Structure Rings of Singularities and Differential Equations

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Abstract

Local function rings near singular points of a manifold are considered. On this basis, the classification of singular points of the underlying manifold is introduced. The questions concerning behavior of solutions near singular points of the manifold are discussed.

Keywords: structure ring, manifolds with singularities, singular points, conical points, cuspidal points, resurgent asymptotics.

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Contents

Introduction

The investigation of partial differential equations on manifolds with singularities, originated from the well-known paper [1] by Agmon and Nirenberg on equation on the infinite cylinder being published as early as in 1963 have been developed further in papers of a lot of mathematicians. Having no aim of presenting a full history of the question, we mention here the paper [2] published in 1967 which has originated the investigation of differential equations in domains with conical points, as well as in a lot of other works (see [3] - [10], and the bibliography therein). However, these investigation were mainly concerned with singular points of the conical type (clearly, there were exceptions from this rule; we mention here the papers [11], [12], where boundary value problems with cusp-type points on the boundary are treated in some particular case).

The appearence in the recent time of a resurgent analysis — a powerful tool of investigation of asymptotic behavior of solutions to differential equations (both ordinary and in partial derivatives) allowed to begin the serious investigation of elliptic partial differential equations on manifolds with cusp-type singularities ([13], [14], [15].) However, the description of the type of singular points of the manifold was not quite satisfactory. The matter is that the considered manifolds are topologically equivalent to a cone

$$K = \{[0,1] \times \Omega\} \,/\, \{\{0\} \times \Omega\} \tag{1}$$

with a smooth base Ω in a neighborhood of all such points (we consider here isolated singularities), and the only difference between conical and cuspidal singularities lies in the form of the differential operator in question. So, the operators with *conical degeneracy* are of the form

$$\hat{H} = r^{-m} H\left(r, \omega, -ir\frac{\partial}{\partial r}, -i\frac{\partial}{\partial \omega}\right), \tag{2}$$

near the corresponding singular point of the manifold M, where (r, ω) are coordinates on K corresponding to representation (1) and m is an order of the operator $\hat{H}$; in this case such a point is referred as a point of *conical singularity*. On the contrary, the operators of *cuspidal degeneracy* of order k are given by

$$\hat{H} = r^{-m(k+1)} H\left(r, \omega, -ir^{k+1}\frac{\partial}{\partial r}, -i\frac{\partial}{\partial \omega}\right) \tag{3}$$

with some integer $k > 0$; here we refer this point as a singular point of *cuspidal type of order k*.

So, the type of singularitiy is defined, in fact, by the class of differential operators considered in a neighborhood of the point in question. Clearly, such a description is possible, but it is preferrable to have a direct (structure) description of singularities of a manifold such that all the analysis and, in particular, the theory of differential equations is a consequence of this structure description.

This paper is an attempt to answer the question: what a point with the given type of singularity is? Clearly, if one considers a Riemannian manifold, then the type of singularity can be characterized by the type of degeneration of the considered Riemannian metrics in a neighborhood of a singular point of this manifold. However, the Riemannian structure is, clearly, an additional structure, and cannot be used as a basis of the description of singularities. This concernes also to the theory of differential equations. The Riemannian structure takes part only in the description of concrete differential operators connected with geometry of Riemannian manifolds such that the Beltrami-Laplace operator. Later on, another approach to the definition of a type of singular point is the notion of an embedded submanifold with singularities of some smooth manifold also is not quite satisfactory since it essentially depends on the embedding. At the same time, for investigation of manifolds with singularities by themselves one needs internal description of this notion.

Certainly, the most general description of a manifold (smooth or with singularities) is a description with the help of the *structure sheaf* of this manifold. All information about the manifold in question, including the form of differential operators, admissible Riemannian structures, and so on, must be expressed in terms of the structure sheaf.

Since we are intended to consider manifolds smooth everywhere except for some discrete set of singular points, a ring of germs of the structure ring at some (arbitrary) singular point is of main interest for us. In what follows, this ring will be referred as a *structure ring* of the singular point in question or even simply as structure ring if the point in question is clear from the context.

Evidently, the structure ring determines all the analysis in a neighborhood of the given point: the class of Riemannian metrices, the ring of differential equations, etc. In particular, the different choice of structure rings leads to different classes of differential equations. Namely, for different structure rings we shall arrive at equations with "conical degeneracy", "cuspidal degeneracy", and a lot of other types of degeneracy which up to now have not been considered in the theory of differential equations. Some of them are considered in the present paper.

And the last remark. As we have already mentioned, in this paper we consider only isolated singularities. This is connected with the fact that the consideration of isolated singularities is crucial for constructing a general manifold with singularities of nonisolated type. The latter can be obtained from the formers with the help of direct product operation, as it will be explained below.

1 Examples and Motivations

To motivate the approach to the description of singular points of the manifolds with singularities, we consider two examples.

1.1 Circular cone

Let us consider a circular cone K with the vertex in the origin and opening θ embedded in the three-dimensional Carthesian space $\mathbf{R}^3$ (see Figure 1.) It seems to be natural to consider the restriction of the local structure ring $[C^\infty(\mathbf{R}^3)]_0$ as a structure ring $[C^\infty(K)]_0$ of the cone K at its vertex. So, the elements of $[C^\infty(K)]_0$ will be germs of functions in

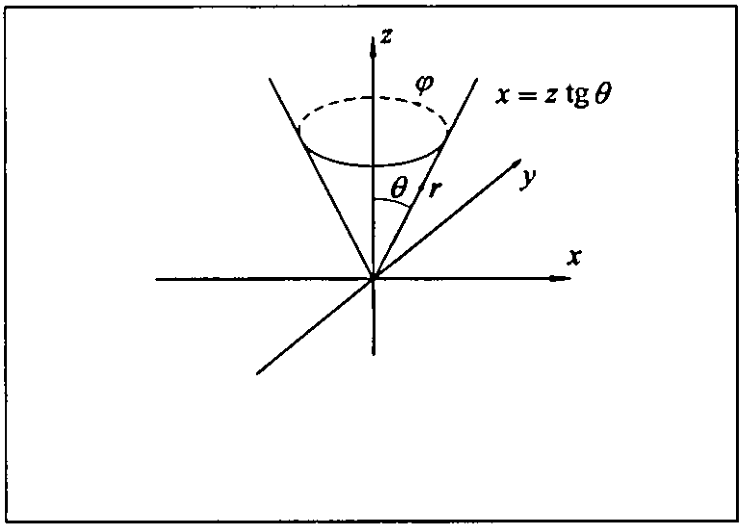

Figure 1. Circular cone.

the coordinates (r, φ) having the form

$$F(r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta)$$

for different smooth functions $F(x, y, z)$ in a neighborhood of the origin in $\mathbf{R}^3$.

However, there exists much better description of the above introduced local ring. To obtain this description, let us consider the Riemannian metrics

$$ds^2 = dr^2 + r^2\sin^2\theta d\varphi^2 \tag{4}$$

on K induced by the standard Riemannian metrics of the space $\mathbf{R}^3$. One can see that this metrics is degenerated at the point $r = 0$. To resolve this degeneration, one can perform the variable change

$$r = e^{-t}, \tag{5}$$

transforming metrics (4) to the form

$$ds^2 = e^{-2t}\left(dt^2 + \sin^2\theta d\varphi^2\right),$$

which becomes to be nondegenerated after dividing by an inessentail factor e^{-2t}. So, the more convenient representation of the cone K is not a representation of the form (1), but the representation of K as a one-point compactification of the infinite cylinder

$$K = [0, +\infty] \times S^1 \tag{6}$$

with the coordinates (t,φ) on it. For such representation, the elements of the local structure ring have the form

$$F\left(e^{-t}\sin\theta\cos\varphi, e^{-t}\sin\theta\sin\varphi, e^{-t}\cos\theta\right),\ F\in\left[C^{\infty}\left(\mathbf{R}^{3}\right)\right]_{0}$$

of smooth functions on the infinite cylinder given by (6) having *exponential stabilization at infinity.*

We remark also that such representation of the local ring gives an opportunity to define naturally the class of *differential operators* to be considered on the cone K. Namely, they must be obtained by the variable change (5) from differential operators of the canonical form

$$\hat{H}=H\left(e^{-t},\varphi,-i\frac{\partial}{\partial t},\frac{\partial}{\partial\varphi}\right)$$

with coeficients stabilizing with exponential speed (that is, belonginig to the above defined local ring). This leads us to a class of differential operators of the form

$$\hat{H}=H\left(r,\varphi,-ir\frac{\partial}{\partial r},\frac{\partial}{\partial\varphi}\right)$$

exactly coinciding with operators (2) with conical degeneracy.

1.2 Circular cusp

Let us consider the circular cusp C_0 obtained by rotation of the parabola

$$x=z^{k+1}$$

of order k around the axis Oz (see Figure 2). Then elements of the restriction of the ring $\left[C^{\infty}\left(\mathbf{R}^{3}\right)\right]_{0}$ to C_0 have the form

$$F\left(z^{k+1}\cos\varphi, z^{k+1}\sin\varphi, z\right).$$

The corresponding Riemannian metrics is

$$ds^{2}=\left(1+(k+1)^{2}z^{2k}\right)dz^{2}+z^{2k+2}d\varphi^{2}. \tag{7}$$

The variable change eliminating the degeneracy of the metrics (7) is

$$t=\frac{1}{kz^{k}},\ z=\frac{1}{k^{1/k}t^{1/k}}. \tag{8}$$

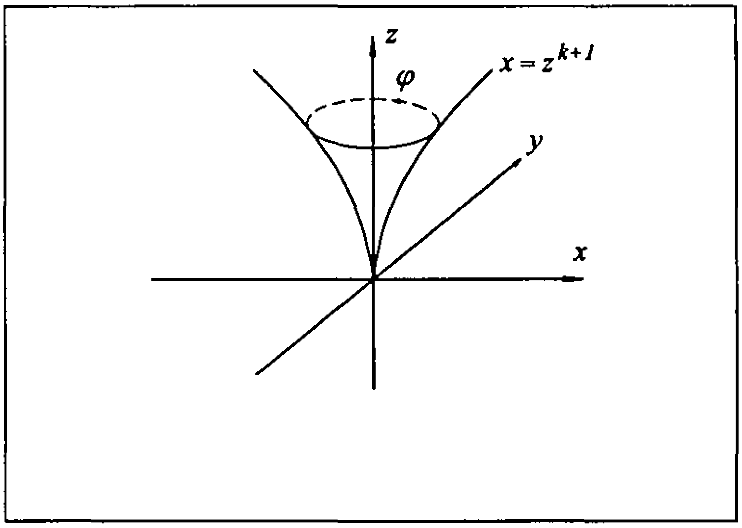

Figure 2. Circular cusp.

The latter variable change transforms the metrics (7) to the form

$$ds^2 = \left(\frac{1}{kt}\right)^{2(1+1/k)} \left[\left(1 + \frac{(k+1)^2}{k^2t^2}\right) dt^2 + d\varphi^2\right]$$

which is evidently nondegenerated in a neighborhood of $t = \infty$.

Now the local ring $[C^\infty(C_0)]_0$ can be described as a ring of functions

$$F\left(t^{-1/k}, \varphi\right), \; F \in C^\infty$$

with *power stabilization at infinity* (with the speed $t^{-1/k}$.)

This description of the local ring leads us again to the description of differential operators which are naturally defined on the manifold near the cusp point. Namely, these must be the images of the operators

$$\hat{H} = H\left(\frac{1}{t^{1/k}}, \varphi, -i\frac{\partial}{\partial t}, -i\frac{\partial}{\partial \varphi}\right)$$

under the action of variable change (8). These operators have the form

$$\hat{H} = H\left(k^{1/k}z, \varphi, -iz^{k+1}\frac{\partial}{\partial z}, -i\frac{\partial}{\partial \varphi}\right),$$

which again coincide with (3).

1.3 Conclusions

The examples considered in the Introduction show that:

- The convenient *topological model* of the isolated singular point of a manifold is a one-point compactification
$$M_{\mathrm{loc}} = C \cup \{\infty\}$$
of a infinite half-cylinder
$$C = \overline{\mathbf{R}_+} \times \Omega \tag{9}$$
with a smooth compact base Ω without boundary rather than a topological cone
$$K = \{[0,1] \times \Omega\} / \{\{0\} \times \Omega\} . \tag{10}$$
Topological models (9) and (10) are homeomorphic to each other since a segment $[0,1]$ is homeomorphic to the compactificated half-axis $\mathbf{R}_+ \cup \{\infty\}$ with 0 corresponding to ∞, but the description of local rings looks more clear on the model M_{loc}.

- The *local rings determining a type* of an isolated singular point of M are described with the help of the speed of stabilization of functions given on the local model M_{loc} as $t \to \infty$. More exactly, if $\varphi(t)$ is a smooth function on $\mathbf{R}_+$ such that
$$\lim_{t \to 0} \varphi(t) = 0,$$
(in what follows we refer $\varphi(t)$ as a *weight function*) then the appropriate candidate for the role of a local ring describing the type of the singular point under consideration is a set of functions
$$u(t,\omega) = F(\varphi(t), \omega), \; F(r,\omega) \in C^\infty([0,1] \times \Omega) . \tag{11}$$
Here by ω we denote local coordinates on the manifold Ω. Clearly, the set of functions of the type (11) have to be modified up to a ring of functions closed with respect to differentiation so that this modified ring could serve as a local ring for the manifold M at the singular point in question.

- The ring of local differential operators on the manifold M_{loc} must be defined now as a *ring of operators* of the form
$$\boxed{\sum_{j=0}^{m} \sum_{|\alpha| \le m-l} a_{j\alpha}(t,\omega) \left(-i\frac{\partial}{\partial t}\right)^j \left(-i\frac{\partial}{\partial \omega}\right)^\alpha} \tag{12}$$

with coefficients from the local structure ring at the singular point. One more reason for the requirement that the local ring must be closed under the differentiation is that this requirement guarantees that the set of operators (12) form a ring (algebra.)

In the following section we construct some scale of local rings on M_{loc} describing a scale of types of isolated singularities of the manifold M. Clearly, the scale presented below does not pretend to be a complete classification of types of singular points of a manifold M with singularities. For example, one can consider singular points for which the speed of stabilization depends on the point ω of the base Ω. Besides it seems to be impossible to write down all weight functions $\varphi(t)$ of the above described type, so we restrict ourselves by consideration of the most interesting cases which appear in applications.

Further, as it was already mentioned in the Introduction, we consider here only isolated singularities of the underlying manifold M. More complicated singularities can be obtained by iterations with the help of direct product operations. For example, we can consider a wedge of the type $\varphi(t)$ as a direct product of the model manifold $(M_{\text{loc}}, \varphi(t))$ with a smooth manifold X or a corner of the type $(\varphi(t), \psi(t))$ as a product of $(M_{\text{loc}}, \varphi(t))$ by some manifold N having isolated singularities of the type $(M_{\text{loc}}, \psi(t))$.

2 Structure rings of singular points

2.1 Power stabilization

Let us consider the case of power stabilization, that is

$$\varphi(t) = t^{-\gamma}$$

for some positive value of γ (this value is fixed for each given type of singularity.) Denote by $\mathcal{R}^\gamma$ the local ring

$$\mathcal{R}^\gamma = \left\{u(t,\omega) = F\left(t^{-\gamma},\omega\right) \left| F(r,\omega) \in C^\infty([0,1]\times\Omega)\right.\right\}$$

(we consider only germs of functions $u(t,\omega)$ at infinity.) This ring is not in general closed with respect to differentiation since

$$\frac{\partial}{\partial t}F\left(t^{-\gamma},\omega\right) = -\gamma t^{-\gamma-1}\frac{\partial F}{\partial r}\left(t^{-\gamma},\omega\right)$$

cannot be represented in the form $G\left(t^{-\gamma},\omega\right)$ for a C^∞-function $G(r,\omega)$ on $[0,1]\times\Omega$. So, to obtain a structure ring describing a singular point with power stabilization, one have to extend this ring up to a ring closed under differentiation. Such an extension is given by

$$\left\{u(t,\omega) = F\left(t^{-\gamma},\omega\right) + t^{-\gamma}G\left(t^{-\gamma},t^{-1},\omega\right) \left| F(r,\omega), G(r,\rho,\omega) \in C^\infty\right.\right\};$$

we denote this ring by $\mathcal{R}^{\gamma}_{\text{loc}}$. The corresponding singular point of the manifold considered with such local ring will be called a *singular point of power stabilization*; in Section 4 we shall show that geometrically such point is a cusp point of degree $\beta = 1 + \gamma^{-1}$.

Remark 1 There is an important particular case of a singular point of power stabilization. This case corresponds to the values of γ of the form $\gamma = k^{-1}$ for *positive integers* $k \in \mathbf{Z}_{+}$. In this case one has

$$\mathcal{R}^{1/k} = \mathcal{R}^{1/k}_{\text{loc}}$$

since the power t^{-1} can be represented as a C^{∞}-function of the variable $t^{-1/k}$:

$$t^{-1} = \left(t^{-1/k}\right)^{k}.$$

As we shall see below, geometrically this casse corresponds to the caspidal point of integer degree $k+1$.

2.2 Exponential stabilization of first order

This case (which is in some sense a limit case of power stabilization as $\alpha \to \infty$) is determined by the weight function

$$\varphi(t) = e^{-t}.$$

As above, we consider a local ring

$$\mathcal{R}^{\exp} = \left\{u(t,\omega) = F\left(e^{-t},\omega\right) | F(r,\omega) \in C^{\infty}([0,1]\times\Omega)\right\}.$$

Since

$$\frac{\partial}{\partial t}F\left(e^{-t},\omega\right) = -e^{-t}\frac{\partial F}{\partial r}\left(e^{-t},\omega\right),$$

this ring is closed under differentiations and, hence, can serve as a local ring on manifold with singularities corresponding to a singular point. It will be denoted also by

$$\mathcal{R}^{\exp}_{\text{loc}} = \mathcal{R}^{\exp}.$$

The corresponding singular point of the manifold considered with such structure ring will be called a *singular point of (simple) exponential stabilization.* As we shall see in Section 4, such point is none more than a *conical point* of singularity of the manifold M under consideration.

Remark 2 With the same function $\varphi(t) = e^{-t}$, one can consider also a ring

$$\mathcal{R}^{\exp,1}_{\text{loc}} = \left\{u(t,\omega) = F\left(e^{-t},t^{-1},\omega\right) | F(r,\rho,\omega) \in C^{\infty}([0,1]\times[0,1]\times\Omega)\right\}.$$

Functions from this ring possess a power, not exponential, stabilization, but the consideration of singular points of this type is not quite senseless.

2.3 Exponential stabilization of an arbitrary order

For construction of such a ring, one have to consider the weight function

$$\varphi(t) = e^{-t^\gamma} \tag{13}$$

for a real positive γ. The local ring of functions with this type of stabilization can be described as follows:

$$\mathcal{R}^{\gamma,\exp} = \left\{ u(t,\omega) = F\left(e^{-t^\gamma},\omega\right) \left| F(r,\omega) \in C^\infty([0,1]\times\Omega)\right.\right\}.$$

This ring is not closed with respect to the differentiation since

$$\frac{\partial}{\partial t} F\left(e^{-t^\gamma},\omega\right) = -\gamma t^{\gamma-1} e^{-t^\gamma} \frac{\partial F}{\partial r}\left(e^{-t^\gamma},\omega\right).$$

The closure of this ring is given by

$$\begin{aligned} \mathcal{R}^{\gamma,\exp}_{\mathrm{loc}} &= \left\{ u(t,\omega) = F\left(e^{-t^\gamma},\omega\right) + t^{N\gamma} e^{-t^\gamma} G\left(t^{-1}, t^{-\gamma}, e^{-t^\gamma},\omega\right) \right. \\ &\qquad \left. \left| F(r,\omega), G(r_1,r_2,r_3,\omega) \in C^\infty \right.\right\}. \end{aligned}$$

This is exactly the structure ring corresponding to speed of decay (13). The corresponding singular point of the manifold considered with such local ring will be called a *singular point of exponential stabilization of degree γ.*

The important particular case of this singularity type corresponds to integer values of $\gamma = k$. Since in this case the power t^γ is a C^∞-function of the variable t^{-1}, the description of the structure ring for such values of γ can be simplified. Namely, we have

$$\begin{aligned} \mathcal{R}^{k,\exp}_{\mathrm{loc}} &= \left\{ u(t,\omega) = F\left(e^{-t^k},\omega\right) + t^{N} e^{-t^k} G\left(t^{-1}, e^{-t^\gamma},\omega\right) \right. \\ &\qquad \left. \left| F(r,\omega), G(r_1,r_2,\omega) \in C^\infty \right.\right\} \end{aligned}$$

The geometry of such kind of singularity will be described in Section 4; as far as we know, the singularities of this type have not been examined in the literature (this concernes also the next type of singularity.)

2.4 Strong exponential stabilization

For constructing *singular points of strong exponential stabilization* we consider the weight function

$$\varphi(t) = e^{-e^t}.$$

Consider the local ring

$$\mathcal{R}^{s,\exp} = \left\{ u(t,\omega) = F\left(e^{-e^t},\omega\right) \left| F(r,\omega) \in C^\infty([0,1]\times\Omega)\right.\right\}.$$

This ring is not closed with respect to the differentiation since

$$\frac{\partial}{\partial t}F\left(e^{-e^t},\omega\right)=-e^te^{-e^t}\frac{\partial F}{\partial r}\left(e^{-e^t},\omega\right).$$

The closure of this ring up to a ring closed under differentiation is given by

$$\begin{aligned}\mathcal{R}^{s,\exp}_{\text{loc}} &= \left\{u\left(t,\omega\right)=F\left(e^{-e^t},\omega\right)+P_N\left(e^t\right)e^{-e^t}G\left(e^{-e^t},\omega\right)\right. \\ &\quad \left|F\left(r,\omega\right),G\left(r,\omega\right)\in C^\infty\right\}.\end{aligned}$$

This is exactly the structure ring corresponding to a singular point of strong exponential stabilization. The reason why we have included into consideration this type of singular points will be clear in the next section.

3 Local rings of differential operators

In this section, we come back to the conical model of a manifold with singularities near an isolated singular point. We recall that topologically the model for singularities of all the above described types is the cone

$$K=\left\{[0,1]\times\Omega\right\}/\left\{\{0\}\times\Omega\right\}. \tag{14}$$

To rewrite the definition of a local ring to this model it suffices to use the variable change

$$r=\varphi\left(t\right),$$

where (r,ω) are coordinates on the cone K corresponding to representation (14). Our aim is to describe the form of differential operators corresponding to different types of singular points in terms of the coordinates (r,ω).

3.1 Power stabilization

First of all we remark that if we use the standard local representation of M near its singular point in the form of one-point compactification of a half-cylinder, then the form of differential operators is quite clear. Namely, any differential operator of order m must be represented in the form

$$\hat{H}=\sum_{j=0}^{m}\sum_{|\alpha|\le m-j}a_{j\alpha}\left(t,\omega\right)\left(-i\frac{\partial}{\partial t}\right)^j\left(-i\frac{\partial}{\partial\omega}\right)^\alpha \tag{15}$$

with the coefficients $a_{j\alpha}(t,\omega)$ from $\mathcal{R}^{\gamma}_{\rm loc}$ near each point of power stabilization of degree γ. The only thing rest is to perform the variable change

$$r = t^{-\gamma}$$

in expression (15) of the differential operator $\hat{H}$. Since for any smooth function $f(t,\omega) = F(t^{-\gamma},\omega)$ we have

$$\frac{\partial f}{\partial t}(t,\omega) = -\gamma t^{-\gamma-1}\frac{\partial F}{\partial r}(t^{-\gamma},\omega) = -\gamma r^{1+1/\gamma}\frac{\partial F}{\partial r}(t^{-\gamma},\omega).$$

So,

$$-i\frac{\partial}{\partial t} \longmapsto i\gamma r^{1+1/\gamma}\frac{\partial}{\partial r},$$

and, hence, operator (15) can be rewritten in the form

$$\hat{H} = \sum_{j=0}^{m}\sum_{|\alpha|\le m-j} b_{j\alpha}(r^{-1/\gamma},\omega)\left(-ir^{1+1/\gamma}\frac{\partial}{\partial r}\right)^{j}\left(-i\frac{\partial}{\partial\omega}\right)^{\alpha},$$

where $b_{j\alpha}(t,\omega) = (-\gamma)^{|\alpha|}a_{j\alpha}(t,\omega)$ are functions from $\mathcal{R}^{\gamma}_{\rm loc}$:

$$\begin{aligned} b_{j\alpha}(r^{-1/\gamma},\omega) &= b'_{j\alpha}(t^{-\gamma},\omega) + t^{-\gamma}b''_{j\alpha}(t^{-\gamma},t^{-1},\omega)\big|_{t=r^{-1/\gamma}} \\ &= b'_{j\alpha}(r,\omega) + rb''_{j\alpha}(r,r^{1/\gamma},\omega) \end{aligned}$$

with some C^{∞}-functions $b'_{j\alpha}(r,\omega)$ and $b''_{j\alpha}(r,\rho,\omega)$. Finally, we obtain a form of differential operator near a singular point of power stabilization as

$$\hat{H} = \sum_{j=0}^{m}\sum_{|\alpha|\le m-j}\left(b'_{j\alpha}(r,\omega) + rb''_{j\alpha}(r,r^{1/\gamma},\omega)\right)\left(-ir^{1+1/\gamma}\frac{\partial}{\partial r}\right)^{j}\left(-i\frac{\partial}{\partial\omega}\right)^{\alpha}. \quad (16)$$

Consider now the particular case $\gamma = 1/k$, $k \in \mathbf{Z}_{+}$. In this case, as it follows from Remark 1, the function $rb''_{j\alpha}$ can be omitted, and we arrive at the representation of a differential operator $\hat{H}$ of the form

$$\hat{H} = \sum_{j=0}^{m}\sum_{|\alpha|\le m-j} b_{j\alpha}(r,\omega)\left(-ir^{1+k}\frac{\partial}{\partial r}\right)^{j}\left(-i\frac{\partial}{\partial\omega}\right)^{\alpha},$$

which coincides with the form (3) of an operator with cuspidal degeneracy up to an inessential factor $r^{-m(k+1)}$. This gives rise to a guess that such points are exactly the singular points of the manifold of the cuspidal type of order k. This guess will be confirmed in the next section.

Remark 3 The above considerations show, that the correct form of a differential operator at a cuspidal point of noninteger order is (16).

3.2 Exponential stabilization of first order

To obtain the form of differential operator for this type of stabilization, we have to perform the variable change

$$r = e^{-t} \tag{17}$$

in expression (15). Since for any function of the form $f(t,\omega) = F(e^{-t},\omega)$ we have

$$\frac{\partial f}{\partial t}(t,\omega) = -e^{-t}\frac{\partial F}{\partial r}(e^{-t},\omega) = -r\frac{\partial F}{\partial r}(r,\omega),$$

hence, the operator $-i\partial/\partial t$ is transformed into the operator

$$-i\frac{\partial}{\partial t} \longmapsto ir\frac{\partial}{\partial r},$$

under the action of the variable change (17). Having in mind the fact that the coefficients of operator (15) must belong to the ring $\mathcal{R}_{\mathrm{loc}}^{\mathrm{exp}}$, we see that this operator can be represented in the form

$$\hat{H} = \sum_{j=0}^{m} \sum_{|\alpha|\leq m-j} b_{j\alpha}(r,\omega)\left(-ir\frac{\partial}{\partial r}\right)^{j}\left(-i\frac{\partial}{\partial \omega}\right)^{\alpha},$$

(where $b_{j\alpha}(r,\omega) \in C^{\infty}([0,1]\times\Omega)$) of an operator with conical degeneracy (up to an inessential factor r^{-m}.) This allows us to refer *singular points with exponential stabilization as conical points* of a manifold M. This will be confirmed from the geometrical viewpoint in the next section.

3.3 Exponential stabilization of an arbitrary order

Let us derive the form of differential operator near a singular point of exponential stabilization of degree γ. Performing the variable change

$$r = e^{-t^{\gamma}}$$

in the expression (15), we see that the operator $-i\partial/\partial t$ is tranformed into the operator

$$-i\frac{\partial}{\partial t} \longmapsto i\gamma\left(\ln\frac{1}{r}\right)^{1-1/\gamma} r\frac{\partial}{\partial r}$$

with some integers $N_{j\alpha}$, and, hence, operator (15) is transformed to an operator

$$\hat{H} = \sum_{j=0}^{m}\sum_{|\alpha|\le m-j} a_{j\alpha}\left(\left(\ln\frac{1}{r}\right)^{1/\gamma},\omega\right)\left(i\gamma r\left(\ln\frac{1}{r}\right)^{1-1/\gamma}\frac{\partial}{\partial r}\right)^{j}\left(-i\frac{\partial}{\partial\omega}\right)^{\alpha},$$

with $a_{j\alpha}(t,\omega)\in\mathcal{R}_{\mathrm{loc}}^{\gamma,\exp}$. This means that

$$a_{j\alpha}(t,\omega) = a'_{j\alpha}\left(e^{-t^{\gamma}},\omega\right) + t^{N_{j\alpha}\gamma}e^{-t^{\gamma}}a''_{j\alpha}\left(t^{-1},t^{-\gamma},e^{-t^{\gamma}},\omega\right),$$

and, hence, operator (15) takes the form

$$\begin{aligned}\hat{H} &= \sum_{j=0}^{m}\sum_{|\alpha|\le m-j}\left(a'_{j\alpha}(r,\omega) + \left(\ln\frac{1}{r}\right)^{N_{j\alpha}} a''_{j\alpha}\left(\left(\ln\frac{1}{r}\right)^{-1/\gamma},\left(\ln\frac{1}{r}\right)^{-1},r,\omega\right)\right)\\ &\quad\times\left(i\gamma r\left(\ln\frac{1}{r}\right)^{1-1/\gamma}\frac{\partial}{\partial r}\right)^{j}\left(-i\frac{\partial}{\partial\omega}\right)^{\alpha}. \qquad (18)\end{aligned}$$

As far as we know, the operators of this kind were not considered in the literature (for $\gamma = 1$ we obtain the simple exponential stabilization considered in the previous section, the case $0 < \gamma < 1$ lyies between simple exponential stabilization and power stabilization, and the case $\gamma > 1$ corresponds to more strong exponential stabilization than a simple one.) The geometry of singular points of this type will be considered in the next section.

3.4 Strong exponential stabilization

This case is interesting mainly due to the fact that it in some sense is a *limit case* of a point with exponential stabilization with degree γ as $\gamma \to \infty$. We shall not present here all the computations for this case and simply write down the answer. Namely, the form of a differential operator in this case is

$$\begin{aligned}\hat{H} &= \sum_{j=0}^{m}\sum_{|\alpha|\le m-j}\left(a'_{j\alpha}(r,\omega) + rP_{N}^{j\alpha}\left(\ln\frac{1}{r}\right)a''_{j\alpha}(r,\omega)\right)\\ &\quad\times\left(ir\left(\ln\frac{1}{r}\right)\frac{\partial}{\partial r}\right)^{j}\left(-i\frac{\partial}{\partial\omega}\right)^{\alpha}.\end{aligned}$$

This operator can be obtained from operator (18) with the help of the limit $\gamma \to \infty$. The corresponding geometry will be considered in the next section.

4 Riemannian metrices and geometry of singularities

In this section, we shall consider how local rings introduced in Section 2 define the class of Riemannian metrices compatible with these rings. First of all, it is clear that admissible metrices on the one-point compactification of the infinite cylinder

$$K = [0, +\infty] \times \Omega$$

have the form

$$ds^2 = c(t) \left(g_{00}(t, \omega)\, dt^2 + \sum_{i,j=1}^{n} g_{ij}(t, \omega)\, d\omega^i d\omega^j \right) \tag{19}$$

(we have taken into account the structure of the direct product on K.) The normalizing factor $c(t)$ must be chosen such that the distance from the infinite point of one-point compactification to any finite point of K is finite. This can be achieved in the following way.

Let us perform the variable change

$$r = \varphi(t),\ t = \varphi^{-1}(r)$$

(we recall that the function φ directly multiplied by the identity diffeomorphism of Ω determines a diffeomorphism between K and the cone

$$K = \{[0, 1] \times \Omega\} / \{\{0\} \times \Omega\}$$

without its vertex such that the vertex of the cone comes to the infinity point of one-point compactification.) Metrics (19) will be transformed into the metrics

$$ds^2 = c(t) \left((\varphi'(t))^{-2} g_{00}(t, \omega)\, dr^2 + \sum_{i,j=1}^{n} g_{ij}(t, \omega)\, d\omega^i d\omega^j \right) \Bigg|_{t=\varphi^{-1}(r)},$$

and the required normalization will be achieved if we put

$$c(t)\,(\varphi'(t))^{-2} = 1. \tag{20}$$

Remark 4 The latter condition can be replaced with the requirement that the coefficient of dr is integrable up to $r = 0$. In most cases condition (20) will be sufficient.

Let us consider this procedure for all above defined types of stabilization.

4.1 Power stabilization

Here we have

$$\varphi(t) = t^{-\gamma},$$

and, hence, the general expression for the Riemannian metrics has the form (19) with $g_{00}(t,\omega)$, $g_{ij}(t,\omega)$ from the ring $\mathcal{R}^{\gamma}_{\mathrm{loc}}$. Hence, this expression has the form

$$\begin{aligned} ds^2 &= c(t)\left[\left(g'_{00}\left(t^{-\gamma},\omega\right) + t^{-\gamma} g''_{00}\left(t^{-\gamma},t^{-1},\omega\right)\right) dt^2 \right. \\ &\left. + \sum_{i,j=1}^{n} g'_{ij}\left(t^{-\gamma},\omega\right) + t^{-\gamma} g''_{ij}\left(t^{-\gamma},t^{-1},\omega\right)\, d\omega^i d\omega^j \right], \end{aligned}$$

or, after the variable change $r = t^{-\gamma}$,

$$\begin{aligned} ds^2 &= c\left(r^{-1/\gamma}\right)\left[\gamma^{-2} r^{-2(1+1/\gamma)} \left(g'_{00}(r,\omega) + r g''_{00}\left(r,r^{\gamma},\omega\right)\right) dr^2 \right. \\ &\left. + \sum_{i,j=1}^{n} \left(g'_{ij}(r,\omega) + r g''_{ij}\left(r,r^{\gamma},\omega\right)\right)\, d\omega^i d\omega^j \right]. \end{aligned}$$

Taking into account relation (20), we obtain finally

$$\begin{aligned} ds^2 &= \left[\left(g'_{00}(r,\omega) + r g''_{00}\left(r,r^{\gamma},\omega\right)\right) dr^2 \right. \\ &\left. + \gamma^2 r^{2(1+1/\gamma)} \sum_{i,j=1}^{n} \left(g'_{ij}(r,\omega) + r g''_{ij}\left(r,r^{\gamma},\omega\right)\right)\, d\omega^i d\omega^j \right]. \end{aligned}$$

In the special case $\gamma = 1/k$, $k \in \mathbf{Z}_+$, we have

$$ds^2 = \left[g_{00}(r,\omega)\, dr^2 + r^{2(1+k)} \sum_{i,j=1}^{n} g_{ij}(r,\omega)\, d\omega^i d\omega^j \right];$$

we remark that the metrix (7) om the cusp of degree k has exactly this form. So, the *power stabilization corresponds geometrically to cusps of power degree.*

4.2 Exponential stabilization of first order

Having in mind that

$$\varphi(t) = e^{-t},$$

we write down the expression of the general form of the metrics near a singular point with exponential stabilization of the manifold M:

$$ds^2 = c(t)\left(g_{00}\left(e^{-t},\omega\right) dt^2 + \sum_{i,j=1}^{n} g_{ij}\left(e^{-t},\omega\right)\, d\omega^i d\omega^j \right)$$

(we have taken into account that the coefficients g_{00}, g_{ij} have to be chosen from the ring $\mathcal{R}^{\exp}$.) After the variable change

$$r = e^{-t},$$

we arrive at the expression

$$ds^2 = c\left(\ln\frac{1}{r}\right)\left(r^{-2}g_{00}\left(e^{-t},\omega\right)dt^2 + \sum_{i,j=1}^{n} g_{ij}\left(e^{-t},\omega\right)\, d\omega^i d\omega^j\right).$$

Using (20), we obtain finally

$$ds^2 = g_{00}\left(e^{-t},\omega\right)dt^2 + r^2\sum_{i,j=1}^{n} g_{ij}\left(e^{-t},\omega\right)\, d\omega^i d\omega^j,$$

which includes, in particular, the expression (4) for a metrix on the circular cone. Hence, we see that the *exponential stabilization corresonds to the conical singularities.*

4.3 Exponential stabilization of an arbitrary order

Let us find out what geometry corresponds to the exponential stabilization with an arbitrary degree $\gamma > 1$. First, let us derive the general expression for a Riemannian metrics near a singular point of this type. Since the coefficients g_{ij} of this metrics must be chosen from $\mathcal{R}_{\mathrm{loc}}^{\gamma,\exp}$, we have

$$\begin{aligned} ds^2 &= c(t)\left[\left(g'_{00}\left(e^{-t^\gamma},\omega\right) + t^{N_{00}\gamma}e^{-t^\gamma}g''_{00}\left(t^{-1},t^{-\gamma},e^{-t^\gamma},\omega\right)\right)dt^2\right. \\ &\quad \left.+\sum_{i,j=1}^{n}\left(g'_{ij}\left(e^{-t^\gamma},\omega\right) + t^{N_{ij}\gamma}e^{-t^\gamma}g''_{ij}\left(t^{-1},t^{-\gamma},e^{-t^\gamma},\omega\right)\right)d\omega^i d\omega^j\right], \end{aligned}$$

where N_{ij} are some nonnegative integers. Having in mind that $r = e^{-t^\gamma}$, and using the relation (20), we obtain

$$\begin{aligned} ds^2 &= \left[\left(g'_{00}(r,\omega) + r\left(\ln\frac{1}{r}\right)^N g''_{00}\left(\left(\ln\frac{1}{r}\right)^{-1/\gamma},\left(\ln\frac{1}{r}\right)^{-1},r,\omega\right)\right)dr^2\right. \\ &\quad +\gamma^2r^2\left(\ln\frac{1}{r}\right)^{2(1-1/\gamma)}\sum_{i,j=1}^{n}\left(g'_{ij}(r,\omega)\right. \\ &\quad \left.\left.+r\left(\ln\frac{1}{r}\right)^N g''_{ij}\left(\left(\ln\frac{1}{r}\right)^{-1/\gamma},\left(\ln\frac{1}{r}\right)^{-1},r,\omega\right)\right)d\omega^i d\omega^j.\right. \end{aligned} \tag{21}$$

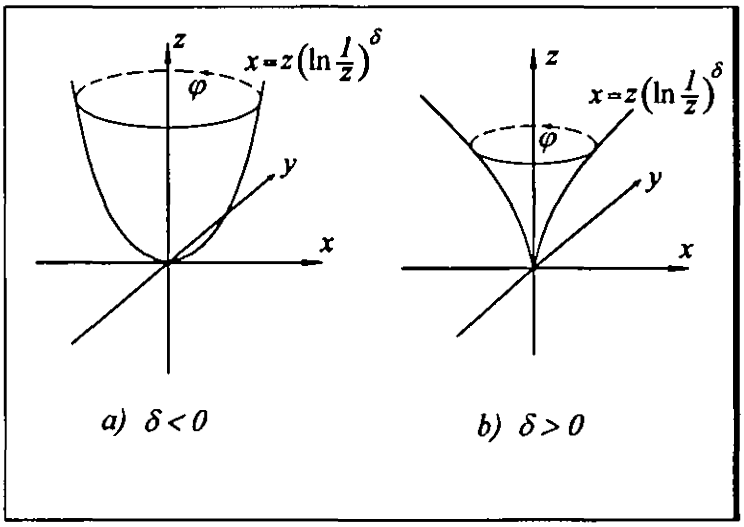

Figure 3. Week smoothness and week cusp.

Now let us try to construct an embedding of some two-dimenional surface into $\mathbf{R}^3$ for which the induced metrics have the form (21). To do this, we consider the surface S in $\mathbf{R}^3$ obtained by rotation of the curve

$$x = z\left(\ln\frac{1}{z}\right)^{\delta}$$

with some real δ around the Oz axis (see Figure 3). One can easily compute the expression for the Riemannian metrics on S inducer from $\mathbf{R}^3$:

$$ds^2 = \left\{1 + \left[\left(\ln\frac{1}{z}\right)^{\delta} - \delta\left(\ln\frac{1}{z}\right)^{\delta-1}\right]\right\} dz^2 + z^2\left(\ln\frac{1}{z}\right)^{2\delta} d\varphi^2,$$

which coincides with (21) for $\delta = 1 - 1/\gamma$. So, the considered type of singularity corresponds to a surface with week smoothness at its singular point for $\delta < 0$ (see Figure 3 a)), and with week cusp for $\delta > 0$ (see Figure 3 b)); in the last case the latter expression needs some renormalization (cf. Remark 4 above.)

4.4 Strong exponential stabilization

For this case one has

$$ds^2 \;=\; \left\{\left[g'_{00}(r,\omega) + rP^{(00)}_{N_{00}}\left(\ln\frac{1}{r}\right) g''_{00}(r,\omega)\right]\right.$$

$$+\sum_{i,j=1}^{n}\left[g'_{ij}(r,\omega)+rP^{(ij)}_{N_{ij}}\left(\ln\frac{1}{r}\right)g''_{ij}(r,\omega)\right]d\omega^i d\omega^j,$$

where $P^{(00)}_{N_{00}}$, $P^{(ij)}_{N_{ij}}$ are polynomials of degree N_{00}, N_{ij}, respectively. The corresponding embedded surface can be obtained as a rotation of the curve

$$x = z\ln\frac{1}{z}$$

around the Oz axis. The Riemannian metrics induced from $\mathbf{R}^3$ is

$$ds^2=\left[1+\left(\ln\frac{1}{z}-1\right)^2\right]dz^2+z^2\left(\ln\frac{1}{z}\right)^2 d\varphi^2$$

(we leave all the computations to the reader.) This surface is drawn on Figure 3 b) with $\delta = 1$.

5 Concluding remarks

5.1 Week power tangency

Here we consider the case of a two-dimensional surface in $\mathbf{R}^3$ obtained by the rotation of the curve

$$x = z^{\beta}$$

for $0<\beta<1$. It is easy to see that the induced metrics in this case has the form

$$ds^2=\left[1+\beta^2 z^{2(\beta-1)}\right]dz^2+z^{2\beta}d\varphi^2.$$

Since $\beta - 1 < 0$, we transform this expression in the following way:

$$ds^2 = z^{2(\beta-1)}\left\{\left[z^{2(1-\beta)}+\beta^2\right]dz^2+z^2 d\varphi^2\right\},$$

which corresponds to the stabilization speed

$$\varphi(t) = e^{-\beta t}.$$

This can be confirmed also by a form of the Beltrami-Laplace operator on this surface:

$$\begin{aligned}\Delta_g &= z^{-2\beta}\left\{\frac{1}{z^{2(1-\beta)}+\beta^2}\left(z\frac{\partial}{\partial z}\right)^2-\frac{(1-\beta)\,z^{2(1-\beta)}}{\left(z^{2(1-\beta)}+\beta^2\right)^2}\right.\\ &\quad\left.\times\left(z\frac{\partial}{\partial z}\right)+\frac{\partial^2}{\partial\varphi^2}\right\}.\end{aligned}$$

The details of the consideration of this example are left to the reader.

5.2 Types of asymptotics

Here we consider the types of asymptotic expansions for solutions to differential equations

$$\hat{H}u = 0 \tag{22}$$

for differential operator $\hat{H}$ near points of singularity of underlying manifold M of different types. It will be convenient to deal with these operators written on the one-point compactification of a half-cylinder

$$C = \overline{\mathbf{R}_+} \times \Omega$$

as on the local model. Then equation (22) can be rewritten in the form

$$\sum_{j=0}^{m} \sum_{|\alpha| \leq m-j} a_{j\alpha}(t,\omega) \left(-i\frac{\partial}{\partial t}\right)^j \left(-i\frac{\partial}{\partial \omega}\right)^\alpha u = 0,$$

where the coefficients belong to the corresponding local ring. Roughly speeking, these functions are functions of the type

$$a_{j\alpha}(t,\omega) = A_{j\alpha}(\varphi(t),\omega)$$

(the closing of the ring in question with respect to the differentiation will not affect our conclusions) with some C^∞-functions $A_{j\alpha}(r,\omega)$. The soluions to equation (22) with constant coefficients are exponentials $e^{\lambda t}$. Hence, as it follows from the paper [16], solutions to equation (22) will have resurgent type (see [13], [14]) if and only if the decay of the function $\varphi(t)$ is weaker than exponential one. So, it is clear that one should obtain resurgent asymptotics in the caspidal types of singularities and only conormal type asymptotics in all cases with stronger speed of stabilization. These asymptotic expansions can be obtained by the scheme described in [17], [13], and the finiteness theorems (Fredholm property) — by noncommutative analysis methods (see [14], [15]).

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