# Quantization of orbit bundles in $\mathfrak{gl}_n^*(\mathbb{C})$

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**Abstract**

Let $G$ be the complex general linear group and $\mathfrak{g}$ its Lie algebra equipped with a factorizable Lie bialgebra structure; let $\mathrm{U}_\hbar(\mathfrak{g})$ be the corresponding quantum group. We construct explicit $\mathrm{U}_\hbar(\mathfrak{g})$-equivariant quantization of Poisson orbit bundles $O_\lambda \longrightarrow O_\mu$ in $\mathfrak{g}^*$.

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# Introduction

Let $G$ be a Poisson Lie group and $\mathfrak{g}$ the Lie algebra of $G$. Let $\mathrm{U}_\hbar(\mathfrak{g})$ be the quantization of the universal enveloping algebra $\mathrm{U}(\mathfrak{g})$ along the corresponding Lie bialgebra structure on $\mathfrak{g}$. Consider two Poisson homogeneous $G$-manifolds, $M := G/G_M$ and $N := G/G_N$ such that the stabilizer $G_M$ is a subgroup in $G_N$. Then there exists a natural projection $M \longrightarrow N$

of $G$-spaces, and it makes $M$ into a $G$-bundle over $N$ with the fiber $G_N/G_M$. Suppose this projection is a Poisson map. It is natural to consider the problem of equivariant quantization of such a bundle. By this we understand a $\mathrm{U}_\hbar(\mathfrak{g})$-equivariant quantization of the function algebras together with the co-projection $F(M) \hookleftarrow F(N)$, to a morphism $F_\hbar(M) \hookleftarrow F_\hbar(N)$ of $\mathrm{U}_\hbar(\mathfrak{g})$-algebras.

In the present paper, we quantize orbit bundles for the case when $G = GL_n(\mathbb{C})$, and $\mathfrak{g}$ is equipped with a factorizable Lie bialgebra structure. We assume the stabilizers $G_M$ and $G_N$ to be Levi subgroups of $G$. Specifically for the $GL_n(\mathbb{C})$-case, those are precisely reductive subgroups of maximal rank. Then the $G$-varieties $M$ and $N$ can be realized as semisimple coadjoint orbits $O_1, O_2 \subset \mathfrak{g}^*$.

The Poisson structure on $\mathcal{O}_i$ is obtained by restriction from a Poisson structure on $\mathfrak{g}^*$. The latter is a linear combination of the $G$-invariant Kostant-Kirillov-Lie-Souriau bracket (KKLS) and the Semenov-Tian-Shansky bracket (STS). In our case, $G = GL_n(\mathbb{C})$, they are compatible, i.e. the Schouten bracket between the two is equal to zero. The Poisson bracket on $\mathcal{O}_i$ is not $G$-invariant, however, it makes $\mathcal{O}_i$ Poisson-Lie manifolds over the Poisson-Lie group $G$. Moreover, it is the only such bracket on $\mathcal{O}_i$ obtained by restriction from $\mathfrak{g}^*$.

Explicit quantization of semisimple orbits has been constructed in [4] for the special case of the standard, or Drinfeld-Jimbo, quantum group $\mathrm{U}_\hbar(\mathfrak{g})$. In the present paper, we extend that quantization to the case of any factorizable Lie bialgebra structure on $\mathfrak{g}$ and the corresponding quantum group. We also describe all semisimple Poisson-Lie orbit bundles $O_1 \longrightarrow O_2$ in $\mathfrak{g}^*$. In particualar, we show that $O_2$ is necessarily symmetric. We explicitly construct a $\mathrm{U}_\hbar(\mathfrak{g})$-equivariant quantization of the projection map $P$ for all orbit bundles.

The paper is organized as follows.
In Section 1 we recall some definitions concerning equivariant quantization.
In Section 2 we establish necessary and sufficient conditions for an orbit bundle to be Poisson.
In Section 3 we study the behavior of algebras defined by a modified (quadratic-linear) Reflection Equation under twist of quantum groups.
In Section 4 we use results of the previous section to extend the double quantization of orbits [4] to the case of the quantum group defined by an arbitrary factorizable classical r-matrix.
In Section 5 we prove that any Poisson orbit bundle admits a $\mathrm{U}_\hbar(\mathfrak{g})$-equivariant quantization, so the conditions of Section 2 are also sufficient. We give an explicit formula for the quantized bundle map.
There is an Appendix at the end of the paper where we study certain properties of the q-trace functions.

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# 1 Generalities on equivariant quantization

## 1.1 Deformation quantization of Poisson varieties

Let $M$ be a variety with a Poisson bracket $p$ and $A = \mathbb{C}[M]$ be the algebra of polynomial functions $M \longrightarrow \mathbb{C}$. Recall the following definition (see e.g. [10]):

**Definition 1** An algebra $(A_\hbar, \star)$ over the ring $\mathbb{C}[[\hbar]]$ of formal power series is called *quantization* of $(A, p)$ if:

(i) $A_\hbar$ is a free $\mathbb{C}[[\hbar]]$-module;

(ii) As a $\mathbb{C}$-algebra, the quotient $A_\hbar/\hbar A_\hbar$ is isomorphic to $A$;

(iii) If $a, b \in A$ then $\dfrac{a \star b - b \star a}{\hbar} \equiv p(a, b)$ modulo $\hbar$.

The Poisson bracket $p$ is called *the infinitesimal* of $(A_\hbar, \star)$.

**Remark 1** The deformed multiplication is expanded as an $\hbar$-series: $a \star b = \sum_{k=0}^{\infty} m_k(a \otimes b)\hbar^k$ for $a, b \in A \subset A_\hbar$. Therefore one has $p(a, b) = m_1(a \otimes b) - m_1(b \otimes a)$.

Let $(M, p_M)$ and $(N, p_N)$ be two Poisson varieties, $A_\hbar$ and $B_\hbar$ some quantizations of the function algebras $A = \mathbb{C}[M]$ and $B = \mathbb{C}[N]$ respectively, $f \colon B \longrightarrow A$ a morphism of Poisson algebras.

**Definition 2** A homomorphism $f_\hbar \colon B_\hbar \longrightarrow A_\hbar$ of $\mathbb{C}[[\hbar]]$-algebras is called a *quantization of the map* $f$ if the induced morphism $f_0 \colon B_\hbar/\hbar B_\hbar \longrightarrow A_\hbar/\hbar A_\hbar$ of $\mathbb{C}$-algebras coincides with $f$.

**Proposition 1** *Suppose there exists a quantization of $f \colon B \longrightarrow A$. Then $f$ is a Poisson map.*

Proof: Denote by $m_{A_\hbar}$ the multiplication in $A_\hbar$ and by $m_A$ the undeformed multiplication in $A$. The map $f_\hbar$ is supposed to be an algebra homomorphism, i.e. $f_\hbar \circ m_{B_\hbar} = m_{A_\hbar} \circ (f_\hbar \otimes f_\hbar)$. Consider the infinitesimal part of this equality:

$$f \circ m_{B_\hbar,1} + f_1 \circ m_B = m_A \circ (f_1 \otimes f) + m_A \circ (f \otimes f_1) + m_{A_\hbar,1} \circ (f \otimes f)\,.$$

Applying this equality first to $a \otimes b \in A \otimes A$, then to $b \otimes a$ and taking the difference, one obtains $f\big(p_M(a,b)\big) = p_N\big(f(a), f(b)\big)$ where $p_M(a,b) = m_{A_\hbar,1}(a,b) - m_{A_\hbar,1}(b,a)$ and $p_N(a,b) = m_{B_\hbar,1}(a,b) - m_{B_\hbar,1}(b,a)$. ■

## 1.2 Quantization of $G$-varieties

Consider a simple complex algebraic group $G$ and its Lie algebra $\mathfrak{g}$. Suppose $G$ is a Poisson group; then $\mathfrak{g}$ is equipped with a quasitriangular Lie bialgebra structure. Denote by $r \in \wedge^2\mathfrak{g}$ the corresponding classical r-matrix. We consider only the factorizable case, when $r$ satisfies the *modified* classical Yang-Baxter equation. By $\mathrm{U}_\hbar(\mathfrak{g}, r)$ we denote the quantization of the universal enveloping algebra $\mathrm{U}(\mathfrak{g})$ along $r$.

Consider the variety $M = K\backslash G$ where $K \subset G$ is a reductive subgroup in $G$ of maximal rank; let $\mathfrak{k}$ denote its Lie algebra (we prefer to work with right coset spaces, so that the right $G$-action induces a left action on functions). Suppose $M$ is a Poisson $G$-variety, that is to say, the action $G \times M \longrightarrow M$ is Poisson. Set $A = \mathbb{C}[M]$ and let $A_\hbar$ be its quantization. We expect the deformed multiplication in $A_\hbar$ to be equivariant with respect to an action of $\mathrm{U}_\hbar(\mathfrak{g}, r)$. In other words, this multiplication should obey the "Leibniz rule"

$$x.(a \star b) = (x^{(1)}.a) \star (x^{(2)}.b)$$

for all $a, b \in A$ and $x \in \mathrm{U}_\hbar(\mathfrak{g}, r)$. We use the standard Sweedler notation $x^{(1)} \otimes x^{(2)}$ for the coproduct $\Delta(x)$.

The infinitesimal of a $\mathrm{U}_\hbar(\mathfrak{g}, r)$-equivariant quantization $A_\hbar$ is always of the form (see [8])

$$p = \overleftarrow{r} + \overrightarrow{s}. \tag{1}$$

Here $\overleftarrow{r}$ denotes the bivector field on $M$ generated by $r \in \wedge^2\mathfrak{g}$ via the action of $G$, and $\overrightarrow{s}$ denotes the invariant bivector field on $M$ generated by an element $s \in (\wedge^2\mathfrak{g}/\mathfrak{k})^{\mathfrak{k}}$. The latter should satisfy certain conditions, and the bivector field $\overrightarrow{s}$ is called *quasi-Poisson structure* on $M$. For simplicity, we call the generator $s$ a quasi-Poisson structure as well.

Recall the constructions of the bivector fields $\overleftarrow{r}$ and $\overrightarrow{s}$. For $r \in \wedge^2\mathfrak{g}$ and $g \in G$, consider the bivector field $(R_g)_* r$ on $G$, where $R_g : G \longrightarrow G$ is the right translation $x \mapsto xg$.

This bivector field is left $G$-invariant, so it is left $K$-invariant. Hence it is projectable to $M = K\backslash G$, and we denote the projection by $\overleftarrow{r}$. Note that $\overleftarrow{r}$ is not $G$-invariant.

To describe the bivector field $\overrightarrow{s}$, lift $s$ from $\wedge^2(\mathfrak{g}/\mathfrak{k})^{\mathfrak{k}}$ to $(\wedge^2\mathfrak{g})^{\mathfrak{k}}$ and consider a left $G$ invariant bivector field $(L_g)_* s$, where $L_g : G \longrightarrow G$ is the left translation $x \mapsto gx$. It is also left $K$-invariant, hence it is projectable to $M = K\backslash G$. We denote the projection by $\overrightarrow{s}$. Any $G$ invariant bivector field on $M$ is obtained in this way.

Left and right invariant vector fields on $G$ commute with each other, hence the Schouten bracket $[\overleftarrow{r}, \overrightarrow{s}]$ vanishes. This implies that $p = \overleftarrow{r} + \overrightarrow{s}$ is a Poisson bracket if and only if $[\overleftarrow{r}, \overleftarrow{r}] = -[\overrightarrow{s}, \overrightarrow{s}]$.

Recall that two Poisson brackets are called compatible if their any linear combination is again a Poisson bracket. Suppose that the new bracket makes the variety $M$ Poisson over the Poisson group $G$. Then the formula (1) suggests that a Poisson bracket $\kappa$ on $M$ is compatible with $p$ if and only if it is $G$-invariant and $[\overrightarrow{s}, \kappa] = 0$. Next we recall the notion of 2-parameter, or double quantization (see, for example, [2]).

**Definition 3** Suppose that the commutative algebra $A$ is endowed with two compatible Poisson brackets, $p$ and $\kappa$, such that $\kappa$ is $G$-invariant. An algebra $(A_{\hbar,t}, \star)$ over the ring $\mathbb{C}[\![\hbar, t]\!]$ of formal power series in two variables is called *equivariant quantization* of $(A, p, \kappa)$ if

- (i) $A_{\hbar,t}$ is a free module over $\mathbb{C}[\![\hbar, t]\!]$;
- (ii) The first order term of the deformed multiplication $\star$ is $\hbar p + t\kappa$;
- (iii) The algebra $A_{\hbar,t}$ is $\mathrm{U}_\hbar(\mathfrak{g}, r)$-equivariant;
- (iv) The quotient $A_{\hbar,t}/\hbar A_{\hbar,t}$ is a $G$-equivariant (one-parameter) quantization of $(A, \kappa)$.

# 2 Poisson-Lie orbit bundles

## 2.1 General remarks on coadjoint orbits in $\mathfrak{gl}_n^*(\mathbb{C})$

Fix $G = GL_n(\mathbb{C})$ and put $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$. Choose the algebra of diagonal matrices as a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. The non-degenerate $G$-invariant bilinear form $\mathrm{Trace}(XY)$ on $\mathfrak{g}$ allows us to think of $\mathfrak{h}^*$ as a subset in $\mathfrak{g}^*$, the dual vector space for $\mathfrak{g}$. Denote by $O_\lambda$ the coadjoint orbit of a semisimple element $\lambda \in \mathfrak{h}^* \subset \mathfrak{g}^*$. As a $G$-variety, $O_\lambda$ is isomorphic to $G^\lambda\backslash G$ where $G^\lambda$ is a Levi subgroup of $G$. That is, the Lie algebra $\mathfrak{g}^\lambda$ of $G^\lambda$ is a Levi subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$.

The trace bilinear form on $\mathfrak{g}$ establishes a canonical isomorphism between coadjoint orbits in $\mathfrak{g}^*$ and adjoint orbits in $\mathfrak{g}$. We will use this isomorphism without further noticing. In the same way, we identify an element $\lambda \in \mathfrak{h}^*$ with the corresponding diagonal matrix in $\mathfrak{h}$.

Two diagonal matrices with different order of the entries belong to the same $G$-orbit. Hence we can choose a representative of the orbit in which all equal entries are grouped up together. In other words, we can think that $\lambda = \mathrm{diag}(\Lambda_1, \dots, \Lambda_l)$ where $\Lambda_i$ is the scalar $n_i \times n_i$ matrix with $\lambda_i$ on its diagonal. In particular, the orbit $O_\lambda$ is determined by a pair $(\boldsymbol{\lambda}, \mathbf{n})$ where $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_l)$ is the row of pairwise distinct eigenvalues of $\lambda$, and $\mathbf{n} = (n_1, \dots, n_l)$ and $\lambda_1, \dots, \lambda_l$ is the row of their multiplicities. Note that the orbit $O_\lambda$ does not depend on simultaneous permutations of the entries of $\boldsymbol{\lambda}$ and $\mathbf{n}$.

## 2.2 The related Poisson structures on $O_\lambda$

The admissible quasi-Poisson bracket on a semisimple orbit has the form (see [3], [7], [8] and [14] for details):

$$\sum_{1 \leqslant i < j \leqslant l} c_{ij} \xi_{ij},$$

where $c_{ij}$ are some coefficients depending on the eigenvalues of $\lambda$ (see below for explicit formulas), and $\xi_{ij}$ are defined as follows:

$$\xi_{ij} = \sum_{s,t} E_{st} \wedge E_{ts} \mod \mathfrak{g} \wedge \mathfrak{g}^\lambda. \tag{2}$$

Here $E_{st}$ is the $(s,t)$-th matrix unit, $l$ is the number of different eigenvalues of $\lambda$, and the sum is taken over $n_1 + n_2 + \dots + n_{i-1} < s \leqslant n_1 + n_2 + \dots + n_i$, $n_1 + n_2 + \dots + n_{j-1} < t \leqslant n_1 + n_2 + \dots + n_j$, where $n_i$ denotes the multiplicity of the eigenvalue $\lambda_i$.

### The KKSL Poisson bracket $\kappa_\lambda$

The Kirillov-Kostant-Lie-Souriau bracket $\kappa_\lambda$ is induced on $O_\lambda$ from the Lie structure on $\mathfrak{g}$. For a semisimple (co)adjoint $GL_n(\mathbb{C})$-orbit $O_\lambda$ it is given by the following expression:

$$\kappa_\lambda = \sum_{1 \leqslant i < j \leqslant l} \frac{1}{\lambda_i - \lambda_j} \xi_{ij}.$$

This is a $G$-invariant non-degenerate Poisson bracket.

**The quasi-Poisson bracket $s^0_\lambda$.**

Consider the bivector field on $O_\lambda$ restricted from the Semenov-Tian-Shansky (STS) bracket on $\mathrm{End}(\mathbb{C}^n)$, see [17]. Its quasi-Poisson part is generated by an element $s^0_\lambda \in \wedge^2(\mathfrak{g}/\mathfrak{g}^\lambda)^{\mathfrak{g}^\lambda}$. Specifically for the case $G = GL_n(\mathbb{C})$, it takes the form:

$$s^0_\lambda = \sum_{1 \leqslant i < j \leqslant l} \frac{\lambda_i + \lambda_j}{\lambda_i - \lambda_j}\, \xi_{ij},$$

see [5].

**The brackets admitting $\mathrm{U}_\hbar(\mathfrak{g}, r)$-equivariant quantization**

In our case $G = GL_n(\mathbb{C})$, the Schouten bracket between $s^0_\lambda$ and $\kappa_\lambda$ vanishes. Thus, by adding a multiple of the KKLS bracket to that generated by $s^0_\lambda$, one obtains the general form for a quasi-Poisson bracket on $O_\lambda$ admitting a $\mathrm{U}_\hbar(\mathfrak{g}, r)$-equivariant quantization:

$$s_\lambda = s^0_\lambda + a\kappa_\lambda = \sum_{1 \leqslant i < j \leqslant l} \frac{\lambda_i + \lambda_j + a}{\lambda_i - \lambda_j}\, \xi_{ij}, \text{ where } a \in \mathbb{C}. \tag{3}$$

Recall once again that we consider only those Poisson structures that are restricted from $\mathfrak{g}^*$.

## 2.3 The structure of Poisson orbit bundles

Here we give the necessary and sufficient conditions for an orbit map $P\colon (O_\lambda, s_\lambda) \longrightarrow (O_\mu, s_\mu)$ to be Poisson. Recall that we do not distinguish between an element $\lambda \in \mathfrak{g}^*$ and the corresponding diagonal matrix $\mathrm{diag}(\Lambda_1, \ldots, \Lambda_l)$.

**Proposition 2** *Any semisimple orbit bundle $O_\lambda \longrightarrow O_\mu$ is of the form $X \mapsto P(X)$ where $P$ is a polynomial in one variable with complex coefficients.*

Proof: If an equivariant map $(O_\lambda, s_\lambda) \longrightarrow (O_\mu, s_\mu)$ brings $\lambda$ to $\mu$, then the isotropy group $G^\lambda$ is a subgroup in $G^\mu$; this gives an inclusion $\mathfrak{g}^\lambda \subset \mathfrak{g}^\mu$ of their Lie algebras.

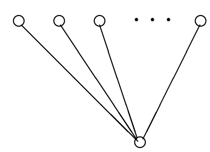
Consider $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$ as an associative algebra, $\mathfrak{g} \cong \mathrm{End}(\mathbb{C}^n)$, and denote by $\mathrm{Z}(\mathfrak{g}^\lambda) \subset \mathfrak{g}$ the centralizer of $\mathfrak{g}^\lambda$. Since $\mathfrak{g}^\lambda$ is a Levi subalgebra, $\mathrm{Z}(\mathfrak{g}^\lambda)$ is a semisimple commutative associative algebra generated by $\lambda$ and by the unit matrix. For any polynomial $P$ in one variable the mapping $X \mapsto P(X)$, $X \in \mathfrak{g}$, is $G$-equivariant. Hence it suffices to check that $\mu = P(\lambda)$ for some polynomial $P$. The inclusion $\mathfrak{g}^\lambda \subset \mathfrak{g}^\mu$ implies the inclusion $Z(\mathfrak{g}^\mu) \subset Z(\mathfrak{g}^\lambda)$. Therefore the matrix $\mu$ is a polynomial in $\lambda$. ∎

**Remark 2** We will use the same symbol $P$ for both the orbit map $O_\lambda \longrightarrow O_\mu$ and for the corresponding polynomial.

### The graph of an orbit bundle

It is convenient to use the following graphical presentation of an orbit bundle $P\colon O_\mu \longrightarrow O_\lambda$. The orbit $O_\lambda$ has a representative in the form of diagonal matrix $\lambda = \mathrm{diag}(\Lambda_1, \dots, \Lambda_l)$. Set $\mu = P(\lambda)$, then $\mu$ also has the form $\mu = \mathrm{diag}(\mathrm{M}_1, \dots, \mathrm{M}_m)$, where $\mathrm{M}_j$ denotes the scalar block corresponding to the eigenvalue $\mu_j$ of $\mu$.

Using this, denote by $\Gamma_P$ the bipart type graph whose upper nodes have labels $1, \dots, l$ corresponding to the blocks $\Lambda_1, \dots, \Lambda_l$, and the lower nodes are labeled by $1, \dots, m$ corresponding to the blocks $\mathrm{M}_1, \dots, \mathrm{M}_m$. The $i$-th node of the upper part of $\Gamma_P$ is connected to the $\alpha$-th node of the lower part if and only if $P(\lambda_i) = \mu_\alpha$. Note that each upper node has exactly one edge. Since the map $P$ is surjective, each lower node is connected to some upper node. Therefore the graph $\Gamma_P$ is a disjoint union of trees of the form:




Each tree is a connected component of $\Gamma_P$ and it is labeled by the blocks of $\mu$. The graph $\Gamma_P$ gives a complete description of the map $P$.

### The graph $\Gamma_P$ for $P$ Poisson

Both $O_\lambda$ and $O_\mu$ are endowed with Poisson structures (3). The tangent space of $O_\lambda$ at the point $\lambda$ is isomorphic as a vector space to the quotient $\mathfrak{g}/\mathfrak{g}^\lambda$. Recall that $\mathfrak{g}^\lambda \subset \mathfrak{g}^\mu$. The tangent map $P_* : \mathfrak{g}/\mathfrak{g}^\lambda \longrightarrow \mathfrak{g}/\mathfrak{g}^\mu$ of $P$ is given by the formula:

$$P_*(X) = \begin{cases} 0, \text{ if } X \in \mathfrak{g}^\mu/\mathfrak{g}^\lambda, \\ \\ X \text{ otherwise.} \end{cases}$$

An element $X \in \mathfrak{g} \setminus \mathfrak{h}$ cannot generate a vector field on $O_\mu$ since it is not even $\mathfrak{h}$-invariant. However, the element like $\xi_{ij}$ (see formula (2)) does generate a bivector field on $O_\mu$. The map $P$ is Poisson if and only if $P_*(s_\lambda) = s_\mu$. The tangent map $P_*$ is determined by its values

$P_*(\xi_{ij}) = \xi_{\alpha\beta}$ where $i, j$ run over the upper nodes of the graph $\Gamma_P$ while $\alpha, \beta$ run over the lower nodes of $\Gamma_P$.

**Lemma 1** *Let $P\colon O_\lambda \longrightarrow O_\mu$ be a Poisson orbit bundle w.r.t. the Poisson structure on $O_\lambda$ determined by $s_\lambda = \sum c_{ij}\xi_{ij}$ and some Poisson structure on $O_\mu$. Suppose that $i \leqslant j < p \leqslant s$ are such* upper *nodes that $i$ and $j$ belong to the same connected component $\alpha$, and $p$, $s$ also belong to the same connected component $\beta$ of $\Gamma_P$. Then $c_{ip} = c_{js}$.*

PROOF: If $P$ is Poisson then $s_\lambda$ is projectable under $P_*$. Thus if $P_*(\xi_{ip})$ and $P_*(\xi_{js})$ enter the same basis bivector $\xi_{\alpha\beta}$, then $c_{ip} = c_{js}$. ■

**Classification of Poisson bundles**

We call a coadjoint orbit $O_\mu$ *symmetric* if the corresponding matrix M has exactly two different eigenvalues.

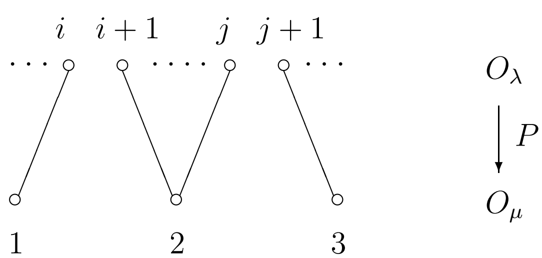
**Theorem 1** *A $GL_n(\mathbb{C})$-equivariant map $P\colon O_1 \longrightarrow O_2$ is Poisson if and only if the following three conditions are satisfied:*

(a) *The orbit $O_2$ is symmetric;*

(b) *There exist $\lambda \in O_1$ and $\mu \in O_2$ such that $P(\lambda) = \mu$ and the multiplicity $n_1$ of the eigenvalue $\lambda_1$ is equal to the multiplicity $m_1$ of the eigenvalue $\mu_1$;*

(c) *The Poisson structures on $O_1$ and $O_2$ are defined by $s_\lambda = s^0_\lambda - 2\lambda_1\kappa_\lambda$ and $s_\mu = s^0_\mu - 2\mu_1\kappa_\mu$ respectively.*

PROOF: Show first that if $P\colon O_1 \longrightarrow O_2$ is a Poisson map, then the orbit $O_2$ is symmetric, i.e. the graph $\Gamma_P$ consists of exactly two connected components. Indeed, suppose that $\Gamma_P$ has the form




Since $P$ is Poisson, Lemma 1 leads to the following system of conditions:

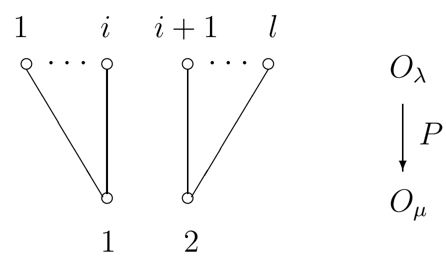
$$\begin{cases} c_{i,i+1} = c_{i,j} \\ c_{i+1,j+1} = c_{j,j+1}, \end{cases}$$

where $c_{ij} = \dfrac{\lambda_i + \lambda_j + a}{\lambda_i - \lambda_j}$, see formula (3). Thus we should solve the following overdefined system of linear equations in $a$:

$$\begin{cases} \dfrac{\lambda_i + \lambda_{i+1} + a}{\lambda_i - \lambda_{i+1}} = \dfrac{\lambda_i + \lambda_j + a}{\lambda_i - \lambda_j} \\[2ex] \dfrac{\lambda_{i+1} + \lambda_{j+1} + a}{\lambda_{i+1} - \lambda_{j+1}} = \dfrac{\lambda_j + \lambda_{j+1} + a}{\lambda_j - \lambda_{j+1}} \end{cases}$$

This system is inconsistent, thus our hypothesis is wrong, and $\Gamma_P$ consists of exactly two connected trees. This means that $\mu$ has exactly two different eigenvalues: $\mu = \mathrm{diag}(\mathrm{M}_1, \mathrm{M}_2)$, i.e. it is symmetric.

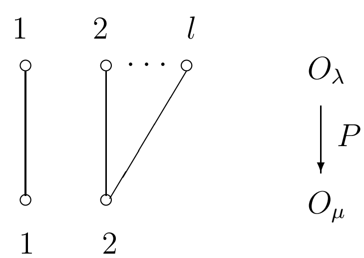
We now show that if $P$ is Poisson, then either mult $\lambda_1 =$ mult $\mu_1$ or mult $\lambda_l =$ mult $\mu_2$, where mult means the multiplicity of the corresponding eigenvalue. In other words, we need to prove that if $P$ is Poisson, then one of the two connected components of the graph $\Gamma_P$ contains precisely one edge. Suppose, to the contrary, that $\Gamma_P$ is of the form:




Again, by Lemma 1 the following system of conditions should be satisfied: $c_{1,i+1} = c_{i,i+1} = c_{i,l}$. This is a system of equations in $a$:

$$\frac{\lambda_1 + \lambda_{i+1} + a}{\lambda_1 - \lambda_{i+1}} = \frac{\lambda_i + \lambda_{i+1} + a}{\lambda_i - \lambda_{i+1}} = \frac{\lambda_i + \lambda_l + a}{\lambda_i - \lambda_l},$$

which obviously has no solution. Thus one of the two connected components of $\Gamma_P$ contains one edge. By appropriate choice of the representatives $\lambda \in O_\lambda$ and $\mu \in O_\mu$, one can assume that to be the left component:




i.e. that the multiplicities of $\lambda_1$ and $\mu_1$ coincide.

Lemma 1 applied to the above graph gives rise to the overdefined system of $l-1$ linear equations $c_{12} = c_{13} = \ldots = c_{1l}$. This system is consistent, and the solution is $a = -2\lambda_1$. Hence the quasi-Poisson brackets on $O_i$ are as in (c).

Now we prove the sufficiency of the conditions (a)–(c). Indeed, consider an invariant mapping $P\colon O_1 \longrightarrow O_2$ corresponding to the above graph. Suppose that $P(\lambda) = \mu$. Then for $s_\lambda = s^0_\lambda - 2\lambda_1\kappa_\lambda$ and $s_\mu = s^0_\mu - 2\mu_1\kappa_\mu$ one has $P_*(s_\lambda) = s_\mu$, i.e. $P$ is a Poisson map with respect to those brackets. ■

## 2.4 Explicit formula for the map $P$

Denote by $\lambda$ and $\mu$ elements of $\mathfrak{h}$ satisfying the conditions of Theorem 1 and by $O_\lambda$ and $O_\mu$ their adjoint orbits. For the purpose of quantization, we need an explicit expression for $P$. Recall that $\lambda_1, \lambda_2, \ldots, \lambda_l$ denote all the distinct eigenvalues of $\lambda \in \mathfrak{g}$, so we need to find a polynomial $P(x)$ of degree $l-1$ in one variable with complex coefficients such that $P(\lambda_1) = \mu_1$ and $P(\lambda_2) = \ldots = P(\lambda_l) = \mu_2$. These $l$ equations determine $P$ uniquely:

$$P(x) = (\mu_1 - \mu_2) \prod_{i=2}^{l} \frac{x - \lambda_i}{\lambda_1 - \lambda_i} + \mu_2. \tag{4}$$

# 3 Reflection equation algebras

An equivariant two-parameter quantization of coadjoint orbits is constructed in [4] for the special case of the standard, or Drinfeld-Jimbo, quantum group $\mathrm{U}_\hbar(\mathfrak{g})$. In this section we extend the quantization of [4] for an arbitrary quantum group $\mathrm{U}_\hbar(\mathfrak{g}, r)$, not necessarily the standard. Note that possible factorizable Lie bialgebra structures on $\mathfrak{g}$ are parameterized by Belavin-Drinfeld triples and a subspace in $\mathfrak{h}$, [1]. Quantization of the universal enveloping algebra along any Lie bialgebra structure has been constructed in [12].

To describe quantized coadjoint orbits explicitly, we need some facts about the so called *modified reflection equation* (mRE) algebra. It is a two parameter quantization of the polynomial ring on the vector space of $n \times n$-matrices. The quantized orbits will be presented as quotients of the mRE algebra by certain ideals which are deformations of the classical ideals of the orbits. In the present section we study how mRE algebras transform under twist of quantum groups.

## 3.1 The definition and basic facts

Let $\mathrm{U}_\hbar(\mathfrak{g}, r)$ be a quantization of the universal enveloping algebra of $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$ along a classical r-matrix $r$. Let $R \in \mathrm{End}(\mathbb{C}^n \otimes \mathbb{C}^n)[\![\hbar]\!]$ be the image of its universal R-matrix in the basic representation in $\mathbb{C}^n[\![\hbar]\!]$. Denote by $S := \sigma R \in \mathrm{End}(\mathbb{C}^n \otimes \mathbb{C}^n)[\![\hbar]\!]$ the quantum permutation, where $\sigma$ designates the usual flip $\mathbb{C}^n \otimes \mathbb{C}^n \longrightarrow \mathbb{C}^n \otimes \mathbb{C}^n$, $u \otimes v \mapsto v \otimes u$.

**Definition 4** The mRE algebra $\mathcal{L}$ is an associative unital algebra over the ring $\mathbb{C}[\![\hbar]\!][t]$ generated by the entries of the matrix $L = (L_{ij})_{i,j=1}^n$ modulo the relations

$$[SL_2S, L_2] = -qt[S, L_2] \in \mathrm{End}(\mathbb{C}^n \otimes \mathbb{C}^n) \otimes \mathcal{L}, \tag{5}$$

where $L_2 := 1 \otimes L$ and $q := e^\hbar$.

The action of $\mathrm{U}_\hbar(\mathfrak{g}, r)$ on the algebra $\mathcal{L}$ is given by the formula

$$x \triangleright L = \rho\big(\gamma(x^{(1)})\big) L \rho(x^{(2)}), \tag{6}$$

where $\gamma$ is the antipode of $\mathrm{U}_\hbar(\mathfrak{g}, r)$ and $\rho$ is the representation $\mathrm{U}_\hbar(\mathfrak{g}, r) \longrightarrow \mathrm{End}(\mathbb{C}^n)[\![\hbar]\!]$. The algebra $\mathcal{L}$ is a $\mathrm{U}_\hbar(\mathfrak{g}, r)$-equivariant quantization of the polynomial ring $\mathbb{C}[\mathrm{End}(\mathbb{C}^n)]$ with $\hbar s^0_\lambda - t\kappa_\lambda$ being the linear term of the deformed multiplication, [4].

**Remark 3** The relations (5) are called the *modified reflection equation* (mRE). These relations become quadratic when $t = 0$. The corresponding quotient $\mathcal{L}/t\mathcal{L}$ is called *quadratic* or simply *reflection equation* (RE) algebra. This algebra can be defined for any quasitriangular Hopf algebra $\mathcal{H}$ and its representation. When $\mathcal{H}$ is a quantized universal enveloping algebra of an algebraic matrix group $G$, then a certain quotient of the quadratic RE algebra yields a (one-parameter) quantization of $\mathbb{C}[G]$. The mRE algebra $\mathcal{L}$ as a two-parameter quantization of the coordinate ring on the matrix space is special for the case $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$.

Let us describe the center $Z(\mathcal{L})$ of the algebra $\mathcal{L}$. First of all, $Z(\mathcal{L})$ coincides with the subalgebra of $\mathrm{U}_\hbar(\mathfrak{g}, r)$-invariants and it is isomorphic to $Z_0 \otimes \mathbb{C}[\![\hbar]\!][t]$, where $Z_0 \subset \mathbb{C}[\mathrm{End}(\mathbb{C}^n)]$ is the subalgebra of classical invariants. To describe $Z(\mathcal{L})$, consider the matrix $R^* := \big((R^{t_1})^{-1}\big)^{t_1} = R^*_1 \otimes R^*_2$, where $t_1$ means transposition of the first tensor component. This matrix is equal to $\mathcal{R}_1 \otimes \gamma(\mathcal{R}_2)$ evaluated in the basic representation. Define the matrix $D := \nu(R^*_1 R^*_2) \in \mathrm{End}(\mathbb{C}^n)[\![\hbar]\!]$, where $\nu$ is a scalar. It is convenient to choose $\nu$ such that $\mathrm{Trace}(D) = \dfrac{1 - q^{-2n}}{1 - q^{-2}}$. Put $\tau_m = \mathrm{Trace}_q(L^m) := \mathrm{Trace}(DL^m) \in Z(\mathcal{L})$ for $m = 1, 2, \ldots$. Then the $Z(\mathcal{L})$ is a polynomial $\mathbb{C}[\![\hbar]\!]$-algebra generated by $\tau_1, \ldots, \tau_{n-1}$.

### 3.2 mRE algebras and twist

The quantum group $\mathrm{U}_\hbar(\mathfrak{g}, r)$ is a twist of the standard quantization $\mathrm{U}_\hbar(\mathfrak{g})$. Let $\mathcal{F} \in \mathrm{U}_\hbar(\mathfrak{g})^{\otimes 2}$ be the corresponding twisting cocycle. It is an invertible element satisfying the identities

$$(\Delta \otimes \mathrm{id})(\mathcal{F})\mathcal{F}_{12} = (\mathrm{id} \otimes \Delta)(\mathcal{F})\mathcal{F}_{23}, \tag{7}$$

$$(\varepsilon \otimes \mathrm{id})(\mathcal{F}) = 1 \otimes 1 = (\mathrm{id} \otimes \varepsilon)(\mathcal{F}), \tag{8}$$

where $\varepsilon$ is the counit in $\mathrm{U}_\hbar(\mathfrak{g})$. As an associative algebra, $\mathrm{U}_\hbar(\mathfrak{g}, r)$ coincides with $\mathrm{U}_\hbar(\mathfrak{g})$ but has a different comultiplication, $x \mapsto \mathcal{F}^{-1}\Delta(x)\mathcal{F}$. The antipode is transformed accordingly, see [11].

Recall that a twist of Hopf algebras induces a transformation of module algebras, which we also call twist. Given a $\mathrm{U}_\hbar(\mathfrak{g})$-algebra $\mathcal{A}$ one gets an algebra over $\mathrm{U}_\hbar(\mathfrak{g}, r)$ with the new multiplication $a \otimes b \mapsto (\mathcal{F}_1 a)(\mathcal{F}_2 b)$ for $a, b \in \mathcal{A}$.

Applying $\mathcal{F}$ to the mRE algebra, one apparently destroys the form of relations (5). Nevertheless, the mRE algebras corresponding to $\mathrm{U}_\hbar(\mathfrak{g}, r)$ and $\mathrm{U}_\hbar(\mathfrak{g})$ are still related by $\mathcal{F}$, as we now demonstrate.

Let $\mathcal{A}$ and $\widetilde{\mathcal{A}}$ denote the quadratic RE algebras corresponding to quantum groups $\mathrm{U}_\hbar(\mathfrak{g})$ and $\mathrm{U}_\hbar(\mathfrak{g}, r)$ respectively. We assume that they are extended trivially to $\mathbb{C}[t]$-algebras. Let $\widetilde{\mathcal{A}}'$ be the twist of the algebra $\mathcal{A}$ by the cocycle $\mathcal{F}$.

Denote by $\{Q_{ij}\} \subset \mathcal{A}$ and $\{\widetilde{Q}_{ij}\} \subset \widetilde{\mathcal{A}}$ the generators satisfying the quadratic RE, i.e. the equation similar to (5) but with zero in the r.h.s. These algebras are quantizations of $\mathbb{C}[\mathrm{End}(\mathbb{C}^n)]$ along the STS brackets. As was shown in [15], $\widetilde{\mathcal{A}}'$ is isomorphic to $\widetilde{\mathcal{A}}$ as $\mathrm{U}_\hbar(\mathfrak{g}, r)$-module algebras, and the isomorphism $\phi\colon \widetilde{\mathcal{A}} \longrightarrow \widetilde{\mathcal{A}}'$ is given by the formula

$$\widetilde{Q} \mapsto (\rho \circ \gamma)(\mathcal{F}_1 \zeta) Q \rho(\mathcal{F}_2). \tag{9}$$

Here $\zeta := \mathcal{F}_2^{-1}\gamma^{-1}(\mathcal{F}_1^{-1}) \in \mathrm{U}_\hbar(\mathfrak{g})$ is the element which participates in definition of the antipode $\tilde{\gamma}$ of $\mathrm{U}_\hbar(\mathfrak{g}, r)$, namely $\tilde{\gamma}(x) = \gamma(\zeta^{-1} x \zeta)$ for all $x \in \mathrm{U}_\hbar(\mathfrak{g}, r)$.

Choose the new generators $\{K_{ij}\} \subset \widetilde{\mathcal{A}}'$ by setting $K_{ij} := Q_{ij} - t\delta_{ij}$ and similarly for $\{\widetilde{K}_{ij}\} \subset \widetilde{\mathcal{A}}$. Note that $K_{ij}$ are also generators of $\mathcal{A}$.

**Lemma 2** *The isomorphism $\phi$ given by the formula (9) defines a linear map* $\mathrm{Span}\big(\widetilde{K}_{ij}\big) \longrightarrow \mathrm{Span}(K_{ij})$ *through the formula*

$$(\mathrm{id} \otimes \phi)(\widetilde{K}) = (\Phi \otimes \mathrm{id})(K), \tag{10}$$

*where $\Phi$ is an invertible linear operator* $\mathrm{End}(V) \longrightarrow \mathrm{End}(V)$ *acting by the rule* $\Phi(X) = (\rho \circ \gamma)(\mathcal{F}_1 \zeta) X \rho(\mathcal{F}_2)$

PROOF: Evaluating $\phi$ on the generators we find

$$\phi(\widetilde{K}) = \rho\big(\gamma(\mathcal{F}_1\zeta)\big)Q\rho(\mathcal{F}_2) - t = \rho\big(\gamma(\mathcal{F}_1\zeta)\big)K\rho\big(\mathcal{F}_2\big) + t\rho\big(\gamma(\mathcal{F}_1\zeta)\mathcal{F}_2\big) - t.$$

The assertion will be proved if we show that $\gamma(\zeta)\gamma(\mathcal{F}_1)\mathcal{F}_2 = 1$. But this is a well known fact from the twist theory, see [11]. ■

Denote by $\mathcal{L}$ and by $\widetilde{\mathcal{L}}$ the mRE algebras corresponding to $\mathrm{U}_\hbar(\mathfrak{g})$ and $\mathrm{U}_\hbar(\mathfrak{g}, r)$, respectively.

**Proposition 3** *The algebra $\widetilde{\mathcal{L}}$ is isomorphic to the twist of $\mathcal{L}$ by the cocycle $\mathcal{F}$.*

PROOF: Let $\widetilde{\mathcal{A}}'$ and $\widetilde{\mathcal{L}}'$ be respectively the twists of the algebras $\mathcal{A}$ and $\mathcal{L}$ by the cocycle $\mathcal{F}$. The algebra $\mathcal{A}$ admits an embedding in $\mathcal{L}$ through the assignment

$$K \mapsto (1 - q^{-2})L. \tag{11}$$

This embedding induces an embedding $\widetilde{\mathcal{A}}' \hookrightarrow \widetilde{\mathcal{L}}'$ of the twisted algebras. Let us prove that the isomorphism (9) extends to an isomorphism $\widetilde{\mathcal{L}} \longrightarrow \widetilde{\mathcal{L}}'$.

Denote by $\mathbb{C}(\!(\hbar)\!)$ the field of Laurent formal series in $\hbar$. First of all notice that the mapping (11) is invertible over $\mathbb{C}(\!(\hbar)\!)$. Further, the mapping (9) induces the isomorphism

$$\widetilde{\mathcal{L}} \otimes_{\mathbb{C}[\![\hbar]\!]} \mathbb{C}(\!(\hbar)\!) \simeq \widetilde{\mathcal{A}} \otimes_{\mathbb{C}[\![\hbar]\!]} \mathbb{C}(\!(\hbar)\!) \longrightarrow \widetilde{\mathcal{A}}' \otimes_{\mathbb{C}[\![\hbar]\!]} \mathbb{C}(\!(\hbar)\!) \simeq \widetilde{\mathcal{L}}' \otimes_{\mathbb{C}[\![\hbar]\!]} \mathbb{C}(\!(\hbar)\!).$$

of $\mathbb{C}(\!(\hbar)\!)$-algebras, which we denote by $\hat{\phi}$. Since $\widetilde{\mathcal{L}}$ and $\widetilde{\mathcal{L}}'$ are free over $\mathbb{C}[\![\hbar]\!]$, we have the inclusions $\widetilde{\mathcal{L}} \subset \widetilde{\mathcal{L}} \otimes_{\mathbb{C}[\![\hbar]\!]} \mathbb{C}(\!(\hbar)\!)$ and $\widetilde{\mathcal{L}}' \subset \widetilde{\mathcal{L}}' \otimes_{\mathbb{C}[\![\hbar]\!]} \mathbb{C}(\!(\hbar)\!)$. It is therefore sufficient to check that the image of $\widetilde{\mathcal{L}}$ under $\hat{\phi}$ lies in $\widetilde{\mathcal{L}}'$ and similarly for the inverse of $\hat{\phi}$.

Introduce the linear operator $\Phi^*\colon \mathrm{Span}(K_{ij}) \longrightarrow \mathrm{Span}(K_{ij})$ through the equality $(\Phi \otimes \mathrm{id})(K) = (\mathrm{id} \otimes \Phi^*)(K)$ (the dual conjugate of $\Phi$). Evaluate $\hat{\phi}$ on a monomial in the generators $\widetilde{L}_{ij}$:

$$\hat{\phi}(\widetilde{L}_{i_1j_1} \dots \widetilde{L}_{i_kj_k}) = \hat{\phi}\left(\frac{1}{\omega}\widetilde{K}_{i_1j_1} \dots \frac{1}{\omega}\widetilde{K}_{i_kj_k}\right) = \frac{1}{\omega}\Phi^*(K_{i_1j_1}) \dots \frac{1}{\omega}\Phi^*(K_{i_kj_k}),$$

where $\omega = 1 - q^{-2}$. The last equality is obtained using Lemma 2. But the rightmost expression is $\Phi^*(L_{i_1j_1}) \dots \Phi^*(L_{i_kj_k}) \in \widetilde{\mathcal{L}}'$. In the same fashion, one can check that $\hat{\phi}^{-1}(\widetilde{\mathcal{L}}') \subset \widetilde{\mathcal{L}}$. ■

**Corollary 1** *Let $O_\lambda$ be a semisimple coadjoint orbit. For any quantum group $\mathrm{U}_\hbar(\mathfrak{g}, r)$ there exists a $\mathrm{U}_\hbar(\mathfrak{g}, r)$-equivariant quantization of $\mathbb{C}[O_\lambda]$ which is a quotient of the mRE algebra associated with $\mathrm{U}_\hbar(\mathfrak{g}, r)$.*

PROOF: Let $\mathcal{B}$ be the quantization of $\mathbb{C}[O_\lambda]$ corresponding to the standard quantum group $\mathrm{U}_\hbar(\mathfrak{g})$. It is a quotient of the mRE algebra $\mathcal{L}$. The twisted module algebra $\widetilde{\mathcal{B}}'$ is a $\mathrm{U}_\hbar(\mathfrak{g}, r)$-quantization of $\mathbb{C}[O_\lambda]$. It is a quotient of the algebra $\widetilde{\mathcal{L}}'$, which is isomorphic to $\widetilde{\mathcal{L}}$ by Proposition 3. ■

## 3.3 More on RE algebras and twists

We are going to derive a description of quantum orbits for an arbitrary quantum group from that corresponding to the Drinfeld-Jimbo quantum group. To this end, we need some facts about Hopf algebras.

As we argued in the previous section (see Proposition 3), the twist of the (modified) reflection equation algebra associated with a quantum group is isomorphic to the (modified) reflection equation algebra associated with the twisted quantum group. In this section we obtain a more detailed information about that isomorphism. We start with the following auxiliary algebraic assertion.

**Lemma 3** *Let $\mathcal{H}$ be a Hopf algebra with multiplication $m$, comultiplication $\Delta$, and invertible antipode $\gamma$. Suppose $\mathcal{F} \in \mathcal{H} \otimes \mathcal{H}$ is a twisting cocycle. Then*

$$m_{23} \circ \gamma_3 \Big( (\Delta \otimes \Delta)(\mathcal{F})(\mathcal{F} \otimes \mathcal{F})(\zeta \otimes 1 \otimes \zeta \otimes 1) \Big) = \mathcal{F}_1 \zeta \otimes 1 \otimes \mathcal{F}_2,$$

*where the argument in the left-hand-side belongs to $\mathcal{H}^{\otimes 4}$.*

PROOF: Applying the cocycle equation (7) to $(\Delta \otimes \Delta)(\mathcal{F})\mathcal{F}_{34}$, we obtain for the left-hand side the expression

$$\mathcal{F}_1^{(1)} \mathcal{F}_{1'}^{(1)} \mathcal{F}_{1''} \zeta \otimes \mathcal{F}_1^{(2)} \mathcal{F}_{1'}^{(2)} \mathcal{F}_{2''} \gamma(\mathcal{F}_1^{(3)} \mathcal{F}_{2'} \zeta) \otimes \mathcal{F}_2.$$

In order to distinguish between different copies of $\mathcal{F}$, the subscripts are marked with dashes. We apply the cocycle equation to $\mathcal{F}_{1'}^{(1)} \mathcal{F}_{1''} \otimes \mathcal{F}_{1'}^{(2)} \mathcal{F}_{2''} \otimes \mathcal{F}_{2'}$ and obtain

$$\mathcal{F}_1^{(1)} \mathcal{F}_{1'} \zeta \otimes \mathcal{F}_1^{(2)} \mathcal{F}_{2'}^{(1)} \mathcal{F}_{1''} \gamma(\mathcal{F}_1^{(3)} \mathcal{F}_{2'}^{(2)} \mathcal{F}_{2''} \zeta) \otimes \mathcal{F}_2.$$

Now the statement immediately follows from the equalities $\mathcal{F}_1 \gamma(\zeta) \gamma(\mathcal{F}_2) = 1$ and (8). ■

Suppose that $\mathcal{H}$ is a quasitriangular Hopf algebra and let $(V, \rho)$ be a finite dimensional representation of $\mathcal{H}$. We say that a matrix $A \in \mathrm{End}(V) \otimes \mathcal{A}$ is *invariant*, if $h \triangleright A = \rho\big(\gamma(h^{(1)})\big) A \rho(h^{(2)})$ for all $h \in \mathcal{H}$, where $h \triangleright A$ denotes the action (6). Let $\mathcal{A}$ and $\widetilde{\mathcal{A}}$ be the (quadratic) RE algebras corresponding to the Hopf algebras $\mathcal{H}$ and $\widetilde{\mathcal{H}}$, where $\widetilde{\mathcal{H}}$ is the twist of $\mathcal{H}$ by the cocycle $\mathcal{F}$. The map (9) implements an equivariant isomorphism of $\widetilde{\mathcal{H}}$-module algebras $\widetilde{\mathcal{A}} \longrightarrow \widetilde{\mathcal{A}}'$ where $\widetilde{\mathcal{A}}'$ is the twist of $\mathcal{A}$ by $\mathcal{F}$. We can also consider $\phi$ as an isomorphism $\widetilde{\mathcal{A}} \longrightarrow \mathcal{A}$ of $\mathcal{H}$-modules.

For an invariant matrix $\widetilde{A} \in \mathrm{End}(V) \otimes \widetilde{\mathcal{A}}$ we have

$$(\mathrm{id} \otimes \phi)(\widetilde{A}) = (\Phi \otimes \mathrm{id})(A), \tag{12}$$

where $A$ is an invariant matrix in $\mathrm{End}(V) \otimes \mathcal{A}$. (For the definition of the operator $\Phi$, see Lemma 2).

**Proposition 4** *Suppose that* $\widetilde{A}$ *and* $\widetilde{B}$ *are invariant matrices from* $\mathrm{End}(V) \otimes \widetilde{\mathcal{A}}$. *Then* $(\mathrm{id} \otimes \phi)(\widetilde{A}\widetilde{B}) = (\Phi \otimes \mathrm{id})(AB)$, *where* $A$ *and* $B$ *are invariant matrices from* $\mathrm{End}(V) \otimes \mathcal{A}$ *defined by (12).*

PROOF: Follows from Lemma 3. ■

For any invariant matrix $A \in \mathrm{End}(V) \otimes \mathcal{A}$ we define an invariant (hence central) element $\mathrm{Trace}_q(A) := \mathrm{Trace}_V\big(\mathcal{R}_1 \gamma(\mathcal{R}_2) A\big) \in \mathcal{A}$. Note that here we suppress the representation symbol and we do not care about the normalizing scalar, contrary to Section 3.1.

**Proposition 5** *Suppose that* $\widetilde{A}$ *and* $A$ *are invariant matrices with coefficients in* $\widetilde{\mathcal{A}}$ *and* $\mathcal{A}$, *respectively, related by (12). Then*

$$(\mathrm{Trace}_q \otimes \phi)(\widetilde{A}) = \mathrm{Trace}_q(A).$$

PROOF: Suppressing the representation symbol $\rho$, we find

$$(\mathrm{Trace}_q \otimes \phi)(\widetilde{A}) = \mathrm{Trace}\Big(\widetilde{\mathcal{R}}_1 \widetilde{\gamma}(\widetilde{\mathcal{R}}_2) \gamma(\mathcal{F}_1 \zeta) A \mathcal{F}_2\Big),$$

where $\widetilde{\gamma}$ is the antipode in $\widetilde{\mathcal{H}}$, $\widetilde{\gamma}(x) = \gamma(\zeta^{-1} x \zeta)$. But

$$\mathcal{F}_2 \widetilde{\mathcal{R}}_1 \widetilde{\gamma}(\widetilde{\mathcal{R}}_2) \gamma(\mathcal{F}_1 \zeta) = \mathcal{F}_2 \widetilde{\mathcal{R}}_1 \gamma(\mathcal{F}_1 \widetilde{\mathcal{R}}_2 \zeta) = \mathcal{R}_1 \mathcal{F}_1 \gamma(\mathcal{R}_2 \mathcal{F}_2 \zeta) = \mathcal{R}_1 \gamma(\mathcal{R}_2)$$

because of the equality $\mathcal{F}_1 \gamma(\zeta) \gamma(\mathcal{F}_2) = 1$. This proves the assertion. ■

Denote by $\{Q_{ij}\} \subset \mathcal{A}$ the RE generators considered simultaneously as generators for $\widetilde{\mathcal{A}}'$ (the latter coincides with $\mathcal{A}$ as an $\mathcal{H}$-module and has the same system of generators as an algebra). Let $\{\widetilde{Q}_{ij}\}$ denote the RE generators of $\widetilde{\mathcal{A}}$. The matrices $Q$ and $\widetilde{Q}$ are invariant and so are their powers relative to the multiplications in $\mathcal{A}$ and $\widetilde{\mathcal{A}}$, respectively. The isomorphism $\phi$ relates $\widetilde{Q}$ and $Q$ by the formula (12). The following result is an immediate corollary of Propositions 4 and 5.

**Proposition 6** *Regard the algebra isomorphism* $\phi\colon \widetilde{\mathcal{A}} \longrightarrow \widetilde{\mathcal{A}}'$ *as an isomorphism* $\widetilde{\mathcal{A}} \longrightarrow \mathcal{A}$ *of vector spaces. Then* $(\mathrm{Trace}_q \otimes\phi)(\widetilde{Q}^m) = \mathrm{Trace}_q(Q^m)$.

Now let $\widetilde{\mathcal{L}}$ and $\mathcal{L}$ be the mRE algebras corresponding to $\widetilde{\mathcal{H}}$ and $\mathcal{H}$. Let $\{\widetilde{L}_{ij}\} \subset \widetilde{\mathcal{L}}$ and $\{L_{ij}\} \subset \mathcal{L}$ be their mRE generators. Put $\widetilde{\mathcal{L}}'$ to be the twist of $\mathcal{L}$ by $\mathcal{F}$. Regard the algebra isomorphism $\phi\colon \widetilde{\mathcal{L}} \longrightarrow \widetilde{\mathcal{L}}'$ extending the isomorphism $\phi\colon \widetilde{\mathcal{A}} \longrightarrow \widetilde{\mathcal{A}}'$ as an isomorphism $\widetilde{\mathcal{L}} \longrightarrow \mathcal{L}$ of vector spaces.

**Proposition 7** (a) *The map* $\phi$ *preserves q-traces:* $(\mathrm{Trace}_q \otimes\phi)(\widetilde{L}^m) = \mathrm{Trace}_q(L^m)$.

(b) *For any polynomial* $P$ *in one variable,* $(\mathrm{id} \otimes\phi)\big(P(\widetilde{L})\big) = (\Phi \otimes \mathrm{id})\big(P(L)\big)$.

Proof: The proof readily follows from Propositions 4 and 6 and the fact that the twist extends from the quadratic RE algebras to the modified RE algebras, by Proposition 3. ■

# 4 $\mathrm{U}_\hbar(\mathfrak{g}, r)$-equivariant quantization of orbits

For the first time, a 2-parameter quantization of symmetric orbits was constructed in [9], in the form of star-product. An algebraic description of semi-simple orbits for the Drinfeld–Jimbo quantum group was given in [4]. In this section we give a description of a 2-parameter quantization of the function algebra $\mathbb{C}[O_\lambda]$ starting from an *arbitrary* factorizable classical r-matrix. This generalizes the construction of [4]. The linear term of this quantization (or, more precisely, the "quasi-Poisson part" of it (see formula (1)) is $\hbar s^0_\lambda + t\kappa_\lambda$ where $\hbar$ and $t$ are formal parameters. Reducing this to a one-parameter quantization corresponding to the curve $t = \lambda_1\left(e^{-2\hbar} - 1\right)$ on the plane $(\hbar, t)$, we get a quantization $\mathbb{C}_\hbar[O_\lambda]$ with the linear term $\hbar\,(s^0_\lambda - 2\lambda_1\kappa_\lambda)$.

## 4.1 Algebraic description of coadjoint orbits

Organize the generators of the symmetric algebra $\mathrm{S}(\mathfrak{g})$ in an $n\times n$ matrix $L = (L_{ij})$, then $\mathrm{S}(\mathfrak{g}) = \mathbb{C}[L_{ij}]$. The algebra $\mathbb{C}[O_\lambda]$ of polynomial functions on $O_\lambda$ is a quotient of $\mathbb{C}[L_{ij}]$ by two sets of relations. The first set of $n^2$ relations can be written in the matrix form as

$$(L - \lambda_1)\dots(L - \lambda_l) = 0, \tag{13}$$

where $(x - \lambda_1)\dots(x - \lambda_l)$ is the minimal polynomial for $\lambda$. To distinguish the orbits corresponding to the same eigenvalues with different multiplicities, one should impose the following *trace conditions*:

$$\mathrm{Trace}\,(L^r) = \sum_{j=1}^{l} n_j \lambda_j^r, \quad r = 1, \ldots, l-1, \tag{14}$$

where $\sum_{j=1}^{l} n_j = n$. It is known that the ideal generated by (13) and (14) is radical, hence it is precisely the ideal of functions vanishing on $O_\lambda$.

## 4.2 On central characters of the mRE algebra

To describe quantum orbits explicitly, we need $q$-analogs of the polynomials in the right-hand side of (14), i.e. quantum trace functions. For every $m \in \mathbb{N}$ put $\hat{m} := \dfrac{1-q^{-2n}}{1-q^{-2}}$. Fix $\boldsymbol{\lambda} := (\lambda_1, \ldots, \lambda_l)$ and $\hat{\mathbf{n}} := (\hat{n}_1, \ldots, \hat{n}_l)$ assuming $\lambda_i$ pairwise distinct; put also $\tilde{\boldsymbol{\lambda}} = (\tilde{\lambda}_1, \ldots, \tilde{\lambda}_l)$, where $\widetilde{\lambda}_i = \lambda_i - \dfrac{t}{\omega}$. Consider the family of functions $\vartheta_r(\boldsymbol{\lambda}, \hat{\mathbf{n}}, q^{-2}, t)$, $r = 0, \ldots, \infty$, defined by

$$\vartheta_r(\boldsymbol{\lambda}, \hat{\mathbf{n}}, q^{-2}, t) := \sum_{i=1}^{l} C_i(\widetilde{\boldsymbol{\lambda}}, \hat{\mathbf{n}}, \omega) \lambda_i^r, \tag{15}$$

where

$$C_i(\boldsymbol{\lambda}, \hat{\mathbf{n}}, \omega) := \hat{n}_i \prod_{j \neq i} \Bigl(1 + \omega \frac{\hat{n}_j \lambda_j}{\lambda_i - \lambda_j}\Bigr). \tag{16}$$

(Recall that we use the notation $\omega = 1 - q^{-2}$). Although manifestly rational, the functions $\vartheta_r$ are in fact polynomials in all arguments, see [4]. In the classical limit $\omega \longrightarrow 0$, the function $\vartheta_r$ turns into the classical trace function $\sum_{i=1}^{l} n_i \lambda_i^r$.

Fix a polynomial $P$ in one variable with coefficients in $\mathbb{C}$. Consider the quotient of $\mathcal{L}$ by the $\mathrm{U}_\hbar(\mathfrak{g}, r)$-invariant ideal of relations $P(L) = 0$. Denote by $Z_P$ its subalgebra of invariants. We call a homomorphism $Z_P \longrightarrow \mathbb{C}[\![\hbar]\!][t]$ *a character* of $Z_P$. The meaning of the functions $\vartheta_r(\boldsymbol{\lambda}, \hat{\mathbf{n}}, q^{-2}, t)$ is explained by the following proposition.

**Proposition 8** *The algebra $Z_P$ is a free module over $\mathbb{C}[\![\hbar]\!][t]$. The characters of $Z_P$ are given by the formulas*

$$\chi_{\hat{\mathbf{n}}} \colon \tau_r \mapsto \vartheta_r(\boldsymbol{\lambda}, \hat{\mathbf{n}}, q^{-2}, t), \quad r = 1, \ldots, \infty, \tag{17}$$

*and define an embedding of $Z_P$ in the direct sum $\bigoplus_{\hat{\mathbf{n}}} \mathbb{C}[\![\hbar]\!][t]$. This embedding becomes an isomorphism over $\mathbb{C}[\![\hbar, t]\!]$.*

This proposition is proved in [4] for the case of standard quantum group. One can prove it for the general quantum group $\mathrm{U}_\hbar(\mathfrak{g}, r)$ using similar arguments.

### 4.3 The DM quantization of coadjoint orbits

Now we are in possession of all ingredients for construction of quantum orbits. We will work over the ring of scalars being $\mathbb{C}[\![\hbar, t]\!]$.

**Theorem 2** *Let* $\mathrm{U}_\hbar(\mathfrak{g}, r)$ *be* any *quasitriangular quantization of* $\mathrm{U}(\mathfrak{g})$ *along a factorizable Lie bialgebra* $\mathfrak{g}$*. Let let* $\mathbb{C}_{\hbar,t}[\mathfrak{g}^*]$ *be the corresponding mRE algebra generated by* $n^2$ *entries of the matrix* $L$*. Then the quotient of* $\mathbb{C}_{\hbar,t}[\mathfrak{g}^*]$ *by the ideal of relations*

$$(L - \lambda_1) \dots (L - \lambda_l) = 0, \tag{18}$$

$$\mathrm{Trace}_q(L^r) = \vartheta_r(\boldsymbol{\lambda}, \hat{\mathbf{n}}, q^{-2}, t, ), \quad m = 1, \dots, l-1, \tag{19}$$

*is a* $\mathrm{U}_\hbar(\mathfrak{g}, r)$*-equivariant quantization of the orbit of matrices with eigenvalues* $\boldsymbol{\lambda}$ *of multiplicities* $\mathbf{n}$.

Proof: The description of the quantized ideal of the orbit can be deduced from Corollary 1 and Proposition 8 using deformation arguments. We will give an alternative proof based on the results of Section 3.3, deriving the quantized ideal of the orbit from the Drinfeld-Jimbo case.

Let $\widetilde{\mathcal{L}}$ and $\mathcal{L}$ denote the mRE algebras corresponding to $\mathrm{U}_\hbar(\mathfrak{g}, r)$ and $\mathrm{U}_\hbar(\mathfrak{g})$, respectively. The quantum group $\mathrm{U}_\hbar(\mathfrak{g}, r)$ is the twist of $\mathrm{U}_\hbar(\mathfrak{g})$ by a cocycle $\mathcal{F}$. Denote by $\widetilde{\mathcal{L}}'$ the corresponding twist of $\mathcal{L}$; that is a module algebra over $\mathrm{U}_\hbar(\mathfrak{g}, r)$. By Proposition 3, there is an equivariant isomorphism of algebras $\phi\colon \widetilde{\mathcal{L}} \longrightarrow \widetilde{\mathcal{L}}'$. The map $\phi$ is determined by formula (10), where the matrices $\widetilde{K}$ and $K$ should be replaced by $\widetilde{L}$ and $L$, respectively.

Denote by $\mathcal{B}$ the quantization of the orbit $O_\lambda$ which is equivariant under $\mathrm{U}_\hbar(\mathfrak{g})$. It is a quotient of $\mathcal{L}$ by the ideal $\mathcal{J}$ of relations (18) and (19). The twist $\widetilde{\mathcal{B}}'$ of the algebra $\mathcal{B}$ by $\mathcal{F}$ is a quantization of $O_\lambda$ which is equivariant under $\mathrm{U}_\hbar(\mathfrak{g}, r)$. It is a quotient of $\widetilde{\mathcal{L}}'$ by the ideal $\widetilde{\mathcal{J}}'$ which coincides with $\mathcal{J}$ as a vector space. Moreover, $\widetilde{\mathcal{J}}'$ is generated by the same submodule as $\mathcal{J}$ in $\mathcal{L}$. In our case that submodule is spanned by the elements of the matrix $P(L)$ and the kernel of the central character of $\mathcal{L}$. Consider the equivariant isomorphism $\phi^{-1}\colon \widetilde{\mathcal{L}}' \longrightarrow \widetilde{\mathcal{L}}$. By Proposition 7, it sends $\mathrm{Span}\big(P(L)_{ij}\big)$ to $\mathrm{Span}\big(P(\widetilde{L})_{ij}\big)$ and preserves the q-traces. This proves the theorem. ■

**Remark 4** In [16], a description similar to Theorem 2 of semisimple quantum conjugacy classes of the Drinfeld-Jimbo matrix quantum groups is given. Using the same arguments as in the proof of Theorem 2 and the results of Section 3.3, the quantization of [16] extends to arbitrary quantum groups of the classical series.

# 5 Quantization of orbit bundles in $\mathfrak{gl}_n^*(\mathbb{C})$

In this section we prove that all orbit bundles admit $\mathrm{U}_\hbar(\mathfrak{g}, r)$-equivariant quantization and give the explicit construction. We start with the following algebraic lemma [5] which we prove here for the sake of completeness.

**Lemma 4** *Let $Q(x)$ be a polynomial over a field $F$ of zero characteristic, $\alpha, \beta$ some elements of $F$, and $L, S$ elements of an associative algebra with unit over $F$ satisfying the following conditions:*

(a) $[SLS, L] = 0$,

(b) $S^2 = \alpha S + 1$,

(c) $LQ(L) = \beta L$.

*Then one has* $[SQ(L)S, Q(L)] = 0$.

**Remark 5** The algebra generated by $S$ and $L$ subject to conditions (a)–(c) is a special case of cyclotomic affine Hecke algebra of rank 1.

PROOF: Prove, using the induction on $m \geq 1$, that $[SL^mS, Q(L)] = 0$. The induction base, $m = 1$, holds true for one checks readily that (a) implies $[SLS, L^k] = 0$ for any $k$. Now, suppose $[SL^mS, Q(L)] = 0$, then one has using (b):

$$\begin{aligned}[SL^{m+1}S, Q(L)] = [SL1L^mS, Q(L)] = [SL(S^2 - \alpha S)L^mS, Q(L)] = \\ = [SLS^2L^mS, Q(L)] - \alpha[SLSL^mS, Q(L)].\end{aligned} \tag{20}$$

According to the induction assumption, both $SLS$ and $SL^mS$ commute with $Q(L)$, thus

$$[SLS^2L^mS, Q(L)] = [(SLS)(SL^mS), Q(L)] = 0.$$

The last term in (20) is treated as follows:

$$\begin{aligned}[SLSL^mS, Q(L)] &= SL\big(SL^mSQ(L)\big) - \big(Q(L)SLS\big)L^mS \\ &= SLQ(L)SL^mS - SLSL^mQ(L)S \\ &= \beta SLSL^mS - \beta SLSL^mS = 0,\end{aligned}$$

where the induction assumption and (c) were used. ■

Recall from see Proposition 2 that any orbit map is determined by a polynomial $P$ in one variable.

**Theorem 3** *Fix a factorizable quantum group $\mathrm{U}_\hbar(\mathfrak{g}, r)$, where $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$. Let $O_\lambda$ and $O_\mu$ be two orbits in $\mathfrak{g}$ satisfying the condintions of Theorem 1, and denote by $\mathbb{C}_\hbar[O_\lambda] = \mathbb{C}_\hbar[L_{\lambda,ij}]$ and $\mathbb{C}_\hbar[O_\mu] = \mathbb{C}_\hbar[L_{\mu,ij}]$ their quantizations from Theorem 2 with $t = \lambda_1\left(e^{-2\hbar} - 1\right)$. Then the assignment $L_\mu \mapsto P(L_\lambda)$, where the polynomial $P$ is given by (4), is a $\mathrm{U}_\hbar(\mathfrak{g}, r)$-equivariant quantization of the orbit bundle $O_\lambda \longrightarrow O_\mu$ determined by $P$.*

Proof: Denote by $P^*$ the algebra monomorphism $\mathbb{C}[O_\mu] \longrightarrow \mathbb{C}[O_\lambda]$ corresponding to the map $P$. Both $L_\lambda$ and $L_\mu$ are subject to the relations (13) and (14). The algebra homomorphism $P^*\colon \mathbb{C}[O_\mu] \longrightarrow \mathbb{C}[O_\lambda]$ is determined by the correspondence $L_\mu \mapsto P(L_\lambda)$. We need to prove that the same correspondence defines a $\mathbb{C}[\![\hbar]\!]$-algebra monomorphism $\mathbb{C}_\hbar[O_\mu] \longrightarrow \mathbb{C}_\hbar[O_\lambda]$, i.e. that the matrix $P(L_\lambda)$ satisfies the same relations as the matrix $L_\mu$.

1. Check the relation: $[SP(L_\lambda)S, P(L_\lambda)] = \mu_1(q - q^{-1})[S, P(L_\lambda)]$.

It can be written in the form:

$$[S\left(P(L_\lambda) - \mu_1\right)S, P(L_\lambda) - \mu_1] = 0, \tag{21}$$

as $S$ is a Hecke matrix. It is easy to check that $(L_\lambda - \lambda_1)\left(P(L_\lambda) - \mu_1\right) = (\mu_2 - \mu_1)(L_\lambda - \lambda_1)$. Now set $\beta := \mu_2 - \mu_1$, $L := L_\lambda - \lambda_1$, $Q(x) := P(x + \lambda_1) - \mu_1$, then (21) follows from Lemma 4.

2. Check the relation:

$$(P(L_\lambda) - \mu_1)(P(L_\lambda) - \mu_2) = 0. \tag{22}$$

Substituting (4) into the l.h.s. of (22), one gets

$$\prod_{i=2}^{l}(L_\lambda - \lambda_i)\left(\prod_{i=2}^{l}(L_\lambda - \lambda_i) - \prod_{i=2}^{l}(\lambda_1 - \lambda_i)\right) \tag{23}$$

up to a constant multiple. The expression in the big brackets is divisible by $L_\lambda - \lambda_1$. Indeed, for any polynomial $f(x)$, the polynomial in two variables $F(x, y) := f(x) - f(y)$ is divisible by $x-y$. This implies that (23) is divisible by the minimal polynomial of $\lambda$, so it is equal to zero.

3. In order to check the q-Trace Condition,

$$\mathrm{Trace}_q\, P(L_\lambda) = \mathrm{Trace}_q(L_\mu), \tag{24}$$

we put $\boldsymbol{\nu} = \hat{\mathbf{n}} = (\hat{n}_1, \ldots, \hat{n}_l)$ and $\omega := 1 - q^{-2}$ in the functions $C_i(\boldsymbol{\lambda}, \boldsymbol{\nu}, \omega)$, see Appendix, formula (28). (As above, $\hat{n}_i = \frac{1-q^{-2n_i}}{1-q^{-2}}$).

Replacing $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$, $L_\lambda$, $L_\mu$ and $P(x)$ by $(0, \lambda_2-\lambda_1, \ldots, \lambda_l-\lambda_1)$, $(0, \mu_2-\mu_1)$, $L_\lambda-\lambda_1$, $L_\mu-\mu_1$ and $P(x+\lambda_1)-\mu_1$ respectively, one reduces the problem to the case $\lambda_1 = \mu_1 = 0$. So, it suffices to prove that the condition (24) is satisfied when $\lambda_1 = 0$ and $P(0) = 0$.

By assumption, $\mu_1 = P(0) = 0$, therefore one has:

$$\mathrm{Trace_q}\, L_\mu = C_2(\boldsymbol{\mu}, \hat{\mathbf{m}}, q)\mu_2 = \hat{n}'\mu_2, \tag{25}$$

where $\mathbf{m} = (n_1, n')$ and $n' := \sum_{i=2}^{l} n_i$. On the other hand,

$$\mathrm{Trace_q}\big(P(L_\lambda)\big) = \sum_{i=1}^{l} P(\lambda_i) C_i(\boldsymbol{\lambda}, \hat{\mathbf{n}}, q) = \mu_2 \sum_{i=2}^{l} C_j(\boldsymbol{\lambda}, \hat{\mathbf{n}}, q), \tag{26}$$

because $P(\lambda_1) = P(0) = 0$. By Corollary 2 (see Appendix),

$$\sum_{i=2}^{l} C_i(\boldsymbol{\lambda}, \hat{\mathbf{n}}, \omega) = \hat{n}', \tag{27}$$

since $1 - \omega\hat{n}_i = q^{-2n_i}$. Substituting (27) into (26), one concludes that the latter is equal to (25). ■

# Appendix

In this section, we study some properties of the coefficients $C_i$ in (15), which were announced without proof in [6]. For $1 \leqslant i \leqslant l$, define a function of $2l+1$ variables $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_l)$, $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_l)$ and $\omega$:

$$C_i(\boldsymbol{\lambda}, \boldsymbol{\nu}, \omega) := \nu_i \prod_{\substack{1 \leqslant j \leqslant l \\ j \neq i}} \left(1 + \omega \frac{\nu_j \lambda_j}{\lambda_i - \lambda_j}\right), \tag{28}$$

and also another function of the same variables:

$$S(\boldsymbol{\lambda},\boldsymbol{\nu},\omega)=\sum_{i=1}^{l}C_i(\boldsymbol{\lambda},\boldsymbol{\nu},\omega).$$

These functions were introduced in [4] and [6]. Our goal is to prove Proposition 9 below.

Obviously, $S(\boldsymbol{\lambda},\boldsymbol{\nu},\omega)$ is stable under simultaneous permutations of the entries of $\boldsymbol{\lambda}$ and the entries of $\boldsymbol{\nu}$. In fact, a stronger statement is true:

**Lemma 5** *$S(\boldsymbol{\lambda},\boldsymbol{\nu},\omega)$ is a symmetric function of $\boldsymbol{\lambda}$.*

Proof: It suffices to show that $S(\boldsymbol{\lambda},\boldsymbol{\nu},\omega)$ is stable under the transposition $\lambda_1\leftrightarrow\lambda_2$. First, opening the brackets in (28) one gets

$$C_i(\boldsymbol{\lambda},\boldsymbol{\nu},\omega)=\nu_i+\nu_i\sum_{k=1}^{l-1}\omega^k\sum_{j_1<\ldots<j_k}\frac{\nu_{j_1}\lambda_{j_1}}{\lambda_i-\lambda_{j_1}}\cdots\frac{\nu_{j_k}\lambda_{j_k}}{\lambda_i-\lambda_{j_k}}. \tag{29}$$

In this form, the functions $C_i$ were introduced in [4]. The multiplicative form (28) appeared in [13]. All the terms in (29) containing $\lambda_1$ and $\lambda_2$ can be arranged into sums of the following three forms:

$$\nu_j\frac{\nu_1\lambda_1}{\lambda_j-\lambda_1}\frac{\nu_2\lambda_2}{\lambda_j-\lambda_2}f=\nu_j\nu_1\nu_2\frac{\lambda_1}{\lambda_j-\lambda_1}\frac{\lambda_2}{\lambda_j-\lambda_2}f,$$

$$\nu_1\frac{\nu_2\lambda_2}{\lambda_2-\lambda_1}f+\nu_2\frac{\nu_1\lambda_1}{\lambda_1-\lambda_2}f=\nu_1\nu_2f,$$

$$\nu_j\frac{\nu_1\lambda_1}{\lambda_j-\lambda_1}f+\nu_j\frac{\nu_2\lambda_2}{\lambda_j-\lambda_2}f+\nu_1\frac{\nu_j\lambda_j}{\lambda_1-\lambda_j}f+\nu_2\frac{\nu_j\lambda_j}{\lambda_2-\lambda_j}f=-(\nu_j\nu_1+\nu_j\nu_2)f,$$

with $j\neq 1,2$, and $f$ being independent on $\lambda_1$ and $\lambda_2$. It is seen that the expressions in the right hand sides are stable under the transposition $\lambda_1\leftrightarrow\lambda_2$. ■

**Proposition 9** $\omega S(\boldsymbol{\lambda},\boldsymbol{\nu},\omega)=1-\prod_{i=1}^{l}(1-\omega\nu_i)$.

Proof: Prove first that $S(\boldsymbol{\lambda},\boldsymbol{\nu},\omega)$ does not actually depend on $\boldsymbol{\lambda}$. Fix $\boldsymbol{\nu}$ and $\omega$, and consider $S(\boldsymbol{\lambda},\boldsymbol{\nu},\omega)$ as a rational function of $\boldsymbol{\lambda}$ only. This function is homogeneous of degree zero. Reducing $S(\boldsymbol{\lambda},\boldsymbol{\nu},\omega)$ to the common denominator $\prod_{i<j}(\lambda_i-\lambda_j)$ we obtain a ratio of two homogeneous polynomials of the same degree. Since $S(\boldsymbol{\lambda},\boldsymbol{\nu},\omega)$ is a symmetric function

of $\boldsymbol{\lambda}$, the numerator is divisible by $\prod_{i<j}(\lambda_i - \lambda_j)$ because the ring of polynomials is a unique factorization domain. Since the numerator of $S(\boldsymbol{\lambda}, \boldsymbol{\nu}, \omega)$ has the same degree as the denominator, $S(\boldsymbol{\lambda}, \boldsymbol{\nu}, \omega)$ is independent on $\boldsymbol{\lambda}$.

Now put $\lambda_l = 0$, then it follows from (28) that $S(\boldsymbol{\lambda}, \boldsymbol{\nu}, \omega) = S(\boldsymbol{\lambda}', \boldsymbol{\nu}', \omega) + \nu_l \prod_{i=1}^{l-1}(1 - \omega\nu_i)$, where $\boldsymbol{\lambda}' = (\lambda_1, \ldots, \lambda_{l-1})$ and $\boldsymbol{\nu}' = (\nu_1, \ldots, \nu_{l-1})$. Finally, one applies the induction on $l$. ■

**Corollary 2** (a) *If* $\lambda_1 = 0$ *then* $\omega \sum_{i=2}^{l} C_i(\boldsymbol{\lambda}, \boldsymbol{\nu}, \omega) = 1 - \prod_{i=2}^{l}(1 - \omega\nu_i)$.

(b) *Denote* $\boldsymbol{\lambda}' = (\lambda_2, \ldots, \lambda_l)$, $\boldsymbol{\nu}' = (\nu_2, \ldots, \nu_l) \in \mathbb{C}^{l-1}$, *and suppose that* $\lambda_1 = 0$. *Then* $\sum_{i=2}^{l} C_i(\boldsymbol{\lambda}, \boldsymbol{\nu}, \omega) = \sum_{i=2}^{l} C_i(\boldsymbol{\lambda}', \boldsymbol{\nu}', \omega)$.

(c) *One has* $\sum_{i=1}^{l} C_i(\boldsymbol{\lambda}, \hat{\mathbf{n}}, \omega) = \hat{n}$.

Proof: (a) Denote $S'(\boldsymbol{\lambda}, \boldsymbol{\nu}, \omega) = \omega \sum_{i=2}^{l} C_i(\boldsymbol{\lambda}, \boldsymbol{\nu}, \omega)$. Then

$$\omega S'(\boldsymbol{\lambda}, \boldsymbol{\nu}, \omega) = \omega\big(S(\boldsymbol{\lambda}, \boldsymbol{\nu}, \omega) - C_1(\boldsymbol{\lambda}, \boldsymbol{\nu}, \omega)\big) =$$

$$= 1 - \prod_{i=1}^{l}(1 - \omega\nu_i) - \omega\nu_1 \prod_{i=2}^{l}(1 - \omega\nu_i) = 1 - \prod_{i=2}^{l}(1 - \omega\nu_i).$$

(b) Obvious.

(c) Note that $1 - \omega\hat{n}_i = q^{-2n_i} = e^{-2n_i\hbar}$, recall that $n = \sum_{i=1}^{l} n_i$ and use Proposition 9. ■

# References


[1] A. A. Belavin and V. G. Drinfeld. On solutions of Yang—Baxter equation. *Functional Analysis and Applications*, 16(3):1–29, 1982. *English translation: Functional Analysis and Applications*, vol.32 (1985), p.p. 254–255.

[2] J. Donin. Double quantization on coadjoint representations of simple Lie groups and its orbits. *MPIM*, September 1999.

[3] J. Donin. Quantum $g$-manifolds. *Contemp. Math.*, 315:47–60, 2002.

[4] J. Donin and A. Mudrov. Explicit quantization on coadjoint orbits of $gl(n, \mathbb{C})$. *Lett. Math. Phys*, 62(1):17–32, 2002.

[5] J. Donin and A. Mudrov. Method of quantum characters in equivariant quantization. *Comm. Math. Phys.*, 234:533–555, 2003.

[6] J. Donin and A. Mudrov. Quantum coadjoint orbits of $GL(n)$ and generalized Verma modules. *Lett. Math. Phys.*, 67:167–184, 2004.

[7] J. Donin and V. Ostapenko. Equivariant quantization on quotients of simple Lie groups by reductive subgroups. *Czech. Journ. of Phys.*, 52(11):1213–1218, 2002.

[8] J. Donin, D. Gurevich and S. Shnider. Double quantization on some orbits in the coadjoint representations of simple Lie groups. *Comm. Math. Phys.*, 204(1):39–60, 1999. math.QA/9807159.

[9] J. Donin and S. Shnider. Quantum symmetric spaces. *J. Pure Appl. Algebra*, 100(1):103–115, 1995. hep-th/9412031.

[10] V. G. Drinfeld. Quantum groups. In *Proceedings of International Congress of Mathematicians, Berkley 1986*, volume 1, pages 798–820. AMS, Providence, 1986.

[11] V. G. Drinfeld. Almost cocommutative Hopf algebras. *Algebra i Analys*, 1(2):321–342, 1989. *English translation: Leningrad Journal of Mathematics*, vol. 1 No. 2 (1990), p.p. 321–342.

[12] P. Etingoff and D. Kazhdan. Quantization of Lie bialgebras. *Selecta Math.*, 2(1):1–41, 1996.

[13] D. Gurevich and P. Saponov. Geometry of non-commutative orbits related to Hecke symmetries. math.QA/0411579.

[14] E. Karolinskii. A classification of Poisson homogeneous spaces of complex reductive Poisson–Lie groups. In P. Urbanski J. Grabowski, editor, *Poisson geometry*, volume 51, pages 103–108. Banach Center, Warsaw, 2000.

[15] P. Kulish and A. Mudrov. Dynamical reflection equation. math.QA/0405556.

[16] A. Mudrov. Quantum conjugacy classes of simple matrix groups. math.QA/0412538.

[17] M. Semenov-Tian-Shansky. Poisson-Lie groups, quantum duality principle, and the quantum double. *Contemp. Math.*, 175:219–248, 1994.