Max-Planck-Institut für Mathematik
Bonn

Existence of metrics maximizing the first eigenvalue on closed surfaces

by

Henrik Matthiesen
Anna Siffert

Max-Planck-Institut für Mathematik
Preprint Series 2017 (19)

Existence of metrics maximizing the first eigenvalue on closed surfaces

Henrik Matthiesen
Anna Siffert

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

MPIM 17-19

EXISTENCE OF METRICS MAXIMIZING THE FIRST EIGENVALUE ON CLOSED SURFACES

HENRIK MATTHIESEN AND ANNA SIFFERT

Abstract. We prove that for closed surfaces of fixed topological type, orientable or non-orientable, there exists a unit volume metric, smooth away from finitely many conical singularities, that maximizes the first eigenvalue of the Laplace operator among all unit volume metrics. The key ingredient are several monotonicity results, which have partially been conjectured to hold before.

1. Introduction

For a closed Riemannian surface (Σ, g) the spectrum of the Laplace operator acting on smooth functions, is purely discrete and can be written as

$$0 = \lambda_0 < \lambda_1(\Sigma, g) \leq \lambda_2(\Sigma, g) \leq \lambda_3(\Sigma, g) \leq \cdots \to \infty,$$

where we repeat an eigenvalue as often as its multiplicity requires.

The pioneering work of Hersch [11] and Yang–Yau [33] raised the natural question, whether there are metrics g that maximize the quantities

$$\lambda_1(\Sigma, g)\,\mathrm{area}(\Sigma, g)$$

if Σ is a closed surface of fixed topological type (see also [17, 19] for the case of non-orientable surfaces). Such maximizers have remarkable properties. In fact, they always arise as immersed minimal surfaces (of possibly high codimension) in a sphere [12] and are unique in their conformal class [21]. By a slight abuse of notation, we also call Σ, endowed with a maximizing metric, a 'maximizer'.

The purpose of the present article is to settle the existence of rather regular maximizers for the first eigenvalue.

For the statement of our results and related work, we need to introduce some notation. We write Σ_γ for a closed orientable surface of genus γ. Similarly, Σ_δ^K denotes a closed non-orientable surface of non-orientable genus δ. We briefly elaborate on these notions in Appendix B. Furthermore, we use the common notation

$$\Lambda_1(\gamma) = \sup_g \lambda_1(\Sigma_\gamma, g)\,\mathrm{area}(\Sigma_\gamma, g),$$

Date: March 13, 2017.

2010 *Mathematics Subject Classification.* 35P15, 58E20.

Key words and phrases. Laplace operator, topological spectrum, harmonic map.

and similarly,

$$\Lambda_1^K(\delta) = \sup_g \lambda_1(\Sigma_\delta^K, g) \operatorname{area}(\Sigma_\delta^K, g),$$

with the supremum taken over all smooth metrics on Σ_γ, respectively Σ_δ^K. It is convenient to use the notation

$$\Lambda_1(\Sigma) = \sup_g \lambda_1(\Sigma, g) \operatorname{area}(\Sigma, g),$$

where Σ is a closed surface and the supremum is taken over all smooth metrics g on Σ. If Σ is orientable and has genus γ, then $\Lambda_1(\Sigma) = \Lambda_1(\gamma)$. If Σ is non-orientable and has non-orientable genus δ, then $\Lambda_1(\Sigma) = \Lambda_1^K(\delta)$.

Explicit values for $\Lambda_1(\gamma)$ or $\Lambda_1^K(\delta)$ are only known in very few cases. However, in all of these cases not only the values but also the maximizing metrics are known.

The case of the sphere is due to Hersch. We have $\Lambda_1(\mathbb{S}^2) = 8\pi$ with unique maximizer the round metric [11]. His arguments are very elegant and a cornerstone in the development of the subject.

For the real projective plane, we have $\Lambda_1(\mathbb{RP}^2) = 12\pi$ with unique maximizer the round metric [19]. The proof extends the ideas from [11] in a conceptually very nice way.

The first result for higher genus surfaces is due to Nadirashvili, namely $\Lambda_1(T^2) = 8\pi^2/\sqrt{3}$ with unique maximizer the flat equilateral torus [23]. Nadirashvili's arguments are very different from the previously employed methods. The crucial step in his proof is to obtain the existence of a maximizer. Using [21] it follows that such a maximizer necessarily has to be flat. The sharp bound follows then from earlier work of Berger [4].

Finally, for the Klein bottle, we have $\Lambda_1(K) = 12\pi E(2\sqrt{2}/3)$ with unique maximizer a metric of revolution [9]. Here E is the complete elliptic integral of the second kind.

Since Nadirashvili's paper [23] there was growing interest in finding maximizers for eigenvalue functionals on surfaces. No doubt partly because of their connection to minimal surfaces. For the Steklov eigenvalue problem, there is a connection to free boundary minimal surfaces in Euclidean balls. Fraser and Schoen showed the existence of maximizers for the first Steklov eigenvalue on bordered surfaces of genus 0 [10]. Recently, Petrides used many of the ideas in [10] to prove the following beautiful result concerning metrics realizing $\Lambda_1(\gamma)$.

Theorem 1.1 (Theorem 2 in [25])**.** *If $\Lambda_1(\gamma - 1) < \Lambda_1(\gamma)$, there is a metric g on $\Sigma = \Sigma_\gamma$, which is smooth away from finitely many conical singularities, such that*

$$\lambda_1(\Sigma, g) \operatorname{area}(\Sigma, g) = \Lambda_1(\gamma).$$

Our main result extends this result in two directions. Firstly, we show that the assumption on $\Lambda_1(\gamma)$ is superfluous. Secondly, we consider also non-orientable surfaces.

Theorem 1.2. *For any closed surface Σ, there is a metric g on Σ, smooth away from finitely many conical singularities, achieving $\Lambda_1(\Sigma)$, i.e.*

$$\Lambda_1(\Sigma) = \lambda_1(\Sigma, g)\,\mathrm{area}(\Sigma, g).$$

A natural question is, whether the maximizing metrics are also unique. As mentioned above, they are always unique in their conformal class [21]. Moreover, they are unique in all known examples. The general case remains open, but it is conjectured, that there is no uniqueness in general [15].

Another important question is, whether extremal metrics indeed possess conical singularities. In all known examples the maximizing metrics are smooth. However, it was conjectured in [15] that a metric with 6 conical singularities on the Bolza surface achieves $\Lambda_1(2)$. If the conjecture holds true, our result is optimal with respect to the regularity of the extremizing metric.

There are two steps in the proof of Theorem 1.2. In a first step, we extend Petrides' arguments to the non-orientable setting, which gives the existence of a maximizer on Σ_δ^K provided that $\Lambda_1(\lfloor(\delta-1)/2\rfloor) < \Lambda_1^K(\delta)$, and $\Lambda_1^K(\delta-1) < \Lambda_1(\delta)$.

The strategy behind the first step is to take a sequence of maximizing conformal classes $c_k = [h_k]$ represented by hyperbolic metrics h_k on Σ_δ^K. Thanks to Theorem 1 in [25], we can find metrics that maximize the first eigenvalue in c_k for any k. We want to show that the conformal classes c_k lie in a compact part of the moduli space. First we use the Mumford compactness criterion to exhibit a compact subspace of the moduli space of non-orientable hyperbolic surfaces in terms of a lower bound on the injectivity radius. In a next step, we use that the maximizers in c_k can be studied using sphere valued harmonic maps. Extending the arguments from [25] and [34] (which is used in [25]) to non-orientable surfaces we can then complete the proof of the first step mentioned above.

The second step consists in proving some monotonicity results for Λ_1 and Λ_1^K. More precisely, we will prove that

$$\begin{aligned} \Lambda_1(\gamma) &< \Lambda_1(\gamma+1), \\ \Lambda_1(\gamma) &< \Lambda_1^K(2\gamma+1), \\ \Lambda_1^K(\delta) &< \Lambda_1^K(\delta+1), \end{aligned}$$

provided that in each case a metric realizing the left hand side exists. The weak form of the above inequalities (at least of the first one) has been known to hold before [8].

The proof of the monotonicity results follows in parts ideas from [10]. We attach a thin handle respectively a thin cross cap, both of definite height, to the given maximizers. We then show that as the handle respectively cross cap collapses to an interval, we can choose the height parameter in such a way, that the first eigenvalue of the new surface is well controlled. More precisely, if ε denotes the radius of the handle respectively cross cap, the gain in area is linear while the eigenvalue can loose at most $o(1)\varepsilon$. This completes the second step.

Combining these two steps, the main result follows by an induction.

We remark that in order to obtain the existence result for non-orientable surfaces, we need to obtain the existence result for orientable surfaces first. The reason is that non-orientable surfaces can degenerate also to orientable surfaces. To rule out this behavior for maximizing sequences, we use the second monotonicity result above, which requires the a priori existence of a maximizer for the orientable setting.

Finally, note that a result analogous to Theorem 1.2 does not hold in general for Λ_i with $i > 1$. See for example [24], where it is proven that the unqiue maximizer for Λ_2 on $\mathbb{S}^2$ has a one-dimensional set of singular points.

The structure of the article is as follows. In Section 2, we prove a version of the Mumford compactness criterion for non-orientable hyperbolic surfaces. Section 3 explains the necessary extensions of Petrides' arguments in order to handle also non-orientable surfaces. The monotonicity results and Theorem 1.2 are proved in Section 4.

Acknowledgments. The authors are grateful to Colette Anné for sending them a copy of her manuscript 'Écrasement d'anses et spectre du Laplacien' [1]. The first named author is grateful to Bogdan Georgiev for discussions during the early stages of this project. He is also grateful for the hospitality of the MIT mathematics department. Furthermore, both authors want to thank Werner Ballmann for proofreading and some valuable comments. Last, but not least, both authors would like to thank the Max Planck Institute for Mathematics in Bonn for the support and the excellent working conditions.

2. Compactness for non-orientable surfaces

The Mumford compactness criterion [22] states that the set of orientable, hyperbolic surfaces with injectivity radius bounded below is a compact subset of Teichmüller space. In this section we show that this also holds for non-orientable surfaces. Probably, this is well-known, but for the sake of completeness and since we will use the arguments from our proof again, we include a proof below.

Given any Riemannian metric g_0 on $\Sigma = \Sigma_\delta^K$, the Poincaré Uniformization theorem asserts that we can find a new metric on Σ, which is pointwise conformal to g_0 and has constant curvature $+1, 0$, or -1, depending on the sign of $\chi(\Sigma)$. Assuming $\delta \geq 3$, these metrics have curvature -1. Let h_k be a sequence of such metrics on Σ with injectivity radius bounded uniformly from below, $\operatorname{inj}(\Sigma, h_k) \geq c > 0$. The goal is to prove that there exist diffeomorphisms σ_k of Σ and a hyperbolic metric h of Σ, such that $\sigma_k^* h_k$ converges smoothly to h as $k \to \infty$. Our strategy is to apply the Mumford compactness criterion to the orientation double covers of the surfaces (Σ, h_k).

So consider the orientation double cover $\hat{\Sigma} = \Sigma_{\delta-1}$ of Σ endowed with the pullback metrics of h_k, denoted by $\hat{h}_k$. Since $\delta \geq 3$, these are orientable hyperbolic surface of genus $\delta - 1$ and may thus be regarded as elements in Teichmüller space $\mathcal{T}_{\delta-1}$, which in addition admit fixed point free, isometric, orientation reversing involutions ι_k.

We have the following lemma.

Lemma 2.1. *Assume that* $\inf_k \operatorname{inj}(\Sigma_\delta^K, h_k) > 0$. *Then there are diffeomorphisms* $\tau_k \colon \Sigma_{\delta-1} \to \Sigma_{\delta-1}$, *such that, up to taking a subsequence,* $\tau_k^* \hat{h}_k \to \hat{h}$ *in* C^∞. *Moreover,* $(\Sigma_{\delta-1}, \hat{h})$ *admits a fixed point free, isometric, orientation reversing involution* ι, *which is obtained as the* C^0*-limit of the involutions* $\tau_k^{-1} \circ \iota_k \circ \tau_k$.

Proof. As above, we simply write Σ instead of Σ_δ^K, and $\hat{\Sigma}$ instead of $\Sigma_{\delta-1}$. It is elementary to see that $\operatorname{inj}(\hat{\Sigma}, \hat{h}_k) \geq \operatorname{inj}(\Sigma, h_k)$. Therefore, we can apply the Mumford compactness criterion [22] and find diffeomorphisms τ_k and a limit metric $\hat{h}$ as asserted.

It remains to show that we can find the involution ι. Since $\tau_k^* \hat{h}_k \to \hat{h}$ in C^∞, we have the uniform Lipschitz bound

$$\begin{aligned} d_{\hat{h}}((\tau_k^{-1} \circ \iota_k \circ \tau_k)(p), (\tau_k^{-1} \circ \iota_k \circ \tau_k)(q)) &\\ \leq C d_{\tau_k^* \hat{h}_k}((\tau_k^{-1} \circ \iota_k \circ \tau_k)(p), (\tau_k^{-1} \circ \iota_k \circ \tau_k)(q)) &\\ = C d_{\tau_k^* \hat{h}_k}(p, q) &\\ \leq C d_{\hat{h}}(p, q). & \end{aligned}$$

Since $\hat{\Sigma}$ is compact, it follows from Arzela–Ascoli, that, up to taking a subsequence, $\tau_k^{-1} \circ \iota_k \circ \tau_k \to \iota$ in $C^0(\hat{\Sigma}, \hat{h})$. We have

$$\begin{aligned} d_{\hat{h}}(\iota(p), \iota(q)) &\leq \lim_{k\to\infty} d_{\tau_k^* \hat{h}_k}(\iota(p), (\tau_k^{-1} \circ \iota_k \circ \tau_k)(p)) \\ &\quad + \lim_{k\to\infty} d_{\tau_k^* \hat{h}_k}((\tau_k^{-1} \circ \iota_k \circ \tau_k)(p), (\tau_k^{-1} \circ \iota_k \circ \tau_k)(q)) \\ &\quad + \lim_{k\to\infty} d_{\tau_k^* \hat{h}_k}((\tau_k^{-1} \circ \iota_k \circ \tau_k)(q), \iota(q)) \\ &\leq C \lim_{k\to\infty} d_{C^0(\hat{\Sigma}, \hat{h})}(\tau_k^{-1} \circ \iota_k \circ \tau_k, \iota) \\ &\quad + \lim_{k\to\infty} d_{\tau_k^* \hat{h}_k}((\tau_k^{-1} \circ \iota_k \circ \tau_k)(p), (\tau_k^{-1} \circ \iota_k \circ \tau_k)(q)) \\ &= d_{\hat{h}}(p, q), \end{aligned} \tag{2.2}$$

using that $\tau_k^*\hat{h}_k \to \hat{h}$ in C^∞, and $\tau_k^{-1} \circ \iota_k \circ \tau_k \to \iota$ in $C^0(\hat{\Sigma}, \hat{h})$. Observe that ι is an involution again, hence (2.2) implies that actually

$$d_{\hat{h}}(\iota(p), \iota(q)) = d_{\hat{h}}(p, q).$$

By the Myers–Steenrod theorem it thus follows that ι is a smooth, isometric involution.

We need to show that ι does not have any fixed points. But this is a consequence of the general bound $d_{\tau_k^*\hat{h}_k}((\tau_k^{-1} \circ \iota_k \circ \tau_k)(p), p) \geq c > 0$ for some uniform c. To prove this let $c > 0$ be such that $B_{\hat{h}}(x, 2c) \subset \hat{\Sigma}$ is strictly geodesically convex for any $x \in \hat{\Sigma}$. Then $B_{\tau_k^*\hat{h}_k}(x, c)$ is strictly geodesically convex for $k \geq K$ sufficiently large. Assume now that there is $k \geq K$, such that $d_{\hat{h}_k}((\tau_k^{-1} \circ \iota_k \circ \tau_k)(p), p) < c$. Let γ be the unique minimizing geodesic connecting p to $(\tau_k^{-1} \circ \iota_k \circ \tau_k)(p)$. Since $\tau_k^{-1} \circ \iota_k \circ \tau_k$ is an isometry, we need to have $\operatorname{im}(\tau_k^{-1} \circ \iota_k \circ \tau_k \circ \gamma) = \operatorname{im} \gamma$. Since ι_k is fixed point free, γ is non-constant. Therefore, $\tau_k^{-1} \circ \iota_k \circ \tau_k$ restricted to $\operatorname{im} \gamma$ induces an involution of the interval $[0, 1]$ mapping 0 to 1 and vice versa. But such an involution needs to have a fixed point. It follows that ι_k has a fixed point for large k, which is a contradiction.

Finally, note that ι is orientation reversing by C^0-convergence. □

It follows that the metric $\hat{h}$ on $\hat{\Sigma}$ is ι-invariant. Therefore, it induces a smooth hyperbolic metric h on Σ. Moreover, the hyperbolic metrics on Σ induced from $\tau_k^* h_k$ and $\tau_k^{-1} \circ \iota_k \circ \tau_k$ converge smoothly to h on Σ. Finally, observe that the diffeomorphisms τ_k induce diffeomorphisms σ_k of Σ, such that $\sigma_k^* h_k$ are the metrics described above and converge smoothly to h.

Thus we have proved the following proposition.

Proposition 2.3. *Let (h_k) be a sequence of hyperbolic metrics on Σ_δ^K such that* $\operatorname{inj}(\Sigma_\delta^K, h_k) \geq c > 0$. *Then there are diffeomorphisms σ_k of Σ_δ^K and a hyperbolic metric h, such that $\sigma_k^* h_k \to h$ smoothly.*

3. Maximizing the first eigenvalue

In this section we extend [25, Theorem 2] to the non-orientable case. The strategy is the same as in [25]. That is, we first use that we can maximize the first eigenvalue in each conformal class. We then pick a maximizing sequence, consisting of maximizers in their own conformal class. This has the advantage, that these metrics can be studied in terms of sphere valued harmonic maps. Using these harmonic maps it is possible to estimate the first eigenvalue along the maximizing sequence in case that the conformal class degenerates. To do so, we extend the results from [34] to non-orientable surfaces.

For fixed non-orientable genus $\delta \geq 3$, let c_k be a sequence of conformal classes on $\Sigma = \Sigma_\delta^K$ represented by hyperbolic metrics h_k, such that

$$\lim_{k\to\infty} \sup_{g\in c_k} \lambda_1(\Sigma, g)\,\mathrm{area}(\Sigma, g) = \Lambda_1^K(\delta).$$

We will now use the following result due to Petrides.

Theorem 3.1 ([25, Theorem 1])**.** *For each conformal class c_k as above, there is a metric g_k, which is smooth away from finitely many conical singularities such that*

$$\lambda_1(\Sigma, g_k)\,\mathrm{area}(\Sigma, g_k) = \sup_{g\in c_k} \lambda_1(\Sigma, g)\,\mathrm{area}(\Sigma, g).$$

From now on we assume that $g_k \in c_k$ is picked as in the preceding theorem. Moreover, we assume that they are normalized to have

$$\mathrm{area}(\Sigma, g_k) = 1.$$

Since these metrics are maximizers, there is a family of first eigenfunctions $u_1^k, \dots u_{\ell(k)+1}^k$, such that $\Phi_k = (u_1^k, \dots u_{\ell(k)+1}^k)\colon (\Sigma, h_k) \to \mathbb{S}^{\ell(k)}$ is a harmonic map [13]. Since the multiplicity of λ_1 is uniformly bounded in terms of the topology of Σ [5, 7], we may pass to a subsequence, such that $\ell(k)$ is some constant number l. Moreover, in this situation the maximizing metrics can be recovered via

$$g_k = \frac{|\nabla\Phi_k|_{h_k}}{\lambda_1(\Sigma, g_k)} h_k.$$

In view of Proposition 2.3, we want to show the following proposition.

Proposition 3.2. *The injectivity radius of g_k is uniformly bounded from below, provided that $\Lambda_1^K(\delta) > \Lambda_1(\delta - 1)$, and $\Lambda_1^K(\delta) > \Lambda_1^K(\delta - 1)$.*

We will argue by contradiction and assume $\mathrm{inj}\,(\Sigma, g_k) \to 0$. The Margulis lemma implies that we can find closed geodesics $\gamma_1^k, \dots, \gamma_s^k$ in (Σ, h_k), such that their lengths go to zero, i.e. $l_{h_k}(\gamma_i^k) \to 0$, as $k \to \infty$. We assume that s is chosen maximal with this property.

Each of these geodesics is either one-sided or two-sided. If a such a geodesic is two-sided, tubular neighborhoods are just described by the classical collar lemma for hyperbolic surfaces [6]. In the second case we may apply the collar lemma to the orientation double cover as follows.

Let c be a one-sided closed geodesic in Σ. We write $\hat{\Sigma}$ for the orientation double cover and τ for the non-trivial deck transformation. The lifts of c to $\hat{\Sigma}$ can not be closed, since in this case they would be disjoint and it would follow that c is two-sided. Thus the lifts c_1 and c_2 are geodesic segments with $\tau \circ c_1 = c_2$. Let $\mathcal{C}$ be a collar around the closed geodesic $c_2 * c_1$. It is not very difficult to see that the action of τ near $c_2 * c_1$ is just given by rotation about π and reflection at $c_2 * c_1$. Therefore, τ maps $\mathcal{C}$ to itself (by the explicit construction of $\mathcal{C}$), so that we can use $\mathcal{C}/\tau$ as a tubular neighborhood of c.

Our first goal is to prove that for the situation at hand the volume, measured with respect to g_k, either concentrates in the neighborhood of a pinching geodesic, or in one connected component of the complement of these neighborhoods. Before stating and proving this result we need to introduce some notation, which we borrow from Section 4 in [25].

We write s_1 for the number of one-sided closed geodesics with length going to 0. Moreover, we denote by s_2 the number of such geodesics that are two-sided. Clearly, $s = s_1 + s_2$ and $0 \leq s_1, s_2 \leq s$. From now on we assume that the closed geodesics γ_k^i are ordered such that the first s_1 geodesics are one-sided.

For all $s_1 + 1 \leq i \leq s$ the collar theorem [6] asserts the existence of an open neighborhood P_k^i of γ_k^i isometric to the following truncated hyperbolic cylinder

$$\mathcal{C}_k^i = \left\{(t,\theta) \mid -w_k^i < t < w_k^i,\, 0 \leq \theta < 2\pi\right\}$$

with

$$w_k^i = \frac{\pi}{l_k^i}\left(\pi - 2\arctan\left(\sinh\frac{l_k^i}{2}\right)\right)$$

endowed with the metric

$$\left(\frac{l_k^i}{2\pi\cos\left(\frac{l_k^i}{2\pi}t\right)}\right)^2 (dt^2 + d\theta^2).$$

Below we identify $(\theta, t) = (0, t)$ with $(\theta, t) = (2\pi, t)$. Thus the closed geodesic γ_α^i corresponds to $\{t = 0\}$.

By the discussion above and the the collar theorem again, we get that for all $1 \leq i \leq s_1$, there exists an open neighborhood P_k^i of γ_α^i isometric to the following truncated Möbius strip

$$\mathcal{M}_k^i = \left\{(t,\theta) \mid -w_k^i < t < w_k^i,\, 0 \leq \theta < 2\pi\right\} / \sim$$

with

$$w_k^i = \frac{\pi}{2l_k^i}\left(\pi - 2\arctan\left(\sinh l_k^i\right)\right)$$

endowed with the metric

$$\left(\frac{2l_k^i}{2\pi\cos\left(\frac{2l_k^i}{2\pi}t\right)}\right)^2 \left(dt^2 + d\theta^2\right).$$

Moreover, the equivalence relation $\sim$ is given by identifying $(t, \theta,) \sim (-t, \theta + \pi)$, where $\theta + \pi \in \mathbb{R}/2\pi\mathbb{R}$. Hence, the closed geodesic γ_α^i corresponds to $\{t = 0\}$.

We denote by $\Sigma_k^1, \cdots, \Sigma_k^r$ the connected components of $\Sigma \setminus \bigcup_{i=1}^s P_k^i$. Consequently, Σ can be written as the disjoint union

$$\Sigma = \left(\bigcup_{i=1}^s P_k^i\right) \bigcup \left(\bigcup_{j=1}^r \Sigma_k^j\right).$$

For $s_1 + 1 \leq i \leq s$ and $0 < b < w_k^i$ we denote by $P_k^i(b)$ the truncated hyperbolic cylinder whose length, compared to P_k^i, is reduced by b, i.e.,

$$P_k^i(b) = \left\{(t,\theta),\, -w_k^i + b < t < w_k^i - b\right\}.$$

Analogously, for $1 \leq i \leq s_1$ and $0 < b < w_k^i$, we introduce

$$P_k^i(b) = \left\{(t,\theta),\, -w_k^i + b < t < w_k^i - b\right\} / \sim .$$

Finally, we denote by $\Sigma_k^j(b)$ the connected components of $\Sigma \setminus \bigcup_{i=1}^{s} P_k^i(b)$ which contains Σ_k^j.

We are now ready to prove the above mentioned result, namely, that the volume either concentrates in the neighborhood of a pinching geodesic P_k^i, or in one connected component Σ_k^j of the complement of these neighborhoods.

Lemma 3.3. *There exists $D > 0$ such that one of the two following assertions is true:*

(1) There exists an $i \in \{1, \ldots, s\}$ such that

$$\operatorname{area}_{g_k}\left(P_k^i(a_k)\right) \geq 1 - \frac{D}{a_k}$$

for all sequences $a_k \to +\infty$ with $\frac{a_k}{w_k^i} \to 0$ as $k \to +\infty$ for all $1 \leq i \leq s$.

(2) There exists a $j \in \{1, \ldots, r\}$ such that

$$\operatorname{area}_{g_k}\left(\Sigma_k^j(9a_k)\right) \geq 1 - \frac{D}{a_k}$$

for all sequences $a_k \to +\infty$ with $\frac{a_k}{w_k^i} \to 0$ as $k \to +\infty$ for all $1 \leq i \leq s$.

Proof. The proof of Claim 11 in [25] can easily be adapted to the present situation. First recall the rough strategy of the proof: construct suitable test functions for $\lambda_1(\Sigma, g_k)$ in the P_k^i and the Σ_k^j's, apply the min-max formula for the first eigenvalue and prove the claim by contradiction. More precisely, on $\hat{\Sigma}$, the test functions are constructed with linear decay in the t variable in neck regions of the type $\hat{P}_k^i(2a_k) \setminus \hat{P}_k^i(3a_k)$ and $\hat{P}_k^i(1a_k) \setminus \hat{P}_k^i(2a_k)$, respectively, where the hat indicates that we consider the preimages under the covering map $\hat{\Sigma} \to \Sigma$. By conformal invariance, the Dirichlet energy of these can be estimated using the hyperbolic metric and decays like a_k^{-1}. From the construction it is clear that these functions are invariant under the relevant involutions. From this point on, one can just follow the arguments in [25]. □

Below we consider the two possible cases of the preceding lemma separately. The following lemma deals with the first case, i.e. when the volume concentrates in one of the P_k^i. We show that in this case we would have $\Lambda_1^K(\delta) \leq 8\pi$ if γ_k^i is 2-sided; and $\Lambda_1^K(\delta) \leq 12\pi$ if γ_k^i is 1-sided.

Lemma 3.4. *Suppose that there exists an $i \in \{1, ..., s\}$ such that*

$$\text{area}_{g_k}(P_k^i(a_k)) \geq 1 - \tfrac{D}{a_k}$$

for all sequences $a_k \to \infty$ which satisfy $\lim_{k\to\infty} \frac{a_k}{w_k^i} = 0$ for all $1 \leq i \leq s$.

(1) If γ_k^i is 2-sided, then $\Lambda_1^K(\delta) \leq 8\pi$.
(2) If γ_k^i is 1-sided, then $\Lambda_1^K(\delta) \leq 12\pi$.

Proof. In [25], Petrides proved the first statement by following ideas of Girouard [14]. The proof of the second statement is carried out analogously.

By assumption, there exists an $i \in \{1, ..., s\}$, such that the volume concentrates on $P_k := P_k^i$. On P_k we have coordinates (t, θ) as above (on $\mathcal{M}_k$). By the assumptions on the volume and a_k, we can find cut-off functions η_k which are 1 on $P_k(a_k)$ and 0 outside P_k, and satisfy

$$\int_\Sigma |\nabla \eta_k|^2 dv_{g_k} \to 0.$$

We denote by $\mathcal{C} = (-\infty, \infty) \times \mathbb{S}^1$ the infinite cylinder with its canonical coordinates $(t, \theta) \in (-\infty, \infty) \times [0, 2\pi)$. Let $\phi : \mathcal{C} \to \mathbb{S}^2 \subset \mathbb{R}^3$ be given by

$$\phi(t, \theta) = \frac{1}{e^{2t} + 1}(2e^t \cos(\theta), 2e^t \sin(\theta), e^{2t} - 1).$$

Observe that this induces a map $\psi \colon \mathcal{M} \to \mathbb{RP}^2(\sqrt{3})$ if we divide by the $\mathbb{Z}/2$ actions that we have on both sides. More precisely, $\mathcal{M} = \mathcal{C}/\sim$, where $(t, \theta) \sim (-t, \theta + \pi)$ as above, and on $\mathbb{S}^2$ we simply take the antipodal map. If we denote by $v : \mathbb{RP}^2(\sqrt{3}) \to \mathbb{S}^4$ the Veronese map, the concatenation $v \circ \phi : \mathcal{M} \to \mathbb{S}^4$ is a conformal map [14]. We may regard $\mathcal{M}_k \subset \mathcal{M}$ using Fermi coordinates as introduced above.

By a theorem of Hersch [11], there exists a conformal diffeomorphism τ_k of $\mathbb{S}^4$, such that

$$\int_{P_k} (\pi \circ \tau_k \circ v \circ \phi)\eta_k dv_{g_k} = 0,$$

where $\pi : \mathbb{S}^4 \hookrightarrow \mathbb{R}^5$ is the standard embedding. Set $u_k^i = (\pi_i \circ \tau_k \circ v \circ \phi)\eta_k$. By construction, we have

$$\sum_{i=1}^5 \int_{\mathcal{M}_k} (u_k^i)^2 dv_{g_k} \geq 1 - \frac{D}{a_k},$$

since $\text{area}_{g_k}(P_\alpha^i(a_k)) \geq 1 - \frac{D}{a_k}$. Using conformal invariance, one easily finds that

$$\int_\Sigma |\nabla u_k|_{g_k}^2 dv_{g_k} \leq 12\pi + o(1).$$

For details we refer to [14]. Consequently, there is $i = i(k) \in \{1, \dots, 5\}$, such that

$$\lambda_1(\Sigma, g_k) \leq \frac{\int_M |\nabla u_k^i|_{g_k}^2 dv_{g_k}}{\int_M (u_k^i)^2 dv_{g_k}} \leq 12\pi + o(1).$$

This finally implies

$$\Lambda_1^K(\delta) \leq \limsup_{k\to\infty} \lambda_1(\Sigma, g_k) \leq 12\pi,$$

which establishes the claim. □

We are thus left with the case second case from Lemma 3.3. In this case, we have the following lemma, which concludes the proof of Proposition 3.2.

Lemma 3.5. *Suppose that the second alternative from Lemma 3.3 holds, then either*

(i) $\Lambda_1^K(\delta) \leq \Lambda_1^K(\delta - 1)$, *or*
(ii) $\Lambda_1^K(\delta) \leq \Lambda_1(\gamma)$,

where $\gamma = \lfloor (\delta-1)/2 \rfloor$.

Proof. Again, we apply the machinery from [25] to the orientation cover. The essential point is to keep track of the geometry of the corresponding involutions. Denote by $(\hat{\Sigma}, \hat{h}_k)$ the orientation covers of (Σ, h_k), and by ι_k the corresponding deck transformations.

We can then identify the spectrum of the Laplacian for any metric g in $[h_k]$ with the spectrum of the Laplacian acting only on the even functions on $(\hat{\Sigma}, \hat{g})$. We consider the associated harmonic maps $\Phi_k \colon (\Sigma, g_k) \to \mathbb{S}^l$. By conformal invariance, we can also view these as harmonic maps from (Σ, h_k) to $\mathbb{S}^l$ In this situation, the metric can be recovered by

$$g_k = \frac{|\nabla \Phi_k|_{h_k}^2}{\lambda_1(\Sigma, g_k)} h_k,$$

see [25, Proof of Theorem 1]. By pulling back the Φ_k's to $\hat{\Sigma}$, we obtain even harmonic maps $\hat{\Phi}_k \colon (\hat{\Sigma}, \hat{h}_k) \to \mathbb{S}^l$, such that

$$\hat{g}_k = \frac{|\nabla \hat{\Phi}_k|_{\hat{h}_k}}{\lambda_1(\Sigma, g_k)} \hat{h}_k.$$

With out loss of generality, we may assume that the volume concentrates in $\Sigma_\alpha^1(9a_k)$. Denote by $\hat{\Sigma}_\alpha^1(9a_k)$ its preimage under the covering projection. Note that this preimage might be disconnected. As in [25, Sect. 4], there are a compact Riemann surface $\bar{\Sigma}$ and diffeomorphisms $\tau_k \colon \bar{\Sigma} \setminus \{p_1, \dots, p_r\} \to \hat{\Sigma}_k^1(9a_k)$. Moreover, the hyperbolic metrics $\bar{h}_k = \tau_k^* \hat{h}_k$ converge in $C^\infty_{loc}(\bar{\Sigma} \setminus \{p_1 \dots, p_r\})$ to a hyperbolic metric $\bar{h}$.

Observe, that we can restrict and pullback the involutions ι_k to get involutions $\bar{\iota}_k$ of $\bar{\Sigma} \setminus \{p_1, \dots p_r\}$. Clearly, these involutions are isometric with respect to the hyperbolic metrics $\bar{h}_k$.

In a next step, we construct a fixed point free limit involution on $\bar{\Sigma}$. For the compact subsets $\bar{\Sigma}_c := \{x \in \bar{\Sigma} \mid \operatorname{inj}_x(\bar{\Sigma}, \bar{h}) \geq c\}$, we can argue exactly as in the proof of Lemma 2.1 to get limit involutions $\bar{\iota}_n$ on $\bar{\Sigma}_{1/n}$. Since any isometric involution must map $\bar{\Sigma}_c$ to itself, we may take subsequences, such that for $m \geq n$, we have $\bar{\iota}_m|_{\bar{\Sigma}_{1/n}} = \bar{\iota}_n$. Using a standard diagonal argument, we find a limit involution on $\bar{\Sigma} \setminus \{p_1, \dots, p_k\}$. Clearly, this involution extends to all of $\bar{\Sigma}$. Moreover, $\bar{\iota}$ is fixed point free: Arguing again as in Lemma 2.1, we can not have fixed points different from the p_i's. If say p_1 is fixed under $\bar{\iota}$, the involution is just rotation by π in a disc centered at p_1. By C^0-convergence away from p_1, we see that the involutions $\hat{\iota}_k$ act just via rotation on the collars around the degenerating geodesic. But this is impossible, since this implies that $\hat{\iota}_k$ is orientation preserving.

By [34], the pullbacks $\bar{\Phi}_k$ of the harmonic maps $\hat{\Phi}_k$ along the diffeomorphisms τ_k are then harmonic maps that converge in $C^\infty_{loc}(\hat{\Sigma} \setminus \{x_1, \dots, x_s\})$ to a limit harmonic map $\bar{\Phi}$. Clearly, $\bar{\Phi}$ is invariant under $\bar{\iota}$. Note, that no energy can be lost at the points x_i. To see this observe first, that such points always come in pairs by the invariance of the harmonic maps. Moreover, from the construction of the limit involution, it is clear, that two such points are bounded away from each other. Therefore, energy concentration of the harmonic maps in a point x_i implies that the volume with respect to the metric g_k concentrates at a point in Σ. But by [18, Lemma 2.1 and 3.1] this implies

$$\Lambda_1^K(\delta) = \lim_{k \to \infty} (\Sigma, g_k) \leq 8\pi.$$

Since it has finite energy $\bar{\Phi}$ extends to a harmonic map $\hat{\Sigma} \to \mathbb{S}^l$ [29, Theorem 3.6]. Moreover, this extension is certainly invariant under $\bar{\iota}$.

We consider the metric

$$\bar{g} = \frac{|\nabla \bar{\Phi}|^2_{\bar{h}}}{\Lambda_1^K(\delta)} \bar{h}$$

and observe that it is invariant under the involution $\bar{\iota}$, so that it descends to a metric g on $\bar{\Sigma}/\bar{\iota}$. Since there is no energy lost along the sequence $\bar{\Phi}_k$ of harmonic maps, we have

$$\operatorname{area}(\Sigma, g) = 1.$$

Let u be the lift of a first eigenfunction of $(\bar{\Sigma}/\bar{\iota}, g)$ and let η_ε be cut-off functions which are 0 near all $p_i's$, and satisfy $\int |\nabla \eta_\varepsilon|^2 dv_{\bar{g}} \to 0$ (which is possible, since the capacity of a point relative to any ball is 0 [20, Chapter 2.2.4]). Using $\eta_\varepsilon u$ as a test function on $\hat{\Sigma}_k(9a_k)$ for k large enough, we find

$$\begin{aligned}\Lambda_1^K(\delta) &= \lim_{k\to\infty} \lambda_1(\Sigma, g_k)\\ &\leq \limsup_{\varepsilon\to 0} \lim_{k\to\infty} \frac{\int_{\hat\Sigma} |\nabla(\eta_\varepsilon u)|^2 dv_{\hat g_k}}{\int_{\hat\Sigma} |\eta_\varepsilon u|^2 dv_{\hat g_k}}\\ &\leq \lambda_1(\bar\Sigma/\bar\iota, g).\end{aligned}$$

If $\bar\Sigma$ is disconnected, it has two connected components and the genus of each component is at most $\lfloor(\delta-1)/2\rfloor$. Therefore, the quotient $\bar\Sigma/\bar\iota$ is an orientable surface of genus at most $\lfloor(\delta-1)/2\rfloor$ in this case. In case $\bar\Sigma$ is connected, the quotient is non-orientable of non-orientable genus at most $\delta-1$. □

Since $\Lambda_1^K(2) > 12\pi$, we can always rule out the first scenario from Lemma 3.3. The following theorem extends Theorem 1.1 to the non-orientable setting.

Theorem 3.6. *Let $\delta \geq 3$. If $\Lambda_1^K(\delta) > \max\{\Lambda_1^K(\delta-1), \Lambda_1(\lfloor(\delta-1)/2\rfloor)\}$, there is a metric smooth away from finitely many singularities on Σ_δ^K that achieves $\Lambda_1^K(\delta)$.*

Proof. By the assumptions, Proposition 3.2, and Proposition 2.3, we can take hyperbolic metrics $h_k \to h$ in C^∞, such that

$$\lim_{k\to\infty} \sup_{g\in[h_k]} \lambda_1(\Sigma, g)\,\mathrm{area}(\Sigma, g) = \Lambda_1^K(\delta)$$

As above, we take unit volume metrics $g_k \in [h_k]$, such that $\lambda_1(\Sigma, g_k) = \sup_{g\in[h_k]} \lambda_1(\Sigma, g)\,\mathrm{area}(\Sigma, g)$. For the corresponding sequence of harmonic maps $\Phi_k\colon (\Sigma, h_k) \to \mathbb{S}^l$ no bubbling can occur since this would imply $\Lambda_1^K(\delta) \leq 8\pi$, by the same argument as above. Therefore, we can take a subsequence such that $\Phi_k \to \Phi$ in C^∞, which implies that $g_k \to g = \frac{|\nabla\Phi|_h^2}{\Lambda_1^K(\delta)}$ in C^∞. In particular,

$$\lambda_1(\Sigma, g)\,\mathrm{area}(\Sigma, g) = \Lambda_1^K(\delta)$$

and g is smooth away from the branch points of Φ. The number of branch points is finite and the branch points correspond to conical singularities of g [30]. □

4. Monotonicity

In this section we provide three new monotonicity results, which are the main ingredient for the proof of Theorem 1.2.

Theorem 4.1. *Let g be a metric on $\Sigma = \Sigma_\delta^K$, smooth away from finitely many conical singularities, such that $\Lambda_1^K(\delta) = \lambda_1(\Sigma, g)\,\mathrm{area}(\Sigma, g)$. Then we have the strict inequality*

$$\Lambda_1^K(\delta) < \Lambda_1^K(\delta+1). \tag{4.2}$$

If we start with a metric, that maximizes λ_1 on Σ_γ, we have the following two spectral gap results.

Theorem 4.3. *Let g be a metric on $\Sigma = \Sigma_\gamma$, smooth away from finitely many conical singularities, such that $\Lambda_1(\gamma) = \lambda_1(\Sigma, g)\operatorname{area}(\Sigma, g)$. Then we have the strict inequalities*

$$\Lambda_1(\gamma) < \Lambda_1(\gamma+1) \tag{4.4}$$

and

$$\Lambda_1(\gamma) < \Lambda_1^K(2\gamma+1). \tag{4.5}$$

The proofs of all these monotonicity results are very similar. In fact, the proof of (4.2) and (4.5) is the same and (4.4) uses essentially the same estimates.

The strategy is as follows: Given a maximizer Σ, we attach a thin handle respectively a thin cross cap, of definite height h to it. We then compute the spectrum of the new surface as the thin handle, respectively the thin cross cap, collapses to an interval on scale ε. Furthermore, we show that as many eigenfunctions as possible converge in a sufficiently strong sense to eigenfunctions on Σ. This allows us to choose for ε small the height parameter $h = h(\varepsilon)$ in such a way, that the first eigenvalue of the new surface has multiplicity at least two and h is bounded away from 0. For such a choice of height parameter, we denote these new surfaces by Σ_ε.

Using that the multiplicity of the first eigenvalue of Σ_ε is at least two, we can show that its first eigenvalue is bounded from below by the first non-trivial Neumann eigenvalue of $\Sigma \setminus B_\varepsilon$. In the next step we derive good estimates on a Neumann eigenfunction on $\Sigma \setminus B_\varepsilon$ near ∂B_ε. We then use the functions, which are obtained by extending first Neumann eigenfunctions harmonically to B_ε, as test functions to obtain a lower bound for the Neumann eigenvalues in terms of the first eigenvalue of Σ. To conclude, we use that the gain in area for Σ_ε is linear in ε, whereas the possible loss in the eigenvalue is at most $o(1)\varepsilon$.

4.1. **Attaching cross caps and small handles.** To show Theorem 4.1 and the second assertion of Theorem 4.3 we glue a cross cap along its boundary. Write

$$M_{\varepsilon,h} = \mathbb{S}^1(\varepsilon) \times [0, 2h]/\sim,$$

where $(\theta, t) \sim (\theta + \pi, 2h - t)$, and endow this with its canonical flat metric $f_{\varepsilon,h}$. Let $x_0 \in \Sigma$ be such that g is smooth near x_0. Let U be a coordinate neighborhood containing x_0, such that g is conformal to the Euclidean metric in U, that is $g = fg_e$ with f a smooth, positive function and g_e the Euclidean metric. Let $B_\varepsilon = B_{g_e}(x_0, \varepsilon)$ be a ball centered at x_0 with radius equals ε with respect to g_e. We then consider the surface

$$\Sigma_{\varepsilon,h} := (\Sigma \setminus B_\varepsilon) \cup_{\partial B_\varepsilon} M_{\varepsilon,h},$$

which we endow with the (non-smooth) metric $g_{\varepsilon,h}$ given by g on $\Sigma \setminus B_\varepsilon$ and by the flat metric $f_{\varepsilon,h}$ on $M_{\varepsilon,h}$. For ε small, the metrics $g_{\varepsilon,h}$ have conical singularities at the singular points of g and, in addition, a 1-dimensional singular set along ∂B_ε. We will show below that for ε small, there is a choice of $h \in [h_0, h_1] \subset (0, \infty)$, such that

$$\lambda_1(\Sigma_{\varepsilon,h}) \operatorname{area} \Sigma_{\varepsilon,h} > \lambda_1(\Sigma). \tag{4.6}$$

For ε and h such that the above holds, we can smooth the metric $g_{\varepsilon,h}$ in such a way, that we still have the strict inequality above.

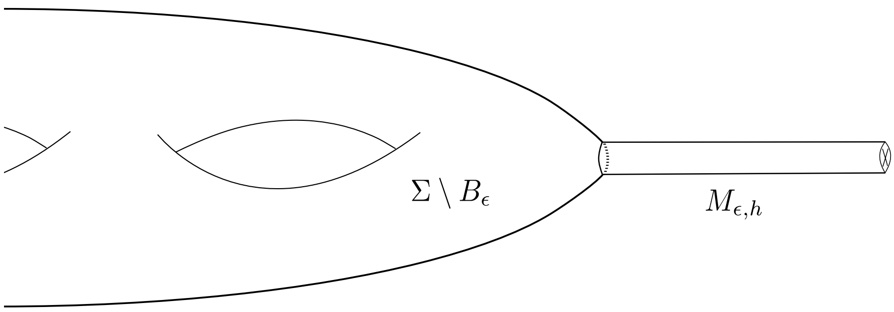

FIGURE 1. A part of the surface $\Sigma_{\varepsilon,h}$

To show the first assertion of Theorem 4.3, we glue a flat cylinder along its two boundary components. More precisely, we take

$$C_{\varepsilon,h} = \mathbb{S}^1(\varepsilon) \times [0, h]$$

endowed with its canonical flat metric. For two points $x_1, x_2 \in \Sigma$, such that g is smooth near both of these points, we take neighborhoods as above. We then consider the surface

$$\Sigma_{\varepsilon,h} = (\Sigma \setminus (B_\varepsilon(x_1) \cup B_\varepsilon(x_2))) \cup_{\partial B_\varepsilon(x_1) \cup \partial B_\varepsilon(x_2)} C_{\varepsilon,h},$$

where the balls $B_\varepsilon(x_i)$ are again with respect to the Euclidean metric. Following the same steps as above, we will be able to show that (4.6) holds here as well for suitable choices of ε and h. As in the first construction we can then smooth the metric such that strict inequality remains valid.

Finally, we recall the following elementary facts, since they will be needed throughout the present section. For a compact manifold with boundary, we denote by λ_0 its smallest Dirichlet eigenvalue and by μ_1 is smallest non-zero Neumann eigenvalue. Recall that $\lambda_0(M_{\varepsilon,h}) = \pi^2/(4h^2)$ and $\lambda_0(C_{\varepsilon,h}) = \pi^2/h^2$ for any $h > 0$. Moreover, for ε such that $\lambda_1(\mathbb{S}^1(\varepsilon)) > \mu_1(M_{\varepsilon,h})$, we have $\lambda_0(M_{\varepsilon,h}) = \pi^2/(4h^2) \leq \pi^2/h^2 = \mu_1(M_{\varepsilon,h})$. Similarly, we have $\lambda_0(C_{\varepsilon,h}) = \pi^2/h^2 = \mu_1(M_{\varepsilon,h})$ provided that $\lambda_1(\mathbb{S}^1(\varepsilon)) > \lambda_0(C_{\varepsilon,h})$.

4.2. **The limit spectrum.** We mainly restrict our discussion in the following sections to the surfaces $\Sigma_{\varepsilon,h} = (\Sigma \setminus B_\varepsilon) \cup_{\partial B_\varepsilon} M_{\varepsilon,h}$. The discussion for glueing handles is similar or identical. We will indicate the necessary changes.

The first thing we need is the computation of the spectrum of $\Sigma_{\varepsilon,h}$ as $\varepsilon \to 0$. We will prove that the spectrum of $\Sigma_{\varepsilon,h}$ converges locally uniformly in the height h to the reordered union of the spectrum of Σ and the spectrum of the interval to which the handle respectively cross cap collapses to. In the case of attached handles and fixed height h, this is due to Anné [2], see also [1, 26, 27]. The arguments for the non-orientable case are essentially along the same lines.

For the precise statement of our result we first need to introduce some notation. Denote by $\sigma_D^{\mathbb{Z}/2}([0,2h])$ the $\mathbb{Z}/2$-invariant Dirichlet spectrum of the interval $[0,2h]$, i.e. the spectrum of the Laplace operator acting on $\left(W_0^{1,2}([0,2h]) \cap W^{2,2}[0,2h]\right)^{\mathbb{Z}/2}$. The superscript indicates that we consider only those functions which are invariant under the involution $t \mapsto 2h - t$. For us the spectrum will always be a weakly increasing sequence, rather than just a set. (All operators we consider have purely discrete spectrum.) For fixed $h > 0$ denote by

$$0 = \nu_0^h < \nu_1^h \leq \nu_2^h \leq \dots$$

the reordered union of $\sigma(\Sigma)$ and $\sigma_D^{\mathbb{Z}/2}([0,2h])$.

The second thing we discuss is the convergence of the eigenfunctions on $\Sigma_{\varepsilon,h}$. The introduction of the following notation is convenient for this purpose. For $u \in W^{1,2}(\Sigma \setminus B_\varepsilon)$, we write $\tilde{u} \in W^{1,2}(\Sigma)$ for the function which is given by u in $\Sigma \setminus B_\varepsilon$ and by the harmonic extension of $u|_{\partial B_\varepsilon}$ to B_ε.

We are now ready to state the above mentioned results.

Theorem 4.7. *The spectrum of $\Sigma_{\varepsilon,h}$ converges locally uniformly in h to $(\nu_i^h)_{i\in\mathbb{N}}$, i.e. for any a, b with $0 < a < b$, any $\delta > 0$ and $k \in \mathbb{N}$ there is $\varepsilon_0 > 0$ such that for any $h \in [a,b]$ and any $\varepsilon < \varepsilon_0$*

$$|\lambda_k(\Sigma_{\varepsilon,h}) - \nu_k^h| < \delta.$$

Let ε_l be a null sequence and $h_l \to h$. For any sequence u_l, of normalized eigenfunctions with bounded eigenvalues on $\Sigma_{\varepsilon_l,h_l}$, we have subsequential convergence in the following ways

(1) *$r_l := u_l|_{\Sigma\setminus B_{\varepsilon_l}}$ satisfies $\tilde{r}_l \to u$ in $L^2(\Sigma)$, where u is a eigenfunction on Σ; or*
(2) *$\int_{M_{\varepsilon_l,h_l}} |(u_l - v_l) - \varepsilon_l^{-1/2} u_0|^2 \to 0$, where v_l denotes the harmonic extension of $u_l|_{\partial M_{\varepsilon_l,h-l}}$ and u_0 is a eigenfunction corresponding to an eigenvalue in $\sigma_D^{\mathbb{Z}/2}([0,2h])$.*

Moreover, for a sequence u_l such that we have convergence of both types as above, we have $\|u_0\|_{L^2([0,2h])} + \|u\|_{L^2(\Sigma)} = 1$.

Remark 4.8. If we attach a collapsing handle as described in Section 4.1 instead of a cross cap, the analogous statement for the spectrum and eigenfunctions holds.

The important implication we need to draw from the behavior of the eigenfunctions is that as many eigenfunctions as possible converge in a sufficiently strong sense to eigenfunctions on Σ; see the proof of Proposition 4.9.

A proof of Theorem 4.7 can be found in Appendix A. It is a modification of the arguments in [26], which apparently do not immediately imply that the convergence is uniform in the height parameter.

4.3. **Choice of the height parameter.** In the following proposition we show that we can find ε and h, such that the multiplicity of $\lambda_1(\Sigma_{\varepsilon,h})$ is at least two. This result is used crucially in the next section to bound $\lambda_1(\Sigma_{\varepsilon,h})$ from below in terms of $\mu_1(\Sigma\setminus B_\varepsilon)$, the first Neumann eigenvalue of $\Sigma \setminus B_\varepsilon$.
Although the rough idea of the proof is inspired by Proposition 4.3 in [10], new ideas and methods are necessary to accomplish it. For example, Fraser and Schoen can write down test functions right away, whereas we need to make use of Theorem 4.7 in order to do so.

Proposition 4.9. *For sufficiently small but fixed ε and $|h_1 - h_0|$ small enough, there is $h = h_\varepsilon \in (h_0, h_1)$, such that the multiplicity of $\lambda_1(\Sigma_{\varepsilon,h})$ is at least two.*

Proof. We choose $0 < h_0 < h_1$, such that $\lambda_0^{\mathbb{Z}/2}([0, 2h_0]) > \lambda_1(\Sigma)$ and $\lambda_0^{\mathbb{Z}/2}([0, 2h_1]) < \lambda_1(\Sigma)$. Note that then we may choose h_0 and h_1 such that we also have

$$\lambda_1(\Sigma) < \lambda_0^{\mathbb{Z}/2}([0, 2h_0]) < \lambda_1^{\mathbb{Z}/2\mathbb{Z}}([0, 2h_1]) < \lambda_2(\Sigma).$$

It then follows from Theorem 4.7 that for ε small enough and $h \in [h_0, h_1]$ there are exactly $\mathrm{mult}(\lambda_1(\Sigma)) + 1$ eigenvalues of $\Sigma_{\varepsilon,h}$ contained in the interval $(0, \lambda_1^{\mathbb{Z}/2}([0, 2h_1]) - \delta)$, for some small $\delta > 0$.

Denote the direct sum of the eigenspaces, associated to these eigenvalues, by $E_{\varepsilon,h}$. We write

$$\Psi_{\varepsilon,h} \colon W^{1,2}(\Sigma_{\varepsilon,h}) \to L^2(\Sigma)$$

for the map given by composing the restriction map $u \mapsto u|_{\Sigma\setminus B_\varepsilon}$ with the harmonic extension map $W^{1,2}(\Sigma \setminus B_\varepsilon) \to W^{1,2}(\Sigma)$.

We claim that

$$\liminf_{\varepsilon\to 0} \sup_{\{h\in[h_0,h_1]\}} \inf_{\{v\in E_{\varepsilon,h} \,:\, \|v\|_{L^2(\Sigma_{\varepsilon,h})}=1\}} \int_\Sigma |\Psi_{\varepsilon,h}(v)|^2 = 0 \tag{4.10}$$

If this does not hold, there is a sequence h_l and $c > 0$, such that

$$\int_\Sigma |\Psi_{\varepsilon,h_l}(v)|^2 \geq c \tag{4.11}$$

for any $v \in E_{\varepsilon,h_l}$ with $\|v\|_{L^2(\Sigma_{\varepsilon,h_l})} = 1$. This implies that $\Psi_{\varepsilon,h_l}(E_{\varepsilon,h_l}) \subset W^{1,2}(\Sigma)$ is $(k+1)$-dimensional. Moreover, it also implies that any $w \in \Psi_{\varepsilon,h}(E_{\varepsilon_l,h_l})$ with $\|w\|_{L^2(\Sigma)} = 1$ satisfies the bound

$$\int_\Sigma |\nabla w|^2 \leq C c^{-1} \lambda_0^{\mathbb{Z}/2}([0, 2h_1]),$$

where C is the constant bounding the harmonic extension operator $W^{1,2}(\Sigma \setminus B_\varepsilon) \to W^{1,2}(\Sigma)$. In particular, we can pick sequences $u^i_{\varepsilon_l,h_l}$ such that $(u^1_{\varepsilon_l,h_l}, \dots, u^{k+1}_{\varepsilon_l,h_l})$ is an orthonormal basis of $\Psi_{\varepsilon,_l,h_l}(E_{\varepsilon_l,h_l})$ (as a subspace of $L^2(\Sigma)$) and $u^i_{\varepsilon,_l,h_l} \rightharpoonup u^i$ in $W^{1,2}(\Sigma)$. One easily checks that $(u^1, \dots, u^{k+1})$ is an orthonormal set of $\lambda_1(\Sigma)$ eigenfunctions. This is a contradiction, since the multiplicity of $\lambda_1(\Sigma)$ is only k.

From here on, let $w_{\varepsilon,h} \in E_{\varepsilon,h}$ with $\|w_{\varepsilon,h}\|_{L^2(\Sigma_{\varepsilon,h})} = 1$ such that

$$\int_\Sigma |\Psi_{\varepsilon,h}(w_{\varepsilon,h})|^2 = \min_{\{v \in E_{\varepsilon,h} \,:\, \|v\|_{L^2(\Sigma_{\varepsilon,h})}=1\}} \int_\Sigma |\Psi_{\varepsilon,h}(v)|^2.$$

We now apply the same type of argument that lead to (4.10) to the second type of convergence in Theorem 4.7. Using the fact that the eigenvalue $\lambda_0^{\mathbb{Z}/2}([0, 2h])$ is simple, we thus get that any $u_{\varepsilon,h} \in \langle w_{\varepsilon,h} \rangle^\perp \subset L^2(\Sigma_{\varepsilon,h})$ has the property

$$\int_{M_{\varepsilon,h}} |u_{\varepsilon,h}|^2 \to 0. \tag{4.12}$$

Here, the analogue of (4.11) is used to ensure that the harmonic extension of an appropriate rescaling of $u_{\varepsilon,h}|_{\partial M_{\varepsilon,h}}$ to $M_{\varepsilon,h}$ goes to 0 in a suitable sense. By Theorem 4.7, this implies that (up to subsequence) any such $u_{\varepsilon,h}$ converges in $C^\infty_{loc}(\Sigma \setminus \{x\})$ to a normalized $\lambda_1(\Sigma)$-eigenfunction on Σ.

Let $h_* \in [h_0, h_1]$ be the unique height parameter with $\lambda_0^{\mathbb{Z}/2}([0, 2h_*]) = \lambda_1(\Sigma)$. The next step is to show that for $h \neq h_*$ there are k orthonormal eigenfunctions on $\Sigma_{\varepsilon,h}$ close to $\langle w_{\varepsilon,h} \rangle^\perp$ locally uniformly in $h \in [h_0, h_1] \setminus \{h_*\}$. In order to show this let $u_{\varepsilon,h} \in \langle w_{\varepsilon,h} \rangle^\perp$. We take some $\delta > 0$ and fix $\Omega \subset\subset \Sigma \setminus \{x\}$ such that

$$\int_{\Sigma_{\varepsilon,h} \setminus \Omega} |u_{\varepsilon,h}|^2 \leq \delta, \tag{4.13}$$

for ε small enough. This is possible due to (4.12) and subsequential convergence in $L^2(\Sigma)$ of $\Psi_{\varepsilon,h}(u_{\varepsilon,h})$ to a uniformly bounded eigenfunction. Note also, that given $\delta > 0$, we can chose Ω and ε as above uniformly in h.

We expand in an orthonormal basis of $E_{\varepsilon,h}$ consisting of eigenfunctions, $\Delta \phi^i_{\varepsilon,h} = \lambda^i_{\varepsilon,h} \phi^i_{\varepsilon,h}$, i.e., we have

$$u_{\varepsilon,h} = \sum_i \alpha^i_{\varepsilon,h} \phi^i_{\varepsilon,h}.$$

For ε small enough, we have

$$|\Delta u_{\varepsilon,h} - \lambda_1(\Sigma) u_{\varepsilon,h}| \leq \delta$$

pointwise in Ω, since $u_{\varepsilon,h}$ converges smoothly to a $\lambda_1(\Sigma)$-eigenfunction in Ω.

Let $\eta \in C^\infty(\Sigma_{\varepsilon,h})$ be some test function. By using the above expansion of $u_{\varepsilon,h}$, we find that

$$\left| \int_{\Sigma_{\varepsilon,h}} (\nabla u_{\varepsilon,h} \cdot \nabla \eta - \lambda_1(\Sigma) u_{\varepsilon,h} \eta) \right| = \left| \sum_i \int_{\Sigma_{\varepsilon,h}} \alpha^i_{\varepsilon,h} \phi^i_{\varepsilon,h} (\lambda^i_{\varepsilon,h} - \lambda_1(\Sigma)) \eta \right|.$$

We split the last integral into the integral over Ω and the rest of $\Sigma_{\varepsilon,h}$ and estimate these two integrals separately. For the integral over Ω we find, using Hölder's inequality, that

$$\begin{aligned} \left| \sum_i \int_{\Omega} \alpha^i_{\varepsilon,h} \phi^i_{\varepsilon,h} (\lambda^i_{\varepsilon,h} - \lambda_1(\Sigma)) \eta \right| &\leq \|\Delta u_{\varepsilon,h} - \lambda_1(\Sigma) u_{\varepsilon,h}\|_{L^2(\Omega)} \|\eta\|_{L^2(\Omega)} \\ &\leq \delta \|\eta\|_{L^2(\Sigma_{\varepsilon,h})}. \end{aligned}$$

The integral over the rest of $\Sigma_{\varepsilon,h}$ can easily be estimated using (4.13). Indeed, using Hölder's inequality once again, we obtain

$$\begin{aligned} \left| \sum_i \int_{\Sigma_{\varepsilon,h} \setminus \Omega} \alpha^i_{\varepsilon,h} \phi^i_{\varepsilon,h} (\lambda^i_{\varepsilon,h} - \lambda_1(\Sigma)) \eta \right| &\leq C \|u_{\varepsilon,h}\|_{L^2(\Sigma_{\varepsilon,h} \setminus \Omega)} \|\eta\|_{L^2(\Sigma_{\varepsilon,h} \setminus \Omega)} \\ &\leq C\delta \|\eta\|_{L^2(\Sigma_{\varepsilon,h})}. \end{aligned}$$

In conclusion, we get

$$\left| \int_{\Sigma_{\varepsilon,h}} (\nabla u_{\varepsilon,h} \cdot \nabla \eta - \lambda_1(\Sigma) u_{\varepsilon,h} \eta) \right| \leq C\delta \|\eta\|_{L^2(\Sigma_{\varepsilon,h})}.$$

By [3, Proposition 1], this in turn implies that there need to be k-orthonormal eigenfunctions each of which is close to some $u_{\varepsilon,h}$ for δ small compared to $|\lambda_0^{\mathbb{Z}/2}([0, 2h_*]) - \lambda_1(\Sigma)|$. Then the remaining eigenfunction needs to be close to $w_{\varepsilon,h}$. In particular, for ε small enough and $h \neq h_*$ there is a unique normalized eigenfunction $v_{\varepsilon,h}$ such that

$$\int_{M_{\varepsilon,h}} |v_{\varepsilon,h}|^2 = 1 - o(1), \tag{4.14}$$

where the $o(1)$-term is locally uniform in $h \neq h_*$

We now show, that this actually holds locally uniformly in $h \in (h_0, h_1)$. For ε fixed, the family $h \mapsto g_{\varepsilon,h}$ is an analytic family of metrics. In particular (see [16, Chapter 7], and also [4, p. 137]), we can find $\phi^1_\varepsilon(h), \dots, \phi^{k+1}_\varepsilon(h)$ such that

(i) $\phi^i_\varepsilon(h)$ is an eigenfunction on $\Sigma_{\varepsilon,h}$ with eigenvalue $\lambda^i_\varepsilon(h)$,
(ii) $(\phi^1_\varepsilon(h), \dots, \phi^{k+1}_\varepsilon(h))$ is orthonormal basis of $E_{\varepsilon,h}$; and
(iii) each ϕ^i_ε depends analytically on h.

Let $v_\varepsilon(h)$ be such an analytic branch that agrees for $h = h_0$ with v_{ε,h_0} as above and denote by $\lambda_\varepsilon(h)$ the branch of corresponding eigenvalues. Consider the function

$$m_\varepsilon(h) := \int_{M_{\varepsilon,h}} |v_\varepsilon(h)|^2.$$

By analyticity of $v_\varepsilon(h)$, also $m_\varepsilon(h)$ is analytic in h. Moreover it is uniformly bounded in ε. Therefore, it converges (up to subsequence) locally uniformly in (h_0, h_1) to an analytic function $m(h)$ as $\varepsilon \to 0$. But for h close to h_0 it follows from the arguments above, that $m_\varepsilon(h) \to 1$. Since m is analytic, this implies that $m \equiv 1$. In conclusion, the bound (4.14) holds locally uniformly in (h_0, h_1).
This implies that $\lambda_\varepsilon(h_0+\delta)$ is close to $\lambda_0^{\mathbb{Z}/2}([0, 2h_0+\delta])$, and similarly, $\lambda_\varepsilon(h_1-\delta)$ is close to $\lambda_0^{\mathbb{Z}/2}([0, 2h_1-\delta])$ for δ small.

In particular, for ε small enough, $v_\varepsilon(h)$ is a $\lambda_1(\Sigma_{\varepsilon,h})$-eigenfunction for $h = h_1$ and not a $\lambda_1(\Sigma_{\varepsilon,h})$-eigenfunction for $h = h_0$. If we choose

$$h_\varepsilon = \inf\{h \mid v_\varepsilon(h) \text{ is a } \lambda_1(\Sigma_{\varepsilon,h})\text{-eigenfunction}\},$$

we need to have $\operatorname{mult}\lambda_1(\Sigma_{\varepsilon,h_\varepsilon}) \geq 2$. Indeed, if $\operatorname{mult}\lambda_1(\Sigma_{\varepsilon,h_\varepsilon}) = 1$ the first eigenspace is spanned by $v_{\varepsilon,h_\varepsilon}$. But this remains the case for h slightly smaller than h_ε, which contradicts the definition of h_ε. □

4.4. **Reduction to Neumann boundary conditions.** As above, we pick a neighborhood U of x_0, such that $g = fg_e$ in U with g_e the Euclidean metric and f a smooth positive function. For ease of notation, we assume that $U = B(0,1) \subset \mathbb{R}^2$ endowed with the corresponding metric, and $x_0 = 0$ under this identification. For $\varepsilon > 0$ sufficiently small we apply Proposition 4.9 and choose $h = h_\varepsilon$ with $\operatorname{mult}(\lambda_1(\Sigma_{\varepsilon,h})) \geq 2$, and $h_\varepsilon \in [h_0, h_1]$. From now on we simply write $M_\varepsilon := M_{\varepsilon,h_\varepsilon}$, $\Sigma_\varepsilon := \Sigma_{\varepsilon,h_\varepsilon}$, and $\lambda_\varepsilon := \lambda_1(\Sigma_\varepsilon)$. By u_ε we denote a λ_ε-eigenfunction with

$$\int_{\Sigma\setminus B_\varepsilon} |u_\varepsilon|^2 = 1.$$

For a domain $\Omega \subset \Sigma$ with smooth boundary, we denote by $\mu_1(\Omega)$ its first non-zero Neumann eigenvalue and by $\lambda_0(\Omega)$ its first Dirichlet eigenvalue.

Lemma 4.15. *For ε sufficiently small, we have*

$$\lambda_\varepsilon \leq \mu_1(M_\varepsilon).$$

Proof. Observe, that for ε sufficiently small, we have $\lambda_0(M_\varepsilon) \leq \mu_1(M_\varepsilon)$. Below, we prove the stronger inequality $\lambda_\varepsilon \leq \lambda_0(M_\varepsilon)$. Let v_ε be a $\lambda_0(M_\varepsilon)$-eigenfunction, which we extend by 0 to all of Σ_ε. Furthermore, let $\psi\colon \Sigma \setminus B_\varepsilon \to [0,1]$ be a cut-off function, such that $\psi = 0$ near ∂B_ε

and

$$\frac{\int_{\Sigma \setminus B_\varepsilon} |\nabla \psi|^2}{\int_{\Sigma \setminus B_\varepsilon} |\psi|^2} \leq \frac{\lambda_\varepsilon}{2}.$$

We extend ψ by 0 to all of Σ_ε and still denote this by ψ. For ε small enough, we can find such ψ, since the capacity of $\{x_0\}$ relative to any ball centered at x_0 is 0 (see e.g. [20, Chapter 2.2.4]). Consider the two dimensional space spanned by v_ε and ψ. Since these two functions have disjoint supports, one easily checks that

$$\frac{\int_{\Sigma_\varepsilon} |\nabla \varphi|^2}{\int_{\Sigma_\varepsilon} |\varphi|^2} \leq \max\{\lambda_\varepsilon/2, \mu_1(M_\varepsilon)\}$$

for any function φ in this space. Thus, we need to have

$$\lambda_\varepsilon \leq \mu_1(M_\varepsilon). \qquad \square$$

Having the estimate for $\mu_1(M_\varepsilon)$ at hand, we can obtain the reverse estimate on $\mu_1(\Sigma \setminus B_\varepsilon)$.

Lemma 4.16. *For ε sufficiently small, we have*

$$\mu_1(\Sigma \setminus B_\varepsilon) \leq \lambda_\varepsilon.$$

Proof. Since $\operatorname{mult}(\lambda_1(\Sigma_\varepsilon)) \geq 2$, we can choose a λ_ε-eigenfunction u_ε on Σ_ε satisfying

$$\int_{M_\varepsilon} u_\varepsilon = 0.$$

In this case, we also have

$$\int_{\Sigma \setminus B_\varepsilon} u_\varepsilon = 0.$$

Thus we can use the restriction of u_ε to $\Sigma \setminus B_\varepsilon$ as a test function for $\mu_1(\Sigma \setminus B_\varepsilon)$. We have

$$\int_{\Sigma \setminus B_\varepsilon} |\nabla u_\varepsilon|^2 = \lambda_\varepsilon \int_{\Sigma \setminus B_\varepsilon} |u_\varepsilon|^2 + \lambda_\varepsilon \int_{M_\varepsilon} |u_\varepsilon|^2 - \int_{M_\varepsilon} |\nabla u_\varepsilon|^2. \tag{4.17}$$

Since $\int_{M_\varepsilon} u_\varepsilon = 0$, we have

$$\lambda_\varepsilon \int_{M_\varepsilon} |u_\varepsilon|^2 \leq \frac{\lambda_\varepsilon}{\mu_1(M_\varepsilon)} \int_{M_\varepsilon} |\nabla u_\varepsilon|^2 \leq \int_{M_\varepsilon} |\nabla u_\varepsilon|^2, \tag{4.18}$$

where we use Lemma 4.16. Inserting (4.17) into (4.18) implies

$$\int_{\Sigma \setminus B_\varepsilon} |\nabla u_\varepsilon|^2 \leq \lambda_\varepsilon \int_{\Sigma \setminus B_\varepsilon} |u_\varepsilon|^2,$$

which clearly gives the assertion. $\square$

In order to find a good lower bound on λ_ε in terms of $\lambda_1(\Sigma)$, it hence suffices to obtain the same kind of lower bound on $\mu_1(\Sigma \setminus B_\varepsilon)$. This is what we are going to do now.

4.5. **Estimates on Neumann eigenfunctions.** From here on u_ε will no longer denote a λ_ε-eigenfunction but instead a normalized $\mu_1(\Sigma \setminus B_\varepsilon)$-eigenfunction. From standard elliptic estimates we get the following bounds for u_ε and its gradient.

Lemma 4.19. *Let u_ε be a normalized $\mu_1(\Sigma \setminus B_\varepsilon)$-eigenfunction. If we use Euclidean polar coordinates (r, θ) centered at x_0, we have the uniform pointwise bounds*

$$|u_\varepsilon|(r,\theta) \leq C \log\left(\frac{1}{r}\right), \tag{4.20}$$

and

$$|\nabla u_\varepsilon|(r,\theta) \leq \frac{C}{r} \tag{4.21}$$

for any $r \leq 1/2$.

Proof. Recall that we have identified a conformally flat neighborhood of x_0 with $B_1 = B(0,1) \subset \mathbb{R}^2$, such that $x_0 = 0$. First, observe that (4.20) is a direct consequence of (4.21). In fact, by the standard elliptic estimates [31, Chapter 5.1], the functions u_ε are uniformly bounded in C^∞ within compact subsets of $\Sigma \setminus \{x_0\}$. Given this, we can integrate the bound (4.21) from $\partial B_{1/2}$ to ∂B_r and find (4.20).

The bound (4.21) follows from standard elliptic estimates after rescaling the scale r to a fixed scale. More precisely, we consider the rescaled functions $w_r(z) := u_\varepsilon(rz)$. On $B_1 \setminus B_\varepsilon$ the metric of Σ is uniformly bounded from above and below by the Euclidean metric. Hence we can perform all computations in the Euclidean metric. We have

$$\int_{B_3 \setminus B_{1/2}} |\nabla w_r|^2 = \int_{B_{3r} \setminus B_{r/2}} |\nabla u_\varepsilon|^2, \tag{4.22}$$

since the Dirichlet energy is conformally invariant in dimension two.

Since the Laplace operator is conformally covariant in dimension two, w_r solves the equation

$$\Delta_e w_r = r^2 f_r \lambda_\varepsilon w_r, \tag{4.23}$$

with $f_r(z) = f(rz)$ a smooth function and Δ_e the Euclidean Laplacian. Since $f \in C^\infty$, we have uniform C^∞-bounds on f_r for $r \leq 1$. Taking derivatives, we find that

$$\Delta_e \nabla w_r = r^2 \lambda_\varepsilon \nabla (f_r w_r), \tag{4.24}$$

where also the gradient is taken with respect to the Euclidean metric. The bound (4.22) implies that the right hand side of this equation is bounded by Cr^2 in $L^2(B_3 \setminus B_{1/2})$. Therefore, by elliptic estimates [31, Chapter 5.1] we have

$$\sup_{\{1 \leq s \leq 2\}} |\nabla w_r|(s,\theta) \leq Cr^2 + C|\nabla w_r|_{L^2(B_3 \setminus B_{1/2})} \leq C,$$

which scales to

$$\sup_{\{r\leq s\leq 2r\}} |\nabla u_\varepsilon|(s,\theta) \leq \frac{C}{s},$$

with C independent of r. This proves the estimate for $r \geq 2\varepsilon$. For the remaining radii, we use the same argument but apply elliptic boundary estimates [31, Chapter 5.7]. □

The last preparatory lemma we need before we can turn to the proof of the main result is a good bound on the L^2-norm of the tangential gradient of u_ε along ∂B_ε. We denote by $\partial_T u_\varepsilon$ the gradient of $u_\varepsilon|_{\partial B_\varepsilon}$.

Lemma 4.25. *We have*

$$\int_{\partial B_\varepsilon} |\partial_T u_\varepsilon|^2 d\mathcal{H}^1 \leq C\varepsilon. \tag{4.26}$$

Proof. As above, we denote by $\tilde{u}_\varepsilon$ the function obtained by extending u_ε harmonically to B_ε, where u_ε denotes a normalized $\mu_1(\Sigma\setminus B_\varepsilon)$-eigenfunction. By a scaling argument, $\tilde{u}_\varepsilon$ is uniformly bounded in $W^{1,2}(\Sigma)$ in terms of the $W^{1,2}$-norm of u_ε [28, p. 40].

Let w_ε be the unique weak solution to

$$\begin{cases} \Delta w_\varepsilon = \mu_1(\Sigma\setminus B_\varepsilon)\tilde{u}_\varepsilon & \text{in } B_1 \\ w_\varepsilon = 0 & \text{on } \partial B_1. \end{cases}$$

By elliptic estimates, w_ε is bounded in $W^{3,2}(B_{1/2})$, which embeds into $C^{1,\alpha}(B_{1/2})$ for any $\alpha < 1$. We can then write

$$u_\varepsilon = w_\varepsilon + v_\varepsilon,$$

with $v_\varepsilon \in W^{1,2}(B_1\setminus B_\varepsilon)$ a harmonic function.

Note that the bound (4.26) clearly holds for w_ε, so it suffices to consider v_ε. If we denote by ν the inward pointing normal of B_ε, we have

$$|\partial_\nu v_\varepsilon| = |\partial_\nu u_\varepsilon - \partial_\nu w_\varepsilon| = |\partial_\nu w_\varepsilon| \leq C \tag{4.27}$$

along ∂B_ε, since w_ε is bounded in $C^{1,\alpha}(B_{1/2})$. Since the Laplace operator is conformally covariant in dimension two, v_ε is also harmonic with respect to the Euclidean metric. Therefore, it follows from separation of variables, that we can expand v_ε in Fourier modes, where we suppress the index ε.

$$v = a + b\log(r) + \sum_{n\in\mathbb{Z}^*} (c_n r^n + d_n r^{-n})e^{in\theta}.$$

Using the L^2-normalization of u_ε and orthogonality, we can show that

$$\sum_{n>0} \frac{c_n^2}{2n+2} + \sum_{n<0} \frac{d_n^2}{2n+2} \leq C. \tag{4.28}$$

Indeed, we have that

$$\sum_{n>0} \int_{\varepsilon}^{1} (c_n r^n + d_n r^{-n})^2 r dr \leq C,$$

and for $\varepsilon \leq 1/2$ we can use Young's inequality to find

$$\begin{aligned}
&\int_{\varepsilon}^{1} (c_n r^n + d_n r^{-n})^2 r dr \\
&\quad = c_n^2 \int_{\varepsilon}^{1} r^{2n+1} dr + 2c_n d_n \int_{\varepsilon}^{1} r dr + d_n^2 \int_{\varepsilon}^{1} r^{-2n+1} dr \\
&\quad = \frac{c_n^2}{2n+2}(1 - \varepsilon^{2n+2}) + c_n d_n (1 - \varepsilon^2) + \frac{d_n^2}{2n-2}(\varepsilon^{-2n+2} - 1) \\
&\quad \geq \frac{c_n^2}{2n+2}(1 - \varepsilon^{2n+2} - (n+1)\delta_n) + \frac{d_n^2}{2n-2}\left(\varepsilon^{-2n+2} - 1 - \frac{n-1}{\delta_n}\right) \\
&\quad \geq \frac{c_n^2}{8(n+1)} + \frac{d_n^2}{2n-2}(\varepsilon^{-2n+2} - 2) \\
&\quad \geq \frac{c_n^2}{8(n+1)},
\end{aligned}$$

with $\delta_n = 1/(2(n+1))$. Of course, the same computation applies to negative n, so that we obtain the same kind of bound for the d_n's. From (4.28), we find that

$$h_1 = \sum_{n>0} c_n r^n e^{in\theta} + \sum_{n<0} d_n r^{-n} e^{in\theta}$$

extends to a harmonic function on all of B_1, which is bounded in L^2, whence in $C^\infty(B_{1/2})$.
Therefore, we are left with bounding the tangential derivative of the harmonic function

$$h_2 = v - h_1 - a = b\log(r) + \sum_{n<0} c_n r^n e^{in\theta} + \sum_{n>0} d_n r^{-n} e^{in\theta}.$$

In a first step, we use that the quantity

$$\rho \int_{\partial B_\rho} \left((\partial_T h_2)^2 - (\partial_r h_2)^2\right) d\mathcal{H}^1$$

is independent of ρ, what can be verified by a straightforward computation. For $\rho \to \infty$ the term $\rho \int_{\partial B_\rho} (\partial_T h_2)^2 d\mathcal{H}^1$ vanishes, since the integrand decays at least like ρ^{-3}. For the other term, note that $\partial_r \log(r)$ and $\partial_r(h_2 - b\log(r))$ are orthogonal in $L^2(\partial B_\rho)$. Therefore, we have

$$\begin{aligned}
\int_{\partial B_\rho} (\partial_r h_2)^2 d\mathcal{H}^1 &= b^2 \int_{\partial B_\rho} (\partial_r \log(r))^2 d\mathcal{H}^1 + \int_{\partial B_\rho} (\partial_r (h_2 - b\log(r)))^2 d\mathcal{H}^1 \\
&= \frac{2\pi b^2}{\rho} + O(\rho^{-3})
\end{aligned}$$

since the integrand of the second summand decays at least like ρ^{-4} as $\rho \to \infty$. In conclusion,

$$\rho \int_{\partial B_\rho} ((\partial_T h_2)^2 - (\partial_r h_2)^2) d\mathcal{H}^1 = 2\pi b^2 \tag{4.29}$$

for any $\rho \geq \varepsilon$. In order to obtain a bound on b, we estimate the L^2-norm of the radial derivative of h_2 on ∂B_ε from below. Using orthogonality as above, we find that

$$\frac{2\pi b^2}{\varepsilon} = \int_{\partial B_\varepsilon} (b\,\partial_r \log(r))^2 d\mathcal{H}^1 \leq \int_{\partial B_\varepsilon} (\partial_r h_2)^2 d\mathcal{H}^1 \leq C\varepsilon, \tag{4.30}$$

where the last inequality makes use of the bound (4.27). Combining (4.29) and (4.30) yields

$$\int_{\partial B_\varepsilon} (\partial_T h_2)^2 d\mathcal{H}^1 = \int_{\partial B_\varepsilon} (\partial_r h_2)^2 d\mathcal{H}^1 + \frac{2\pi b^2}{\varepsilon} \leq C\varepsilon. \qquad \square$$

Corollary 4.31. *We have*

$$\int_{B_\varepsilon} |\nabla \tilde{u}_\varepsilon|^2 \leq C\varepsilon^2. \tag{4.32}$$

Proof. Without loss of generality, we may assume that

$$\int_{\partial B_\varepsilon} u_\varepsilon d\mathcal{H}^1 = 0, \tag{4.33}$$

since subtracting a constant only results in subtracting a constant from $\tilde{u}_\varepsilon$ in B_ε. In particular, it does not change the energy of $\tilde{u}_\varepsilon$ in B_ε. We use $\hat{u}_\varepsilon(r,\theta) = \frac{r}{\varepsilon} u(\varepsilon,\theta)$ as a competitor. In order to estimate its energy, we use that (4.33), the Poincaré inequality, and Lemma 4.25 imply

$$\int_{\partial B_\varepsilon} |u_\varepsilon|^2 d\mathcal{H}^1 \leq C\varepsilon^2 \int_{\partial B_\varepsilon} (\partial_T u_\varepsilon)^2 d\mathcal{H}^1 \leq C\varepsilon^3. \tag{4.34}$$

Therefore, we get

$$\begin{aligned} \int_{B_\varepsilon} |\nabla \hat{u}_\varepsilon|^2 &\leq \frac{C}{\varepsilon^2} \int_0^{2\pi} \int_0^\varepsilon \big(|u_\varepsilon|^2(\varepsilon,\theta) + (\partial_\theta u_\varepsilon)^2(\varepsilon,\theta) \big)\, r dr d\theta \\ &\leq \frac{C}{\varepsilon} \int_{\partial B_\varepsilon} |u_\varepsilon|^2 d\mathcal{H}^1 + C\varepsilon \int_{\partial B_\varepsilon} (\partial_T u_\varepsilon)^2 d\mathcal{H}^1 \\ &\leq C\varepsilon^2, \end{aligned}$$

where we have used (4.34) and Lemma 4.25. $\square$

4.6. **Proofs of the monotonicity results.** In this subsection we provide the proofs of Theorem 4.1 and Theorem 4.3.

Proof of Theorem 4.1. We use the function $\tilde{u}_\varepsilon$ from above as a test function for $\lambda_1(\Sigma)$. From the maximum principle and the bound (4.20), we find that

$$\left|\int_\Sigma \tilde{u}_\varepsilon\right| \leq \int_{B_\varepsilon} |\tilde{u}_\varepsilon| \leq C|\log(\varepsilon)|\varepsilon^2 = o(1)\varepsilon.$$

From Corollary 4.31, we find

$$\int_\Sigma |\nabla \tilde{u}_\varepsilon|^2 = \mu_1(\Sigma \setminus B_\varepsilon) \int_{\Sigma\setminus B_\varepsilon} |\tilde{u}_\varepsilon|^2 + \int_{B_\varepsilon} |\nabla \tilde{u}_\varepsilon|^2 \leq \mu_1(\Sigma \setminus B_\varepsilon) + C\varepsilon^2,$$

using the normalization $\int_{\Sigma\setminus B_\varepsilon} |\tilde{u}_\varepsilon|^2 = 1$. Therefore, we can estimate $\lambda_1(\Sigma)$ from above by

$$\lambda_1(\Sigma) \leq \frac{\int_\Sigma |\nabla \tilde{u}_\varepsilon|^2}{\int_\Sigma |\tilde{u}_\varepsilon|^2 - \left(\int_\Sigma \tilde{u}_\varepsilon\right)^2} \leq \frac{\mu_1(\Sigma \setminus B_\varepsilon) + o(1)\varepsilon}{1 - o(1)\varepsilon}.$$

That is,

$$\begin{aligned}\lambda_\varepsilon \geq \mu_1(\Sigma \setminus B_\varepsilon) &\geq \lambda_1(\Sigma)(1 - o(1)\varepsilon) - o(1)\varepsilon \\ &= \lambda_1(\Sigma) - o(1)\varepsilon,\end{aligned}$$

thanks to Lemma 4.16. Combining this with the linear gain in area, we arrive at

$$\begin{aligned}\lambda_\varepsilon \operatorname{area}(\Sigma_\varepsilon) &\geq (\lambda_1(\Sigma) - o(1)\varepsilon)(1 + \Theta(\varepsilon)) \\ &\geq \lambda_1(\Sigma) + \Theta(\varepsilon).\end{aligned}$$

Therefore, for ε small enough, the singular metric $g_\varepsilon = g_{\varepsilon,h_\varepsilon}$ on $\Sigma^K_{\delta+1}$ has

$$\lambda_1(\Sigma^K_{\delta+1}, g_\varepsilon) \operatorname{area}(\Sigma^K_{\delta+1}, g_\varepsilon) > \Lambda^K_1(\delta).$$

In a final step, we perturb g_ε to a smooth metric, that still satisfies the above inequality. Fix some small $\delta > 0$ and choose a cut-off function η_δ on $[\varepsilon + \delta, \varepsilon + 2\delta]$. We modify g_ε on $B_{\varepsilon+2\delta} \setminus B_{\varepsilon+\delta}$ by defining

$$g'_\varepsilon(\theta, r) = (1 - \eta(r))g_\varepsilon(\theta, r) + \eta(r)(r^2 d\theta^2 + dr^2),$$

so that g'_ε is Euclidean near $\partial B_{\varepsilon+\delta}$. We extend g'_ε by the Euclidean metric to $B_{\varepsilon+\delta} \setminus B_\varepsilon$. In a second step, we modify g'_ε on $B_{\varepsilon+\delta} \setminus B_\varepsilon$ via

$$g''_\varepsilon(\theta, r) = (1 - \eta'(r))g'_\varepsilon(\theta, r) + \eta'(r)(\varepsilon^2 d\theta^2 + dr^2),$$

where η' is a cut-off function on $[\varepsilon, \varepsilon + \delta]$. By construction, for ε small, the metric on $\Sigma^K_{\delta+1}$ given by g''_ε on $\Sigma \setminus B_\varepsilon$ and by the flat metric on $M_{\varepsilon,h}$ is smooth away from the conical singularities of g. It is easy to check, that with respect to some smooth background metric $g''_\varepsilon \to g_\varepsilon$ in L^2 as $\delta \to 0$. In particular, this implies that the volume and all eigenvalues of g''_ε converge to those of g_ε as $\delta \to 0$. We then pick δ small enough, such that g''_ε still has $\lambda_1(\Sigma^K_{\delta+1}, g''_\varepsilon) \operatorname{area}(\Sigma^K_{\delta+1}, g''_\varepsilon) > \Lambda^K_1(\delta)$. The only singularities of the new metric g''_ε are conical. In particular, g''_ε defines a smooth conformal structure. We can then smooth g''_ε to a smooth metric in its conformal class using the heat kernel. Namely, we take

$(K_t)_{t>0}$ the heat semigroup of a smooth metric g_0 in the conformal class of g''_ε. If we write $g''_\varepsilon = \phi g_0$, we define $g'''_\varepsilon = K_t[\phi]g_0$. As above, for t small enough, we will still have $\lambda_1(\Sigma^K_{\delta+1}, g'''_\varepsilon)\,\mathrm{area}(\Sigma^K_{\delta+1}, g'''_\varepsilon) > \Lambda^K_1(\delta)$. □

Proof of Theorem 4.3. The proof of the inequality

$$\Lambda_1(\gamma) < \Lambda^K_1(2\gamma+1)$$

is exactly as the proof of Theorem 4.1 above. The proof of

$$\Lambda_1(\gamma) < \Lambda_1(\gamma+1)$$

is analogous to what we have done above, using the calculation of the limit spectrum of Σ with a collapsing handle attached. As remarked above, for the flat cylinder $C_{\varepsilon,h} = \mathbb{S}^1(\varepsilon) \times [0,h]$, the values of $\lambda_0(C_{\varepsilon,h})$ and $\mu_1(C_{\varepsilon,h})$ coincide. Thus, by the same arguments used above, the analogue of Lemma 4.15 and hence also of Lemma 4.16 holds. Therefore, it suffices to estimate $\lambda_1(\Sigma)$ in terms of $\mu_1(\Sigma \setminus (B(x_1,\varepsilon) \cup B(x_2,\varepsilon)))$ and a good error term. Using Lemma 4.19 and Lemma 4.25 in the annular regions $B(x_i,1/2) \setminus B(x_i,\varepsilon)$, we find also in this case that

$$\lambda_1(\Sigma) \le \mu_1(\Sigma \setminus (B(x_1,\varepsilon) \cup B(x_2,\varepsilon))) \le \lambda_\varepsilon + o(1)\varepsilon.$$

Since the gain in area is linear, we can smooth the metric as above and the assertion follows. □

4.7. **Proof of the main result.**

Proof of Theorem 1.2. This is now an easy induction argument. By the results from [9, 11, 19, 23], it suffices to consider the cases $\gamma \ge 2$ and $\delta \ge 3$. Assume that there exists a metric, smooth away from finitely many conical singularities, achieving $\Lambda_1(\gamma)$. Clearly, the combination of the first monotonicity result from Theorem 4.3 and the main result from [25] (see Theorem 1.1) gives the existence of metrics, smooth away from finitely many conical singularities, achieving $\Lambda_1(\gamma+1)$. This establishes the assertion of Theorem 1.2 for orientable surfaces. Having this at hand, we find from the second assertion of Theorem 4.3 that $\Lambda_1(\gamma) < \Lambda^K_1(2\gamma+1)$ for any $\gamma > 0$. Assume now, that there exists a metric, smooth away from finitely many conical singularities, achieving $\Lambda^K_1(\gamma)$. Using Theorem 4.1 and Theorem 3.6, we conclude that $\Lambda^K_1(\delta) < \Lambda^K_1(\delta+1)$ for any $\delta \ge 0$, and $\Lambda^K_1(\delta+1)$ is achieved by a metric, which is smooth away from finitely many conical singularities. □

As already mentioned in the introduction, there is a close connection between maximizing metrics and minimal immersions into spheres. For a more detailed introduction, we refer the reader to [25] and the references therein. From Theorem 1.2 we obtain the following Corollary.

Corollary 4.35. *For each $\gamma \ge 1$, there exists a minimal immersion of a compact surface of orientable genus γ into some sphere by first eigenfunctions. Furthermore, for each $\delta \ge 1$, there exists a minimal* *immersion of a compact surface of non-orientable genus δ into some sphere by first eigenfunctions.*

This result clearly generalizes the corresponding corollary of Petrides in [25], page 1338.

Appendix A. Proof of Theorem 4.7

For sake of completeness we give a proof of Theorem 4.7 in this appendix. Essentially all this material is contained in [2, 26]. We give proofs here mainly for three reasons. Firstly, we need to have some uniformity in the height parameter h — this does not seem to follow directly from the arguments in the above mentioned articles. Moreover, some of the statements we use are a bit hidden in the proofs in [2, 26]. Last but not least, the mentioned articles only cover the case of attaching handles, but we also need the case of attaching cross caps.

We need to provide three preparatory lemmas. In the first one we show that the Neumann spectrum of $\Sigma \setminus B_\varepsilon$ converges to the spectrum of Σ.

Lemma A.1. *The spectrum of $\Sigma \setminus B_\varepsilon$ with Neumann boundary conditions converges to the spectrum of Σ. Moreover, for any sequence $\varepsilon_l \to 0$ and orthonormal eigenfunctions $u_1^{\varepsilon_l}, \dots u_k^{\varepsilon_l}$ on $\Sigma \setminus B_{\varepsilon_l}$, with uniformly bounded eigenvalues, we have subsequential convergence $\tilde{u}_i^{\varepsilon_l} \to u_i$ in $L^2(\Sigma)$, where $u_1, \dots, u_k$ are orthonormal eigenfunctions on Σ.*

Proof. First, note that a simple cut-off argument using that the capacity of $\{x_0\}$ in any ball is 0 yields

$$\lim_{\varepsilon \to 0} \mu_k(\Sigma \setminus B_\varepsilon) \leq \lambda_k(\Sigma). \tag{A.2}$$

To obtain the revers bound, let u_ε be a normalized μ_k-eigenfunction and let $\tilde{u}_\varepsilon$ be the function, that is obtained by extending u_ε harmonically to B_ε. By (A.2), u_ε is bounded in $W^{1,2}(\Sigma \setminus B_\varepsilon)$, thus $\tilde{u}_\varepsilon$ is bounded in $W^{1,2}(\Sigma)$ and we may extract a subsequence $\varepsilon_l \to 0$, such that for $u_l = u_{\varepsilon_l}$ we have $u_l \rightharpoonup u$ in $W^{1,2}(\Sigma)$. By the compact Sobolev embedding we thus get $u_l \to u$ in $L^2(\Sigma)$. Hence, from standard elliptic estimates, we obtain $u_l \to u$ in $C^\infty_{loc}(\Sigma \setminus \{x_0\})$. If $\phi \in C^\infty_c(\Sigma \setminus \{x_0\})$, we find $\rho > 0$, such that $\operatorname{supp} \phi \subset \Sigma \setminus B_\rho$. By extracting a further subsequence if necessary, we may assume $\mu_1(\Sigma \setminus B_{\varepsilon_l}) \to \lambda$, using (A.2) another time. Then we have

$$\begin{aligned}
\int_\Sigma \nabla u \cdot \nabla \phi &= \lim_{l \to \infty} \int_{\Sigma \setminus B_\rho} \nabla u_l \cdot \nabla \phi \\
&= \lim_{l \to \infty} \mu_k(\Sigma \setminus B_{\varepsilon_l}) \int_{\Sigma \setminus B_\rho} u_l \phi \\
&= \lambda \int_\Sigma u \phi.
\end{aligned}$$

Since $C_c^\infty(\Sigma \setminus \{x_0\}) \subset W^{1,2}(\Sigma)$ is dense, it follows that u is an eigenfunction on Σ with eigenvalue λ. Thus we have that all accumulations of points of $(\mu_1(\Sigma \setminus B_{\varepsilon_l}))_l$ are contained in the spectrum of Σ. Moreover, we also have convergence of the eigenfunctions as claimed.

A simple argument using the ordering of the eigenvalues implies then that we actually have convergence $\mu_k(\Sigma \setminus B_\varepsilon) \to \lambda_k(\Sigma)$ (and not only subsequential convergence).

The assertion concerning the convergence of the eigenfunctions follows from the arguments above, combined with Lemma 4.19 and the maximum principle. □

Remark A.3. The same arguments as above give the same result if we remove a larger number of balls instead of just a single one.

Next we prove that the Dirichlet spectrum of $M_{\varepsilon,h}$ converges to the spectrum of the interval to which $M_{\varepsilon,h}$ collapses to.

Lemma A.4. *The Dirichlet spectrum of $M_{\varepsilon,h}$ converges locally uniformly in $h > 0$ to $\sigma_D^{\mathbb{Z}/2}([0, 2h])$. Moreover, any sequence of eigenfunctions u^{ε_l} for $\varepsilon_l \to 0$ with uniformly bounded eigenvalue consists of horizontal functions for ε_l sufficiently small.*

Proof. This is obvious since $M_{\varepsilon,h}$ is covered by a product, one of whose factors shrinks at rate ε. □

For the proof of Theorem 4.7, we need a result relating the spectra of quadratic forms on different Hilbert spaces in the presence of a so-called coupling map. This result generalizes the 'Main Lemma' in [26], since we have to take care of the additional parameter h.

Suppose we are given separable Hilbert spaces $\mathcal{H}_{\varepsilon,h}$ and $\mathcal{H}'_{\varepsilon,h}$, equipped with quadratic forms $q_{\varepsilon,h}$ and $q'_{\varepsilon,h}$, respectively. We assume that these quadratic forms are non-negative and closed. Then there is a unique self-adjoint operator associated to $q_{\varepsilon,h}$ which will henceforth be referred to as $Q_{\varepsilon,h}$, similarly we have $Q'_{\varepsilon,h}$ associated to $q'_{\varepsilon,h}$. Note, that the spectrum of $Q_{\varepsilon,h}$ and $Q_{\varepsilon,h'}$ is purely discrete.

The k-th eigenvalues of $q_{\varepsilon,h}$ and $q'_{\varepsilon,h}$ are henceforth denoted by $\lambda_k(\varepsilon, h)$ and $\lambda_k(\varepsilon, h)'$, respectively. Let $L_k(\varepsilon, h)$ denote the direct sum of the eigenspaces of $Q_{\varepsilon,h}$ corresponding to the first $(k + 1)$-eigenvalues. Finally, we denote by $\operatorname{dom}(q_{\varepsilon,h})$ the domain of $q_{\varepsilon,h}$.

Lemma A.5. *For each $\varepsilon, h > 0$ let $\Phi_{\varepsilon,h} \colon \operatorname{dom}(q_{\varepsilon,h}) \to \operatorname{dom}(q'_{\varepsilon,h})$ be a linear map such that all $u_\varepsilon \in L_k(\varepsilon, h)$ with $\sup_\varepsilon(\|u_\varepsilon\|_{\mathcal{H}_{\varepsilon,h}} + q_{\varepsilon,h}(u_\varepsilon)) < \infty$ satisfy the following two conditions.*

(1) $\lim_{\varepsilon\to 0}(\|\Phi_{\varepsilon,h}u_\varepsilon\|_{\mathcal{H}'_{\varepsilon,h}} - \|u_\varepsilon\|_{\mathcal{H}_{\varepsilon,h}}) = 0$, *locally uniformly in h,*

(2) $q'_{\varepsilon,h}(\Phi_{\varepsilon,h}u_\varepsilon) \le q_{\varepsilon,h}(u_\varepsilon)$.

Moreover, assume that $\lambda_k(\varepsilon, h) \le C$ for any $\varepsilon > 0$, fixed k, and $h \in [h_0, h_1] \subset (0, \infty)$. Then we have

$$\lambda'_k(\varepsilon) \le \lambda_k(\varepsilon) + o(1),$$

where the $o(1)$ term is locally uniform in k and $h \in (0,\infty)$.

Proof. We just repeat the proof from [26], where the result is proved without the additional parameter h.

Denote by $\phi^i_{\varepsilon,h}$ orthonormal bases of $\mathcal{H}_{\varepsilon,h}$ consisting of eigenfunctions of $Q_{\varepsilon,h}$. Given any $u \in L_k(\varepsilon,h)$, we can expand this as $u = \sum_{i=0}^k \alpha_i^{\varepsilon,h}\phi^i_{\varepsilon,h}$. Then, suppressing the indices ε and h whenever it is clear what they are, we get

$$\begin{aligned}\|u\|^2 - \|\Phi_{\varepsilon,h}u\|^2 &= \sum_{i,j=0}^k \alpha_i\alpha_j(\delta_{ij} - \langle \Phi_{\varepsilon,h}\phi^i_{\varepsilon,h}, \Phi_{\varepsilon,h}\phi^j_{\varepsilon,h}\rangle) \\ &\leq \delta_k'(\varepsilon,h)\sum_{j=1}^k |\alpha_j|^2 = \delta_k'(\varepsilon,h)\|u\|^2,\end{aligned}$$

where $\delta_k'(\varepsilon,h) = k\max_{i,j\leq k}|\delta_{ij} - \langle \Phi_{\varepsilon,h}\phi^i_{\varepsilon,h}, \Phi_{\varepsilon,h}\phi^j_{\varepsilon,h}\rangle|$. Assumption (1) combined with polarization implies that $\delta_k'(\varepsilon,h) \to 0$ locally uniformly in h. In particular, we find that

$$\|\Phi_{\varepsilon,h}u\|^2 \geq (1-\delta_k'(\varepsilon,h))\|u\|^2, \tag{A.6}$$

which also implies that $\Phi_{\varepsilon,h}$ is injective on $L_k(\varepsilon,h)$ for ε small enough. An easy computation then leads to

$$\frac{q'_{\varepsilon,h}(\Phi_{\varepsilon,h}u)}{\|\Phi_{\varepsilon,h}u\|^2} - \frac{q_{\varepsilon,h}(u)}{\|u\|^2} \leq \frac{C\delta_k'(\varepsilon,h)}{1-\delta_k'(\varepsilon,h)}.$$

Applying the min-max characterization of eigenvalues to the above estimate establishes the claim. □

Combining the previous lemmata we can now prove Theorem 4.7.

Proof of Theorem 4.7. Clearly, we have an upper bound

$$\lambda_k(\Sigma_{\varepsilon,h}) \leq \nu_k^h + o(1), \tag{A.7}$$

using extensions of Dirichlet eigenfunctions of $\Sigma\setminus B_\varepsilon$ and $M_{\varepsilon,h}$ as test functions, and the fact that the Dirichlet spectrum of $\Sigma\setminus B_\varepsilon$ converges to the spectrum of Σ (this is similar to, but easier than Lemma A.1 above). In particular, we see that the $o(1)$ term is independent of h (but of course might depend on k).

For the lower bound and the assertion concerning the behavior of the eigenfunctions we use Lemma A.5. Our first family of Hilbert spaces is $\mathcal{H}_{\varepsilon,h} = L^2(\Sigma_{\varepsilon,h})$ with quadratic forms $q_\varepsilon(u) = \int_{\Sigma_{\varepsilon,h}}|\nabla u|^2$. The second family is given by $\mathcal{H}'_{\varepsilon,h} = L^2(\Sigma\setminus B_\varepsilon)\oplus L^2(M_{\varepsilon,h})$, with quadratic forms $q'_{\varepsilon,h}(u) = \int_{\Sigma\setminus B_\varepsilon}|\nabla u_1|^2 + \int_{M_{\varepsilon,h}}|\nabla u_2|^2$. Here the first summand is subject to Neumann boundary conditions and the second one to Dirichlet boundary conditions. The coupling map $\Phi_{\varepsilon,h}\colon \mathcal{H}_{\varepsilon,h}\to\mathcal{H}'_{\varepsilon,h}$ is defined as follows

$$\Phi_{\varepsilon,h}(u) = u|_{\Sigma\setminus B_\varepsilon} \oplus (u|_{M_{\varepsilon,h}} - v_{\varepsilon,h}),$$

where $v_{\varepsilon,h} \in L^2(M_{\varepsilon,h})$ is the harmonic extension of $u|_{\partial M_{\varepsilon,h}}$ to $M_{\varepsilon,h}$. Next, we verify assumptions (1) and (2) from Lemma A.5.

To check the first condition, we need to show $v_{\varepsilon,h} \to 0$ in L^2, meaning that

$$\int_{M_{\varepsilon,h}} |v_{\varepsilon,h}|^2 \to 0.$$

It suffices to check this in the case that u is an eigenfunction. The general case follows since the harmonic extension operator is linear. If u_ε is an eigenfunction, this follows from the maximum principle once we can show that $|v_{\varepsilon,h}| \leq C|\log(\varepsilon)|$ on $\partial M_{\varepsilon,h}$. The proof of this inequality is similar to the proof of Lemma 4.19, so we omit some details. Let $A^1_\varepsilon = \{x \in M_{\varepsilon,h} \mid \operatorname{dist}(x, \partial M_{\varepsilon,h}) \leq \varepsilon\}$ and $A_\varepsilon = A^1_\varepsilon \cup (B_{3\varepsilon} \setminus B_\varepsilon) \subset \Sigma_{\varepsilon,h}$. We rescale the metric on A_ε by ε^{-1} and consider the function $w_\varepsilon = u_\varepsilon - (u_\varepsilon)_{A_\varepsilon}$, where $(u_\varepsilon)_{A_\varepsilon}$ denotes the mean value of u_ε on A_ε with respect to the rescaled metric. As in the proof of Lemma 4.19, we find that w_ε has gradient bounded in L^2 with respect to the rescaled metric. Since w_ε has mean value 0, it follows from the Poincaré inequality, that we get an L^2 bound on w_ε (again with respect to the rescaled metric). By applying the inhomogeneous De Giorgi–Nash–Moser estimates (see e.g. [32, Chapter 14.9]) to the rescaled equations, we find that $\sup_{p \in \partial B_2, q \in \partial B_1} |w_\varepsilon(p) - w_\varepsilon(q)| \leq C$. But this scales back to $\sup_{p \in \partial B_{2\varepsilon}, q \in \partial B_\varepsilon} |u_\varepsilon(p) - u_\varepsilon(q)| \leq C$. Since the L^∞ estimate from Lemma 4.19 holds for u_ε up to radius 2ε, this implies that $|u_\varepsilon| \leq C|\log(\varepsilon)|$ on $\partial M_{\varepsilon,h}$.

In order to prove that the second condition is satisfied, observe that $u|_{M_{\varepsilon,h}} - v_{\varepsilon,h} \in W^{1,2}_0(M_{\varepsilon,h})$. Consequently, we have

$$\int_{M_{\varepsilon,h}} \nabla(u - v_{\varepsilon,h}) \cdot \nabla v_{\varepsilon,h} = 0.$$

This is turn implies that

$$\int_{M_{\varepsilon,h}} |\nabla(u - v_{\varepsilon,h})|^2 = \int_{M_{\varepsilon,h}} |\nabla u|^2 - \int_{M_{\varepsilon,h}} |\nabla v_{\varepsilon,h}|^2 \leq \int_{M_{\varepsilon,h}} |\nabla u|^2$$

so that $q'_{\varepsilon,h}(\Phi_{\varepsilon,h} u) \leq q_{\varepsilon,h}(u)$.

Trivially, the convergence of the Neumann spectrum of $\Sigma \setminus B_\varepsilon$ to the spectrum of Σ is uniform in h. Therefore it follows from (A.7) and Lemma A.5 that the converge is locally uniform in h and k as claimed.

The assertion concerning the convergence of the eigenfunctions follows from the fact that the quantity $\delta'_k(\varepsilon)$ in the proof of Lemma A.5 converges to zero. Indeed, let u_l be a normalized sequence of eigenfunctions corresponding to the eigenvalue $\lambda_k(\Sigma_{\varepsilon_l,h})$. From the bound (A.6), we can infer that we can extract a subsequence, such that either $\|u_l\|_{L^2(\Sigma \setminus B_\varepsilon)}$ or $\|u_l - v_l\|_{L^2(M_{\varepsilon,h})}$ is bounded away from zero. In the first case, we find that the sequence of harmonic extension $\tilde{u}_l$ is bounded in $W^{1,2}(\Sigma)$ and by the arguments from the proof of Lemma A.1 we have subsequential convergence to a non-trivial eigenfunction on Σ in $L^2$and $C^\infty_{loc}(\Sigma \setminus \{x_0\})$. In the second case, we use that we know the Dirichlet spectrum and eigenfunctions of $M_{\varepsilon,h}$ explicitly. If one expands $u_{\varepsilon,h} - v_{\varepsilon,h}$ in the eigenfunctions, it is easily checked, that it becomes more and more horizontal, since the energy of the vertical eigenmodes explodes. Given this, the assertion follows easily by an argument similar to that of the first case. $\square$

Appendix B. Topology of surfaces

For convenience of the reader and the authors, we review here the notion of non-orientable genus.

Recall the classification of closed surfaces. The classes of closed orientable and non-orientable surfaces are both uniquely described up to diffeomorphism by the Euler characteristic. More precisely, any closed orientable surface is diffeomorphic to a surface of the form

$$\Sigma_\gamma = \mathbb{S}^2 \# \underbrace{T^2 \# \ldots \# T^2}_{\gamma-\text{times}},$$

and any closed non-orientable surface is diffeomorphic to a surface of the form

$$\Sigma^K_\delta = \mathbb{S}^2 \# \underbrace{\mathbb{R}P^2 \# \ldots \# \mathbb{R}P^2}_{\delta-\text{times}}.$$

These two families provide – up to diffeomorphism – a complete list of all orientable respectively non-orientable surfaces. We call γ the *genus* of Σ_γ and δ the *non-orientable genus* of Σ^K_δ. Note that with this convention, the real projective plane has non-orientable genus 1. We have $\chi(\Sigma_\gamma) = 2 - 2\gamma$ and $\chi(\Sigma^K_\delta) = 2 - \delta$, so that the orientation cover of Σ^K_δ is given by $\Sigma_{\delta-1}$. Some authors prefer to refer to the genus of the orientation cover as the non-orientable genus. As explained above these two definitions differ. Moreover, recall that we have the relation

$$\mathbb{S}^2 \# \underbrace{\mathbb{R}P^2 \# \ldots \# \mathbb{R}P^2}_{\delta-\text{times}} \cong \mathbb{S}^2 \# \underbrace{T^2 \# \ldots \# T^2}_{k-\text{times}} \# \underbrace{\mathbb{R}P^2 \# \ldots \# \mathbb{R}P^2}_{(\delta-2k)-\text{times}},$$

if $2k < \delta$.

References

[1] C. Anné, *Ecrasement d'anses et spectre du Laplacien*, Prépublications de l'Institut Fourier, **67**, 1986.

[2] C. Anné, *Spectre du laplacien et écrasement d'anses*, Ann. Sci. École Norm. Sup. (4) **20**, 1987, 271–280.

[3] C. Anné, *Fonctions propres sur des variétés avec des anses fines, application à la multiplicité*, Comm. Partial Differential Equations **12**, no.11, 1990, 1617-1630.

[4] M. Berger, *Sur les premières valeurs propres des variétés riemanniennes*, Compositio Math. **26**, 1973, 129–149.

[5] G. Besson, *Sur la multiplicité de la première valeur propre des surfaces riemanniennes*, Ann. Inst. Fourier (Grenoble) **30** (1), 1980, 109–128.

[6] P. Buser, *Geometry and Spectra of Compact Riemann Surfaces*, Birkhäuser Verlag, Basel - Boston - New-York, 1992.

[7] S.Y. Cheng, *Eigenfunctions and nodal sets*, Comment. Math. Helv. **51**, 1976, 43–55.

[8] B. Colbois, A. El Soufi, *Extremal eigenvalues of the Laplacian in a conformal class of metrics: the 'conformal spectrum'* , Ann. Global Anal. Geom. **24**, 2003, 337–349.

[9] A. El Soufi, H. Giacomini, J. Mustapha, *A unique extremal metric for the least eigenvalue of the Laplacian on the Klein bottle*, Duke Math. J. **135**, 2006, 181-202.

[10] A. Fraser, R. Schoen, *Sharp eigenvalue bounds and minimal surfaces in the ball*, Invent. Math. **203**, 2016, 823–890.

[11] J. Hersch, *Quatre propriétés isopérimétriques de membranes sphériques homogènes*, C.R. Acad. Sci. Paris Sér. A-B **270**, 1970, A1645–A1648.

[12] A. El Soufi, S. Ilias, *Riemannian manifolds admitting isometric immersions by their first eigenfunctions*, Pacific J. Math. **195**, 2000, 91–99.

[13] A. El Soufi, S. Ilias, *Extremal metrics for the first eigenvalue of the Laplacian in a conformal class*, Proc. Amer. Math. Soc. **131**, 2003, 1611–1618.

[14] A. Girouard, *Fundamental tone, concentration of density to points and conformal degeneration on surfaces*, Canad. J. Math. **61**, 2009, 548–565.

[15] D. Jakobson, M. Levitin, N. Nadirashvili, N. Nigam, I. Polterovich, *How large can the first eigenvalue be on a surface of genus two?*, Int. Math. Res. Not. bf 63, 2005, 3967–3985.

[16] T. Kato, *Perturbation theory for linear operators*, Classics in Mathematics, Reprint of the 1980 edition, 1995, xxii+619pp.

[17] M.A. Karpukhin, *Upper bounds for the first eigenvalue of the Laplacian on non-orientable surfaces*, Int. Math. Res. Not. IMRN **20**, 2016, 6200-6209.

[18] G. Kokarev, *Variational aspects of Laplace eigenvalues on Riemannian surfaces*, Adv. Math. **258**, 2014, 191–239.

[19] P. Li, S.T. Yau, *A new conformal invariant and its applications to the Willmore conjecture and the first eigenvalue of compact surfaces*, Invent.. Math. **69**, 1982, 269–291.

[20] V. Maz'ya, *Sobolev spaces with applications to elliptic partial differential equations*, Grundlehren der Mathematischen Wissenschaften **342**, 2nd edition, 2011

[21] S. Montiel, A. Ros, *Minimal immersions of surfaces by the first eigenfunctions and conformal area*, Invent. Math. **83**, 1996, no.1, 153–166

[22] D. Mumford, *A remark on Mahler's compactness theorem*, Proc. Am. Math. Soc. **28**, 1971, 289–294.

[23] N. Nadirashvili, *Berger's isoperimetric problem and minimal immersions of surfaces*, Geom. Func. Anal. **6**, 1996, 877-897.

[24] N. Nadirashvili, *Isoperimetric inequality for the second eigenvalue of a sphere*, J. Differential Geom. **61**, 2002, 335–340.

[25] R. Petrides, *Existence and regularity of maximal metrics for the first Laplace eigenvalue on surfaces*, Geom. Funct. Anal. **24**, 2014, 1336–1376.

[26] O. Post, *Periodic manifolds with spectral gaps*, J. Differential Equations **187**, 2003, 23–45.

[27] O. Post, *Periodic manifolds, spectral gaps, and eigenvalues in gaps*, Ph.D. Thesis, Technische Universität Braunschweig, 2000.

[28] J. Rauch, M. Taylor, *Potential and scattering theory on wildly perturbed domains* J. Funct. Anal. **18**, 1975, 27–59.

[29] J. Sacks, K. Uhlenbeck, *The existence of minimal immersions of* 2*-spheres*, Ann. of Math. **113**, 1981, no.1, 1–24.

[30] S. Salamon, *Harmonic and holomorphic maps*, In: *G*eometry Seminar Luigi Bianchi II1984. Lecture Notes in Mathematics, Vol. 1164. Springer, Berlin (1985), pp. 161-224.

[31] M.E. Taylor, *Partial differential equations I. Basic Theory*, Applied Mathematical Sciences **115**, 2nd edition, 2011.

[32] M.E. Taylor, *Partial differential equations III. Nonlinear equations*, Applied Mathematical Sciences **117**, 2nd edition, 2011.

[33] P.C. Yang, S.T. Yau, *Eigenvalues of the Laplacian of compact Riemannian surfaces and minimal submanifolds*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **7**, 1980, no.1, 53–63.

[34] M. Zhu, *Harmonic maps form degenerating Riemann surfaces*, Math. Z. **262**, 2010, no.1, 63–85.

HM: Max Planck Institute for Mathematics, Vivatsgasse 7, 53111 Bonn.

E-mail address: hematt@mpim-bonn.mpg.de

AS: Max Planck Institute for Mathematics, Vivatsgasse 7, 53111 Bonn.

E-mail address: siffert@mpim-bonn.mpg.de