Rational and homological equivalence of real algebraic cycles

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1 Introduction and main results

Throughout this paper X denotes a smooth n-dimensional quasiprojective variety over $\mathbb{R}$ (that is an integral smooth n-dimensional quasiprojective scheme over $\mathbb{R}$). We endow the set $X(\mathbb{R})$ of $\mathbb{R}$-rational points of X with the topology induced by the usual metric topology on $\mathbb{R}$, and assume that $X(\mathbb{R})$ is nonempty and compact. Thus $X(\mathbb{R})$ is a closed n-dimensional $\mathcal{C}^\infty$ manifold. Since there is an affine neighborhood of $X(\mathbb{R})$ in X, we can regard $X(\mathbb{R})$ as an algebraic subset of $\mathbb{R}^N$ for some N. Given a nonnegative integer k, we let $Z_k(X)$ denote the group of algebraic k-cycles on X (that is, the free Abelian group on the set of closed k-dimensional subvarieties of X). There is a unique group homomorphism

$$c\ell_k^{\mathbb{R}} : Z_k(X) \to H_k(X(\mathbb{R}), \mathbb{Z}/2)$$

such that for every closed k-dimensional subvariety Y of X, the homology class $c\ell_k^{\mathbb{R}}(Y)$ is represented by $Y(\mathbb{R})$ assuming $\dim Y(\mathbb{R}) = k$, and $c\ell_k^{\mathbb{R}}(Y) = 0$ otherwise [3] ($\dim Y(\mathbb{R})$ refers to the dimension of $Y(\mathbb{R})$ regarded as an algebraic subset of $X(\mathbb{R})$; of course, $X(\mathbb{R}) = X(\mathbb{R}) \cap Y$ and $\dim Y(\mathbb{R}) \leq k$, with $\dim Y(\mathbb{R}) = k$ if and only if $Y(\mathbb{R})$ is dense in Y). A k-cycle z on X is said to be *homologous to zero* (or *homologically equivalent to zero*) if $c\ell_k^{\mathbb{R}}(z) = 0$. Below we recall a beautiful characterization of such cycles due to Ischebeck and Schülting

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[11]. Following [11], we say that a k-cycle z on X is *thin*, provided

$$\dim(X(\mathbb{R}) \cap \operatorname{supp} z) < k$$

($\operatorname{supp} z$ stands for the support of z : if $z = n_1 Y_1 + \cdots + n_r Y_r$, where $n_1, \ldots, n_r$ are nonzero integers and $Y_i \neq Y_j$ for $i \neq j$, then $\operatorname{supp} z = Y_1 \cup \ldots \cup Y_r$, while $\operatorname{supp} 0 = \emptyset$). We also say that z *supports real points* (resp. *supports no real points*) if $X(\mathbb{R}) \cap \operatorname{supp} z$ is nonempty (resp. empty). Clearly, z is thin if it supports no real points. By definition of $c\ell_k^{\mathbb{R}}$, every thin cycle is homologous to zero. Moreover, it is well known that every cycle on X rationally equivalent to zero is also homologous to zero [3].

We can now state the main result of [11].

Theorem A. *A given k-cycle on X, where $0 \leq k \leq n-1$, is homologous to zero if and only if it is rationally equivalent to a thin cycle.*

In [11] X is assumed to be a complete variety. However, Theorem A is true as stated since, by virtue of [9], X has a smooth projectivization with the set of $\mathbb{R}$-rational points equal to $X(\mathbb{R})$. Some special cases of Theorem A had been known before [11] was published. If $k = 0$ and $n = 1$, the result follows from [16], while for $k = 0$ and $k = 1$ it is contained in [6] and [10], respectively. Note that thin cycles in $Z_0(X)$ support no real points. This remark corresponds to an earlier result for $k = n-1$, due to Bröcker [4], asserting that every homologous to zero $(n-1)$-cycle on X is rationally equivalent to a cycle supporting no real points (cf. also [7] for a related theorem formulated in terms of vector bundles). As we demonstrate below, the cases $k = 0$ and $k = n-1$ are somewhat misleading since for many values of k, $1 \leq k \leq n-2$, a k-cycle on X that is homologous to zero need not be rationally equivalent to any cycle supporting no real points. We actually prove much more (cf. also [15]).

Theorem 1.1 *Let n and ℓ be positive integers. If $4\ell \leq n$, then there exist a smooth projective variety $X_{n\ell}$ over $\mathbb{R}$ and a cycle $z_{n\ell}$ in $Z_{n-2\ell}(X_{n\ell})$ such that $\dim X_{n\ell} = n$, $X_{n\ell}(\mathbb{R})$ is nonempty, $z_{n\ell}$ is homologous to zero, and every cycle rationally equivalent to $r z_{n\ell}$, where r is an arbitrary* *nonzero integer, supports real points. Furthermore, $X_{n\ell}$ and $z_{n\ell}$ can be described by explicitly given equations.*

In the light of Theorem 1.1, the following problem becomes natural.

Problem B. Assuming $1 \leq k \leq n-2$, identify cases in which one of the following two conditions is satisfied:

(i) Every homologous to zero k-cycle on X is rationally equivalent to a cycle supporting no real points.

(ii) For every k-cycle z on X, there exists a nonzero integer r such that rz is rationally equivalent to a cycle supporting no real points (note that $2z$ is always homologous to zero).

The only contribution we can make to Problem B(i) is the following.

Theorem 1.2 *If $n = 3$, then every homologous to zero 1-cycle on X is rationally equivalent to a cycle supporting no real points.*

Our next result deals with Problem B(ii) and covers a wider range of cases.

Theorem 1.3 *There is a positive integer r, depending only on X, such that for every k-cycle z on X, where $2k+1 \leq n$, the multiple rz is rationally equivalent to a cycle supporting no real points.*

If $n-k$ is even, then in view of Theorem 1.1, the assumption $2k+1 \leq n$ in Theorem 1.3 is optimal and cannot be changed without imposing some extra conditions on X.

We precede our last result with a definition of a technical nature. Regard $X(\mathbb{R})$ as a subset of the set $X(\mathbb{C})$ of $\mathbb{C}$-rational points of X. Let $e : X(\mathbb{R}) \hookrightarrow X(\mathbb{C})$ be the inclusion map and

$$e^* : H^*(X(\mathbb{C}), \mathbb{Z}) \to H^*(X(\mathbb{R}), \mathbb{Z})$$

the induced homomorphism. The variety X is said to have *property* (D), provided that the image $e^*(H^n(X(\mathbb{C}),\mathbb{Z}))$ is a torsion group. Note that $H^n(X(\mathbb{R}),\mathbb{Z}) \cong \mathbb{Z}^p \oplus (\mathbb{Z}/2)^q$ for some nonnegative integers p and q. If each connected component of $X(\mathbb{R})$ is nonorientable, then $p = 0$, and hence X has property (D). Another example of X having property (D) is the product of n smooth projective curves $C_1, \dots, C_n$, where $C_i(\mathbb{R})$ is connected and $C_i(\mathbb{C}) \backslash C_i(\mathbb{R})$ is disconnected for each i.

Theorem 1.4 *Assume that X has property (D). If $2k = n$ and $k \geq 2$, then there is a positive integer r, depending only on X, such that for every k-cycle z on X, the multiple rz is rationally equivalent to a cycle supporting no real points. Furthermore, if $n = 4$ and a 2-cycle z on X is homologous to zero, then $2z$ is rationally equivalent to a cycle supporting no real points.*

Theorem 1.1 implies that one cannot drop the assumption about X having property (D) if k is even.

2 Proofs

We need some facts from algebraic geometry over $\mathbb{C}$. Let V be a smooth n-dimensional quasiprojective variety over $\mathbb{C}$. The set $V(\mathbb{C})$ of $\mathbb{C}$-rational points of V is an n-dimensional complex manifold. There is a unique homomorphism

$$c\ell_k^{\mathbb{C}} : Z_k(V) \to H_{2k}^{BM}(V(\mathbb{C}),\mathbb{Z})$$

which assigns to each closed k-dimensional subvariety W of V the homology class represented by $W(\mathbb{C})$ [3, 8] ($H_{2k}^{BM}(V(\mathbb{C}),\mathbb{Z})$ stands for the Borel-Moore homology, $H_{2k}^{BM}(V(\mathbb{C}),\mathbb{Z}) = H_{2k}(V(\mathbb{C}),\mathbb{Z})$ if $V(\mathbb{C})$ is compact). This homomorphism respects rational equivalence, that is, $c\ell_k^{\mathbb{C}}(u) = 0$ assuming u is rationally equivalent to zero [3, 8].

For any variety Y over $\mathbb{R}$, we set $Y_{\mathbb{C}} = Y \times_{\mathbb{R}} \mathbb{C}$. In order to ease the notation, we will always identify $Y_{\mathbb{C}}(\mathbb{C})$ with $Y(\mathbb{C})$. Thus $Y(\mathbb{R}) \subset Y(\mathbb{C}) = Y_{\mathbb{C}}(\mathbb{C})$. The canonical projection

$\pi : Y_{\mathbb{C}} \to Y$ gives rise to the pullback homomorphism

$$\pi^* : Z_k(Y) \to Z_k(Y_{\mathbb{C}}),$$
$$\pi^*(n_1Y_1 + \cdots + n_rY_r) = n_1Y_{1\mathbb{C}} + \cdots + n_rY_{r\mathbb{C}}.$$

If a k-cycle z on Y is rationally equivalent to zero, then so is the cycle $\pi^*(z)$ on $Y_{\mathbb{C}}$.

The following example will play a crucial role in the proof of Theorem 1.1.

Example 2.1. For any positive integer n denote by X_n the subvariety of $\mathbb{P}^{n+1}$ (projective $(n+1)$-space over $\mathbb{R}$) defined by the equation

$$x_1^2 + \cdots + x_{n+1}^2 = x_{n+2}^2.$$

Clearly, X_n is smooth of dimension n and $X_n(\mathbb{R})$ is the unit n-sphere in $\mathbb{R}^{n+1} \subset \mathbb{P}^{n+1}(\mathbb{R})$. In particular, $H_i(X_n(\mathbb{R}), \mathbb{Z}/2) = 0$ and hence every cycle in $Z_i(X_n)$ is homologous to zero for $1 \leq i \leq n-1$.

Assertion. Assume $n = 2k$, where k is even and positive. Let Y_k be the reduced subscheme of $\mathbb{P}^{n+1}$ defined by the system of equations

$$x_1^2 + x_2^2 = 0, \ldots, x_{n-1}^2 + x_n^2 = 0, \quad x_{n+1} = x_{n+2}.$$

Then Y_k is a closed k-dimensional subvariety of X_n and the k-cycle $z_k = Y_k$ on X_n (we introduce z_k here just to emphasize that Y_k is regarded as a cycle) has the following property: given a nonzero integer r, every cycle rationally equivalent to rz_k supports real points.

We prove the assertion as follows. Note that

$$Y_k(\mathbb{C}) = L \cup \sigma(L) \text{ and } L \cap \sigma(L) = \{p\},$$

where L is the linear subspace of $\mathbb{P}^{n+1}(\mathbb{C})$ defined by the system of equations

$$x_1 - \sqrt{-1}x_2 = 0, \ldots, x_{n-1} - \sqrt{-1}x_n = 0, \quad x_{n+1} = x_{n+2},$$

$\sigma : \mathbb{P}^{n+1}(\mathbb{C}) \to \mathbb{P}^{n+1}(\mathbb{C})$ is the complex conjugation, and $p = (0 : \ldots : 0 : 1 : 1)$. It follows that Y_k is irreducible of dimension k. Obviously, Y_k is contained in X_n. In other words, Y_k is a closed k-dimensional subvariety of X_n and $z_k = Y_k$ is a cycle in $Z_k(X_n)$.

Observe that

$$X_n(\mathbb{R}) \cap L = \{p\} \text{ and } X_n(\mathbb{R}) \cap \sigma(L) = \{p\}.$$

Moreover, $X_n(\mathbb{R})$ and L (resp. $X_n(\mathbb{R})$ and $\sigma(L)$) are transverse $\mathcal{C}^\infty$ submanifolds of $X_n(\mathbb{C})$. The manifolds L, $\sigma(L)$, and $X_n(\mathbb{C})$ come with the canonical orientations determined by their complex structure. We fix an orientation on $X_n(\mathbb{R})$ so that the intersection number of $X_n(\mathbb{R})$ and L in $X_n(\mathbb{C})$ is equal to 1. One readily shows that the intersection number of $X_n(\mathbb{R})$ and $\sigma(L)$ in $X_n(\mathbb{C})$ is also 1 (this is the only place, where k even is used). It follows that if ξ, η, and α are the homology classes in $H_n(X_n(\mathbb{C}), \mathbb{Z})$ represented by L, $\sigma(L)$, and $X_n(\mathbb{R})$, respectively, then the intersection number $\alpha \bullet (\xi + \eta)$ of α and $\xi + \eta$ is equal to 2.

Let $\pi : X_{n\mathbb{C}} = X_n \times_{\mathbb{R}} \mathbb{C} \to X_n$ be the canonical projection. By definition of $c\ell_k^{\mathbb{C}}$, we have $c\ell_k^{\mathbb{C}}(\pi^*(z_k)) = \xi + \eta$. Hence for any integer r,

$$(*) \qquad \alpha \bullet c\ell_k^{\mathbb{C}}(\pi^*(rz_k)) = 2r.$$

Suppose now that rz_k is rationally equivalent to a cycle w which supports no real points. Since $\pi^*(rz_k)$ and $\pi^*(w)$ are also rationally equivalent, we obtain $c\ell_k^{\mathbb{C}}(\pi^*(rz_k)) = c\ell_k^{\mathbb{C}}(\pi^*(w))$. The fact that w supports no real points means $X_n(\mathbb{R}) \cap \operatorname{supp} \pi^*(w) = \emptyset$, and therefore $\alpha \bullet c\ell_k^{\mathbb{C}}(\pi^*(w)) = 0$. Thus $\alpha \bullet c\ell_k^{\mathbb{C}}(\pi^*(rz_k)) = 0$, which in view of $(*)$ implies $r = 0$. The assertion is proved. □

Henceforth X is as in Section 1. Denote by $A_k(X)$ the kth Chow group of X ($Z_k(X)$ modulo rational equivalence). As usual, we set $A^i(X) = A_{n-i}(X)$. For any k-cycle z on X, we let $[z]$ denote its class in $A_k(X)$. The following simple observation will be useful.

Lemma 2.2 *A given k-cycle z on X is rationally equivalent to a cycle supporting no real points if and only if there exists an open neighborhood U of $X(\mathbb{R})$ in X such that the restriction homomorphism $A_k(X) \to A_k(U)$ sends $[z]$ to 0.*

Proof. It is obvious that such a neighborhood U exists if z is rationally equivalent to a cycle supporting no real points. The converse holds since for any open neighborhood U of

$X(\mathbb{R})$ in X, the sequence

$$A_k(X\backslash U) \to A_k(X) \to A_k(U) \to 0,$$

where the first homomorphism is induced by the inclusion $X\backslash U \hookrightarrow X$, is exact [8, Proposition 1.8]. $\square$

Proof of Theorem 1.1. We will refer to X_n, Y_k, and z_k constructed in Example 2.1.

If $n = 4\ell$, it suffices to take $X_{n\ell} = X_n$ and $z_{n\ell} = z_{2\ell}$. Suppose $n \geq 4\ell + 1$. We set

$$X_{n\ell} = X_{4\ell} \times_{\mathbb{R}} X_{n-4\ell} \text{ and } z_{n\ell} = Y_{2\ell} \times_{\mathbb{R}} X_{n-4\ell}.$$

Since the 2ℓ-cycle $z_{2\ell} = Y_{2\ell}$ on $X_{4\ell}$ is homologous to zero, it follows that the $(n-2\ell)$-cycle $z_{n\ell}$ on $X_{n\ell}$ is also homologous to zero. Suppose that for some integer r the cycle $rz_{n\ell}$ on $X_{n\ell}$ is rationally equivalent to a cycle supporting no real points. In order to complete the proof it suffices to show $r = 0$. We argue as follows. Choose any point p in $X_{n-4\ell}(\mathbb{R})$. Let $i : X_{4\ell} \to X_{n\ell}$ be the obvious embedding identifying $X_{4\ell}$ with $X_{4\ell} \times_{\mathbb{R}} \{p\}$. We have $\alpha = i^*(\beta)$, where $\alpha = [z_{2\ell}]$ in $A_{2\ell}(X_{4\ell}) = A^{2\ell}(X_{4\ell})$, $\beta = [z_{n\ell}]$ in $A_{n-2\ell}(X_{n\ell}) = A^{2\ell}(X_{n\ell})$, and $i^* : A^{2\ell}(X_{n\ell}) \to A^{2\ell}(X_n)$ is the pullback homomorphism. Since $rz_{n\ell}$ is rationally equivalent to a cycle supporting no real points, one can find an open neighborhood U of $X_{n\ell}(\mathbb{R})$ in $X_{n\ell}$ such that $r\beta$ restricted to U is 0 in $A^{2\ell}(U)$. Then $r\alpha = i^*(r\beta)$ restricted to $i^{-1}(U)$ is 0 in $A^{2\ell}(i^{-1}(U))$. By Lemma 2.2, the cycle $rz_{2\ell}$ on $X_{4\ell}$ is rationally equivalent to a cycle supporting no real points. This implies $r = 0$ in view of Example 2.1. The proof is complete. $\square$

In the proofs of other theorems from Section 1 we will need some facts concerning vector bundles. Any vector bundle E on X determines in a canonical way a real vector bundle on $X(\mathbb{R})$, which we denote by $E(\mathbb{R})$.

Lemma 2.3 *If the vector bundle $E(\mathbb{R})$ on $X(\mathbb{R})$ is topologically stably trivial, then there exists an open neighborhood U of $X(\mathbb{R})$ in X such that the restriction $E|U$ of E to U is algebraically stably trivial.*

Proof. Replacing X by an affine neighborhood of $X(\mathbb{R})$, we may assume that X itself is affine. Thus the lemma follows from either of [7] or [2, Theorem 12.3.3] (recall that $X(\mathbb{R})$ is compact). □

By abuse of notation, we let

$$c\ell_k^{\mathbb{R}} : A_k(X) \to H_k(X(\mathbb{R}), \mathbb{Z}/2)$$

denote the homomorphism induced by $c\ell_k^{\mathbb{R}}$ defined in Section 1. Let

$$c\ell_{\mathbb{R}}^i : A^i(X) \to H^i(X(\mathbb{R}), \mathbb{Z}/2)$$

be the composite homomorphism $D^{-1} \circ c\ell_{n-i}^{\mathbb{R}}$, where

$$D : H^i(X(\mathbb{R}), \mathbb{Z}/2) \to H_{n-i}(X(\mathbb{R}), \mathbb{Z}/2)$$

is the Poincaré duality and

$$c\ell_{n-i}^{\mathbb{R}} : A^i(X) = A_{n-i}(X) \to H_{n-i}(X(\mathbb{R}), \mathbb{Z}/2).$$

Given a vector bundle E on X, denote by $c_i(E)$ its ith Chern class in $A^i(X)$ [3, 8]. According to [3],

$$w_i(E(\mathbb{R})) = c\ell_{\mathbb{R}}^i(c_i(E)) \text{ for all } i \geq 0, \tag{2.4}$$

where $w_i(-)$ is the ith Stiefel-Whitney class.

Proof of Theorem 1.2. Replacing X by an affine neighborhood of $X(\mathbb{R})$, we may assume that X is affine. Let z be a 1-cycle on X that is homologous to zero, that is, satisfies $c\ell_1^{\mathbb{R}}(z) = 0$. Then the class $\alpha = [z]$ is in $A_1(X) = A^2(X)$ and

$$c\ell_{\mathbb{R}}^2(\alpha) = 0. \tag{1}$$

By Grothendieck's formula [8, Example 15.3.6], there exists a vector bundle E on X with

$$c_1(E) = 0 \text{ and } c_2(E) = \alpha. \tag{2}$$

It follows from (1), (2), and (2.4) that

(3)
$$w_1(E(\mathbb{R})) = 0 \text{ and } w_2(E(\mathbb{R})) = 0.$$

Since E is of constant rank (X being irreducible) and $n = 3$, condition (3) implies that the vector bundle $E(\mathbb{R})$ on $X(\mathbb{R})$ is topologically stably trivial, cf. for example [5, Lemma 1.2]. Thus, by Lemma 2.3, there exists an open neighborhood U of $X(\mathbb{R})$ in X such that the restriction $E|U$ is algebraically stably trivial. As a consequence, $c_2(E|U) = 0$ in $A^2(U)$. This means, in view of the second equality in (2), that the restriction homomorphism $A^2(X) \to A^2(U)$ sends α to 0. Therefore, according to Lemma 2.2, z is rationally equivalent to a cycle supporting no real points. $\square$

We will again consider varieties over $\mathbb{C}$. Let V be a smooth n-dimensional quasiprojective variety over $\mathbb{C}$. Denote by

$$c\ell_k^{\mathbb{C}} : A_k(V) \to H_{2k}^{BM}(V(\mathbb{C}), \mathbb{Z})$$

the homomorphism induced by the previously defined

$$c\ell_k^{\mathbb{C}} : Z_k(V) \to H_{2k}^{BM}(V(\mathbb{C}), \mathbb{Z}).$$

Let

$$c\ell_{\mathbb{C}}^i : A^i(V) \to H^{2i}(V(\mathbb{C}), \mathbb{Z})$$

be the composite homomorphism $D^{-1} \circ c\ell_{n-i}^{\mathbb{C}}$, where

$$D : H^{2i}(V(\mathbb{C}), \mathbb{Z}) \to H_{2n-2i}^{BM}(V(\mathbb{C}), \mathbb{Z})$$

is the Poincaré duality and

$$c\ell_{n-i}^{\mathbb{C}} : A^i(V) = A_{n-i}(V) \to H_{2n-2i}^{BM}(V(\mathbb{C}), \mathbb{Z}).$$

Any vector bundle F on V determines in a canonical way a complex vector bundle $F(\mathbb{C})$ on $V(\mathbb{C})$. By [3, 8],

$$c_i^{\text{top}}(F(\mathbb{C})) = c\ell_{\mathbb{C}}^i(c_i(F)) \text{ for all } i \geq 0,$$

where $c_i(F)$ is the ith Chern class of F in $A^i(V)$ and $c_i^{\text{top}}(F(\mathbb{C}))$ is the ith topological Chern class of $F(\mathbb{C})$ in $H^{2i}(V(\mathbb{C}), \mathbb{Z})$.

Any vector bundle E on X determines in a canonical way a complex vector bundle on $X(\mathbb{C})$, denoted $E(\mathbb{C})$. The restriction $E(\mathbb{C})|X(\mathbb{R})$, regarded as a real vector bundle, is isomorphic to $E(\mathbb{R}) \oplus E(\mathbb{R})$. If $\pi : X_{\mathbb{C}} \to X$ is the canonical projection, then π^*E is a vector bundle on $X_{\mathbb{C}}$. We identify $(\pi^*E)(\mathbb{C})$ and $E(\mathbb{C})$ as vector bundles on $X(\mathbb{C}) = X_{\mathbb{C}}(\mathbb{C})$. Since $c_i(\pi^*E) = \pi^*(c_i(E))$, we have

$$c_i^{\text{top}}(E(\mathbb{C})) = c\ell_{\mathbb{C}}^i(\pi^*(c_i(E))) \text{ for all } i \geq 0. \tag{2.5}$$

We also need the following topological result.

Lemma 2.6 *Let M be a compact $\mathcal{C}^\infty$ manifold. There exists a positive integer $s = s(M)$ such that for every complex vector bundle ξ on M, which is of constant rank and satisfies $c_i^{\text{top}}(\xi) = 0$ for all $i \geq 1$, the direct sum $\xi \oplus \cdots \oplus \xi$ of s copies of ξ is topologically stably trivial.*

Proof. The Grothendieck group $K(M)$ of complex vector bundles on M is a finitely generated Abelian group, cf. for example [1, p. 110]. Thus there is a positive integer $s = s(M)$ such that $sa = 0$ for every torsion element a of $K(M)$. If ξ is a complex vector bundle of constant rank on M and with $c_i(\xi) = 0$ for all $i \geq 1$, then the Chern character of ξ is equal to 0, and hence the class of ξ in $K(M)$ is a torsion element [12, p. 283, Theorem 3.2.5]. This implies that the direct sum of s copies of ξ is stably trivial. □

Proofs of Theorems 1.3 and 1.4. Let us set $r = 2s(n-k-1)!$, where $s = s(X(\mathbb{R}))$ is as in Lemma 2.6 with $M = X(\mathbb{R})$. We claim that r has the required property if the hypothesis of either Theorem 1.3 or Theorem 1.4, in the latter case with $n \geq 6$, are satisfied. The special case of Theorem 1.4 with $n = 4$ will be handled at the end of this proof.

The claim is proved as follows. We may assume that X is affine. Let z be a k-cycle on X and let $\alpha = [z]$ be its class in $A_k(X) = A^{n-k}(X)$. By Grothendieck's formula [8, Example 15.3.6], there exists a vector bundle E on X satisfying

(1)
$$\begin{gathered} c_i(E) = 0 \ \text{ for } \ 1 \leq i \leq n-k-1, \\ c_{n-k}(E) = (n-k-1)!\, \alpha. \end{gathered}$$

Making use of (1) and (2.5), we get

$$\begin{aligned} c_i^{\mathrm{top}}(E(\mathbb{C})) &= 0 \ \text{ for } \ 1 \leq i \leq n-k-1, \\ c_{n-k}^{\mathrm{top}}(E(\mathbb{C})) &= (n-k-1)!\, c\ell_{\mathbb{C}}^{n-k}(\pi^*(\alpha)), \end{aligned}$$

where $\pi : X_{\mathbb{C}} \to X$ is the canonical projection. For the restriction $E(\mathbb{C})|X(\mathbb{R})$ this yields

(2)
$$\begin{aligned} c_i^{\mathrm{top}}(E(\mathbb{C})|X(\mathbb{R})) &= 0 \ \text{ for } \ 1 \leq i \leq n-k-1, \\ c_{n-k}^{\mathrm{top}}(E(\mathbb{C})|X(\mathbb{R})) &= (n-k-1)!\, e^*(c\ell_{\mathbb{C}}^{n-k}(\pi^*(\alpha))), \end{aligned}$$

where $e : X(\mathbb{R}) \hookrightarrow X(\mathbb{C})$ is the inclusion map and

$$e^* : H^{2n-2k}(X(\mathbb{C}), \mathbb{Z}) \to H^{2n-2k}(X(\mathbb{R}), \mathbb{Z})$$

is the induced homomorphism.

If $2k+1 \leq n$, then $2n-2k \geq n+1$ and hence $H^{2n-2k}(X(\mathbb{R}), \mathbb{Z}) = 0$. If $2k = n$, $k \geq 3$, and X has property (D), then $(n-k-1)!$ is divisible by 2, $2n-2k = n$, and $e^*(H^{2n-2k}(X(\mathbb{C}), \mathbb{Z}))$ is a torsion subgroup of $H^{2n-2k}(X(\mathbb{R}), \mathbb{Z}) \cong \mathbb{Z}^p \oplus (\mathbb{Z}/2)^q$. Therefore in either of these two cases

$$(n-k-1)!\, e^*(c\ell_{\mathbb{C}}^{n-k}(\pi^*(\alpha))) = 0.$$

In view of (2), $c_j^{\mathrm{top}}(E(\mathbb{C})|X(\mathbb{R})) = 0$ for $1 \leq j \leq n-k$. This actually yields

(3)
$$c_j^{\mathrm{top}}(E(\mathbb{C})|X(\mathbb{R})) = 0 \ \text{ for all } \ j \geq 0.$$

Indeed, in the cases under consideration, $2k \leq n$. Hence for $j \geq n-k+1$ we always have $2j \geq n+1$, and therefore $c_j^{\mathrm{top}}(E(\mathbb{C})|X(\mathbb{R})) = 0$.

By Lemma 2.6 and (3), if $F = E \oplus \cdots \oplus E$ is the direct sum of $s = s(X(\mathbb{R}))$ copies of E, then the complex vector bundle $F(\mathbb{C})|X(\mathbb{R})$ on $X(\mathbb{R})$ is topologically stably trivial (note that F is of constant rank, X being irreducible). Since $F(\mathbb{C})|X(\mathbb{R})$ and $F(\mathbb{R}) \oplus F(\mathbb{R})$ are isomorphic as real vector bundles, it follows that $F(\mathbb{R}) \oplus F(\mathbb{R})$ is topologically stably trivial on $X(\mathbb{R})$. The vector bundle $G = F \oplus F$ on X is the direct sum of $2s$ copies of E, and $G(\mathbb{R})$ is topologically stably trivial on $X(\mathbb{R})$. By Lemma 2.3, there exists an open neighborhood U of $X(\mathbb{R})$ in X such that the restriction $G|U$ is algebraically stably trivial on U. In particular,

(4)
$$c_{n-k}(G|U) = 0 \text{ in } A^{n-k}(U).$$

On the other hand, (1) yields

(5)
$$c_{n-k}(G) = 2s(n-k-1)!\,\alpha = r\alpha \text{ in } A^{n-k}(X).$$

By (4) and (5), the restriction of $r\alpha$ to U is 0 in $A^{n-k}(U)$. Hence, according to Lemma 2.2, rz is rationally equivalent to a cycle supporting no real points. The claim is proved.

The proof of Theorems 1.3 and 1.4 is now complete, except the case $n = 4$ in Theorem 1.4.

Suppose then that $n = 4$, X has property (D), and z is a 2-cycle on X homologous to zero. We will show that $2z$ is rationally equivalent to a cycle supporting no real points.

Let $\alpha = [z]$ be the class of z in $A_2(X) = A^2(X)$. By assumption, $c\ell^2_{\mathbb{R}}(\alpha) = 0$. One still can find a vector bundle E on X for which (1) and (2) hold with $n = 4$ and $k = 2$. In other words,

(6)
$$c_1(E) = 0, \quad c_2(E) = \alpha$$

(7)
$$c_1^{\mathrm{top}}(E(\mathbb{C})|X(\mathbb{R})) = 0, \quad c_2^{\mathrm{top}}(E(\mathbb{C})|X(\mathbb{R})) = e^*(c\ell^2_{\mathbb{C}}(\pi^*(\alpha))).$$

We assert that $c_2^{\mathrm{top}}(E(\mathbb{C})|X(\mathbb{R}))=0$. Indeed, in view of (6) and (2.4), we obtain

$$w_1(E(\mathbb{R}))=0 \ \text{ and } \ w_2(E(\mathbb{R}))=c\ell^2_{\mathbb{R}}(\alpha)=0,$$

which in turn imply $w_4(E(\mathbb{R})\oplus E(\mathbb{R}))=0$. Since $E(\mathbb{R})\oplus E(\mathbb{R})$ and $E(\mathbb{C})|X(\mathbb{R})$ are isomorphic as real vector bundles, the last equality yields

$$w_4(E(\mathbb{C})|X(\mathbb{R}))=0.$$

If $\rho : H^4(X(\mathbb{R}),\mathbb{Z})\to H^4(X(\mathbb{R}),\mathbb{Z}/2)$ is the homomorphism induced by the canonical projection $\mathbb{Z}\to\mathbb{Z}/2$, then $\rho(c_2(E(\mathbb{C})|X(\mathbb{R})))=w_4(E(\mathbb{C})|X(\mathbb{R}))$ [13, p. 171, Problem 14-B] and hence

$$\rho(c_2(E(\mathbb{C})|X(\mathbb{R})))=0. \tag{8}$$

Since $X(\mathbb{R})$ is a closed 4-dimensional manifold, ρ is injective on the torsion part of $H^4(X(\mathbb{R}),\mathbb{Z})$. It follows from the second equality in (7) and the fact that X has property (D) that $c_2(E(\mathbb{C})|X(\mathbb{R}))$ is a torsion element of $H^4(X(\mathbb{R}),\mathbb{Z})$. Hence (8) implies $c_2(E(\mathbb{C})|X(\mathbb{R}))=0$ and the assertion is proved.

One readily sees (but it is also a special case of [14]) that the conditions $n=4$, E is of constant rank, and $c_i(E(\mathbb{C})|X(\mathbb{R}))=0$ for $i=1,2$ imply that the complex vector bundle $E(\mathbb{C})|X(\mathbb{R})$ is topologically stably trivial. Thus the real vector bundle $E(\mathbb{R})\oplus E(\mathbb{R})$ is also topologically stably trivial on $X(\mathbb{R})$. By Lemma 2.3, there exists an open neighborhood U of $X(\mathbb{R})$ in X for which the restriction $(E\oplus E)|U$ is algebraically stably trivial on U. In particular, $c_2((E\oplus E)|U)=0$ in $A^2(U)$. On the other hand, using (6), we get $c_2(E\oplus E)=2\alpha$ in $A^2(X)$. The last two equalities imply that the restriction of 2α to U is 0 in $A^2(U)$ and hence, according to Lemma 2.2, $2z$ is rationally equivalent to a cycle supporting no real points.

If now z is an arbitrary 2-cycle on X, then $2z$ is homologous to zero and therefore, according to what we just proved, $4z$ is rationally equivalent to a cycle supporting no real points. Thus one can take $r = 4$ for $n = 4$ in Theorem 1.4. The proof is complete in all cases. $\square$

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