The Nielsen-Thurston classification and automorphisms of a free group I

Z. Sela

Columbia University
New York NY 10027

USA

and

Hebrew University
Jerusalem 91904

Israel

Max-Planck-Institut für Mathematik
Gottfried-Claren-Straße 26
53225 Bonn

Germany

MPI / 94-83

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Z. SELA[1]

In a sequence of 2 papers we construct a hierarchical decomposition of a free group with respect to a given automorphism of it. This hierarchical decomposition is shown to be the analogue of the Nielsen-Thurston classification for automorphisms of a free group. In the first paper we introduce a dynamical-algebraic commutative diagram and use Rips' analysis of (stable) group actions on real trees to get a "uniform" approach to the Nielsen-Thurston classification and the Scott conjecture for automorphisms of a free group. In the second paper we introduce further dynamical invariants which allow us to obtain the hierarchical decomposition using the methods described in this paper.

Using train tracks and invariant laminations, Thurston has developed a whole theory to understand the dynamics and geometry of diffeomorphisms of surfaces ([Th], [Ca-Bl]). By introducing a combinatorial analogue of train tracks M. Bestvina and M. Handel [Be-Ha] have managed to analyse irreducible automorphisms of a free group, and using this analysis to bound the rank of the fixed subgroup of an automorphism by the rank of the ambient group, which was known before as the Scott conjecture.

In [Sel] borrowing Jaco-Shalen-Johannson theory of the characteristic submanifold, the author introduced a cononical decomposition for freely indecomposable (Gromov) hyperbolic groups, which serves as a fundamental object for generalizing results from the mapping class group of surfaces to automorphisms and the automorphism group of freely indecomposable hyperbolic groups, and in particular to generalize Thurston's work to this class of groups. This JSJ decomposition was later generalized to single ended f.p. groups in [Ri-Sel].

The construction of the JSJ decomposition, which uses extensively Rips' work on real trees, succeeds in generalizing Thurston's theory but has not been able to suggest an alternative approach to Thurston's original work. In addition the whole construction relies extensively on the groups in question being freely indecomposable and above all having no free factors.

In this sequence of papers we construct a hierarchical decomposition of a free group with respect to a given automorphism of it. Our construction which is dynamical in nature suggests a "unified" approach to Thurston's theory on the dynamics of diffeomorphisms of surfaces and to the study of the dynamics of automorphisms of a free group and may sometimes serve as a complementary to the Bestvina-Handel train tracks and invariant laminations. We also believe that besides the

[1] Columbia University, New York NY 10027, USA, and Hebrew University, Jerusalem 91904, Israel.

applicability of our methods to the study of the dynamics of automorphisms of a free group, it can be applied in various other cases. In particular we have already used it to obtain the Hopf property for hyperbolic groups [Se2].

In this paper we introduce the basic tools needed for the construction of the hierarchical decomposition. We start by introducing a dynamical-algebraic commutative diagram which is the basis for our whole approach. This commutative diagram allows us to interpret dynamical invariants in algebraic terms and vice versa. Using this diagram we show how to obtain the Nielsen-Thurston classification of automorphisms of surfaces on the algebraic level in the third chapter and the Scott conjecture for automorphisms of a free group in the fourth one.

Our arguments makes an extensive use of Rips' work on stable actions of groups on real trees. Since a preliminary version of our work appeared, F. Paulin has shown how to replace the bi-Lipschitz equivariant map appears in our commutative diagram by an equivariant isometry. This may indeed be helpful for further applications. M. Lustig has managed to obtain somewhat stronger closely related structural results using elements from Bestvina-Handel's work.

1. A Dynamical-Algebraic Commutative Diagram.

From a sequence of actions of a hyperbolic group on its Cayley graph via powers of a non-periodic automorphism, it is possible to extract a subsequence converging to an action of the group on a real tree by a theorem of F. Paulin [Pa]. We start this section by reviewing this construction and stating Rips' structurte theorem which allows us to analyse the obtained action later in the sequel. Having the action we construct a commutative diagram which allows one to relate algebraic assumptions with dynamical assertions and vice versa. Later on we will see how to use this commutative diagram together with Rips' classification to obtain both the Nielsen-Thurston classification for automorphisms of surfaces and a generalized version of the Scott conjecture to automorphisms of a free group. In a continuation paper the same commutative diagram will allow us to obtain a hierarchical decomposition of a free group with respect to a given automorphism of it.

Let $\Gamma = < G|R > = < g_1, \dots, g_t | r_1, \dots, r_s >$ be a torsion-free δ-hyperbolic group, X its Cayley graph with respect to the given presentation and φ an infinite order auotmorphism in Out (Γ). Since φ is not a periodic automorphism the t-tuple $(\varphi^{m_1}(g_1), \dots, \varphi^{m_1}(g_t))$ is not conjugate to the t-tuple $(\varphi^{m_2}(g_1), \dots, \varphi^{m_2}(g_t))$ for $m_1 \neq m_2$. For each m we pick an element $\gamma_m \in \Gamma$ which is translated minimally under the action of $\varphi^m(g_1), \dots, \varphi^m(g_t)$ and set μ_m to be that minimal translation, i.e.:

$$\mu_m = \max_{1 \leq j \leq t} |\gamma_m \, \varphi^m(g_j) \gamma_m^{-1}| = \min_{\gamma \in \Gamma} \max_{1 \leq j \leq t} |\gamma \, \varphi^m(g_j) \gamma^{-1}|$$

Picking γ_m we define $\hat{\varphi}_m$ to be the automorphism given by $\hat{\varphi}_m(\gamma) = \gamma_m \varphi^m(\gamma) \gamma_m^{-1}$. Since $\hat{\varphi}_m$ is determined by the image of the generators $\{g_j\}_{j=1}^t$ and since these images are not conjugate for $m_1 \neq m_2$, necessarily $\mu_m \to \infty$. Let $\{(X_m, id.)\}_{m=1}^{\infty}$ be the pointed metric spaces obtained from the Cayley graph X by dividing the metric d_X by $\mu_m \cdot (X_m, id)$ is endowed with a left isometric action of Γ via $\hat{\varphi}_m$. At this stage we can apply the following.

Theorem 1.1 ([Pa], **2.3**) *Let $\{X_m\}_{m=1}^{\infty}$ be a sequence of δ_m-hyperbolic spaces with $\delta_\infty = \underline{\lim}\, \delta_m < \infty$. Let G be a countable group isometrically acting on X_m. Suppose that for each m there exists a base point u_m in X_m such that for every finite subset P of G, the union of the geodesic segments between the images of u_m under P is compact and these unions are a sequence of totally bounded metric spaces. Then there is a subsequence converging in the Gromov topology to a $50\,\delta_\infty$-hyperbolic space X_∞ endowed with a non-trivial isometric action of G.*

Our pointed metric spaces $(X_m, id.)$ clearly satisfy the assumptions of the theorem (see [Pa] for details) and they are $\frac{\delta}{\mu_m}$ hyperbolic, hence, there exists a subsequence $\{(X_{m_k}, id.)\}_{k=1}^{\infty}$ converging into a real tree $Y = X_\infty$ which is endowed with an isometric action of Γ. By our construction Y is minimal under the action of Γ, i.e., Y contains no Γ-invariant proper subtree.

To analyse the action of Γ on the real tree Y, we need to study some of its basic properties. We start by showing the action is small and stable which will allow us to use Rips' classification of such actions in the sequel. The elementary properties we need are standard and appear in [Ri-Se2].

Proposition 1.2 ([Ri-Se2], **4.1 - 4.2**) *With the notations above:*

(i) *Stabilizers of segments of Y are cyclic.*
(ii) *Stablizers of tripods (convex hull of 3 points which are not on a segment) are trivial.*
(iii) *Let $[y_1, y_2] \subset [y_3, y_4]$ be segments of Y and assume $stab\,([y_3, y_4]) \neq 1$. Then $stab\,([y_1, y_2]) = stab\,([y_3, y_4])$.*

Proposition 1.2 shows the action of Γ on the real tree Y satisfies Rips' Ascending chain condition ([Ri],[Be-Fel],([Ri-Se2],10.2)), so it enables analysing the action using Rips' classification of stable actions of f.p. groups on real trees. In ([Ri] and [Be-Fel]) the real tree Y is divided into distinct components, where on each component a subgroup of Γ acts according to one of several canonical types of actions. We bring the version of this analysis appears in the appendix of [Ri-Se2] which is going to be used extensively both in this paper and its consecutive one. For the notions and basic definitions used in the statement of the following theorem we refer the reader to the appendix of [Ri-Se2] and to [Be-Fel]

Theorem 1.3 (cf. [Ri],([Be-Fel],**15**)([Ri-Se2], **10.8**)*Let G be a f.p. group which admits a small isometric action on a real tree Y that satisfies the ACC condition. Assume the stabilizer of each tripod in Y is trivial.*

1) *There exist canonical subtrees of Y: $Y_1, \dots Y_k$ with the following properties:*
    (i) gY_i intersects Y_j at most in one point if $i \neq j$.
    (ii) gY_i is either identical with Y_i or it intersects it at most in one point.
    (iii) The action of $stab\,(Y_i)$ on Y_i is either discrete or it is of axial type or IET type.
2) *The group G admits a (canonical) graph of groups with:*
    (i) Vertices corresponding to branching points with non-trivial stabilizer in Y.
    (ii) Vertices corresponding to orbits of the canonical subtrees $Y_1, \dots, Y_k$ which are of axial or IET type. The groups associated with these

vertices are conjugates of the stabilizers of these components. To a stabilizer of an IET componenet there exists an associate 2-orbifold. All boundary componenets and branching points in this associated 2-orbifold stabilize points in Y. For each such stabilizer we add edges that connect the vertex stabilized by it and the vertices stabilized by its boundary componenets and branching points.

(iii) *A (possible) vertex stabilized by a free factor of G and connected to the other parts of the graph of groups by a unique edge with trivial stabilizer.*

(iv) *Edges corresponding to orbits of edges between branching points with non-trivial stabilizer in the discrete part of Y with edge groups which are conjugates of the stabilizers of these edges.*

(vi) *Edges corresponding to orbits of poins of intersection between the orbits of $Y_1, \ldots, Y_k$.*

Before concluding our preliminary study of the limit real tree and start constructing our dynamical-algebraic commutative diagram, we exclude axial components isometric to a real line appear in the statement of theorem 1.3 above.

Proposition 1.4. *With the notations above:*

(i) *Y does not contain a minimal axial component isometric to a real line.*

(ii) *Stabilizers of non-degenerate segments which lie in the complement of the discrete parts of Y are trivial. Stabilizers of segments in the discrete components of Y are trivial or maximal cyclic.*

Proof: If Y contains an axial component isometric to a real line, it contains a solvable subgroup with Z^2 as a quotient. But the only solvable subgroups of Γ are virtualy cyclic, so we have (i). Since Y does not have componenets isometric to a real line, the ACC condition implies that the stabilizer of a non-degenerate segment in the complement of the discrete parts of Y, stabilize a tripod in Y. Since stabilizers of tripods are trivial by lemma 1.2, stabilizers of segments in the complement of the discrete parts of Y are trivial. Stabilizers of segments in the discrete part of Y are trivial or maximal cyclic by lemma 1.2. □

Having a subsequence of powers of an automorphism φ of a hyperbolic group Γ converging into a real tree Y with the above properties, we are able to introduce our approach to the study of the dynamics of φ. The approach adopted in this paper is much weaker than the one introduced in [Sel] for freely indecomposable hyperbolic groups which are not surface groups, but as we will see it gives the Nielsen-Thurston classification and can serve as a basis for understanding the dynamics of automorphisms of a free group. Let $\{\psi_k | \psi_k = \hat{\varphi}_{m_k} = \gamma_{m_k} \varphi^{m_k} \gamma_{m_k}^{-1}\}_{k=1}^{\infty}$ be a subsequence of automorphisms obtained by theorem 1.1, namely a subsequence for which the metric spaces $\{(X_{m_k}, id)\}_{k=1}^{\infty}$ equipped with a left isometric action of Γ via ψ_k converges into our real tree Y equipped with a left isometric Γ action. We start with the following rather immediate commutative diagram which connects algebraically two Γ actions on real trees.

Proposition 1.5 *For each k let $\left(\mathrm{X}^1_{m_k}, id.\right)$ be the pointed metric space $\left(\mathrm{X}_{m_k}, id.\right)$ equipped with a left isometric action of Γ via the automorphisms $\psi^1_k = \psi_k \circ \varphi$. Then the sequence of pointed metric spaces $\left\{\left(\mathrm{X}^1_{m_k}, id.\right)\right\}_{k=1}^{\infty}$ converges in the Gromov topology on metric spaces to a pointed real tree $\left(Y^1, y^1_0\right)$ which is isometric to the pointed real tree $\left(Y, y_0\right)$ via an equivariant isometry $\tau : \left(Y^1, y^1_0\right) \to \left(Y, y_0\right)$ such that the following diagram is commutative:*

$$\begin{array}{ccc} \Gamma \times \left(Y^1, y^1_0\right) & \longrightarrow & \left(Y^1, y^1_0\right) \\ \Big\downarrow {\scriptstyle (\varphi,\tau)} & & \Big\downarrow {\scriptstyle \tau} \\ \Gamma \times \left(Y, y_0\right) & \longrightarrow & \left(Y, y_0\right) \end{array}$$

(i.e. $\forall \gamma \in \Gamma \quad \forall \hat{y} \in Y^1 \ \tau\big(\gamma(\hat{y})\big) = \varphi(\gamma)\ \big(\tau(\hat{y})\big)$) In particular Y^1 is minimal under the action of Γ.

Proof: The convergence of the sequence of the Γ-spaces $\left\{\left(\mathrm{X}^1_{m_k}, id.\right)\right\}_{k=1}^{\infty}$ to a real tree Y^1 follows from the convergence of the sequence $\left\{\left(\mathrm{X}_{m_k}, id.\right)\right\}_{k=1}^{\infty}$ to the real tree Y. The commutative diagram in the limit follows from the existence of corresponding commutative diagrams between the Γ spaces $\left(\mathrm{X}^1_{m_k}, id.\right)$ and $\left(\mathrm{X}_{m_k}, id.\right)$ at any finite step k. Y^1 is minimal, since the identity moves minimaly by the image under ψ^1_k of the generating system $(\varphi^{-1}(g_1), \ldots, \varphi^{-1}(g_\ell))$. □

Having composed ψ_k with our initial automorphism φ, we compose them in a different order. Unlike the composition from the left, when we compose on the right we need to take a subsequence in order to get convergence into a real tree equipped with a Γ action.

Proposition 1.6 *For each k let $\left(\mathrm{X}^2_{m_k}, id.\right)$ be the pointed metric space $\left(\mathrm{X}_{m_k}, id.\right)$ equipped with a left isometric action of Γ via the automorphism $\psi^2_k = \varphi \circ \psi_k$. Then there exists a subsequence of the pointed metric spaces $\left\{\left(\mathrm{X}^2_{m_k}, id.\right)\right\}_{k=1}^{\infty}$ converges in the Gromov topology on metric spaces to a pointed real tree $\left(Y^2, y^2_0\right)$ equipped with a minimal left isometric action of Γ, and such that there exists a bi-Lipschitz equivariant homeomorphism $\mu : \left(Y, y_0\right) \to \left(Y^2, y^2_0\right)$ and the following diagram is commutative:*

$$\begin{array}{ccc} \Gamma \times Y & \longrightarrow & Y \\ \Big\downarrow {\scriptstyle (id.,\ \mu)} & & \Big\downarrow {\scriptstyle \mu} \\ \Gamma \times Y^2 & \longrightarrow & Y^2 \end{array}$$

(i.e., $\mu\big(\gamma(y)\big) = \gamma\big(\mu(y)\big)$).

Proof: φ acts on the Cayley graph X as a quasi-isometry and so does φ^{-1}. For each k and ℓ, let $\psi_k(B_\ell)$ be given by:

$$\psi_k\left(B_\ell\right) = \left\{\psi_k(v) \middle| v \in \mathrm{X};\ |v| \le \ell\right\}$$

Then clearly φ acts on $\psi_k(B_\ell)$ as a bi-Lipschitz map with bi-Lipschitz coefficients independent of k and ℓ. This clearly guarantees that the metric spaces $\{(X^2_{m_k}, id.)\}_{k=1}^{\infty}$ satisfy the assumptions of theorem 1.1, which gives a convergent subsequence in the Gromov topology. Moreover, the independence of the bi-Lipschitz coefficients in k and ℓ give us the existence of the (equivariant) bi-Lipschitz homeomorphism μ in the above commutative diagram. Since (Y, y_o) is minimal under the action of Γ, and (Y^2, y_0^2) is obtained from (Y, y_0) by an equivariant bi-Lipschitz homeomorphism, (Y^2, y_0^2) is minimal under the action of Γ as well. $\square$

Proposition 1.5 connects the Γ action on the real tree (Y^1, y_0^1) on the algebraic level and proposition 1.6 connects the Γ action on the real tree (Y^2, y_0^2) on the space level. In order to deduce algebraic assertions from dynamical data and vice versa, we need to put the two diagrams obtained in the above propositions into a one diagram. To do that we show that there exists a Γ equivariant isometry between the real trees (Y^1, y_0^1) and (Y^2, y_0^2). For simplicity we continue to denote the subsequence obtained in proposition 1.6 by $\{\psi_k^2\}_{k=1}^{\infty}$ and the corresponding subsequence appears in proposition 1.5 by $\{\psi_k^1\}_{k=1}^{\infty}$.

Proposition 1.7 *With the notations above there exists a bi-Lipschitz homeomorphism $\sigma : Y^1 \to Y$ for which the following diagram is commutative:*

$$\begin{array}{ccc} \Gamma \times Y^1 & \longrightarrow & Y^1 \\ \big\downarrow (id., \sigma) & & \big\downarrow \sigma \\ \Gamma \times Y & \longrightarrow & Y \end{array}$$

(i.e. $\forall \hat{y} \in Y^1 \qquad \gamma(\sigma(\hat{y})) = \sigma(\gamma(\hat{y}))$).

Proof: The real tree (Y^1, y_0^1) is obtained as the limit of the sequence of actions of Γ via the automorphisms ψ_k^1 on the metric spaces $(X_{m_k}, id.)$ and the real tree (Y^2, y_0^2) is obtained as the limit of the sequence of actions of Γ via the automorphisms ψ_k^2 on the metric spaces $(X_{m_k}, id.)$. The automorphisms ψ_k^1 and ψ_k^2 are both conjugate to φ^{m_k+1}, hence, by changing base points, we may view the real tree (Y^1, y_0^1) as the limit of the sequence of actions of Γ via the automorphisms φ^{m_k+1} on the metric spaces (X_{m_k}, γ_{m_k}) and the real tree (Y^2, y_0^2) as the limit of the sequence of actions of Γ via the automorphisms φ^{m_k+1} on the metric spaces $(X_{m_k}, \varphi(\gamma_{m_k}))$.

Every non-elementary hyperbolic group contains two elements which generate a free group, hence, we may assume that our generating set $g_1, \dots, g_t$ contains such a pair. The point γ_{m_k} was chosen so that its maximal displacement by the t-tuple $(\varphi^{m_k}(g_1), \dots, \varphi^{m_k}(g_t))$ is minimal and equals μ_{m_k}. Therefore, there exist contstants $c_1, \dots, c_4$ for which:

$$c_1\mu_{m_k} \leq \max_{1 \leq j \leq t} d_X(\varphi^{m_k+1}(g_j)(\gamma_{m_k}), \gamma_{m_k}) \leq c_2\mu_{m_k}$$

$$c_3\mu_{m_k} \leq \max_{1 \leq j \leq t} d_X(\varphi^{m_k+1}(g_j)(\varphi(\gamma_{m_k})), \varphi(\gamma_{m_k})) \leq c_4\mu_{m_k}$$

Let $c = \max c_i$ and let g_1 and g_2 be the pair of elements from our generating set which generate a free group. Since g_1 and g_2 generate a free group, the commutators $[g_1^r, g_2^r]$ are distinct for different r's. If the axes for $\varphi^{m_k+1}(g_1)$ and for $\varphi^{m_k+1}(g_2)$ remain in distance not more than 2δ in a ball of diameter $2\ell c\mu_{m_k}$ in X, then there exists a point in one of these axes so that for all $r = 0, 1, \dots, \ell$ this point is being moved by $\varphi^{m_k+1}([g_1^r, g_2^r])$ a distance bounded by 8δ in X - the Cayley graph of Γ. Hence, the axes of $\varphi^{m_k+1}(g_1)$ and $\varphi^{m_k+1}(g_2)$ can remain at distance not bigger than $2c\mu_{m_k}$ only in a set of diameter not exceeding $2(v_{8\delta}+5)c\mu_{m_k}$ where $v_{8\delta}$ is the volume of a ball with radius 8δ in X. Since both points γ_{m_k} and $\varphi(\gamma_{m_k})$ have to lie in this set by the above inequalities, the distance between these points is bounded by $2(v_{8\delta}+5)c\mu_{m_k}$ as well.

Having a bound on the distance between γ_{m_k} and $\varphi(\gamma_{m_k})$ in terms of μ_{m_k} we can define a new limit procedure which will give us a new Γ real tree T which includes both real trees Y^1 and Y^2. To do that we need a minor modification of theorem 1.1 that can be proved by exactly the same arguments to those appear in ([Pa],2.3).

Lemma 1.8 *Let $\{X_m\}_{m=1}^{\infty}$ be a sequence of δ_m-hyperbolic spaces with $\delta_\infty = \underline{\lim}\,\delta_m < \infty$. Let G be a countable group isometrically acting on X_m. Suppose that for each m there exist a couple of points u_m and v_m in X_m such that for every finte subset P of G, the union of the geodesic segments between the images of u_m and v_m under P is compact and these unions are a sequence of totally bounded metric spaces. Then there is a subsequence converging in the Gromov topology to a $50\,\delta_\infty$-hyperbolic space X_∞ endowed with a non-trivial isometric action of G and the sequence of points u_m and v_m subconverge into a couple of points in this limit space.*

By the above arguments the metric spaces X_{m_k} together with the points γ_{m_k} and $\varphi(\gamma_{m_k})$ satisfy the assumptions of the lemma, hence, these metric spaces subconverge into a double pointed real tree (T, t_1, t_2). Clearly, both Y^1 and Y^2 are subtrees of the real trees T, and since the action of Γ on T is non-trivial and both trees Y^1 and Y^2 are minimal for that action - $Y^1 = Y^2$ and in particular there exists a Γ-equivariant isometry between them. Composing this equivariant isometry with the bi-Lipschitz equivariant homeomorphism from Y^2 to Y we get the bi-Lipschitz equivariant map from Y^1 to Y. □

Propositon 1.7 together with propositon 1.5 give us the commutative diagram which is the key point in our whole approach to the dynamics of automorphisms of hyperbolic groups. This commutative diagram will allow us to relate algebraic and dynamical properties of automorphisms of hyperbolic groups and in particular of free groups.

(1)
$$\begin{array}{ccc} \Gamma \times Y^1 & \longrightarrow & Y^1 \\ {\scriptstyle (id.,\sigma)}\downarrow\ \downarrow{\scriptstyle (\varphi,\tau)} & & {\scriptstyle \sigma}\downarrow\ \downarrow{\scriptstyle \tau} \\ \Gamma \times Y & \longrightarrow & Y \end{array}$$

$\forall \gamma \in \Gamma \quad \forall \hat{y} \in Y^1 \quad \sigma(\gamma(\hat{y})) = \gamma(\sigma(\hat{y}));\ \tau(\gamma(\hat{y})) = \varphi(\gamma)\ (\tau(\hat{y})).$

2. Conservation Laws for the Basic Commutative Diagram.

The commuatative diagram (1) gives us a linkage between the algebraic automorphism φ and its action on the hyperbolic group Γ and a Γ-equivariant bi-Lipschitz map σ between the real trees Y and Y^1. Having such a diagram we naturally continue by studying some of the (dynamical) invariants of the map σ and relate these invariants with the algebraic structure of the automorphism φ. To get the dynamical-algebraic linkage we make an extensive use of Rips' classification of stable actions of f.p. groups on real trees (see theorem 1.3 above). In [Se2], in the coarse of proving the Hopf property for hyperbolic groups, some of the conservation laws obtained in this section for bi-Lipschitz equivariant maps are generalized to equivariant Lipschitz ones.

Our main goal in studying dynamical invariants of the bi-Lipschitz equivariant map σ is showing that parts of the graph of groups associated with the action of Γ on the real tree Y remain invariant under the automorphism φ. This invariance can be obtained immediately from Rips' construction of the graph of groups, since the whole Rips' machine (cf. [Be-Fel]) is invariant under bi-Lipschitz equivariant maps. For those who are familiar with Bestvina-Feighn approach to Rips' work, let us note that the lamination structure remains invariant under bi-Lipschitz equivariant map, although the transverse measure changes. Eventhough relying on Rips' proof is the quickest approach, we have prefered to get our conservation laws from Rips' theorem and not from his proof. This is partly because this approach is better for generalizations (cf. [Se2]) and mainly because we want this paper to be accessible also for people who are not familiar with the proof of Rips' theorem. A knowledge of Rips' final theorem is crucial though, and the interested reader is refered to either [Ri],[Be-Fel] or the appendix of [Ri-Se2]. We will also assume the reader is familiar with the basics of the Bass-serre theory for actions of groups on simplicial trees. In our continuation paper we extend the list of conservation laws which allow us to enlarge and refine the parts of the graph of groups which remain invariant under the automorphism φ. This refinement will also allow us to show that our whole construction is canonical.

Lemma 2.1 *With the notations of the commutative diagram (1), the automorphism φ gives a morphism between the (Rips') graph of groups associated by theorem 1.3 with the action of Γ on the real tree Y^1 and the graph of groups associated with the action of Γ on the real tree Y. In particular the number of IET components is identical for these two graphs of groups as well as the number of orbits of points stabilized by a non-elementary subgroup of Γ, the number of orbits of edges in the discrete parts of Y and Y^1 and the number of orbits of edges stabilized by a (maximal) cyclic subgroup of Γ.*

Proof: By the commutative diagram (1), Y is isometric to Y^1 via τ, and $\gamma \in \Gamma$ acts on Y^1 exactly in the same way $\varphi(\gamma)$ acts on Y. This clearly gives the morphism between the graphs of groups associated with these two actions by theorem 1.3 as well as the equality in the number of components and orbits stated in the lemma. □

Observing that the Rips' graph of groups associated with the action of Γ on Y is similar to the one associated with the action of Γ on Y^1, we start studying properties of the Γ-equivariant bi-Lipschitz map σ between these two Γ-real trees. Following J. Morgan [Mo] we will need the following notion:

Definition 2.2 *A subtree (or forest) T_1 of a Γ-real tree T is called mixing if for every two closed non-degenerate segments I and J in T_1, there exists a finite cover of J with closed intervals $J_1, \dots, J_n$ and elements $\gamma_1, \dots, \gamma_n \in \Gamma$ so that $\gamma_i(J_i) \subset I$ for $i = 1, \dots, n$. Note that a mixing subtree of a Γ-real tree contains, in particular, a dense orbit. In fact the orbit of every point in a mixing subtree is dense in it.*

Lemma 2.1 shows the correspondance we get from the isometry τ between the graphs of groups associated with the actions of Γ on Y and Y^1. To use these correspondance we need to look for properties of these actions which are preserved under the action of the bi-Lipschitz equivariant map σ. The following invariants of σ are immediate from its definition.

Lemma 2.3 *With the notations of the commutative diagram (1):*

(i) *If T is a mixing subtree of Y^1 then $\sigma(T)$ is a mixing subtree of Y.*
(ii) *If T is a subtree of Y^1 on which Γ acts discretely, then Γ acts discretely on $\sigma(T)$.*
(iii) *If T is a subtree of Y^1 in which Γ has a dense orbit, then Γ has a dense orbit in $\sigma(T)$.*
(iv) *Let H be a subgroup of Γ that fixes a point (segment) in Y^1, then H fixes a point (segment) in Y.*

Lemma 2.3 gives us some of the elementary invariants of equivariant bi-Lipschitz maps. Using them we can start looking for parts of the graph of groups associated with the action of Γ on Y^1 which remain invariant under the action of the automorphism φ. In this paper we study the invariance of the IET components, in its consecutive one we study mostly the structure of components with indiscrete action of free factors. Both of these studies use the notion of a mixing subtree in an essential way.

Lemma 2.4 *If T is an interval exchange type subtree of Y^1, then $\sigma(T)$ is contained in orbit of an interval exhange type subtree of Y.*

Proof: A IET subtree is in particular mixing (cf. [Mo]), so by lemma 2.3 $\sigma(T)$ is mixing as well. Since T contains a dense orbit, $\sigma(T)$ does not intersect the interior of the discrete parts of Y. Furtheremore, a mixing subtree of Y is either contained in an orbit of a IET subtree, or it does not cut any IET component in a non-degenerate segment.

Let Q be the subgroup that maps T into itself (Q appears as a vertex group in the Rips' graph of groups associated to the action of Γ on Y^1). Since Γ is assumed torsion-free, Q is the fundamental group of a surface (cf. [Ri] or [Be-Fel]) or a free group where the cyclic subgroups corresponding to punctures of this surface fix points in T. We have also associated a graph of groups with the action of Γ on the real tree Y, from which Q being a subgroup of Γ inherits a small splitting.

If Q is a surface group it is freely indecomposable so T must be mapped into the orbit of an interval exhange component of Y^1 by σ. If Q is the fundamental group of a punctured surface all its cyclic boundary components fix points in Y by lemma

2.3, hence, they are all contained in vertex groups Q inherits from its action on Y. Once again the fundamental group of a punctured surface can not be written as a free product where all boundary components can be conjugated into one of the factors. Therefore, $\sigma(T)$ is contained in the orbit of a IET component of the real tree Y. □

Showing that an IET component of Y^1 is mapped by the bi-Lipschitz equivariant map σ into an orbit of an IET component of Y we show that the stabilizer of such a component in Y^1 is mapped by the automorphism φ into a stabilizer of an IET component in Y. This will suffice to get the Nielsen-Thurston classification of automorphisms of surfaces in the next section and the Scott conjecture in the fourth one. To get our hierarchical decomposition one needs to look more closely into the (dynamical) structure of components with an indiscrete action of a free factor. This will be done in our continuation paper.

Proposition 2.5 *Let T be an interval exchange type subtree of Y^1 and let Q be the subgroup that maps T into itself. Then $\sigma(T)$ is an interval exchange type subtree of Y and Q is its stabilizer. In particular, Q is conjugate to a (IET) vertex group in the graph of groups associated with the action of Γ on Y where boundary elements of Q are conjugate to boundary elements of this IET vertex group.*

Proof: First, suppose Q is a surface group. In this case all its elements act hyperbolically on Y^1, hence, they all act hyperbolically on Y. If Q can not be conjugated into a (IET) vertex group of the graph of groups associated with the action of Γ from its action on Y, Q inherits a non-trivial small splitting from this graph of groups. In this inherited small splitting all edge groups fix points in Y, which is a contradiction since all elements of Q act hyperbolically on Y.

If Q is the fundamental group of a punctured surface, the only elements in Q which fix points in Y_1 are conjugate to a power of a boundary element, hence, these are also the only elements in Q which fix points in Y by lemma 2.3. Repeating the argument for a surface group, if Q can not be conjugated into a (IET) vertex of the graph of groups associated with the action of Γ on Y, Q inherits a non-trivial small splitting from this graph of groups. In this small splitting every edge group fix a point in Y. But in such a splitting of a punctured surface, each edge group is generated by an element corersponding to a s.c.c. which is not a boundary element, a contradiction.

So far we have shown that Q can be conjugated into a (IET) vertex group U in Y. σ^{-1} gives us a bi-Lipschitz Γ-equivariant map from Y to Y^1, hence, we may repeat all our arguments and conclude that U can be conjugated into a (IET) vertex group of the graph of groups associated with the action of Γ on Y^1. Since Q is a vertex group in this graph of groups and since Q can be conjugated into U, necessarily Q is a conjugate of U. □

Proposition 2.5 shows that an IET vertex group in the graph of groups associated with the action of Γ on Y^1 is a conjugate of an IET vertex group in the graph of groups associated with the action of Γ on Y. By the commuatative diagram (1) an IET vertex group in the first graph of groups is mapped by the automorphism φ to an IET vertex group in the second graph of groups. Therefore, the automorphism

φ acts as a permutation on conjugacy classes of IET vertex groups of the two graphs of groups associated with the action of Γ on Y^1 and Y.

3. The Nielsen-Thurston Classification.

The basic conservation laws derived in the previous section shows the automorphism φ acts as a permutation on conjugacy classes of various vertex and edge groups in the graphs of groups associated with the actions of Γ on the real trees Y and Y^1. As we will see, this is enough to get the Nielsen-Thurston classification on the algebraic level, and in fact even to generalize it to any torsion-free, freely indecomposable hyperbolic group. For a freely indecomposable group which is not a surface group, what we are getting follows easily from the canonical JSJ decomposition [Sel] which is a much stronger structural result - the decomposition we will get is with respect to a single automorphism, whereas the JSJ is associated with the ambient group and preserved by all automorphisms of it.

Throughout this section we will use the notations of the commutative diagram (1) and assume Γ is a torsion-free, non-elementary, freely indecomposable hyperbolic group and φ is not a periodic automorphism (by periodic automorphism we always mean an automorphism of finite order in $\text{Out}(\Gamma)$). We start with the pseudo-Anosov case and some of its properties - all are well known and follow from Thurston's work ([Th], [Ca-Bl]). Our aim in bringing them is mainly showing the applicablity of the commutative diagram (1) to derive algebraic information on automorphisms.

Proposition 3.1 *With the above notations and assumptions, if Γ acts freely on Y then:*

(i) *Γ is a surface group.*
(ii) *φ does not have any periodic conjugacy classes.*
(iii) *the growth rate of elements in Γ is uniform. i.e., for any two non-trivial elements $\gamma_1, \gamma_2 \in \Gamma$ there exist constants c_1, c_2 such that for all positive n:*

$$c_1|\varphi^n(\gamma_1)| < |\varphi^n(\gamma_2)| < c_2|\varphi^n(\gamma_1)|$$

and the same holds for the growth rate of their conjugacy classes.

Proof: If the action of Γ on Y is free, and Γ is a non-elementary freely-indecomposable hyperbolic group, then Γ is a surface group by Rips' theorem (theorem 1.3 above). Every periodic conjugacy class of φ has to fix a point in the limit tree Y. Since the action of Γ is free, φ has no periodic conjugacy classes.

Let $\gamma_1, \gamma_2 \in \Gamma$ be a pair of non-trivial elements, and suppose that $|\varphi^{m_k}(\gamma_1)| < c_k|\varphi^{m_k}(\gamma_2)|$ where $c_k \to 0$. In this case we can extract a pointed limit tree (T, t) as the limit of a subsequence of the actions of Γ on the metric spaces $(X_{m_k}, id.)$ via the automorphisms φ^{m_k} (i.e., we don't compose the powers of φ with an inner automorphism as we did in our original construction). The action of Γ on T is small and it may be trivial. Still, because of the non-uniform growth, γ_1 fixes a point in T which is not fixed by the entire group Γ. The whole construction of the commutative diagram (1) works for our new construction, hence, we get such a commuative diagram for Γ real trees (T^1, t^1) and (T, t).

Suppose the action of Γ on T is trivial. The base points t and t_1 are stabilized by γ_1 and $\sigma(t^1) = \tau(t^1) = t$ by our construction. Let $q \in T$ be the point stabilized by the entire group Γ. Any element in Γ that stabilize t stabilize the entire non-degenerate segment $[t, q] \subset T$ and since the action of Γ on T is small by proposition 1.2, the stabilizer of the base point t is a cyclic group containing γ_1.

Now, $\sigma(t^1) = \tau(t^1) = t$ by construction, so by the commutative diagram (1) the stabilizer of t is mapped to itself by φ, and since we already know it is cyclic, φ^2 acts trivialy on it and we have contradicted (ii).

If Γ acts non-trivially on T, Γ is freely indecomposable and stabilizers of tripods in T are trivial, so the action of Γ on T is discrete or T contains an IET component by theorem 1.3 above. If the action of Γ on T is discrete, φ acts as a permutation on conjugacy classes of segment stabilizers in T. If T contains a IET component then the conjugacy classes of boundary elements in the stabilizer of an IET component are periodic under φ by proposition 2.5. Again, a contradiction to (ii).

The lengths of non-trivial conjugacy classes is uniform by exactly the same argument applied to our original construction extracted from a non-uniform subsequence. □

Analysing the case of a free action, we continue by associating to φ a canonical graph of groups with fundamental group Γ. The edges of this graph of groups are all cyclic, the automorphism premutes the conjugacy classes of vertex and edge groups, and a power of it acts either as a periodic automorphism or as a pseudo-Anosov of a punctured surface on each of the vertex groups. This power of the automorphism decomposes into its (canonical) actions on the vertex groups composed with (possible) Dehn twists along the edges.

Theorem 3.2 *Let Γ be a torsion-free, freely indecomposable hyperbolic group. With the notations of the commutative diagram (1), if φ is not a periodic automorphism and the action of Γ on the real tree Y is not free there exists a (canonical) graph of groups Λ_φ with the following properties:*

- (i) *edge groups of Λ_φ are cyclic.*
- (ii) *φ permutes the conjugacy classes of vertex and edge groups in Λ_φ.*
- (iii) *There exists $k(\Gamma)$ so that $\varphi^{k(\Gamma)}$ composed with an approperiate inner automorphism acts on each of the vertex groups either as a periodic automorphism or as a pseudo-Anosov of a punctured surface.*
- (iv) *$\varphi^k(\Gamma)$ can be written as a composition of its canonical actions on the vertex groups composed with (possible) Dehn twists along the edges.*

Before we prove theorem 3.2 note that if Γ is a surface group it is exactly the reducible case in the Nielsen-Thurston classification.

Corollary 3.3 *Let Γ be the fundamental group of a closed surface S and let φ be an automorphism of Γ. Then either:*

- (i) *φ is a periodic automorphism.*
- (ii) *φ is a pseudo-Anosov. In this case there are no periodic conjugacy classes and growth rate of elements and conjugacy classes is uniform.*
- (iii) *there exists a collection of non-homotopic essential s.c.c. on S, so that φ permutes the conjugacy classes of the punctured surfaces obtained by cutting S along these s.c.c., and a power of φ acts on these punctured surfaces either as a periodic automorphism or as a pseudo-Anosov of a punctured surface.*

This power decomposes into its canonical actions on the punctured surfaces composed with (possible) Dehn twists along the s.c.c. .

Proof: With the notations of the commutative diagram (1) if φ is not periodic and Γ acts freely on Y, φ is a pseudo-Anosov and its basic properties are given in proposition 3.1. If the action of Γ on Y is not free, Γ is the fundamental group of the graph of groups Λ_φ with cyclic edge stabilizers and with properties given in theorem 3.2 above. The edge groups in every such graph of groups with a surface fundamental group correspond to disjoint non-homotopic s.c.c. by [ZVC]. Hence, from theorem 3.2 we get the reducible case in the Nielsen-Thurston classification (case (iii) above). □

Proof of theorem 3.2: Γ is a torsion-free, freely indecomposable group and by proposition 1.4 Y does not contain axial components isometric to the real line, so Y contains only IET and discrete components according to thoerem 1.3 above. By the same theorem we can associate to the action of Γ on Y a graph of groups with fundamental group Γ. Let Λ_1 be this graph of groups. Vertex groups in Λ_1 are either stabilizers of points or stabilizers of IET components in Y. Edge groups are either boundary subgroups of stabilizers of IET components, or stabilizers of edges in the discrete part of Y.

By lemma 2.3 and the commutative diagram (1) φ permutes the conjugacy classes of point and edge stabilizers in Y and by proposition 2.5 φ permutes the conjugacy classes of stabilizers of IET components in Y. Hence, in the case of a torsion-free, freely indecomposable group φ permutes the conjugacy classes of all vertex and edge groups in the graph of groups Λ_1.

At this point we start refining Λ_1 to obtain eventualy the canonical decomposition Λ_φ. We take a fixed power k_1 of φ that fixes the conjugacy classes of all vertex and edge groups in Λ_1. If φ^{k_1} acts on a vertex group of Λ_1 either as a periodic automorphim or as a pseudo-Anosov of a punctured surface we leave this vertex as it is. Suppose A is a vertex group of Λ_1 on which φ^{k_1} does not act in one of these two canonical forms. All edge groups connected to A are cyclic and their conjugacy classes are fixed under the action of φ^{k_1}.

Up to conjugation we can assume that φ^{k_1} maps A to itself so we can obtain a new commutative diagram from the action of φ^{k_1} on the subgroup A. This gives us a new graph of groups Δ_A with fundamental group A. The conjugacy classes of all the original edge groups of edges connected to the vertex stabilized by A in Λ_1 are periodic, hence, these edge groups are subgroups of a vertex group in Δ_A. Therefore, we can replace the vertex stabilized by A in Λ_1 by Δ_A and get a new graph of groups with fundamental group Γ which we denote Λ_2. The number of edges in Λ_2 is strictly bigger than the number of edges in Λ_1. Applying lemma 2.3 and proposition 2.5 to Δ_A we conclude that the conjugacy classes of all vertex and edge groups in Δ_A and hence in Λ_2 are periodic under φ^{k_1}, so they are periodic under φ.

As long as we have vertices on which a power of φ does not act as either a periodic automorphism or a pseudo-Anosov of a punctured surface we can continue the refinement process and get graphs of groups for Γ with more and more edges. By generalized accessibility [Be-Fe2] or by acylindrical accessibility [Se3] this refinement has to terminate and the final refined graph of groups which we denote by Λ_φ satisfies the conclusion of the theorem. □

Note that the steps in the refinement procedure in which we have obtained Λ_φ are precisely the different growth rates of conjugacy classes in Γ under the action of φ. A similar refinement procedure in the case of a free group is sufficient for obtaining the Scott conjecture as we will see in the next section. To analyse automorphisms of a free group in general we need to look more closely at automorphisms with a uniform growth rate. This will be done in our continuation paper.

4. The Scott Conjecture.

The commutative diagram (1) joined with the conservation laws proven in section 2 and the refinement procedure described in the 3rd section while obtaining an algebraic version of the Nielsen-Thurston classification give a basic "scheme" for studying automorphisms. For various applications this "scheme" needs to be elaborate, but as we will see in this section, it is enough for obtaining a somewhat stronger form of the Scott conjecture on the rank of the fixed subgroup of an automorphism of a free group. The Scott conjecture was originally proven by M. Bestvina and M. Handel in [Be-Ha].

Like in obtaining a version of the Nielsen-Thurston classification for freely indecomposable hyperbolic groups, given an automorphism φ of a free group F_n we associate to φ a graph of groups Λ_φ with fundamental group F_n. In this graph of groups the fixed subgroup of φ is a subgroup of a vertex group of Λ_φ on which φ acts as a periodic automorphism. Later, this graph of groups together with Culler's analysis of periodic automorphisms of a free group [Cu] will prove the Scott conjecture.

Theorem 4.1 *Let φ be a non-periodic automorphism of a free group F_n. There exists a graph of groups Λ_φ with fundamental group F_n and with the following properties:*

- (i) *edge groups of Λ_φ are either cyclic or trivial. φ permutes the conjugacy classes of the edge groups.*
- (ii) *vertex groups are either:*
  - *(1) free factors of F_n which are connected to the other parts of Λ_φ by edges with trivial stabilizers.*
  - *(2) fundamental groups of punctured surfaces. These are connected to other parts of Λ_φ by edges stabilized by their (cyclic) boundary subgroups. φ permutes the conjugacy classes of these vertices and a composition of a power of φ with an approperiate inner automorphism acts on each such vertex group as a pseudo-Anosov of a punctured surface.*
  - *(3) the "remaining" vertices. φ permutes the conjugacy classes of the remaining vertices, and a composition of a power of it with an approperiate inner automorphism acts on each of them as a periodic automorphism.*
- (iii) *if the fixed subgroup of φ is not trivial, $Fix(\varphi)$ is either a cyclic edge group in Λ_φ or a subgroup of a vertex group on which φ acts as a periodic automorphism.*

Proof: We apply again the refinement scheme presented in the proof of theorem 3.2. Using the notations of the commuatative diagram (1) if φ is not periodic and F_n acts freely on Y, our graph of groups Λ_φ is degenerate - it is a unique vertex stabilized by F_n. Clearly, in this case there are no periodic conjugacy classes under the action of φ and in particular the fixed subgroup is trivial.

If the action of F_n on Y is not free, we get a non-trivial graph of groups Λ_1 with fundamental group F_n associated with this action by theorem 1.3 above. By proposition 1.4 Y does not contain axial components isometric to the real line so Y contains only IET and discrete components. Hence, by theorem 1.3 above, vertex groups in Λ_1 are either stabilizers of points, stabilizers of IET componenets, or free factors of F_n connected to the other parts of Λ_1 by a single edge with trivial stabilizer. Edge groups are either boundary subgroups of stabilizers of IET componenets, or stabilizers of edges in the discrete part of Y. By lemma 2.3 and the commutative diagram (1) φ permutes the conjugacy classes of point and edge stabilizers in Y and by proposition 2.5 φ permutes the conjugacy classes of stabilizers of IET components in Y.

As we did in the case of freely indecomposable groups, at this point we start refining Λ_1 to eventually get the decomposition Λ_φ. We take a fixed power k_1 that fixes the conjugacy classes of all vertex groups in Λ_1 corresponding to stabilizers of IET components and point stabilizers in Y and conjugacy classes of all edge groups in Λ_1. Our refinement procedure will deal only with vertices corresponding to point stabilizers in Y - the rest of the vertices already satisfy the conclusion of the theorem. Let A be a vertex group in Λ_1 that fixes a point in Y. If φ^{k_1} composed with an approperiate inner automorphism acts on A as a periodic automorphism we leave this vertex as it is. Otherwise, up to composing with an inner automorphism we may assume that φ^{k_1} maps A to itself so we can obtain a new commutative diagram with A-real trees Y_A^1 and Y_A from the action of φ^{k_1} on the subgroup A.

Since the conjugacy classes of all edge groups in Λ_1 are fixed under φ, if A acts freely on Y_A all the edges connected to the vertex stabilized by A in Λ_1 have trivial stabilizers. Hence, A is a free factor of F_n and we leave it as is. If A does not act freely on Y_A we get a new graph of groups Δ_A with fundamental group A. All the original edge groups of edges connected to the vertex stabilized by A have to stabilize points in Y_A since their conjugacy class is fixed by φ^{k_1}, so each such edge group can be conjugated into a vertex group in Δ_A. Therefore, we can replace the vertex stabilized by A in Λ_1 by the graph of groups Δ_A and get a new graph of groups Λ_2 with fundamental group F_n. The number of edges in this new graph of groups is strictly bigger than the number of edges in Λ_1. Applying lemma 2.3 and proposition 2.5 to Δ_A we may conclude that all conjugacy classes of edge groups and vertex groups corresponding to point stabilizers and stabilizers of IET components in Δ_A are periodic under the action of φ^{k_1}, hence, the same holds for all conjugacy classes of these vertex and edge stabilizers in Λ_2.

As long as we have a vertex group B in our refined graph of groups on which a power of φ does not act as either a periodic automorphism or a pseudo-Anosov of a punctured surface, and B does not act freely on a limit tree Y_B obtained from a sequence of actions of B on its Cayley graph via powers of φ, we can continue our refinement procedure and obtain graphs of groups with more and more edges. By generalized accessibility [Be-Fe2] this refinement has to terminate and we get a graph of groups which we denote by Λ_φ which satisfies properties (i) and (ii) of the theorem.

The whole limit procedure by which one obtains an action of a hyperbolic group on a real tree from a sequence of powers of an automorphism of the group, forces the fixed subgroup to fix a point in the limit tree. Hence, either the fixed subgroup fixes a point in the interior of an edge in the discrete part of one of the real trees constructed during our refinement procedure - in which case Fix(φ) is a cyclic edge group in Λ_φ, or Fix(φ) is a subgroup of a vertex group A on which a power of φ composed with an approperiate inner automorphism acts as a periodic automorphism and φ (possibly) permutes the conjugacy class of A with the conjugacy classes of other vertices in Λ_φ.

Let T_{Λ_φ} be the Bass-Serre tree corresponding to the graph of groups Λ_φ. By the properties of the graph of groups Λ_φ both A and $\varphi(A)$ fix vertices in the Bass-Serre tree T_{Λ_φ}. If Fix(φ) is not a cyclic edge group in Λ_φ it fixes only the vertex stabilized by A in the Bass-Serre tree T_{Λ_φ}, hence, $A = \varphi(A)$ and φ acts as a periodic automorphism on A, so we have (iii). □

Theorem 4.1 already gives a rather specific characterization of the fixed subgroup of an automorphism of a free group. We already know that the fixed subgroup of φ is either a cyclic edge group or a subgroup of a vertex group in a small splitting of F_n and the automorphism φ acts on this vertex group as a periodic automorphism. To get the Scott conjecture from it we will need to use M. Culler's analysis of periodic automorphisms.

Theorem 4.2 ([Cu],3.2) *Let F_k be a free group and let α be a periodic automorphism of it. Then the fixed subgroup of α is either cyclic or is a free factor in F_k.*

From the existence of the small splitting Λ_φ and Culler's theorem we immediately obtain the Scott conjecture.

Corollary 4.3. *Let φ be an automorphism of a free group. Then $rk(Fix(\varphi)) \leq n$.*

Proof: By part (iii) of theorem 4.1 if Fix(φ) is not trivial then it is either a cyclic edge group or a subgroup of a vertex group A of Λ_φ on which φ acts as a periodic automorphism. By a standard homological argument, the rank of each vertex group in a small splitting of F_n is bounded by n - the rank of the ambient free group. Hence, $rk(A) \leq n$. Applying theorem 4.2 to the action of φ on A, Fix(φ) is either cyclic or is a free factor of A, so in particular $rk(\text{Fix}(\varphi)) \leq n$. □

Our description of the fixed subgroup as cyclic or a free factor of a vertex group in Λ_φ can be applied to generalize the Scott conjecture in various ways. To demonstrate that we bring the Scott conjecture for subgroups of automorphisms which has been proven recently by W. Dicks and F. Ventura.

Corollary 4.4. *Let H be a f.g. subgroup of $Aut(F_n)$. Let $Fix(H)$ denotes the subgroup of F_n which is fixed by all elements of H. Then $rk(H) \leq n$.*

Proof: Let H be generated by the automorphisms $\varphi_1, \dots, \varphi_s$, and let K_1 be the fixed subgroup of φ_1. By corollary 4.3 $rk(K_1) \leq n$. If φ_2 is a non-trivial periodic automorphism, its fixed subgroup K_2 is either cyclic or a free factor of F_n. In either case $rk(K_1 \cap K_2) \leq n$.

By theorems 4.1 and 4.2 if φ_2 is not a periodic automorphism and its fixed subgroup K_2 is not trivial, either K_2 is cyclic or there exists a small splitting Λ_{φ_2} of F_n in which K_2 is a free factor of a vertex group A_2. K_1 being a subgroup of F_n inherits a small splitting from Λ_{φ_2}. If the vertex group A_2 intersects K_1 non-trivially, $A_2 \cap K_1$ is a vertex group in the small splitting K_1 inherits from Λ_{φ_2}. So in particular $rk(A_2 \cap K_1) \leq rk(K_1) \leq n$. K_2 is a free factor of A_2, so if the intersection between the fixed subgroups K_1 and K_2 is not trivial, $K_1 \cap K_2$ is a free factor of $A_2 \cap K_1$, and $rk(K_1 \cap K_2) \leq rk(A_2 \cap K_1) \leq n$. A finite induction argument finishes the proof of the corollary. □

To get the commutative diagram (1) we do not necessarily need our group to act on its Cayley graph. We may as well get such a diagram from a sequence of actions of a group on a δ-hyperbolic space via powers of a given automorphism. This diagram together with the "scheme" for getting the Nielsen-Thurston classification and the Scott conjecture may serve to get various generalizations of this conjecture. Let us note that the generalized Scott conjecture for an automorphism of a free product proven by D. Collins and E. Turner [Co-Tu] can be proven using our scheme applied to the actions of a group on the Bass-Serre tree corresponding to a free product.

The small splitting Λ_φ of a free group associated with an automorphism of it in theorem 4.1, gives us an understanding of all periodic conjugacy classes of such automorphism. It is not sufficient for understanding the structure of periodic conjugacy classes of free factors. To get a better understanding of these, we need to further refine the splitting. This will be done by introducing new dynamical invariants which give rise to new conservation laws for our commutative diagram. Having such a refinement we will also be able to show it is canonical.

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