Lagrange's four squares theorem with variables of special type

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1 Introduction and statement of the results.

In this paper we study the equation of Lagrange

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = N \tag{1}$$

with multiplicative restrictions imposed on the variables.

It is expected that every sufficiently large integer N, satisfying the congruence condition $N \equiv 4 \pmod{24}$, can be represented in the form (1) with prime variables x_i. This conjecture has not been proved so far. We should mention, however, that Hua [6] proved that all large integers N such that $N \equiv 5 \pmod{24}$, are sums of five squares of primes.

Greaves [3], Plaksin [10] and Shields [11] established the solvability of (1) with two prime and two integer variables provided that N is sufficiently large and satisfies a natural congruence condition. Brüdern and Fouvry [1] proved that any sufficiently large integer $N \equiv 4 \pmod{24}$, can be represented in the form (1), where each variable is a number of type P_{34} (as usual, by P_r we denote any integer having at most r prime factors, counted according to multiplicity).

D.R. Heath–Brown and the author considered recently the equation (1) with more restrictive conditions on the variables than in the paper of Brüdern and Fouvry. In [5] two theorems were proved. The first of them states that every sufficiently large $N \equiv 4 \pmod{24}$ can be represented in the form (1), where x_1 is a prime and where each of x_2, x_3, x_4 is a number of type P_{101}. The most important part of the proof of this result is the establishment of Propositions 1 and 2. The second of them asserts, that the sum $\mathcal{L}(\mathbf{k}, N)$, defined by (3), can be approximated in some average sense to the expected main term, which we can find by a formal application of the circle method. After that the proof of Theorem 1 of [5] can be established by using the vector sieve. This sieve method was proposed by Iwaniec [7] and was also used by Brüdern and Fouvry [1], [2] and by the author [12] – [14].

The second theorem of [5] states that for every sufficiently large $N \equiv 4 \pmod{24}$ the equation (1) is solvable in variables of type P_{25}. To prove this result we use Proposition 3 of [5]. It, roughly speaking, states that the number of the solutions of (1) in integers lying in progressions, can be approximated on average by the expected value with better level of distribution than in the relevant theorem from the paper [1]. Then the result follows again by application of the vector sieve.

The aim of the present paper is to show that improvements upon the results of [5] (of about 20%, in some sense) can be achieved by attaching Kuhn's weight to one of the variables. The following theorems hold:

Theorem 1. *Every sufficiently large integer N, satisfying $N \equiv 4 \pmod{24}$, can be represented in the form*

$$q^2 + x_1^2 + x_2^2 + x_3^2 = N, \tag{2}$$

where $x_1 = \mathrm{P}_{78}, x_2 = \mathrm{P}_{80}, x_3 = \mathrm{P}'_{80}$ and where q is a prime.

Theorem 2. *Every sufficiently large integer N, satisfying $N \equiv 4 \pmod{24}$, can be represented in the form (1), where each of the variables is of type P_{21}.*

We present only the proof of Theorem 1. Theorem 2 can be proved in the same way. We omit the calculations because they are similar to those in the relevant parts of [5], [12] – [14].

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2 Notations.

We assume that the integer N is sufficiently large and satisfies the congruence $N \equiv 4 \pmod{24}$. Denote $P = N^{1/2}$. Let $\varepsilon \in (0, 10^{-6})$ be arbitrarily small and let $A > 10^6$ be arbitrarily large number (they may not be the same in different occurrences). The constants in $\mathcal{O}$–terms and $\ll$–symbols are absolute or depend on ε and A. For positive U and V we write $U \asymp V$ as an abbreviation of $U \ll V \ll U$. The letters p and q are reserved for prime numbers. As usual, $\mu(n)$, $\tau(n)$, $\Omega(n)$ denote, respectively, the Möbius function, the number of divisors of n and the number of prime factors of n, counted according to multiplicity. Instead of $m \equiv n \pmod{k}$ we write for simplicity $m \equiv n\ (k)$. If $p^l \mid m$, but $p^{l+1} \nmid m$ then we write $p^l \,||\, m$.

We denote by (m_1, m_2) and $[m_1, m_2]$ the greatest common divisor and, respectively, the least common multiple of the integers of m_1, m_2. However, if u, v are real numbers then (u, v) means the interval with endpoints u and v. Finally, by bold style letters we denote three–dimensional vectors in the following way: $\mathbf{d} = \langle d_1, d_2, d_3 \rangle$. The meaning is always clear from the context.

3 Proposition.

First we state a proposition, which is an analog of Proposition 2 of [5]. Consider the function

$$\omega_0(t) = \begin{cases} \exp\left(\frac{1}{(20t-10)^2-1}\right) & \text{if} \quad t \in \left(\frac{9}{20}, \frac{11}{20}\right), \\ 0 & \text{otherwise} \end{cases}$$

and denote $\omega(\mathbf{x}) = \prod_{i=1}^{3} \omega_0(x_i P^{-1})$. For any vector $\mathbf{d}$ with squarefree odd components we consider the sum

$$\mathcal{L}(\mathbf{d}, N) = \sum_{\substack{q^2+x_1^2+x_2^2+x_3^2=N \\ x_i \equiv 0\ (d_i),\ i=1,2,3}} \omega(\mathbf{x})\,. \tag{3}$$

In order to prove Theorem 1 we have to approximate this sum by another expression, which is easier to work with. The most difficult part of the proof is to establish that this expression approximates $\mathcal{L}(\mathbf{d}, N)$ in some average sense indeed. We have the following:

Proposition. *Suppose that K_1, K_2, K_3 are positive numbers, satisfying*

$$K_1 \le K_2 K_3\,, \quad K_2 \le K_1 K_3\,, \quad K_3 \le K_1 K_2\,, \quad K_1 K_2 K_3 \le P^{2/23-\varepsilon}\,.$$

Let $\beta_i(d)$ be real functions, supported on the set of positive squarefree odd integers, and such that

$$|\beta_i(d)| \le \tau^2(d)\,, \qquad \beta_i(d) = 0 \quad \text{if} \quad d > K_i\,, \qquad i = 1, 2, 3\,.$$

There exist quantities $\mathcal{N}_0 = \mathcal{N}_0(N)$, $\xi_0 = \xi_0(N)$, $\mathcal{R}(\mathbf{d}, N)$ with the following properties:

For $\mathcal{N}_0$ and ξ_0 we have

$$\mathcal{N}_0 \asymp \frac{P^2}{\log P}\,, \qquad 1 \ll \xi_0 \ll \log\log P\,. \tag{4}$$

The quantity $\mathcal{R}$ is defined for vectors $\mathbf{d}$ with squarefree odd components and can be decomposed in the following way

$$\mathcal{R}(\mathbf{d}, N) = \prod_{p \| d_1 d_2 d_3} \psi_1(p, N) \prod_{p^2 \| d_1 d_2 d_3} \psi_2(p, N) \prod_{p^3 \| d_1 d_2 d_3} \psi_3(p, N)\,, \tag{5}$$

where $\psi_i(p, N)$, $i = 1, 2, 3$, are functions defined for primes $p > 2$ and such that

$$\begin{gathered} \psi_1(3, N) = 1/3\,, \quad 1/4 \le \psi_1(p, N) \le 4\,, \quad \psi_1(p, N) = 1 + \mathcal{O}(p^{-1})\,, \\ 0 \le \psi_i(p, N) \le 4\,, \qquad i = 2, 3\,. \end{gathered} \tag{6}$$

We also have

$$\mathcal{R}(\mathbf{d}, N) \ll \tau^2(d_1)\tau^2(d_2)\tau^2(d_3)\,. \tag{7}$$

Furthermore, if $2 \nmid k_i l_j$ and $\mu(k_i l_j) \ne 0$ for $1 \le i, j \le 3$ then

$$\mathcal{R}(\langle k_1 l_1, k_2 l_2, k_3 l_3 \rangle, N) = R(\mathbf{k}, N)\, R(\mathbf{l}, N)\,. \tag{8}$$

Finally, the following estimate holds:

$$\sum_{d_1, d_2, d_3} \beta_1(d_1)\beta_2(d_2)\beta_3(d_3) \Big(\mathcal{L}(\mathbf{d}, N) - \frac{\mathcal{N}_0\, \xi_0\, \mathcal{R}(\mathbf{d}, N)}{d_1 d_2 d_3} \Big) \ll P^2 (\log P)^{-A}\,. \tag{9}$$

Let us notice, that the terms $\mathcal{N}_0$ and $\xi_0 \mathcal{R}(\mathbf{d}, N)/(d_1 d_2 d_3)$ can be considered as the “singular integral” and, respectively, the “singular series”, for the sum $\mathcal{L}(\mathbf{d}, N)$. Hence the second term in the brackets in (9) is the expected approximation to $\mathcal{L}(\mathbf{d}, N)$, which we can find by a formal application of the circle method. For the exact definitions of $\mathcal{N}_0$, ξ_0 and $\mathcal{R}(\mathbf{d}, N)$ we refer the reader to [5].

The present proposition is a straightforward generalization of Proposition 2 from [5] (where we have $K_1 = K_2 = K_3$). The proof is long and complicated, but differs slightly from the proof of the corresponding assertion in [5], so we omit it. We also notice, that in Proposition 2 from [5] the "singular series" is given in the form of series indeed, and its decomposition as a product and the properties of the factors are established in [5], Section 4.

4 Proof of Theorem 1.

4.1 Beginning of the proof.

Suppose that $\alpha_1, \alpha_2, \alpha_3$ and θ are constants, such that

$$0 < \theta < 1\,, \qquad 0 < \alpha_1 < \alpha_3 < 1\,, \qquad \alpha_1 < \alpha_2 < 1\,. \tag{10}$$

Define

$$z_0 = (\log P)^{1000}\,, \qquad z_i = P^{\alpha_i}\,, \qquad i = 1, 2, 3\,, \tag{11}$$

and let

$$\mathfrak{P}_0 = \prod_{2<p<z_0} p\,, \qquad \mathfrak{P}_i = \prod_{z_0 \le p < z_i} p\,, \qquad i = 1, 2\,. \tag{12}$$

Consider the sum

$$\Gamma = \sum_{\substack{q^2+x_1^2+x_2^2+x_3^2=N \\ (x_1, \mathfrak{P}_0\mathfrak{P}_1)=(x_2x_3, \mathfrak{P}_0\mathfrak{P}_2)=1}} \omega(\mathbf{x}) \left(1 - \theta \sum_{\substack{z_1 \le p < z_3 \\ p \mid x_1}} \left(1 - \frac{\log p}{\log z_3}\right)\right). \tag{13}$$

Using the condition $N \equiv 4\ (24)$ and the definition of $\omega(\mathbf{x})$ we find that the solutions of (2), such that $2 \mid qx_1x_2x_3$, are not counted in Γ.

Our aim is to show that for suitable constants $\alpha_1, \alpha_2, \alpha_3, \theta$, satisfying (10), we have

$$\Gamma \gg P^2 (\log P)^{-4}\,. \tag{14}$$

Having proved this, consider the part Γ' of Γ, consisting of all terms, such that $x_1 \equiv 0\ (p^2)$ for some prime $p \in [z_1, z_3)$. Using (10) and (11) we get

$$\Gamma' \ll \sum_{z_1 \le p < z_3} \sum_{\substack{q^2+x_1^2+x_2^2+x_3^2=N \\ x_1 \equiv 0(p^2)}} 1 = \sum_{z_1 \le p < z_3} \sum_{k \le N} \Big(\sum_{q^2+x_3^2=k} 1 \Big) \sum_{\substack{x_1^2+x_2^2=N-k \\ x_1 \equiv 0(p^2)}} 1 \tag{15}$$

$$\ll P^{\varepsilon} \sum_{z_1 \le p < z_3} \sum_{\substack{x_1, x_2 \le P \\ x_1 \equiv 0(p^2)}} 1 \ll P^{1+\varepsilon}\big(P z_1^{-1} + z_3\big) \ll P^{2-\varepsilon}\,.$$

From (13) – (15) we conclude that there are $\gg P^2(\log P)^{-4}$ quadruples q, x_1, x_2, x_3, satisfying (2) and such that q is a prime, x_1 has no multiple prime factors $p \in [z_1, z_3)$,

$$(x_1, 2\mathfrak{P}_0\mathfrak{P}_1) = (x_2x_3, 2\mathfrak{P}_0\mathfrak{P}_2) = 1 \tag{16}$$

and

$$1 - \theta \sum_{\substack{z_1 \le p < z_3 \\ p \mid x_1}} \Big(1 - \frac{\log p}{\log z_3}\Big) > 0\,. \tag{17}$$

From (11) and (16) we get

$$\Omega(x_2),\ \Omega(x_3) \le \alpha_2^{-1}\,. \tag{18}$$

Furthermore, from (16) and (17) we easily obtain

$$\Omega(x_1) < \theta^{-1} + \alpha_3^{-1}\,. \tag{19}$$

The arguments are similar to those in [4], Chapter 9, § 2, for example, so we omit them.

We see that our aim is to choose constants $\alpha_1, \alpha_2, \alpha_3, \theta$, satisfying (10), in such a way, that the estimate (14) holds, and at the same time the number

$$\max\big(\alpha_2^{-1}, \theta^{-1} + \alpha_3^{-1}\big)$$

should be as small as possible.

It is clear that

$$\Gamma = \mathcal{F} - \theta\,\mathcal{G}\,, \tag{20}$$

where

$$\mathcal{F} = \sum_{\substack{q^2+x_1^2+x_2^2+x_3^2=N \\ (x_1,\mathfrak{P}_0\mathfrak{P}_1)=(x_2x_3,\mathfrak{P}_0\mathfrak{P}_2)=1}} \omega(\mathbf{x})\,, \tag{21}$$

$$\mathcal{G} = \sum_{z_1\le p<z_3} \left(1-\frac{\log p}{\log z_3}\right) \sum_{\substack{q^2+x_1^2+x_2^2+x_3^2=N \\ (x_1,\mathfrak{P}_0\mathfrak{P}_1)=(x_2x_3,\mathfrak{P}_0\mathfrak{P}_2)=1 \\ x_1\equiv 0\,(p)}} \omega(\mathbf{x})\,. \tag{22}$$

In order to prove (14) we have to estimate $\mathcal{F}$ from below and $\mathcal{G}$ from above.

4.2 The estimation of $\mathcal{F}$.

To study the sum $\mathcal{F}$, defined by (21), we apply the vector sieve. We proceed exactly as in Section 4 of [5]. For reader's convenience we present the main points, but omit the calculations.

Using the fundamental property of the Möbius function we represent $\mathcal{F}$ in the form

$$\mathcal{F} = \sum_{q^2+x_1^2+x_2^2+x_3^2=N} \omega(\mathbf{x})\,\Phi_1\,\Phi_2\,\Phi_3\,\Lambda_1\,\Lambda_2\,\Lambda_3\,, \tag{23}$$

where

$$\Phi_i = \sum_{d|(x_i,\mathfrak{P}_0)} \mu(d)\,, \qquad i=1,2,3\,; \tag{24}$$

$$\Lambda_1 = \sum_{d|(x_1,\mathfrak{P}_1)} \mu(d)\,; \qquad \Lambda_i = \sum_{d|(x_i,\mathfrak{P}_2)} \mu(d)\,, \qquad i=2,3\,. \tag{25}$$

Define

$$D_0 = P^{\varepsilon}\,; \qquad D_i = P^{\eta_i}\,, \qquad i=1,2\,, \tag{26}$$

where η_1,η_2 are constants such that

$$2 \le s_2 = \frac{\eta_2}{\alpha_2} \le 3 \le s_1 = \frac{\eta_1}{\alpha_1} \le 4\,, \qquad \alpha_1+\alpha_3 \le \eta_1\,. \tag{27}$$

Let $\lambda_i^{\pm}$, $i=0,1,2$, be the Rosser functions of orders D_i, respectively. Denote

$$\Phi_i^{\pm} = \sum_{d|(x_i,\mathfrak{P}_0)} \lambda_0^{\pm}(d)\,, \qquad i=1,2,3\,; \tag{28}$$

$$\Lambda_1^{\pm} = \sum_{d|(x_1,\mathfrak{P}_1)} \lambda_1^{\pm}(d)\,; \qquad \Lambda_i = \sum_{d|(x_i,\mathfrak{P}_2)} \lambda_2^{\pm}(d)\,, \qquad i=2,3\,. \tag{29}$$

The definition and the properties of the Rosser weights can be found in Iwaniec [8], [9]. In particular, we have

$$|\lambda_i^{\pm}(d)| \le 1\,, \quad \lambda_i^{\pm}(d)=0 \ \text{ if } \ \mu(d)=0 \ \text{ or } \ d>D_i\,, \quad i=0,1,2\,; \tag{30}$$

$$\Phi_i^- \le \Phi_i \le \Phi_i^+\,, \qquad \Lambda_i^- \le \Lambda_i \le \Lambda_i^+\,, \qquad i=1,2,3\,. \tag{31}$$

The lower estimate for $\mathcal{F}$ is based on the elementary inequality

$$\begin{aligned}\Phi_1\Phi_2\Phi_3\Lambda_1\Lambda_2\Lambda_3 \ge{}& \Phi_1^-\Phi_2^+\Phi_3^+\Lambda_1^+\Lambda_2^+\Lambda_3^+ \ + \ \Phi_1^+\Phi_2^-\Phi_3^+\Lambda_1^+\Lambda_2^+\Lambda_3^+ \\ &+\Phi_1^+\Phi_2^+\Phi_3^-\Lambda_1^+\Lambda_2^+\Lambda_3^+ \ + \ \Phi_1^+\Phi_2^+\Phi_3^+\Lambda_1^-\Lambda_2^+\Lambda_3^+ \\ &+\Phi_1^+\Phi_2^+\Phi_3^+\Lambda_1^+\Lambda_2^-\Lambda_3^+ \ + \ \Phi_1^+\Phi_2^+\Phi_3^+\Lambda_1^+\Lambda_2^+\Lambda_3^- \\ &-5\,\Phi_1^+\Phi_2^+\Phi_3^+\Lambda_1^+\Lambda_2^+\Lambda_3^+\,,\end{aligned} \tag{32}$$

which proof is similar to the proof of Lemma 13 of [1]. From (23) and (32) we get

$$\mathcal{F} \ge \mathcal{F}_1 + \cdots + \mathcal{F}_6 - 5\mathcal{F}_7\,, \tag{33}$$

where $\mathcal{F}_i$ are the contributions arising from the consecutive terms from the right–hand side of (32).

Consider, for example, $\mathcal{F}_1$. Using (28), (29) and changing the order of summation we write it in the form

$$\mathcal{F}_1 = \sum_{d_1,d_2,d_3} \beta_1(d_1)\,\beta_2(d_2)\,\beta_3(d_3)\,\mathcal{L}(\mathbf{d},N)\,,$$

where

$$\beta_1(d) = \sum_{\substack{k|\mathfrak{P}_0\,,\ l|\mathfrak{P}_1\\ kl=d}} \lambda_0^-(k)\,\lambda_1^+(l)\,, \tag{34}$$

$$\beta_i(d) = \sum_{\substack{k|\mathfrak{P}_0\,,\ l|\mathfrak{P}_2\\ kl=d}} \lambda_0^+(k)\,\lambda_2^+(l)\,, \quad i=2,3\,. \tag{35}$$

From this point onwards we assume that

$$\eta_1 < 2\eta_2\,, \qquad \eta_1 + 2\eta_2 < 2/23\,. \tag{36}$$

We take ε sufficiently small and use (26), (30) and (36) to verify that the functions $\beta_i(d)$ satisfy the requirements of our Proposition (with $K_1 = D_0D_1$, $K_2 = K_3 = D_0D_2$). Hence we have

$$\mathcal{F}_1 = \mathcal{N}_0\xi_0\mathcal{F}_1' + \mathcal{O}(P^2(\log P)^{-A}) \tag{37}$$

where

$$\mathcal{F}_1' = \sum_{d_1,d_2,d_3} \frac{\beta_1(d_1)\,\beta_2(d_2)\,\beta_3(d_3)}{d_1d_2d_3}\mathcal{R}(\mathbf{d}, N)\,. \tag{38}$$

From (8), (34), (35) and (38) we get

$$\mathcal{F}_1' = \mathcal{H}^-\mathcal{V}^+\,, \tag{39}$$

where

$$\mathcal{H}^\pm = \sum_{k_1,k_2,k_3|\mathfrak{P}_0} \frac{\lambda_0^\pm(k_1)\lambda_0^+(k_2)\lambda_0^+(k_3)}{k_1k_2k_3}\,\mathcal{R}(\mathbf{k}, N)\,, \tag{40}$$

$$\mathcal{V}^+ = \sum_{l_1|\mathfrak{P}_1\,;\,l_2,l_3|\mathfrak{P}_2} \frac{\lambda_1^+(l_1)\lambda_2^+(l_2)\lambda_2^+(l_3)}{l_1l_2l_3}\,\mathcal{R}(\mathbf{l}, N)\,. \tag{41}$$

From (37) and (39) an asymptotic formula for $\mathcal{F}_1$ follows.

We proceed with $\mathcal{F}_i\,,\ 2 \le i \le 7$ in the same way and use (33) to find

$$\mathcal{F} \ge \mathcal{N}_0\xi_0(3\mathcal{H}^-\mathcal{V}^+ + \mathcal{H}^+\mathcal{V}_1^- + 2\mathcal{H}^+\mathcal{V}_2^- - 5\mathcal{H}^+\mathcal{V}^+) + \mathcal{O}(P^2(\log P)^{-A})\,, \tag{42}$$

where

$$\mathcal{V}_1^- = \sum_{l_1|\mathfrak{P}_1\,;\,l_2,l_3|\mathfrak{P}_2} \frac{\lambda_1^-(l_1)\lambda_2^+(l_2)\lambda_2^+(l_3)}{l_1l_2l_3}\,\mathcal{R}(\mathbf{l}, N)\,, \tag{43}$$

$$\mathcal{V}_2^- = \sum_{l_1|\mathfrak{P}_1\,;\,l_2,l_3|\mathfrak{P}_2} \frac{\lambda_1^+(l_1)\lambda_2^-(l_2)\lambda_2^+(l_3)}{l_1l_2l_3}\,\mathcal{R}(\mathbf{l}, N)\,. \tag{44}$$

We note that from (7) and (30) follows

$$\mathcal{V}^+,\ \mathcal{V}_1^-,\ \mathcal{V}_2^- \ll (\log P)^{12}\,. \tag{45}$$

Consider $\mathcal{H}^{\pm}$. Obviously $s_0 = \log D_0/\log z_0 \to \infty$ as $P \to \infty$. Hence we may expect that the Rosser functions $\lambda_0^{\pm}$ behave like the Möbius function. Indeed, if we substitute $\mu(k_1)\mu(k_2)\mu(k_3)$ for $\lambda_0^{\pm}(k_1)\lambda_0^{+}(k_2)\lambda_0^{+}(k_3)$ and denote the new sum by $\mathcal{H}_0$ then we can find that

$$\mathcal{H}^{\pm} = \mathcal{H}_0 + \mathcal{O}(\exp(-\sqrt{\log P}))\,. \tag{46}$$

A detailed proof of this asymptotic formula is available in Section 3.7 of [14].

Furthermore, it is easy to prove that

$$H_0 \asymp (\log\log P)^{-3}\,. \tag{47}$$

From (4), (42) and (45) – (47) we get

$$\mathcal{F} \ge \mathcal{N}_0\xi_0\mathcal{H}_0\,(\mathcal{V}_1^- + 2\mathcal{V}_2^- - 2\mathcal{V}^+) \,+\, \mathcal{O}(P^2(\log P)^{-A})\,. \tag{48}$$

4.3 The estimation of $\mathcal{G}$.

We write the sum $\mathcal{G}$, defined by (22), in the form

$$\mathcal{G} = \sum_{z_1 \le p < z_3} \left(1 - \frac{\log p}{\log z_3}\right) \sum_{\substack{q^2+x_1^2+x_2^2+x_3^2=N \\ x_1 \equiv 0\,(p)}} \omega(\mathbf{x})\ \Phi_1\Phi_2\Phi_3\Lambda_1\Lambda_2\Lambda_3\,, \tag{49}$$

where Φ_i, Λ_i are specified by (24) and (25).

For any prime $p \in [z_1, z_3)$ consider the upper Rosser function $\lambda^{(p)}$ of order D_1/p, where D_1 is given by (26). Let

$$\Lambda^{(p)} = \sum_{d|(x_1,\mathfrak{P}_1)} \lambda^{(p)}(d)\,. \tag{50}$$

We have

$$|\lambda^{(p)}(d)| \le 1\,, \quad \lambda^{(p)}(d) = 0 \ \text{ if } \ \mu(d) = 0 \ \text{ or } \ d > D_1/p \tag{51}$$

and also

$$\Lambda_1 \le \Lambda^{(p)}\,. \tag{52}$$

Using (31), (49) and (52) we find that

$$\mathcal{G} \le \mathcal{G}_1 , \tag{53}$$

where

$$\mathcal{G}_1 = \sum_{z_1 \le p < z_3} \Big(1 - \frac{\log p}{\log z_3}\Big) \sum_{\substack{q^2+x_1^2+x_2^2+x_3^2=N \\ x_1 \equiv 0\,(p)}} \omega(\mathbf{x})\ \Phi_1^+\Phi_2^+\Phi_3^+\Lambda^{(p)}\Lambda_2^+\Lambda_3^+ . \tag{54}$$

From (28), (29), (50) and (54) after some rearrangements we obtain

$$\mathcal{G}_1 = \sum_{d_1,d_2,d_3} \beta_1^*(d_1)\, \beta_2(d_2)\, \beta_3(d_3)\, \mathcal{L}(\mathbf{d}, N) ,$$

where

$$\beta_1^*(d) = \sum_{\substack{z_1 \le p < z_3 \\ k \mid \mathfrak{P}_0\,,\, l \mid \mathfrak{P}_1 \\ klp=d}} \Big(1 - \frac{\log p}{\log z_3}\Big)\, \lambda_0^+(k)\, \lambda^{(p)}(l) \tag{55}$$

and where $\beta_i(d)$, $i = 2, 3$, are defined by (35).

From (30), (36) and (51) follows that we can apply our Proposition again (with $K_1 = D_0 D_1$, $K_2 = K_3 = D_0 D_2$) and we find

$$\mathcal{G}_1 = \mathcal{N}_0 \xi_0 \mathcal{G}_2 \ +\ \mathcal{O}(P^2(\log P)^{-A}) , \tag{56}$$

where

$$\mathcal{G}_2 = \sum_{d_1,d_2,d_3} \frac{\beta_1^*(d_1)\beta_2(d_2)\beta_3(d_3)}{d_1 d_2 d_3}\, \mathcal{R}(\mathbf{d}, N) . \tag{57}$$

From (5), (8), (35), (55) and (57) we get

$$\mathcal{G}_2 = \mathcal{H}^+ \mathcal{G}_3 , \tag{58}$$

where $\mathcal{H}^+$ is defined by (40),

$$\mathcal{G}_3 = \sum_{z_1 \le p < z_3} \frac{\psi_1(p, N)}{p} \Big(1 - \frac{\log p}{\log z_3}\Big)\, \mathcal{V}^{(p)} \tag{59}$$

and where

$$(60)\qquad \mathcal{V}^{(p)} = \sum_{l_1|\mathfrak{P}_1\,;\,l_2,l_3|\mathfrak{P}_2} \frac{\lambda^{(p)}(l_1)\lambda_2^+(l_2)\lambda_2^+(l_3)}{l_1l_2l_3}\,\mathcal{R}(\mathbf{l},N)\,.$$

From (7), (30) and (51) we get

$$(61)\qquad \mathcal{V}^{(p)} \ll (\log P)^{12}\,.$$

Furthermore, using (6), (11), (59) and (61) we find that

$$(62)\qquad \mathcal{G}_3 = \mathcal{G}_4 + \mathcal{O}(P^{-\varepsilon})\,,$$

where

$$(63)\qquad \mathcal{G}_4 = \sum_{z_1\le p<z_3} \frac{1}{p}\Big(1-\frac{\log p}{\log z_3}\Big)\,\mathcal{V}^{(p)}\,.$$

Finally, from (4), (46), (47), (53), (56), (58), (61) and (62) we obtain

$$(64)\qquad \mathcal{G} \le \mathcal{N}_0\xi_0\mathcal{H}_0\mathcal{G}_4 \;+\; \mathcal{O}(P^2(\log P)^{-A}))\,.$$

4.4 The lower bound for Γ and the end of the proof.

We take into account (20), (48) and (64) to get

$$(65)\qquad \Gamma \ge \mathcal{N}_0\xi_0\mathcal{H}_0\mathcal{M} \;+\; \mathcal{O}(P^2(\log P)^{-A}))\,,$$

where

$$(66)\qquad \mathcal{M} = \mathcal{V}_1^- + 2\mathcal{V}_2^- - 2\mathcal{V}^+ - \theta\mathcal{G}_4\,.$$

We shall find approximations for the expressions $\mathcal{V}^+$, $\mathcal{V}_1^-$, $\mathcal{V}_2^-$ and $\mathcal{V}^{(p)}$, defined, respectively, by (41), (43), (44) and (60).

Consider, for example, $\mathcal{V}^+$. Suppose that $l_1 \mid \mathfrak{P}_1$ and $l_2, l_3 \mid \mathfrak{P}_2$. Using (12) we find that the condition $(l_i, l_j) > 1$ implies $(l_i, l_j) \ge z_0$. From this fact and (7) we easily obtain

$$(67)\qquad \mathcal{V}^+ = \widetilde{\mathcal{V}^+} + \mathcal{O}((\log P)^{-100})\,,$$

where in $\widetilde{\mathcal{V}^+}$ the summation is restricted to l_i such that $(l_1, l_2) = (l_1, l_3) = (l_2, l_3) = 1$. In this case $\mathcal{R}(\mathbf{l}, N) = \psi_1(l_1, N)\psi_1(l_2, N)\psi_1(l_3, N)$, where we have defined $\psi_1(l, N) = \prod_{p|l} \psi_1(p, N)$. Using the fundamental property of the Möbius function and changing the order of summation we find that

$$\widetilde{\mathcal{V}^+} = \sum_{h_1,h_2,h_3|\mathfrak{P}_2} \mu(h_1)\mu(h_2)\mu(h_3) \times \sum_{\substack{l_1|\mathfrak{P}_1\,,\, l_2,l_3|\mathfrak{P}_2 \\ l_1\equiv 0\ ([h_2,h_3]) \\ l_2\equiv 0\ ([h_1,h_3]) \\ l_3\equiv 0\ ([h_1,h_2])}} \frac{\lambda_1^+(l_1)\lambda_2^+(l_2)\lambda_2^+(l_3)}{l_1 l_2 l_3} \psi_1(l_1, N)\psi_1(l_2, N)\psi_1(l_3, N)\,. \tag{68}$$

If $h_i \mid \mathfrak{P}_2$ and $h_i > 1$ then $h_i \geq z_0$. So, after some standard calculations, which we leave to the reader, we get

$$\widetilde{\mathcal{V}^+} = \overline{\mathcal{V}^+} + \mathcal{O}((\log P)^{-100})\,, \tag{69}$$

where $\overline{\mathcal{V}^+}$ is the contribution of the terms with $h_1 = h_2 = h_3 = 1$. From (67), (69) and from the definition of $\overline{\mathcal{V}^+}$ we obtain

$$\mathcal{V}^+ = \mathcal{T}_1^+ \, (\mathcal{T}_2^+)^2 \;+\; \mathcal{O}((\log P)^{-100})\,, \tag{70}$$

where

$$\mathcal{T}_i^\pm = \sum_{l|\mathfrak{P}_i} \frac{\lambda_i^\pm(l)}{l} \psi_1(l, N)\,, \qquad i = 1, 2\,. \tag{71}$$

Similarly we find that

$$\mathcal{V}_1^- = \mathcal{T}_1^- \, (\mathcal{T}_2^+)^2 \;+\; \mathcal{O}((\log P)^{-100})\,, \tag{72}$$

$$\mathcal{V}_2^- = \mathcal{T}_1^+ \, \mathcal{T}_2^- \, \mathcal{T}_2^+ \;+\; \mathcal{O}((\log P)^{-100})\,, \tag{73}$$

$$\mathcal{V}^{(p)} = \mathcal{T}^{(p)} \, (\mathcal{T}_2^+)^2 \;+\; \mathcal{O}((\log P)^{-100})\,, \tag{74}$$

where

$$\mathcal{T}^{(p)} = \sum_{l|\mathfrak{P}_1} \frac{\lambda^{(p)}(l)}{l} \psi_1(l, N)\,. \tag{75}$$

From (63), (66), (70) and (72) – (74) we get

$$\mathcal{M} = \mathcal{T}_2^+ \mathcal{M}^* + \mathcal{O}((\log P)^{-100}), \tag{76}$$

where

$$\mathcal{M}^* = \mathcal{T}_1^- \mathcal{T}_2^+ + 2\mathcal{T}_1^+ \mathcal{T}_2^- - 2\mathcal{T}_1^+ \mathcal{T}_2^+ - \theta \mathcal{T}_2^+ \mathcal{G}_5,$$

and

$$\mathcal{G}_5 = \sum_{z_1 \le p < z_3} \frac{1}{p} \Big(1 - \frac{\log p}{\log z_3} \Big) \mathcal{T}^{(p)}. \tag{77}$$

We write $\mathcal{M}^*$ in the form

$$\mathcal{M}^* = \mathcal{M}_1 \mathcal{T}_2^+ + 2\, \mathcal{M}_2 \mathcal{T}_1^+, \tag{78}$$

where

$$\mathcal{M}_1 = \mathcal{T}_1^- - 2\theta_1 \mathcal{T}_1^+ - \theta \mathcal{G}_5, \qquad \mathcal{M}_2 = \mathcal{T}_2^- - \theta_2 \mathcal{T}_2^+ \tag{79}$$

and where $\theta_1, \theta_2 > 0$ are constants such that

$$\theta_1 + \theta_2 = 1. \tag{80}$$

Suppose that $F(s)$ and $f(s)$ are the functions of the linear sieve. We have

$$f(s) = 2e^{\gamma} s^{-1} \log(s-1) \qquad \text{for} \qquad 2 \le s \le 4; \tag{81}$$

$$F(s) = \begin{cases} 2e^{\gamma} s^{-1} & \text{for} \qquad 1 \le s \le 3, \\ 2e^{\gamma} s^{-1} \Big(1 + \int_2^{s-1} t^{-1} \log(t-1)\, dt \Big) & \text{for} \qquad 3 \le s \le 4. \end{cases} \tag{82}$$

Here $\gamma = 0.577\ldots$ is the Euler constant.

Denote

$$\mathfrak{N}_i = \prod_{z_0 \le p < z_i} \Big(1 - \frac{\psi_1(p, N)}{p} \Big), \qquad i = 1, 2. \tag{83}$$

Using (6) and (11) we get

$$\mathfrak{N}_i \asymp \frac{\log\log P}{\log P}, \qquad i = 1, 2. \tag{84}$$

From the theory of the linear sieve (see Iwaniec [8], [9]) follows that if the conditions (10), (11), (26) and (27) hold and if $p \in [z_1, z_3)$ then the quantities $\mathcal{T}_i^{\pm}$ and $\mathcal{T}^{(p)}$, defined by (71) and (75), satisfy

$$\mathfrak{N}_i \le \mathcal{T}_i^{+} \le \mathfrak{N}_i \left\{ F(s_i) + \mathcal{O}((\log P)^{-1/3}) \right\}, \qquad i = 1, 2, \tag{85}$$

$$\mathcal{T}_i^{-} \ge \mathfrak{N}_i \left\{ f(s_i) + \mathcal{O}((\log P)^{-1/3}) \right\}, \qquad i = 1, 2, \tag{86}$$

$$\mathcal{T}^{(p)} \le \mathfrak{N}_1 \left\{ F\Big(\frac{\log D_1/p}{\log z_1} \Big) + \mathcal{O}((\log P)^{-1/3}) \right\}. \tag{87}$$

Consider the sum $\mathcal{G}_5$, defined by (77). From (87) we get

$$\mathcal{G}_5 \le \mathfrak{N}_1 \left\{ \mathcal{G}_6 + \mathcal{O}((\log P)^{-1/4}) \right\}, \tag{88}$$

where

$$\mathcal{G}_6 = \sum_{z_1 \le p < z_3} \frac{1}{p} \Big(1 - \frac{\log p}{\log z_3} \Big) F\Big(\frac{\log D_1/p}{\log z_1} \Big).$$

It follows from (27) that the argument of the function $F(s)$ in the last formula belongs to the segment $[1, 3]$, so we can apply (82). Using Abel's summation formula and the Prime Number Theorem after some calculations we obtain

$$\mathcal{G}_6 = 2e^{\gamma} \kappa \, + \, \mathcal{O}((\log P)^{-1}), \tag{89}$$

where

$$\kappa = \kappa(\alpha_1, \alpha_3, s_1) = \int_{\alpha_3^{-1}}^{\alpha_1^{-1}} \frac{t - \alpha_3^{-1}}{s_1 t^2 - \alpha_1^{-1} t} \, dt.$$

From (27), (79), (81), (82), (85), (86), (88) and (89) we get

$$\mathcal{M}_i \ge 2e^{\gamma} \mathfrak{N}_i \left\{ \kappa_i + \mathcal{O}((\log P)^{-1/4}) \right\}, \qquad i = 1, 2, \tag{90}$$

where

$$\kappa_1 = s_1^{-1} \Big(\log(s_1 - 1) - 2\theta_1 - 2\theta_1 \int_2^{s_1 - 1} t^{-1} \log(t - 1) \, dt \Big) - \theta \kappa,$$
$$\kappa_2 = s_2^{-1} \big(\log(s_2 - 1) - \theta_2 \big).$$

We choose

$$\alpha_1 = 0.004764, \qquad \alpha_2 = 0.012346, \qquad \alpha_3 = 0.014291,$$
$$\theta = 0.12, \qquad \theta_1 = 0.4405, \qquad \theta_2 = 0.5595,$$
$$\eta_1 = 0.019056, \qquad \eta_2 = 0.03395.$$

It is easy to see that all conditions (10), (27), (36), (80) are satisfied. Moreover, using numerical integration we can verify that in this case

$$\kappa_1\,,\ \kappa_2 > 0\,. \tag{91}$$

Using (84), (90) and (91) we find that $\mathcal{M}_i \gg (\log\log P)\,(\log P)^{-1}$ for $i = 1, 2$. Having in mind (76), (78), (84) and (85) we get $\mathcal{M} \gg (\log\log P)^3\,(\log P)^{-3}$. Now we apply (4), (47) and (65) and the estimate (14) follows.

We have chosen our constants in such a way as to minimize the number $\max\left(\alpha_2^{-1}\,,\ \theta^{-1} + \alpha_3^{-1}\right)$. It remains to note that our choice gives

$$80 < \alpha_2^{-1} < 81\,, \qquad 78 < \theta^{-1} + \alpha_3^{-1} < 79\,.$$

We take into account (18), (19) and Theorem 1 is proved.

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