad \tilde{b}_{\infty,\pm}^{[m]} = \tilde{b}_{\infty}^{[m]} \cap \mathfrak{gl}_{\pm}^{[m]} \quad \tilde{b}_{\infty 0}^{[m]} = \tilde{b}_{\infty}^{[m]} \cap \mathfrak{gl}_0^{[m]}.
\end{aligned}$$

Remark 1.1 The Lie algebra $\tilde{b}_\infty^{[m]}$ is isomorphic to $b_\infty^{[m]}$ by sending the elements $u^k E_{ij} - (-u)^k E_{-j,-i}$ to $u^k E_{ij} + (-1)^{i+j+1}(-u)^k E_{-j,-i}$, $i,j \in \mathbb{Z}, k \in \mathbb{Z}_+$. Their Cartan subalgebras coincide.

Given $\Lambda \in \left(b_\infty^{[m]}{}_0\right)^*$, we let

$$\begin{aligned}
c_j &= \Lambda(u^j),\\
{}^b\lambda_0^{(j)} &= \Lambda\left(2u^j E_{00}\right) \quad (j \text{ odd}),\\
{}^b\lambda_i^{(j)} &= \Lambda\left(u^j E_{ii} - (-u)^j E_{-i,-i}\right),\\
{}^bH_i^{(j)} &= u^j E_{ii} - u^j E_{i+1,i+1} + (-u)^j E_{-i-1,-i-1} - (-u)^j E_{-i,-i},\\
{}^bH_0^{(j)} &= 2\left(u^j E_{-1,-1} - u^j E_{1,1}\right) + 2u^j \quad (j \text{ even}),\\
{}^bH_0^{(j)} &= \left(2u^j E_{0,0} - u^j E_{-1,-1} - u^j E_{1,1}\right) + u^j \quad (j \text{ odd}),\\
{}^bh_i^{(j)} &= \Lambda({}^bH_i^{(j)}) = {}^b\lambda_i^{(j)} - {}^b\lambda_{i+1}^{(j)},\\
{}^bh_0^{(j)} &= \Lambda({}^bH_0^{(j)}) = -2\,{}^b\lambda_1^{(j)} + 2c_j \quad (j \text{ even}),\\
{}^bh_0^{(j)} &= \Lambda({}^bH_0^{(j)}) = {}^b\lambda_0^{(j)} - {}^b\lambda_1^{(j)} + c_j \quad (j \text{ odd}),
\end{aligned} \tag{1.7}$$

where $i \in \mathbb{N}$ and $j = 0, \ldots, m$. The superscript b here means B type. Denote by $L(b_\infty^{[m]};\Lambda)$ (resp. $L(\tilde{b}_\infty^{[m]};\Lambda)$) the highest weight module over $b_\infty^{[m]}$ (resp. $\tilde{b}_\infty^{[m]}$) with highest weight Λ. The c_j are called the *central charges* and ${}^b\lambda_i^{(j)}$ are called the *labels* of $L(b_\infty^{[m]};\Lambda)$ or $L(\tilde{b}_\infty^{[m]};\Lambda)$.

In particular when $m = 0$ we have the usual Lie subalgebras of $\mathfrak{gl}$ and $\widehat{\mathfrak{gl}}$, denoted by $\overline{b}_\infty^-$ and b_∞ [K] (resp. $\overline{b}_\infty^+$ and $\tilde{b}_\infty$ [W2]). Denote by ${}^b\widehat{\Lambda}_i$ the i-th *fundamental weight* of b_∞ (and $\tilde{b}_\infty$), namely ${}^b\widehat{\Lambda}_i({}^bh_j) = \delta_{ij}$.

The set of simple coroots of b_∞ (and $\tilde{b}_\infty$), denoted by $\Pi^\vee$, can be described as follows:

$$\begin{aligned}
\Pi^\vee = \{\alpha_0^\vee &= 2(E_{-1,-1} - E_{1,1}) + 2C,\\
\alpha_i^\vee &= E_{i,i} + E_{-i-1,-i-1} - E_{i+1,i+1} - E_{-i,-i}, i \in \mathbb{N}\}.
\end{aligned}$$

The set of roots is:

$$\Delta = \{\pm\epsilon_0, \pm\epsilon_i, \pm\epsilon_i \pm \epsilon_j, i \neq j, i,j \in \mathbb{N}\}.$$

The set of positive coroots is:

$$\begin{aligned}
\Delta_+^\vee &= \{\alpha_i^\vee + \alpha_{i+1}^\vee + \ldots + \alpha_j^\vee, \quad 1 \leq i \leq j\}\\
&\quad \cup \{\alpha_0^\vee + 2\alpha_1^\vee + \ldots + 2\alpha_i^\vee + \alpha_{i+1}^\vee + \ldots + \alpha_{j-1}^\vee, \quad 0 \leq i < j\}.
\end{aligned}$$

The set of simple roots is:

$$\Pi = \{\alpha_0 = -\epsilon_1, \alpha_i = \epsilon_i - \epsilon_{i+1}, i \in \mathbb{N}\}.$$

Here ϵ_i are viewed restricted to the restricted dual of the Cartan subalgebra of b_∞, so that $\epsilon_i = -\epsilon_{-i}$.

Given $\Lambda = {}^b\widehat{\Lambda}_{n_1} + {}^b\widehat{\Lambda}_{n_2} + \ldots + {}^b\widehat{\Lambda}_{n_k} + {}^bh\,{}^b\widehat{\Lambda}_0, \quad n_1 \geq n_2 \geq \ldots \geq n_k \geq 1, {}^bh \in \mathbb{Z}_+$, the b_∞-module $L(b_\infty;\Lambda)$ has central charge $c = k + {}^bh/2$. Here and further we let ${}^bh = {}^bh^{(0)}$ and ${}^b\lambda_i = {}^b\lambda_i^{(0)}$, see (1.7). Let

$$\varphi_i(q) = \prod_{j=1}^{i}(1-q^j).$$

Further on we shall use the following notations: $\overline{m} = 1$ for $m \in 2\mathbb{Z}$ and $\overline{m} = 0$ for $m \in 2\mathbb{Z}+1$; for a real number x denote by $[x]$ the integer no greater than and closest to x.

Proposition 1.1 *The q-character formula of $L(b_\infty;\Lambda)$ (the same for $L(\tilde{b}_\infty;\Lambda)$) corresponding to the principal gradation of b_∞ is*

$$\begin{aligned} ch_q L(b_\infty;\Lambda) \;=\; & \frac{\prod_{1\leq i<j\leq k}(1-q^{n_i-n_j+j-i})}{\prod_{1\leq i\leq k}\varphi_{n_i+k-i}(q)} \cdot \frac{\varphi(q^2)^{\overline{2c+1}}\prod_{j>0}\varphi_{2c-2j}(q)}{\varphi(q)^{[\frac{2c+1}{2}]}} \cdot \\ & \prod_{i=0}^{n_1-1}\frac{\varphi_{2c+i+n_1}(q)}{\varphi_{2c+n_1+i-{}^b\lambda_{i+1}}(q)} \cdot \prod_{0\leq i<j\leq n_1}\frac{1-q^{2c+j+i-{}^b\lambda_{i+1}-{}^b\lambda_j}}{1-q^{2c+j+i}}. \end{aligned} \tag{1.8}$$

Proof. Applying formula (1.4) to $L(b_\infty;\Lambda)$, we get

$$\mathrm{ch}_q L(b_\infty;\Lambda) = \prod_{1\leq i<j}\frac{1-q^{{}^b\lambda_i-{}^b\lambda_j+j-i}}{1-q^{j-i}} \cdot \prod_{0\leq i<j}\frac{1-q^{2c-{}^b\lambda_{i+1}-{}^b\lambda_j+j+i}}{1-q^{i+j}}. \tag{1.9}$$

Denote by the first (resp. second) product on the right hand side of (1.9) to be A (resp. B). Consider the Young diagram Y_Λ corresponding to the partition $(n_1, n_2, \ldots, n_k)$, it is easy to see that $({}^b\lambda_1, {}^b\lambda_2, \ldots, {}^b\lambda_{n_1})$ is its conjugate (cf. [M] for terminology). Also ${}^b\lambda_i = 0$ for $i > n_1$. Thus

$$A = \prod_{1\leq i<j}\frac{1-q^{{}^b\lambda_i-{}^b\lambda_j+j-i}}{1-q^{j-i}} = \prod_{j\in Y_\Lambda}(1-q^{h_j})^{-1} = \frac{\prod_{1\leq i<j\leq k}(1-q^{n_i-n_j+j-i})}{\prod_{1\leq i\leq k}\varphi_{n_i+k-i}(q)} \tag{1.10}$$

where h_j are the hook numbers of the Young diagram Y_Λ (cf. [M]).

On the other hand, B can be computed as follows.

$$\begin{aligned} B \;&=\; \prod_{0\leq i<j}\frac{1-q^{2c+j+i-{}^b\lambda_{i+1}-{}^b\lambda_j}}{1-q^{i+j}} \\ &=\; \prod_{0\leq i<j}\frac{1-q^{2c+j+i}}{1-q^{i+j}} \cdot \frac{1-q^{2c+j+i-{}^b\lambda_{i+1}-{}^b\lambda_j}}{1-q^{2c+j+i}} \end{aligned}$$

$$= \prod_{0\le i<j} \frac{1-q^{2c+j+i}}{1-q^{i+j}} \prod_{0\le i<n_1<j} \frac{1-q^{2c+j+i-{}^b\lambda_{i+1}-{}^b\lambda_j}}{1-q^{2c+j+i}} \cdot \prod_{0\le i<j\le n_1} \frac{1-q^{2c+j+i-{}^b\lambda_{i+1}-{}^b\lambda_j}}{1-q^{2c+j+i}}.$$

A little further manipulation shows that the first, second and third products of the right hand side of the equation above give rise to the second, third and fourth products of the right hand side of (1.8) respectively. □

1.3 Lie algebra $c_\infty^{[m]}$

As before we take a basis $v_i = t^i$ $(i \in \mathbb{Z})$ of $R_m[t,t^{-1}]$ over R_m. Consider on this space the following $\mathbb{C}$-bilinear form

$$C(u^m v_i, u^n v_j) = u^m(-u)^n(-1)^i \delta_{i,1-j}.$$

Denote by $\overline{c}_\infty^{[m]}$ the Lie subalgebra of $\mathfrak{gl}^{[m]}$ which preserves this bilinear form:

$$\begin{aligned}\overline{c}_\infty^{[m]} &= \{g \in \mathfrak{gl}^{[m]} \mid C(a(u),v) + C(u,a(v)) = 0\}\\ &= \{(a_{ij}(u))_{i,j\in\mathbb{Z}} \in \mathfrak{gl}^{[m]} \mid a_{ij}(u) = (-1)^{i+j+1}a_{1-j,1-i}(-u)\}.\end{aligned}$$

Denote by $c_\infty^{[m]} = \overline{c}_\infty^{[m]} \bigoplus R_m$ the central extension of $\overline{c}_\infty^{[m]}$ given by the 2-cocycle (1.2) restricted to $\overline{c}_\infty^{[m]}$. Then $c_\infty^{[m]}$ inherits from $\widehat{\mathfrak{gl}}^{[m]}$ the natural $\mathbb{Z}$-gradation and the triangular decomposition:

$$c_\infty^{[m]} = \bigoplus_{j\in\mathbb{Z}} c_\infty^{[m]}{}_j, \quad c_\infty^{[m]} = c_\infty^{[m]}{}_+ \bigoplus c_\infty^{[m]}{}_0 \bigoplus c_\infty^{[m]}{}_-,$$

where $c_\infty^{[m]}{}_j = c_\infty^{[m]} \cap \widehat{\mathfrak{gl}}_j^{[m]}$, $c_{\infty\pm}^{[m]} = c_\infty^{[m]} \cap \mathfrak{gl}_\pm^{[m]}$ and $c_\infty^{[m]}{}_0 = c_\infty^{[m]} \cap \mathfrak{gl}_0^{[m]}$.

Given $\Lambda \in \left(c_{\infty 0}^{[m]}\right)^*$, we let

$$\begin{aligned}
c_j &= \Lambda(u^j),\\
{}^c\lambda_i^{(j)} &= \Lambda\left(u^jE_{ii} - (-u)^jE_{1-i,1-i}\right),\\
{}^cH_i^{(j)} &= u^jE_{ii} - u^jE_{i+1,i+1} + (-u)^jE_{-i,-i} - (-u)^jE_{1-i,1-i},\\
{}^cH_0^{(j)} &= \left(u^jE_{0,0} - u^jE_{1,1}\right) + u^j \quad (j \text{ even}),\\
{}^ch_i^{(j)} &= \Lambda({}^cH_i^{(j)}) = {}^c\lambda_i^{(j)} - {}^c\lambda_{i+1}^{(j)},\\
{}^ch_0^{(j)} &= \Lambda({}^cH_0^{(j)}) = -{}^c\lambda_1^{(j)} + c_j \quad (j \text{ even}),
\end{aligned} \tag{1.11}$$

where $i \in \mathbb{N}$ and $j = 0,\dots,m$. For later purpose, it is convenient to put ${}^ch_0^{(j)} = c_j$ (j odd), $j = 0,\dots,m$. The superscript c here denotes c_∞ which is of C type. Denote by $L(c_\infty^{[m]};\Lambda)$ the highest weight $c_\infty^{[m]}$–module with highest weight Λ. The c_j are called the *central charges* and ${}^c\lambda_i^{(j)}$ are called the *labels* of $L(c_\infty^{[m]};\Lambda)$.

In particular, when $m=0$ we have the usual $\overline{c}_\infty = \overline{c}_\infty^{[0]}$, $c_\infty = c_\infty^{[0]}$ [K]. In this case, we drop the superscript [0]. Then we denote by ${}^c\widehat{\Lambda}_i$ the i-th fundamental weight of c_∞, namely ${}^c\widehat{\Lambda}_i({}^ch_j) = \delta_{ij}$.

The set of simple coroots of c_∞, denoted by $\Pi^\vee$, can be described as follows:

$$\begin{aligned}\Pi^\vee = \{&\alpha_0^\vee = (E_{0,0} - E_{1,1}) + C,\\ &\alpha_i^\vee = E_{i,i} + E_{-i,-i} - E_{i+1,i+1} - E_{1-i,1-i}, i \in \mathbb{N}\}.\end{aligned}$$

The set of roots is:

$$\Delta = \{\pm\epsilon_i \pm \epsilon_j, \pm 2\epsilon_i, i \neq j, i, j \in \mathbb{N}\}.$$

The set of positive coroots is:

$$\begin{aligned}\Delta_+^\vee &= \{\alpha_i^\vee + \alpha_{i+1}^\vee + \ldots + \alpha_j^\vee, \quad 0 \leq i \leq j\}\\ &\cup \quad \{2\alpha_0^\vee + \ldots + 2\alpha_i^\vee + \alpha_{i+1}^\vee + \ldots + \alpha_j^\vee, \quad 0 \leq i < j\}.\end{aligned}$$

The set of simple roots is:

$$\Pi = \{\alpha_0 = -2\epsilon_1, \alpha_i = \epsilon_i - \epsilon_{i+1}, i = 1, 2, \ldots\}.$$

Here ϵ_i are viewed restricted to $\mathfrak{h}'$, so that $\epsilon_i = -\epsilon_{-i+1}$.

Given $\Lambda = {}^c\widehat{\Lambda}_{n_1} + {}^c\widehat{\Lambda}_{n_2} + \ldots + {}^c\widehat{\Lambda}_{n_k} + {}^ch\,{}^c\widehat{\Lambda}_0, \quad n_1 \geq n_2 \geq \ldots \geq n_k \geq 1, {}^ch \in \mathbb{Z}_+$, the c_∞-module $L(c_\infty;\Lambda)$ has central charge $c = k + {}^ch$.

Proposition 1.2 *The q-character formula of $L(c_\infty;\Lambda)$ corresponding to the principal gradation of c_∞ is*

$$\begin{aligned} ch_q L(c_\infty;\Lambda) &= \frac{\prod_{1\leq i<j\leq k}(1-q^{n_i-n_j+j-i})}{\prod_{1\leq i\leq k}\varphi_{n_i+k-i}(q)} \cdot \frac{\prod_{1\leq j\leq c-1}\varphi_{2j}(q)}{\varphi(q)^c} \cdot \\ &\quad \prod_{i=0}^{n_1}\frac{\varphi_{2c+n_1+i}(q)}{\varphi_{2c+n_1+i-{}^c\lambda_i}(q)} \cdot \prod_{0\leq i<j\leq n_1}\frac{1-q^{2c+i+j-{}^c\lambda_{i+1}-{}^c\lambda_j}}{1-q^{2c+i+j}}. \end{aligned} \tag{1.12}$$

Proof. Applying formula (1.4) to $L(c_\infty;\Lambda)$, we get

$$\begin{aligned} \mathrm{ch}_q L(c_\infty;\Lambda) &= \prod_{1\leq i<j}\frac{1-q^{{}^c\lambda_i-{}^c\lambda_j+j-i}}{1-q^{j-i}} \cdot \\ &\quad \prod_{0\leq i<j}\frac{1-q^{2c-{}^c\lambda_{i+1}-{}^c\lambda_{j+1}+j+i+2}}{1-q^{i+j+2}} \cdot \prod_{j>0}\frac{1-q^{c-{}^c\lambda_j+j}}{1-q^j} \\ &= \prod_{1\leq i<j}\frac{1-q^{{}^c\lambda_i-{}^c\lambda_j+j-i}}{1-q^{j-i}} \cdot \prod_{0\leq i<j}\frac{1-q^{2c-{}^c\lambda_i-{}^c\lambda_j+j+i}}{1-q^{i+j}} \end{aligned} \tag{1.13}$$

with the convention ${}^c\lambda_0 = 0$. The first product on the right hand side of (1.13) is already given by formula (1.10) with ${}^b\lambda_i$ substituted by ${}^c\lambda_i$. Denote by C the second product on the right hand side of (1.13); it can be calculated in a similar way as calculating B in the proof of Proposition 1.1:

$$\begin{aligned} C &= \prod_{0\le i<j} \frac{1-q^{2c-{}^c\lambda_i-{}^c\lambda_j+i+j}}{1-q^{i+j}} \\ &= \frac{\prod_{1\le j\le c-1}\varphi_{2j}(q)}{\varphi(q)^c}\cdot\prod_{i=0}^{n_1}\frac{\varphi_{2c+n_1+i}(q)}{\varphi_{2c+n_1+i-{}^c\lambda_i}(q)}\cdot\prod_{0\le i<j\le n_1}\frac{1-q^{2c+i+j-{}^c\lambda_{i+1}-{}^c\lambda_j}}{1-q^{2c+i+j}}. \end{aligned}$$

□

1.4 Lie algebra $d_\infty^{[m]}$

As before we take a basis $v_i = t^i$ $(i \in \mathbb{Z})$ of $R_m[t,t^{-1}]$ over R_m. Consider the following $\mathbb{C}$-bilinear form

$$D(u^m v_i, u^n v_j) = u^m(-u)^n \delta_{i,1-j}, i, j \in \mathbb{Z}.$$

Denote by $\overline{d}_\infty^{[m]}$ the Lie subalgebra of $\mathfrak{gl}^{[m]}$ which preserves this bilinear form:

$$\begin{aligned} \overline{d}_\infty^{[m]} &= \{g \in \mathfrak{gl}^{[m]} \mid D(a(u),v)+D(u,a(v))=0\} \\ &= \{(a_{ij}(u))_{i,j\in\mathbb{Z}} \in \mathfrak{gl}^{[m]} \mid a_{ij}(u) = (-1)^{i+j+1}a_{1-j,1-i}(-u)\}. \end{aligned}$$

Denote by $d_\infty^{[m]} = \overline{d}_\infty^{[m]} \bigoplus R_m$ the central extension of $\overline{d}_\infty^{[m]}$ given by the 2-cocycle (1.2) restricted to $\overline{d}_\infty^{[m]}$. Then $d_\infty^{[m]}$ inherits from $\widehat{\mathfrak{gl}}^{[m]}$ a natural $\mathbb{Z}$-gradation and the triangular decomposition:

$$\begin{aligned} d_\infty^{[m]} &= \bigoplus_{j\in\mathbb{Z}} {d_\infty^{[m]}}_j \\ d_\infty^{[m]} &= {d_\infty^{[m]}}_+ \bigoplus {d_\infty^{[m]}}_0 \bigoplus {d_\infty^{[m]}}_- \end{aligned}$$

where ${d_\infty^{[m]}}_j = d_\infty^{[m]} \cap \widehat{\mathfrak{gl}}_j^{[m]}$, ${d_\infty}_\pm^{[m]} = d_\infty^{[m]} \cap \mathfrak{gl}_\pm^{[m]}$ and ${d_\infty^{[m]}}_0 = d_\infty^{[m]} \cap \mathfrak{gl}_0^{[m]}$.

Given $\Lambda \in \left({d_\infty}_0^{[m]}\right)^*$, we let

$$\begin{aligned} c_j &= \Lambda(u^j), \\ {}^d\lambda_i^{(j)} &= \Lambda\left(u^j E_{ii} - (-u)^j E_{1-i,1-i}\right), \\ {}^dH_i^{(j)} &= u^j E_{ii} - u^j E_{i+1,i+1} + (-u)^j E_{-i,-i} - (-u)^j E_{1-i,1-i}, \\ {}^dH_0^{(j)} &= \left((-u)^j E_{0,0} + (-u)^j E_{-1,-1} - u^j E_{2,2} - u^j E_{1,1}\right) + 2u^j, \qquad (1.14) \\ {}^dh_i^{(j)} &= \Lambda({}^dH_i^{(j)}) = {}^d\lambda_i^{(j)} - {}^d\lambda_{i+1}^{(j)}, \\ {}^dh_0^{(j)} &= \Lambda({}^dH_0^{(j)}) = -{}^d\lambda_1^{(j)} - {}^d\lambda_2^{(j)} + 2c_j, \end{aligned}$$

where $i \in \mathbb{N}$ and $j = 0, \ldots, m$. The superscript d here denotes d_∞ which is of D type. Denote by $L(d_\infty^{[m]}; \Lambda)$ the highest weight $d_\infty^{[m]}$–module with highest weight Λ. The c_j are called the central charges and ${}^d\lambda_i^{(j)}$ are called the labels of $L(d_\infty^{[m]}; \Lambda)$.

In particular, when $m = 0$ we have the usual $\overline{d}_\infty = \overline{d}_\infty^{[0]}$, $d_\infty = d_\infty^{[0]}$, cf. [K1]. In this case, we drop the superscript [0]. Denote by $\widehat{\Lambda}_i^d$ the i-th fundamental weight of d_∞, namely ${}^d\widehat{\Lambda}_i({}^d h_j) = \delta_{ij}$.

The set of simple coroots of d_∞, denoted by $\Pi^\vee$, can be described as follows:

$$\begin{aligned}\Pi^\vee = \{&\alpha_0^\vee = (E_{0,0} + E_{-1,-1} - E_{2,2} - E_{1,1}) + 2C,\\ &\alpha_i^\vee = E_{i,i} + E_{-i,-i} - E_{i+1,i+1} - E_{1-i,1-i}, i \in \mathbb{N}\}.\end{aligned}$$

The set of roots is:

$$\Delta = \{\pm\epsilon_i \pm \epsilon_j, i \neq j, i, j \in \mathbb{N}\}.$$

The set of positive coroots is:

$$\begin{aligned}\Delta_+^\vee &= \{\alpha_i^\vee + \alpha_{i+1}^\vee + \cdots + \alpha_j^\vee \quad (0 \le i \le j), \quad \alpha_0^\vee + \alpha_2^\vee + \cdots + \alpha_j^\vee \quad (j \ge 2)\}\\ &\cup \{\alpha_0^\vee + \alpha_1^\vee + 2\alpha_2^\vee + \cdots + 2\alpha_i^\vee + \alpha_{i+1}^\vee + \cdots + \alpha_j^\vee, \quad 1 < i < j\}.\end{aligned}$$

The set of simple roots is:

$$\Pi = \{\alpha_0 = -\epsilon_1 - \epsilon_2, \alpha_i = \epsilon_i - \epsilon_{i+1}, i \in \mathbb{N}\}.$$

Here ϵ_i are viewed restricted to $\mathfrak{h}'$, so that $\epsilon_i = -\epsilon_{-i+1}$. The root vectors corresponding to $\pm\alpha_0$ are $e_0 = E_{0,2} - E_{-1,1}$ and $f_0 = E_{2,0} - E_{1,-1}$. Introduce $\rho \in \left(d_{\infty 0}^{[m]}\right)^*$ such that $(\rho, \alpha_i^\vee) = 1, i \in \mathbb{Z}_+$.

Remark 1.2 The q-character formula (corresponding to the induced $\mathbb{Z}$-gradation on d_∞ from $\widehat{\mathfrak{gl}}$) of a d_∞-module is given in the Appendix. Note that such a $\mathbb{Z}$-gradation on d_∞ is of type $(2, 1, 1, \ldots)$ rather than the principal $\mathbb{Z}$-gradation of type $(1, 1, \ldots)$ as for $\widehat{\mathfrak{gl}}$, b_∞ and c_∞ since $f_0 = E_{2,0} - E_{1,-1}$ has degree -2 instead of -1.

Example 1.1 Take a parabolic subalgebra of $\overline{d}_\infty$

$$\mathcal{P}_0 = \{(a_{ij}) \in \overline{d}_\infty \mid a_{ij} = 0 \text{ if } i > 0 \ge j\},$$

and let $\widehat{\mathcal{P}}_0 = \mathcal{P}_0 \bigoplus \mathbb{C}C$. The 2-cocycle (1.2) when restricted to $\mathcal{P}_0$ is trivial. Consider the so-called vacuum module $M_c(d_\infty)$, i.e. the induced d_∞-module from a 1-dimensional $\mathcal{P}_0$-module on which $\mathcal{P}_0$ acts as zero and C as $c \in \mathbb{C}$. The irreducible quotient of $M_c(d_\infty)$ is $L(d_\infty; 2c\,{}^d\widehat{\Lambda}_0)$. It follows by a standard argument (cf. [K1], Chapter 10) that

$$L(d_\infty; 2c\,{}^d\widehat{\Lambda}_0) = M_c(d_\infty)/\langle f_0^{2c+1}\rangle \text{ if } c \in \mathbb{Z}_+/2.$$

2 Anti-involutions of $\widehat{\mathcal{D}}$ preserving its principal gradation

Let $\mathcal{D}_{as}$ be the associative algebra of regular differential operators on the circle. The elements

$$J_k^l = -t^{l+k}(\partial_t)^l \quad (l \in \mathbb{Z}_+, k \in \mathbb{Z})$$

form its basis, where ∂_t denotes $\frac{d}{dt}$. Another basis of $\mathcal{D}$ is

$$L_k^l = -t^k D^l \quad (l \in \mathbb{Z}_+, k \in \mathbb{Z})$$

where $D = t\partial_t$. It is easy to see that $J_k^l = -t^k[D]_l$. Here and further we use the notation

$$[x]_l = x(x-1)\dots(x-l+1). \tag{2.15}$$

Let $\mathcal{D}$ denote the Lie algebra obtained from $\mathcal{D}_{as}$ by taking the usual bracket $[a,b] = ab - ba$. Denote by $\widehat{\mathcal{D}}$ the central extension of $\mathcal{D}$ by a one-dimensional center with a generator C: $\widehat{\mathcal{D}} = \mathcal{D} + \mathbb{C}C$. The Lie algebra $\widehat{\mathcal{D}}$ has the following commutation relations [KR1]

$$\begin{aligned} [t^r f(D), t^s g(D)] &= t^{r+s}\left(f(D+s)g(D) - f(D)g(D+r)\right) \\ &+ \Psi\left(t^r f(D), t^s g(D)\right) C \end{aligned} \tag{2.16}$$

where

$$\Psi\left(t^r f(D), t^s g(D)\right) = \begin{cases} \sum\limits_{-r \le j \le -1} f(j)g(j+r), & r = -s \ge 0 \\ 0, & r+s \ne 0. \end{cases} \tag{2.17}$$

The 2-cocycle Ψ can be equivalently given in terms of another formula [KP]:

$$\Psi\left(f(t)(\partial_t)^m, g(t)(\partial_t)^n\right) = \frac{m!n!}{(m+n+1)!}\mathrm{Res}_{t=0} f^{(n+1)}(t) g^{(m)}(t) dt. \tag{2.18}$$

Let the weight of J_k^l be k and the weight of C be 0. This defines the principal $\mathbb{Z}$-gradations of $\mathcal{D}_{as}, \mathcal{D}$ and $\widehat{\mathcal{D}}$:

$$\mathcal{D} = \bigoplus_{j\in\mathbb{Z}} \mathcal{D}_j, \quad \widehat{\mathcal{D}} = \bigoplus_{j\in\mathbb{Z}} \widehat{\mathcal{D}}_j$$

and so we have the triangular decomposition

$$\widehat{\mathcal{D}} = \widehat{\mathcal{D}}_+ \bigoplus \widehat{\mathcal{D}}_0 \bigoplus \widehat{\mathcal{D}}_-,$$

where

$$\widehat{\mathcal{D}}_{\pm} = \bigoplus_{j \in \pm\mathbb{N}} \hat{\mathcal{D}}_j, \quad \widehat{\mathcal{D}}_0 = \mathcal{D}_0 \bigoplus \mathbb{C}C.$$

An *anti-involution* σ of $\mathcal{D}_{as}$ is an involutive anti-automorphism of $\mathcal{D}$, i.e. $\sigma^2 = I, \sigma(aX + bY) = a\sigma(X) + b\sigma(Y)$ and $\sigma(XY) = \sigma(Y)\sigma(X)$, where $a, b \in \mathbb{C}, X, Y \in \mathcal{D}$.

Proposition 2.1 *Any anti-involution σ of $\mathcal{D}_{as}$ which preserves the principal $\mathbb{Z}$-gradation is one of the following:*

1) $\sigma_{-,b}(t) = -t, \quad \sigma_{-,b}(D) = -D + b;$

2) $\sigma_{+,b}(t) = t, \quad \sigma_{+,b}(D) = -D + b, \quad b \in \mathbb{C}.$

Proof. Since σ preserves the principal $\mathbb{Z}$-gradation, we can assume that $\sigma(t) = tf(D)$, and $\sigma(D) = g(D)$, where f and g are polynomials. Since σ is an anti-involution we have

$$\begin{aligned} g(g(D)) &= D, && (2.19) \\ f(g(D)) \cdot (tf(D)) &= t. && (2.20) \end{aligned}$$

Equation (2.19) means that the polynomial $g(w)$ satisfies $g(g(w)) = w$, which can only be possible when $g(w)$ is linear in w. Hence $g(w) = w$ or $g(w) = -w + b$ for some $b \in \mathbb{C}$.

It follows from equation (2.20) that

$$t = f(g(D)) \cdot (tf(D)) = tf(g(D + 1))f(D)$$

which implies $f(w) = \pm 1$. By applying the anti-involution σ to the equation $[t, D] = -t$, we have $[\sigma(D), \sigma(t)] = -\sigma(t)$, which excludes the possibility $g(w) = w$.

An anti-involution σ of $\mathcal{D}_{as}$ is completely determined by $\sigma(t)$ and $\sigma(D)$. On the other hand, it is straightforward to check the $\sigma_{\pm,b}$ listed in the proposition are indeed anti-involutions of $\mathcal{D}_{as}$. This completes the proof of the proposition. □

It follows immediately that $\sigma_{\pm,b}(\partial_t) = \mp\left(\partial_t - (b+1)t^{-1}\right)$. Given $s \in \mathbb{C}$, denote by Θ_s the automorphism of $\mathcal{D}_{as}$ which sends t to t and D to $D + s$. Equivalently Θ_s is given by sending $a \in \mathcal{D}$ to $t^{-s}at^s$, the conjugate of a by t^s. Clearly Θ_s preserves the principal $\mathbb{Z}$-gradation of $\mathcal{D}_{as}$. We have

$$\sigma_{\pm,b} \cdot \Theta_s = \sigma_{\pm,b+s}, \quad \Theta_{-s} \cdot \sigma_{\pm,b} = \sigma_{\pm,b+s}. \tag{2.21}$$

Denote by $\mathcal{D}^{\pm,b}$ the fixed Lie subalgebra of $\mathcal{D}$ by $-\sigma_{\pm,b}$, namely

$$\mathcal{D}^{\pm,b} = \{a \in \mathcal{D} \mid \sigma_{\pm,b}(a) = -a\}.$$

It inherits a $\mathbb{Z}$-gradation from $\mathcal{D}$ since $\sigma_{\pm,b}$ preserves the principal $\mathbb{Z}$-gradation of $\mathcal{D}$: $\mathcal{D}^{\pm,b} = \bigoplus_{j\in\mathbb{Z}} \mathcal{D}_j^{\pm,b}$, where

$$\mathcal{D}_j^{\pm,b} = \{\ t^j f(D) \mid f(w) \in \mathbb{C}[w] \text{ and } \sigma_{\pm,b}(t^j f(D)) = -t^j f(D)\}.$$

Let us denote by $\mathbb{C}[w]^{(1)}$ the set of all odd polynomials in $\mathbb{C}[w]$, and by $\mathbb{C}[w]^{(0)}$ the set of all even polynomials in $\mathbb{C}[w]$. As before we let $\overline{k} = 0$ if k is an odd integer and $\overline{k} = 1$ if k is even. The following lemma gives a complete description of $\mathcal{D}_j^{\pm,b}$.

Lemma 2.1 *We have*

$$\begin{aligned} \mathcal{D}_j^{-,b} &= \left\{ t^j g\left(D + (j-b)/2\right) \mid g(w) \in \mathbb{C}[w]^{(\bar{j})},\ j \in \mathbb{Z} \right\}, \\ \mathcal{D}_j^{+,b} &= \left\{ t^j g\left(D + (j-b)/2\right) \mid g(w) \in \mathbb{C}[w]^{(1)},\ j \in \mathbb{Z} \right\}. \end{aligned}$$

Proof. Given $t^j f(D) \in \mathcal{D}_j^{-,b}$, we have $\sigma_{-,b}(t^j f(D)) = -t^j f(D)$, which means $(-1)^j t^j f(-D - j + b) = -t^j f(D)$. Equivalently we have $(-1)^j f(-w - j + b) = -f(w)$. Letting $g(w) = f(w - \frac{j-b}{2})$, we have $g(-w) = (-1)^{j+1} g(w)$. □

The relation among $\mathcal{D}^{\pm,b}$ for different $b \in \mathbb{C}$ is given by the following lemma which follows from Lemma 2.1.

Lemma 2.2 *The Lie algebras $\mathcal{D}^{+,b}$ (resp. $\mathcal{D}^{-,b}$) for different $b \in \mathbb{C}$ are all isomorphic. More precisely, we have $\Theta_s(\mathcal{D}^{\pm,b}) = \mathcal{D}^{\pm,b-2s}$.*

Due to Lemma 2.2 we may choose among $\mathcal{D}^{-,b}$ a Lie algebra, say $\mathcal{D}^- = \mathcal{D}^{-,0}$. We see from Lemma 2.1 that $D^n \in \mathcal{D}^-$ for $n \in 2\mathbb{Z}+1$, and $D^n \notin \mathcal{D}^-$ for $n \in 2\mathbb{Z}$. Let (see notation (2.15))

$$T_k^{n,s} = -t^k \left([D-s]_n + (-1)^{k+1} [-D-k-s]_n \right) \quad (k \in \mathbb{Z}, n \in \mathbb{Z}_+, s \in \mathbb{C}). \tag{2.22}$$

Clearly $T_k^{n,s} \in \mathcal{D}^-$. It is also clear that $T_k^{n,s}$ $(k \in \mathbb{Z}, n \in 2\mathbb{Z}_+ + 1)$ form a $\mathbb{C}$-basis of $\mathcal{D}^-$ for a fixed $s \in \mathbb{C}$. We denote $T_k^{n,0}$ by T_k^n. A straightforward computation shows

$$T_k^{n,s} = -\left(t^{k+n+s} \partial_t^n t^{-s} + (-1)^{k+n+1} t^{1-s} \partial_t^n t^{k+n+s-1} \right). \tag{2.23}$$

Although $t^{m\pm s}$ $(m \in \mathbb{Z}, s \in \mathbb{C})$ does not lie in $\mathcal{D}$ in general, the above expression (2.23) does lie in $\mathcal{D}^-$.

We denote again by Ψ the restriction of the 2-cocycle Ψ to $\mathcal{D}^-$, namely

$$\begin{aligned} &\Psi\left(t^r f(D + r/2), t^s g(D + s/2)\right) \\ &\quad = \begin{cases} \displaystyle\sum_{-r \le j \le -1} f(j + r/2) g(j + r/2), & r = -s \ge 0 \\ 0, & r + s \ne 0 \end{cases} \end{aligned}$$

where $f(w) \in \mathbb{C}[w]^{(\bar{r})}, g(w) \in \mathbb{C}[w]^{(\bar{s})}$. Denote by $\widehat{\mathcal{D}}^-$ the central extension of $\mathcal{D}^-$ by $\mathbb{C}C$ corresponding to the 2-cocycle Ψ. $\widehat{\mathcal{D}}^-$ is a Lie subalgebra of $\widehat{\mathcal{D}}$ by definition.

The most convenient choice for the other family of Lie algebras is $b = -1$ $(\sigma_{+,-1}(\partial_t) = -\partial_t)$. Denote $\mathcal{D}^+ = \mathcal{D}^{+,-1}$. We shall see that there is a canonical structure of a vertex algebra on the vacuum module of the central extension of $\mathcal{D}^+$. Let

$$W_k^{n,s} = -\frac{1}{2}t^k\left([D+s]_n - [-D-k-1+s]_n\right) \quad (k \in \mathbb{Z}, n \in \mathbb{N}, s \in \mathbb{C}). \tag{2.24}$$

An important linear basis of $\mathcal{D}^+$ is $W_k^{n,s}$ $(k \in \mathbb{Z}, n \in 2\mathbb{Z}_+ + 1)$ for a fixed $s \in \mathbb{C}$. In particular we denote $W_k^{n,0}$ by W_k^n. A straightforward computation shows that

$$W_k^n = -\frac{1}{2}\left(t^{k+n}\partial_t^n + (-1)^{n+1}\partial_t^n t^{k+n}\right). \tag{2.25}$$

Note that the W_k^1 $(k \in \mathbb{Z})$ span a Virasoro algebra, namely we have

$$\left[W_m^1, W_n^1\right] = (m-n)W_{m+n}^1 + \delta_{m,-n}\frac{m^3-m}{12}C.$$

By abuse of notation we again denote by Ψ the restriction of the 2-cocycle Ψ to $\mathcal{D}^+$:

$$\begin{aligned}&\Psi\left(t^r f(D+(r+1)/2), t^s g(D+(s+1)/2)\right)\\ &= \begin{cases}\sum\limits_{-r\le j\le -1} f(j+(r+1)/2)g(j+(r+1)/2), & r=-s\ge 0\\ 0, & r+s\neq 0,\end{cases}\end{aligned}$$

where $f(w), g(w) \in \mathbb{C}[w]^{(1)}$. Denote by $\widehat{\mathcal{D}}^+$ the central extension of $\mathcal{D}^+$ by $\mathbb{C}C$ corresponding to the 2-cocycle Ψ. The Lie algebra $\widehat{\mathcal{D}}^+$ is a subalgebra of $\widehat{\mathcal{D}}$ by definition.

3 Structure of parabolic subalgebras of $\widehat{\mathcal{D}}^{\pm}$

3.1 $\widehat{\mathcal{D}}^-$ case

We define a *parabolic subalgebra* $\mathcal{P}$ of $\widehat{\mathcal{D}}^-$ as a subalgebra of the form $\mathcal{P} = \oplus_{j\in\mathbb{Z}}\mathcal{P}_j$, where $\mathcal{P}_j = \widehat{\mathcal{D}}_j^-$ if $j \geq 0$, and $\mathcal{P}_j \neq 0$ for some $j < 0$.

For each positive integer k we have $\mathcal{P}_{-k} = \{t^{-k}h(D-k/2) \mid h(w) \in I_{-k}\}$, where I_{-k} is some subspace of $\mathbb{C}[w]^{(\bar{k})}$. Given $p(w) \in \mathbb{C}[w]^{(\bar{k})}$ and $f(w) \in \mathbb{C}[w]^{(1)}$, we have $f(D), t^{-k}p(D-k/2) \in \widehat{\mathcal{D}}^-$. One calculates that

$$\begin{aligned}\left[f(D), t^{-k}p(D-k/2)\right] &= t^{-k}(f(D-k)-f(D))p(D-k/2)\\ &= t^{-k}g(D-k/2)p(D-k/2)\end{aligned} \tag{3.26}$$

where $g(w) = f(w-k/2) - f(w+k/2)$. As f ranges over all odd polynomials, $g(w)$ ranges over all even polynomials. Thus equation (3.26) implies that if $p(w) \in I_{-k}$ then $\mathbb{C}[w]^{(0)}p(w) \subset I_{-k}$. This means that I_{-k} is a submodule of $\mathbb{C}[w]^{(\bar{k})}$ over $\mathbb{C}[w]^{(0)}$, where $\mathbb{C}[w]^{(\bar{k})}$ is regarded as a module over $\mathbb{C}[w]^{(0)}$ by multiplication. Clearly every non-zero submodule of $\mathbb{C}[w]^{(\bar{k})}$ over $\mathbb{C}[w]^{(0)}$ is a free rank 1 submodule generated by a monic polynomial. Denote by $b_k(w)$ such a generator for I_{-k} if $I_{-k} \neq 0$, and let $b_k(w) = 0$ if $I_{-k} = 0$. We call $b_k(w)$ $(k = 1, 2, \ldots)$ the *characteristic polynomials* of $\mathcal{P}$.

Lemma 3.1 *Let $\{b_k, k \in \mathbb{N}\}$ be the sequence of characteristic polynomials of a parabolic subalgebra $\mathcal{P}$ of the Lie algebra $\widehat{\mathcal{D}}^-$. Then*

1) $b_k(w) \in \mathbb{C}[w]^{(\bar{k})}$;

2) $b_k(w)$ divides $wb_{k+1}(w-1/2)$ and $wb_{k+1}(w+1/2)$ for all $k \in \mathbb{N}$;

3) $b_{k+l}(w)$ divides $wb_k(w-l/2)b_l(w+k/2)$ for all $l, k \in \mathbb{N}$;

4) $\mathcal{P}_{-k} \neq 0$ for all $k \in \mathbb{N}$.

In particular, all $b_k(w)$ are non-zero.

Proof. Part 1) follows from the definition of the characteristic polynomials. From the commutation relation

$$\begin{aligned}&\left[t, t^{-k-1}b_{k+1}\left(D-(k+1)/2\right)\right]\\ &\qquad = t^{-k}\left(b_{k+1}\left(D-(k+1)/2\right) - b_{k+1}\left(D-(k-1)/2\right)\right)\end{aligned}$$

we see that $b_k(w-k/2)$ divides

$$b_{k+1}\left(w-(k+1)/2\right) - b_{k+1}\left(w-(k-1)/2\right). \tag{3.27}$$

In particular $b_{k+1} \neq 0$ implies $b_k \neq 0$. From the commutation relation

$$\begin{aligned}&\left[t,\ t^{-k-1}\left(D-(k+1)/2\right)^2 b_{k+1}\left(D-(k+1)/2\right)\right]\\ &\quad = t^{-k}\Big\{\left\{D-(k-1)/2\right)^2\left(b_{k+1}\left(D-(k+1)/2\right) - b_{k+1}\left(D-(k-1)/2\right)\right\}\\ &\qquad -2\left(D-k/2\right)b_{k+1}\left(D-(k+1)/2\right)\Big\}\end{aligned}$$

we see that $b_k\left(w-k/2\right)$ divides

$$\begin{aligned}&\left(w-(k-1)/2\right)^2\left[b_{k+1}\left(w-(k+1)/2\right) - b_{k+1}\left(w-(k-1)/2\right)\right]\\ &\qquad\qquad -2\left(w-k/2\right)b_{k+1}\left(w-(k+1)/2\right).\end{aligned} \tag{3.28}$$

From (3.27) and (3.28), we see that $b_k(w)$ divides $wb_{k+1}(w-1/2)$. Noting that $b_k(w) = \pm b_k(-w)$ and $b_{k+1}(w) = \pm b_{k+1}(-w)$, $b_k(w)$ also divides $wb_{k+1}(w+1/2)$. This proves part 2).

Part 3) can be similarly proved by calculating the following two commutators

$$\left[t^{-k}b_k(D-k/2), t^{-l}b_l(D-l/2)\right]$$

and

$$\left[t^{-k}(D-k/2)^2 b_k(D-k/2), t^{-l}b_l(D-l/2)\right].$$

In particular, $b_k \neq 0$ and $b_l \neq 0$ imply $b_{k+l} \neq 0$. Part 4) follows from 2) and 3). $\square$

Given a monic even polynomial $b = b(w)$, denote by $\widehat{\mathcal{D}}_0^-(b)$ the subspace of $\widehat{\mathcal{D}}_0^-$ spanned by

$$f(D-1/2)b(D-1/2) - f(D+1/2)b(D+1/2) + f(-1/2)b(-1/2)C$$

where $f(w) \in \mathbb{C}[w]^{(0)}$. We have the following proposition.

Proposition 3.1 *Let $\mathcal{P}$ be a parabolic subalgebra of $\widehat{\mathcal{D}}^-$ and let $b = b_1(w)$ be its first characteristic polynomial. Then*

$$[\mathcal{P}, \mathcal{P}] = \widehat{\mathcal{D}}_0^-(b) \bigoplus \bigoplus_{k \neq 0} \mathcal{P}_k.$$

Proof. It follows from $\left[D, t^k f(D)\right] = kt^k f(D)$ that $\mathcal{P}_k \subset [\mathcal{P}, \mathcal{P}]$ for $k \neq 0$. Next we claim $\left[\mathcal{D}_1^-, \mathcal{D}_k^-\right] = \mathcal{D}_{k+1}^-$ for $k \in \mathbb{N}$. Indeed, first consider the case when k is an even integer.

$$\begin{aligned} &\left[t(D+1/2)^{2l}, t^k(D+k/2)\right] \\ &= t^{k+1}\left((D+k+1/2)^{2l}(D+k/2) - (D+1/2)^{2l}(D+k/2+1)\right). \end{aligned} \tag{3.29}$$

Note that the leading term of $(D+k+1/2)^{2l}(D+k/2) - (D+1/2)^{2l}(D+k/2+1)$ is $(2kl-1)D^{2l}$. It is clear that when l ranges over $\mathbb{Z}_+$, the right hand side of (3.29) form a basis of $\widehat{\mathcal{D}}_{k+1}^-$. The case when k is odd can be treated similarly.

Now from the fact that $\mathcal{P}_k = \mathcal{D}_k^-$ for $k \geq 0$, we have

$$\begin{aligned} &[\mathcal{P}_{k+1}, \mathcal{P}_{-k-1}] \subset [[\mathcal{P}_1, \mathcal{P}_k], \mathcal{P}_{-1-k}] \\ &\subset \ [[\mathcal{P}_k, \mathcal{P}_{-k-1}], \mathcal{P}_1] + [[\mathcal{P}_{-k-1}, \mathcal{P}_1], \mathcal{P}_k] \subset [\mathcal{P}_{-1}, \mathcal{P}_1] + [\mathcal{P}_{-k}, \mathcal{P}_k]. \end{aligned}$$

Hence by induction, $[\mathcal{P}, \mathcal{P}]_0 = [\mathcal{P}_1, \mathcal{P}_{-1}]$. A direct computation shows that $[\mathcal{P}_1, \mathcal{P}_{-1}]$ is exactly $\widehat{\mathcal{D}}_0^-(b)$. $\square$

3.2 $\widehat{\mathcal{D}}^{+}$ case

Given a parabolic subalgebra $\mathcal{P} = \bigoplus_{j\in\mathbb{Z}} \mathcal{P}_j$ of $\widehat{\mathcal{D}}^{+}$, we have:

$$\mathcal{P}_{-k} = \{t^{-k}h(D + (-k+1)/2) \mid h(w) \in I_{-k}\} \quad (k \in \mathbb{N}),$$

where I_{-k} is some subspace of $\mathbb{C}[w]^{(1)}$. Given $p(w), f(w) \in \mathbb{C}[w]^{(1)}$ we have: $t^{-k}p(D + (-k+1)/2)$, $f(D+1/2) \in \widehat{\mathcal{D}}^{+}$. One calculates that

$$\begin{aligned} &\left[f\left(D+1/2\right), t^{-k}p\left(D+(1-k)/2\right)\right] \\ &\quad = \; t^{-k}g\left(D+(1-k)/2\right)p\left(D+(1-k)/2\right), \end{aligned} \tag{3.30}$$

where $g(w) = f(w-k/2) - f(w+k/2)$. As f ranges over all odd polynomials, $g(w)$ ranges over all even polynomials. Thus (3.30) implies that if $p(w) \in I_{-k}$ then $p(w)$ multiplied by any even polynomial belongs to I_{-k}. Let $b_k(w)$ $(k \in \mathbb{N})$ be the unique monic odd polynomial in I_{-k} of minimal degree when $I_{-k} \neq 0$ and let $b_k(w) = 0$ when $I_{-k} = 0$. We call $b_k(w)$ $(k = 1, 2, \ldots)$ the *characteristic polynomials* of $\mathcal{P}$.

Lemma 3.2 *Let $\{b_k, k \in \mathbb{N}\}$ be the sequence of characteristic polynomials of a parabolic subalgebra $\mathcal{P}$ of the Lie algebra $\widehat{\mathcal{D}}^{+}$. Then*

1) $b_k(w)$ divides $w(w + (k+1)/2)b_{k+1}(w + 1/2)$ for all $k \in \mathbb{N}$;

2) $b_{k+l}(w)$ divides $wb_k(w + l/2)b_l(w - k/2)$ for all $l, k \in \mathbb{N}$;

3) $\mathcal{P}_{-k} \neq 0$ for all $k \in \mathbb{N}$.

Proof. It follows from

$$\begin{aligned} &[t(D+1), t^{-k-1}b_{k+1}(D-k/2)] \\ &\quad = \; t^{-k}\left((D-k)b_{k+1}(D-k/2) - (D+1)b_{k+1}(D-k/2+1)\right) \end{aligned}$$

that $b_k(w + (-k+1)/2)$ divides

$$(w-k)b_{k+1}(w-k/2) - (w+1)b_{k+1}(w-k/2+1). \tag{3.31}$$

We see that $b_k(w + (-k+1)/2)$ divides

$$(w-k)(w-k/2)^2b_{k+1}(w-k/2) - (w+1)(w-k/2+1)^2b_{k+1}(w-k/2+1) \tag{3.32}$$

by computing $[t(D+1), t^{-k-1}(D-k/2)^2b_{k+1}(D-k/2)]$. Thus $b_k(w + (-k+1)/2)$ divides (3.32) subtracted by (3.31) multiplied with $(w-k/2)^2$ which is equal to $(w+1)(w+(-k+1)/2)b_{k+1}(w-k/2+1)$. This proves 1). The above computation shows that $b_{k+1}(w) \neq 0$ implies $b_k(w) \neq 0$.

Part 2) can be similarly proved by computing the following two commutators

$$[t^{-k}b_k(D + (-k+1)/2), t^{-l}b_l(D + (-l+1)/2)],$$

$$[t^{-k}(D + (-k+1)/2)^2b_k(D + (-k+1)/2), t^{-l}b_l(D + (-l+1)/2)].$$

Similarly it follows that $b_k(w), b_l(w) \neq 0$ implies $b_{k+l}(w) \neq 0$. Now part 3) follows from 1) and 2). $\square$

Lemma 3.3 $\left[\widehat{\mathcal{D}}_1^+, \widehat{\mathcal{D}}_k^+\right] = \widehat{\mathcal{D}}_{k+1}^+ \;\; (k > 1),\; \widehat{\mathcal{D}}_2^+ = \left[\widehat{\mathcal{D}}_1^+, \widehat{\mathcal{D}}_1^+\right] \bigoplus \mathbb{C}\, t^2(D+3/2).$

Proof. First we have

$$\begin{aligned}&\left[t(D+1)^l, t^k(D+(k+1)/2)^m\right] \\ &\quad = \; t^{k+1}\Big((D+k+1)^l(D+(k+1)/2)^m - (D+1)^l(D+(k+3)/2)^m\Big). \end{aligned} \tag{3.33}$$

For odd positive integers l and m, $t(D+1)^l, t^k(D+(k+1)/2)^m \in \widehat{\mathcal{D}}^+$. For $m = 1$, the leading term of the right hand side is $(lk-1)D^l$. When $k > 1$, $(lk-1) \neq 0$ for $l \in \mathbb{N}$. Thus in the case $k > 1$ the right hand side of (3.33) span the whole $\widehat{\mathcal{D}}_{k+1}^+$ when l ranges over all odd positive integers. In the case $k = 1$, the right hand side of (3.33) when l ranges over all odd positive integers together with $t^2(D+1)$ span the whole $\widehat{\mathcal{D}}_2^+$.

On the other hand, putting $k = 1$ in equation (3.33) we see that the right hand side always contains a factor $(D+1)(D+2)$. So $[\widehat{\mathcal{D}}_1^+, \widehat{\mathcal{D}}_1^+]$ does not contain $t^2(D+3/2)$. □

Let $\widehat{\mathcal{D}}_0^+(b_1, b_2)$ denote the subspace of $\widehat{\mathcal{D}}_0^+$ spanned by

$$\begin{aligned}\{&g(D-1/2)b_2(D-1/2) - g(D+3/2)b_2(D+3/2) + 2g(1/2)b_2(1/2)C, \\ &\qquad f(D)b_1(D) - f(D+1)b_1(D+1),\; f, g \in \mathbb{C}[w]^{(1)}\}.\end{aligned}$$

Proposition 3.2 *Let $\mathcal{P}$ be a parabolic subalgebra of $\widehat{\mathcal{D}}^+$ and let $b_i = b_i(w)$ $(i = 1, 2)$ be its first and second characteristic polynomials. Then*

$$[\mathcal{P}, \mathcal{P}] = \widehat{\mathcal{D}}_0^+(b_1, b_2) \bigoplus \bigoplus_{k \neq 0} \mathcal{P}_k.$$

Proof. It follows from $\left[D, z^k f(D)\right] = k z^k f(D)$ and $D \in \widehat{\mathcal{D}}^+$ that $\mathcal{P}_k = [\mathcal{P}, \mathcal{P}]_k$ for $k \neq 0$. Since $\mathcal{P}_k = \mathcal{D}_k^-$ for $k \geq 0$, it follows from Lemma 3.3 that for $k > 1$

$$\begin{aligned}&[\mathcal{P}_{k+1}, \mathcal{P}_{-k-1}] \subset [[\mathcal{P}_1, \mathcal{P}_k], \mathcal{P}_{-1-k}] \\ &\quad \subset \; [[\mathcal{P}_k, \mathcal{P}_{-k-1}], \mathcal{P}_1] + [[\mathcal{P}_{-k-1}, \mathcal{P}_1], \mathcal{P}_k] \subset [\mathcal{P}_{-1}, \mathcal{P}_1] + [\mathcal{P}_{-k}, \mathcal{P}_k].\end{aligned}$$

Hence it follows by induction that

$$[\mathcal{P}, \mathcal{P}]_0 = [\mathcal{P}_1, \mathcal{P}_{-1}] + [\mathcal{P}_2, \mathcal{P}_{-2}]. \tag{3.34}$$

A direct computation shows the right hand side of (3.34) is indeed $\widehat{\mathcal{D}}_0^+(b_1, b_2)$. □

Example 3.1 Let $\mathcal{P} = \{W_k^n \mid n+k \geq 0, k \in \mathbb{Z}, n \in 2\mathbb{Z}_+ + 1\}$ and let $\widehat{\mathcal{P}} = \mathcal{P} \bigoplus \mathbb{C}C$. By using (2.25) it is easy to see that $\mathcal{P}$ is closed under the Lie bracket and thus is a parabolic subalgebra of $\mathcal{D}^+$. Geometrically, $\mathcal{P}$ consists of those differential operators in $\widehat{\mathcal{D}}^+$ which extend to the interior of the circle. It is clear from formula (2.18) that the 2-cocycle Ψ when restricted to $\mathcal{P}$ is trivial. Denote by M_c the generalized Verma module $M(\widehat{\mathcal{D}}^+, \mathcal{P}, \xi_0)$ where ξ_0 has labels $\Delta_i = 0$ for all i and central charge c. Denote by V_c the irreducible quotient of the $\widehat{\mathcal{D}}^+$-module M_c.

4 Characterization of quasifiniteness of HWM's of $\widehat{\mathcal{D}}^-$

Let $\mathfrak{g} = \oplus_{j\in\mathbb{Z}}\mathfrak{g}_j$ (possibly $\dim \mathfrak{g}_j = \infty$) be a $\mathbb{Z}$-graded Lie algebra over $\mathbb{C}$ satisfying $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$, and let $\mathfrak{g}_+ = \oplus_{j>0}\mathfrak{g}_j$. A $\mathfrak{g}$-module V is called *$\mathbb{Z}$-graded* if $V = \oplus_{j\in\mathbb{Z}}V_j$, and $\mathfrak{g}_i V_j \subset V_{j-i}$. A graded $\mathfrak{g}$-module is called *quasifinite* if $\dim V_j < \infty$ for all j.

Given $\Lambda \in \mathfrak{g}_0^*$, a *highest weight module* (HWM) is a $\mathbb{Z}$-graded $\mathfrak{g}$-module $V(\mathfrak{g}, \Lambda) = \oplus_{j\in\mathbb{Z}_+}V_j$ generated by a highest weight vector $v_\Lambda \in V_0$ which satisfies

$$hv_\Lambda = \Lambda(h)v_\Lambda \quad (h \in \mathfrak{g}_0), \qquad \mathfrak{g}_+ v_\Lambda = 0.$$

A non-zero vector $v \in V(\mathfrak{g}, \Lambda)$ is called *singular* if $\mathfrak{g}_+ v = 0$.

A Verma module is defined as

$$M(\mathfrak{g}; \Lambda) = \mathcal{U}(\mathfrak{g}) \bigotimes_{\mathcal{U}(\mathfrak{g}_0 \oplus \mathfrak{g}_+)} \mathbb{C}_\Lambda$$

where $\mathbb{C}_\Lambda$ is the 1-dimensional $(\mathfrak{g}_0 \oplus \mathfrak{g}_+)$-module given by $h \mapsto \Lambda(h)$ if $h \in \mathfrak{g}_0$ and $\mathfrak{g}_+ \mapsto 0$. Here and further $\mathcal{U}(\mathfrak{s})$ stands for the universal enveloping algebra of the Lie algebra $\mathfrak{s}$. Any highest weight module $V(\mathfrak{g}, \Lambda)$ is a quotient module of $M(\mathfrak{g}; \Lambda)$. The irreducible module $L(\mathfrak{g}; \Lambda)$ is the quotient of $M(\mathfrak{g}; \Lambda)$ by the maximal proper graded submodule.

Let $\mathcal{P} = \oplus_j \mathcal{P}_j$ be a parabolic subalgebra of $\mathfrak{g}$, and let $\Lambda \in \mathfrak{g}_0^*$ be such that $\Lambda|_{\mathfrak{g}_0\cap[\mathcal{P},\mathcal{P}]} = 0$. Then the $(\mathfrak{g}_0 \oplus \mathfrak{g}_+)$-module $\mathbb{C}_\Lambda$ extends to a $\mathcal{P}$-module by letting $\mathcal{P}_j$ act as 0 for $j < 0$, and we may construct the highest weight module

$$M(\mathfrak{g}, \mathcal{P}, \Lambda) = \mathcal{U}(\mathfrak{g}) \bigotimes_{\mathcal{U}(\mathcal{P})} \mathbb{C}_\Lambda$$

which is usually referred to as the *generalized Verma module*. Clearly all the highest weight modules are graded.

In the following we consider $\mathfrak{g} = \widehat{\mathcal{D}}^\pm$ and $\xi \in (\widehat{\mathcal{D}}^\pm)_0^*$. Let $b(w)$ be a monic even polynomial (resp. $b_1(w), b_2(w)$ be two monic odd polynomials). Let $\xi \in \left(\widehat{\mathcal{D}}_0^-\right)^*$ be such that $\xi\, |_{\widehat{\mathcal{D}}_0^-(b)}= 0$, (resp. $\xi \in \left(\widehat{\mathcal{D}}_0^+\right)^*$ be such that $\xi\, |_{\widehat{\mathcal{D}}_0^+(b_1,b_2)}= 0$). Consider a parabolic subalgebra $\mathcal{P}$ of $\widehat{\mathcal{D}}^-$ (resp. $\widehat{\mathcal{D}}^+$) whose first characteristic polynomial is $b(w)$ (resp. whose first and second characteristic polynomials are $b_1(w), b_2(w)$). Denote by $M(\widehat{\mathcal{D}}^-; \xi, b)$ (resp. $M(\widehat{\mathcal{D}}^+; \xi, b_1, b_2)$) the generalized Verma module $M(\widehat{\mathcal{D}}^-, \mathcal{P}, \xi)$ (resp. $M(\widehat{\mathcal{D}}^+, \mathcal{P}, \xi)$). The polynomial b (resp. b_1, b_2) does not determine $\mathcal{P}$ uniquely, but for our need, any corresponding parabolic $\mathcal{P}$ will do.

Proposition 4.1 *The following conditions on $\xi \in \left(\widehat{\mathcal{D}}_0^\pm\right)^*$ are equivalent:*

1) *$M(\widehat{\mathcal{D}}^-; \xi, b)$ (resp. $M(\widehat{\mathcal{D}}^+; \xi, b_1, b_2)$) contains a singular vector in its first graded subspace;*

2) $L(\widehat{\mathcal{D}}^{\pm};\xi)$ is quasifinite;

3) $L(\widehat{\mathcal{D}}^{\pm};\xi)$ is a quotient of a generalized Verma module $M(\widehat{\mathcal{D}}^{-};\xi,b)$ (resp. $M(\widehat{\mathcal{D}}^{+};\xi,b_1,b_2)$) for some monic even polynomial b (resp. some monic odd polynomials b_1,b_2).

Proof. We give the proof for the $-$ case. The proof for the $+$ case is essentially the same. 1) $\Rightarrow$ 3). Denote by $(t^{-1}b(D-1/2))v_\xi$ the singular vector where $b(w)$ is a monic even polynomial of minimal degree (note that $t^{-1}b(D-1/2)\in\widehat{\mathcal{D}}^{-}_{-1}$). Then it is easy to see that 3) holds for this particular monic even polynomial b. 3) $\Rightarrow$ 2) and 2) $\Rightarrow$ 1) follow by Lemma 3.1. □

Let $L(\xi)$ be an irreducible QHWM over $\widehat{\mathcal{D}}^{\pm}$. According to Proposition 4.1, we have $(t^{-1}b(D-1/2))v_\xi=0$ for some monic even polynomial $b(w)$ in the $-$ case, and $(t^{-1}b(D))v_\xi=0$ for some monic odd polynomial $b(w)$ in the $+$ case. Such a monic polynomial of minimal degree is uniquely determined by ξ and is called the *characteristic polynomial* of $L(\xi)$.

We shall characterize a weight $\xi\in(\widehat{\mathcal{D}}^{-}_0)^*$ (resp. $\xi\in(\widehat{\mathcal{D}}^{+}_0)^*$) by its labels $\Delta^{-}_n=-\xi(D^n)$ (resp. $\Delta^{+}_n=-\xi((D+1/2)^n)$), where $n\in\mathbb{N}_{odd}=\{1,3,5,\ldots\}$, and the central charge $c=\xi(C)$. Introduce the generating series

$$\Delta^{\pm}_\xi(x)=\sum_{n\in\mathbb{N}_{odd}}\frac{x^n}{n!}\Delta^{\pm}_n.$$

Sometimes we simply write $\Delta^{\pm}(x)$ instead of $\Delta^{\pm}_\xi(x)$, or even drop $\pm$ when no confusion may arise. Clearly we have

$$\Delta^{-}(x) = -\frac{1}{2}\xi\left(e^{xD}-e^{-xD}\right), \tag{4.35}$$

$$\Delta^{+}(x) = -\frac{1}{2}\xi\left(e^{x(D+1/2)}-e^{-x(D+1/2)}\right). \tag{4.36}$$

A *quasipolynomial* is a finite linear combination of functions of the form $p(x)e^{\alpha x}$, where $p(x)$ is a polynomial and $\alpha\in\mathbb{C}$. Quasipolynomials have the following well-known simple characterization: a formal power series is a quasipolynomial if and only if it satisfies a non-trivial linear differential equation with constant coefficients. We have the following characterization of quasi-finiteness of an irreducible module $L(\xi)$.

Theorem 4.1 *A $\widehat{\mathcal{D}}^{\pm}$-module $L(\widehat{\mathcal{D}}^{\pm};\xi)$ is quasifinite if and only if*

$$\Delta^{\pm}(x)=\frac{F(x)}{2\sinh(x/2)}$$

where $F(x)$ is an even quasipolynomial such that $F(0)=0$.

Proof. We prove the + case first. It follows from Propositions 3.2 and 4.1 that $L(\widehat{\mathcal{D}}^+;\xi)$ is quasifinite if and only if there exist two monic odd polynomials $b_1(w)$ and $b_2(w)$ such that for all $l \geq 1$ the following two equations hold:

$$\xi(D^{2l-1}b_1(D) - (D+1)^{2l-1}b_1(D+1)) \quad = \quad 0, \tag{4.37}$$

$$\begin{aligned} \xi\Big((D-1/2)^{2l-1}b_2(D-1/2) \\ -(D+3/2)^{2l-1}b_2(D+3/2) + 2(1/2)^{2l-1}b_2(1/2)C\Big) = 0. \end{aligned} \tag{4.38}$$

Let

$$b_1(w) = \sum_{n=0}^{M} a_n w^{2n+1}, \quad b_2(w) = \sum_{n=0}^{N} c_n w^{2n+1}.$$

Then we can rewrite equations (4.37) and (4.38) as

$$\sum_{n=0}^{M} a_n \left(\sum_{i\in\mathbb{N}_{odd}} \binom{2n+2l}{i} \left(\frac{1}{2}\right)^{2n+2l-i} \Delta_i^+ \right) \quad = \quad 0 \tag{4.39}$$

$$\sum_{n=0}^{M} c_n \left(\sum_{i\in\mathbb{N}_{odd}} \binom{2n+2l}{i} \Delta_i^+ + c\left(\frac{1}{2}\right)^{2n+2l} \right) \quad = \quad 0. \tag{4.40}$$

Let $F(x) = \Delta^+(x)\sinh\frac{x}{2}$ and $G(x) = \Delta^+(x)\sinh x + c\cosh\frac{x}{2}$. It is straightforward to check that equations (4.39) and (4.40) can be equivalently reformulated as follows:

$$\begin{aligned} (\sum_{n=0}^{M} a_n (\frac{d}{dx})^{2n+1})F(x) &= 0 \\ (\sum_{n=0}^{N} c_n (\frac{d}{dx})^{2n+1})G(x) &= 0. \end{aligned}$$

Since $G(x) = (2F(x)+c)\cosh\frac{x}{2}$, we see that $L(\widehat{\mathcal{D}}^+;\xi)$ is quasifinite if and only if $F(x)$ is an even quasipolynomial.

In the $-$ case, it follows from Propositions 3.1 and 4.1 that $L(\widehat{\mathcal{D}}^-;\xi)$ is quasifinite if and only if there exists a monic even polynomial $b(w) = \sum_{n=0}^{M} \beta_n w^{2n}$ such that

$$\xi\left((D-1/2)^{2l}b(D-1/2) - (D+1/2)^{2l}b(D+1/2) + (-1/2)^{2l}b(-1/2)C\right) = 0$$

for all $l \in \mathbb{Z}_+$. As in the + case, one can show that this condition can be reformulated as

$$\left(\sum_{n=0}^{M} \beta_n \left(d^2/dx^2\right)^n\right) H(x) = 0.$$

where

$$H(x) = 2\Delta^-(x)\sinh(x/2) + c\cosh(x/2), \quad H(0) = c. \tag{4.41}$$

Thus $L(\xi)$ is quasifinite if and only if $H(x)$ is an even quasipolynomial such that $H(0)=0$. By (4.41), letting $F(x)=H(x)-c\cosh(x/2)$ completes the proof. □

¿From the proof of Theorem 4.1 we obtain the following corollary.

Corollary 4.1 *Let $L(\widehat{\mathcal{D}}^+;\xi)$ (resp. $L(\widehat{\mathcal{D}}^-;\xi)$) be an irreducible quasifinite highest weight module over $\widehat{\mathcal{D}}^+$ (resp. $\widehat{\mathcal{D}}^-$) with $b(w)$ (resp. $b_1(w)$) as its first characteristic polynomial. Then*

$$F(x)=\Delta^+(x)\sinh\frac{x}{2},\quad H(x)=2\Delta^-(x)\sinh(x/2)+c\cosh(x/2)$$

are even quasipolynomials. Let $F^{(M)}+a_{M-1}F^{(M-1)}+\cdots+a_0=0$ be the minimal order linear differential equation with constant coefficients satisfied by $F(x)$ such that $w^M+a_{M-1}w^{M-1}+\ldots+a_0$ is an odd polynomial. Let $H^{(N)}+\beta_{N-1}H^{(N-1)}+\cdots+\beta_0=0$ be the minimal order linear differential equation with constant coefficients satisfied by $H(x)$ such that $w^N+\beta_{N-1}w^{N-1}+\ldots+\beta_0$ is an even polynomial. Then $b(w)=w^N+\beta_{N-1}w^{N-1}+\ldots+\beta_0, b_1(w)=w^M+a_{M-1}w^{M-1}+\ldots+a_0$.

Given a quasifinite irreducible highest weight $\widehat{\mathcal{D}}^{\pm}$-module V with central charge $c\in\mathbb{C}$ and with $\Delta(x)$ as in Theorem 4.1, write $F(x)+c$ in the $+$ case or $F(x)+c\cosh(x/2)$ in the $-$ case as a finite sum of the form

$$\sum_i p_i(x)\cosh(e_i^+x)+\sum_j q_j(x)\sinh(e_j^-x), \tag{4.42}$$

where $p_i(x)$ (resp. $q_j(x)$) are non-zero even (resp. odd) polynomials and e_i^+ (resp. e_j^-) are distinct complex numbers. Clearly

$$\sum_i p_i(0)=c. \tag{4.43}$$

The expression (4.42) is unique up to a sign of e_i^+ or a simultaneous change of signs of e_j^- and $q_j(x)$. We call e_i^+ (resp. e_j^-) the *even type* (resp. *odd type*) *exponents* of V with *multiplicities* $p_i(x)$ (resp. $q_j(x)$). We denote by e^+ the set of even type exponents e_i^+ with multiplicity $p_i(x)$ and by e^- the set of odd type exponents e_j^- with multiplicity $q_j(x)$. Then the pair (e^+,e^-) determines V uniquely. We shall therefore denote this module by $L(\widehat{\mathcal{D}}^{\pm};e^+,e^-)$. As we shall see, the following class of $\widehat{\mathcal{D}}^{\pm}$-modules is especially important.

Definition 4.1 *A quasifinite irreducible highest weight $\widehat{\mathcal{D}}^{\pm}$-module V with central charge $c\in\mathbb{C}$ is called* primitive *if the multiplicities of its exponents e_i^+ are nonzero constants $n_i\in\mathbb{C}$ and $e^-=\varnothing$. A primitive $\widehat{\mathcal{D}}^-$-module V is called* positive *if $n_i\in\mathbb{N}$ when $e_i\neq\pm\frac{1}{2}$ and $n_i\in\frac{1}{2}\mathbb{N}$ when $e_i=\pm\frac{1}{2}$. A primitive $\widehat{\mathcal{D}}^+$-module V is called* positive *if $n_i\in\mathbb{N}$ when $e_i\neq 0$ and $-\frac{1}{2}n_{i_0}\leq n_i\in\frac{1}{2}\mathbb{Z}$ when $e_i=0$, where i_0 is the index such that $e_{i_0}=1$. (In both $\pm$ cases the central charge $c=\sum_i n_i\in\frac{1}{2}\mathbb{N}$).*

It is convenient to make the following convention.

Convention 4.1 *For a primitive module V of $\widehat{\mathcal{D}}^{\pm}$, we let e stand for the set of (even type) exponents with their multiplicities in e^+ which are not equal to $\pm\frac{1}{2}$ (resp. 0) in the $-$ (resp. $+$) case. The pair (e, c) determines uniquely the module V. We will denote this primitive module by $L(\widehat{\mathcal{D}}^{\pm}; e, c)$.*

5 Embedding of $\widehat{\mathcal{D}}^-$ into infinite rank classical Lie algebras

Let $\mathcal{O}$ be the algebra of all holomorphic functions on $\mathbb{C}$ with topology of uniform convergence on compact sets. Denote

$$\begin{aligned}\mathcal{O}^{(1)} &= \{f \in \mathcal{O} \mid f(w) = -f(-w)\}\\ \mathcal{O}^{(0)} &= \{f \in \mathcal{O} \mid f(w) = f(-w)\}.\end{aligned}$$

We define a completion $\mathcal{D}^{\mathcal{O}}$ of $\mathcal{D}$ consisting of all differential operators of the form $t^j f(D)$ where $f \in \mathcal{O}$ and $j \in \mathbb{Z}$. We similarly define a completion $\mathcal{D}^{\mathcal{O},-}$ (resp. $\mathcal{D}^{\mathcal{O},+}$) of $\mathcal{D}^-$ (resp. $\mathcal{D}^+$) consisting of all differential operators of the form $t^j f(D)$ (resp. $t^j f(D + j/2)$) where $f \in \mathcal{O}$ (resp. $\mathcal{O}^{(\bar{j})}$) and $j \in \mathbb{Z}$.

Note that $t^j f(D)$ acts on $\mathbb{C}[t, t^{-1}]$ by

$$t^j f(D) t^k = f(k) t^{k+j}.$$

Formula (2.17) for the 2-cocycle Ψ on $\mathcal{D}$ (resp. $\mathcal{D}^-$) extends to a 2-cocycle on $\mathcal{D}^{\mathcal{O}}$ (resp. $\mathcal{D}^{\mathcal{O},-}$). We denote the corresponding central extension by $\widehat{\mathcal{D}}^{\mathcal{O}} = \mathcal{D}^{\mathcal{O}} \bigoplus \mathbb{C}C$ and $\widehat{\mathcal{D}}^{\mathcal{O},-} = \mathcal{D}^{\mathcal{O},-} \bigoplus \mathbb{C}C$. The commutation relations (2.16) extend as well.

The vector space $R_m[t, t^{-1}]t^s$ ($s \in \mathbb{C}$) has a basis $v_i = t^{-i+s}$ ($i \in \mathbb{Z}$) over R_m. The Lie algebra $\mathfrak{gl}^{[m]}$ acts on this vector space by (1.6). The Lie algebras $\mathcal{D}$ and $\mathcal{D}^{\mathcal{O}}$ also act on $R_m[t, t^{-1}]$ naturally as differential operators. In this way we obtain a family of embeddings $\phi_s^{[m]}$ of the Lie algebra $\mathcal{D}$ (resp. $\mathcal{D}^{\mathcal{O}}$) to $\mathfrak{gl}^{[m]}$ defined by

$$\phi_s^{[m]}(t^k f(D)) = \sum_{j \in \mathbb{Z}} f(-j + s + u) E_{j-k,j} = \sum_{i=0}^{m} \sum_{j \in \mathbb{Z}} \frac{f^{(i)}(-j + s)}{i!} u^i E_{j-k,j}$$

where $f^{(i)}$ denotes the i-th derivative. When restricted to $\mathcal{D}^-$ and $\mathcal{D}^{\mathcal{O},-}$, we have

$$\phi_s^{[m]}\left(t^k f(D + k/2)\right) = \sum_{i=0}^{m} \sum_{j \in \mathbb{Z}} \frac{f^{(i)}(-j + k/2 + s)}{i!} u^i E_{j-k,j} \text{ for } f \in \mathbb{C}[w]^{(\bar{k})}. \quad (5.44)$$

When restricted to $\mathcal{D}^+$ and $\mathcal{D}^{\mathcal{O},+}$, we have

$$\phi_s^{[m]}\left(t^k f(D + (k+1)/2)\right) = \sum_{i=0}^{m} \sum_{j \in \mathbb{Z}} \frac{f^{(i)}(-j + (k+1)/2 + s)}{i!} u^i E_{j-k,j} \quad (5.45)$$

for $f \in \mathcal{O}^{(1)}$.

Remark 5.1 The principal $\mathbb{Z}$-gradations on $\mathcal{D}^{\pm}$ and $\mathfrak{gl}^{[m]}$ are compatible under the homomorphisms $\phi_s^{[m]}$.

Let

$$\begin{aligned} I_{s,k}^{[m],-} &= \{f \in \mathcal{O}^{(\bar{k})} \mid f^{(i)}(n+k/2+s) = 0 \text{ for all } n \in \mathbb{Z}, i = 0,1,...,m\} \\ I_{s,k}^{[m],+} &= \{f \in \mathcal{O}^{(1)} \mid f^{(i)}(n+(k+1)/2+s) = 0 \text{ for all } n \in \mathbb{Z}, i = 0,1,...,m\} \end{aligned}$$

and let

$$\begin{aligned} J_s^{[m],-} &= \bigoplus_{k\in\mathbb{Z}} \{t^k f(D+k/2) \mid f \in I_{s,k}^{[m],-}\}, \\ J_s^{[m],+} &= \bigoplus_{k\in\mathbb{Z}} \{t^k f(D+(k+1)/2) \mid f \in I_{s,k}^{[m],+}\}. \end{aligned}$$

Now fix $\vec{s} = (s_1, s_2, \ldots, s_N) \in \mathbb{C}^N$, $s_i - s_j \notin \mathbb{Z}$ if $i \neq j$, $s_i + s_j \notin \mathbb{Z}$ for all i, j. Also fix $\vec{m} = (m_1, m_2, \ldots, m_N) \in \mathbb{Z}_+^N$. Let

$$\mathfrak{gl}^{[\vec{m}]} = \bigoplus_{i=1}^{N} \mathfrak{gl}^{[m_i]}$$

and consider the homomorphism

$$\phi_{\vec{s}}^{[\vec{m}]} = \bigoplus_{i=1}^{N} \phi_{s_i}^{[m_i],\pm} : \mathcal{D}^{\mathcal{O},\pm} \longrightarrow \mathfrak{gl}^{[\vec{m}]}.$$

Proposition 5.1 *Given $\vec{s}$ and $\vec{m}$ as above, we have the following exact sequence of Lie algebras:*

$$0 \longrightarrow J_{\vec{s}}^{[\vec{m}],\pm} \longrightarrow \mathcal{D}^{\mathcal{O},\pm} \xrightarrow{\phi_{\vec{s}}^{[\vec{m}]}} \mathfrak{gl}^{[\vec{m}]} \longrightarrow 0$$

where $J_{\vec{s}}^{[\vec{m}],\pm} = \cap_{i=1}^{N} J_{s_i}^{[m_i],\pm}$.

Proof. We will prove the proposition only in the $-$ case. The proof in the $+$ case is parallel. For the sake of simplicity of notations, we prove it in the case $N = 1$: $\vec{m} = m \in \mathbb{Z}_+$ and $\vec{s} = s \in \mathbb{C}$ ($s \notin \mathbb{Z}/2$ by the assumption on $\vec{s}$). The general case is similar.

It is clear from the definition of $J_s^{[m]}$ that $\ker \phi_s^{[m]} = J_s^{[m]}$. To show the surjectivity of $\phi_s^{[m]}$, it suffices to find a preimage of

$$g = \sum_{i=0}^{m} \sum_{j\in\mathbb{Z}} \frac{p_{ij}}{i!} u^j E_{j-k,j} \in \mathfrak{gl}^{[m]} \quad (p_{ij} \in \mathbb{C})$$

for a fixed $k \in \mathbb{Z}$. We need to quote the following well-known theorem: for every discrete sequence of points in $\mathbb{C}$ and a non-negative integer m there exists $p(w) \in \mathcal{O}$ having prescribed values of its first m derivatives at these points.

Since $s \notin \mathbb{Z}/2$, the two sequences $\{-j+\frac{k}{2}+s\}_{j\in\mathbb{Z}}$ and $\{j-\frac{k}{2}-s\}_{j\in\mathbb{Z}}$ are disjoint. Thus there exists $p(w) \in \mathcal{O}$ such that

$$p^{(i)}(-j+k/2+s) = p_{ij}, \quad p^{(i)}(j-k/2-s) = (-1)^{k+i+1}p_{ij}.$$

Now let

$$f(w) = \frac{p(w)-(-1)^k p(-w)}{2} \in \mathcal{O}^{(\bar{k})}.$$

Then $t^k f(D+k/2)$ is the preimage of g via $\phi_s^{[m]}$. □

Now we want to extend the homomorphism $\phi_s^{[m]}$ to a homomorphism between the central extensions of the corresponding Lie algebras. Introduce the following functions:

$$\eta_j(x;\mu) = \frac{1}{2j!}x^j(e^{\mu x}+(-1)^j e^{-\mu x}) \quad (j \in \mathbb{Z}_+, \mu \in \mathbb{C}).$$

The functions $\eta_j(x;\mu)$ satisfy:

$$\eta_j(-x;\mu) = \eta_j(x;\mu), \quad \eta_j(x;-\mu) = (-1)^j\eta_j(x;\mu), \quad \eta_0(x;\mu) = \cosh(\mu x). \quad (5.46)$$

Note that, being an even quasipolynomial, $F(x)$ in Theorem 4.1 is a finite linear combination of the functions $\eta_j(x;\mu)$.

Proposition 5.2 *1) The $\mathbb{C}$-linear map $\widehat{\phi}_s^{[m]} : \widehat{\mathcal{D}}^+ \longrightarrow \widehat{\mathfrak{gl}}^{[m]}$ defined by*

$$\begin{aligned} \phi_s^{[m]}(C) &= 1 & (5.47)\\ \widehat{\phi}_s^{[m]}|_{\widehat{\mathcal{D}}_j^+} &= \phi_s^{[m]}|_{\mathcal{D}_j^+} \quad \text{if } j \neq 0 \\ \widehat{\phi}_s^{[m]}\left(e^{x(D+1/2)} - e^{-x(D+1/2)}\right) &= \phi_s^{[m]}\left(e^{x(D+1/2)} - e^{-x(D+1/2)}\right) \\ & \quad -\frac{\cosh sx - 1}{\sinh\frac{x}{2}}\mathbf{1} - \sum_{j=1}^{m}\frac{u^j\eta_j(x;s)}{2\sinh\frac{x}{2}} & (5.48) \end{aligned}$$

is a homomorphism of Lie algebras over $\mathbb{C}$.

2) The $\mathbb{C}$-linear map $\widehat{\phi}_s^{[m]} : \widehat{\mathcal{D}}^- \longrightarrow \widehat{\mathfrak{gl}}^{[m]}$ defined by

$$\widehat{\phi}_s^{[m]}(C) = \mathbf{1} \in R_m, \quad \widehat{\phi}_s^{[m]}|_{\widehat{\mathcal{D}}_j^-} = \phi_s^{[m]}|_{\mathcal{D}_j^-} \quad (j \neq 0), \quad (5.49)$$

$$\begin{aligned} \widehat{\phi}_s^{[m]}\left(e^{xD} - e^{-xD}\right) &= \phi_s^{[m]}\left(e^{xD}-e^{-xD}\right) - \frac{\cosh(s-1/2)x - \cosh(x/2)}{\sinh(x/2)}\mathbf{1} \\ & \quad -\sum_{j=1}^{m}\frac{u^j\eta_j(x;s-1/2)}{\sinh(x/2)} & (5.50) \end{aligned}$$

is a homomorphism of Lie algebras over $\mathbb{C}$.

Proof. We will prove part 1). The proof of part 2) is similar. Part 1) follows directly from the computation of the difference

$$\widehat{\phi}_s^{[m]}\left(e^{x(D+1/2)} - e^{-x(D+1/2)}\right)$$

by using the following lemma (also cf. Proposition 4.4, [KR1]). □

Lemma 5.1 *The $\mathbb{C}$-linear map $\widehat{\phi}_s^{[m]} : \widehat{\mathcal{D}} \longrightarrow \widehat{\mathfrak{gl}}^{[m]}$ defined by*

$$\begin{aligned}
\phi_s^{[m]}(C) &= 1 \\
\widehat{\phi}_s^{[m]}|_{\widehat{\mathcal{D}}_j} &= \phi_s^{[m]}|_{\mathcal{D}_j} \qquad \text{if } j \neq 0 \\
\widehat{\phi}_s^{[m]}\left(e^{x(D+1/2)}\right) &= \phi_s^{[m]}\left(e^{x(D+1/2)}\right) \\
&\quad - \frac{e^{(s+1/2)x} - e^{x/2}}{e^x - 1}\mathbf{1} - \sum_{i=1}^{m} \frac{x^i e^{(s+1/2)x}}{e^x - 1} u^i/i!.
\end{aligned} \tag{5.51}$$

is a homomorphism of Lie algebras over $\mathbb{C}$.

Proof. Introduce two formal variables α, β and let $x = \alpha + \beta$. It suffices to check that for a given $k \in \mathbb{N}$

$$\left[\widehat{\phi}_s^{[m]}(t^k e^{\alpha(D+1/2)}), \widehat{\phi}_s^{[m]}(t^{-k} e^{\beta(D+1/2)})\right] = \widehat{\phi}_s^{[m]}\left[(t^k e^{\alpha(D+1/2)}), (t^{-k} e^{\beta(D+1/2)})\right]. \tag{5.52}$$

By a straightforward computation using (5.45) we obtain

$$\begin{aligned}
&\left[\widehat{\phi}_s^{[m]}(t^k e^{\alpha(D+1/2)}), \widehat{\phi}_s^{[m]}(t^{-k} e^{\beta(D+1/2)})\right] \\
&= \left[\phi_s^{[m]}(t^k e^{\alpha(D+1/2)}), \phi_s^{[m]}(t^{-k} e^{\beta(D+1/2)})\right] \\
&= \left(e^{-\alpha k} - e^{\beta k}\right)\left(\textstyle\sum_{j\in\mathbb{Z}} e^{x(-j+s+1/2)} E_{j,j} + e^{xu} e^{x(s+1/2)} \frac{1}{1-e^x}\right) \\
&= \left(e^{-\alpha k} - e^{\beta k}\right)\left(\phi_s^{[m]}\left(e^{x(D+1/2)}\right) + \frac{e^{x(s+1/2)}}{1-e^x}\mathbf{1} + \textstyle\sum_{j+1}^{m} \frac{x^i e^{x(s+1/2)}}{1-e^x} u^i/j!\right).
\end{aligned} \tag{5.53}$$

On the other hand, using (2.16) we have

$$\left[t^k e^{\alpha(D+1/2)}, t^{-k} e^{\beta(D+1/2)}\right] = \left(e^{-\alpha k} - e^{\beta k}\right)\left(e^{x(D+1/2)} + \frac{e^{x/2}}{1-e^x} C\right). \tag{5.54}$$

By applying $\widehat{\phi}_s^{[m]}$ to (5.54) and then comparing with (5.53) with the help of (5.51), we obtain (5.52). □

The homomorphism $\phi_s^{[m]} : \mathcal{D}^{\mathcal{O},-} \longrightarrow \mathfrak{gl}^{[m]}$ is defined for any $s \in \mathbb{C}$. However for $s \in \mathbb{Z}/2$, it is no longer surjective. The case $s = 0$ is described by the following proposition.

Proposition 5.3 *We have the following exact sequence of Lie algebras:*

$$0 \longrightarrow J_0^{[m],-} \longrightarrow \mathcal{D}^{\mathcal{O},-} \xrightarrow{\phi_0^{[m]}} \overline{b}_\infty^{-[m]} \longrightarrow 0$$
$$0 \longrightarrow J_0^{[m],+} \longrightarrow \mathcal{D}^{\mathcal{O},+} \xrightarrow{\phi_0^{[m]}} \overline{d}_\infty^{[m]} \longrightarrow 0.$$

Proof. By the definition of $\phi_0^{[m]}$, it is easy to see that the image of $\phi_0^{[m]}$ lies in $\overline{b}_\infty^{-[m]}$ (resp. $\overline{d}_\infty^{[m]}$). The proof of the rest of the proposition is similar to that of Proposition 5.1. □

Similarly for $s = \frac{1}{2}$ we have the following proposition.

Proposition 5.4 *We have the following exact sequence of Lie algebras:*

$$0 \longrightarrow J_{1/2}^{[m],-} \longrightarrow \mathcal{D}^{\mathcal{O},-} \xrightarrow{\phi_{1/2}^{[m]}} \overline{c}_\infty^{[m]} \longrightarrow 0$$
$$0 \longrightarrow J_{-1/2}^{[m],+} \longrightarrow \mathcal{D}^{\mathcal{O},+} \xrightarrow{\phi_{-1/2}^{[m]}} \overline{b}_\infty^{+[m]} \longrightarrow 0.$$

Remark 5.2 For $s \in \mathbb{Z}$, the image of $\widehat{\mathcal{D}}^-$ (resp. $\widehat{\mathcal{D}}^+$) under the homomorphism $\phi_s^{[m]}$ is $\nu^s(\overline{b}_\infty^{-[m]})$ (resp. $\nu^s(\overline{d}_\infty^{[m]})$) (recall that ν is defined in (1.1)). For $s \in \frac{1}{2} + \mathbb{Z}$, the image of $\widehat{\mathcal{D}}^-$ (resp. $\widehat{\mathcal{D}}^+$) under $\phi_s^{[m]}$ is $\nu^{s-1/2}(\overline{c}_\infty^{[m]})$ (resp. $\nu^{s+1/2}(\overline{b}_\infty^{+[m]})$). Hence we will only need to consider $s = 0, \frac{1}{2}$ in the $-$ case and $s = 0, -\frac{1}{2}$ in the $+$ case whenever $s \in \mathbb{Z}/2$ throughout the paper. Note that the principal $\mathbb{Z}$-gradation of $\widehat{\mathcal{D}}^+$ is compatible with the gradation of type $(2, 1, 1, \ldots)$ on $\overline{d}_\infty$ via the homomorphism ϕ_0.

Definition 5.1 *We say that the vector $\vec{s} = (s_1, s_2, \ldots, s_N) \in \mathbb{C}^N$ satisfies the $(\star-)$ (resp. $(\star+)$) condition if $s_i \in \mathbb{Z}$ implies $s_i = 0$, $s_i \in \frac{1}{2} + \mathbb{Z}$ implies $s_i = \frac{1}{2}$ (resp. $-\frac{1}{2}$), and $s_i \neq \pm s_j \mod \mathbb{Z}$ for $i \neq j$.*

Given $\vec{m} = (m_1, \ldots, m_N) \in \mathbb{Z}_+^N$ and $\vec{s} = (s_1, s_2, \ldots, s_N) \in \mathbb{C}^N$ satisfying the $(\star-)$ (resp. $(\star+)$) condition, combining Propositions 5.1, 5.3 and 5.4, we obtain a homomorphism of Lie algebras over $\mathbb{C}$:

$$\widehat{\phi}_{\vec{s}}^{[\vec{m}]} = \bigoplus_{i=1}^N \widehat{\phi}_{s_i}^{[m_i]} : \widehat{\mathcal{D}}^\pm \longrightarrow \mathfrak{g}^{[\vec{m}]} := \bigoplus_{i=1}^N \mathfrak{g}^{[m_i]}, \qquad (5.55)$$

where **the following consistency condition is always assumed throughout the paper**: in the $-$ case,

$$\mathfrak{g}^{[m_i]} = \begin{cases} b_\infty^{[m_i]}, & \text{if } s_i = 0 \\ c_\infty^{[m_i]}, & \text{if } s_i = \frac{1}{2} \\ \widehat{\mathfrak{gl}}^{[m_i]}, & \text{if } s_i \notin \mathbb{Z}/2, \end{cases}$$

while in the $+$ case,

$$\mathfrak{g}^{[m_i]} = \begin{cases} d_\infty^{[m_i]}, & \text{if } s_i = 0 \\ \tilde{b}_\infty^{[m_i]}, & \text{if } s_i = -\frac{1}{2} \\ \widehat{\mathfrak{gl}}^{[m_i]}, & \text{if } s_i \notin \mathbb{Z}/2. \end{cases}$$

Furthermore, we can prove the following proposition in the same way as Proposition 5.1.

Proposition 5.5 *The homomorphism $\widehat{\phi}_{\vec{s}}^{[\vec{m}]}$ extends to a surjective homomorphism of Lie algebras over $\mathbb{C}$ which is denoted again by $\widehat{\phi}_{\vec{s}}^{[\vec{m}]}$:*

$$\widehat{\phi}_{\vec{s}}^{[\vec{m}]} = \bigoplus_{i=1}^{N} \widehat{\phi}_{s_i}^{[m_i]} : \widehat{\mathcal{D}}^{\mathcal{O},\pm} \longrightarrow \mathfrak{g}^{[\vec{m}]}.$$

6 Realization of QHWM's of $\widehat{\mathcal{D}}^{\pm}$

Let $\mathfrak{g}^{[m]}$ stand for $\widehat{\mathfrak{gl}}^{[m]}$, or one of its classical Lie subalgebras. The proof of the following simple proposition is standard.

Proposition 6.1 *The $\mathfrak{g}^{[m]}$-module $L(\mathfrak{g}^{[m]};\Lambda)$ is quasifinite if and only if all but finitely many of the ${}^*h_k^{(j)}$ are zero, where $*$ represents a, b, c or d depending on whether $\mathfrak{g}^{[m]}$ is $\widehat{\mathfrak{gl}}^{[m]}$, $b_\infty^{[m]}$, $\tilde{b}_\infty^{[m]}$, $c_\infty^{[m]}$ or $d_\infty^{[m]}$.*

Take a quasifinite $\Lambda(i) \in (\mathfrak{g}_0^{[m_i]})^*$ for each $i = 1, \ldots, N$, and let $L\left(\mathfrak{g}^{[m_i]};\Lambda(i)\right)$ be the corresponding irreducible $\mathfrak{g}^{[m_i]}$-module. Then the outer tensor product

$$L\left(\mathfrak{g}^{[\vec{m}]};\vec{\Lambda}\right) \equiv \bigotimes_{i=1}^{N} L\left(\mathfrak{g}^{[m_i]};\Lambda(i)\right)$$

is an irreducible $\mathfrak{g}^{[\vec{m}]}$-module. The module $L\left(\mathfrak{g}^{[\vec{m}]};\vec{\Lambda}\right)$ can be regarded as a $\widehat{\mathcal{D}}^{\pm}$-module via the homomorphism $\widehat{\phi}_{\vec{s}}^{[\vec{m}]}$ given by (5.55), which we shall denote by $L_{\vec{s}}^{[\vec{m}],\pm}(\vec{\Lambda})$.

We will need a technical lemma whose proof is analogous to that of Proposition 4.3 in [KR1].

Lemma 6.1 *Let V be a quasifinite $\widehat{\mathcal{D}}^{\pm}$-module. Then the action of $\widehat{\mathcal{D}}^{\pm}$ on V naturally extends to the action of $\widehat{\mathcal{D}}_k^{\mathcal{O},\pm}$ on V for any $k \neq 0$.*

Theorem 6.1 *Let V be a quasifinite $\mathfrak{g}^{[\vec{m}]}$-module, which can be regarded as a quasifinite $\widehat{\mathcal{D}}^{\pm}$-module via the homomorphism $\widehat{\phi}_{\vec{s}}^{[\vec{m}]}$. Then any $\widehat{\mathcal{D}}^{\pm}$-submodule of V is also a $\mathfrak{g}^{[\vec{m}]}$-submodule. In particular, the $\widehat{\mathcal{D}}^{\pm}$-modules $L_{\vec{s}}^{[\vec{m}],\pm}(\vec{\Lambda})$ are irreducible if $\vec{s} = (s_1, s_2, \ldots, s_N)$ satisfies the $(\star\pm)$ condition.*

Proof. Take any $\widehat{\mathcal{D}}^{\pm}$-submodule W of V. By the Lemma 6.1 we can extend the action of $\widehat{\mathcal{D}}^{\pm}$ to $\widehat{\mathcal{D}}_j^{\mathcal{O},\pm}$ $(j \neq 0)$. Then by Proposition 5.5, we see that the subspace W is preserved by the action of the graded subspace $\mathfrak{g}_j^{[\vec{m}]}$ $(j \neq 0)$ of $\mathfrak{g}^{[\vec{m}]}$. Since $\mathfrak{g}^{[\vec{m}]}$ coincides with its derived algebra, W is preserved by the action of the whole $\mathfrak{g}^{[\vec{m}]}$. $\square$

We will show that in fact all the quasifinite $\widehat{\mathcal{D}}^{\pm}$-modules can be realized as some $L_{\vec{s}}^{[\vec{m}],\pm}(\vec{\Lambda})$. But first let us calculate the generating function $\Delta_{m,s,\Lambda}^{\pm}(x)$ of highest weight for $L_{\vec{s}}^{[\vec{m}],\pm}(\vec{\Lambda})$ in some typical cases.

Proposition 6.2 *Consider the embedding $\widehat{\phi}_s^{[m]} : \widehat{\mathcal{D}}^{-} \longrightarrow \widehat{\mathfrak{gl}}^{[m]}$ with $s \notin \mathbb{Z}/2$. The $\widehat{\mathfrak{gl}}^{[m]}$-module $L(\widehat{\mathfrak{gl}}^{[m]}; \Lambda)$ regarded as a $\widehat{\mathcal{D}}^{-}$-module is isomorphic to $L(\widehat{\mathcal{D}}^{-}; e^{+}, e^{-})$ where e^{+} and e^{-} consist of the exponents $s - i - 1/2$ $(i \in \mathbb{Z})$ with multiplicities*

$$\sum_{0 \le j \le m,\, j \text{ even}} {}^a h_i^{(j)} x^j / j! \quad \textit{and} \quad \sum_{0 \le j \le m,\, j \text{ odd}} {}^a h_i^{(j)} x^j / j! \tag{6.56}$$

respectively (see Section 1.1 for notations; the exponents with zero multiplicities are dropped).

Proof. By Theorem 6.1 and Proposition 6.1 the $\mathcal{D}^{-}$-module $L_{\vec{s}}^{[\vec{m}],-}(\vec{\lambda})$ is an irreducible quasifinite highest weight module. By formula (5.49) we see the central charge $c = c_0$. By applying Λ to (5.50) and using formulas (4.36) and (5.44) we obtain:

$$\begin{aligned}
2\Delta_{m,s,\Lambda}^{-}(x) &= -\sum_{j=0}^{m} \sum_{i \in \mathbb{Z}} ({}^a\lambda_i^{(j)}/j!) x^j \left(e^{(s-i)x} - (-1)^j e^{(i-s)x} \right) \\
&\quad + c_0 \frac{\cosh(s-1/2)x - \cosh(x/2)}{\sinh(x/2)} + \sum_{j=1}^{m} \frac{c_j \eta_j(x; s-1/2)}{\sinh(x/2)} \\
&= -\sum_{j=0}^{m} \sum_{i \in \mathbb{Z}} ({}^a\lambda_i^{(j)}/j!) \frac{(\eta_j(x; s-i+1/2) - \eta_j(x; s-i-1/2))}{\sinh(x/2)} \\
&\quad + c_0 \frac{\cosh(s-1/2)x - \cosh(x/2)}{\sinh(x/2)} + \sum_{j=1}^{m} \frac{c_j \eta_j(x; s-1/2)}{\sinh(x/2)} \\
&\overset{(1)}{=} \frac{\sum_{j=0}^{m} \sum_{i \in \mathbb{Z}} {}^a h_i^{(j)} \eta_j(x; s-i-1/2)}{\sinh(x/2)} - \frac{c_0 \cosh(x/2)}{\sinh(x/2)}.
\end{aligned}$$

The identity (1) above is obtained by shifting the index i to $i+1$ in the first half of the first summation of the left hand side of (1). Now the proposition follows from the definition of exponents and their multiplicities. $\square$

Proposition 6.3 *Consider the embedding* $\widehat{\phi}_0^{[m]} : \widehat{\mathcal{D}}^- \longrightarrow b_\infty^{[m]}$. *The* $b_\infty^{[m]}$*-module* $L(b_\infty^{[m]}; \Lambda)$ *regarded as a* $\widehat{\mathcal{D}}^-$*-module is isomorphic to* $L(\widehat{\mathcal{D}}^-; e^+, e^-)$ *where* e^+ *and* e^- *consist of the exponents* $-i-1/2$ $(i \in \mathbb{Z}_+)$ *with multiplicities*

$$\sum_{0 \le j \le m,\, j\ even} {}^b\tilde{h}_i^{(j)} x^j/j! \quad and \quad \sum_{0 \le j \le m,\, j\ odd} {}^b\tilde{h}_i^{(j)} x^j/j! \tag{6.57}$$

respectively, where ${}^b\tilde{h}_i^{(j)} = {}^b h_i^{(j)}$ $(i > 0)$ *and* ${}^b\tilde{h}_0^{(j)} = \frac{1}{2}\, {}^b h_0^{(j)}$ *(see Section 1.2 for notations).*

Proof. We will only need to calculate $\Delta^-_{m,s,\Lambda}(x)$. The rest of the statement is clear, cf. the proof of Proposition 6.2. We have:

$$\begin{aligned}
2\Delta^-_{m,s,\Lambda}(x) \overset{(1)}{=}\ & -\sum_{j=0}^{m}\sum_{i\in\mathbb{Z}}(\Lambda(u^jE_{ii})/j!)\frac{(\eta_j(x;-i+1/2)-\eta_j(x;-i-1/2))}{\sinh(x/2)} \\
& + \sum_{j=1}^{m}\frac{c_j\eta_j(x;-1/2)}{\sinh(x/2)} \\
\overset{(2)}{=}\ & \frac{\sum_{j=0}^{m}\sum_{i\in\mathbb{Z}}\Lambda(u^jE_{ii}-u^jE_{i+1,i+1})\eta_j(x;-i-1/2)}{\sinh(x/2)} \\
& + \sum_{j=1}^{m}\frac{c_j\eta_j(x;-1/2)}{\sinh(x/2)} \\
\overset{(3)}{=}\ & \frac{\sum_{j=0}^{m}\sum_{i\in\mathbb{Z}_+}{}^b\tilde{h}_i^{(j)}\eta_j(x;-i-1/2)}{\sinh(x/2)} - \frac{c_0\cosh(x/2)}{\sinh(x/2)}.
\end{aligned}$$

The identity (1) is obtained from a similar identity in the proof of Proposition 6.2 by putting $s = 0$. The identity (2) is obtained by shifting the index i to $i+1$ in the first half of the first summation of the left hand side of (2). The identity (3) is obtained by splitting the summation into two, $\sum_{i\in\mathbb{Z}} = \sum_{i\in\mathbb{Z}_+} + \sum_{i\in-\mathbb{N}}$, changing the index i to $-i-1$ in the second summation, and then using formulas (1.7) and (5.46). $\square$

Proposition 6.4 *Consider the embedding* $\widehat{\phi}_{1/2}^{[m]} : \widehat{\mathcal{D}}^- \longrightarrow c_\infty^{[m]}$. *The* $c_\infty^{[m]}$*-module* $L(c_\infty^{[m]}; \Lambda)$ *regarded as a* $\widehat{\mathcal{D}}^-$*-module is isomorphic to* $L(\widehat{\mathcal{D}}^-; e^+, e^-)$ *where* e^+ *and* e^- *consist of the exponents* $s-i-1/2$ $(i \in \mathbb{Z}_+)$ *with multiplicities*

$$\sum_{0 \le j \le m,\, j\ even} {}^c h_i^{(j)} x^j/j! \quad and \quad \sum_{0 \le j \le m,\, j\ odd} {}^c h_i^{(j)} x^j/j! \tag{6.58}$$

respectively (see Section 1.3 for more notations).

Proof. It suffices to calculate $\Delta^-_{m,s,\Lambda}(x)$:

$$\begin{aligned}
2\Delta^-_{m,s,\Lambda}(x) &\overset{(1)}{=} -\sum_{j=0}^{m}\sum_{i\in\mathbb{Z}}(\Lambda(u^jE_{ii})/j!)\frac{(\eta_j(x;-i+1)-\eta_j(x;-i))}{\sinh(x/2)} \\
&\quad +c_0\frac{1-\cosh(x/2)}{\sinh(x/2)}+\sum_{j=1}^{m}\frac{c_j\eta_j(x;0)}{\sinh(x/2)} \\
&\overset{(2)}{=} \frac{\sum_{j=0}^{m}\sum_{i\in\mathbb{Z}}\Lambda(u^jE_{ii}-u^jE_{i+1,i+1})\eta_j(x;-i)}{\sinh(x/2)} \\
&\quad +c_0\frac{1-\cosh(x/2)}{\sinh(x/2)}+\sum_{j=1}^{m}\frac{c_j\eta_j(x;0)}{\sinh(x/2)} \\
&\overset{(3)}{=} \frac{\sum_{j=0}^{m}\sum_{i\in\mathbb{Z}_+}{}^{c}h_i^{(j)}\eta_j(x;-i)}{\sinh(x/2)}-\frac{c_0\cosh(x/2)}{\sinh(x/2)}.
\end{aligned}$$

The identity (1) is obtained from a similar identity in the proof of Proposition 6.2 by putting $s=\frac{1}{2}$. The identity (2) is obtained by shifting the index i to $i+1$ in the first half of the first summation of the left hand side of (2). The identity (3) is obtained by splitting the summation into two, $\sum_{i\in\mathbb{Z}}=\sum_{i\in\mathbb{Z}_+}+\sum_{i\in-\mathbb{N}}$, changing the index i to $-i$ in the second summation, and then using formulas(1.11) and (5.46). □

Proposition 6.5 *Consider the embedding $\widehat{\phi}_s^{[m]}:\widehat{\mathcal{D}}^+\longrightarrow\widehat{\mathfrak{gl}}^{[m]}$ with $s\notin\mathbb{Z}/2$. The $\widehat{\mathfrak{gl}}^{[m]}$-module $L(\widehat{\mathfrak{gl}}^{[m]};\Lambda)$ regarded as a $\widehat{\mathcal{D}}^+$-module is isomorphic to $L(\widehat{\mathcal{D}}^+;e^+,e^-)$ where e^+ and e^- consist of the exponents $s-i$ $(i\in\mathbb{Z})$ with multiplicities*

$$\sum_{0\le j\le m,\, j\ even}{}^{a}h_i^{(j)}x^j/j! \quad and \quad \sum_{0\le j\le m,\, j\ odd}{}^{a}h_i^{(j)}x^j/j! \tag{6.59}$$

respectively (see Section 1.1 for notations).

Proof. By Theorem 6.1 and Proposition 6.1 $L_{\vec{s}}^{[\vec{m}],+}(\vec{\lambda})$ is an irreducible quasifinite highest weight $\widehat{\mathcal{D}}^+$-module. By formula (5.47) we see the central charge $c=c_0$. By applying Λ to (5.48) and using formulas (4.36) and (5.45) we obtain

$$\begin{aligned}
2\Delta^+_{m,s,\Lambda}(x) &= -\sum_{j=0}^{m}\sum_{i\in\mathbb{Z}}({}^{a}\lambda_i^{(j)}/j!)x^j\left(e^{(s-i+1/2)x}-(-1)^je^{(i-s-1/2)x}\right) \\
&\quad +c_0\frac{\cosh(sx)-1}{\sinh(x/2)}+\sum_{j=1}^{m}\frac{c_j\eta_j(x;s)}{\sinh(x/2)} \\
&= -\sum_{j=0}^{m}\sum_{i\in\mathbb{Z}}({}^{a}\lambda_i^{(j)}/j!)\frac{(\eta_j(x;s-i+1)-\eta_j(x;s-i))}{\sinh(x/2)} \\
&\quad +c_0\frac{\cosh(sx)-1}{\sinh(x/2)}+\sum_{j=1}^{m}\frac{c_j\eta_j(x;s)}{\sinh(x/2)}
\end{aligned}$$

$$\overset{(1)}{=} \frac{\sum_{j=0}^m \sum_{i\in\mathbb{Z}} {}^a h_i^{(j)} \eta_j(x; s-i)}{\sinh(x/2)} - \frac{c_0}{\sinh(x/2)}.$$

The identity (1) above is obtained by shifting the index i to $i+1$ in the first half of the first summation of the left hand side of (1). This completes the proof of this proposition. $\square$

Proposition 6.6 *Consider the embedding $\widehat{\phi}_0^{[m]} : \widehat{\mathcal{D}}^+ \longrightarrow d_\infty^{[m]}$. The $d_\infty^{[m]}$-module $L(d_\infty^{[m]}; \Lambda)$ regarded as a $\widehat{\mathcal{D}}^+$-module is isomorphic to $L(\widehat{\mathcal{D}}^-; e^+, e^-)$ where e^+ and e^- consist of the exponents $-i$ ($i \in \mathbb{Z}_+$) with multiplicities*

$$\sum_{0\le j\le m,\, j \text{ even}} {}^d\tilde{h}_i^{(j)} x^j/j! \quad \text{and} \quad \sum_{0\le j\le m,\, j \text{ odd}} {}^d\tilde{h}_i^{(j)} x^j/j! \tag{6.60}$$

respectively, where ${}^d\tilde{h}_i^{(j)} = {}^d h_i^{(j)}$ ($i>0$) and ${}^d\tilde{h}_0^{(j)} = \frac{1}{2}\left({}^d h_0^{(j)} - {}^d h_1^{(j)}\right)$ (see Section 1.4 for notations).

Proof. We will only need to calculate $\Delta^+_{m,s,\Lambda}(x)$. The rest of the statement is clear, cf. the proof of Proposition 6.2.

$$\begin{aligned} 2\Delta^+_{m,s,\Lambda}(x) &\overset{(1)}{=} -\sum_{j=0}^m \sum_{i\in\mathbb{Z}} (\Lambda(u^j E_{ii})/j!) \frac{(\eta_j(x; -i+1) - \eta_j(x; -i))}{\sinh(x/2)} + \sum_{j=1}^m \frac{c_j \eta_j(x;0)}{\sinh(x/2)} \\ &\overset{(2)}{=} \frac{\sum_{j=0}^m \sum_{i\in\mathbb{Z}} \Lambda(u^j E_{ii} - u^j E_{i+1,i+1}) \eta_j(x; -i)}{\sinh(x/2)} + \sum_{j=1}^m \frac{c_j \eta_j(x;0)}{\sinh(x/2)} \\ &\overset{(3)}{=} \frac{\sum_{j=0}^m \left(\sum_{i\in\mathbb{N}} {}^d h_i^{(j)} \eta_j(x;-i) + \frac{1}{2}(h_0^{(j)} - h_1^{(j)}) \eta_j(x;0) \right)}{\sinh(x/2)} - \frac{c_0}{\sinh(x/2)}. \end{aligned}$$

The identity (1) is obtained from a similar identity in the proof of Proposition 6.2 by putting $s=0$. The identity (2) is obtained by shifting the index i to $i+1$ in the first half of the first summation of the left hand side of (2). The identity (3) is obtained by splitting the summation into two, $\sum_{i\in\mathbb{Z}} = \sum_{i\in\mathbb{Z}_+} + \sum_{i\in-\mathbb{N}}$, changing the index i to $-i$ in the second summation, and then using formulas (1.14) and (5.46). $\square$

Proposition 6.7 *Consider the embedding $\widehat{\phi}_{-1/2}^{[m]} : \widehat{\mathcal{D}}^+ \longrightarrow \tilde{b}_\infty^{[m]}$. The $\tilde{b}_\infty^{[m]}$-module $L(\tilde{b}_\infty^{[m]}; \Lambda)$ regarded as a $\widehat{\mathcal{D}}^+$-module is isomorphic to $L(\widehat{\mathcal{D}}^-; e^+, e^-)$ where e^+ and e^- consist of the exponents $-i-1/2$ ($i \in \mathbb{Z}_+$) with multiplicities*

$$\sum_{0\le j\le m,\, j \text{ even}} {}^b\tilde{h}_i^{(j)} x^j/j! \quad \text{and} \quad \sum_{0\le j\le m,\, j \text{ odd}} {}^b\tilde{h}_i^{(j)} x^j/j! \tag{6.61}$$

respectively, where ${}^b\tilde{h}_i^{(j)} = {}^b h_i^{(j)}$ ($i>0$) and ${}^b\tilde{h}_0^{(j)} = \frac{1}{2}\, {}^b h_0^{(j)}$ (see Section 1.2 for notations).

Proof. Again it suffices to calculate $\Delta^+_{m,s,\Lambda}(x)$.

$$\begin{aligned}
2\Delta^+_{m,s,\Lambda}(x) &\overset{(1)}{=} -\sum_{j=0}^{m}\sum_{i\in\mathbb{Z}}(\Lambda(u^jE_{ii})/j!)\frac{(\eta_j(x;-i+1/2)-\eta_j(x;-i-1/2))}{\sinh(x/2)}\\
&\quad +c_0\frac{\cosh(x/2)-1}{\sinh(x/2)}+\sum_{j=1}^{m}\frac{c_j\eta_j(x;-1/2)}{\sinh(x/2)}\\
&\overset{(2)}{=} \frac{\sum_{j=0}^m\sum_{i\in\mathbb{Z}}\Lambda(u^jE_{ii}-u^jE_{i+1,i+1})\eta_j(x;-i-1/2)}{\sinh(x/2)}\\
&\quad +c_0\frac{\cosh(x/2)-1}{\sinh(x/2)}+\sum_{j=1}^{m}\frac{c_j\eta_j(x;-1/2)}{\sinh(x/2)}\\
&\overset{(3)}{=} \frac{\sum_{j=0}^m\sum_{i\in\mathbb{Z}_+}{}^b\tilde{h}_i^{(j)}\eta_j(x;-i-1/2)}{\sinh(x/2)}-\frac{c_0}{\sinh(x/2)}.
\end{aligned}$$

The identity (1) is obtained from a similar identity in the proof of Proposition 6.2 by putting $s=-\frac{1}{2}$. The identity (2) is obtained by shifting the index i to $i+1$ in the first half of the first summation of the left hand side of (2). The identity (3) is obtained by splitting the summation into two, $\sum_{i\in\mathbb{Z}}=\sum_{i\in\mathbb{Z}_+}+\sum_{i\in-\mathbb{N}}$, changing the index i to $-i-1$ in the second summation, and then using formulas(1.7) and (5.46). $\square$

Take an irreducible quasifinite highest weight $\widehat{\mathcal{D}}^-$-module V with central charge c and

$$\Delta(x)=\frac{F(x)}{2\sinh(x/2)}$$

where $F(x)$ is an even quasipolynomial such that $F(0)=0$ (see Theorem 4.1). We may write

$$F(x)+c\cosh(x/2)=\sum_{s\in\mathbb{C}}\sum_{j=0}^{m_s}a_{s,j}\eta_j(x;s-1/2) \tag{6.62}$$

where $a_{s,j}\in\mathbb{C}$ and $a_{s,j}\neq 0$ for only finitely many $s\in\mathbb{C}$. We can fix the ambiguities in expressing $F(x)$ in the form (6.62) caused by the symmetries (5.46) by the following rules in choosing the parameter s: when $s\in\mathbb{Z}$, we require $s\leq 0$; when $s\in\frac{1}{2}+\mathbb{Z}$, we require $s\leq\frac{1}{2}$; when $s\notin\frac{1}{2}\mathbb{Z}$, we require that Im $s>0$ if Im $s\neq 0$, or $s-[s]<\frac{1}{2}$ if $s\in\mathbb{R}$, where Im s denotes the imaginary part of s and $[s]$ the closest integer to s which is not larger than s.

Decompose the set $\{s\in\mathbb{C}\mid a_{s,j}\neq 0 \text{ for some } j\}$ into a disjoint union of equivalence classes under the equivalence condition: $s\sim s'$ if and only if $s=\pm s' \pmod{\mathbb{Z}}$. Pick a representative s in an equivalence class S such that $s=0$ if the equivalence class lies in $\mathbb{Z}$ and $s=\frac{1}{2}$ if the equivalence class lies in $\frac{1}{2}+\mathbb{Z}$. Let $S=\{s,s-k_1,s-k_2,\ldots\}$ be such an equivalence class and let $m=\max_{s\in S}m_s$. Put $k_0=0$. It is easy to see by the rules in picking the parameter s that if $s=0$ or $\frac{1}{2}$, then $k_1,k_2,\dots\in\mathbb{N}$.

We associate to S the $\mathfrak{g}^{[m]}$-module $L_s^{[m]}(\lambda_s)$ in one of the following three ways:

First, if $s \notin \mathbb{Z}/2$, let ${}^a h_{k_r}^{(j)} = a_{s+k_r,j}$ $(j = 0, \ldots, m_s, r = 0, 1, 2, \ldots)$. We associate to S a $\widehat{\mathfrak{gl}}^{[m]}$-module $L_s^{[m]}(\lambda_S)$ with central charges and labels

$$c_j = \sum_{k_r} {}^a h_{k_r}^{(j)}, \quad {}^a\lambda_i^{(j)} = \sum_{k_r \geq i} {}^a\tilde{h}_{k_r}^{(j)}$$

where ${}^a\tilde{h}_k^{(j)} = {}^a h_k^{(j)} - c_j \delta_{k,0}$.

Secondly, if $s = 0$, let ${}^b\tilde{h}_{k_r}^{(j)} = a_{k_r,j}(j = 0, \ldots, m_0, r \in \mathbb{Z}_+)$. We associate to S a $b_\infty^{[m_0]}$-module $L_0^{[m_0]}(\lambda_S)$ with central charges and labels

$$c_j = \sum_{k_r} {}^b\tilde{h}_{k_r}^{(j)} \ (j \text{ even}), \qquad c_j = 0 \ (j \text{ odd})$$

$${}^b\lambda_0^{(j)} = \sum_{k_r \geq 0} h_{k_r}^{(j)} \ (j \text{ odd}), \qquad {}^b\lambda_i^{(j)} = \sum_{k_r \geq i} h_{k_r}^{(j)}$$

where $i \in \mathbb{N}, j = 0, \ldots, m_0$.

Thirdly, if $s = \frac{1}{2}$, let ${}^c h_{k_r}^{(j)} = a_{1/2+k_r,j}(j = 0, \ldots, m_{1/2}, r \in \mathbb{Z}_+)$. We associate to S the $c_\infty^{[m_{1/2}]}$-module $L_{\frac{1}{2}}^{[m_{1/2}]}(\lambda_S)$ with central charges and labels

$$c_j = \sum_{k_r} {}^c h_{k_r}^{(j)} \ (j \text{ even}), \quad c_j = 0 \ (j \text{ odd}), \quad {}^c\lambda_i^{(j)} = \sum_{k_r \geq i} {}^c h_{k_r}^{(j)}$$

where $i \in \mathbb{N}, j = 0, \ldots, m_{1/2}$.

Denote by $\{s_1, s_2, \ldots, s_N\}$ a set of representatives of equivalence classes in the set $\{s \in \mathbb{C} \mid a_{s,j} \neq 0 \text{ for some } j\}$. By Theorem 6.1, the $\widehat{\mathcal{D}}^-$-module $L_{\vec{s}}^{[\vec{m}],-}(\vec{\lambda})$ is irreducible for $\vec{s} = (s_1, s_2, \ldots, s_N)$ satisfying the $(\star-)$ condition. Then we have

$$\Delta^-_{\vec{m},\vec{s},\vec{\lambda}}(x) = \sum_i \Delta^-_{m_i,s_i,\lambda(i)}(x), \quad c = \sum_i c_0(i).$$

Summarizing the above, together with Theorem 6.1, Propositions 6.2, 6.3 and 6.4, we have proved the following theorem for the $-$ case. Similarly one can prove the $+$ case with the help of Theorem 6.1, Propositions 6.5, 6.6 and 6.7.

Theorem 6.2 *Let V be an irreducible quasifinite highest weight $\widehat{\mathcal{D}}^{\pm}$-module with central charge c and*

$$\Delta^{\pm}(x) = \frac{F(x)}{2\sinh(x/2)}$$

for some even quasipolynomial $F(x)$ which is written in the form (6.62). Then V is isomorphic to the tensor product of the modules $L_s^{[m],\pm}(\Lambda_S)$ for different equivalence classes S.

Remark 6.1 A different choice of representative $s \notin \frac{1}{2}\mathbb{Z}$ in an equivalence class S has the effect of shifting $\widehat{\mathfrak{gl}}^{[m]}$ via the automorphism ν^i for some i. It is not difficult to see that any irreducible quasifinite highest weight module $L(\widehat{\mathcal{D}}^{\pm};\xi)$ can be obtained as above in an essentially unique way up to the shift. Note that we have always put $c_j = 0$ (j odd) when $s = 0$ or $\frac{1}{2}$ in the $-$ case (resp. $-\frac{1}{2}$ in the $+$ case). We could have defined $b_\infty^{[m]}$, $\tilde{b}_\infty^{[m]}$, $c_\infty^{[m]}$ and $d_\infty^{[m]}$ to be the central extensions of $\overline{b}_\infty^{-[m]}$, $\overline{b}_\infty^{+[m]}$, $\overline{c}_\infty^{[m]}$ and $\overline{d}_\infty^{[m]}$ by $R'_m = \bigoplus_{j=1}^{[m/2]} \mathbb{C}u^{2j} \subset R_m$. We have made our choice for the convenience of notations.

7 Unitary QHWM's of $\widehat{\mathcal{D}}^{\pm}$

Consider the following anti-linear anti-involution ω of the Lie algebra $\widehat{\mathcal{D}}$:

$$\omega(t^k f(D)) = t^{-k}\overline{f}(D-k), \quad \omega(C) = C,$$

where $\overline{f}$ denotes the complex conjugate. Note that $\widehat{\mathcal{D}}^{\pm}$ is ω-invariant:

$$\begin{aligned} \omega(t^k f(D+k/2)) &= t^{-k}\overline{f}(D-k/2) \\ \omega(t^k f(D+(k+1)/2)) &= t^{-k}\overline{f}(D+(-k+1)/2). \end{aligned}$$

The goal of this section is to classify all unitary quasifinite highest weight modules over $\widehat{\mathcal{D}}^{\pm}$ with respect to the anti-involution ω.

Remark 7.1 1) The anti-involution ω on $\mathcal{D}$ is compatible with the standard anti-involution ω' of $\mathfrak{gl}$ (given by the complex conjugate transpose of a matrix) under the homomorphism $\phi_s = \phi_s^{[0]} : \mathcal{D}^{\mathcal{O},-} \longrightarrow \mathfrak{gl}$:

$$\omega'(\phi_s(t^k f(D))) = \phi_{\bar{s}}(\omega(t^k f(D))),$$

where $s \in \mathbb{C}$ and $\bar{s}$ denotes its complex conjugate.

2) With respect to the anti-involution ω' of $\widehat{\mathfrak{gl}}$ defined by $\omega'(A) = {}^t\overline{A}$ and $\omega'(c) = \overline{c}$, a highest weight $\widehat{\mathfrak{gl}}$-module with highest weight Λ and central charge c is unitary if and only if all ${}^a h_i$ are non-negative integers and $c = \sum_i {}^a h_i$. This follows from the unitarity of finite dimensional modules over $\mathfrak{gl}_n$.

Similarly, a highest weight module of b_∞ or $\tilde{b}_\infty$ with highest weight Λ and central charge c is unitary with respect to ω' if and only if the numbers ${}^b h_i$ ($i \in \mathbb{Z}_+$) are non-negative integers and $c = \frac{1}{2}{}^b h_0 + \sum_{i \geq 1} {}^b h_i$.

A highest weight c_∞-module with highest weight Λ and central charge c is unitary if and only if the numbers ${}^c h_i$ ($i \in \mathbb{Z}_+$) are non-negative integers and $c = \sum_{i \geq 0} {}^c h_i$.

A highest weight d_∞-module with highest weight Λ and central charge c is unitary if and only if the numbers ${}^d h_i$ ($i \in \mathbb{Z}_+$) are non-negative integers and $c = ({}^d h_0 + {}^d h_1)/2 + \sum_{i \geq 2} {}^d h_i$.

Remark 7.2 *A highest weight d_∞-module with highest weight Λ and central charge c is unitary if and only if the numbers ${}^d h_i \in \mathbb{Z}_+$ $(i \in \mathbb{N})$, $c \in \frac{1}{2}\mathbb{Z}_+$ and $c \geq {}^d h_1/2 + \sum_{i \geq 2} {}^d h_i$.*

Lemma 7.1 *1) Let V be a unitary quasifinite highest weight module over $\widehat{\mathcal{D}}^-$, and let $b(w)$ be its characteristic polynomial. Then $b(w)$ has only real roots and any non-zero root of $b(w)$ is simple; 0 is a double root of $b(w)$ if it is a root.*

2) Let V be a unitary quasifinite highest weight module over $\widehat{\mathcal{D}}^+$, and let $b(w)$ be its characteristic polynomial. Then $b(w)$ has only real simple roots.

Proof. We will prove part 1). The proof of part 2) is similar.

Let v_Λ be a highest weight vector of V. Since $b(w)$ is an even polynomial, we may assume that $\deg b(w) = 2n$, and $b(w) = g(w^2)$ for some polynomial $g(w)$. Then the first graded subspace V_1 of V has a basis

$$\{t^{-1}(D - 1/2)^{2l} v_\Lambda, \quad 0 \leq l < n\}.$$

Consider the action of $S = -\frac{1}{12}(4D^3 + 4\Delta_3 + 1) \in \widehat{\mathcal{D}}^-$ on End V_1. It is straightforward to check that

$$S^j \left(t^{-1} v_\Lambda\right) = \left(t^{-1}(D - 1/2)^{2j}\right) v_\Lambda.$$

It follows that $g(S)(t^{-1}v_\Lambda) = 0$ and $\{S^j(t^{-1}v_\Lambda), \quad 0 \leq j < n\}$ is a basis of V_1. We conclude from the above that $g(w)$ is the characteristic polynomial of the operator S on V_1. Since the operator S is self-adjoint, all the roots of $g(w)$ are real.

Similarly, let $b_2(w)$ be the polynomial of minimal degree such that

$$t^{-2} b_2(D - 1) v_\Lambda = 0.$$

Since $b_2(w)$ is an odd polynomial, we may assume $b_2(w) = w g_2(w^2)$ for some polynomial $g_2(w)$ of degree k. Consider the action of $T = -\frac{1}{6}(D^3 + \Delta_3 + 2) \in \widehat{\mathcal{D}}^-$ on the second graded subspace V_2 of V. One can check that

$$T^j \left(\left(t^{-2}(D - 1)\right) v_\Lambda\right) = \left(t^{-2}(D - 1)^{2j+1}\right) v_\Lambda.$$

It follows that $g_2(T)((t^{-2}(D-1))v_\Lambda) = 0$ and $\{T^j(t^{-2}(D-1))v_\Lambda), \quad 0 \leq j < n\}$ is a basis of V_2.

We then conclude that $g_2(w)$ is the characteristic polynomial of the operator T on V_2. Since the operator T is self-adjoint, all the roots of $g_2(w)$ are real. By Lemma 3.1, $b(w)$ divides $w b_2(w - 1/2)$. Given a root a of $b(w)$, it follows that

$$a(a - 1/2) g_2 \left((a - 1/2)^2\right) = 0.$$

So either a is equal to $0, \frac{1}{2}$, or $(a-1/2)^2$ is real. But we know already that a^2 is real as well since a^2 is a root of $g(w)$. Thus a is real.

Now let r be a root of $g(w)$ of multiplicity m, then we may write $g(w) = h(w)(w-r)^m$ for some polynomial $h(w)$. Denote $v = (S-r)^{m-1}h(S)(t^{-1}v_\lambda)$. It is a non-zero vector in V_1. However, on the other hand,

$$(v, v) = (c(S)(t^{-1}v_\lambda), (S-r)^{2m-2}c(S)(t^{-1}v_\lambda)) = 0 \quad \text{if } m \geq 2.$$

Hence the unitarity condition forces $m = 1$. Consequently, any non-zero root of $b(w) = g(w^2)$ has multiplicity 1 and 0 is a double root if it is a root. □

Lemma 7.2 *If $L_{\vec{s}}^{[\vec{m}],\pm}(\vec{\Lambda})$ is a unitary quasifinite irreducible highest weight module over $\widehat{\mathcal{D}}^{\pm}$, then $m = 0$.*

Proof. By Theorem 6.2, Corollary 4.1 and Lemma 7.1, we see that $a_{s,j} = 0$ $(j \geq 1)$ in (6.62) in the $-$ case. The argument for the $+$ case is the same. □

Theorem 7.1 *1) An irreducible quasifinite highest weight $\widehat{\mathcal{D}}^-$-module with central charge c is unitary if and only if it is positive primitive with real exponents e_i, or equivalently if and only if*

$$\Delta(x) = \frac{\sum_i n_i(\cosh(e_i x) - \cosh(x/2))}{2\sinh(x/2)} \tag{7.63}$$

where e_i are distinct real numbers different from $\pm\frac{1}{2}$, $n_i \in \mathbb{N}$, $c \in \frac{1}{2}\mathbb{Z}_+$ and $c \geq \sum_i n_i$.

2) A quasifinite irreducible highest weight $\widehat{\mathcal{D}}^+$-module with central charge c is unitary if and only if it is positive primitive with real exponents e_i or equivalently if and only if

$$\Delta(x) = \frac{\sum_i n_i(\cosh(e_i x) - 1)}{2\sinh(x/2)}$$

where e_i are distinct non-zero real numbers, $n_i \in \mathbb{N}$, $c \in \frac{1}{2}\mathbb{Z}_+$ and $c \geq \sum_{i\neq i_0} n_i + n_{i_0}/2$. Here i_0 is the index such that $e_{i_0} = 1$.

3) Any unitary quasifinite highest weight $\widehat{\mathcal{D}}^{\pm}$-module can be obtained by taking a tensor product of N unitary irreducible highest weight modules over $\mathfrak{g}^{[\vec{0}]}$ and restricting to $\widehat{\mathcal{D}}^{\pm}$ via $\widehat{\phi}_{\vec{s}}^{[\vec{0}]}$, where $\vec{s} = (s_1, s_2, \ldots, s_N)$ $(s_i \in \mathbb{R})$ satisfies the $(\star\pm)$ condition.

Proof. The "only if" part of 1) was implied by the Lemma 7.2. The part 3) follows from Remark 7.1. Now the "if" part of 1) follows from part 3) since by Theorem 6.2 we have realized all irreducible unitary quasifinite highest weight $\widehat{\mathcal{D}}^-$-modules with $\Delta(x)$ of the form (7.63). The proof of part 2) is similar to part 1) (cf. Remark 7.2). $\square$

By Theorem 6.2, a positive primitive $\widehat{\mathcal{D}}^{\pm}$-module is the pullback of a tensor product of N unitary irreducible highest weight modules V_i over $\mathfrak{g}^{[0]}$ via a homomorphism $\widehat{\phi}_{\vec{s}}^{[0]}$, where $\vec{s} = (s_1, s_2, \ldots, s_N)$ ($s_i \in \mathbb{C}$) satisfies the $(\star\pm)$ condition. Since the $\mathbb{Z}$-gradations of $\widehat{\mathcal{D}}^{\pm}$ and $\mathfrak{g}^{[0]}$ are compatible under the homomorphism $\widehat{\phi}_{\vec{s}}^{[0]}$, the q-character formula of a positive primitive module of $\widehat{\mathcal{D}}^{\pm}$ is given by the product of the q-character formulas of V_i, which are in turn given by formulas (1.5), (1.8), and (1.12) in $-$ case (resp. (15.115) in $+$ case).

Remark 7.3 One can introduce a tensor category $\mathcal{O}_{b,\pm}$ consisting of all positive primitive modules of $\widehat{\mathcal{D}}^{\pm}$ with non-negative half-integral central charge as was done for $\mathcal{W}_{1+\infty}$ in [W1] (also cf. [W2]) equipped with the usual tensor product. One can show that $\mathcal{O}_{b,\pm}$ is a semisimple tensor category. A tensor product of two irreducible modules in such a category is decomposed into an (infinite) sum of irreducibles with finite multiplicities. Certain reciprocity laws can be established between these multiplicities and some branching coefficients in finite dimensional Lie groups as a formal consequence of various duality results we will establish in the next sections.

8 FFR's of QHWM's over $\widehat{\mathcal{D}}^-$ with $c \in -\mathbb{N}/2$

8.1 Dual pair $(O(2l), \widehat{\mathcal{D}}^-)$

In this subsection we study the free field realizations (FFR's) of certain primitive $\widehat{\mathcal{D}}^-$-modules with negative integral central charges in some bosonic Fock spaces and establish certain duality results. These duality results are intimately related to duality results obtained in [W2]. We refer the reader to [W2] for more detail. In the case of $\widehat{\mathcal{D}}$ similar results were obtained in [KR2].

Let us take a pair of bosonic ghost fields

$$\gamma^+(z) = \sum_{n \in \frac{1}{2}+\mathbb{Z}} \gamma_n^+ z^{-n-\frac{1}{2}}, \quad \gamma^-(z) = \sum_{n \in \frac{1}{2}+\mathbb{Z}} \gamma_n^- z^{-n-\frac{1}{2}}$$

with the following commutation relations

$$[\gamma_m^+, \gamma_n^-] = \delta_{m+n,0}.$$

We define the Fock space $\mathcal{F}^{\bigotimes -1}$ of the fields $\gamma^+(z)$ and $\gamma^-(z)$, generated by the vacuum $|0\rangle$, satisfying

$$\gamma_n^+|0\rangle = \gamma_n^-|0\rangle = 0, \quad n \in \frac{1}{2} + \mathbb{Z}_+.$$

Now we take l pairs of bosonic ghost fields $\gamma^{+,p}(z), \gamma^{-,p}(z)$ $(p = 1, \dots, l)$ and consider the corresponding Fock space $\mathcal{F}^{\bigotimes -l}$.

It is well known and straightforward to verify that

$$E(z,w) \equiv \sum_{i,j\in\mathbb{Z}} E_{ij} z^{i-1} w^{-j} = -\sum_{p=1}^{l} :\gamma^{+,p}(z)\gamma^{-,p}(w): \tag{8.64}$$

defines an action of $\widehat{\mathfrak{gl}}$ on $\mathcal{F}^{\bigotimes -l}$ with central charge $-l$.

It is known [FF] that the Fourier components of the following generating functions

$$e^{pq}_{**}(z) \equiv \sum_{i,j\in\mathbb{Z}} e^{pq}_{**}(n) z^{-n-1} =: \gamma^{+,p}(z)\gamma^{+,q}(-z): \quad (p \neq q) \tag{8.65}$$

$$e^{pq}(z) \equiv \sum_{i,j\in\mathbb{Z}} e^{pq}(n) z^{-n-1} =: \gamma^{-,p}(z)\gamma^{-,q}(-z): \quad (p \neq q) \tag{8.66}$$

$$e^{pq}_{*}(z) \equiv \sum_{i,j\in\mathbb{Z}} e^{pq}_{*}(n) z^{-n-1} =: \gamma^{+,p}(z)\gamma^{-,q}(z): \quad (p, q = 1, \dots, l) \tag{8.67}$$

span an affine algebra $\mathfrak{gl}^{(2)}(2l)$ of type $A^{(2)}_{2l-1}$ of central charge -1. The horizontal subalgebra of the affine algebra $\mathfrak{gl}^{(2)}(2l)$ spanned by $e^{pq}_{**}(0), e^{pq}_{*}(0)$, $e^{pq}(0)$ $(p, q = 1, \dots, l)$ is isomorphic to the Lie algebra $\mathfrak{so}(2l)$. The Borel subalgebra of $\mathfrak{so}(2l)$ is taken to be the one spanned by $e^{pq}_{*}(p \leq q), e^{pq}_{**}, p, q = 1, \dots, l$ and the Cartan subalgebra is spanned by $e^{pp}_{*}, p = 1, \dots, l$. The action of $\mathfrak{so}(2l)$ can be lifted to an action of $SO(2l)$ on $\mathcal{F}^{\bigotimes -l}$ and then extends to $O(2l)$ naturally. For example the operator which commutes with $\psi^{\pm,k}(z)$ $(k = 1, \dots, l-1)$ and sends $\psi^{\pm,l}(z)$ to $\psi^{\mp,l}(z)$ lies in $\in O(2l) - SO(2l)$. Put

$$\begin{aligned}&\sum_{i,j\in\mathbb{Z}} (E_{ij} - (-1)^{i+j} E_{1-j,1-i}) z^{i-1} w^{-j} \\ &= \textstyle\sum_{k=1}^{l} \left(\gamma^{+,k}(z)\gamma^{-,k}(w) - \gamma^{+,k}(-w)\gamma^{-,k}(-z)\right).\end{aligned} \tag{8.68}$$

The operators $E_{ij} - (-1)^{i+j} E_{1-j,1-i}$ $(i, j \in \mathbb{Z})$ span c_∞ with central charge $-l$. It is known [W2] that the actions of $O(2l)$ and c_∞ commute with each other and form a dual pair on $\mathcal{F}^{\bigotimes -l}$ in the sense of Howe [H1, H2].

$O(2l)$ is a semi-direct product of $SO(2l)$ by $\mathbb{Z}_2$. If λ is a representation of $SO(2l)$ of highest weight $(m_1, m_2, \dots, m_l)$ $(m_l \neq 0)$, then the induced representation of $(m_1, m_2, \dots, m_l)$ to $O(2l)$ is irreducible and its restriction to $SO(2l)$ is a sum of $(m_1, m_2, \dots, m_l)$ and $(m_1, m_2, \dots, -m_l)$. We denote this irreducible representation λ of $O(2l)$ by $(m_1, m_2, \dots, \overline{m_l})$, where m_l is chosen to be greater than 0. If $m_l = 0$, the representation $\lambda = (m_1, m_2, \dots, m_{l-1}, 0)$ extends to two different representations of $O(2l)$, denoted by λ and $\lambda \otimes det$, where det is the 1-dimensional non-trivial representation of $O(2l)$. We denote

$$\begin{aligned}\Sigma(D) = \quad & \{(m_1, m_2, \dots, \overline{m_l}) \mid m_1 \geq m_2 \geq \dots \geq m_l > 0, m_i \in \mathbb{Z}; \\ & (m_1, m_2, \dots, m_{l-1}, 0) \bigotimes det, \\ & (m_1, m_2, \dots, m_{l-1}, 0) \mid m_1 \geq m_2 \geq \dots \geq m_{l-1} \geq 0, m_i \in \mathbb{Z}\}.\end{aligned}$$

Define a map $\Lambda^{\mathfrak{d}\mathfrak{c}}$ from $\Sigma(D)$ to $c_{\infty 0}^{*}$ by sending $\lambda = (m_1, \cdots, \overline{m_l})$ $(m_l > 0)$ to

$$\Lambda^{\mathfrak{d}\mathfrak{c}}(\lambda) = (-l - m_1)^c\widehat{\Lambda}_0 + \sum_{k=1}^{l}(m_k - m_{k+1})^c\widehat{\Lambda}_k \quad (m_{l+1} = 0),$$

sending $(m_1, \cdots, m_j, 0, \cdots, 0)$ $(j < l)$ to

$$\Lambda^{\mathfrak{d}\mathfrak{c}}(\lambda) = (-l - m_1)^c\widehat{\Lambda}_0 + \sum_{k=1}^{j}(m_k - m_{k+1})^c\widehat{\Lambda}_k,$$

and sending $(m_1, \cdots, m_j, 0, \cdots, 0) \bigotimes det$ to

$$\Lambda^{\mathfrak{d}\mathfrak{c}}(\lambda) = (-l - m_1)^c\widehat{\Lambda}_0 + \sum_{k=1}^{j-1}(m_k - m_{k+1})^c\widehat{\Lambda}_k + (m_j - 1)^c\widehat{\Lambda}_j + {}^c\widehat{\Lambda}_{2l-j},$$

if $m_1 \geq \ldots \geq m_j > m_{j+1} = \ldots = m_l = 0$.

The following theorem was proved in [W2].

Theorem 8.1 *We have the following $(O(2l), c_\infty)$-module decomposition:*

$$\mathcal{F}^{\bigotimes -l} = \bigoplus_{\lambda \in \Sigma(D)} V(O(2l); \lambda) \otimes L\left(c_\infty; \Lambda^{\mathfrak{d}\mathfrak{c}}(\lambda), -l\right) \tag{8.69}$$

where $V(O(2l); \lambda)$ is the irreducible $O(2l)$-module parametrized by $\lambda \in \Sigma(D)$ and $L\left(c_\infty; \Lambda^{\mathfrak{d}\mathfrak{c}}(\lambda), -l\right)$ is the irreducible c_∞-module of highest weight $\Lambda^{\mathfrak{d}\mathfrak{c}}(\lambda)$ and central charge $-l$.

Note that $\widehat{\mathcal{D}}^-$ acts on $\mathcal{F}^{\bigotimes -l}$ via the composition of the homomorphism $\widehat{\phi}_{1/2}$ and the action of c_∞ given by (8.68). Introduce the following generating function

$$T^n_{1/2}(z) = \sum_{k \in \mathbb{Z}} T^{n,1/2}_k z^{-k-n-1} \quad (n \in 2\mathbb{Z}_+ + 1)$$

for the basis $T^{n,1/2}_k$ defined in (2.22) of $\widehat{\mathcal{D}}^-$. It follows from Proposition 5.2 that

$$\begin{aligned} c &= -l & (8.70)\\ \widehat{\phi}_{1/2}\left(e^{xD} - e^{-xD}\right) &= \phi_{1/2}\left(e^{xD} - e^{-xD}\right) - l\tanh(x/4). & (8.71)\end{aligned}$$

It follows that

$$\widehat{\phi}_{1/2}(T^{n,1/2}_0) = \phi_{1/2}(T^{n,1/2}_0) - \alpha_n l \tag{8.72}$$

for some constant α_n determined by (8.71) depending on n only.

Lemma 8.1 *One can recast the action of $\widehat{\mathcal{D}}^-$ in terms of $T^n_{1/2}(z)$ as*

$$T^n_{1/2}(z) = -\sum_{k=1}^{l}\left(: \gamma^{+,k}(z)\partial_z^n\gamma^{-,k}(z) : + : \partial_z^n\gamma^{+,k}(-z)\gamma^{-,k}(-z) :\right) - \alpha_n l z^{-n-1}.$$

Proof. We have the following calculations.

$$\begin{aligned}
T^n_{1/2}(z) &= \sum_{k\in\mathbb{Z}} T^{n,1/2}_k z^{-k-n-1} \\
&\overset{(1)}{=} -\sum_{k\in\mathbb{Z}}([-j]_n + (-1)^{k+1}[j-k-1]_n)E_{j-k,j}z^{-k-n-1} - \alpha_n l z^{-n-1} \\
&\overset{(2)}{=} -\sum_{k,j\in\mathbb{Z}}[-j]_n\left(E_{j-k,j} + (-1)^{k+1}E_{1-j,k-j+1}\right)z^{-k-n-1} - \alpha_n l z^{-n-1} \\
&\overset{(3)}{=} -\sum_{k=1}^{l}\left(:\gamma^{+,k}(z)\partial_z^n\gamma^{-,k}(z): + :\partial_z^n\gamma^{+,k}(-z)\gamma^{-,k}(-z):\right) - \alpha_n l z^{-n-1},
\end{aligned}$$

where equation (1) is given by the homomorphism $\widehat{\phi}_{1/2}$ (cf. (5.44) and (8.72)), (2) is obtained by change of variables in the second summation (replacing $j-k-1$ with j), and (3) follows by taking the n-th derivative of (8.68) with respect to w and then putting $w=z$. □

For $\lambda = (m_1, \cdots, \overline{m_l}) \in \Sigma(D)$ $(m_l > 0)$ we let $e(\lambda)$ be the set of the exponents k with multiplicity $m_k - m_{k+1}$ $(k = 1, \ldots, l)$ and the exponent 0 with multiplicity $-l-m_1$, where $m_{l+1} = 0$; for $\lambda = (m_1, \cdots, m_j, 0, \cdots, 0) \in \Sigma(D)$ $(j < l)$, we let $e(\lambda)$ be the set of the exponents k with multiplicity $m_k - m_{k+1}$ $(k = 1, \ldots, j)$ and the exponent 0 with multiplicity $-l-m_1$, where $m_{j+1} = 0$; for $(m_1, \cdots, m_j, 0, \cdots, 0) \bigotimes det \in \Sigma(D)$ $(j < l)$ we let $e(\lambda)$ be the set of the exponents k with multiplicity $m_k - m_{k+1}$ $(k = 1, \ldots, j-1)$, the exponent j with multiplicity $m_j - 1$, the exponent $2l - j$ with multiplicity 1 and the exponent 0 with multiplicity $-l - m_1$.

Now we can state a duality theorem between $O(2l)$ and $\widehat{\mathcal{D}}^-$ (recall Convention 4.1).

Theorem 8.2 *$O(2l)$ and $\widehat{\mathcal{D}}^-$ form a dual pair on $\mathcal{F}^{\bigotimes -l}$. More explicitly we have the following $(O(2l), \widehat{\mathcal{D}}^-)$-module decomposition:*

$$\mathcal{F}^{\bigotimes -l} = \bigoplus_{\lambda\in\Sigma(D)} I_\lambda \equiv \bigoplus_{\lambda\in\Sigma(D)} V(O(2l);\lambda) \otimes L\left(\widehat{\mathcal{D}}^-; e(\lambda), -l\right). \tag{8.73}$$

Proof. Since the actions of $O(2l)$ commutes with c_∞, the action of $O(2l)$ commutes with the action of $\widehat{\mathcal{D}}^-$ given in Lemma 8.1 by Proposition 5.4. So the decomposition of the Fock space into isotypic subspaces with respect to the dual pair $(Sp(2l), c_\infty)$ can be regarded as decomposition with respect to the dual pair $(Sp(2l), \widehat{\mathcal{D}}^-)$ as well. By Theorems 8.1 and 6.1 each isotypic subspace is irreducible under the joint action of $Sp(2l)$ and $\widehat{\mathcal{D}}^-$. By Proposition 6.4 and Theorem 8.1 the highest weight of the representation I_λ with respect to $\widehat{\mathcal{D}}^-$ is given as follow.

$$\Delta_{m,s,\Lambda}(x) = \frac{(-l-m_1) + \sum_{k=1}^{j}(m_k - m_{k+1})\cosh(kx)}{2\sinh(x/2)} - \frac{-l\cosh(x/2)}{2\sinh(x/2)}$$

for $\lambda = (m_1, \cdots, \overline{m_l}) \in \Sigma(D)$ $(m_l > 0)$.

The cases $\lambda = (m_1, \cdots, m_j, 0, \cdots, 0)$ $(j < l)$ and $(m_1, \cdots, m_j, 0, \cdots, 0) \bigotimes det$ can be treated similarly. □

8.2 Dual pair $(O(2l+1), \widehat{\mathcal{D}}^-)$

In this subsection we will realize certain primitive $\widehat{\mathcal{D}}^-$-modules with negative half-odd-integral central charges in some Fock spaces and establish some duality theorems. We need a bosonic field $\chi(z) = \sum_{n \in \frac{1}{2}+\mathbb{Z}} \chi_n z^{-n-\frac{1}{2}}$ which satisfies the following commutation relations:

$$[\chi_m, \chi_n] = (-1)^{m+\frac{1}{2}} \delta_{m,-n}, \quad m, n \in \frac{1}{2} + \mathbb{Z}.$$

Let $\mathcal{F}^{\bigotimes -l-\frac{1}{2}}$ be the tensor product of the Fock space of l pairs of bosonic ghost fields $\gamma^{\pm,k}(z)$ $(k = 1, \ldots, l)$ and the Fock space $\mathcal{F}^{\bigotimes -\frac{1}{2}}$ of $\chi(z)$.

It is known [FF] that the Fourier components of the generating functions (8.65), (8.66), (8.67) and the following generating functions

$$\begin{aligned}
\zeta(z) \equiv \sum_{n\in\mathbb{Z}} \zeta(n) z^{-n-1} &= \ :\chi(z)\chi(-z): \\
e^p(z) \equiv \sum_{n\in\mathbb{Z}} e^p(n) z^{-n-1} &= \ :\gamma^{-,p}(z)\chi(-z): \\
e^p_*(z) \equiv \sum_{n\in\mathbb{Z}} e^p_*(n) z^{-n-1} &= \ :\gamma^{+,p}(z)\chi(z):
\end{aligned}$$

span an affine algebra $A^{(2)}_{2l}$ of central charge -1 when acting on $\mathcal{F}^{\bigotimes -\frac{1}{2}}$. Its horizontal subalgebra is isomorphic to $\mathfrak{so}(2l+1)$. We take the Borel subalgebra $\mathfrak{b}(\mathfrak{so}(2l+1))$ to be the one spanned by $e^{pq}_*(0) (p \leq q), e^p_*(0), e^{pq}_{**}(0), p, q = 1, \ldots, l$. The Cartan subalgebra $\mathfrak{h}(\mathfrak{so}(2l+1))$ is spanned by $e^{pp}_*(0), p = 1, \ldots, l$.

The action of $\mathfrak{so}(2l+1)$ can be lifted to $SO(2l+1)$ on $\mathcal{F}^{\bigotimes -l-\frac{1}{2}}$ and then extends naturally to $O(2l+1)$. For example the operator which commutes with $\Psi^{\pm,k}(z), k = 1, \ldots, l$ and sends $\chi(z)$ to $-\chi(z)$ lies in $O(2l+1) - SO(2l+1)$. Let

$$\begin{aligned}
&\sum_{i,j\in\mathbb{Z}} (E_{ij} - (-1)^{i+j} E_{1-j,1-i}) z^{i-1} w^{-j} \\
&= \ \textstyle\sum_{k=1}^{l} \left(:\gamma^{+,k}(z)\gamma^{-,k}(w): + :\gamma^{+,k}(-w)\gamma^{-,k}(-z): \right) + :\chi(z)\chi(-w): .
\end{aligned} \tag{8.74}$$

The operators $E_{ij} - (-1)^{i+j} E_{1-j,1-i}$ $(i, j \in \mathbb{Z})$ span c_∞ with central charge $-l-1/2$. It is shown [W2] that $O(2l+1)$ and c_∞ form a dual pair on $\mathcal{F}^{\bigotimes -l-\frac{1}{2}}$.

An irreducible modules of $SO(2l+1)$ is parametrized by its highest weight $\lambda = (m_1, \ldots, m_l), m_1 \geq \ldots \geq m_l \geq 0, m_i \in \mathbb{Z}$. It is well known that $O(2l+1)$ is isomorphic to the direct product $SO(2l+1) \times \mathbb{Z}_2$ by sending the minus identity matrix to $-1 \in \mathbb{Z}_2 = \{\pm 1\}$. Denote by det the non-trivial one-dimensional representation of $O(2l+1)$. An irreducible representation λ of $SO(2l+1)$ extends to two different irreducible representations of $O(2l+1)$ by tensoring with the trivial representation and non-trivial representation of $\mathbb{Z}_2$, denoted by λ and $\lambda \bigotimes det$. All irreducible representations of $O(2l+1)$ can be obtained this way. Then we can parametrize irreducible representations of $O(2l+1)$ by $\Sigma(B)$ consisting of highest weights $(m_1, \ldots, m_l)$ and $(m_1, \ldots, m_l) \bigotimes det$.

Define a map $\Lambda^{\mathfrak{bc}}$ from $\Sigma(B)$ to ${c_\infty}_0^*$ by sending $\lambda = (m_1, m_2, \ldots, m_l)$ to

$$\Lambda^{\mathfrak{bc}}(\lambda) = (-l - m_1 - 1/2)^c\widehat{\Lambda}_0 + \sum_{k=1}^{j}(m_k - m_{k+1})^c\widehat{\Lambda}_k$$

and sending $\lambda = (m_1, m_2, \ldots, m_l) \bigotimes det$ to

$$\Lambda^{\mathfrak{bc}}(\lambda) = (-l - m_1 - 1/2)^c\widehat{\Lambda}_0 + \sum_{k=1}^{j-1}(m_k - m_{k+1})^c\widehat{\Lambda}_k + (m_j - 1)^c\widehat{\Lambda}_j + {}^c\widehat{\Lambda}_{2l-j+1},$$

where $m_1 \geq \ldots \geq m_j > m_{j+1} = \ldots = m_l = 0$. The following theorem is proved in [W2].

Theorem 8.3 *We have the following $(O(2l+1), c_\infty)$-module decomposition:*

$$\mathcal{F}^{\bigotimes -l-\frac{1}{2}} = \bigoplus_{\lambda \in \Sigma(B)} V(O(2l+1); \lambda) \otimes L\left(c_\infty; \Lambda^{\mathfrak{bc}}(\lambda), -l - 1/2\right)$$

where $V(O(2l+1); \lambda)$ is the irreducible $O(2l+1)$-module parametrized by $\lambda \in \Sigma(B)$ and $L(c_\infty; \Lambda^{\mathfrak{bc}}(\lambda), -l - 1/2)$ is the irreducible highest weight c_∞-module of highest weight $\Lambda^{\mathfrak{bc}}(\lambda)$ and central charge $-l - 1/2$.

Note that $\widehat{\mathcal{D}}^-$ now acts on $\mathcal{F}^{\bigotimes -l-\frac{1}{2}}$ via the composition of the homomorphism $\widehat{\phi}_{1/2}$ and the action of c_∞ given by formula (8.74). The following lemma is analogous to Lemma 8.1.

Lemma 8.2 *We can recast the action of $\widehat{\mathcal{D}}^-$ in terms of generating function $T^n_{1/2}(z) = \sum_{k \in \mathbb{Z}} T^{n,1/2}_k z^{-k-n-1}$ as*

$$\begin{aligned} T^n_{1/2}(z) &= -\sum_{k=1}^{l}\left(:\gamma^{+,k}(z)\partial_z^n\gamma^{-,k}(z): + :\partial_z^n\gamma^{+,k}(-z)\gamma^{-,k}(-z):\right) \\ &\quad - :\chi(z)\partial_z^n\chi(-z): - \alpha_n(l+1/2)z^{-k-n-1}, \quad n \in 2Z_+ + 1. \end{aligned}$$

For $\lambda = (m_1, \cdots, m_j, 0, \cdots, 0) \in \Sigma(B)$, we let $e(\lambda)$ be the set of the exponents k with multiplicity $m_k - m_{k+1}$ $(k = 1, \ldots, j)$, where $m_{j+1} = 0$, and the exponent 0 with multiplicity $-l - m_1 - 1/2$; for $\lambda = (m_1, \cdots, m_j, 0, \cdots, 0) \bigotimes det \in \Sigma(B)$ we let $e(\lambda)$ be the set of the exponents k with multiplicity $m_k - m_{k+1}$ $(k = 1, \ldots, j-1)$, the exponent j with multiplicity $m_j - 1$, the exponent $2l - j + 1$ with multiplicity 1, and the exponent 0 with multiplicity $-l - m_1 - 1/2$.

Now we have the following duality theorem on the joint action on $\mathcal{F}^{\bigotimes -l-\frac{1}{2}}$ of the dual pair $(O(2l+1), \widehat{\mathcal{D}}^-)$. The proof is similar to that of Theorem 8.2 which is now based on Theorem 8.3 and Proposition 6.4.

Theorem 8.4 *We have the following* $(O(2l+1), \widehat{\mathcal{D}}^-)$*-module decomposition:*

$$\mathcal{F}^{\bigotimes -l-\frac{1}{2}} = \bigoplus_{\lambda \in \Sigma(B)} V(O(2l+1); \lambda) \otimes L\left(\widehat{\mathcal{D}}^-; e(\lambda), -l-1/2\right).$$

9 FFR's of QHWM's over $\widehat{\mathcal{D}}^-$ with $c \in \mathbb{N}$

In this section we will realize certain primitive $\widehat{\mathcal{D}}^-$-modules with positive integral central charges in some fermionic Fock spaces. Similar results for $\widehat{\mathcal{D}}$ were obtained in [FKRW] (see also [KR2]). Let us take a pair of fermionic fields

$$\psi^+(z) = \sum_{n \in \underline{\mathbb{Z}}} \psi_n^+ z^{-n-\frac{1}{2}+\epsilon}, \quad \psi^-(z) = \sum_{n \in \underline{\mathbb{Z}}} \psi_n^- z^{-n-\frac{1}{2}+\epsilon}, \quad \underline{\mathbb{Z}} = \frac{1}{2} + \mathbb{Z} \text{ or } \mathbb{Z}$$

with the following anti-commutation relations

$$\begin{aligned} [\psi_m^+, \psi_n^-]_+ &= \delta_{m+n,0} \\ [\psi_m^\pm, \psi_n^\pm]_+ &= 0. \end{aligned}$$

We take the convention here and below that $\epsilon = 0$ if $\underline{\mathbb{Z}} = \frac{1}{2} + \mathbb{Z}$; and $\epsilon = \frac{1}{2}$ if $\underline{\mathbb{Z}} = \mathbb{Z}$. Denote by $\mathcal{F}$ the Fock space of the fields $\psi^-(z)$ and $\psi^+(z)$, generated by the vacuum $|0\rangle$, satisfying

$$\begin{aligned} \psi_n^+|0\rangle = \psi_n^-|0\rangle = 0 \quad (n \in \frac{1}{2} + \mathbb{Z}_+), \qquad &\text{when } \underline{\mathbb{Z}} = \frac{1}{2} + \mathbb{Z}, \\ \psi_n^+|0\rangle = \psi_{n+1}^-|0\rangle = 0 \quad (n \in \mathbb{Z}_+), \qquad &\text{when } \underline{\mathbb{Z}} = \mathbb{Z}. \end{aligned}$$

Now we take l pairs of fermionic fields, $\psi^{\pm,p}(z)$ $(p = 1, \ldots, l)$ and consider the corresponding Fock space $\mathcal{F}^{\bigotimes l}$.

Introduce the following "twisted" generating functions

$$\begin{aligned} E(z,w) &\equiv \sum_{i,j \in \mathbb{Z}} E_{ij} z^{i-1+2\epsilon} w^{-j} = \sum_{k=1}^{l} :\psi^{+,k}(z)\psi^{-,k}(w):, && (9.75) \\ e^{pq}(z) &\equiv \sum_{n \in \mathbb{Z}} e^{pq}(n) z^{-n-1+2\epsilon} =: \psi^{-,p}(z)\psi^{-,q}(-z):, && (9.76) \\ e^{pq}_{**}(z) &\equiv \sum_{n \in \mathbb{Z}} e^{pq}_{**}(n) z^{-n-1+2\epsilon} =: \psi^{+,p}(z)\psi^{+,q}(-z):, && (9.77) \\ e^{pq}_{*}(z) &\equiv \sum_{n \in \mathbb{Z}} e^{pq}_{*}(n) z^{-n-1+2\epsilon} =: \psi^{+,p}(z)\psi^{-,q}(z): + \delta_{p,q}\epsilon z^{-1}, && (9.78) \end{aligned}$$

where $p, q = 1, \ldots, l$. It is known [FF] that the Fourier components of these generating functions span a representation of the twisted affine algebra $\mathfrak{gl}^{(2)}(2l)$ of type $A^{(2)}_{2l-1}$ with central charge 1.

I. Case $\underline{\mathbb{Z}} = \frac{1}{2} + \mathbb{Z}$: dual pair $(\mathbf{Sp(2l)}, \widehat{\mathcal{D}}^-)$

The horizontal subalgebra of $\mathfrak{gl}^{(2)}(2l)$ spanned by the operators $e^{pq}(0), e_*^{pq}(0)$, $e_{**}^{pq}(0)$, $(p, q = 1, \cdots, l)$ is isomorphic to Lie algebra $\mathfrak{sp}(2l)$. We identify the Borel subalgebra $\mathfrak{b}(\mathfrak{sp}(2l))$ with the one generated by $e_*^{pq}(0)$ $(p \leq q)$, $e_{**}^{pq}(0)$ $(p, q = 1, \ldots, l)$ and the Cartan subalgebra with the one generated by $e_*^{pp}(0)$ $(p = 1, \ldots, l)$. Let

$$\begin{aligned} &\sum_{i,j \in \mathbb{Z}} (E_{i,j} - (-1)^{i+j} E_{1-j,1-i}) z^{i-1} w^{-j} \\ &= \textstyle\sum_{k=1}^{l} \left(: \psi^{+,k}(z) \psi^{-,k}(w) : + : \psi^{+,k}(-w) \psi^{-,k}(-z) : \right). \end{aligned} \tag{9.79}$$

The operators $E_{i,j} - (-1)^{i+j} E_{1-j,1-i}$ $(i, j \in \mathbb{Z})$ span c_∞ with central charge l. The action of $\mathfrak{sp}(2l)$ on $\mathcal{F}^{\bigotimes l}$ can be integrated to $Sp(2l)$. It is known [W2] that the actions of $Sp(2l)$ and c_∞ commute with each other on $\mathcal{F}^{\bigotimes l}$ and they form a dual pair.

Finite dimensional irreducible modules of $Sp(2l)$ are parametrized by the highest weights in

$$\Sigma(C) = \{\lambda = (m_1, \ldots, m_l), m_1 \geq \ldots \geq m_l, m_i \in \mathbb{Z}_+\}.$$

We define a map Λ^{cc} from $\Sigma(C)$ to ${c_\infty}_0^*$ by sending $(m_1, \ldots, m_l)$ to

$$\Lambda^{cc}(\lambda) = (l-j)^c\widehat{\Lambda}_0 + \sum_{k=1}^{j} {}^c\widehat{\Lambda}_{m_k},$$

where $m_1 \geq \ldots \geq m_j > m_{j+1} = \ldots = m_l = 0$. We quote the following theorem from [W2].

Theorem 9.1 *We have the following $(Sp(2l), c_\infty)$-module decomposition:*

$$\mathcal{F}^{\bigotimes l} = \bigoplus_{\lambda \in \Sigma(C)} V(Sp(2l); \lambda) \otimes L\left(c_\infty; \Lambda^{cc}(\lambda), l\right) \tag{9.80}$$

where $V(Sp(2l); \lambda)$ is the irreducible $Sp(2l)$-module parametrized by $\lambda \in \Sigma(C)$, and $L\left(c_\infty; \Lambda^{cc}(\lambda), l\right)$ is the irreducible highest weight c_∞-module of highest weight $\Lambda^{cc}(\lambda)$ and central charge l.

Note that $\widehat{\mathcal{D}}^-$ acts on $\mathcal{F}^{\bigotimes l}$ via the composition of the homomorphism $\widehat{\phi}_{1/2}$ and the action of c_∞ given by formula (9.79). Similarly as in Lemma 8.1 we can rewrite the action of $\widehat{\mathcal{D}}^-$ in terms of generating function $T_{1/2}^n(z)$ as

$$\begin{aligned} T_{1/2}^n(z) &= -\sum_{k=1}^{l} \left(: \psi^{+,k}(z) \partial_z^n \psi^{-,k}(z) : + : \partial_z^n \psi^{+,k}(-z) \psi^{-,k}(-z) : \right) \\ &\quad + \alpha_n l z^{-k-n-1}. \end{aligned} \tag{9.81}$$

Given $\lambda = (m_1, \ldots, m_l) \in \Sigma(C)$, where $m_1 \geq \ldots \geq m_j > m_{j+1} = \ldots = m_l = 0$, let $e(\lambda)$ be the set of exponents m_k $(k = 1, \ldots, j)$ with multiplicity 1 and the exponent 0 with multiplicity $l - j$.

Theorem 9.2 *$Sp(2l)$ and $\widehat{\mathcal{D}}^-$ form a dual pair on $\mathcal{F}^{\bigotimes l}$. Furthermore we have the following $(Sp(2l), \widehat{\mathcal{D}}^-)$-module decomposition:*

$$\mathcal{F}^{\bigotimes l} = \bigoplus_{\lambda \in \Sigma(C)} I_\lambda \equiv \bigoplus_{\lambda \in \Sigma(C)} V(Sp(2l); \lambda) \otimes L\left(\widehat{\mathcal{D}}^-; e(\lambda), l\right).$$

Proof. Since the actions of $Sp(2l)$ commutes with c_∞, the action of $Sp(2l)$ commutes with the action of $\widehat{\mathcal{D}}^-$ given by formula (9.81) by Proposition 5.4. So the decomposition of the Fock space into isotypic subspaces with respect to the dual pair $(Sp(2l), c_\infty)$ can be regarded as decomposition with respect to the dual pair $(Sp(2l), \widehat{\mathcal{D}}^-)$ as well. By Theorem 6.1 each isotypic subspace is irreducible under the joint action of $Sp(2l)$ and $\widehat{\mathcal{D}}^-$. By Proposition 6.4 and Theorem 9.1 the highest weight of the representation I_λ with respect to $\widehat{\mathcal{D}}^-$ is given as follows.

$$\Delta_{m,s,\Lambda}(x) = \frac{(l-j) + \sum_{k=1}^{j} \cosh(m_k x)}{2\sinh(x/2)} - \frac{l\cosh(x/2)}{2\sinh(x/2)}.$$

□

II. Case $\underline{\mathbb{Z}} = \mathbb{Z}$: dual pair $(\mathbf{Pin(2l)}, \widehat{\mathcal{D}}^-)$

In this case the horizontal subalgebra of $\mathfrak{gl}^{(2)}(2l)$ spanned by the operators $e^{pq}, e^{pq}_*, e^{pq}_{**}$, $(p, q = 1, \cdots, l)$ is isomorphic to Lie algebra $\mathfrak{so}(2l)$. In particular, the operators e^{pq}_* $(p, q = 1, \cdots l)$ form a subalgebra $\mathfrak{gl}(l)$ in the horizontal $\mathfrak{so}(2l)$. We identify the Borel subalgebra $\mathfrak{b}(\mathfrak{so}(2l))$ with the one spanned by e^{pq}_* $(p \leq q), e^{pq}_{**}$, $p, q = 1, \cdots, l$. The action of $\mathfrak{so}(2l)$ can be lifted to $Pin(2l)$ on $\mathcal{F}^{\bigotimes l}$. Recall that $Pin(2l)$ (resp. $Spin(2l)$) is the double covering group of $O(2l)$ (resp. $SO(2l)$).

Denote $\mathbf{1}_l = (1, 1, \ldots, 1) \in \mathbb{Z}^l$ and $\bar{\mathbf{1}}_l = (1, 1, \ldots, 1, -1) \in \mathbb{Z}^l$. An irreducible representation of $Spin(2l)$ which does not factor to $SO(2l)$ is an irreducible representation of $\mathfrak{so}(2l)$ given by its highest weight

$$\lambda = \frac{1}{2}\mathbf{1}_l + (m_1, m_2, \ldots, m_l) \tag{9.82}$$

or

$$\lambda = \frac{1}{2}\bar{\mathbf{1}}_l + (m_1, m_2, \ldots, -m_l) \tag{9.83}$$

where $m_1 \geq \ldots \geq m_l \geq 0, m_i \in \mathbb{Z}$. A representation of $Pin(2l)$ induced from λ of $Spin(2l)$ of the form (9.82) is decomposed into a sum of two irreducible $Spin(2l)$-modules of highest weights (9.82) and (9.83). We use $\lambda = \frac{1}{2}|\mathbf{1}_l| + (m_1, m_2, \ldots, \overline{m_l}), m_l \geq 0$ to denote this irreducible representation of $Pin(2l)$. Denote

$$\Sigma(Pin) = \{\frac{1}{2}|\mathbf{1}_l| + (m_1, m_2, \ldots, \overline{m_l}), m_1 \geq \ldots \geq m_l \geq 0, m_i \in \mathbb{Z}\}. \tag{9.84}$$

Let

$$\begin{aligned} &\sum_{i,j\in\mathbb{Z}} (E_{i,j} - (-1)^{i+j} E_{-j,-i}) z^i w^{-j} \\ &= \sum_{k=1}^{l} \left(: \psi^{+,k}(z)\psi^{-,k}(w) : - : \psi^{+,k}(-w)\psi^{-,k}(-z) : \right). \end{aligned} \tag{9.85}$$

It is known [W2] that the operators $E_{i,j} - (-1)^{i+j}E_{-j,-i} (i,j \in \mathbb{Z})$ span b_∞ with central charge l and $Pin(2l)$ and b_∞ form a dual pair on $\mathcal{F}^{\bigotimes l}$. We define a map $\Lambda^{\mathfrak{d}\mathfrak{b}}$ from $\Sigma(Pin)$ to $b_{\infty 0}^{*}$ by sending $\lambda = (m_1, \ldots, \overline{m_l})$ to

$$\Lambda^{\mathfrak{d}\mathfrak{b}}(\lambda) = (2l-2j)^b\widehat{\Lambda}_0 + \sum_{k=1}^{j} {}^b\widehat{\Lambda}_{m_k}$$

if $m_1 \geq \ldots \geq m_j > m_{j+1} = \ldots = m_l = 0$.

We need to quote the following theorem from [W2].

Theorem 9.3 *We have the following $(Pin(2l), b_\infty)$-module decomposition:*

$$\mathcal{F}^{\bigotimes l} = \bigoplus_{\lambda\in\Sigma(Pin)} V(Pin(2l);\lambda) \otimes L\left(b_\infty; \Lambda^{\mathfrak{d}\mathfrak{b}}(\lambda), l\right)$$

where $V(Pin(2l);\lambda)$ is the irreducible $Pin(2l)$-module parametrized by $\lambda \in \Sigma(Pin)$, and $L(b_\infty; \Lambda^{\mathfrak{d}\mathfrak{b}}(\lambda), l)$ is the irreducible highest weight b_∞-module of highest weight $\Lambda^{\mathfrak{d}\mathfrak{b}}(\lambda)$ and central charge l.

Note that $\widehat{\mathcal{D}}^-$ acts on $\mathcal{F}^{\bigotimes l}$ via the composition of the homomorphism $\widehat{\phi}_0$ and the action of b_∞ given by (9.85). We can rewrite the action of $\widehat{\mathcal{D}}^-$ in terms of generating function $T^n(z)$ $(n \in 2\mathbb{Z}_+ + 1)$ as

$$T^n(z) = -\sum_{k=1}^{l} \left(: \psi^{+,k}(z)\partial_z^n\psi^{-,k}(z) : - : \partial_z^n\psi^{+,k}(-z)\psi^{-,k}(-z) : \right).$$

Given $\lambda = (m_1, \ldots, \overline{m_l}) \in \Sigma(Pin)$, where

$$m_1 \geq \ldots \geq m_j > m_{j+1} = \ldots = m_l = 0,$$

let $e(\lambda)$ be the set of exponents $m_k + 1/2$ $(k = 1, \ldots, j)$ with multiplicity 1. The following theorem can be proved in the same way as Theorem 9.2, based on Theorem 9.3 and Proposition 6.3.

Theorem 9.4 *$Pin(2l)$ and $\widehat{\mathcal{D}}^-$ form a dual pair on $\mathcal{F}^{\bigotimes l}$. More explicitly we have the following $(Pin(2l), \widehat{\mathcal{D}}^-)$-module decomposition:*

$$\mathcal{F}^{\bigotimes l} = \bigoplus_{\lambda\in\Sigma(Pin)} V(Pin(2l);\lambda) \otimes L\left(\widehat{\mathcal{D}}^-; e(\lambda), l\right).$$

10 FFR's of QHWM's over $\widehat{\mathcal{D}}^-$ with $c \in \mathbb{N} - 1/2$

In this section we will realize certain primitive $\widehat{\mathcal{D}}^-$-modules with positive half-integral central charges in some Fock spaces.

I. Case $\underline{\mathbb{Z}} = \frac{1}{2} + \mathbb{Z}$: dual pair $(\mathfrak{osp}(1, 2l), \widehat{\mathcal{D}}^-)$

We need a bosonic field $\chi(z) = \sum_{n \in \frac{1}{2}+\mathbb{Z}} \chi_n z^{-n-\frac{1}{2}}$ which satisfies the following commutation relations:

$$[\chi_m, \chi_n] = (-1)^{m+\frac{1}{2}} \delta_{m,-n}, \quad m, n \in \frac{1}{2} + \mathbb{Z}.$$

Denote by $\mathcal{F}^{\bigotimes -\frac{1}{2}}$ the Fock space of $\chi(z)$ generated by a vacuum vector which is annihilated by $\chi_n, n \in \frac{1}{2} + \mathbb{Z}_+$. Let $\mathcal{F}^{\bigotimes l-\frac{1}{2}}$ be the tensor product of the Fock space of l pairs of fermionic fields $\psi^{\pm,k}(z)$ $(k = 1, \ldots, l)$ and the Fock space $\mathcal{F}^{\bigotimes -\frac{1}{2}}$ of $\chi(z)$.

It is known [FF] that the Fourier components of the following "twisted" generating functions together with (9.76), (9.77) and (9.78)

$$\begin{aligned}
\zeta(z) &\equiv \sum_{n \in \mathbb{Z}} \zeta(n) z^{-n-1} =: \chi(z)\chi(-z) :, \\
e^p(z) &\equiv \sum_{n \in \mathbb{Z}} e^p(n) z^{-n-1} =: \psi^{-,p}(z)\chi(-z) :, \\
e^p_*(z) &\equiv \sum_{n \in \mathbb{Z}} e^p_*(n) z^{-n-1} =: \psi^{+,p}(z)\chi(z) :,
\end{aligned}$$

generate a representation of the affine superalgebra $\mathfrak{gl}^{(2)}(1, 2l)$ of type $A^{(2)}(0, 2l-1)$ [K1] with central charge 1. Denote

$$e^p \equiv e^p(0), e^p_* \equiv e^p_*(0), e^{pq} \equiv e^{pq}(0), e^{pq}_* \equiv e^{pq}_*(0), e^{pq}_{**} \equiv e^{pq}_{**}(0),\ p, q = 1, \cdots, l.$$

The horizontal subalgebra in $\mathfrak{gl}^{(2)}(1, 2l)$ spanned by the operators $e^p, e^p_*, e^{pq}, e^{pq}_*, e^{pq}_{**}$, $(p, q = 1, \cdots, l)$ is isomorphic to Lie superalgebra $\mathfrak{osp}(1, 2l)$. We identify the Borel subalgebra $\mathfrak{b}(\mathfrak{osp}(1, 2l))$ with the one generated by e^{pq}_{**}, e^{pq}_* $(p \leq q)$, $p, q = 1, \cdots, l$. Let

$$\begin{aligned}
&\sum_{i,j \in \mathbb{Z}} (E_{i,j} - (-1)^{i+j} E_{1-j,1-i}) z^{i-1} w^{-j} \\
&= \sum_{k=1}^{l} \Big(: \psi^{+,k}(z)\psi^{-,k}(w) : + : \psi^{+,k}(-w)\psi^{-,k}(-z) : \Big) + : \chi(z)\chi(-w) :
\end{aligned} \tag{10.86}$$

It is shown in [W2] that $E_{i,j} - (-1)^{i+j} E_{1-j,1-i}$ $(i, j \in \mathbb{Z})$ span c_∞ with central charge $l - \frac{1}{2}$ and $(\mathfrak{osp}(1, 2l), c_\infty)$ form a dual pair on $\mathcal{F}^{\bigotimes l-\frac{1}{2}}$.

Finite dimensional irreducible representations of $\mathfrak{osp}(1,2l)$ are parametrized by the highest weights [K1]

$$\Sigma(Osp) = \{(m_1, m_2, \ldots, m_l) \mid \quad m_1 \geq m_2 \geq \ldots \geq m_l \geq 0, m_i \in \mathbb{Z}\}.$$

Define a map $\Lambda^{\mathfrak{ospc}}$ from $\Sigma(Osp)$ to $c_{\infty}{}_0^*$ by sending $\lambda = (m_1, \ldots, m_l)$ to

$$\Lambda^{\mathfrak{ospc}}(\lambda) = (l - j - 1/2)^c\widehat{\Lambda}_0 + \sum_{k=1}^{j} {}^c\widehat{\Lambda}_{m_k},$$

if $m_1 \geq \ldots \geq m_j > m_{j+1} = \ldots = m_l = 0$. We quote the following theorem from [W2].

Theorem 10.1 *We have the following* $(\mathfrak{osp}(1,2l), c_\infty)$*-module decomposition:*

$$\mathcal{F}^{\bigotimes l - \frac{1}{2}} = \bigoplus_{\lambda \in \Sigma(Osp)} V(\mathfrak{osp}(1,2l); \lambda) \otimes L\left(c_\infty; \Lambda^{\mathfrak{ospc}}(\lambda), l - 1/2\right)$$

where $V(\mathfrak{osp}(1,2l); \lambda)$ *is the irreducible module of* $\mathfrak{osp}(1,2l)$ *parametrized by* $\lambda \in \Sigma(Osp)$*, and* $L\left(c_\infty; \Lambda^{\mathfrak{ospc}}(\lambda), l - 1/2\right)$ *is the irreducible highest weight* c_∞*-module of highest weight* $\Lambda^{\mathfrak{ospc}}(\lambda)$ *and central charge* $l - 1/2$*.*

Note that $\widehat{\mathcal{D}}^-$ acts on $\mathcal{F}^{\bigotimes l}$ via the the composition of homomorphism $\widehat{\phi}_{1/2}$ and the action of c_∞ given by (10.86). We can rewrite the action of $\widehat{\mathcal{D}}^-$ in terms of generating function $T^n_{1/2}(z)$ $(n \in 2\mathbb{Z}_+ + 1)$ as

$$\begin{aligned} T^n_{1/2}(z) &= -\sum_{k=1}^{l} \Big(: \psi^{+,k}(z)\partial_z^n\psi^{-,k}(z) : - : \partial_z^n\psi^{+,k}(-z)\psi^{-,k}(-z) : \Big) \\ &\quad - : \chi(z)\partial_z^n\chi(-z) : . \end{aligned}$$

Given $\lambda = (m_1, m_2, \ldots, m_l) \in \Sigma(Osp)$, where $m_1 \geq \ldots \geq m_j > m_{j+1} = \ldots = m_l = 0$, let $e(\lambda)$ be the set of exponents m_k $(k = 1, \ldots, j)$ with multiplicity 1 and the exponent 0 with multiplicity $l - j - \frac{1}{2}$. The following duality theorem can be proved in a similar way as Theorem 9.2, based on Theorem 10.1 and Proposition 6.4.

Theorem 10.2 $\mathfrak{osp}(1,2l)$ *and* $\widehat{\mathcal{D}}^-$ *form a dual pair on* $\mathcal{F}^{\bigotimes l - \frac{1}{2}}$*. Furthermore we have the following* $(\mathfrak{osp}(1,2l), \widehat{\mathcal{D}}^-)$*-module decomposition:*

$$\mathcal{F}^{\bigotimes l - \frac{1}{2}} = \bigoplus_{\lambda \in \Sigma(Osp)} V(\mathfrak{osp}(1,2l); \lambda) \otimes L\left(\widehat{\mathcal{D}}^-; e(\lambda), l - 1/2\right).$$

II. Case $\underline{\mathbb{Z}} = \mathbb{Z}$: dual pair $(\mathbf{Spin(2l+1)}, \widehat{\mathcal{D}}^-)$

Recall that a fermionic field $\varphi(z) = \sum_{n\in\mathbb{Z}} \varphi_n z^{-n}$ which satisfy the following commutation relations:

$$[\varphi_m, \varphi_n]_+ = (-1)^m \delta_{m,-n}, \quad m, n \in \mathbb{Z}.$$

In this case the Fock space $\mathcal{F}^{\bigotimes l+\frac{1}{2}}$ is the tensor product of the Fock space of l pairs of fermionic fields $\psi^{\pm,k}(z), k = 1, \ldots, l$ and the Fock space $\mathcal{F}^{\bigotimes \frac{1}{2}}$ of $\varphi(z)$ generated by a vacuum vector which is annihilated by $\varphi_m, m \in \mathbb{N}$.

The Fourier components of the following generating functions together with (9.76), (9.77) and (9.78)

$$\begin{aligned} \eta(z) &\equiv \sum_{n\in\mathbb{Z}} \eta(n) z^{-n} =: \varphi(z)\varphi(-z) :, \\ e^p(z) &\equiv \sum_{n\in\mathbb{Z}} e^p(n) z^{-n} =: \psi^{-,p}(z)\varphi(-z) :, \\ e^p_*(z) &\equiv \sum_{n\in\mathbb{Z}} e^p_*(n) z^{-n} =: \psi^{+,p}(z)\varphi(z) : \end{aligned}$$

span an affine algebra of type $A^{(2)}_{2l}$ on $\mathcal{F}^{\bigotimes l+\frac{1}{2}}$. The horizontal subalgebra of $A^{(2)}_{2l}$ is isomorphic to $\mathfrak{so}(2l+1)$. The action of $\mathfrak{so}(2l+1)$ can be lifted to an action of $Spin(2l+1)$ on $\mathcal{F}^{\bigotimes l+\frac{1}{2}}$. It is well known that an irreducible representation of $Spin(2l+1)$ which does not factor to $SO(2l+1)$ is an irreducible representation of $\mathfrak{so}(2l+1)$ parametrized by its highest weight

$$\lambda = \frac{1}{2}\mathbf{1}_l + (m_1, m_2, \ldots, m_l), \quad m_1 \geq \ldots \geq m_l \geq 0. \tag{10.87}$$

Denote

$$\Sigma(PB) = \left\{ \frac{1}{2}\mathbf{1}_l + (m_1, m_2, \ldots, m_l) \mid m_1 \geq \ldots \geq m_l \geq 0, m_i \in \mathbb{Z} \right\}.$$

Let

$$\begin{aligned} &\sum_{i,j\in\mathbb{Z}} (E_{i,j} - (-1)^{i+j} E_{-j,-i}) z^i w^{-j} \\ &= \sum_{k=1}^l \Big(: \psi^{+,k}(z)\psi^{-,k}(w) : + : \psi^{+,k}(-w)\psi^{-,k}(-z) : \Big) + : \varphi(z)\varphi(-w) : \end{aligned} \tag{10.88}$$

The Fock space $\mathcal{F}^{\bigotimes l+\frac{1}{2}}$ splits into a sum of two subspaces $\mathcal{F}_e^{\bigotimes l+\frac{1}{2}}$ and $\mathcal{F}_o^{\bigotimes l+\frac{1}{2}}$, where $\mathcal{F}_e^{\bigotimes l+\frac{1}{2}}$ consists all even vectors while $\mathcal{F}_o^{\bigotimes l+\frac{1}{2}}$ consists all odd vectors according to the $\mathbb{Z}_2$ gradation on the vector superspace $\mathcal{F}^{\bigotimes l+\frac{1}{2}}$. The action of $\mathfrak{so}(2l+1)$ can be lifted to $Spin(2l)$ on $\mathcal{F}_e^{\bigotimes l+\frac{1}{2}}$ and $\mathcal{F}_o^{\bigotimes l+\frac{1}{2}}$. $Spin(2l)$ and b_∞ form a dual pair on $\mathcal{F}_e^{\bigotimes l+\frac{1}{2}}$ and $\mathcal{F}_o^{\bigotimes l+\frac{1}{2}}$ [W2].

Note that $\widehat{\mathcal{D}}^-$ acts on $\mathcal{F}^{\bigotimes l+\frac{1}{2}}$ via the composition of the homomorphism $\widehat{\phi}_0$ and the action of b_∞ given by (10.88). We can rewrite the action of $\widehat{\mathcal{D}}^-$ in terms of generating function $T^n(z)$ ($n \in 2\mathbb{Z}_+ + 1$) as

$$\begin{aligned} T^n(z) &= -\sum_{k=1}^{l}\Big(: \psi^{+,k}(z)\partial_z^n\psi^{-,k}(z) : +(-1)^n : \partial_z^n\psi^{+,k}(-z)\psi^{-,k}(-z) :\Big) \\ &\quad -(-1)^n : \varphi(z)\partial_z^n\varphi(-z) : . \end{aligned}$$

Given $\lambda = \frac{1}{2}\mathbf{1}_l + (m_1, m_2, \ldots, m_l) \in \Sigma(PB)$, where

$$m_1 \geq \ldots \geq m_j > m_{j+1} = \ldots = m_l = 0,$$

let $V(Spin(2l+1); \lambda)$ be the irreducible $Spin(2l+1)$-module parametrized by λ and let $e(\lambda)$ be the set of exponents $m_k + 1/2$ ($k = 1, \ldots, j$) with multiplicity 1 (as in Section 9). The following duality theorem on the commuting actions of $Pin(2l)$ and $\widehat{\mathcal{D}}^-$ follows from a corresponding duality theorem of a dual pair $(Spin(2l+1), b_\infty)$ in [W2] and similar argument as in Theorem 9.2.

Theorem 10.3 *We have the following $(Spin(2l+1), \widehat{\mathcal{D}}^-)$-module decomposition:*

$$\begin{aligned} \mathcal{F}_e^{\bigotimes l+\frac{1}{2}} &= \bigoplus_{\lambda\in\Sigma(PB)} V(Spin(2l+1); \lambda) \otimes L\left(\widehat{\mathcal{D}}^-; e(\lambda), l+1/2\right) \\ \mathcal{F}_o^{\bigotimes l+\frac{1}{2}} &= \bigoplus_{\lambda\in\Sigma(PB)} V(Spin(2l+1); \lambda) \otimes L\left(\widehat{\mathcal{D}}^-; e(\lambda), l+1/2\right). \end{aligned}$$

11 FFR's of QHWM's over $\widehat{\mathcal{D}}^+$ with $c \in \mathbb{N}$

Let us take a pair of fermionic fields

$$\psi^\pm(z) = \sum_{n\in\underline{\mathbb{Z}}} \psi_n^\pm z^{-n-\frac{1}{2}+\epsilon}, \quad \underline{\mathbb{Z}} = \frac{1}{2} + \mathbb{Z} \text{ or } \mathbb{Z}.$$

In the case $\underline{\mathbb{Z}} = \frac{1}{2} + \mathbb{Z}$ the anti-commutation relations among $\psi_n^\pm$ is equivalent to the following operator product expansions (OPE)

$$\psi^+(z)\psi^-(w) \sim \frac{1}{z-w}, \quad \psi^+(z)\psi^+(w) \sim 0, \quad \psi^-(z)\psi^-(w) \sim 0.$$

Take l pairs of fermionic fields, $\psi^{\pm,p}(z)$ ($p = 1, \ldots, l$) and consider the corresponding Fock space $\mathcal{F}^{\bigotimes l}$. Introduce the following generating functions

$$E(z, w) \equiv \sum_{i,j\in\mathbb{Z}} E_{ij} z^{i-1+2\epsilon} w^{-j} = \sum_{p=1}^{l} : \psi^{+,p}(z)\psi^{-,p}(w) : \qquad (11.89)$$

$$\begin{aligned} e^{pq}(z) &\equiv \sum_{n\in\mathbb{Z}} e^{pq}(n) z^{-n-1+2\epsilon} =: \psi^{-,p}(z)\psi^{-,q}(z) : \quad (p \neq q) && (11.90)\\ e^{pq}_{**}(z) &\equiv \sum_{n\in\mathbb{Z}} e^{pq}_{**}(n) z^{-n-1+2\epsilon} =: \psi^{+,p}(z)\psi^{+,q}(z) : \quad (p \neq q) && (11.91)\\ e^{pq}_{*}(z) &\equiv \sum_{n\in\mathbb{Z}} e^{pq}_{*}(n) z^{-n-1+2\epsilon} =: \psi^{+,p}(z)\psi^{-,q}(z) : +\delta_{p,q}\epsilon && (11.92) \end{aligned}$$

where $p, q = 1, \ldots, l$, and the normal ordering :: means that the operators annihilating $|0\rangle$ are moved to the right and multiplied by -1.

It is well known [F1, KP] that the operators $e^{pq}(n), e^{pq}_*(n), e^{pq}_{**}(n)$, $p, q = 1, \ldots, l$, $n \in \mathbb{Z}$ form a representation of the affine algebra $\widehat{so}(2l)$ of level 1. The operators $e^{pq}(0), e^{pq}_*(0), e^{pq}_{**}(0)$ $(p, q = 1, \cdots, l)$ form the horizonal subalgebra $\mathfrak{so}(2l)$ in $\widehat{\mathfrak{so}}(2l)$. We identify the Borel subalgebra $\mathfrak{b}(\mathfrak{so}(2l))$ with the one generated by e^{pq}_{**} $(p \neq q), e^{pq}_*$ $(p \leq q), p, q = 1, \cdots, l$.

¿From now on we need to treat the two cases $\underline{\mathbb{Z}} = \frac{1}{2} + \mathbb{Z}$ or $\mathbb{Z}$ separately. First consider the case $\underline{\mathbb{Z}} = \frac{1}{2} + \mathbb{Z}$. It follows from (11.89) that

$$\begin{aligned} &\sum_{i,j\in\mathbb{Z}} (E_{ij} - E_{1-j,1-i}) z^{i-1} w^{-j} = \\ &\qquad \textstyle\sum_{k=1}^{l} (: \psi^{+,k}(z)\psi^{-,k}(w) : - : \psi^{+,k}(w)\psi^{-,k}(z) :). \end{aligned} \tag{11.93}$$

One can show that $E_{ij} - E_{1-j,1-i}$ $(i, j \in \mathbb{Z})$ span d_∞. The action of the horizontal subalgebra $\mathfrak{so}(2l)$ can be integrated into an action of $SO(2l)$ and extended to an action of $O(2l)$ naturally. It is known [W2] that the action of d_∞ commutes with the action of $O(2l)$ on $\mathcal{F}^{\bigotimes l}$ and moreover d_∞ and $O(2l)$ form a dual pair[1] by the same argument as in finite dimensional dual pairs case [H1, H2].

We define a map $\Lambda^{\mathfrak{d}\mathfrak{d}} : \Sigma(D) \longrightarrow {d_\infty}_0^*$ by sending $\lambda = (m_1, \cdots, \overline{m_l})$ $(m_l > 0)$ to

$$\Lambda^{\mathfrak{d}\mathfrak{d}}(\lambda) = (l-i)^d\widehat{\Lambda}_0 + (l-i)^d\widehat{\Lambda}_1 + \sum_{k=1}^{i} {}^d\widehat{\Lambda}_{m_k},$$

sending $(m_1, \cdots, m_j, 0, \ldots, 0)$ $(j < l)$ to

$$\Lambda^{\mathfrak{d}\mathfrak{d}}(\lambda) = (2l-i-j)^d\widehat{\Lambda}_0 + (j-i)^d\widehat{\Lambda}_1 + \sum_{k=1}^{i} {}^d\widehat{\Lambda}_{m_k},$$

and sending $(m_1, \ldots, m_j, 0, \ldots, 0) \bigotimes det$ $(j < l)$ to

$$\Lambda^{\mathfrak{d}\mathfrak{d}}(\lambda) = (j-i)^d\widehat{\Lambda}_0 + (2l-i-j)^d\widehat{\Lambda}_1 + \sum_{k=1}^{i} {}^d\widehat{\Lambda}_{m_k},$$

if $m_1 \geq \ldots m_i > m_{i+1} = \ldots = m_j = 1 > m_{j+1} = \ldots = m_l = 0$.

The following theorem was proved in [W2].

---

[1] This fact was also known to V. Kac and C. Yan.

Theorem 11.1 *We have the following $(O(2l), d_\infty)$-module decomposition:*

$$\mathcal{F}^{\bigotimes l} = \bigoplus_{\lambda \in \Sigma(D)} V(O(2l);\lambda) \otimes L\left(d_\infty; \Lambda^{\mathfrak{d}\mathfrak{d}}(\lambda), l\right)$$

where $V(O(2l);\lambda)$ is the irreducible $O(2l)$-module parametrized by $\lambda \in \Sigma(D)$ and $L\left(d_\infty; \Lambda^{\mathfrak{d}\mathfrak{d}}(\lambda), l\right)$ is the irreducible highest weight d_∞-module of highest weight $\Lambda^{\mathfrak{d}\mathfrak{d}}(\lambda)$ and central charge l.

We can obtain the action of $\widehat{\mathcal{D}}^{+}$ on $\mathcal{F}^{\bigotimes l}$ by composing the action of d_∞ and the homomorphism $\widehat{\phi}_0$ given by the formula (5.45). Introduce the generating function

$$W^n(z) = \sum_{k \in \mathbb{Z}} W_k^n z^{-k-n-1} \quad (n \in 2\mathbb{Z}_+ + 1). \tag{11.94}$$

Lemma 11.1 *On $\mathcal{F}^{\bigotimes l}$ we have*

$$W^n(z) = \frac{1}{2}\sum_{k=1}^{l}(: \partial_z^n \psi^{-,k}(z)\psi^{+,k}(z) : + : \partial_z^n \psi^{+,k}(z)\psi^{-,k}(z) :). \tag{11.95}$$

Proof. We calculate $W^n(z)$ as follows.

$$\begin{aligned}
W^n(z) &= -\frac{1}{2}\sum_{k,j\in\mathbb{Z}}([-j]_n - [j-k-1]_n)E_{j-k,j}z^{-k-n-1} && (11.96)\\
&= -\frac{1}{2}\sum_{k,j\in\mathbb{Z}}[-j]_n(E_{j-k,j} - E_{1-j,k-j+1})z^{-k-n-1} && (11.97)\\
&= \frac{1}{2}\sum_{k=1}^{l}(- : \psi^{+,k}(z)\partial_z^n\psi^{-,k}(z) : + : \partial_z^n\psi^{+,k}(z)\psi^{-,k}(z) :). && (11.98)
\end{aligned}$$

Here (11.96) is given by (11.94) and (5.45), (11.97) is obtained by shifting the indices from $j-k-1$ to j in the second part of the right hand side of (11.96), and (11.98) is obtained by taking n-th derivatives of (11.93) with respect to w. It is clear that (11.98) is the same as (11.95). □

For $\lambda = (m_1, \cdots, \overline{m_l}) \in \Sigma(D)$ where $m_1 \geq \ldots \geq m_i > m_{i+1} = \ldots = m_l = 1$, we let $e(\lambda)$ be the set of exponents m_k $(k = 1, \ldots, i)$ with multiplicity 1 and the exponent 1 with multiplicity $l - i$; for $\lambda = (m_1, \cdots, m_l) \in \Sigma(D)$ where

$$m_1 \geq \ldots m_i > m_{i+1} = \ldots = m_j = 1 > m_{j+1} = \ldots = m_l = 0 \ (j < l),$$

we let $e(\lambda)$ be the set of exponents m_k $(k = 1, \ldots, i)$ of multiplicity 1, exponent 1 of multiplicity $j - i$; for $(m_1, \cdots, m_l) \bigotimes det \in \Sigma(D)$ where

$$m_1 \geq \ldots m_i > m_{i+1} = \ldots = m_j = 1 > m_{j+1} = \ldots = m_l = 0 \ (j < l),$$

we let $e(\lambda)$ be the set of exponents m_k $(k = 1, \ldots, i)$ of multiplicity 1, exponent 1 of multiplicity $2l - i - j$. We will simply write $(0, \cdots, 0)$ and $(0, \cdots, 0) \bigotimes det$ as 0 and *det* respectively.

Theorem 11.2 *$O(2l)$ and $\widehat{\mathcal{D}}^+$ form a dual pair on $\mathcal{F}^{\bigotimes l}$. Moreover we have the following $(O(2l), \widehat{\mathcal{D}}^+)$-module decomposition:*

$$\mathcal{F}^{\bigotimes l} = \bigoplus_{\lambda \in \Sigma(D)} V(O(2l); \lambda) \otimes L\left(\widehat{\mathcal{D}}^+; e(\lambda), l\right). \tag{11.99}$$

Proof. By Theorem 6.1 the d_∞-module $L\left(d_\infty; \Lambda^{\mathfrak{d}\mathfrak{d}}(\lambda), l\right)$ regarded as a $\widehat{\mathcal{D}}^+$-module via the pullback by $\widehat{\phi}_0$ remains irreducible. Then this theorem follows from Theorem 11.1 once we determine the corresponding $\Delta(x)$ for this $\widehat{\mathcal{D}}^+$-module. It follows from the definition of $\Lambda^{\mathfrak{d}\mathfrak{d}}(\lambda)$ by using Proposition 6.6 that for $\lambda = (m_1, \cdots, \overline{m_l}) \in \Sigma(D)$ where $m_1 \geq \ldots \geq m_i > m_{i+1} = \ldots = m_l = 1$,

$$\Delta(x) \;=\; \sum_{k=1}^{i} \frac{\cosh(m_k x) + (l-i)\cosh x + \frac{1}{2}((l-i)-(l-i))}{2\sinh(x/2)} - \frac{l}{2\sinh(x/2)}.$$

The computations of $\Delta(x)$ for the remaining $\lambda \in \Sigma(D)$ are similar. □

We have an immediate corollary.

Corollary 11.1 *The space of invariants of $O(2l)$ in the Fock space $\mathcal{F}^{\bigotimes l}$ is naturally isomorphic to the irreducible module $L(d_\infty; 2l\ {}^d\widehat{\Lambda}_0)$ of central charge l with highest weight vector $|0\rangle$, or equivalently to the irreducible module $L\left(\widehat{\mathcal{D}}^+; e(0), l\right)$.*

Remark 11.1 The Dynkin diagram of d_∞ admits an automorphism of order 2 denoted by σ. σ induces naturally an automorphism of order 2 of d_∞, which is denoted again by σ by abuse of notation. σ acts on the set of highest weights of d_∞ by mapping $\lambda = {}^dh_0\ {}^d\widehat{\Lambda}_0 + {}^dh_1\ {}^d\widehat{\Lambda}_1 + \sum_{i\geq 2} {}^dh_i\ {}^d\widehat{\Lambda}_i$ to $\sigma(\lambda) = {}^dh_1\ {}^d\widehat{\Lambda}_0 + {}^dh_0\ {}^d\widehat{\Lambda}_1 + \sum_{i\geq 2} {}^dh_i\ {}^d\widehat{\Lambda}_i$. In this way one can obtain an irreducible module of the semi-product $\sigma \ltimes d_\infty$ on $L(d_\infty;\lambda) \bigoplus L(d_\infty;\sigma(\lambda))$ if $\sigma(\lambda) \neq \lambda$ and on $L(d_\infty;\lambda)$ if $\sigma(\lambda) = \lambda$.

It was noted in [W2] that $(SO(2l), \sigma \ltimes d_\infty)$ form a dual pair on $\mathcal{F}^{\bigotimes l}$. In particular the space of invariants of $\mathcal{F}^{\bigotimes l}$ under the action fo $SO(2l)$ is isomorphic to the d_∞-module $L(d_\infty; 2l\ {}^d\widehat{\Lambda}_0) \bigoplus L(d_\infty; 2l\ {}^d\widehat{\Lambda}_1)$ or equivalently the $\widehat{\mathcal{D}}^+$-module $L\left(\widehat{\mathcal{D}}^+; e(0), l\right) \bigoplus L\left(\widehat{\mathcal{D}}^+; e(det), l\right)$.

Now we consider the case $\underline{\mathbb{Z}} = \mathbb{Z}$. It follows from (11.89) that

$$\begin{aligned} &\sum_{i,j\in\mathbb{Z}} (E_{ij} - E_{-j,-i}) z^i w^{-j} = \\ &\qquad \textstyle\sum_{k=1}^{l} (: \psi^{+,k}(z)\psi^{-,k}(w) : - : \psi^{+,k}(w)\psi^{-,k}(z) :). \end{aligned} \tag{11.100}$$

One can show that $E_{ij} - E_{1-j,1-i}$ $(i,j \in \mathbb{Z})$ span $\tilde{b}_\infty$. The action of the horizontal subalgebra $\mathfrak{so}(2l)$ can be integrated into an action of $Spin(2l)$ (cf. e.g. [BtD] for more on spin groups) and then extended naturally to $Pin(2l)$. $Pin(2l)$ and $\tilde{b}_\infty$ form a dual pair on $\mathcal{F}^{\bigotimes l}$ [W2].

We define a map $\Lambda^{\mathfrak{ob}}$ from $\Sigma(Pin)$ (see (9.84) for notation) to $b_{\infty}{}_0^*$ by sending

$$\lambda = (m_1, \ldots, \overline{m_l}), \quad m_1 \geq m_2 \geq \ldots \geq m_l \geq 0$$

to

$$\Lambda^{\mathfrak{ob}}(\lambda) = (2l - 2j)^b\widehat{\Lambda}_0 + \sum_{k=1}^{j} {}^b\widehat{\Lambda}_{m_k}$$

if $m_1 \geq \ldots m_j > m_{j+1} = \ldots = m_l = 0$.

The following theorem was proved in [W2].

Theorem 11.3 *We have the following $(Pin(2l), \tilde{b}_\infty)$-module decomposition:*

$$\mathcal{F}^{\bigotimes l} = \bigoplus_{\lambda \in \Sigma(Pin)} V(Pin(2l); \lambda) \otimes L\left(\tilde{b}_\infty; \Lambda^{\mathfrak{ob}}(\lambda), l\right)$$

where $V(Pin(2l); \lambda)$ is the irreducible $Pin(2l)$-module parametrized by $\lambda \in \Sigma(Pin)$, and $L\left(\tilde{b}_\infty; \Lambda^{\mathfrak{ob}}(\lambda), l\right)$ is the irreducible highest weight $\tilde{b}_\infty$-module of highest weight $\Lambda^{\mathfrak{ob}}(\lambda)$ and central charge l.

We define the action of $\widehat{\mathcal{D}}^+$ on $\mathcal{F}^{\bigotimes l}$ by the composition of the action of $\tilde{b}_\infty$ and the homomorphism $\widehat{\phi}_{-1/2}$ given by (5.45). It follows that the action of $Pin(2l)$ commutes with that of $\widehat{\mathcal{D}}^+$. Introduce the following generating function

$$W^n_{1/2}(z) = \sum_{k \in \mathbb{Z}} W^{n,1/2}_k z^{-k-n} \tag{11.101}$$

It follows from Proposition 5.2 that the representation of $\widehat{\mathcal{D}}^+$ on $\mathcal{F}^{\bigotimes l}$ has central charge l and

$$\widehat{\phi}_{-1/2}\left(e^{xD} - e^{-xD}\right) = \phi_{-1/2}\left(e^{xD} - e^{-xD}\right) - l\tanh(x/4). \tag{11.102}$$

Therefore

$$\widehat{\phi}_{-1/2}(W^{n,1/2}_0) = \phi_{-1/2}(W^{n,1/2}_0) - \alpha_n l$$

for some constant α_n determined by (11.102) depending on n only. Similarly as in Lemma 11.1 one can recast the action of $\widehat{\mathcal{D}}^+$ in terms of $W^n_{1/2}(z)$ as

$$\begin{aligned} &W^n_{1/2}(z) \\ &= \tfrac{1}{2}\textstyle\sum_{k=1}^l \left(: \partial_z^n \psi^{-,k}(z)\psi^{+,k}(z) : + : \partial_z^n \psi^{+,k}(z)\psi^{-,k}(z) :\right) - \alpha_n l z^{-n}. \end{aligned} \tag{11.103}$$

Given $\lambda = (m_1, \ldots, \overline{m_l}) \in \Sigma(Pin)$ where $m_1 \geq m_2 \geq \ldots \geq m_l \geq 0$, we let $e(\lambda)$ be the set of exponents $m_k + \frac{1}{2}$ $(k = 1, \ldots, j)$ of multiplicity 1 and exponent $\frac{1}{2}$ of multiplicity $l - j$.

The proof of the following theorem is obtained in an analogous way as for Theorem 11.2 by using Theorems 6.1, 11.3 and Proposition 6.7.

Theorem 11.4 *We have the following $(Pin(2l), \widehat{\mathcal{D}}^+)$-module decomposition:*

$$\mathcal{F}^{\bigotimes l} = \bigoplus_{\lambda \in \Sigma(Pin)} V(Pin(2l); \lambda) \otimes L\left(\widehat{\mathcal{D}}^+; e(\lambda), l\right).$$

12 FFR's of QHWM's over $\widehat{\mathcal{D}}^{+}$ with $c \in \frac{1}{2} + \mathbb{Z}_{+}$

Introduce a neutral fermionic field $\phi(z) = \sum_{n \in \underline{\mathbb{Z}}} \phi_n z^{-n-\frac{1}{2}+\epsilon}$ which satisfies the following commutation relations:

$$[\phi_m, \phi_n]_+ = \delta_{m,-n}, \quad m, n \in \underline{\mathbb{Z}}.$$

Denote by $\mathcal{F}^{\bigotimes \frac{1}{2}}$ the Fock space of $\phi(z)$ generated by a vacuum vector $|0\rangle$, which is annihilated by $\phi_n, n \in \underline{\mathbb{Z}}_+$. Denote by $\mathcal{F}^{\bigotimes l+\frac{1}{2}}$ the $\mathbb{Z}_2$-graded tensor product of $\mathcal{F}^{\bigotimes \frac{1}{2}}$ and the Fock space $\mathcal{F}^{\bigotimes l}$ of l pairs of fermionic fields $\psi^{\pm,k}(z)(k = 1, \dots, l)$.

Denote by

$$\begin{aligned} e^p(z) \equiv \sum_{n \in \mathbb{Z}} e^p(n) z^{-n-1+2\epsilon} &= \ :\psi^{-,p}(z)\phi(z): \\ e^p_*(z) \equiv \sum_{n \in \mathbb{Z}} e^p_*(n) z^{-n-1+2\epsilon} &= \ :\psi^{+,p}(z)\phi(z): \quad (p = 1, \dots, l). \end{aligned} \tag{12.104}$$

Then the Fourier components of $e^p(z), e^p_*(z)$, together with generating functions (11.90), (11.91) and (11.92)

$$e^{pq}_*(n),\ e^{pq}(n)(p \neq q),\ e^{pq}_{**}(n)(p \neq q),\ e^p(n), e^p_*(n) \quad (n \in \mathbb{Z},\ p, q = 1, \dots, l)$$

generate an affine algebra $\widehat{\mathfrak{so}}(2l+1)$ of level 1 [F1, KP]. $e^{pq}(0)$ $(p \neq q), e^{pq}_{**}(0)$ $(p \neq q)$, $e^{pq}_*(0), e^p(0), e^p_*(0)$ $(p, q = 1, \dots, l)$ generate the horizontal subalgebra $\mathfrak{so}(2l+1)$ of $\widehat{\mathfrak{so}}(2l+1)$. In particular, we identify the Borel subalgebra $\mathfrak{b}(\mathfrak{so}(2l+1))$ with the one generated by $e^{pq}_{**}(0)(p \neq q), e^{pq}_*(0)(p \leq q),\ e^p_*(0), p, q = 1, \cdots, l$.

¿From now on we need to consider the two cases $\underline{\mathbb{Z}} = \frac{1}{2} + \mathbb{Z}$ and $\mathbb{Z}$ separately.

First consider the case $\underline{\mathbb{Z}} = \frac{1}{2} + \mathbb{Z}$. Introduce a generating function

$$\begin{aligned} &\sum_{i,j \in \mathbb{Z}} (E_{ij} - E_{1-j,1-i}) z^{i-1} w^{-j} \\ &\quad = \textstyle\sum_{k=1}^{l} (:\psi^{+,k}(z)\psi^{-,k}(w): - :\psi^{+,k}(w)\psi^{-,k}(z):) + :\phi(z)\phi(w): . \end{aligned}$$

One can show that this defines an action of d_∞ on $\mathcal{F}^{\bigotimes l+\frac{1}{2}}$ with central charge $l + \frac{1}{2}$. The action of the horizontal subalgebra $\mathfrak{so}(2l+1)$ can be lifted to an action of $O(2l+1)$. The action of $O(2l+1)$ commutes with that of d_∞ generated by $E_{ij} - E_{1-j,1-i}$ $(i, j \in \mathbb{Z})$ on $\mathcal{F}^{\bigotimes l+\frac{1}{2}}$.

Define a map $\Lambda^{\mathfrak{bd}}$ from $\Sigma(B)$ to $d_{\infty 0}^{*}$ by sending

$$\lambda = (m_1, m_2, \dots, m_l)$$

to

$$\Lambda^{\mathfrak{bd}} = (2l + 1 - i - j)^d \widehat{\Lambda}_0 + (j - i)^d \widehat{\Lambda}_1 + \sum_{k=1}^{i} {}^d\widehat{\Lambda}_{m_k}$$

and sending

$$\lambda = (m_1, m_2, \ldots, m_l) \bigotimes det$$

to

$$\Lambda^{\mathfrak{bd}} = (j-i)^d \widehat{\Lambda}_0 + (2l+1-i-j)^d \widehat{\Lambda}_1 + \sum_{k=1}^{i} {}^d\widehat{\Lambda}_{m_k}$$

assuming that

$$m_1 \geq \ldots \geq m_i > m_{i+1} = \ldots = m_j = 1 > m_{j+1} = \ldots = m_l = 0.$$

The following theorem is quoted from [W2].

Theorem 12.1 *We have the following $(O(2l+1), d_\infty)$-module decomposition:*

$$\mathcal{F}^{\bigotimes l+\frac{1}{2}} = \bigoplus_{\lambda \in \Sigma(B)} V(O(2l+1); \lambda) \otimes L\left(d_\infty; \Lambda_+^{\mathfrak{bd}}(\lambda), l+1/2\right)$$

where $V(O(2l+1);\lambda)$ is the irreducible $O(2l+1)$-module parametrized by $\lambda \in \Sigma(B)$ and $L\left(d_\infty; \Lambda^{\mathfrak{bd}}(\lambda), l+1/2\right)$ is the irreducible highest weight d_∞-module with highest weight $\Lambda^{\mathfrak{bd}}(\lambda)$ and central charge $l+1/2$.

The action of $\widehat{\mathcal{D}}^+$ on $\mathcal{F}^{\bigotimes l+\frac{1}{2}}$ is given by the composition of the action of d_∞ on $\mathcal{F}^{\bigotimes l+\frac{1}{2}}$ and the homomorphism $\widehat{\phi}_0$. Similarly as in Lemma 11.1 we can show that

$$W^n(z) = \frac{1}{2}\sum_{k=1}^{l}\left(: \partial_z^n \psi^{-,k}(z)\psi^{+,k}(z) : + : \partial_z^n \psi^{+,k}(z)\psi^{-,k}(z) :\right) + \frac{1}{2} : \partial_z^n \phi(z)\phi(z) : . \quad (12.105)$$

For $\lambda = (m_1, m_2, \ldots, m_l) \in \Sigma(B)$ we let $e(\lambda)$ be the set of exponents m_k ($k = 1, \ldots, i$) of multiplicity 1, the exponent 1 of multiplicity $j-i$ (the exponent 0 has multiplicity $l-j+\frac{1}{2}$); and for $\lambda = (m_1, m_2, \ldots, m_l) \bigotimes det$ we let $e(\lambda)$ be the set of exponents m_k ($k = 1, \ldots, i$) of multiplicity 1, the exponent 1 of multiplicity $2l+1-i-j$ (the exponent 0 has multiplicity $-l+j-\frac{1}{2}$), where

$$m_1 \geq \ldots \geq m_i > m_{i+1} = \ldots = m_j = 1 > m_{j+1} = \ldots = m_l = 0.$$

The following theorem can be proved in an analogous way as Theorem 11.2 by using Theorems 6.1, 12.1 and Proposition 6.6.

Theorem 12.2 *We have the following $(O(2l+1), \widehat{\mathcal{D}}^+)$-module decomposition:*

$$\mathcal{F}^{\bigotimes l+\frac{1}{2}} = \bigoplus_{\lambda \in \Sigma(B)} V(O(2l+1); \lambda) \otimes L\left(\widehat{\mathcal{D}}^+; e(\lambda), l+1/2\right).$$

The following corollary is immediate.

Corollary 12.1 *The space of invariants of $O(2l+1)$ in $\mathcal{F}^{\bigotimes l+\frac{1}{2}}$ is naturally isomorphic to the irreducible d_∞-module $L(d_\infty; (2l+1)\ {}^d\widehat{\Lambda}_0)$ or equivalently to the irreducible $\widehat{\mathcal{D}}^+$-module $L\left(\widehat{\mathcal{D}}^+; e(0), l+1/2\right)$.*

Remark 12.1 $(SO(2l+1), \sigma \ltimes d_\infty)$ form a dual pair on $\mathcal{F}^{\bigotimes l+\frac{1}{2}}$. In particular the space of invariants of $\mathcal{F}^{\bigotimes l+\frac{1}{2}}$ with respect to $SO(2l+1)$ is isomorphic to the d_∞-module $L(d_\infty; (2l+1)\ {}^d\widehat{\Lambda}_0) \bigoplus L(d_\infty; (2l+1)\ {}^d\widehat{\Lambda}_1)$ or equivalently the $\widehat{\mathcal{D}}^+$-module $L(\widehat{\mathcal{D}}^+; e(0), l+1/2) \bigoplus L(\widehat{\mathcal{D}}^+; e(det), l+1/2)$.

Now we consider the case $\underline{\mathbb{Z}} = \mathbb{Z}$. Introduce the following generating function

$$\begin{aligned}&\sum_{i,j\in\mathbb{Z}} (E_{i,j} - E_{-j,-i}) z^i w^{-j} \\ &\quad = \textstyle\sum_{k=1}^l \Big(:\psi^{+,k}(z)\psi^{-,k}(w): - :\psi^{+,k}(w)\psi^{-,k}(z):\Big) + :\phi(z)\phi(w):.\end{aligned}$$

This defines a representation of $\tilde{b}_\infty$ on $\mathcal{F}^{\bigotimes l+\frac{1}{2}}$ of central charge $l+1/2$.

The Fock space $\mathcal{F}^{\bigotimes l+\frac{1}{2}}$ splits into a sum of two subspaces $\mathcal{F}_e^{\bigotimes l+\frac{1}{2}}$ and $\mathcal{F}_o^{\bigotimes l+\frac{1}{2}}$, where $\mathcal{F}_e^{\bigotimes l+\frac{1}{2}}$ consists all even vectors while $\mathcal{F}_o^{\bigotimes l+\frac{1}{2}}$ consists all odd vectors according to the $\mathbb{Z}_2$ gradation on the vector superspace $\mathcal{F}^{\bigotimes l+\frac{1}{2}}$. The action of $\mathfrak{so}(2l+1)$ can be lifted to $Spin(2l)$ on $\mathcal{F}_e^{\bigotimes l+\frac{1}{2}}$ and $\mathcal{F}_o^{\bigotimes l+\frac{1}{2}}$ respectively. $Spin(2l)$ and b_∞ form a dual pair on $\mathcal{F}_e^{\bigotimes l+\frac{1}{2}}$ and $\mathcal{F}_o^{\bigotimes l+\frac{1}{2}}$ [W2].

Define a map $\Lambda^{\mathfrak{bb}}$ from $\Sigma(PB)$ to ${b_\infty}_0^*$ by sending $\lambda = \frac{1}{2}\mathbf{1}_l + (m_1, m_2, \ldots, m_l)$ to

$$\Lambda^{\mathfrak{bb}}(\lambda) = (2l+1-2j)^b\widehat{\Lambda}_0 + \sum_{k=1}^{j} {}^b\widehat{\Lambda}_{m_k}$$

if $m_1 \geq \ldots \geq m_j > m_{j+1} = \ldots = m_l = 0$.

The following theorem was proved in [W2].

Theorem 12.3 *We have the following $(Spin(2l+1), \tilde{b}_\infty)$-module decomposition:*

$$\begin{aligned}\mathcal{F}_e^{\bigotimes l+\frac{1}{2}} &= \bigoplus_{\lambda\in\Sigma(PB)} V(Spin(2l+1);\lambda) \otimes L\left(\tilde{b}_\infty; \Lambda^{\mathfrak{bb}}(\lambda), l+\frac{1}{2}\right) \\ \mathcal{F}_o^{\bigotimes l+\frac{1}{2}} &= \bigoplus_{\lambda\in\Sigma(PB)} V(Spin(2l+1);\lambda) \otimes L\left(\tilde{b}_\infty; \Lambda^{\mathfrak{bb}}(\lambda), l+\frac{1}{2}\right)\end{aligned}$$

where $V(O(2l+1);\lambda)$ is the irreducible $Spin(2l+1)$-module parametrized by $\lambda = \frac{1}{2}\mathbf{1}_l + (m_1, m_2, \ldots, m_l)$ and $L\left(\tilde{b}_\infty; \Lambda^{\mathfrak{bb}}(\lambda), l+\frac{1}{2}\right)$ is the irreducible highest weight $\tilde{b}_\infty$-module with central charge $l+\frac{1}{2}$.

Now the action of $\widehat{\mathcal{D}}^+$ on $\mathcal{F}_e^{\bigotimes l+\frac{1}{2}}$ and $\mathcal{F}_o^{\bigotimes l+\frac{1}{2}}$ can be obtained by the composition of the action of $\tilde{b}_\infty$ and the homomorphism $\widehat{\phi}_{-1/2}$ given by (5.45). In terms of $W^n_{1/2}(z) = \sum_{k\in\mathbb{Z}} W_k^{n,1/2} z^{-k-n}$ we obtain in an anagolous way as in (11.103) that

$$\begin{aligned} W^n_{1/2}(z) &= \frac{1}{2}\sum_{k=1}^{l}\Big(: \partial_z^n\psi^{-,k}(z)\psi^{+,k}(z) : + : \partial_z^n\psi^{+,k}(z)\psi^{-,k}(z) :\Big) \\ &\quad +\frac{1}{2} : \partial_z^n\phi(z)\phi(z) : -\alpha_n(l+1/2)z^{-n}. \end{aligned}$$

For $\lambda = \frac{1}{2}\mathbf{1}_l + (m_1, m_2, \dots, m_l) \in \Sigma(PB)$ where

$$m_1 \geq \ldots \geq m_j > m_{j+1} = \ldots = m_l = 0,$$

we let $e(\lambda)$ be the set of exponents $m_k + \frac{1}{2}$ $(k = 1, \dots, j)$ of multiplicity 1 and the exponent $\frac{1}{2}$ of multiplicity $l + 1/2 - j$. The following theorem can be proved in an analogous way as Theorem 11.2 by using instead Theorems 6.1, 12.3 and Proposition 6.7.

Theorem 12.4 *We have the following $(Spin(2l+1), \widehat{\mathcal{D}}^+)$-module decomposition:*

$$\begin{aligned} \mathcal{F}_e^{\bigotimes l+\frac{1}{2}} &= \bigoplus_{\lambda\in\Sigma(PB)} V(Spin(2l+1);\lambda)\otimes L\left(\widehat{\mathcal{D}}^+; e(\lambda), l+\frac{1}{2}\right) \\ \mathcal{F}_o^{\bigotimes l+\frac{1}{2}} &= \bigoplus_{\lambda\in\Sigma(PB)} V(Spin(2l+1);\lambda)\otimes L\left(\widehat{\mathcal{D}}^+; e(\lambda), l+\frac{1}{2}\right). \end{aligned}$$

13 FFR's of QHWM's over $\widehat{\mathcal{D}}^+$ with $c \in -\frac{1}{2}\mathbb{N}$

13.1 Case $c \in -\mathbb{N}$

Let us take a pair of bosonic ghost fields

$$\gamma^\pm(z) = \sum_{n\in\frac{1}{2}+\mathbb{Z}} \gamma_n^\pm z^{-n-\frac{1}{2}}.$$

Equivalently we have the following operator product expansions

$$\gamma^+(z)\gamma^-(w) \sim \frac{1}{z-w}, \quad \gamma^+(z)\gamma^+(w) \sim 0, \quad \gamma^-(z)\gamma^-(w) \sim 0.$$

We take l pairs of bosonic ghost fields $\gamma^{\pm,p}(z)$ $(p = 1, \dots, l)$ and consider the corresponding Fock space $\mathcal{F}^{\bigotimes -l}$.

Introduce the following generating functions

$$E(z,w) \equiv \sum_{i,j\in\mathbb{Z}} E_{ij} z^{i-1} w^{-j} = -\sum_{p=1}^{l} :\gamma^{+,p}(z)\gamma^{-,p}(w): \tag{13.106}$$

$$e^{pq}_{**}(z) \equiv \sum_{i,j\in\mathbb{Z}} e^{pq}_{**}(n) z^{-n-1} =: \gamma^{+,p}(z)\gamma^{+,q}(z): \quad (p\neq q)$$

$$e^{pq}(z) \equiv \sum_{i,j\in\mathbb{Z}} e^{pq}(n) z^{-n-1} =: \gamma^{-,p}(z)\gamma^{-,q}(z): \quad (p\neq q) \tag{13.107}$$

$$e^{pq}_{*}(z) \equiv \sum_{i,j\in\mathbb{Z}} e^{pq}_{*}(n) z^{-n-1} =: \gamma^{+,p}(z)\gamma^{-,q}(z): \quad (p,q=1,\ldots,l)$$

where the normal ordering :: means that the operators annihilating $|0\rangle$ are moved to the right.

It is well known that the operators E_{ij} $(i,j\in\mathbb{Z})$ form a representation in $\mathcal{F}^{\bigotimes -l}$ of the Lie algebra $\widehat{\mathfrak{gl}}$ with central charge $-l$; the operators

$$e^{pq}(n), e^{pq}_{*}(n), e^{pq}_{**}(n) \quad (p,q=1,\ldots,l, n\in\mathbb{Z})$$

form an affine algebra $\widehat{\mathfrak{sp}}(2l)$ with central charge -1 [FF]. The operators $e^{pq}(0)$, $e^{pq}_{*}(0)$, $e^{pq}_{**}(0)$ $(p,q=1,\cdots l)$ span the horizontal subalgebra $\mathfrak{sp}(2l)$ in $\widehat{\mathfrak{sp}}(2l)$. In particular, operators e^{pq}_{*} $(p,q=1,\cdots l)$ form a Lie subalgebra $\mathfrak{gl}(l)$ in the horizontal subalgebra $\mathfrak{sp}(2l)$. We identify the Borel subalgebra $\mathfrak{b}(\mathfrak{sp}(2l))$ with the one generated by $e^{pq}_{**}(0), e^{pq}_{*}(0)(p\leq q)$, $p,q=1,\cdots,l$. The action of the horizontal subalgebra $\mathfrak{sp}(2l)$ can be lifted to an action of Lie group $Sp(2l)$ on $\mathcal{F}^{\bigotimes -l}$.

It follows from (13.106) that

$$\sum_{i,j\in\mathbb{Z}} (E_{ij} - E_{1-j,1-i}) z^{i-1} w^{-j} = \sum_{p=1}^{l} (:\gamma^{+,p}(w)\gamma^{-,p}(z): - :\gamma^{+,p}(z)\gamma^{-,p}(w):).$$

The operators $E_{ij} - E_{1-j,1-i}$ $(i,j\in\mathbb{Z})$ span the Lie algebra d_∞ with central charge $-l$. The actions of $Sp(2l)$ and of d_∞ on $\mathcal{F}^{\bigotimes -l}$ commutes with each other and form a dual pair. We now define a map $\Lambda^{\mathfrak{cd}}$ from $\Sigma(C)$ to ${d_\infty}^*_0$ which maps $\lambda=(m_1,\ldots,m_l)$ to

$$\Lambda^{\mathfrak{cd}}(\lambda) = (-2l - m_1 - m_2)^d\widehat{\Lambda}_0 + \sum_{k=1}^{l}(m_k - m_{k+1})^d\widehat{\Lambda}_k$$

with the convention $m_{l+1}=0$ here and below. The following theorem was proved in [W2].

Theorem 13.1 *We have the following $(Sp(2l), d_\infty)$-module decomposition:*

$$\mathcal{F}^{\bigotimes -l} = \bigoplus_{\lambda\in\Sigma(C)} V(Sp(2l);\lambda)\otimes L\left(d_\infty;\Lambda^{\mathfrak{cd}}(\lambda), -l\right)$$

where $V(Sp(2l);\lambda)$ is the irreducible highest weight $Sp(2l)$-module of highest weight λ and $L\left(d_\infty;\Lambda^{\mathfrak{cd}}(\lambda),-l\right)$ the irreducible highest weight d_∞-module of highest weight $\Lambda^{\mathfrak{cd}}(\lambda)$ and central charge $-l$.

The action of $\widehat{\mathcal{D}}^+$ on $\mathcal{F}^{\bigotimes -l}$ is given by the composition of the action of d_∞ and the homomorphism $\widehat{\phi}_{-1/2}$. A similar argument as in Lemma 11.1 shows that

$$W^n(z) = \frac{1}{2}\sum_{p=1}^{l}\left(: \partial_z^n \gamma^{+,p}(z)\gamma^{-,p}(z) : - : \gamma^{+,p}(z)\partial_z^n\gamma^{-,p}(z) :\right). \tag{13.108}$$

Given $\lambda = (m_1, \dots, m_l) \in \Sigma(C)$, we let $e(\lambda)$ be the set of exponents k with multiplicity $m_k - m_{k+1}$ $(k = 1, \dots, l)$ where $m_{l+1} = 0$ (the exponent 0 has multiplicity $-l - m_1$).

Theorem 13.2 *We have the following $(Sp(2l), \widehat{\mathcal{D}}^+)$-module decomposition:*

$$\mathcal{F}^{\bigotimes -l} = \bigoplus_{\lambda \in \Sigma(C)} V(Sp(2l); \lambda) \otimes L\left(\widehat{\mathcal{D}}^+; e(\lambda), -l\right).$$

Proof. By Theorem 6.1 the d_∞-module $L(d_\infty; \Lambda^{\mathfrak{c}\mathfrak{d}}(\lambda), -l)$ regarded as a $\widehat{\mathcal{D}}^+$-module via the pullback by $\widehat{\phi}_0$ remains irreducible. To prove the thereom it suffices to determine the corresponding $\Delta(x)$ for this $\widehat{\mathcal{D}}^+$-module. It follows from the definition of $\Lambda^{\mathfrak{c}\mathfrak{d}}(\lambda)$ by using Proposition 6.6 that for $\lambda = (m_1, \dots, m_l) \in \Sigma(C)$

$$\begin{aligned}\Delta(x) &= \frac{\sum_{k=1}^{l}(m_k - m_{k+1})\cosh(kx) + \frac{1}{2}((-2l - m_1 - m_2) - (m_1 - m_2)) - (-l)}{2\sinh(x/2)} \\ &= \frac{\sum_{k=1}^{l}(m_k - m_{k+1})\cosh(kx) + (-l - m_1)}{2\sinh(x/2)} - \frac{(-l)}{2\sinh(x/2)}.\end{aligned}$$

□

The following corollary is immediate.

Corollary 13.1 *The space of invariants of $Sp(2l)$ in the Fock space $\mathcal{F}^{\bigotimes -l}$ is naturally isomorphic to the irreducible module $L(d_\infty; -2l^d\widehat{\Lambda}_0)$ or equivalently to the irreducible $\widehat{\mathcal{D}}^+$-module $L\left(\widehat{\mathcal{D}}^+; e(0), -l\right)$.*

13.2 Case $c \in -\mathbb{N} + 1/2$

We denote by $\mathcal{F}^{\bigotimes -l+\frac{1}{2}}$ the tensor product of the Fock space $\mathcal{F}^{\bigotimes -l}$ of l pairs of bosonic ghost fields and the Fock space $\mathcal{F}^{\bigotimes \frac{1}{2}}$ of a neutral fermionic field $\phi(z)$. It is known [FF] that the Fourier components of the generating functions $e^{pq}(z), e_*^{pq}(z)$, $e_{**}^{pq}(z)$ in (13.107) and the following generating functions

$$\begin{aligned} e^p(z) \equiv \sum_{n\in\mathbb{Z}} e^p(n)z^{-n-1} &= \ : \gamma^{-,p}(z)\phi(z) : \\ \tilde{e}_*^p(z) \equiv \sum_{n\in\mathbb{Z}} \tilde{e}_*^p(n)z^{-n-1} &= \ : \gamma^{+,p}(z)\phi(z) : \quad (p = 1, \dots, l)\end{aligned}$$

span the affine superalgebra $\widehat{\mathfrak{osp}}(1,2l)$ of level -1. The operators $e^{pq}(0)$, $e_*^{pq}(0)$, $e_{**}^{pq}(0)$, $\tilde{e}^p(0), \tilde{e}_*^p(0)$ $(p,q=1,\ldots,l)$ generate the horizontal subalgebra $\mathfrak{osp}(1,2l)$ of the affine superalgebra $\widehat{\mathfrak{osp}}(1,2l)$. We identify the Borel subalgebra $\mathfrak{b}(\mathfrak{osp}(1,2l))$ with the one generated by $e_*^{pq}(0)(p \le q), e_{**}^{pq}(0), \tilde{e}_*^p(0)$, $p,q=1,\cdots,l$. Introduce the generating function

$$\begin{aligned}
&\sum_{i,j\in\mathbb{Z}} (E_{ij} - E_{1-j,1-i})\, z^{i-1}w^{-j} \\
&\quad = \textstyle\sum_{k=1}^{l} \Big(: \gamma^{+,k}(z)\gamma^{-,k}(w) : - : \gamma^{+,k}(w)\gamma^{-,k}(z) : \Big) + : \phi(z)\phi(w) : .
\end{aligned} \tag{13.109}$$

This defines a representation of d_∞ of central charge $-l+\frac{1}{2}$ on $\mathcal{F}^{\bigotimes -l+\frac{1}{2}}$. It is known [W2] the action of the horizontal subalgebra $\mathfrak{osp}(1,2l)$ commute with that of Lie algebra d_∞ on $\mathcal{F}^{\bigotimes -l+\frac{1}{2}}$. $(\mathfrak{osp}(1,2l), d_\infty)$ form a dual pair on $\mathcal{F}^{\bigotimes -l+\frac{1}{2}}$.

We define a map $\Lambda^{\mathfrak{ospd}}$ from $\Sigma(osp)$ to ${d_\infty}_0^*$ by sending $\lambda = (m_1,\ldots,m_l)$ to

$$\Lambda^{\mathfrak{ospd}}(\lambda) = (-2l+1-m_1-m_2)^d\widehat{\Lambda}_0 + \sum_{k=1}^{l}(m_k - m_{k+1})^d\widehat{\Lambda}_k.$$

The following duality theorem was quoted from [W2].

Theorem 13.3 *We have the following $(\mathfrak{osp}(1,2l), d_\infty)$-module decomposition*

$$\mathcal{F}^{\bigotimes -l+\frac{1}{2}} = \bigoplus_{\lambda\in\Sigma(Osp)} V(\mathfrak{osp}(1,2l);\lambda) \otimes L\left(d_\infty; \Lambda^{\mathfrak{ospd}}(\lambda), -l+1/2\right)$$

where $V(\mathfrak{osp}(1,2l);\lambda)$ is the irreducible module of $\mathfrak{osp}(1,2l)(1,2l)$ of highest weight λ, and $L\left(d_\infty; \Lambda^{\mathfrak{ospd}}(\lambda), -l+1/2\right)$ is the irreducible highest weight d_∞-module of highest weight $\Lambda^{\mathfrak{ospd}}(\lambda)$ and central charge $-l+1/2$.

We define the action of $\widehat{\mathcal{D}}^+$ on $\mathcal{F}^{\bigotimes -l+\frac{1}{2}}$ to be the composition of the action of d_∞ and the homomorphism $\widehat{\phi}_0$ given by (5.45). In a similar way as obtaining (13.108) we have

$$\begin{aligned}
W^n(z) \;&=\; \frac{1}{2}\sum_{k=1}^{l}\Big(: \gamma^{+,k}(z)\partial_z^n\gamma^{-,k}(z) : - : \gamma^{-,k}(z)\partial_z^n\gamma^{+,k}(z) : \Big) \\
&\quad + \frac{1}{2} : \phi(z)\partial_z^n\phi(z) : .
\end{aligned} \tag{13.110}$$

Then of course the action of the horizontal subalgebra $\mathfrak{osp}(1,2l)$ commute with that of Lie algebra $\widehat{\mathcal{D}}^+$ on $\mathcal{F}^{\bigotimes -l+\frac{1}{2}}$.

Given $\lambda = (m_1,\ldots,m_l) \in \Sigma(Osp)$, we let $e(\lambda)$ be the set of exponents k of multiplicity $m_k - m_{k+1}$ $(k=1,\ldots,l)$ where $m_{l+1}=0$ (the exponent 0 has multiplicity $-l-m_1+\frac{1}{2}$). We obtain the following theorem in a similar way as Theorem 13.2 by using Theorems 6.1, 13.3, and Proposition 6.6.

Theorem 13.4 *We have the following* $(\mathfrak{osp}(1,2l), \widehat{\mathcal{D}}^+)$*-module decomposition*

$$\mathcal{F}^{\bigotimes -l+\frac{1}{2}} = \bigoplus_{\lambda \in \Sigma(Osp)} V(\mathfrak{osp}(1,2l);\lambda) \otimes L\left(\widehat{\mathcal{D}}^+; e(\lambda), -l+1/2\right).$$

The following corollary is immediate.

Corollary 13.2 *The space of invariants of* $\mathfrak{osp}(1,2l)$ *in the Fock space* $\mathcal{F}^{\bigotimes -l+\frac{1}{2}}$ *is naturally isomorphic to the irreducible module* $L(d_\infty; (-2l+1)^d\widehat{\Lambda}_0)$ *or equivalently to the irreducible* $\widehat{\mathcal{D}}^+$*-module* $L\left(\widehat{\mathcal{D}}^+; 0, -l+1/2\right)$.

14 Vertex algebra associated to $\widehat{\mathcal{D}}^+$

14.1 Vertex algebra structure on the vacuum module of $\widehat{\mathcal{D}}^+$

In Example 3.1 we have constructed the vacuum $\widehat{\mathcal{D}}^+$-modules M_c and V_c ($c \in \mathbb{C}$). We want to show that M_c and V_c carry a natural structure of a vertex algebra, cf. [B, FLM, DL, K2].

Denote by $|0\rangle$ the highest weight vector of M_c and V_c. M_c (resp. V_c) is spanned by the vectors

$$W^{n_j}_{-k_j-n_j-1} \cdots W^{n_1}_{-k_1-n_1-1}|0\rangle,\ n_i \in 2\mathbb{Z}_+ + 1,\ k_i \in \mathbb{Z}. \tag{14.111}$$

By the definition of M_c the vector $|0\rangle$ is annihilated by $\mathcal{P}$, namely $W^n_k|0\rangle = 0$ for $k+n \geq 0$. Operators from $\mathcal{P}$ is often referred to as annihilation operators and the operators W^n_k from $\widehat{\mathcal{D}}^+ - \mathcal{P}$ as creation operators. We have known that $W^1_k = -t^k(D+(k+1)/2)$ ($k \in \mathbb{Z}$) form the Virasoro algebra with central charge c when acting on M_c or V_c. Also it follows by a direct computation using (2.16) that

$$\left[W^1_{-1}, W^n_k\right] = -(n+k)W^n_{k-1}$$

or equivalently

$$\left[W^1_{-1}, W^n(z)\right] = \partial_z W^n(z). \tag{14.112}$$

Denote $T = W^1_{-1}$. Define a linear map $Y(\cdot, z) : M_c \longrightarrow (\mathrm{End}\ M_c)[[z, z^{-1}]]$ by

$$Y(W^n_{-n-1}|0\rangle, z) = W^n(z) \quad (n \in 2\mathbb{Z}_+ + 1)$$

and in general we associate to a vector a of the form (14.111) the following generating function

$$Y(a,z) =: \partial_z^{(k_j)} W^{n_j}(z) \cdots \partial_z^{(k_1)} W^{n_1}(z) :$$

where $\partial_z^{(k)}$ denotes $\partial_z^k/k!$ and :: denotes the standard normal ordering from the right to left which moves the annihilators to the right. Clearly $Y(a,z)|0\rangle\ |_{z=0} = a$.

Proposition 14.1 $(V_l, |0\rangle, T, Y(\cdot, z))$ *is a vertex algebra isomorphic to the space of invariants of* $\mathcal{F}^{\bigotimes l}$ *for* $\underline{\mathbb{Z}} = \frac{1}{2} + \mathbb{Z}$ *with respect to* $O(2l)$*. The generating fields* $W^n(z)$ *are given by formula (11.94).*

By Corollary 11.1, V_l and the space of invariants of $\mathcal{F}^{\bigotimes l}$ with respect to $O(2l)$ are naturally isomorphic as $\widehat{\mathcal{D}}^+$-modules. Since $\mathcal{F}^{\bigotimes l}$ has a natural vertex algebra structure, the vertex algebra structure on V_l is ensured by the well-known fact that the space of invariants of an automorphism group of a vertex algebra is a vertex algebra. We will give a direct proof in Lemma 14.2 below that the generating fields $W^n(z)$ $(n \in 2\mathbb{Z}_+ + 1)$ are closed under operator product expansions.

Consider the case of $\mathcal{F}^{\bigotimes l}$ with $\underline{\mathbb{Z}} = \frac{1}{2} + \mathbb{Z}$. Denote

$$\Psi^{m,n}(z) = \sum_{k=1}^{l} \Big(: \partial^m \psi^{+,k}(z) \partial^n \psi^{-,k}(z) : + : \partial^m \psi^{-,k}(z) \partial^n \psi^{+,k}(z) : \Big), \quad m, n \in \mathbb{Z}_+.$$

Note the obvious symmetry $\Psi^{m,n}(z) = -\Psi^{n,m}(z)$.

Lemma 14.1 $\Psi^{m,n}(z)$ *is a linear combination of* $\partial^i W^{m+n-i}(z), 0 \le i \le m+n$ *and* $i \equiv m + n - 1 \bmod 2$.

Proof. We prove by induction on $m+n$. When $m+n=0$ or 1, the statement is obvious.

Assume that for $m+n = 2k-1$ $(k \in \mathbb{N})$ the statement is true. Then by this induction assumption, $\partial \Psi^{2k-1-m,m}(z)$ $(m = 0, \ldots, k-1)$ is a linear combination of $\partial^i W^{2k+2-i}(z)$ $(0 \le i \le 2k-1$ and i odd). Since

$$\partial \Psi^{2k-1-m,m}(z) = \Psi^{2k-m,m}(z) + \Psi^{2k-1-m,m+1}(z)$$

it follows by a little algebra that the linear span of $\Psi^{2k-m,m}(z), m = 0, \ldots, k-1$ is equal to the linear span of $\partial \Psi^{2k-1-m,m}(z), m = 0, \ldots, k-1$. This proves that the statement is true for $m+n = 2k$.

Therefore it follows that $\partial \Psi^{2k-m,m}(z), m = 0, \ldots, k-1$ is a linear combination of $\partial^i W^{2k+1-i}(z)$ $(0 \le i \le 2k+1$ and i even). Since

$$\partial \Psi^{2k-m,m}(z) = \Psi^{2k+1-m,m}(z) + \Psi^{2k+1-m,m}(z)$$

by a little linear algebra it is easy to show that the linear span of $\Psi^{2k+1-m,m}(z)$ $(m = 0, \ldots, k)$ is equal to the linear span of $\partial \Psi^{2k-m,m}(z)$ $(m = 0, \ldots, k-1)$ and $W^{2k+1}(z)$. This proves that the statement is true for $m+n = 2k+1$. $\square$

Lemma 14.2 $\partial^i W^j(z), i \in \mathbb{Z}_+, j \in 2\mathbb{Z}_+ + 1$ *are closed under the operator product expansions.*

Proof. For the simplicity of notations, we first assume $l = 1$ and write $\psi^{\pm}(z)$ for $\psi^{\pm,1}(z)$. By Wick's theorem, we have

$$\begin{aligned}
W^m(z)W^n(z) \;\sim\; & \frac{1}{4}\Big(:\partial_z^m\psi^-(z)\psi^+(z):+:\partial_z^m\psi^+(z)\psi^-(z):\Big)\cdot \\
& \Big(:\partial_w^n\psi^-(w)\psi^+(w):+:\partial_w^n\psi^+(w)\psi^-(w):\Big) \\
\sim\; & \frac{1}{4}\Big(\partial_z^m((z-w)^{-1})\Big)\Big(:\psi^+(z)\partial_w^n\psi^-(w):+:\psi^-(z)\partial_w^n\psi^+(w):\Big) \\
& +\frac{1}{4}\Big(\partial_w^n((z-w)^{-1})\Big)\Big(:\partial_z^m\psi^-(z)\psi^+(w):+:\partial_z^m\psi^+(z)\psi^-(w):\Big) \\
& +\frac{1}{2}\partial_z^m((z-w)^{-1})\partial_w^n((z-w)^{-1}) \\
\sim\; & \frac{(-1)^m m!}{4(z-w)^{m+1}}\Big(:\psi^+(z)\partial_w^n\psi^-(w):+:\psi^-(z)\partial_w^n\psi^+(w):\Big) \\
& +\frac{n!}{4(z-w)^{n+1}}\Big(:\partial_z^m\psi^-(z)\psi^+(w):+:\partial_z^m\psi^+(z)\psi^-(w):\Big) \\
& +\frac{(-1)^m(m+n)!}{2(z-w)^{m+n+2}}.
\end{aligned}$$

By taking the Taylor expansions of $\psi^{\pm}(z)$ and $\partial_z^m\psi^{\pm}(z)$ at $z = w$, we can easily see that all the fields appearing on the right hand side of the above OPE are linear combinations of $\Psi^{m,n}(z)$. Thus our lemma follows from Lemma 14.1. For the general case $\mathcal{F}^{\bigotimes l}$, it is clear that the only modification in the final OPE formula above is that the central term should be mutiplied by l. □

The following corollary holds in the cases $c = l$ ($l \in \mathbb{N}$) by the computation of OPE in Lemma 14.2 since the maximal order of poles at $z = w$ appearing there is $m + n + 2$. Since the central term depends on the central charge c linearly, the corollary remains true for an arbitrary central charge.

Corollary 14.1 *For arbitrary central charge c we have as a formal power series*

$$(z-w)^{m+n+2}\,[W^m(z), W^n(w)] = 0.$$

Now the following theorem follows by a general argument in the theory of vertex algebras, cf. [L], Proposition 3.1 in [FKRW], or Theorem 4.5 in [K2], since all the requirements there are satisfied (also cf. [G]).

Theorem 14.1 $(M_c, |0\rangle, T, Y(\cdot, z))$ *and* $(V_c, |0\rangle, T, Y(\cdot, z))$ *are vertex algebras.*

The following proposition follows from Corollaries 12.1, 13.1, 13.2 and Theorem 14.1.

Proposition 14.2 *1)* $(V_{l+1/2}, |0\rangle, \omega, Y(\cdot, z))$ *is a vertex algebra isomorphic to the space of invariants of* $\mathcal{F}^{\bigotimes\, l+\frac{1}{2}}$ *for* $\underline{\mathbb{Z}} = \frac{1}{2} + \mathbb{Z}$ *with respect to* $O(2l+1)$. *The generating fields* $W^n(z)$ *are given by formula (12.105).*

2) $(V_{-l}, |0\rangle, \omega, Y(\cdot, z))$ *is a vertex algebra isomorphic to the space of invariants of* $\mathcal{F}^{\bigotimes\, -l}$ *with respect to* $Sp(2l)$. *The generating fields* $W^n(z)$ *are given by formula (13.108).*

3) $(V_{-l+1/2}, |0\rangle, \omega, Y(\cdot, z))$ *is a vertex algebra isomorphic to the space of invariants of* $\mathcal{F}^{\bigotimes\, -l+\frac{1}{2}}$ *with respect to* $\mathfrak{osp}(1, 2l)$. *The generating fields* $W^n(z)$ *are given by formula (13.110).*

Corollary 14.2 *The irreducible representations of* $\widehat{\mathcal{D}}^+$ *appearing in the decompositions of Fock spaces* $\mathcal{F}^{\bigotimes\, l}, \mathcal{F}^{\bigotimes\, l+\frac{1}{2}}, \mathcal{F}^{\bigotimes\, -l}$ *and* $\mathcal{F}^{\bigotimes\, -l+\frac{1}{2}}$ *are representations of the vertex algebra associated to* $\widehat{\mathcal{D}}^+$ *with central charges* $l, l+1/2, -l$ *and* $-l+1/2$ *respectively.*

Remark 14.1 It is interesting to compare some theorems in [DLM] by Dong-Li-Mason with our Theorems 11.2, 12.2, 13.2 and 13.4 in the framework of vertex algebras. Their results have the virtue of being general while the statements of our theorems are more powerful and precise thanks to being concrete.

14.2 Vertex algebra V_c for $c \in \frac{1}{2}\mathbb{Z}$ and $\mathcal{W}$ algebras

A particular important class of vertex algebras is the so-called $\mathcal{W}$-algebras. One can associate a $\mathcal{W}$-algebra $\mathcal{W}\mathfrak{g}$ to an arbitrary complex simple Lie algebra $\mathfrak{g}$ (see [BS, FeF] and references therein). In general such a $\mathcal{W}$-algebra can be defined in terms of intersections of screening operators, cf. e.g. [FeF]. Denote by $\mathcal{W}(\widehat{\mathfrak{g}}/\mathfrak{g}, k)$ the $\mathcal{W}$-algebra arising as the space of invariants of $\mathfrak{g}$ in the vacuum module of the affine algebra $\widehat{\mathfrak{g}}$ of level k, where $\mathfrak{g}$ is the horizontal subalgebra of $\widehat{\mathfrak{g}}$. In the case when $\mathfrak{g}$ is simply-laced of rank l, $\mathcal{W}\mathfrak{g}$ with central charge l can be shown to be isomorphic to $\mathcal{W}(\widehat{\mathfrak{g}}/\mathfrak{g}, 1)$ cf. e.g. [BS, F2, BBSS, FKRW].

It is known [F1, KP] that the basic representation of $\widehat{\mathfrak{so}}(2l)$ is isomorphic to the even part of the vector superspace $\mathcal{F}^{\bigotimes\, l}$. ¿From Remark 11.1 we see that the space of invariants of $\mathcal{F}^{\bigotimes\, l}$ with respect to $SO(2l)$ is isomorphic to the $\widehat{\mathcal{D}}^+$-module $L\left(\widehat{\mathcal{D}}^+; 0, l\right) \bigoplus L\left(\widehat{\mathcal{D}}^+; e(det), l\right)$ (see Section 11). The highest weight vector of $L\left(\widehat{\mathcal{D}}^+; e(det), l\right)$ is given by $\prod_{k=1}^{l}(\psi^{+,k}_{-1/2}\psi^{-,k}_{-1/2})|0\rangle$ [W2] which lies in the even part of $\mathcal{F}^{\bigotimes\, l}$. Thus $\mathcal{W}(D_l^{(1)}/D_l, 1)$ is isomorphic to the space of invariants of $\mathcal{F}^{\bigotimes\, l}$ with respect to $SO(2l)$. In this way we have reached the following conclusion.

Theorem 14.2 *The* $\mathcal{W}$*-algebra* $\mathcal{W}\mathcal{D}_l$ *with central charge* l *is isomorphic to a sum of the vertex algebra* V_l *and the irreducible module* $L\left(\widehat{\mathcal{D}}^+; e(det), l\right)$ *of* V_l.

As in [FKRW], Theorem 14.2 implies the following corollary.

Corollary 14.3 *All positive primitive modules over $\widehat{\mathcal{D}}^+$ with a positive integral central charge l are irreducible modules over the vertex algebra V_l.*

Remark 14.2 It is an important question to determine whether these positive primitive modules are all irreducible modules over the vertex algebra V_l.

Remark 14.3 Theorem 14.2 provides a new way to compute the q-character of $\mathcal{WD}_l$ with central charge l:

$$\begin{aligned} \mathrm{ch}_q \mathcal{WD}_l &= \mathrm{ch}_q L\left(\widehat{\mathcal{D}}^+; 0, l\right) + q^l \mathrm{ch}_q L\left(\widehat{\mathcal{D}}^+; e(det), l\right) \\ &= \mathrm{ch}_q L\left(d_\infty, 2l\,{}^d\widehat{\Lambda}_0\right) + q^l \mathrm{ch}_q L\left(d_\infty, 2l\,{}^d\widehat{\Lambda}_1\right). \end{aligned} \tag{14.113}$$

Here q^l accounts for the weight l of the highest weight vector $\prod_{k=1}^l (\psi^{+,k}_{-1/2}\psi^{-,k}_{-1/2})|0\rangle$ of $L\left(\widehat{\mathcal{D}}^+; e(det), l\right)$ in $\mathcal{F}^{\bigotimes l}$. The q-character formulas of d_∞-modules $L\left(d_\infty, 2l\,{}^d\widehat{\Lambda}_0\right)$ and $L\left(d_\infty, 2l\,{}^d\widehat{\Lambda}_1\right)$ can be read from the Appendix. A straightforward computation by using (14.113) yields

$$\begin{aligned} \mathrm{ch}_q \mathcal{WD}_l &= \prod_{1\le i<j\le l} (1-q^{j-i})(1-q^{i+j-2})/\varphi(q)^l \\ &= \frac{\prod_{i=1}^l (\prod_{n=1}^{e_i}(1-q^n))}{\varphi(q)^l} \end{aligned}$$

where $\varphi(q) = \prod_{i\ge 1}(1-q^i)$ and $e_i = 2i-1$ $(i = 1, \cdots, l-1)$, $e_l = l-1$ are the exponents of the simple Lie algebra $\mathfrak{so}(2l)$. The same formula was earlier deduced in [K0] and [BS]. The character formula implies that $\mathcal{WD}_l$ with central charge l is freely generated by fields $W^i(z)(i = 1, 3, \ldots, 2l-3)$ of conformal weights $i+1$ and the field $\prod_{i=1}^l : \psi^{+,k}(z)\psi^{-,k}(z) :$ of conformal weight l corresponding to the vector $\prod_{k=1}^l (\psi^{+,k}_{-1/2}\psi^{-,k}_{-1/2})|0\rangle$.

Remark 14.4 Combining Corollary 11.1 with Theorem 14.2, we obtain another dual pair $(SO(2l), \mathcal{WD}_l)$ on the Fock space $\mathcal{F}^{\bigotimes l}$. When restricting to the even part of $\mathcal{F}^{\bigotimes l}$ which is isomorphic to the basic representation of $\widehat{\mathfrak{so}}(2l)$, we recover a duality theorem of I. Frenkel [F2]. Since the language of vertex algebras was not available at the time of [F2], $\mathcal{WD}_l$ was replaced by the Lie algebra of Fourier components of fields in $\mathcal{WD}_l$.

The even part of the fermionic Fock space $\mathcal{F}^{\bigotimes l+\frac{1}{2}}$ for $\underline{\mathbb{Z}} = \frac{1}{2} + \mathbb{Z}$ is isomorphic to the basic representation of $\widehat{\mathfrak{so}}(2l+1)$ [F1, KP]. According to Corollary 12.1, the space of invariants of $\mathcal{F}^{\bigotimes l+\frac{1}{2}}$ with respect to $SO(2l+1)$ is isomorphic to the $\widehat{\mathcal{D}}^+$-module $L(\widehat{\mathcal{D}}^+; 0, l+1/2) \bigoplus L(\widehat{\mathcal{D}}^+; e(det), l+1/2)$. The highest weight vector of $L(\widehat{\mathcal{D}}^+; e(det), l+1/2)$ in $\mathcal{F}^{\bigotimes l+\frac{1}{2}}$ is $\prod_{k=1}^l (\psi^{+,k}_{-1/2}\psi^{-,k}_{-1/2})\phi_{-1/2}|0\rangle$, cf. [W2]. Observe that it is an odd vector in $\mathcal{F}^{\bigotimes l+\frac{1}{2}}$. Thus $\mathcal{W}(B_l^{(1)}/B_l, 1)$ is isomorphic to $L(\widehat{\mathcal{D}}^+; 0, l+1/2)$. Combining with Proposition 14.2 we have proved the following theorem.

Theorem 14.3 *The vertex algebra $V_{l+1/2}$ associated to $\widehat{\mathcal{D}}^+$ with $c = l + 1/2$ is isomorphic to the $\mathcal{W}$-algebra $\mathcal{W}(B_l^{(1)}/B_l, 1)$.*

Corollary 14.4 *All positive primitive modules over $\widehat{\mathcal{D}}^+$ with a central charge $l + 1/2 \in 1/2 + \mathbb{Z}_+$ are irreducible modules over the vertex algebra $V_{l+1/2}$.*

Remark 14.5 It remains to determine whether these positive primitive modules are all irreducible modules over the vertex algebra $V_{l+1/2}$.

It follows that the character formula of the $\mathcal{W}$-algebra $\mathcal{W}(B_l^{(1)}/B_l, 1)$ is the same as the q-character formula of $L\left(d_\infty; (2l+1)^d\widehat{\Lambda}_0\right)$ (cf. Appendix):

$$\frac{\prod_{i=1}^l(\prod_{n=1}^{e_i}(1-q^n))}{\varphi(q)^l} \cdot \frac{1}{2}\left(\prod_{n\geq 1}(1+q^{n+l-1/2}) + \prod_{n\geq 1}(1-q^{n+l-1/2})\right).$$

where $e_i = 2i - 1$ $(i = 1, \cdots, l)$ are the exponents of the Lie algebra B_l. A different method was used in [BS] to obtain the same formula. In particular we have seen that $\mathcal{W}(B_l^{(1)}/B_l, 1)$ lies inside a vertex superalgebra, denoted by $\mathcal{WB}(0, l)$, of central charge $l - 1/2$ which is isomorphic to the space of invariants of $\mathcal{F}^{\bigotimes\, l+\frac{1}{2}}$ with respect to $SO(2l+1)$. The q-character of the latter can be calculated by using the q-character formulas of d_∞-modules given in the Appendix as

$$\begin{aligned}\mathrm{ch}_q L\left(d_\infty; (2l+1)^d\widehat{\Lambda}_0\right) &+ q^{l+1/2}\mathrm{ch}_q L\left(d_\infty; (2l+1)^d\widehat{\Lambda}_1\right) \\ &= \frac{\prod_{i=1}^l(\prod_{n=1}^{e_i}(1-q^n))}{\varphi(q)^l}\prod_{n\geq 1}(1+q^{n+l-1/2}). \qquad (14.114)\end{aligned}$$

Here $q^{l+1/2}$ accounts for the grading of the highest weight vector of the d_∞-module $L\left(d_\infty; (2l+1)^d\widehat{\Lambda}_1\right)$ in $\mathcal{F}^{\bigotimes\, l+\frac{1}{2}}$.

Formula (14.114) indicates that vertex superalgebra $\mathcal{WB}(0, l)$ is freely generated by the bosonic fields $W^i(z)(i = 1, 3, \ldots, 2l-1)$ of conformal weights $i + 1$ and the fermionic field $\prod_{i=1}^l : \psi^{+,k}(z)\psi^{-,k}(z)\varphi(z) :$ of conformal weight $l + 1/2$ corresponding to the vector $\prod_{k=1}^l(\psi^{+,k}_{-1/2}\psi^{-,k}_{-1/2})\varphi_{-1/2})|0\rangle$. The phenomenon that $\mathcal{W}(B_l^{(1)}/B_l, 1)$ lies in such a vertex superalgebra was conjectured in [BS]. Here we find the explicit model for $\mathcal{WB}(0, l)$ with central charge $l - 1/2$. $\mathcal{WB}(0, l)$ were also obtained by the quantized Drinfeld-Sokolov reduction in [Ito]. We summarize these in the following theorem which resembles remarkably Theorem 14.2.

Theorem 14.4 *The vertex superalgebra $\mathcal{WB}(0, l)$ is the sum of the vertex algebra $V_{l+1/2}$ and the irreducible module $L\left(\widehat{\mathcal{D}}^+; e(det), l + 1/2\right)$ of $V_{l+1/2}$.*

The following theorem easily follows from Proposition 14.2 and similar argument which leads to Theorems 14.3.

Theorem 14.5 *The vertex algebra V_{-l} associated to $\widehat{\mathcal{D}}^+$ with central charge $-l$ is isomorphic to the $\mathcal{W}$-algebra $\mathcal{W}(C_l^{(1)}/C_l, -1)$. The vertex algebra $V_{-l+1/2}$ associated to $\widehat{\mathcal{D}}^+$ with central charge $-l+1/2$ is isomorphic to $\mathcal{W}(B^{(1)}(0,l)/B(0,l), -1)$.*

Remark 14.6 Theorems 14.2 and 14.4 tell us what is the minimal set of generating fields for V_c with $c \in \frac{1}{2}\mathbb{N}$ (or rather for the vertex (super)algebra which are the $\mathbb{Z}_2$ extension of V_c). The question remains open for V_c for a general negative half-ingeral central charge. A similar question was addressed in the case of $\mathcal{W}_{1+\infty}$ in [W] (also see [EFH]). The negative central charge case turned out to be more subtle and difficult.

15 Appendix

The following proposition gives the q-character formula (compatible with the $\mathbb{Z}$-gradation of d_∞ induced from that of $\widehat{\mathfrak{gl}}$) of a highest weight representation of d_∞ with highest weight $\Lambda = {}^d\widehat{\Lambda}_{n_1} + {}^d\widehat{\Lambda}_{n_2} + \ldots + {}^d\widehat{\Lambda}_{n_k} + {}^dh_0 {}^d\widehat{\Lambda}_0$, where $n_1 \geq \ldots n_k \geq 1$, ${}^dh_0 \in \mathbb{Z}_+$. Denote ${}^dh_1$ to be the number of n_i's which are equal to 1. Then the central charge $c = \frac{1}{2}({}^dh_0 - {}^dh_1) + k$. We retain the notation in Section 1.

Proposition 15.1 *The q-character formula of $L(d_\infty;\Lambda)$ corresponding to the principal gradation of d_∞ is*

$$\begin{aligned} ch_q L(d_\infty;\Lambda) \quad = \quad & \frac{\prod_{1\leq i<j\leq k}(1-q^{n_i-n_j+j-i})}{\prod_{1\leq i\leq k}\varphi_{n_i+k-i}(q)} \cdot \frac{\varphi(q^2)^{\overline{2c}}\prod_{j>0}\varphi_{2c-2j}(q)}{\varphi(q)^{[c]-\overline{2c}}} \cdot \\ & \prod_{i=0}^{n_1-1}\frac{\varphi_{2c+i+n_1}(q)}{\varphi_{2c+n_1+i-{}^d\lambda_{i+1}}(q)} \cdot \prod_{0\leq i<j\leq n_1}\frac{1-q^{2c+j+i-{}^d\lambda_{i+1}-{}^d\lambda_{j+1}}}{1-q^{2c+j+i}} \cdot \\ & \cdot\frac{1}{2}\left(\prod_{j\in\mathbb{N}}\left(1+q^{-(\Lambda+\rho,\epsilon_j)}\right)+\prod_{j\in\mathbb{N}}\left(1-q^{-(\Lambda+\rho,\epsilon_j)}\right)\right). \end{aligned} \tag{15.115}$$

Proof. Note that the q-characters of d_∞-modules are specialized characters of type $(2,1,1,\ldots)$, in contrast to the q-characters of modules of b_∞, c_∞ which are of type $(1,1,\ldots)$. So the way of computing such a q-character of a d_∞-module will be different. We will proceed as follows. By the Weyl-Kac character formula

$$e(-\Lambda)\mathrm{ch}L(\Lambda) \equiv \frac{\mathcal{N}}{\mathcal{D}} = \frac{\sum_{w\in W}\epsilon(w)e(w(\Lambda+\rho)-(\Lambda+\rho))}{\prod_{\alpha\in\Delta_+}(1-e(-\alpha))}.$$

Here $\mathcal{N}$ (resp. $\mathcal{D}$) denotes the numerator (resp. denominator). Given $\vec{s} = (2,1,1,\ldots)$, define a homomorphism $F_{\vec{s}} : \mathbb{C}[[e(-\alpha_i)]] \longrightarrow \mathbb{C}[[q]]$ by $F_{\vec{s}}(e(-\alpha_i) = q^{s_i}$. Put $\vec{t} = (t_0, t_1, t_2, \ldots)$ where $t_i = \langle \Lambda+\rho, \alpha_i^\vee\rangle$. Then

$$\begin{aligned}
F_{\vec{s}}(\mathcal{N}) &= \sum_{w\in W} \epsilon(w) q^{\langle \Lambda+\rho-w(\Lambda+\rho), \widehat{\Lambda}_0^\vee+\rho^\vee\rangle} \\
&= q^{\langle \Lambda+\rho, \widehat{\Lambda}_0^\vee+\rho^\vee\rangle} \sum_{w\in W} \epsilon(w) q^{\langle \Lambda+\rho, -w(\widehat{\Lambda}_0^\vee+\rho^\vee)\rangle} \\
&= q^{\langle \Lambda+\rho, \widehat{\Lambda}_0^\vee+\rho^\vee\rangle} F_{\vec{t}}\left(\sum_{w\in W} \epsilon(w) e(w(\widehat{\Lambda}_0+\rho))\right) \\
&= q^{\langle \Lambda+\rho, \widehat{\Lambda}_0^\vee+\rho^\vee\rangle} F_{\vec{t}}\left(\mathrm{ch} L(\widehat{\Lambda}_0)\cdot \sum \epsilon(w) e(w(\rho)-\rho)\cdot e(\rho)\right) \\
&= F_{\vec{t}}(e(-\rho-\widehat{\Lambda}_0)) F_{\vec{t}}\left(\mathrm{ch} L(\widehat{\Lambda}_0)\right) F_{\vec{t}}\left(\prod_{\alpha\in\Delta_+}(1-e(-\alpha)\right) F_{\vec{t}}(e(\rho)) \\
&= F_{\vec{t}}\left(e(-\widehat{\Lambda}_0)\mathrm{ch} L(\widehat{\Lambda}_0)\right)\cdot F_{\vec{t}}\left(\prod_{\alpha\in\Delta_+}(1-e(-\alpha))\right). \qquad (15.116)
\end{aligned}$$

By Theorem 12.3 (putting $l=0$ there), the even part of $\mathcal{F}^{\bigotimes \frac{1}{2}}$ is isomorphic to the irreducible d_∞-module with highest weight $\widehat{\Lambda}_0$ and central charge $\frac{1}{2}$, so

$$F_{\vec{s}}(e(-\widehat{\Lambda}_0)\mathrm{ch} L(\widehat{\Lambda}_0)) = \frac{1}{2}\left(\prod_{j\in\mathbb{N}}(1+q^{j-\frac{1}{2}}) + \prod_{j\in\mathbb{N}}(1-q^{j-\frac{1}{2}})\right), \qquad (15.117)$$

$$e(-\widehat{\Lambda}_0)\mathrm{ch} L(\widehat{\Lambda}_0) = \frac{1}{2}\left(\prod_{j\in\mathbb{N}}(1+e(\epsilon_j)) + \prod_{j\in\mathbb{N}}(1-e(\epsilon_j))\right). \qquad (15.118)$$

By combining equations (15.116) and (15.118) we obtain

$$\begin{aligned}
\mathrm{ch}_q L(d_\infty;\Lambda) &= \prod_{\alpha\in\Delta_+}\frac{1-q^{(\Lambda+\rho,\alpha)}}{1-q^{(\widehat{\Lambda}_0+\rho,\alpha)}}\cdot \\
&\quad \cdot\frac{1}{2}\left(\prod_{j\in\mathbb{N}}\left(1+q^{-(\Lambda+\rho,\epsilon_j)}\right)+\prod_{j\in\mathbb{N}}\left(1-q^{-(\Lambda+\rho,\epsilon_j)}\right)\right) \\
&= \prod_{1\le i<j}\frac{1-q^{{}^d\lambda_i-{}^d\lambda_j+j-i}}{1-q^{j-i}}\cdot\prod_{0\le i<j}\frac{1-q^{2c-{}^d\lambda_{i+1}-{}^d\lambda_{j+1}+j+i}}{1-q^{j+i+1}}\cdot \\
&\quad \cdot\frac{1}{2}\left(\prod_{j\in\mathbb{N}}\left(1+q^{-(\Lambda+\rho,\epsilon_j)}\right)+\prod_{j\in\mathbb{N}}\left(1-q^{-(\Lambda+\rho,\epsilon_j)}\right)\right) \qquad (15.119)
\end{aligned}$$

A little manipulation of algebra shows that the first product on the right hand side of (15.119) is the same as the first term in (15.115). We rewrite the second term in (15.119) into the product of three terms as follows:

$$\prod_{0\le i<j}\frac{1-q^{2c+j+i}}{1-q^{j+i+1}}\cdot\prod_{0\le i<n_1<j}\frac{1-q^{2c-{}^d\lambda_{i+1}-{}^d\lambda_{j+1}+j+i}}{1-q^{2c+j+i}}\cdot$$

$$\cdot \prod_{0\le i<j\le n_1} \frac{1-q^{2c-{}^d\lambda_{i+1}-{}^d\lambda_{j+1}+j+i}}{1-q^{2c+j+i}}.$$

A little further manipulation of algebra will show the first, second and third terms in the above formula are equal to the second, third and fourth terms of (15.115).

□

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