On zero-dimensional subschemes of complete intersections

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Introduction

For zero-dimensional schemes in $\mathbf{P}^2$ there are many interesting results such that

- The Hilbert functions of sets of points in uniform position is of decreasing type.
- Cayley-Bacharach theorem on the configuration of points.
- Dubreil theorem on bounds for the minimal number of generators of the defining ideals.

For all these results the Hilbert-Burch structure theorem usually plays an important role. It is well–known that there is no similar structure theorem for zero-dimensional schemes in higher projective spaces. But one may still raise the question whether the above mentioned results can be extended to zero-dimensional schemes in $\mathbf{P}^n$, $n > 2$.

In this paper we will present a method to do that. This method is based on a well-known property of graded artinian Gorenstein rings which gives a sort of duality between an ideal and its annihilator. This idea already appeared in [DGO]. Using this method we will study properties of a zero-dimensional scheme X in $\mathbf{P}^n$ by means of the degree of the hypersurfaces of a complete intersection passing through X. Algebraically, we have to study properties of ideals in a artinian graded complete intersection ring (which is Gorenstein).

The main result of this paper (Theorem 2.1) gives strong information on the bahaviour of the Hilbert function $h_X(t)$ of X. More precisely, we can control the h-vector of $h_X(t)$ for the last part. As applications we present large classes of zero-dimesional schemes X in $\mathbf{P}^n$ for which the h-vector of $h_X(t)$ is of unimodal type or decreasing type. Such behaviours of the Hilbert functions are of interest for different reasons [Ha],

The first author is partially supported by the National Basic Research Program, Vietnam, and, during the completion of this paper, by the Max-Planck-Institut für Mathematik, Bonn, and the second author by M.P.I, Italy.

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[MR], [St], [Hi]. These classes include for example all zero-dimesional schemes in $\mathbf{P}^2$ or those in $\mathbf{P}^3$ which lie on irreducible quadrics or cubics.

Moreover, we also obtain results on the superabundance of linear systems of hypersurfaces passing through X and on upper bounds for the minimal number of generators of the defining ideal of X. From these results we can generalize Cayley–Bacharach theorem (Theorem 3.2) and Dubreil theorem (Theorem 4.2) to zero-dimensional schemes in $\mathbf{P}^n$, $n > 2$.

1. Basic facts

Let k be any infinite field. By a graded ring A we always mean a standard graded k-algebra of finite type, that is, A is the quotient of a polynomial ring over k by an homogeneous ideal. We denote by

$$h_M(t) := dim_k M_t$$

the *Hilbert function* of any finitely generated graded A-module M. The generating function of this numerical function is the formal power series $P_M(z) := \sum_{t\geq 0} h_M(t)z^t$. In the case $M = A$, as a consequence of the Hilbert-Serre theorem, we can write $P_A(z) = H_A(z)/(1-z)^d$, where $H_A(z) \in \mathbf{Z}[z]$ is a polynomial with integer coefficients such that $H_A(1) \neq 0$. The natural number $H_A(1)$ is the *multiplicity* $e(A)$ of A while the degree of $H_A(z)$ is the *socle degree* of A. We will write $s(A)$ to indicate the socle degree of A. From the definition we get that the socle degree of a graded artinian ring A is s if $h_A(s) > 0$ and $h_A(s+1) = 0$. For example, the socle degree of an artinian complete intersection $k[X_1, \dots, X_n]/(F_1, \dots, F_n)$ is $\sum_{i=1}^{n} deg(F_i) - n$.

We shall often use the fact (see e.g. [St]) that if A is a graded artinian Gorenstein ring, then the Hilbert function of A is symmetric, which means that if we let $s := s(A)$ then

$$h_A(t) = h_A(s-t)$$

for every $t = 0, \dots, s$.

The basic result for our investigation is the following well-known property of the Hilbert function of an ideal in a graded artinian Gorenstein ring.

Theorem 1.1. *Let A be a graded artinian Gorenstein ring and let I be an homogeneous ideal of A. Then for every $t = 0, \dots, s(A)$*

$$h_A(t) = h_{A/I}(t) + h_{A/(0:I)}(s(A)-t).$$

A proof for Theorem 1.1 can be found for example in [DGO].

The following consequence of Theorem 1.1 is more or less a well-known result in the ideal theory of Gorenstein rings.

For an homogeneous ideal I of A let $v(I)$ denote the minimal number of generators of I. Let $\tau(A)$ denote the *Cohen-Macaulay type* of A. Note that if A is an artinian graded ring, $\tau(A)$ is the dimension of the socle $0 : A_1$ of A.

Corollary 1.2. *Let A be a graded artinian Gorenstein ring and I an homogeneous ideal of A. Then for every $t = 0, \dots, s(A)$*

$$h_{I/A_1 I}(t) = h_{(0:A_1 I)/(0:I)}(s(A) - t).$$

In particular

$$v(I) = \tau(A/(0:I))$$

Proof. By Theorem 1.1 we have $h_A(t) = h_{A/I}(t) + h_{A/(0:I)}(s(A) - t)$, hence $H_I(t) = h_{A/(0:I)}(s(A) - t)$. In the same way we get $h_{A_1 I}(t) = h_{A/(0:A_1 I)}(s(A) - t)$. Hence

$$h_{I/A_1 I}(t) = h_{A/(0:I)}(s(A) - t) - h_{A/(0:A_1 I)}(s(A) - t).$$

This proves the first assertion. Since $v(I) = dim_k(I/A_1 I)$ and

$$\tau(A/(0:I)) = dim_k((0:I):A_1)/(0:I)) = dim_k((0:A_1 I)/(0:I)),$$

the second assertion follows as well.

2. Hilbert Function

The main result of this paper is the following theorem which gives strong information on the Hilbert function of the graded ring R/I in terms of the degrees of the elements of a regular sequence in I.

Theorem 2.1. *Let $R = k[X_1, \dots, X_n]$ and I be a zero-dimensional homogeneous ideals of R such that I contains a regular sequence $F_1, \dots, F_n$ of forms of degrees $d_1 \leq \dots \leq d_n$. Set $d := \sum_{i=1}^n d_i - n$.*

(a) If i is an integer, $1 \leq i \leq n$, then

$$h_{R/I}(t) \geq h_{R/I}(t+1) + n - i$$

for $d - d_i + 1 \leq t < s(R/I)$. Moreover,

$$h_{R/I}(d - d_i) \geq h_{R/I}(d - d_i + 1) + n - i$$

if $d_i \leq d_1 + \dots + d_{i-1} - i + 1$.

(b) If i is an integer, $1 \leq i \leq n-1$, such that $(F_1, \dots, F_{i-1})$ is a prime ideal, then

$$h_{R/I}(t) \geq \min\{n h_{R/I}(t+1),\ h_{R/I}(t+1) + 2n - i - 2\}$$

for $d - d_i + 1 \leq t < s(R/I)$.

The most interesting cases of Theorem 2.1 are for $i = n,\ n-1$. In these cases, we have the following statements on the behaviour of the Hilbert function of R/I near to $s(R/I)$.

Corollary 2.2. *Let I be a zero-dimensional homogeneous ideal as in Theorem 1.1. Then*

(a) $h_{R/I}(t)$ is not increasing for $t \geq d - d_n + 1$. Moreover,

$$h_{R/I}(d - d_n) \geq h_{R/I}(d - d_n + 1)$$

if and only if there is a form of degree $\leq d_1 + \cdots + d_{n-1} - n + 1$ in I which does not belong to $(F_1, \ldots, F_{n-1})$.

(b) $h_{R/I}(t)$ is decreasing for $d - d_{n-1} + 1 \leq t < s(R/I)$. Moreover,

$$h_{R/I}(d - d_{n-1}) > h_{R/I}(d - d_{n-1} + 1)$$

if $d_{n-1} \leq d_1 + \cdots + d_{n-2} - n + 2$.

(c) If $(F_1, \ldots, F_{n-2})$ is a prime ideal, then

$$h_{R/I}(t) \geq h_{R/I}(t+1) + n - 1$$

for $d - d_{n-1} + 1 \leq t < s(R/I)$.

This corollary covers some interesting results on the postulation of zero-dimesional schemes.

Let X be a zero-dimensional scheme in $\mathbf{P}^n = \mathbf{P}^n(k)$, where k is an algebraically closed field. We denote by $\Delta_X(t)$ the first difference of the Hilbert function of X, which is defined as follows:

$$\Delta_X(t) = \begin{cases} 1 & \text{if} \quad t = 0, \\ h_X(t) - h_X(t-1) & \text{if} \quad t > 0. \end{cases}$$

The non-zero values of $\Delta_X(t)$ form the so called h-vector of $h_X(t)$.

Following [St] and [MG] we say that the Hilbert function $h_X(t)$ of X is of *unimodal type* if $\Delta_X(t)$ is non-decreasing up to the maximum of $\Delta_X(t)$ and then non-increasing (i.e. $\Delta_X(t)$ has only a local maximum) and that $h_X(t)$ is of *decreasing type* if moreover $\Delta_X(t)$ is (strictly) decreasing until it reach zero once it starts to decrease.

The first notion has been studied mainly from the combinatorial point of view (see e.g. [St], [Hi]). It was conjectured that if the homogeneous coordinate ring of X is Gorenstein, then X is of unimodal type [Hi, Conjecture 1.5]. The latter notion plays an important role in the characterization of zero-dimensional schemes in $\mathbf{P}^2$ (e.g. points in uniform psoition) which arise as hyperplane sections of smooth curves in $\mathbf{P}^3$. In fact, Harris [Ha] showed that the Hilbert function of such a zero-dimensional scheme is of decreasing type. By [GP] and [MR], this condition exactly characterizes the Hilbert functions of hyperplane sections of reduced irreducible curves (see also [GM]). Similar results on the postulation of zero-dimensional schemes on smooth quadrics in $\mathbf{P}^3$ have been recently discovered by Raciti, Paxia and Ragusa [R1], [R2], [PRR]. We shall see that for many zero-dimensional schemes this behaviour of the Hilbert functions are easy consequences of Corollary 2.2.

In the sequel we will denote by $a_1(X) \le \ldots \le a_r(X)$ the degrees of the elements of a homogeneous minimal basic of the defining ideal of X arranged in non-decreasing order.

Theorem 2.3. *Let X be a non-degenerate zero-dimesional scheme in $\mathbf{P}^n$ and $a_i = a_i(X)$. Assume that X lies on a complete intersection of $n-1$ hypersurfaces of degree $a_1, \ldots, a_{n-1}$ and $a_n \ge a_1 + \cdots + a_{n-1} - n$. Then $h_X(t)$ is of unimodal type.*

Proof. Let $I(X)$ be the defining ideal of X in $S = k[X_0, \ldots, X_n]$. Without restriction we may assume that X_0 is a non-zerodivisor of $I(X)$. Let I denote the artinian reduction $I(X) + (X_0)/(X_0)$ of $I(X)$ in $R = k[X_1, \ldots, X_n]$. Then $h_{R/I}(t) = \Delta_X(t)$. Moreover, there exists in I a regular sequence $F_1, \ldots, F_n$ with $\deg F_i = a_i$, $i = 1, \ldots, n-1$, and $\deg F_n \ge a_n$. It is clear that for $t \le a_n - 1$, $h_{R/I}(t)$ equals the Hilbert function of $B = R/(F_1, \ldots, F_{n-1})$. Since B is a one-dimensional Cohen-Macaulay ring, the latter function is increasing until it reachs $t = a_1 + \cdots + a_{n-1} - n + 1$ when it remains constant. Put $d_i = \deg F_i$ and $d = \sum_{i=1}^n d_i - n$. Note that $d - d_n = a_1 + \cdots + a_{n-1} - n$. Then $h_{R/I}(t)$ is increasing for $t \le d - d_n$ if $a_n \ge a_1 + \cdots a_{n-1} - n + 1$ or for $t \le d - d_n - 1$ if $a_n = a_1 + \cdots a_{n-1} - n$. On the other hand, the statement (a) of Corollary 2.2 say that $h_{R/I}(t)$ is non-increasing afterwards. Hence $h_X(t)$ is of unimodal type.

In Theorem 2.3 we can replace the condition that X lies on a complete intersection of $n-1$ hypersurfaces of degree $a_1, \ldots, a_{n-1}$ by the stronger condition that X lies on an *irreducible* complete intersection C of $n-2$ hypersurfaces of degree $a_1, \ldots, a_{n-2}$. Moreover, the condition $a_n \ge a_1 + \cdots + a_{n-1} - n$ is satisfied in the following cases:

$$n = 2;$$

$$n = 3,\ a_1 \le 3;$$

$$n = 4,\ a_1 = a_2 = 2.$$

Now we will present some cases where $h_X(t)$ is of decreasing type. Let us assume that X lies on a complete intersection of n hypersurfaces of degree $a_1, \ldots, a_n$. By the proof of Theorem 2.3, $\Delta_X(t)$ is increasing up to $t = \min\{a_1 + \cdots + a_{n-1} - n + 1,\ a_n - 1\}$ when it eventually remains constant until $t = a_n - 1$. Now using the statement (b) of Corollary 2.2 we have that $\Delta_X(t)$ is decreasing for for $t \ge a_n$ if $d - a_{n-1} + 1 \le a_n$ or $d - a_{n-1} = a_n$ and $a_{n-1} \le a_1 + \cdots + a_{n-2} - n + 2$, where $d = \sum_{i=1}^n a_i - n$. It is easy to check that these conditions are satisfied only for the following cases:

$$n = 2;$$

$$n = 3,\ a_1 = 2;$$

$$n = 4,\ a_1 = a_2 = a_3 = 2.$$

Combining the above observations with Theorem 2.3 and Corollary 2.2 (c) we obtain the following results on the postulation of zero-dimensional schemes in $\mathbf{P}^n$, $n = 2, 3, 4$.

Corollary 2.4. (cf. [H]) *Let X be a zero-dimensional scheme in $\mathbf{P}^2$. Then*

(a) $h_X(t)$ is of unimodal type.

(b) $h_X(t)$ is of decreasing type if X lies on a complete intersection C of two curves of degrees $a \leq b$ such that there is no curve of degree $< b$ passing through X but not C.

The assumption of Corollary 2.4 (b) is satisfied if X arises as an hyperplane section of an reduced irreducible curve of $\mathbf{P}^3$. According to [Sau] and [GM] (see also [HTV]) any reduced irreducible curve V in $\mathbf{P}^3$ lies on a complete intersection C of two surfaces of degrees $a \leq b$ such that there is no surface of degree $< b$ passing through V but not C.

Corollary 2.5. (cf. [R2], [PRR]) *Let X be a non-degenerate zero-dimensional scheme in $\mathbf{P}^3$. Then*

(a) $h_X(t)$ is of unimodal type if X lies on an irreducible quadric or cubic.

(b) $h_X(t)$ is of decreasing type if X lies on a complete intersection C of a quadric with two surfaces of degree $a \leq b$ such that there is no surface of degree $< b$ passing through X but not C. Moreover, if the quadric is irreducible, then $\Delta_X(t) \geq \Delta_X(t)+2$ for $t \geq b$.

By the above remark, the assumption of Corollary 2.5 (b) is satisfied if X is the intersection of a quadric with a reduced irreducible curve in $\mathbf{P}^3$.

Corollary 2.6. *Let X be a non-degenerate zero-dimensional scheme in $\mathbf{P}^4$.*

(a) $h_X(t)$ is of unimodal type if X lies on an irreducible complete intersection of two quadrics.

(b) $h_X(t)$ is of decreasing type if X is lies on a complete intersection C of three quadrics and an hypersurface of degree $a \geq 2$ and there is no hypersurface of degree $< a$ passing through X but not C. Moreover, $\Delta_X(t) \geq \Delta_X(t) + 3$ for $t > a$.

For Corollary 2.6 (b) we can show that C lies on an irreducible complete intersection of two quadrics by using a kind of Bertini theorem (see e.g. [T, Hauptsatz]).

Now we want to show that from Theorem 2.1 we easily get a strong version of a classical result which, accordingly to [D], was first noted by Castelnuovo and reborn a number of times since: see [Hn], [GP], [Hs], [DGM] and especially [D] where a new and elementary proof is presented. This classical result is a special case, namely the case $n = 2$, of the following corollary.

Corollary 2.7. *Let $R = k[X_1, \ldots, X_n]$ and V be a proper subspace of R_s whose elements generate an ideal of height n. If t is an integer such that $t \geq ns - 2s - n + 1$ and $codim(R_tV) > 0$, then*

$$codim_k(R_{t+1}V) < codim_k(R_tV).$$

Proof. Let I be the ideal generated by a vector base of V. Then $I_{t+s+1} = R_{t+1}V$ and $I_{t+s} = R_tV$. Hence $codim_k(R_{t+1}V) = h_{R/I}(t+s+1)$ and $codim_k(R_tV) = h_{R/I}(t+s)$. The conclusion now follows from Corollary 2.1 (a) because $sn-s-n+1 \le t+s \le s(R/I)$, where the last inequality follows from the assumption that $codim_k(R_tV) > 0$.

Another easy consequence of Theorem 2.1 is the following result which has been proved in [DGM, Theorem 2.4] by using an anusual genericity argument. Here we write Δh_A for the first difference of the Hilbert function of a graded ring A, which is defined as follows:

$$\Delta h_A(i) = \begin{cases} 1 & \text{if } \; i = 0, \\ h_A(i) - h_A(i-1) & \text{if } \; i > 0. \end{cases}$$

Corollary 2.8. *Let $R = k[X,Y]$ and $J \subseteq I$ be homogeneous ideals of R such that $J = (F,G)$ where F,G is a regular sequence of degree $a \le b$. Then*

$$\Delta h_{R/J}(i) \ge \Delta h_{R/I}(i)$$

for every $i = 0,\dots,s(R/I)+1$.

Proof. It is well known that

$$h_{R/J}(t+1) - h_{R/J}(t) = \begin{cases} 1 & \text{if } \; 0 \le t \le a-2 \\ 0 & \text{if } \; a-1 \le t \le b-2 \\ -1 & \text{if } \; b-1 \le t \le a+b-2. \end{cases}$$

Since $h_{R/I}(n+1) - h_{R/I}(n) \le 1$ for every $n \ge 0$, the conclusion follows immediately in the interval $0 \le t \le a-2$, while in the interval $a-1 \le t \le s(R/I)$, it is a trivial consequence of the Corollary 2.2 (c).

The following example shows that this result does not hold if R has dimension > 2.

Example. Let $R = k[X,Y,Z]$, $J = (X^3, Y^4, Z^5)$ and $I = (X^3, X^2Y, XZ^3, Y^4, Z^5)$. Then $s(R/I) = 7$ but $\Delta h_{R/J}(7) = -3$ while $\Delta h_{R/I}(7) = -2$.

For the proof of Theorem 2.1 we need the following lemma.

Lemma 2.9. *Let A be a graded ring of depth $g \ge 1$ and embedding dimension n. If V is a subspace of A_t of dimension $r > 0$, then*

$$dim_k(A_1V) \ge r + g - 1.$$

Further, if $g \ge 2$ and V contains an element which is a non zero divisor in A, then

$$dim_k(A_1V) \ge \min\{rn, r+n+g-3\}.$$

Proof. Let $F_1, \dots, F_r$ be a vector base of V and let $x_1, \dots, x_g$ be a regular sequence of linear forms in A. Set $x := (x_1, \dots, x_g)$.

Claim 1. The vector space xV generated by the rg vectors $\{x_iF_j\}$, $i = 1, \dots, g$ and $j = 1, \dots, r$, cannot be generated by a set of vectors $\{x_iF_j\}$ involving only $x_1, \dots, x_m$ with $m < g$.

Assume the contrary. Then we have $x_gF_j \in xV \subset (x_1, \dots, x_m)$, hence $F_j \in (x_1, \dots, x_m)$ for every j. Thus $xV \subset (x_1, \dots, x_m)^2$, and, for every j, we get $x_gF_j \in xV \subset (x_1, \dots, x_m)^2$. Since $x_1, \dots, x_g$ is a regular sequence, this implies that for every j, $F_j \in (x_1, \dots, x_m)^2$ so that $xV \subset (x_1, \dots, x_m)^3$. Going on in this way, we get $xV \subset (x_1, \dots, x_m)^{t+2}$, which is impossible. This proves Claim 1.

Now it is clear that $x_1F_1, x_1F_2, \dots, x_1F_r$ are vectors in xV which are linearly independent. We can use $g-1$ times Claim 1 to find vectors $x_2F_{i_2}, \dots, x_gF_{i_g}$ in xV such that

$$x_1F_1, x_1F_2, \dots, x_1F_r, x_2F_{i_2}, \dots, x_gF_{i_g}$$

are linearly independent. This proves the first part of Lemma 2.9.

As for the second assertion, let $x_1, \dots, x_n$ be the linear forms which are a k-basis of A_1. We may assume that $F_1, x_1, \dots, x_{g-1}$ form a regular sequence in A.

Claim 2. $F_1x_1, F_1x_2, \dots, F_1x_n, F_2x_1, \dots, F_rx_1$ are linearly independent vectors in A_1V.

Let

$$\sum_{i=1}^{n} \lambda_i F_1 x_i + \sum_{i=2}^{r} \mu_i F_i x_1 = 0$$

for some $\lambda_i, \mu_i \in k$. Then we get

$$F_1 \left(\sum_{i=1}^{n} \lambda_i x_i \right) + x_1 \left(\sum_{i=2}^{r} \mu_i F_i \right) = 0.$$

This implies $\sum_{i=2}^{r} \mu_i F_i \in (F_1)$. Since $F_1, \dots, F_r$ share the same degree and are linearly independent, we get $\mu_2, \dots, \mu_r = 0$. Since F_1 is a non zero divisor in A, we get $\sum_{i=1}^{n} \lambda_i x_i = 0$ and this implies $\lambda_1 = \dots = \lambda_n = 0$ by the linear independence of $x_1, \dots, x_n$. This proves Claim 2.

Claim 2 implies that we have $n+r-1$ vectors in A_1V which are linearly independent. Hence, if $r = 1$, then $n + r - 1 = rn$ and the conclusion follows.

Claim 3. Let $r \geq 2$ and $x := (x_1, \dots, x_{g-1})$. Then xV cannot be generated by

$$F_1x_1, F_1x_2, \dots, F_1x_n, F_2x_1, \dots, F_rx_1$$

plus a set of vectors $\{x_iF_j\}$ involving only $x_1, \dots, x_m$ with $m < g-1$.

Assume the contrary. Then we have

$$x_{g-1}F_j \in xV \subset (F_1, x_1, \dots, x_m),$$

hence $F_j \in (F_1, x_1, \dots, x_m)$ for every j. Thus $xV \subset (F_1) + (x_1, \dots, x_m)^2$ and, for every j, we get

$$x_{g-1}F_j \in xV \subset (F_1) + (x_1, \dots, x_m)^2.$$

Since $F_1, x_1, \dots, x_m$ is a regular sequence, this implies that for every j, $F_j \in (F_1) + (x_1, \dots, x_m)^2$ so that $xV \subset (F_1) + (x_1, \dots, x_m)^3$. Going on in this way we get $xV \subset (F_1) + (x_1, \dots, x_m)^{t+2}$ so that $xV \subset (F_1)$. This implies for example $x_1F_2 = F_1L$ for some linear form L in A. Hence $F_2 = \alpha F_1$, a contradiction. This proves Claim 3.

We can use $g-2$ times Claim 3 to find vectors $x_2F_{i_2}, \dots, x_{g-1}F_{i_{g-1}}$ in xV such that

$$F_1x_1, F_1x_2, \dots, F_1x_n, F_2x_1, \dots, F_rx_1, x_2F_{i_2}, \dots, x_{g-1}F_{i_{g-1}}$$

are linearly independent. This proves the second part of Lemma 2.9.

Proof of Theorem 2.1. Let $A := R/J$ and $\bar{I} := I/J$, where $J = (F_1, \dots, F_n)$. Using Theorem 1.1 we get

$$\begin{aligned} h_{R/I}(t) = h_{A/\bar{I}}(t) &= H_A(t) - h_{A/0:\bar{I}}(d-t) \\ &= H_A(t) - H_A(d-t) + h_{0:\bar{I}}(d-t) = h_{0:\bar{I}}(d-t) \end{aligned}$$

for every $t \le d$. For convenience we set $m = d - t$.

To prove the first assertion of Theorem 2.1 (a) we need to verify that

$$h_{0:\bar{I}}(m) \ge h_{0:\bar{I}}(m-1) + n - i$$

for $d - s(R/I) < m \le d_i - 1$. Set $B = R/(F_1, \dots, F_{i-1})$ and $Q = (J : I)/(F_1, \dots, F_{i-1})$. For $m \le d_i - 1$, we have $J_m = (F_1, \dots, F_{i-1})_m$ so that we may identify $0 : \bar{I}_m = (J : I/J)_m$ with the subspace Q_m of B_m. For $m > d - s(R/I)$, we have $t < s(R/I)$ so that

$$dim_k Q_{m-1} = h_{0:\bar{I}}(m-1) = h_{R/I}(t+1) > 0.$$

Therefore we can apply the first part of Lemma 2.9 to Q_{m-1}. Note that B has depth $n - i + 1 > 0$. Then we get

$$\begin{aligned} h_{0:\bar{I}}(m) = dim_k Q_m &\ge dim_k Q_{m-1}B_1 \\ &\ge dim_k Q_{m-1} + n - i = h_{0:\bar{I}}(m-1) + n - i, \end{aligned}$$

which proves the first assertion of Theorem 2.1 (a). The same arguments also shows that the second part of Lemma 2.9 gives Theorem 2.1 (b).

It remains to prove the second assertion of Theorem 2.2 (a). For this we need to show that

$$h_{0:\bar{I}}(d_i) \geq h_{0:\bar{I}}(d_i - 1) + n - i$$

if $d_i \leq d_1 + \cdots + d_{i-1} - i + 1$. We have seen that

$$h_{0:\bar{I}}(d_i - 1) = h_Q(d_i - 1).$$

Since $J_{d_i} = (F_1, \ldots, F_i)_{d_i}$, it is easy to check that

$$h_{0:\bar{I}}(d_i) = \dim_k(J : I/J)_{d_i} = h_Q(d_i) - 1.$$

Without restriction we may assume that $X_1, \ldots, X_{n-i+1}$ forms a regular sequence in B. Then $\bar{B} = B/(X_1, \ldots, X_{n-i+1})B$ is a graded artinian Gorenstein ring with $s(\bar{B}) = d_1 + \cdots + d_{i-1} - i + 1$. If $d_i \leq d_1 + \cdots + d_{i-1} - i + 1$, we may further assume that $Q_{d_i - 1} \not\subseteq (X_1, \ldots, X_{n-i+1})B$. Since $s(\bar{B}) > d_i - 1$, the image of Q_{d_i-1} in $\bar{B}$ is not contained in the socle of $\bar{B}$. So we have $Q_{d_i-1}B_1 \not\subseteq (X_1, \ldots, X_{n-i+1})B$. By the proof for the first part of Lemma 2.9 this implies that $\dim_k Q_{d_i-1}B_1 > \dim_k Q_{d_i-1} + n - i$. Since $Q_{d_i-1}B_1 \subseteq Q_{d_i}$, we get

$$h_Q(d_i) > h_Q(d_i - 1) + n - i$$

so that $h_{0:\bar{I}}(d_i) + 1 > h_{0:\bar{I}}(d_i - 1) + n - i$, as required. The proof of Theorem 2.1 is now complete.

Proof of Corollary 2.2. All statements of Corollary 2.2 follow from Theorem 2.1 except the second assertion of Corollary 2.2 (a). To prove this we follow the proof of the second assertion of Theorem 2.1 (a). We need to show that

$$h_Q(d_n) > h_Q(d_n - 1)$$

if and only if there is a form of degree $\leq d_1 + \cdots + d_{n-1} - n + 1$ in I which does not belong to $(F_1, \ldots, F_{n-1})$, where Q is now the ideal $(J : I)/(F_1, \ldots, F_{n-1})$ of the ring $B = R/(F_1, \ldots, F_{n-1})$. We have seen that $h_Q(d_n) > h_Q(d_n - 1)$ if $d_n \leq d_1 + \cdots + d_{n-1} - n + 1$. In this case, F_n is a form in I which does not belong to $(F_1, \ldots, F_{n-1})$. So we may assume from the beginning that $d_n > d_1 + \cdots + d_{n-1} - n + 1$.

If $h_Q(d_n) > h_Q(d_n - 1)$, then we have $Q_{d_n-1} \neq B_{d_n-1}$ so that $h_{B/Q}(d_n - 1) > 0$. Since $R/J : I \simeq B/Q$, we get $h_{R/J:I}(d_n - 1) > 0$. By Theorem 1.1 this implies

$$h_{R/J}(d - d_n + 1) > h_{R/I}(d - d_n + 1).$$

Therefore there exists a form in I_{d-d_n+1} which does not belong to J_{d-d_n+1}. Since $d - d_n + 1 = d_1 + \cdots + d_{n-1} - n + 1 < d_n$, we have $J_{d-d_n+1} = (F_1, \ldots, F_{n-1})$. So we can find a form of degree $d_1 + \cdots + d_{n-1} - n + 1$ in I which does not belong to $(F_1, \dots, F_{n-1})$.

Conversely, if there is a form of degree $\leq d_1 + \cdots + d_{n-1} - n + 1$ in I which does not belong to $(F_1, \dots, F_{n-1})$, then we can always find a form of degree $d - d_n + 1 = d_1 + \cdots + d_{n-1} - n + 1$ in I which does not belong to $(F_1, \dots, F_{n-1})$. Similarly as above, we can show that this implies $h_Q(d_n) > h_Q(d_n - 1)$, as required.

3. Superabundance

Let X be a zero-dimensional scheme in $\mathbf{P}^n$. We know that $h_X(t)$ is strictly increasing until it reaches the degree $\deg(X)$ of X, at which it stabilizes. The number $\omega_X(t) := \deg(X) - h_X(t)$ is called the *superabundance* of the linear system of hypersurfaces of degree t passing through X. Also $\omega_X(t) = h^1\Im_X(t)$, where $\Im_X$ is the ideal sheaf of X.

The next result gives a formula relating the Hilbert function of a zero-dimensional complete intersection X in $\mathbf{P}^n$, that of a proper subschme Y of X, and the superabundance of the residual scheme Z of Y on X. This result extends to arbitrary n a result proved by J.Harris in the case $n = 2$ (see [H2, Lemma] and [C, 3.2]). Recall that if J is the defining ideal of X and I that of Y in $S := k[X_0, \dots, X_n]$, then the *residual scheme* Z of Y on X is defined by the ideal $J : I$.

Proposition 3.1. *Let X be a zero-dimensional scheme in $\mathbf{P}^n$ which is the complete intersection of n hypersurfaces of degree $d_1, \dots, d_n$ and $d := \sum_{i=1}^n d_i - n$. Let Y be a proper subscheme of X and Z the residual scheme of Y on X. Then*

$$\omega_Z(t) = h_X(d - t - 1) - h_Y(d - t - 1)$$

for very integer t such that $0 \leq t \leq d - 1$.

Proof. It is well known that under our assumptions the defining ideals J, I and $J : I$ of X, Y, and Z are perfect ideals of codimension n such that

$$e(S/J) = e(S/I) + e(S/(J : I)).$$

We may assume that X_0 is a non-zerodivisor modulo I, J and $J : I$. Let $\bar{}$ denote reduction modulo X_0 and let $R = k[X_1, \dots, X_n]$. By Theorem 1.1 we get

$$h_{R/\bar{J}}(i) = h_{R/\bar{I}}(i) + h_{R/(\bar{J}:\bar{I})}(d - i)$$

for every $0 \leq i \leq d$. On the other hand, we have

$$e(R/\overline{J : I}) = e(S/(J : I)) = e(S/J) - e(S/I) = e(R/\bar{J}) - e(R/\bar{I}) = e(R/(\bar{J} : \bar{I})).$$

Since $\overline{J:I} \subseteq \bar{J}:\bar{I}$, we get $\overline{J:I} = \bar{J}:\bar{I}$. This implies for every t, $0 \le t \le d-1$,

$$\begin{aligned} h_{S/J}(d-t-1) - h_{S/I}(d-t-1) &= \sum_{i=0}^{d-t-1} \left[h_{R/\bar{J}}(i) - h_{R/\bar{I}}(i)\right] \\ &= \sum_{i=0}^{d-t-1} h_{R/\overline{J:I}}(d-i) \\ &= \sum_{j=t+1}^{d} h_{R/\overline{J:I}}(j) = h_{S/(J:I)}(d) - h_{S/(J:I)}(t). \end{aligned}$$

Since $\bar{J} \subseteq \overline{J:I}$, we have $0 \le h_{R/\overline{J:I}}(s) \le h_{R/\bar{J}}(s) = 0$ for every $s \ge d+1$. This implies $h_{S/(J:I)}(d) = e(S/(J:I))$ and the conclusion follows.

As a trivial application of Proposition 3.1 we get the following very general version of the classical Cayley-Bacharach theorem, which says that every curve of degree $a+b-3$ in $\mathbf{P}^2$ passing through $ab-1$ of ab points of an intersection of two curves of degree a and b has to pass the remaining point of the intersection. The possibility of extending the classical Caley-Bacharach theorem to zero-dimensional subschemes of $\mathbf{P}^n$ has been already observed by Davis, Geramita, and Orecchia [DGO].

Theorem 3.2. *Let X be a closed zero-dimensional subscheme of $\mathbf{P}^n$ which is the complete intersection of n hypersurfaces of degrees $d_1, \ldots, d_n$. Let $d := \sum_{i=1}^n d_i - n$ and t an integer such that $0 \le t \le d-1$. If Y is a proper subscheme of X such that the residual scheme Z has $\omega_Z(t) = 0$, then every hypersurface of degree $d-t-1$ passing through Y has to contain X.*

If Z is a set of points, then $\omega_Z(t) = 0$ if and only if Z imposes $|Z|$ independent conditions on hypersurfaces of degree t. In particular, if $|Z| = \binom{n+t}{n}$, this means that Z is not contained in any hypersurface of degree t. Hence Theorem 3.2 has the following interesting consequence.

Corollary 3.3. *Let X be a set of $d_1 \cdots d_n$ points of the intersection of n hypersurfaces of degrees $d_1, \ldots, d_n$. Let $d := \sum_{i=1}^n d_i - n$ and t an integer such that $0 \le t \le d-1$. If Y is a subset of $|X| - \binom{n+t}{n}$ points of X such that the set of the remaining points of X does not lie on any hypersurface of degree t, then every hypersurface of degree $d-t-1$ passing through Y has to contain X.*

Theorem 3.2 (for points) and Corollary 3.3 have been proved recently by M.-A. Coppo [C]. The classical Cayley-Bacharach theorem is the case $n=2$, $t=0$ of Corollary 3.3.

Another application is the following result of J. Briancon which gives a condition for a zero-dimensional scheme to be a complete intersection.

Corollary 3.4. [B] *Let X be a zero-dimensional scheme in $\mathbf{P}^n$ which is the complete intersection of n hypersurfaces of degree $d_1, \ldots, d_n$. Let $d := \sum_{i=1}^n d_i - n$, and Y a subscheme of X with $\omega_Y(d-1) > 0$, then $Y = X$.*

Proof.. By Proposition 3.1 we have $\omega_Z(0) = H_X(d-1) - H_Y(d-1)$. This implies

$$|X| - |Y| - H_Y(0) = |X| - 1 - (|Y| - \omega_Y(d-1))$$

which gives $H_Z(0) = 1 - \omega_Y(d-1)$ for $Z = X \setminus Y$. Thus we get $H_Z(0) = 0$, hence $I(Z) = (1)$ which implies $Z = \emptyset$ and $Y = X$.

4. Number of generators

In this last section we apply our methods to bound the minimal number of generators $v(I)$ of a homogeneous ideal I of a polynomial ring $R = k[X_1, \ldots, X_n]$.

Our idea is to combine Theorem 1.1 with a simple result of J. Sally [Sa] which says that the minimal number of generators of any ideal in a one-dimensional Cohen-Macaulay local (or homogeneous) ring A is bounded above by the multiplicity $e(A)$ of A (cf. [G, Proposition 1.2]). For this we shall need the following easy lemma.

Lemma 4.1.. *Let A be a graded ring. Then*

$$\max_{I \subseteq A} v(I) = \max_{J \subseteq A} \tau(A/J).$$

Proof. For every homogeneous ideal I we let J to be the ideal $A_1 I$. Then we have

$$v(I) = dim_k(I/J) \leq dim_k(J : A_1/J) = \tau(A/J).$$

On the other hand for every homogeneous ideal J in A we have

$$\tau(A/J) = dim_k(J : A_1/J) \leq dim_k(J : A_1/A_1(J : A_1)) = v(J : A_1).$$

Now the conclusion is immediate.

The main result of this section is a generalization of a classical theorem of Dubreil [D, Theoreme II] which says that for any homogeneous ideal I of height 2 in $k[X_1, X_2]$, $v(I) \leq a + b - s + 1$, where a is the least degree of forms in I, b is the least number such that I contains a regular sequence of two forms of degree a and b, and s is the socle degree of $k[X_1, X_2]/I$.

Theorem 4.2 .. *Let I be a height n homogeneous ideal of $R := k[X_1, \ldots, X_n]$, $n \geq 2$. Assume that there exists in I a regular sequence $F_1, \ldots, F_n$ of forms of degrees*

$d_1 \geq \cdots \geq d_n$ such that $(F_1, \ldots, F_{n-2})$ is a prime ideal. Let $d = \sum_{i=1}^{n} d_i - n$ and s be the socle degree of R/I. Then

$$v(I) \leq (d-s) \prod_{i=1}^{n-2} d_i + n.$$

Proof.. Let $A = R/(F_1, \ldots, F_n)$ and $\bar{I} = I/(F_1, \ldots, F_n)$. Then $s(A) = d$ and

$$v(I) \leq v(\bar{I}) + n.$$

By Theorem 1.1 we get $h_{0:\bar{I}}(d-s) = h_{A/\bar{I}}(s) > 0$. Hence there exists a form $F \in R_{d-s}$ such that the image of F in A is a non-zero element of $0 : I$. We may write

$$v(\bar{I}) = \tau(A/0 : \bar{I}) = \tau(B/Q),$$

where $B = R/(F_1, \ldots, F_{n-2}, F)$ and Q is the preimage of the ideal $0 : \bar{I}$ in B. Since $(F_1, \ldots, F_{n-2})$ is a prime ideal and $F \notin (F_1, \ldots, F_n)$, the sequence $F_1, \ldots, F_{n-2}, F$ is regular, hence B is a one-dimensional Cohen-Macaulay ring. Using Lemma 4.1 and Sally's bound for the number of generators of ideals in B we get

$$\tau(B/Q) \leq e(B) = (d-s) \prod_{i=1}^{n-2} d_i$$

which gives the conclusion.

Note that Theorem 4.2 does not hold if we drop the assumption that $(F_1, \ldots, F_{n-2})$ is a prime ideal. If $d - s > d_{n-1}$, one should use instead of Theorem 4.2 the following trivial application of Sally's bound.

Lemma 4.3. (cf. [G, Proposition 3.7]) *Let I be an homogeneous ideal of $R := k[X_1, \ldots, X_n]$, $n \geq 2$. If there exists a regular sequence $F_1, \ldots, F_{n-1}$ in I of degree $d_1, \ldots, d_{n-1}$, then*

$$v(I) \leq \prod_{i=1}^{n-1} d_i + n - 1.$$

Proof. Let $B = R/(F_1, \ldots, F_{n-1})$ and $Q = I/(F_1, \ldots, F_{n-1})$. Then $v(I) \leq v(Q) + n - 1$. Since $e(B) = \prod_{i=1}^{n-1} d_i$, from Sally's bound we get $v(Q) \leq \prod_{i=1}^{n-1} d_i$.

Note that for $n = 2$, Lemma 4.3 gives another classical result of Dubreil [Du, Theoreme I] (see also [G] and [DGM]). Now we will use Lemma 4.3 to prove a modified version of Theorem 4.1 which sometimes gives a better bound for $v(I)$.

Theorem 4.4. *Let I be a height n homogeneous ideal of $R := k[X_1, \dots, X_n]$, $n \geq 2$. Assume that there exists in I a regular sequence $F_1, \dots, F_n$ of forms of degrees $d_1 \geq \dots \geq d_n$ such that $(F_1, \dots, F_{n-2})$ is a prime ideal. Let $d = \sum_{i=1}^{n} d_i - n$ and m be the largest degree of the elements of a homogeneous minimal basis for I. Then*

$$v(I) \leq (d-m)\prod_{i=1}^{n-2} d_i + n + 1.$$

Proof. Consider the artinian Gorenstein ring $A := R/J$ where $J := (F_1, \dots, F_n)$. If $m \leq d_n$, then

$$\begin{aligned}(d-m)\prod_{i=1}^{n-2} d_i + n + 1 &\geq (d-d_n)\prod_{i=1}^{n-2} d_i + n + 1 \\ &= (\sum_{i=1}^{n-1} d_i - n)\prod_{i=1}^{n-2} d_i + n + 1 \\ &= 2 + (\sum_{i=1}^{n-2} d_i - n)\prod_{i=1}^{n-2} d_i + \prod_{i=1}^{n-1} d_i + n - 1 \\ &\geq \prod_{i=1}^{n-1} d_i + n - 1,\end{aligned}$$

Therefore, if $m \leq d_n$, the conclusion follows from Lemma 4.3.

Let $m > d_n$ and $\bar{I} = I/J$. By Corollary 1.2 we have

$$h_{(0:A_1\bar{I})/(0:\bar{I})}(d-m) = h_{I/A_1\bar{I}}(m) > 0.$$

This means that we can find an element $F \in R_{d-m}$ such that the image f of F in A belongs to $0 : A_1\bar{I}$ but $f \notin 0 : \bar{I}$. Using Corollary 1.2 we also get

$$\begin{aligned}v(I) &\leq v(\bar{I}) + n = \tau(A/0 : \bar{I}) + n \\ &\leq v((0:\bar{I}) : A_1) + n = v(0 : A_1\bar{I}) + n.\end{aligned}$$

If $m = d$, then $0 : A_1\bar{I} = A$ and the conclusion follows. If $m < d$, then since $(F_1, \dots, F_{n-2})$ is a prime ideal which does not contain F, we get that $F_1, \dots, F_{n-2}, F$ form a regular sequence and the ring $B := R/(F_1, \dots, F_{n-2}, F)$ is Gorenstein of dimension 1. Therefore

$$\begin{aligned}v(0 : A_1\bar{I}) &\leq v(0 : A_1\bar{I}/f) + 1 = v((J : R_1I)/(J, F)) + 1 \\ &\leq v((J : R_1I)/(F_1, \dots, F_{n-2}, F)) + 1 \leq (d-m)\prod_{i=1}^{n-2} d_i + 1,\end{aligned}$$

where the last inequality follows again by the quoted Sally's bound for the number of generators of ideals in B. The proof of Theorem 4.4 is now complete.

In general we always have $m \leq s+1$. Hence Theorem 4.4 give a better bound for $v(I)$ than Theorem 4.2 only if $m = s+1$.

The following example shows that in Theorem 4.2 and Theorem 4.4 we can not delete the assumption that $(F_1, \ldots, F_{n-2})$ is a prime ideal in R.

Example.. Let $R = k[X,Y,Z]$ and $I = (X^2, XY^2, XZ^2, XYZ, Y^3, Z^4, Y^2Z^3)$. Then $d_1 = 2$, $d_2 = 3$, $d_3 = 4$, $d = 6$, $s = 4$, and $m = 5$. But $v(I) = 7$, while

$$(d-s)d_1 + 2 - 1 = (d-m)d_1 + 3 + 1 = 6.$$

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