Eisenstein Cohomology of Arithmetic Groups

The Case GL_2

by

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<u>Introduction</u>: This is the first in a series of prospective papers which concern the Eisenstein cohomology of arithmetic groups. The Eisenstein cohomology of an arithmetic group Γ is the part of the cohomology of Γ which in a sense comes - or better is induced - from the cohomology of the boundary of the Borel-Serre compactification of the locally symmetric space attached to Γ (comp. [B-S]). The Eisenstein cohomology classes are represented by differential forms which are Eisenstein series in the sense of Langlands and Selberg (comp. [La]). The Eisenstein cohomology has been studied by several authors in different cases and from a different point of view ([Ha 1-4], [Schw], [Sp]).

The goal of this paper is to give a general and systematic account of the case of the group GL_2 over an arbitrary number field. This case has also been treated in my papers [Ha1], [Ha3] and [Ha4] but always under some special assumptions. This paper here will contain the proofs of the main results announced in [Ha1] more than twelve years ago. One reason for the delay in the publication of the proofs is that it took me some time to find the general framework in which the results can be stated in a somewhat definite form, so that we also have a good starting point for the induction to higher dimensional groups.

This paper will also cover the results in |Ha4| and we shall follow to a large extent the pattern of that paper. There may be also some repetition in the exposition if the generalization from [Ha4] to the situation here is not so obvious.

In the first section we recall some general facts about the cohomology with coefficients. The coefficient systems which we consider are obtained by rational representations of the underlying algebraic group which in our case is GL_2 . We shall also discuss some algebraicity properties of the system of cohomology groups if we vary the coefficients, this means we investigate how the cohomology changes under the action of the Galois group if we change the coefficient system by a Galois automorphism.

In the second section we compute the cohomology of the boundary as a module under the group of finite adeles. The basic result is that the cohomology of the boundary can be described in terms of algebraic Hecke characters on the maximal split torus and the types of these characters can be read of from the data entering into the coefficient system. The cohomology of the boundary as a module under the action of the group of finite adeles is a sum of modules induced from these characters (Thm. 1 , [Ha4], Thm. 1.).

In the fourth section we describe the image of the global cohomology - i.e. the cohomology of the locally symmetric space - in the cohomology of the boundary. We compute this image in terms of the given data, i.e. the algebraic Hecke characters. The image will depend on the types, on special values and on poles of the Hecke L-functions attached to these characters (Thm. 2, and Thm. 2 in [Ha4], Thm. 2.1. [Ha1]).

In section five we construct certain homology classes depending on a quadratic field extension of our ground field and on certain algebraic Hecke characters whose type is again depending on the coefficient system. We can evaluate the Eisenstein cohomology classes on the cycles and the result of this evaluation is essentially given in terms of special values of L-functions attached to the characters - and combinations of them - which provide the classes (Thm. 3).

The results of section four and five have arithmetic applications. The Eisenstein classes are defined over a number field which depends on the algebraic Hecke character which defines the Eisenstein class and we can keep track of the action of the Galoisgroup on the cohomology and the homology classes. This gives us algebraicity results for special values of L-functions (Cor. 4.3.1, Cor. 5.7.2).

In a subsequent paper we shall generalize part of the results of this paper to GL_n . Again we will get algebraicity results about special values of L-functions attached to algebraic Hecke characters. The results combined with the results of Don Blasius [DB] will provide a proof Deligne's conjecture on special values of L-functions attached to algebraic Hecke characters (see [D]). We have to use Blasius' results since in our approach we always get ratios of special values where the periods predicted by Deligne cancel out.

I. Generalities

1.0. Notations and conventions: Throughout this paper $\bar{\mathbb{Q}}$ will be the field of algebraic numbers in the field of complex numbers $\mathbb{C}$.

Let $F/\mathbb{Q}$ be an arbitrary finite extension of $\mathbb{Q}$, we do not fix an embedding of F into $\bar{\mathbb{Q}}$. Let $\mathcal{O} \subset F$ be the ring of integers in F . The places of F will be denoted by v,w and if we refer to finite places we denote them by $\mathfrak{p},\mathfrak{q}$. Let S_∞ be the set of infinite places. The completions of F with respect to these places will be denoted by F_v or $F_{\mathfrak{p}}$. At a finite place $\mathfrak{p}$ we denote the ring of integers of $F_{\mathfrak{p}}$ by $\mathcal{O}_{\mathfrak{p}}$. For any place we denote the normalized absolute value by $|x_v|_v$ for $x_v \in F_v$.

The ring of adeles (resp. the group of ideles) will be denoted by A_F (resp. I_F). We abbreviate $A = A_{\mathbb{Q}}$ and $I = I_{\mathbb{Q}}$. We have the usual decomposition

$$A_F = A_{F,\infty} \times A_{F,f} \quad \text{and} \quad I_F = I_{F,\infty} \times I_{F,f}$$

into the finite and infinite part. We shall denote adelic variables by underlining them and we decompose

$$\underline{x} = (\underline{x}_\infty, \underline{x}_f)$$

A character in $F^\times \backslash I_F$ is a continuous homomorphism $\phi : F^\times \backslash I_F \to \mathbb{C}^\times$, we do not require it to be unitary. For any character we have a decomposition

$$\phi(\underline{x}) = \prod_v \phi_v(x_v) = \phi_\infty(\underline{x}_\infty) \cdot \phi_f(\underline{x}_f) \quad .$$

Occasionally we shall drop the subscript and write $\phi(x_v) = \phi_v(x_v)$, $\phi(\underline{x}_\infty) = \phi_\infty(\underline{x}_\infty)$ and $\phi(\underline{x}_f) = \phi_f(\underline{x}_f)$.

The Tate character is given by the absolute value

$$| \ | : F^\times \backslash I_F \to \mathbb{C}^\times$$

$$| \ | : \underline{x} \to |\underline{x}| = \prod_v |x_v|_v \tag{1.0.1}$$

<u>1.0.2</u> Let $G_o/F = GL_2/F$. We put

$$G/\mathbb{Q} = R_{F/\mathbb{Q}}(G_o/F)$$

where $R_{F/\mathbb{Q}}$ is the restriction of scalars For any subgroup $H_o/F \to G_o/F$ we denote by $H/\mathbb{Q} = R_{F/\mathbb{Q}}(H_o/F)$ the corresponding subgroup of $G/\mathbb{Q}$. Let B_o/F, U_o/F and T_o/F be the standard Borel subgroup of upper triangular matrices, its unipotent radical and the standard maximal torus in B_o/F respectively. Let $T_o^{(1)}/F \to T_o/F$ be the subtorus of elements with determinant 1. We shall identify

$$\begin{aligned} \gamma_o &: G_m/F \to T_o^{(1)}/F \\ \gamma_o &: a \to \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \end{aligned} \qquad (1.0.3)$$

for $a \in G_m(F)$. Then

$$\gamma : R_{F/\mathbb{Q}}(G_m/F) \xrightarrow{\sim} R_{F/\mathbb{Q}}(T_o^{(1)}/F) = T^{(1)}/\mathbb{Q} \quad . \qquad (1.0.4)$$

The positive simple root defines a homomorphism

$$\begin{aligned} \alpha_o &: B_o/F \to G_m/F \\ \alpha_o &: \begin{pmatrix} t_1 & u \\ 0 & t_2 \end{pmatrix} \to t_1/t_2 \quad . \end{aligned}$$

We get a homomorphism

$$\alpha : B/\mathbb{Q} \to R_{F/\mathbb{Q}}(G_m) \quad . \qquad (1.0.5)$$

Restricting this to the adele groups we get $\alpha_A : B(A) \to I_F$ and we put

$$|\alpha| = |\ | \circ \alpha_A \quad . \qquad (1.0.6)$$

<u>1.0.7</u> For any algebraic group $H/\mathbb{Q}$ and any ring A containing $\mathbb{Q}$ we write $H(A)$ for the group of A valued points. We shall abbreviate

$$H_\infty = H(\mathbb{R}) \quad .$$

From the definition of the restriction of scalars we get

$$G_\infty = G(\mathbb{R}) = G_o(\mathbb{R} \otimes F) = \prod_{v \in S_\infty} GL_2(F_v) \quad .$$

We choose a specific subgroup $K_\infty \subset G_\infty$ where $K_\infty = \prod_{v \in S_\infty} K_v$ and

$$K_v = SO(2,\mathbb{R}) \cdot Z_o(\mathbb{R}) = SO(2,\mathbb{R}) \cdot \mathbb{R}^x \quad \text{if} \quad F_v = \mathbb{R}$$

$$K_v = U(2) \cdot Z_o(\mathbb{C}) = U(2) \cdot \mathbb{C}^x \quad \text{if} \quad F_v \cong \mathbb{C} \quad .$$

Here Z_o/F is the centre of G_o/F . We notice that the K_v and hence K_∞ are always connected.

Let $X = G_\infty/K_\infty = \prod_{v \in S_\infty} GL_2(F_v)/K_v$ be the symmetric space associated to G_∞, K_∞ . We have a base point $x_o = K_\infty \in G_\infty/K_\infty$, of course X may have several connected components.

The group $GL_2(F_v)$ has two connected components if $F_v = \mathbb{R}$. We represent the components by matrices

$$\begin{pmatrix} \varepsilon & 0 \\ 0 & 1 \end{pmatrix} \in GL_2(\mathbb{R}) \qquad \varepsilon = \pm 1$$

and using these representatives we may view the group $\pi_o(G_\infty)$ of connected components as a subgroup

$$\pi_o(G_\infty) \subset G_\infty \tag{1.0.8}$$

which normalizes K_∞ .

1.1. Cohomology with coefficients

For any choice of an open compact subgroup $K_f \subset G(A_f)$ we put $K = K_\infty K_f \subset G(A)$ and define the space

$$S_K = G(\mathbb{Q})\backslash G(\mathbb{A})/K$$

(For a geometric description see [Ha4], 1.3.). Let

$$\rho : G \times_{\mathbb{Q}} \bar{\mathbb{Q}} \to GL(M)$$

be an irreducible representation of the algebraic group $G \times_{\mathbb{Q}} \bar{\mathbb{Q}}$, where M of course is a finite dimensional $\bar{\mathbb{Q}}$-vector space. We define a sheaf $\tilde{M}/S_K = \tilde{M}_\rho/S_K$ by describing its section over small open sets in S_K. Let $\pi : G(\mathbb{A})/K \to S_K$ be the projection, then we put for a "small" open set $U \subset S_K$

$$\tilde{M}(U) = \left\{ s : \pi^{-1}(U) \to M \;\middle|\; \begin{array}{l} s \text{ is locally constant and for} \\ \text{all } a \in G(\mathbb{Q}) \text{ and } u \in \pi^{-1}(U) \\ \text{we have } s(au) = \rho(a) \cdot s(u) \end{array} \right\} \quad (1.1.1)$$

We can also describe the stalks of these sheaves. For $x \in S_K$ the stalk is given by (comp. [Gr])

$$(1.1.2) \quad \tilde{M}_x = \{ s : \pi^{-1}(x) \to M \mid s(au) = \rho(a) \cdot s(u) \text{ for } a \in G(\mathbb{Q}), u \in \pi^{-1}(x) \}.$$

<u>1.1.3</u> At this point we have to be aware of the fact that in general the action of $G(\mathbb{Q})$ on $G(\mathbb{A})/K$ will always have fixed points even if we take K_f to be small. The reason for this is that $K_\infty \supset Z_\infty = Z(\mathbb{R})$. Hence we shall always have a subgroup E_{K_f} of the group E_o of units of $Z(\mathbb{Q}) = F^\times$ which is of finite index and which fixes every point in $G(\mathbb{A})/K$. The restriction of ρ to $Z \times_{\mathbb{Q}} \bar{\mathbb{Q}}$ is a one-dimensional character ω_ρ and if we want that $\tilde{M} \neq 0$ for some K_f we must have that $\omega_\rho | E_{K_f} \equiv 1$. This means that ω_ρ has to be the type of an algebraic Hecke character (see discussion in 2.5.). We shall assume that this is always the case.

1.2. We may now define the cohomology groups

$$H^{\bullet}(S_K,\tilde{M}) \quad .$$

If we take the description of S_K in [Ha4], I, 1.3. into account and if we assume that K_f is sufficiently small then this cohomology is the sum of the cohomology groups of the connected components and on a component we get

$$H^{\bullet}(\Gamma^{(i)}\backslash X,\tilde{M}) = H^{\bullet}(\Gamma^{(i)}/E_{K_f},M) \quad .$$

This of course implies also that $\dim H^{\bullet}(S_K,\tilde{M}) < \infty$. If we pass to a smaller compact open subgroup $K_f' \subset K_f$ we get an obvious map

$$H^{\bullet}(S_K,\tilde{M}) \to H^{\bullet}(S_{K'},\tilde{M})$$

and we define the limit

$$H^{\bullet}(\tilde{S},\tilde{M}) = \varinjlim_{K} H^{\bullet}(S_K,\tilde{M}) \quad .$$

1.2.1 As in |Ha4|, I we may consider $\tilde{S}$ to be a "symbolic" letter for a "space" whose cohomology is the right hand side. But it is actually very easy to verify that for

$$\tilde{S} = \varprojlim_{K_f} S_K$$

the above equation becomes true.

We check very easily that for any element (see 1.0.8)

$$\underline{g} = (\varepsilon,\underline{g}_f) \in \pi_0(G_\infty) \times G(A_f)$$

the multiplication from the right by $\underline{g}$ defines a map

$$S_K \to S_{\underline{g}^{-1}K\underline{g}}$$

(ε normalizes K_∞) which extends to a map between the sheaves. Hence we get also a map between the cohomolgy groups

$$H^\bullet(S_K,\tilde{M}) \to H^\bullet(S_{\underline{g}^{-1}K\underline{g}},\tilde{M})$$

and if we pass to the limit we get

$$H^\bullet(\tilde{S},\tilde{M}) \quad \text{is a} \quad \pi_0(G\ )\times G(A_f)\text{-module} \quad .$$

1.3. $\mathbb{Q}$-structures on the cohomology

At this general stage there is no point to assume that the representation

$$\rho \quad : \quad G \times_{\mathbb{Q}} \overline{\mathbb{Q}} \to GL(M)$$

is irreducible. We could as well start from any representation

$$\rho_1 \quad : \quad G/\mathbb{Q} \to GL(V)$$

where V is now a $\mathbb{Q}$-vector space. We gain that the cohomology groups $H^\bullet(\tilde{S},\tilde{V})$ are $\mathbb{Q}$-vector spaces, this fact will be important for our application. But for the statement of our results lateron it will be important to work with absolutely irreducible representations. One way out of course would be to start from an irreducible ρ_1 and to decompose $\rho_1 \times_{\mathbb{Q}} \overline{\mathbb{Q}}$ into its irreducible pieces and to keep track of the action of the Galois group of $\overline{\mathbb{Q}}/\mathbb{Q}$.

Here we propose a slightly different way how to recover the "$\mathbb{Q}$-structure" of the cohomology.

Let Λ be a finite set together with an action of the Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on Λ

$$\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \times \Lambda \to \Lambda$$

$$(\sigma,\lambda) \to \sigma \cdot \lambda \qquad .$$

Moreover, we assume that we have a $\bar{\mathbb{Q}}$-vector space V_λ for each $\lambda \in \Lambda$ and for any $\lambda \in \Lambda$ and $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ we have a σ-linear map

$$\phi_{\lambda,\sigma} : V_\lambda \to V_{\sigma\lambda} \quad .$$

We want this system $\{V_\lambda, \phi_{\lambda,\sigma}\}_{\lambda,\sigma}$ to satisfy the following two conditions.

1) (Continuity) For $v \in V_\lambda$ there exists a finite extension $\mathbb{Q}(v)$ of $\mathbb{Q}$ such that

$$\phi_{\lambda,\sigma}(v) = v \qquad \text{for all} \qquad \sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}(v)) \quad .$$

2) For all $\lambda \in \Lambda$ and $\sigma,\tau \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ we have a commutative diagram

$$\begin{tikzcd}
V_\lambda \arrow[rr, "\phi_{\lambda,\sigma}"] \arrow[dr, "\phi_{\lambda,\sigma\tau}"'] & & V_{\sigma\lambda} \arrow[dl, "\phi^{\sigma}_{\lambda,\tau}"] \\
& V_{\sigma\tau\lambda} &
\end{tikzcd}$$

If these conditions are satisfied then we say that $\{\phi_{\lambda,\sigma}\}_{\lambda,\sigma}$ defines a $\mathbb{Q}$-structure on $\{V_\lambda\}_\lambda$ or that $\{V_\lambda, \phi_{\lambda,\sigma}\}_{\lambda,\sigma}$ has a $\mathbb{Q}$-structure. The $\phi_{\lambda,\sigma}$ are called the transition maps of the $\mathbb{Q}$-structure. It is a simple excercise in Galois theory to prove that

$$W_o = \{(\ldots v_\lambda \ldots)_{\lambda\in\Lambda} \in \oplus V_\lambda \mid \phi_{\lambda,\sigma}(v_\lambda) = v_{\sigma\lambda} \quad \text{for all} \quad \lambda,\sigma\}$$

is a $\mathbb{Q}$-vector space and $W_o \otimes_{\mathbb{Q}} \bar{\mathbb{Q}} \simeq \bigoplus_{\lambda\in\Lambda} V_\lambda$. We call W_o the underlying $\mathbb{Q}$-vector space.

If we have two such systems $\{V_\lambda, \Phi_{\lambda,\sigma}\}_{\lambda\in\Lambda}$ and $\{X_\mu, \Psi_{\mu,\sigma}\}_{\mu\in\underline{M}}$ [1] then a $\mathbb{Q}$-rational map between them is the collection of data

1) A subset $\underline{P} \subset \Lambda\times\underline{M}$ which is invariant under the action of the Galois group.

•2) For each $(\lambda,\mu) \in \underline{P}$ we have a map

$$\Phi_{\lambda,\mu} : V_\lambda \to X_\mu$$

such that for all $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ the diagrams

$$\begin{array}{cccc} \Phi_{\lambda,\mu} : & V_\lambda & \longrightarrow & X_\mu \\ & \Big\downarrow \Phi_{\lambda,\sigma} & & \Big\downarrow \Psi_{\mu,\sigma} \\ & V_{\sigma\lambda} & \longrightarrow & X_{\sigma\mu} \end{array}$$

commutes.

We may extend the system of maps $\Phi_{\lambda,\mu}$ to all of $\Lambda\times\underline{M}$ by putting $\Phi_{\lambda,\mu} = 0$ if $(\lambda,\mu) \notin \underline{P}$. It is clear that then this system of maps defines a linear map

$$\Phi : \bigoplus_{\lambda\in\Lambda} V_\lambda \to \bigoplus_{\mu\in\underline{M}} X_\mu$$

and the commutation condition in 2) is equivalent to the condition that Φ is the extension of a map Φ_o between the underlying $\mathbb{Q}$-vector spaces.

We do not need the finiteness of Λ and $\underline{M}$, we only need that the Galois group has finite orbits on both sets.

<u>1.4.</u> We want to introduce a $\mathbb{Q}$-structure on the system $H^\bullet(\tilde{S},\tilde{M})$ if we conjugate M by

1) Here $\underline{M}$, $\underline{P}$ are the capital greek letters My , Rho .

the action of the Galois group. This is very easy. We observe that

$$G \times_{\mathbb{Q}} \overline{\mathbb{Q}} = \prod_{\tau : F \to \overline{\mathbb{Q}}} GL_2/\overline{\mathbb{Q}}$$

and hence our irreducible representation ρ is a tensor product

$$M = \bigotimes_{\tau : F \to \overline{\mathbb{Q}}} M_\tau$$

where M_τ is an irreducible representation of $GL_2/\overline{\mathbb{Q}}$. We realize these representations explicitely, we write

$$M_\tau = M(d(\tau),\nu(\tau)) \qquad d(\tau) \in \mathbb{N} \quad , \quad \nu(\tau) \in \mathbb{Z}$$

where $M(d(\tau),\nu(\tau))$ is the $\overline{\mathbb{Q}}$-vector space of homogeneous polynomials of degree $d(\tau)$ in two variables X_τ , Y_τ on which $GL_2(\overline{\mathbb{Q}})$ acts by

$$\rho_\tau\begin{pmatrix} a & b \\ c & d \end{pmatrix} \;:\; P(X_\tau,Y_\tau) \to P(aX_\tau+cY_\tau, bX_\tau+dY_\tau)\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\nu(\tau)} \quad .$$

The character module $\mathrm{Hom}(T \times_{\mathbb{Q}} \overline{\mathbb{Q}}, G_m) = X(T \times_{\mathbb{Q}} \overline{\mathbb{Q}})$ is isomorphic to

$$X(T \times_{\mathbb{Q}} \overline{\mathbb{Q}}) = \bigoplus_{\tau : F \to \overline{\mathbb{Q}}} X(T_o)$$

and the pair $(d(\tau),\nu(\tau))$ defines a character

$$\lambda_\tau \;:\; T_o \to G_m$$

$$\lambda_\tau \;:\; \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \to t_1^{d(\tau)} \, (t_1 t_2)^{\nu(\tau)} \quad .$$

Hence we can consider M as determined by the character

$$\lambda = \lambda(M) = (\ldots\lambda_\tau\ldots)_{\tau : F \to \overline{\mathbb{Q}}} \in X(T \times_{\mathbb{Q}} \overline{\mathbb{Q}})$$

On the characters we have an action of the Galois group for $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ we put

$$\lambda^{\sigma} = (\ldots\lambda_{\sigma\tau}\ldots)_{\tau:F\to\bar{\mathbb{Q}}} \quad .$$

So the value of λ^{σ} at τ is $\lambda_{\sigma\tau}$. We have an obvious σ-linear map

$$\phi_{\lambda,\sigma} \; : \; M = M(\lambda) \to M(\lambda^{\sigma})$$

given by

$$\bigotimes_{\tau:F\to\bar{\mathbb{Q}}} P_{\tau}(X_{\tau},Y_{\tau}) \to \bigotimes_{\tau:F\to\bar{\mathbb{Q}}} P^{\sigma}_{\sigma\tau}(X_{\tau},Y_{\tau})$$

where the superscript σ means that we apply σ to the coefficients. If Λ is the orbit of λ under the action of the Galois group then the system

$$\{M(\lambda),\phi_{\lambda,\sigma}\}$$

has a $\mathbb{Q}$-structure. The action of $G(\mathbb{Q})$ on the $M(\lambda)$ commutes with the transition maps, we get also σ-linear maps between the sheaves $\tilde{M}(\lambda)$ and hence we get a system of transition maps

$$\phi^{\bullet}_{\lambda,\sigma} \; : \; H^{\bullet}(\tilde{S},\tilde{M}(\lambda)) \to H^{\bullet}(\tilde{S},\tilde{M}(\lambda^{\sigma}))$$

and this defines a $\mathbb{Q}$-structure on the system of cohomology groups $\{H^{\bullet}(\tilde{S},\tilde{M}(\lambda)),\phi^{\bullet}_{\lambda,\sigma}\}_{\lambda\in\Lambda}$, on which $\pi_0(G_{\infty}) \times G(\mathbb{A}_f)$ acts by $\mathbb{Q}$-rational maps.

II. The cohomology of the boundary

2.1. We look at the embedding

$$S_K \to \bar{S}_K$$

where $\bar{S}_K$ is the Borel-Serre compactification of the space S_K ([B-S], § 9). We want to recall briefly the main properties of this compactification. We consider the projection map

$$p \;:\; B(\mathbb{Q})\backslash G(\mathbb{A})/K_\infty K_f \rightarrow G(\mathbb{Q})\backslash G(\mathbb{A})/K_\infty K_f = S_K \quad .$$

On the left hand side we define a level function

$$n \;:\; B(\mathbb{Q})\backslash G(\mathbb{A})/K_\infty K_f \rightarrow \mathbb{R}_{>0}$$

by the following formula: Write any element $\underline{g} \in G(\mathbb{A})$ in the form

$$\underline{g} = \underline{b}\,\underline{k}$$

where $\underline{b} \in B(\mathbb{A})$ and $\underline{k} \in K_\infty \times \prod_{\mathfrak{p}} GL_2(\mathcal{O}_{\mathfrak{p}})$ and we put

$$n(\underline{g}) = |\alpha|\,(\underline{b})$$

(1.0.6). Then the reduction theory implies that for a suitably large choice of $c > 0$ the map p induces an open inclusion

$$\underline{p} \;:\; n^{-1}((c,\infty)) \rightarrow G(\mathbb{Q})\backslash G(\mathbb{A})/K_\infty K_f$$

(comp. for instance |Ha4|, 3.1 and |Ha2|, § 3), and for any $t_o \in (c,\infty)$ we have by the geodesic action (|B-S|, |Ha2|, § 3)

$$n^{-1}((c,\infty)) = n^{-1}(t_o) \times (c,\infty) \quad .$$

We get the Borel-Serre compactification by embedding

$$n^{-1}((c,\infty)) = n^{-1}(t_o) \times (c,\infty) \rightarrow n^{-1}(t_o) \times (c,\infty] \quad .$$

This allows us to extend the sheaves to the compactification, we have obviously

$$H^\bullet(S_K,\tilde{M}) \simeq H^\bullet(\bar{S}_K,\tilde{M})$$

and

$$H^{\bullet}(\partial\bar{S}_K,\tilde{M}) \simeq H^{\bullet}(B(\mathbb{Q})\backslash G(A)/K_\infty K_f,\tilde{M})$$

wher ∂S_K is the boundary. The last assertion is a generalization of Proposition 3.1 in [Ha4] to this case.

2.2. Again we pass to the limit and write

$$\lim_{\substack{\to \\ K_f}} H^{\bullet}(\partial\bar{S}_K,\tilde{M}) = H^{\bullet}(\partial\tilde{S},\tilde{M}) \quad ,$$

the right hand side is now a $\pi_o(G_\infty) \times G(A_f)$-module and the map

$$r : H^{\bullet}(\tilde{S},\tilde{M}) \to H^{\bullet}(\partial\tilde{S},\tilde{M})$$

is a morphism of $\pi_o(G_\infty)\times G(A_f)$-modules. We may also apply the consideration of 1.5.1 to the right hand side and introduce a $\mathbb{Q}$-structure on the system $\{H^{\bullet}(\partial\tilde{S},\tilde{M}(\lambda))\}_\lambda$ with the same set Λ and the system of maps

$$r_{\lambda,\lambda} : H^{\bullet}(\tilde{S},\tilde{M}(\lambda)) \to H^{\bullet}(\partial\tilde{S},\tilde{M}(\lambda))$$

will be a $\mathbb{Q}$-rational map in the sense of 1.5.

2.3. The aim of this section is to compute the cohomology of the boundary as a $\pi_o(G_\infty)\times G(A_f)$-module. We also have to keep track of the $\mathbb{Q}$-structure of the system of cohomology groups if we vary M over its conjugates under the action of the Galois group. At the end we will have a generalization of Theorem 1 in [Ha4], which will be the Theorem 1 in 2.6.

2.3.1 We need an algebraic version of the van-Est theorem [vE]. We put ourselves in the following general situation: Let $U/\mathbb{Q}$ be a connected unipotent algebraic group, let $\check{\mathfrak{u}}/\mathbb{Q} = \mathrm{Lie}(U/\mathbb{Q})$ be its Lie algebra. Let $L/\mathbb{Q}$ be any extension of $\mathbb{Q}$ and let us assume that we have a representation

$$\rho \;:\; U \times_{\mathbb{Q}} L \to GL(M)$$

where M is a finite dimensional L-vector space. Let $\Gamma_U \subset U(\mathbb{Q})$ be an arithmetic subgroup, then the quotient space $\Gamma_U \backslash U_\infty$ will be a compact nil-manifold. The same construction as the one given in 1.1. gives us a sheaf $\tilde{M}$ on $\Gamma_U \backslash U_\infty$ and the van-Est theorem says that we have a natural isomorphism

$$H^\bullet(\Gamma_U \backslash U_\infty, \tilde{M}) \;\simeq\; H^\bullet(\check{\mu}, M) \quad . \tag{2.3.2}$$

Usually this theorem is stated for the case $L = \mathbb{R}$ or $\mathbb{C}$ and the proof uses the de-Rham isomorphism.

But there is a very simple algebraic way to prove it. At first we notice that

$$H^0(\Gamma_U \backslash U_\infty, \tilde{M}) \;=\; M^{\Gamma_U} \;=\; M^{\check{\mu}} \;=\; H^0(\check{\mu}, M)$$

since Γ_U is Zariski-dense in U . Then one uses the fact that $U/\mathbb{Q}$ and as well M/L can be filtered so that the successive quotients become one dimensional and a simple argument with spectral sequences shows that the functor $H^\bullet(\check{\mu}, M)$ is effaceable ([ ],). Hence we get a natural map from

$$H^\bullet(\check{\mu}, M) \;\to\; H^\bullet(\Gamma_U \backslash U_\infty, \tilde{M}) \quad .$$

Using the filtration argument a second time, now also for the right hand side, we get that this map is an isomorphism.

After this excursion into homological algebra we come back to our original situation. We put $K_\infty^B = B_\infty \cap K_\infty$ and $K_f^B = K_f \cap B(A_f)$. We have an embedding

$$B(\mathbb{Q})\backslash B(A)/K_\infty^B K_f^B \to B(A)\backslash G(A)/K_\infty K_f$$

where the image is a union of connected components of the right hand side ([Ha4], p. 109). We introduce the $\pi_o(B_o)\times G(A_f)$-module

$$H_B^\bullet(\tilde{M}) = \lim_{\substack{\to \\ K_f}} H^\bullet(B(\mathbb{Q})\backslash B(A)/K_\infty^B K_f^B,\tilde{M})$$

and as in [Ha4], p. 117 we show that

$$H^\bullet(\partial\tilde{S},\tilde{M}) = \mathrm{Ind}_{\pi_o(B_\infty)\times B(A_f)}^{\pi_o(G_\infty)\times G(A_f)} H_B^\bullet(\tilde{M}) \quad .$$

Hence we are left with the computation of $H_B^\bullet(\tilde{M})$. We consider the torus $T/\mathbb{Q} = (B/U)/\mathbb{Q}$ and let K_∞^T, K_f^T be the images of K_∞^B, K_f^B in T_∞, $T(A_f)$. We have a projection

$$p : B(\mathbb{Q})\backslash B(A)/K_\infty^B K_f^B \to T(\mathbb{Q})\backslash T(A)/K_\infty^T K_f^T \quad .$$

This is a fibration whose fiber over the point 1 is

$$U(\mathbb{Q})\backslash U(A)/K_f^U = \Gamma_U\backslash U_\infty$$

with $\Gamma_U = U(\mathbb{Q}) \cap U_\infty \cdot K_f^U$. We apply 2.3.1 and get

$$H^\bullet(\Gamma_U\backslash U_\infty,\tilde{M}) = H^\bullet(\check{\mu},M) \quad .$$

The right hand side is a module for the Torus $T\times_{\mathbb{Q}}\bar{\mathbb{Q}}$, we get this module structure since M and $\check{\mu}$ are B-modules. Using the same construction as in 1.1. we get a sheaf

$$H^\bullet(\check{\mu},M) \qquad \text{on} \qquad T(\mathbb{Q})\backslash T(A)/K_\infty^T K_f \quad .$$

It is now a little bit painful but straightforward to check that this sheaf is actually the sheaf of cohomology groups of the fibers the sheaf of cohomology groups of the fibers of the map p with coefficients in M. Hence the spectral sequence for the map p becomes

$$H^\bullet(T(\mathbb{Q})\backslash T(A)/K_\infty^T K^T, \widetilde{H^\bullet(\check{\mu},M)}) \implies H^\bullet(B(\mathbb{Q})\backslash B(A)/K_\infty^B K_f^B,\tilde{M}) \quad .$$

We shall see that this spectral sequence degenerates and we have to do the following:

A) Compute the $T \times_{\mathbb{Q}} \overline{\mathbb{Q}}$-module $H^\bullet(\check{\mu},M)$ and decompose it into one dimensional pieces.

B) For each one dimensional piece we have to compute

$$H^\bullet(T(\mathbb{Q})\backslash T(A)/K_\infty^T K_f^T, \tilde{\overline{\mathbb{Q}}}_\gamma)$$

as a $\pi_0(T_\infty) \times T(A_f)$-module. Here γ is a character on $T \times_{\mathbb{Q}} \overline{\mathbb{Q}}$, $\overline{\mathbb{Q}}_\gamma$ is a one dimensional $\overline{\mathbb{Q}}$-vector space on which $T \times_{\mathbb{Q}} \overline{\mathbb{Q}}$ acts by γ.

We go back to A). To compute the cohomology we have to apply the Künneth-formula and get

$$H^\bullet(\check{\mu},M) = H^\bullet(\check{\mu}\otimes_{\mathbb{Q}} \overline{\mathbb{Q}},M) \xrightarrow{\sim} \bigotimes_{\tau : F \to \overline{\mathbb{Q}}} H^\bullet(\check{\mu}_0,M_\tau)$$

where $\check{\mu}_0$ is the standard unipotent Lie-algebra of U_0/F extended by τ to $\overline{\mathbb{Q}}$. We have to be careful since this isomorphism is not canonical, the identification depends on an ordering of the set $\{\tau | \tau : F \to \overline{\mathbb{Q}}\}$ and may change sign if we choose another order.

But now it is very easy to find the characters of $T \times_{\mathbb{Q}} \overline{\mathbb{Q}}$ occuring in $H^\bullet(\check{\mu},M)$. For each τ the torus $T_0^{(1)}/F$ acts upon M_τ and has a highest and lowest weight vector

$$e_{d(\tau)} = X_\tau^{d(\tau)} \quad , \quad e_{-d(\tau)} = Y_\tau^{d(\tau)}$$

and the weights on the torus are $t^{d(\tau)}$, $t^{-d(\tau)}$ respectively (1.0.3). It acts on $\check{\mu}_o$ by the weight t^2 . In $\check{\mu}_o$ we have the generator

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and let u_α^v be the element in $\mathrm{Hom}(\check{\mu}_o, F)$ which does

$$u_\alpha^v \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 1 \quad .$$

Then it is well known and obvious that

$$H^o(\check{\mu}_o, M_\tau) = \bar{\mathbb{Q}} e_{d(\tau)} \quad , \quad H^1(\check{\mu}_o, M_\tau) = \bar{\mathbb{Q}} e_{-d(\tau)} \otimes u_\alpha^v \quad .$$

Hence the torus $T_o = \left\{ \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \right\} \subset GL_2/F$ is acting on the cohomology $H^\bullet(\check{\mu}_o, M_\tau)$ by the two characters

$$te_{d(\tau)} = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} e_{d(\tau)} = t_1^{d(\tau)} (t_1 t_2)^{\nu(\tau)} e_{d(\tau)}$$

$$te_{-d(\tau)} \otimes u_\alpha^v = t_2^{d(\tau)+\nu(\tau)+1} t_1^{\nu(\tau)-1} e_{-d(\tau)} \otimes u_\alpha^v \quad . \tag{2.3.2}$$

If $\gamma_\tau \in X_o(T)$ is one of the above two characters we put $\deg(\gamma_\tau) = 0$ or 1 according to whether its eigenclass sits in degree 0 or 1 (or whether it's the first or second).

Now we get from the definition of restriction of scalars for the character-modul of $T \times_{\mathbb{Q}} \bar{\mathbb{Q}}$

$$X(T \times \bar{\mathbb{Q}}) = \bigoplus_{\tau : F \to \bar{\mathbb{Q}}} X(T_o)$$

and

$$H^{\bullet}(\check{\mu},M) = \bigoplus_{\gamma \in \mathrm{Coh}(M)} \bar{\mathbb{Q}}_{\gamma} = \bigoplus_{\gamma \in \mathrm{Coh}(M)} H^{\bullet}(\check{\mu},M)(\gamma) \qquad (2.3.3)$$

where $\gamma \in \mathrm{Coh}(M)$ are the characters

$$\gamma = (\ldots\gamma_{\tau}\ldots)_{\tau : F \to \bar{\mathbb{Q}}}$$

and γ_{τ} is one of the two characters in 2.3.2.

2.4. We want to discuss the $\mathbb{Q}$-rationality of the decomposition (2.3.3). Given λ which determines the module M (see 1.4.) we looked at the orbit Λ of λ in $X(T \times \bar{\mathbb{Q}})$ and we have a $\mathbb{Q}$-structure

$$\{M(\lambda),\phi_{\lambda,\sigma}\}_{\lambda \in \Lambda} \quad .$$

For $\lambda \in \Lambda$ we may look at the set of γ in $\mathrm{Coh}(M(\lambda))$ and we define

$$\underline{M} = \{\mu = (\lambda,\gamma) \mid \lambda \in \Lambda \text{ and } \gamma \in \mathrm{Coh}(M(\lambda))\} \quad .$$

We have a Galois action on $\underline{M}$ and the projection

$$\underline{P} : \underline{M} \to \Lambda$$

commutes with the action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. But then it is obvious that the map $\phi_{\lambda,\sigma}$ induces

$$\phi^{\bullet}_{\mu,\sigma} : H^{\bullet}(\check{\mu},M(\lambda))(\gamma) \to H^{\bullet}(\check{\mu},M(\lambda^{\sigma}))(\gamma^{\sigma})$$

and we get a $\mathbb{Q}$-structure on the system

$$\{H^{\bullet}(\check{\mu},M(\lambda))(\gamma),\phi^{\bullet}_{\mu,\sigma}\}_{\mu = (\lambda,\gamma) \in \underline{M}} \quad .$$

The spaces $H^{\bullet}(\check{\mu},M(\lambda))(\gamma)$ are of dimension 1 over $\bar{\mathbb{Q}}$. Hence we can choose a system of generators

$$e(\lambda,\gamma) = e(\mu) \in H^{\bullet}(\check{\mu},M(\lambda))(\gamma)$$

which are mapped onto each other by the transition maps, the means that for $\mu = (\lambda,\gamma)$ and $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$

$$\phi^{\bullet}_{\mu,\sigma}(e(\mu)) = e(\mu^{\sigma}) \quad .$$

The existence of such system is a consequence of Hilbert's theorem 90. We shall call such a system a rational system of generators.

For $\mu \in \underline{M}$ we define the field of definition $\mathbb{Q}(\mu)$ in the obvious way

$$\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}(\mu)) = \{\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \mid \mu^{\sigma} = \mu\} \cdot$$

If we have two rational systems of generators

$$\{e(\mu)\} \quad \text{and} \quad \{e'(\mu)\}$$

then we have $e'(\mu) = a(\mu)e(\mu)$ where $a(\mu) \in \mathbb{Q}(\mu)$ and $a(\mu)^{\sigma} = a(\mu^{\sigma})$.

2.4.1 There is a very explicit choice of a rational system of generators. For any $\mu = (\lambda,\gamma)$ we shall construct a generator

$$e(\lambda,\gamma,\omega) \in H^{\bullet}(\check{\mu},M)(\gamma)$$

which depends on a total ordering $<$ of the set $\{\tau \mid \tau: F \to \bar{\mathbb{Q}}\}$. This generator is defined in the following way: Our character γ is of the form

$$\gamma = (\ldots\gamma_{\tau}\ldots)_{\tau}$$

and for each τ the character γ_{τ} is one of the characters in (2.3.2). We consider the set

$$I(\gamma) = \{\tau \mid \gamma_{\tau} \text{ is the character on } H^{1}(\check{\mu}_{\delta},M_{\tau})\}$$

and we numerate the elements of $I(\gamma)$ according to the total order

$$I(\gamma) = \{\tau_1,\tau_2\ldots,\tau_s\} \quad .$$

Then s is the degree of the cohomology group $H^\bullet(\check{\mu},M)(\gamma) = H^s(\check{\mu},M)(\gamma)$.
We define our generator by giving an explicit representative

$$\xi(\lambda,\gamma,<) \in \mathrm{Hom}(\Lambda^s(\bigoplus_\tau \check{\mu}_o),M)$$

for it. Let $u_{\alpha,\tau} \in \oplus \check{\mu}_o$ be the element whose τ component is $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and whose other components are zero, then

$$\xi(\lambda,\gamma,<)(u_{\alpha,\tau_1},\ldots,u_{\alpha,\tau_s}) = \bigotimes_{\tau\in I(\gamma)} e_{-d(\tau)} \otimes \bigotimes_{\tau\notin I(\gamma)} e_{d(\tau)}$$

and $\xi(\lambda,\gamma,<)$ is zero on all other s-tupels.

Then is obvious that

$$\phi^\bullet_{\mu,\sigma}(e(\lambda,\gamma,<)) = e(\lambda^\sigma,\gamma^\sigma,<^\sigma) \quad .$$

To get rational generators we have to look at the stabilizer of $(\lambda,\gamma) = \mu$

$$\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}(\mu)) \subset \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \quad .$$

The group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}(\mu))$ fixes the set $I(\gamma)$ and $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}(\mu))$ induces a permutation $p(\sigma,\gamma)$ on $I(\gamma)$. Then it is clear that for such a σ

$$e(\lambda,\gamma,<^\sigma) = \mathrm{sgn}(p(\sigma,\gamma)) \cdot e(\lambda,\gamma,<) \quad .$$

We have a homomorphism

$$\begin{aligned} \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}(\mu)) &\to \mathbb{Z}/2\mathbb{Z} \\ \sigma &\to \mathrm{sgn}(p(\sigma,\gamma)) \end{aligned}$$

and this defines for us a square class

$$d(\mu) \in \mathbb{Q}(\mu)^{\times}/(\mathbb{Q}(\mu)^{\times})^2 \quad .$$

If we choose a square root $\delta(\mu) = \sqrt{d(\mu)}$ in $\overline{\mathbb{Q}}$, then the element $\delta(\mu)e(\lambda,\gamma,<)$ will be invariant under $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\mu))$ and the system

$$\{\delta(\mu)^{\sigma} e(\lambda^{\sigma},\gamma^{\sigma},<^{\sigma})\}_{\sigma}$$

will be a rational system of generators.

We have done A) and go to B) and tackle the cohomology groups

$$H^{\bullet}(T(\mathbb{Q})\backslash T(A)/K_{\infty}^{T}K_{f}^{T},\tilde{\mathbb{Q}}_{\gamma})$$

where $\gamma \in \mathrm{Coh}(M)$. But at this point it seems to be reasonable first to study the cohomology of tori from a more general point of view. So we go to

2.5. Intermission: The cohomology of tori

The general situation to be studied is this: Let $T/\mathbb{Q}$ be an arbitrary torus over $\mathbb{Q}$, let $X = X(T \times_{\mathbb{Q}} \overline{\mathbb{Q}})$ be its character module. Any $\gamma \in X$ gives us a representation

$$\gamma \; : \; T \times_{\mathbb{Q}} \overline{\mathbb{Q}} \to GL(\overline{\mathbb{Q}}_{\gamma}) \quad .$$

Here $\overline{\mathbb{Q}}$ is the vector space $\overline{\mathbb{Q}}$ on which $T \times_{\mathbb{Q}} \overline{\mathbb{Q}}$ acts by γ. In $T_{\infty} = T(\mathbb{R})$ we choose a connected subgroup K_{∞}^{T} which is of the form

$$K_{\infty}^{o} \cdot Z(\mathbb{R})^{o} = K_{\infty}^{T,o} \; .$$

Here K_{∞}^{o} is the connected component of the maximal compact subgroup in T_{∞}, the group Z is a subtorus of the torus $T/\mathbb{Q}$ and $Z(\mathbb{R})^{o}$ is the connected component of its real valued points. We define

$$S_{K}^{T} = T(\mathbb{Q})\backslash T(A)/K_{\infty}^{T,o} \cdot K_{f}^{T}$$

for any open compact subgroup $K_f^T \subset T(A_f)$. The representation γ defines a sheaf $\tilde{\tilde{\mathbb{Q}}}_\gamma$ on S^T (see 1.1.) and we are interested in the structure of the $\pi_0(T_\infty) \times T(A_f)$-module

$$H^\bullet(\tilde{S}^T, \tilde{\tilde{\mathbb{Q}}}_\gamma) = \varinjlim_{K_f} H^\bullet(S_K^T, \tilde{\tilde{\mathbb{Q}}}_\gamma) \quad .$$

(Here it is not necessary to pass to the limit in order to get a module structure.) The answer to this question is given in proposition 2.6.1 .

<u>2.5.1 Remark</u>: The choice of the torus $Z/\mathbb{Q}$ will turn out to be rather irrelevant. If we take $Z/\mathbb{Q}$ to be the split component itself then we have the advantage that S_K^T will be compact, if not we get something compact times an uninteresting factor $(\mathbb{R}^\times_{>0})^1$. In general the choice of $Z/\mathbb{Q}$ is dictated by induction hypotheses. In our special case for instance it is the choice of the $K_\infty \subset G(\mathbb{R})$ in 1.0. which will dictate the further choices. We made the choice of K_∞ in 1.0. in order to kill some absolutely uninteresting contributions to the cohomology.

The above cohomology will be described in terms of algebraic Hecke characters of type γ and we want to recall briefly the definition and basic properties of these (see [Se], II, § 3).

<u>2.5.2</u> The (rational) character γ defines a homomorphism

$$\gamma_\infty \;:\; \begin{tikzcd} T(\mathbb{R}) \arrow[rr] \arrow[dr, hook] & & \mathbb{C}^\times \\ & T(\mathbb{C}) \arrow[ur, "\gamma"'] & \end{tikzcd}$$

A continuous homomorphism

$$\phi \;:\; T(\mathbb{Q})\backslash T(A) \to \mathbb{C}^\times$$

is called an algebraic Hecke character of type γ if the infinite component satisfies

$$\phi_\infty\Big|_{T^O(\mathbb{R})} = \gamma_\infty^{-1}\Big|_{T^O(\mathbb{R})} \tag{2.5.2.1}$$

where $T^O(\mathbb{R})$ is the connected component of the identity.

<u>2.5.3</u> We can recognize an algebraic Hecke character of type γ from its restriction

$$\phi_f \; : \; T(A_f) \to \mathbb{C}^\times \quad .$$

First of all we observe the following fact: If $a \in T(\mathbb{Q}) \cap T^O(\mathbb{R}) = T_+(\mathbb{Q})$ (i.e. totally positive) and if $\bar{a}$ is the projection of a to $T(A_f)$ then (2.5.2.1) implies

$$\phi_f(\bar{a}) = \gamma(a) \in \bar{\mathbb{Q}}^\times \quad . \tag{2.5.3.1}$$

If we use the fact that ϕ_f has to be trivial on an open compact subgroup K_f^T and that $T(\mathbb{Q})\backslash T(A_f)/K_f^T$ is finite we get that ϕ_f takes its values in $\bar{\mathbb{Q}}^*$. If we now assume

$$\phi_f \; : \; T(A_f) \to \bar{\mathbb{Q}}^\times$$

is given, continuous and satisfies (2.5.3.1) then we may define ϕ on $T^O(\mathbb{R}) \times T(A_f)$ by

$$\phi(\underline{t}_\infty, \underline{t}_f) = \gamma_\infty^{-1}(\underline{t}_\infty) \cdot \phi_f(\underline{t}_f)$$

and ϕ is a homomorphism from

$$\phi \; : \; T_+(\mathbb{Q})\backslash T^O(\mathbb{R}) \times T(A_f) \to \mathbb{C}^\times$$

which estends by a theorem of Tate ([Kn]), which says that $T(\mathbb{Q}) \to \pi_o(T_\infty)$ is surjective, to an algebraic Hecke character of type γ .

2.5.4 We may read (2.5.2.1) in a different way. Given ϕ and γ we could say that there exists a signature character

$$\varepsilon \; : \; \pi_o(T_\infty) \to \{\pm 1\}$$

such that

$$\varepsilon \cdot \gamma_\infty^{-1} = \phi_\infty \quad . \tag{2.5.4.1}$$

We shall call ε the signature of the pair ϕ,γ . We want to extend ϕ_f by abuse of language to

$$\phi_f \; : \; \pi_o(T_\infty) \times T(A_f) \to \overline{\mathbb{Q}}^\times$$

by the obvious formula

$$\phi_f(\eta,\underline{t}_f) = \varepsilon(\eta) \cdot \Phi_f(t_f) \tag{2.5.4.2}$$

If we have $a \in T(\mathbb{Q})$ and project it to $(\overline{a}_\infty,a_f) \in \pi_o(T_\infty) \times T(A_f)$ then we still have $\Phi_f(\overline{a}_\infty,a_f) = \gamma(a)$.

2.5.5 The (rational) character γ has to satisfy a certain condition if we want to have algebraic Hecke characters of type γ . Since such an algebraic Hecke character has to be trivial on some $K_f^T \quad T(A_f)$ it follows that it has to be trivial on

$$E_+(K_f^T) = \{a \in T_+(\mathbb{Q}) \mid \overline{a} \in K_f^T\}$$

which is subgroup of finite index in the group $E_o(T)$ of units in $T(\mathbb{Q})$ (elements which are in the maximal compact subgroup at all places). Since γ is rational it has to vanish on the Zariski closure $\overline{E(K_f^T)}$. If we use

Chevalley's theorem ([Ch],) which guarantees that every subgroup of finite index in $E_o(T)$ is a congruence subgroup we find the following fact:

2.5.5.1 A character γ admits algebraic Hecke characters of type γ if and only if γ is trivial on the torus $H/\mathbb{Q} \to T/\mathbb{Q}$ which is the connected component of the Zariski closure of the units $E_o(T)$ in $T/\mathbb{Q}$.

2.6. We now come back to the computation of

$$H^\bullet(S_K^T, \tilde{\tilde{\mathbb{Q}}}_\gamma) \quad .$$

Let us assume K_f^T sufficiently small so that $E(K_f^T) = \{a \in T(\mathbb{Q}) \mid \bar{a} \in K_f^T\}$ has no elements of finite order and all its elements are in $T^o(\mathbb{R})$. Then a connected component of S_K^T is of the form

$$T(\mathbb{Q})\backslash T(\mathbb{Q}) \cdot T^o(\mathbb{R}) \cdot \underline{t} \cdot K_\infty^{T,o} K_f^T / K_\infty^T K_f^T = E(K_f^T)\backslash T^o(\mathbb{R}) \cdot \underline{t}/K_\infty^T$$

and we have to compute first the cohomology of such a component

$$H^\bullet(E(K_f^T)\backslash T^o(\mathbb{R}) \cdot \underline{t}/K_\infty^{T,o}, \tilde{\tilde{\mathbb{Q}}}_\gamma) \ .$$

The space $T^o(\mathbb{R})\,\underline{t}/K_\infty^{T,o}$ is contractible. If we put $\Gamma_T = E(K_f^T)$ and $\Gamma_Z = \Gamma_T \cap Z(\mathbb{Q})$ then Γ_T/Γ_Z is a free abelian group which acts without fixed points on $T^o(\mathbb{R})\,\underline{t}/K_\infty^{T,o}$. (This is the place where the choice of Z and hence K_∞^T enters.) It is clear that $\tilde{\tilde{Q}}_\gamma$ is equal to zero unless $\gamma|\Gamma_Z \equiv 1$ (comp. 1.1.3.), so we shall assume this. Then we get

$$H^\bullet(\Gamma_T\backslash T^o(\mathbb{R}) \cdot \underline{t}/K_\infty^{T,o}, \tilde{\tilde{\mathbb{Q}}}_\gamma) = H^\bullet(\Gamma_T/\Gamma_Z, \bar{\mathbb{Q}}_\gamma) \quad .$$

The group Γ_T/Γ_Z acts upon $\bar{\mathbb{Q}}_\gamma$ by the restriction of γ to Γ_T and then it is clear that

$$H^{\bullet}(\Gamma_T/\Gamma_Z,\bar{\mathbb{Q}}_\gamma) = \begin{cases} 0 & \gamma|\Gamma_T \not\equiv 1 \\ \Lambda^{\bullet}(\mathrm{Hom}(\Gamma_T/\Gamma_Z,\bar{\mathbb{Q}})) & \gamma|\Gamma_T \equiv 1 \end{cases} .$$

Of course for small K_f^T the group Γ_T/Γ_Z is simply a subgroup of finite index in the group of units in T/Z, so it is free abelian and if we pass to a smaller K_f^T this group gets changed into a subgroup of finite index. This means that

$$\mathrm{Hom}(\Gamma_T/\Gamma_Z,\bar{\mathbb{Q}}_\gamma) = \mathrm{Hom}(\Gamma_T/\Gamma_Z,\mathbb{Q}) \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}_\gamma$$

is independent of the choice of the congruence condition, i.e. of K_f^T. Let us put

$$\mathrm{Hom}(\Gamma_T/\Gamma_Z,\mathbb{Q}) = \mathcal{H}(T/Z)$$

then we find that in the case $\gamma|\Gamma_T \equiv 1$ the cohomology of a component is

$$H^{\bullet}(\Gamma_T/\Gamma_Z,\bar{\mathbb{Q}}_\gamma) = \Lambda^{\bullet}(\mathcal{H}(T/Z)) \otimes \bar{\mathbb{Q}}_\gamma .$$

Then we get from the definitions

$$H^{\bullet}(\tilde{S}^T,\bar{\bar{\mathbb{Q}}}_\gamma) = H^0(\tilde{S}^T,\bar{\bar{\mathbb{Q}}}_\gamma) \otimes_{\mathbb{Q}} \Lambda^{\bullet}(\mathcal{H}(T/Z))$$

and this an isomorphism of $\pi_0(T_\infty) \times T(A_f)$-modules, the action on the exterior powers is trivial. Let us assume that we select a generator $e(\gamma) \in \bar{\mathbb{Q}}_\gamma$, then I claim that every algebraic Hecke character ϕ of type γ gives us a section

$$e_\phi(e(\gamma)) = e_\phi \in H^0(\tilde{S}^T,\bar{\bar{\mathbb{Q}}}_\gamma)$$

which is defined as a map

$$e_\phi \;:\; T(A)/K_\infty^T \to \bar{\mathbb{Q}}_\gamma$$

$$e_\phi \;:\; \underline{t} \mapsto \phi_f(\bar{\underline{t}}_\infty, \underline{t}_f)e(\gamma)$$

(see 2.5.4.2), where $\bar{\underline{t}}_\infty$ is the image of t_∞ in $\pi_o(T_\infty)$. This map is locally constant and satisfies for $a \in T(\mathbb{Q})$

$$e_\phi(a\underline{t}) = \gamma(a)e_\phi(\underline{t})$$

and hence it is a global section (see 1.1.1). It is clear that the e_ϕ form a basis for $H^o(\tilde{S}^T, \tilde{\bar{\mathbb{Q}}}_\gamma)$ and we have

<u>Proposition 2.6.1</u>. <u>Under the conditions formulated in 2.5 we have</u>

$$H^\bullet(\tilde{S}^T, \tilde{\bar{\mathbb{Q}}}_\gamma) = \bigoplus_{\substack{\phi\ :\ \text{algebraic Hecke} \\ \text{character of type } \gamma}} \bar{\mathbb{Q}}e_\phi \otimes_{\mathbb{Q}} \Lambda^\bullet(\mathcal{X}(T/Z))$$

<u>where</u> $\pi_o(T_\infty) \times T(A_f)$ <u>acts via</u> ϕ_f <u>on</u> $\bar{\mathbb{Q}}e_\phi$.

We apply this to answer question B.). Our torus is $T = R_{F/\mathbb{Q}}(G_m) \times R_{F/\mathbb{Q}}(G_m)$ and $Z = R_{F/\mathbb{Q}}(G_m)$ embedded diagonally. Then $T/Z = R_{F/\mathbb{Q}}(G_m)$ and $\mathcal{X}(T/Z) = \mathrm{Hom}(U(F),\mathbb{Q})$ where $U(F)$ is the group of units of F. Before we apply 2.6.1 we have to take into account, that we divide $T(\mathbb{R})$ by $K_\infty^T = K_\infty^{T,o}Z(\mathbb{R})$ in the situation of question B.). Let $\bar{Z}$ be the image of K_∞^T in $\pi_o(T_\infty)$. If we apply proposition 2.6.1 we sum only over those ϕ which satisfy $\phi_f|\bar{Z} \equiv 1$.

Now we go back to our spectral sequence in 2.3.. We have

$$H^\bullet(\tilde{S}^T, \widetilde{H^\bullet(\mu,M)}) = \bigoplus_{\gamma \in \mathrm{coh}(M)} H^\bullet(\tilde{S}^T, \widetilde{H^\bullet(\mu,M)}(\gamma)) \Rightarrow H_B^\bullet(\tilde{M}) \quad .$$

The $H^{\bullet}(\check{\mu},M)(\gamma) = \bar{\mathbb{Q}}\cdot e(\gamma)$, so they occur with multiplicity one. Therefore the spectral sequence must degenerate. We apply proposition 2.6.1.

<u>Proposition 2.6.2</u>: <u>The action of</u> $\pi_o(B_\infty) \times \mathbb{B}(A_f)$ <u>on</u> $H^{\bullet}_B(\tilde{M})$ <u>factors over</u> $\pi_o(T_\infty) \times T(A_f)$ <u>and as</u> $\pi_o(T_\infty)\times T(A_f)$<u>-modules</u>

$$H^{\bullet}_B(\tilde{M}) = \bigoplus_{\gamma\in \mathrm{Coh}(M)} \bigoplus_{\substack{\phi\,:\,\phi \text{ of type } \gamma\,, \\ \phi_f|\bar{Z} \equiv 1}} \bar{\mathbb{Q}}\, e_\phi \otimes \Lambda^{\bullet}\mathcal{H}(T/Z) .$$

Here we introduce a simplification in the notation. If we choose a generator $e(\gamma) = e(\lambda,\gamma)$ for the space $H^{\bullet}(\mu,M)(\gamma)$ then $e_\phi \cdot e(\lambda,\gamma)$ would be a generator of $H^o(\tilde{S}^T, H^{\bullet}(\mu,M)(\gamma))(\phi)$ and we call the generator e_ϕ again.

Combining this with our considerations in 2.3.1 we get

<u>Theorem 1</u>: <u>As a</u> $\pi_o(G_\infty)\times G(A_f)$<u>-module the cohomology of the boundary is given by</u>

$$H^{\bullet}(\partial\tilde{S},\tilde{M}) = \bigoplus_{\gamma\in \mathrm{Coh}(M)} \bigoplus_{\substack{\phi\,:\,\phi \text{ of type } \gamma \\ \phi_f|\bar{Z} \equiv 1}} \mathrm{Ind}_{\pi_o(B_\infty)\times \mathbb{B}(A_f)}^{\pi_o(G_\infty)\times G(A_f)} \bar{\mathbb{Q}}\, e_\phi \otimes \Lambda^{\bullet}(\mathcal{H}(T/Z)) .$$

<u>2.7.</u> We have to discuss the $\mathbb{Q}$-rationality of this decomposition. Actually it is rather obvious what we have to do. First we observe that we have an action of the Galois group on the algebraic Hecke characters. If we look at the finite part of such an algebraic Hecke character then

$$\phi_f : T(A_f) \to \bar{\mathbb{Q}}^{\times}$$

and it is obvious that for $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ and

$$\phi_f^\sigma(\underline{t}_f) = (\phi_f(\underline{t}_f))^\sigma$$

ϕ_f^σ is again the finite part of an algebraic Hecke character ϕ^σ . It is also clear that this operation commutes with the formation of the type (The Galois group acts trivially on $T(A_f)$.). Now we consider triplets

$$\mu_1 = (\lambda,\gamma,\phi) \qquad \gamma \in \mathrm{Coh}(\lambda) \quad , \quad \text{type of } \phi = \gamma$$

and we have a Galois group action on these triplets. If M_1 is any orbit under the group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ we have maps

$$\underline{M}_1 \xrightarrow{p_1} \underline{M} \xrightarrow{p} \Lambda \ .$$

Let us put

$$V_\phi = \mathrm{Ind}_{\pi_o(B_\infty)\times B(A_f)}^{\pi_o(G_\infty)\times G(A_f)} \bar{\mathbb{Q}}\, e_\phi \quad (= V_{\mu_1})$$

considered as a submodule of $H^\bullet(\partial\tilde{S},\tilde{M})$ then we define a system of transition maps

$$\phi^\sigma_{\mu_1,\sigma} \ : \ V_\phi \to V_{\phi^\sigma}$$

by sending e_ϕ into $e_{\phi\sigma}$, where we assume that the $e(\lambda,\gamma)$ have been chosen as a rational system of generators (see remark after Thm 1 and 2.4.1). Then we have the addendum to Theorem 1:

<u>2.7.1</u> <u>The system of maps</u> $H^\bullet(\partial\tilde{S},\tilde{M}(\lambda)) \to V_\phi$ <u>is a</u> $\mathbb{Q}$<u>-rational system of maps between</u> $\{H^\bullet(\partial\tilde{S},M(\lambda))\}_{\lambda\in\Lambda}$ <u>and</u> $\{V_\phi\}_{\mu_1\in\underline{M}}$.

<u>2.8.</u> We want to discuss Theorem 1 in some special cases. The first case is that the field F is not totally imaginary. In this case it is well known that for $T = R_{F/\mathbb{Q}}(T_o) = R_{F/\mathbb{Q}}(G_m\times G_m)$ the torus H (Zariski closure of the units) is given by the kernel of the norm map, i.e. we have an exact sequence

$$1 \to H \to R_{F/\mathbb{Q}}(G_m\times G_m) \to (G_m\times G_m)/\mathbb{Q} \to 1$$

(comp. |Se|, II, 3.3). This means that for

$$\gamma \in X(T \times_{\mathbb{Q}} \overline{\mathbb{Q}}) = \bigoplus_{\tau : F \to \overline{\mathbb{Q}}} X_0(T)$$

the condition $\gamma|H \equiv 1$ is equivalent to

$$\gamma = (\ldots\gamma\ldots)_{\tau : F \to \overline{\mathbb{Q}}}$$

i.e. all the components must be equal. So if we look at classes

$$e_\gamma \in H^\bullet(\check{\mu}, M)(\gamma) = \bigotimes_{\tau : F \to \overline{\mathbb{Q}}} H^\bullet(\check{\mu}_0, M_\tau)$$

and want them to contribute to the cohomology of the boundary then the following conditions must be satisfied

(2.8.1) $\lambda = (\ldots\lambda_\tau\ldots)_\tau$ must be constant

i.e.

$$\lambda_\tau : \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \to t_1^d \cdot (t_1 t_2)^\nu$$

or stated otherwise $M = \bigotimes_\tau M(d,\nu)$.

<u>2.8.2</u> We have the two possibilities

$$\gamma_\tau = \begin{cases} t_1^d (t_1 t_2)^\nu & \text{for all } \tau \\ t_2^{d+\nu+1} \, t_1^{\nu-1} & \text{for all } \tau \end{cases}$$

If the first case in (2.8.2) holds then

$$\gamma_0 = (\ldots\gamma_\tau\ldots)_\tau$$

gives cohomology in degree zero, in the second case

$$\gamma_n = (\ldots\gamma_\tau\ldots)_\tau$$

gives cohomology in degree n , where $n = |F:\mathbb{Q}|$. Then Theorem 1 simplifies because we have a serious restriction on λ and for any λ there are exactly two possible types in $\mathrm{Coh}(\lambda)$ which admit algebraic Hecke characters.

2.9. The situation is more complicated and also more interesting if the field F is totally imaginary. In this case we put $[F:\mathbb{Q}] = n = 2d$.

Since we assumed $\bar{\mathbb{Q}} \subset \mathbb{C}$ we have the complex conjugation in $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, for $\tau : F \to \bar{\mathbb{Q}}$ let $\bar{\tau}$ be τ followed by the complex conjugation. We group

$$\{\tau \mid \tau : F \to \bar{\mathbb{Q}}\} = \{\ldots\tau,\bar{\tau},\ldots\sigma,\bar{\sigma}\ldots\}$$

into pairs. A character $\gamma \in X(T \times_{\mathbb{Q}} \bar{\mathbb{Q}})$ is given by its components

$$\gamma = (\ldots\gamma_\tau,\gamma_{\bar{\tau}}\ldots\gamma_\sigma,\gamma_{\bar{\sigma}}\ldots)$$

where each γ_τ is a character on $X(T_0)$. Then it is clear that γ can admit algebraic Hecke characters of type γ only if

$$\gamma_\tau + \gamma_{\bar{\tau}} = \gamma_\sigma + \gamma_{\bar{\sigma}} = w(\gamma) \tag{2.9.1}$$

for all pairs $\{\tau,\bar{\tau}\}$, $\{\sigma,\bar{\sigma}\}$ (|Se |, II, 3.3). We call $w(\gamma) \in X(T_0)$ the weight of γ . If our field F is a CM-field then this condition is also sufficient.

If we give ourselves the module

$$M = M(\lambda) = \bigotimes_{\tau : F \to \bar{\mathbb{Q}}} M(d(\tau),\nu(\tau))$$

then 2.9.1 restricted to the centre gives

$$\nu(\tau) + \nu(\bar{\tau}) = \nu \qquad \text{for all} \quad \tau,\bar{\tau}$$

a condition which we always assume, otherwise we have no cohomology at all (see 1.1.3).

If we now want to analyse for which γ a constituent $H^{\bullet}(\check{\mu},M)(\gamma)$ can contribute to the cohomology of the boundary we may restrict to the torus $T^{(1)}/\mathbb{Q}$ which we identified with $R_{F/\mathbb{Q}}(G_m)$ (see 1.0.4). Then $\gamma|T^{(1)} \times_{\mathbb{Q}} \bar{\mathbb{Q}} = \gamma^{(1)}$ is a 2d-tupel of integers

$$\gamma^{(1)} = (\ldots n_{\tau}, n_{\bar{\tau}} \ldots n_{\sigma}, n_{\bar{\sigma}} \ldots)$$

and $w(\gamma^{(1)}) = w = n_{\tau} + n_{\bar{\tau}}$ does not depend on $\{\tau,\bar{\tau}\}$. If we decompose

$$H^{\bullet}(\check{\mu},M)(\gamma) = \bigotimes_{\tau} H^{\bullet}(\check{\mu}_{o},M_{\tau})(\gamma_{\tau})$$

then we know from 2.3 especially 2.3.2 that for each τ the number n_{τ} has to be one of the two numbers

$$\begin{aligned} n_{\tau} &= d(\tau) && \text{if} \quad \deg(\gamma_{\tau}) = 0 \\ n_{\tau} &= -d(\tau)-2 && \text{if} \quad \deg(\gamma_{\tau}) = 1 \quad . \end{aligned}$$

Hence the above weight w has to be taken from the following list

$$Lw = (d(\tau)+d(\bar{\tau}), d(\tau)-d(\bar{\tau})-2, d(\bar{\tau})-d(\tau)-2, -d(\tau)-d(\bar{\tau})-4) \quad .$$

The members of list correspond to classes in $H^{\bullet}(\check{\mu}_{\delta},M_{\tau}) \otimes H^{\bullet}(\check{\mu}_{\delta},M_{\bar{\tau}})$ which sit in degrees 0,1,1,2 respectively.

Now we should distinguish two cases

<u>2.9.2</u> The weight $w = w(\gamma^{(1)}) \neq -2$. In this case we see that for each pair $\{\tau,\bar{\tau}\}$ exactly one member of the above list is equal to w. If $w = -1$ or -3 then this member corresponds to cohomology in degree 1, if

$w \geq 0$ then it corresponds to cohomology in degree 0 or 1 , if $w \leq -4$ then it corresponds to cohomology in degree 1 or 2 . We refer to this case as the non-unitary case.

<u>2.9.3</u> We have $w = -2$. This can only happen if $d(\tau) = d(\bar{\tau})$ for all pairs $\{\tau,\bar{\tau}\}$ and then there are exactly two members in the list which give that weight and they correspond to cohomology in degree 1 . This is the unitary case.

<u>2.9.4</u> Let $s_o = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in G(\mathbb{Q})$ be the non-trivial element in the Weyl group. Then s_o defines an involution on $X(T \times_{\mathbb{Q}} \bar{\mathbb{Q}})$ as well as on the algebraic Hecke characters on $T(\mathbb{Q})\backslash T(A)$ and this involution commutes with the formation of types

$$s_o \cdot \gamma = \alpha + \gamma^{s_o}$$

$$s_o \cdot \phi = |\alpha| \phi^{s_o}$$

(see 1.0.5 and 1.0.6) where γ^{s_o} and ϕ^{s_o} has the obvious meaning. If we restrict this action to the characters (algebraic Hecke characters) on $T^{(1)}$ then

$$s_o \cdot \gamma^{(1)} = \alpha - \gamma^{(1)} \quad ; \quad s_o \cdot \phi^{(1)} = |\alpha| \cdot (\phi^{(1)})^{-1}$$

and the above numbers change as follows: $n_\tau \to -n_\tau + 2$, and $w(s_o \cdot \gamma^{(1)}) = -4 - w(\gamma^{(1)})$.

If the character γ corresponds to cohomology in $\deg(\gamma) = \sum_\tau \deg(\gamma_\tau)$ then $s_o \cdot \gamma$ corresponds to cohomology in

$$\deg(s_o \cdot \gamma) = 2n - \deg(\gamma)$$

We call γ balanced if $\deg(\gamma) = \deg(s_o \cdot \gamma) = n$ which is certainly the case if γ is unitary. If the field is not totally imaginary then we are always in the non-balanced case.

III Cohomology and representations

The material in this section is more or less well known, many of the results are just specialization of general results to the case GL_2 . I want to review it briefly, because it seems to be difficult to find it in the literature and I want to give a rather complete treatment of the case GL_2 in this paper

3.1. We consider the kernel of the map r

$$\tilde{H}^{\bullet}(\tilde{S},\tilde{M}) = \ker(H^{\bullet}(\tilde{S},\tilde{M}) \to H^{\bullet}(\partial\tilde{S},\tilde{M})) \quad .$$

This is a $\pi_0(G_\infty)\times G(A_f)$-module and it can be described in terms of automorphic forms. This description is given by the so called Eichler-Shimura isomorphism. We want to explain this in the given context.

We have $\bar{\mathbb{Q}} \subset \mathbb{C}$ and extend $M_{\mathbb{C}} = M \otimes_{\bar{\mathbb{Q}}} \mathbb{C}$, the corresponding sheaf on S_K is denoted by $\tilde{M}_{\mathbb{C}}$. The cohomology groups $H^{\bullet}(S_K,\tilde{M}_{\mathbb{C}})$ can be computed in terms of the de-Rham-complex and the de-Rham-complex is isomorphic to a relative Lie algebra complex, so the cohomology can be expressed in terms of $(\mathfrak{g}_\infty,K_\infty)$-cohomology, where $\mathfrak{g}_\infty = \mathrm{Lie}(G_\infty)$ (comp. [B-W], VII, 2.7.). We want to state this more precisely. The centre $Z/\mathbb{Q}$ of $G/\mathbb{Q}$ acts on $M_{\mathbb{C}}$ by a character $\omega \in X(Z \times_{\mathbb{Q}} \bar{\mathbb{Q}})$, we define the space

$$\mathcal{C}_\infty(G(\mathbb{Q})\backslash G(A))(\omega^{-1})$$

which consists of all $\mathcal{C}_\infty$-functions on $G(\mathbb{Q})\backslash G(A)$ which satisfy

$$f(\underline{z}_\infty \underline{g}) = \omega^{-1}(\underline{z}_\infty)f(\underline{g})$$

for all $\underline{z}_\infty \in Z_\infty^o \subset Z(A)$ and $\underline{g} \in G(A)$. We recall that $f(\underline{g}) = f(\underline{g}_\infty, \underline{g}_f)$ is called $\mathcal{C}_\infty$ if it is $\mathcal{C}_\infty$ in the variable $\underline{g}_\infty$ and right invariant under the transformations of a suitably small open compact subgroup $K_f \subset G(A_f)$. If $k_\infty = \mathrm{Lie}(K_\infty)$ we can identify the de-Rham complex of forms with coefficients in $\tilde{M}_{\mathbb{C}}$ to the relative Lie algebra complex

$$\Omega^\bullet(S_K, \tilde{M}_{\mathbb{C}}) = \mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}_\infty/k_\infty), \mathcal{C}_\infty(G(\mathbb{Q})\backslash G(A)/K_f)(\omega^{-1}) \otimes M_{\mathbb{C}})$$

(comp. [B-W], VII 2.7. , [M-M], § 1), and passing to the limit we get an isomorphism of $\pi_o(G_\infty) \times G(A_f)$-modules

$$H^\bullet(\tilde{S}, \tilde{M}_{\mathbb{C}}) \xrightarrow{\sim} H^\bullet(\mathfrak{g}_\infty, K_\infty, \mathcal{C}_\infty(G(\mathbb{Q})\backslash G(A))(\omega^{-1}) \otimes M_{\mathbb{C}}) \quad .$$

Borel developed very general methods which allow us to investigate the structure of the subspace $\tilde{H}^\bullet(\tilde{S}, M_{\mathbb{C}})$ by functional analytic methods.

We introduce the Hilbert space

$$L^2(G(\mathbb{Q})\backslash G(A))(\omega^{-1})$$

of those functions which transform under Z_∞^o by ω^{-1} and which satisfy

$$\int_{Z^o G(\mathbb{Q})\backslash G(A)} |f(\underline{g})|^2 |\omega(\underline{g})|^2 d\underline{g} < \infty \quad .$$

This is a unitary $G(A)$-module, the discrete part of this space is the closure of the sum of all irreducible closed subspaces, one knows that this space is a direct sum

$$L_d^2(G(\mathbb{Q})\backslash G(A))(\omega^{-1}) = \overline{\bigoplus_\pi H_\pi}$$

where each isomorphism type of $G(A)$-modules which occurs, occurs with multiplicity one. One knows furthermore

3.1.1 If $\dim H_\pi < \infty$ then $H_\pi = \mathbb{C}\tilde{\phi}$ where $\tilde{\phi}$ factors over the determinant, i.e.

$$\tilde{\phi} \; : \; G(\mathbb{Q})\backslash G(A) \xrightarrow{\det} F^{\times}\backslash I_F \xrightarrow{\phi} \mathbb{C}^{\times}$$

and ϕ is a character on $F^{\times}\backslash I_F$.

3.1.2 If $\dim H_\pi = \infty$ then H_π is contained in the space of cusp forms $L^2_o(G(\mathbb{Q})\backslash G(A))(\omega^{-1})$ (For all this: |G-J|, 4.2.).

It is known that the G(A)-modules H_π can be decomposed into a tensor-product of local G(A)-modules

$$H_\pi = H_{\pi_\infty} \otimes H_{\pi_f} = \bigotimes_{v\in S_\infty} H_{\pi_v} \otimes \otimes H_\pi$$

(comp. [Fl], [Go], § 3, 2). In H_{π_∞} we have the subspace $H_{\pi_\infty}^{(K_\infty)}$ of K_∞-finite vectors, this is a $(\mathfrak{g}_\infty, K_\infty)$-module (Harish-Chandra module) ([B-W], I, 2.2, [Vo], Ch. 0, § 3) and for these modules we have the notion of $(\mathfrak{g}_\infty, K_\infty)$-cohomology with coefficients in $M_{\mathbb{C}}$:

$$H^\bullet(\mathfrak{g}_\infty, K_\infty, H_{\pi_\infty}^{(K_\infty)} \otimes M_{\mathbb{C}}) = H^\bullet(\mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}_\infty/k_\infty), H_{\pi_\infty}^{(K_\infty)} \otimes M_{\mathbb{C}})) \quad .$$

We say that $\pi \in$ Coh(M) if $H^\bullet(\mathfrak{g}_\infty, K_\infty, H_{\pi_\infty}^{(K_\infty)} \otimes M_{\mathbb{C}}) \neq 0$ and we say $\pi \in \mathrm{Coh}_o(M)$ if π in addition occurs in the space of cusp forms.

The condition $\pi \in$ Coh(M) is only a condition at infinity. The problem of determining the unitary $(\mathfrak{g}_\infty, K)$-modules which have cohomology is solved in full generality, this means for all real semi-simple Lie algebras ([Vo] , [VZ],). We shall describe these results in our special situation briefly in 3.3.

We have an inclusion

$$\bigoplus_{\pi \in \mathrm{Coh}(M)} H_{\pi_\infty}^{(K_\infty)} \otimes H_{\pi_f} \to \mathcal{C}_\infty(G(\mathbb{Q})\backslash G(\mathbb{A}))(\omega^{-1})$$

and this induces a map

$$I_d \ : \ \bigoplus_{\pi \in \underline{\mathrm{Coh}}(M)} H^\bullet(\mathfrak{g}_\infty, K_\infty, H_{\pi_\infty}^{(K_\infty)}) \otimes H_{\pi_f} \to H^\bullet(\tilde{S}, \tilde{M}_{\mathbb{C}}) \quad .$$

The following two assertions are special cases of a completely general result of Borel (|Bo1|, Thm. 3.5 , |Bo2|, Cor. 5.5).

<u>3.1.3</u> The image of I_d contains $\tilde{H}^\bullet(\tilde{S}, \tilde{M}_{\mathbb{C}})$.

<u>3.1.4</u> The restriction

$$I_{d,o} \ : \ \bigoplus_{\pi \in \underline{\mathrm{Coh}}_o(M)} H^\bullet(\mathfrak{g}_\infty, K_\infty, H_{\pi_\infty}^{(K_\infty)}) \otimes H_{\pi_f} \to H^\bullet(\tilde{S}, \tilde{M}_{\mathbb{C}})$$

is an injection. Moreover

$$\mathrm{Im}(I_{d,o}) \subset \tilde{H}^\bullet(\tilde{S}, \tilde{M}_{\mathbb{C}}) \quad .$$

One usually calls $\mathrm{Im}(I_{d,o})$ the cuspidal cohomology and it is denoted by $H^\bullet_{cusp}(\tilde{S}, \tilde{M}_{\mathbb{C}})$.

<u>3.2.</u> Now we have three spaces

$$H^\bullet_{cusp}(\tilde{S}, \tilde{M}_{\mathbb{C}}) \subset \tilde{H}^\bullet(\tilde{S}, \tilde{M}_{\mathbb{C}}) \subset \mathrm{Im}(I_d)$$

and we want to study the relationship among these spaces a little bit more closely. We have to understand the contribution of the one dimensional spaces to the cohomology.

If we have

$$H^\bullet(\mathfrak{g}_\infty, K_\infty, \mathbb{C}\tilde{\phi} \otimes M_{\mathbb{C}}) \neq 0$$

then the following two conditions must be satisfied.

3.2.1 $\dim M_{\mathbb{C}} = 1$ (comp. |Bo3|, 1.6).

3.2.2 $\phi(\det(z_\infty)) = \tilde{\phi}(z_\infty) = \omega(z_\infty)^{-1}$ for $z_\infty \in Z_\infty^o$.

Hence our representation $M = M(\lambda)$ is of the form

$$\begin{tikzcd} G \times_{\mathbb{Q}} \overline{\mathbb{Q}} \arrow[rr, "\rho"] \arrow[dr, "\det"'] & & GL(1) = G_m \\ & R_{F/\mathbb{Q}}(G_m) \arrow[ur, "\lambda"'] & \end{tikzcd}$$

and $\lambda = \lambda \circ \det|T$. Of course we shall keep in mind that we always assume, that $\lambda|Z$ is the type of an algebraic Hecke character, this is also our character ω . Now we get a map

3.2.3 $$J_d : \bigoplus_{\substack{\phi: F^* \setminus I_F \to \mathbb{C} \\ \text{type}(\tilde{\phi})_{|Z} = \omega^{-1}}} H^\bullet(\mathfrak{g}_\infty, K_\infty, \mathbb{C}\tilde{\phi} \otimes M_{\mathbb{C}}(\lambda)) \to H^\bullet(\tilde{S}, M_{\mathbb{C}}(\lambda))$$

and hence we get a decomposition

$$\mathrm{Im}(I_d) = \mathrm{Im}(I_{d,o}) \oplus \mathrm{Im}(J_d) \quad .$$

(The fact that the sum is direct follows from multiplicity one (for instance).) Now we want to investigate

$$\mathrm{Im}(J_d) = \bigoplus_{\substack{\phi : F^* \setminus I_F \to \mathbb{C} \\ \text{type}(\tilde{\phi})_{|Z} = \omega^{-1}}} J_d(H^\bullet(\mathfrak{g}_\infty, K_\infty, \mathbb{C}\tilde{\phi} \otimes M_{\mathbb{C}}(\lambda))$$

The $\mathfrak{g}_\infty$-module $\mathbb{C}\tilde{\phi} \otimes M_{\mathbb{C}}(\lambda)$ is trivial and hence we get

$$H^\bullet(\mathfrak{g}_\infty, K_\infty, \mathbb{C}\tilde{\phi} \otimes M_{\mathbb{C}}(\lambda)) = \mathbb{C}\tilde{\phi} \otimes \mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}_\infty/k_\infty), \mathbb{C}) \quad .$$

The Lie algebra $\mathfrak{g}_\infty$ and the group K_∞ decompose into

$$\mathfrak{g}_\infty = \bigoplus_{v \in S_\infty} \mathfrak{g}_v \quad \text{and} \quad K_\infty = \prod_{v \in S_\infty} K_v$$

and with $k_v = \mathrm{Lie}(K_v)$ we get

$$\Lambda^\bullet(\mathfrak{g}_v/k_v)^{K_v} = \Lambda^0(\mathfrak{g}_v/k_v) \oplus \Lambda^{d_v}(\mathfrak{g}_v/k_v)$$

where $d_v = 2$ (resp. 3) if v is real (resp. complex), we have $m = \dim X = \sum_{v \, S_\infty} d_v$.

<u>Proposition 3.2.4:</u> <u>Let</u> λ <u>be as above,</u> <u>let</u> ϕ <u>be an algebraic Hecke character for which</u> $\mathrm{type}(\tilde{\phi})_{|Z} = \lambda^{-1}{}_{|Z}$. <u>Then we have</u>

(i) <u>The image of</u>

$$\mathrm{Hom}_{K_\infty}(\Lambda^0(\mathfrak{g}_\infty/k_\infty), \mathbb{C}) \otimes \mathbb{C}\tilde{\phi}$$

<u>under</u> J_d <u>is not contained in</u> $\tilde{H}^0(\tilde{S}, \tilde{M}_{\mathbb{C}}(\lambda))$.

(ii) $$J_d\Big(\bigoplus_{\nu > 0} \mathrm{Hom}_{K_\infty}(\Lambda^\nu(\mathfrak{g}_\infty/k_\infty, \mathbb{C}) \otimes \mathbb{C}\tilde{\phi}\Big) \subset \tilde{H}^\bullet(\tilde{S}, M_{\mathbb{C}}(\lambda)) \quad .$$

(iii) <u>The kernel of</u> J_d <u>restricted to this component is</u>

$$\mathrm{Hom}(\Lambda^m(\mathfrak{g}_\infty/k_\infty), \mathbb{C}) \otimes \mathbb{C}\tilde{\phi} \quad .$$

This proposition is more or less equivalent to proposition 2.3 in [Ha1], which is stated there without proof. For the purpose of completeness I will prove it here, the proof is rather unpleasant. The proposition is not really needed in this paper.

To prove the proposition we go to a fixed level, i.e. we choose a K_f and we write

$$S_K = \bigsqcup_{i \in \pi_o(S_K)} \Gamma^{(i)} \backslash X$$

(see 1.2.). We look at a single connected component which we will denote by $\Gamma\backslash X$. On this component we have the invariant differential forms

$$\omega_I = \bigwedge_{v \in I} \omega_v$$

where $I \subset S_\infty$ and ω_v is the volume form on the factor X_v. These classes span a graded ring $A(X)$, we are investigating the map

$$A(X) \to H^\bullet(\Gamma\backslash X, \mathbb{C}) \quad .$$

Now we are actually back in the situation of |Ha 1|. The first assertion in the proposition is obvious, to prove (ii) and (iii) we have to work with some considerations on the growth of differential forms.

If we have a boundary component in the Borel-Serre compactification $\Gamma\backslash\overline{X}$ then it is associated to a Borel subgroup $B/\mathbb{Q} \to G/\mathbb{Q}$ (which for this proof is not necessarily the standard one). We put $\Gamma_B = \Gamma \cap B(\mathbb{Q})$ and we have the map

$$p_B : \Gamma_B\backslash X \to \Gamma\backslash X \quad ,$$

the group B_∞ acts transitively on X and we define the level function

$$n(x) = n(b_\infty x_o) = |\alpha|_\infty(b_\infty)$$

(see 1.0.6 and 1.0.7). It is known that for c sufficiently large the set

$$\Gamma_B\backslash X(c) = n^{-1}((c,\infty)) \hookrightarrow \Gamma\backslash X$$

and this set $\Gamma_B\backslash X(c)$ is homotopy equivalent to the boundary component attached to B. We use the usual description of the de-Rham complex

$$\Omega^\bullet(\Gamma_B\backslash X) \simeq \mathrm{Hom}_{K_\infty^B}(\Lambda^\bullet(b_\infty/b_\infty^K), \mathcal{C}_\infty(\Gamma_B\backslash B_\infty))$$

where $K_\infty^B = B_\infty \cap K_\infty$. We may choose a torus $T_\infty \subset B_\infty$ for which $T_\infty \cap K_\infty = K_\infty^T$ is maximal compact in T_∞, and we put $\mathcal{A}_\infty = \mathrm{Lie}(T_\infty)$. Then we get

$$\mathrm{Hom}_{K_\infty^B}(\Lambda^\bullet(b_\infty/b_\infty^K), \mathcal{C}_\infty(\Gamma_\infty\backslash B_\infty)) \simeq$$

$$\mathrm{Hom}_{K_\infty^T}(\Lambda^\bullet(\mathcal{A}_\infty/\mathcal{A}_\infty^K) \otimes \Lambda^\bullet(\check{\mu}_\infty), \mathcal{C}_\infty(\Gamma_\infty\backslash B_\infty)) \quad .$$

Now we prove (ii). We assume that we have an invariant form ω_I with $I \neq \emptyset$ whose restriction to the boundary gives a non-trivial class. We may assume it is non-zero when we restrict it to $\Gamma_B\backslash X$ by p_B, since $\Gamma_B\backslash X$ is homotopy equivalent to the boundary component Y_B. We know that there exists a class

$$\xi \in H^{m-1-d_I}(Y_B,\mathbb{C}) \qquad\qquad d_I = \sum_{v\in I} d_v$$

such that $[\omega_I] \wedge \xi \in H^{m-1}(Y_B,\mathbb{C})$ is non-zero. We computed the cohomology of the boundary from a degenerating spectral sequence (see 2.3 and [Ha1], prop. 1.1) and we may write

$$\xi = \sum_{p+q = m-1-d_I} \xi_{p,q}$$

where $\xi_{p,q} \in H^p((S^1)^{r_1+r_2-1}, H^q(\Gamma_U\backslash U_\infty,\mathbb{C}))$

(see [Ha1] § 1). We shall see in course of the construction of the

Eisenstein classes (see 4.2.2) that we can represent $\xi_{p,q}$ by a form

$$\omega^{\xi}_{p,q} \in \mathrm{Hom}_{K^T_\infty}(\Lambda^p(\mathcal{A}_\infty/\mathcal{A}^K_\infty) \otimes \Lambda^q \check{\mu}_\infty, \mathcal{C}_\infty(\Gamma_B \backslash B_\infty))$$

which satisfies a growth condition if we approach the boundary Y_B namely

$$|\omega^{\xi}_{p,q}(T_1 \dots T_p, U_1 \dots U_q)(b_\infty)| \leq C \cdot (|\alpha|_\infty(b_\infty))^{+q/n}$$

for $|\alpha|_\infty(b_\infty) > c$ and where $T_1 \dots T_p$ and $U_1 \dots U_q$ have to be taken from a fixed basis in $\mathcal{A}_\infty$ and $\check{\mu}_\infty$. (We know that only $q = 0$, $n/2$ and n are possible.) Then the above cup-product is given by the value of the integral

$$[\omega_I] \wedge [\xi] = \int_{n^{-1}(t)} \omega_I \wedge \omega^{\xi} = \sum_q \int_{n^{-1}(t)} \omega_I \wedge \omega^{\xi}_{p,q}$$

and the individual summands are independent of t . But the volume of $n^{-1}(t)$ with respect to the invariant metric is

$$\mathrm{vol}(n^{-1}(t)) = a_0 t^{-1}$$

(comp. [HaO], prop. 1.2.1) and hence we get

$$\lim_{t \to \infty} \int_{n^{-1}(t)} \omega_I \wedge \omega^{\xi}_{p,q} = 0$$

if $q < n = |F:\mathbb{Q}|$. But if $q = n$ then I claim that $\omega_I \wedge \omega^{\xi}_{p,q} \equiv 0$. To see this we look at the value

$$\omega_I(X_1, \dots, X_{d_I}) \cdot \omega^{\xi}_{p,n}(X_{d_I+1}, \dots X_m)$$

where the $X_i \in \mathcal{A}_\infty/\mathcal{A}^K_\infty \oplus \check{\mu}_\infty$. (This splitting is not canonical but $\check{\mu}_\infty$ is an intrinsic submodule of b_∞/b^K_∞ .) For this to be non-zero the $X_1, \dots, X_{d_I}$ have to form a basis of

$$\bigoplus_{v\in I} \mathfrak{b}_v/\mathfrak{b}_v^{K_v} = \bigoplus_{v\in I} \mathfrak{g}_v/\mathfrak{k}_v$$

when we project them into this space. Among the $X_{d_I+1},\dots,X_m$ we must find n vectors lying in $\check{\mu}_\infty$ but then $X_1,\dots,X_m$ cannot be linearly independent, but this shows that

$$\omega_I \wedge \omega^{\xi}_{p,n}(X_1,\dots,X_m) = 0 \quad .$$

This is a contradiction to the assumption that ω_I does not vanish on the boundary.

To prove (iii) we pick a form ω_I with $I \neq \emptyset$, S_∞, let $I' \quad S_\infty$ such that $|I| + |I'| = |S_\infty|$, i.e. a form in the complementary degree. Since we know (ii) we may write

$$\omega_I = \eta_I + d\phi$$

and then we integrate

$$\int_{\Gamma\backslash X(c)} \omega_I \wedge \omega_{I'} = \int_{\Gamma\backslash X(c)} \eta_I \wedge \omega_{I'} + \int_{\Gamma\backslash X(c)} d\phi \wedge \omega_{I'} \quad .$$

Here $\Gamma\backslash X(c)$ is the submanifold with boundary which we get if we chop off at all cusps at a certain level far enough out. The right hand side becomes

$$\int_{\Gamma\backslash X(c)} \eta_I \wedge \omega_{I'} + \int_{\partial(\Gamma\backslash X(c))} \phi \wedge \omega_{I'} \quad .$$

The first term is the cup-product of the compactly supported class $[\eta_I]$ and $[\omega_I]$ evaluated at the top class up to a sign. Let us assume that we know

$$\lim_{c\to\infty} \int_{\partial(\Gamma\backslash X(c))} \phi \wedge \omega_{I'} = 0 \quad . \qquad (*)$$

Then we have

$$\int \omega_I \wedge \omega_{I'} = \int \eta_I \wedge \omega_{I'} \quad .$$

The left hand side is different from zero if and only if $I \cup I' = S_\infty$ and this implies (iii).

It remains to prove (*) which is unfortunately a little bit technical. Again it is sufficient to see what happens in the neighborhood of one cusp, the one corresponding to $B/\mathbb{Q}$. We have to exploit the fact, that we can choose the form ϕ in such a way, that it is of logarithmic growth ([Bo1], 7.2, Thm. 7.4) and then we shall see that (*) holds. The condition for ϕ to be of logarithmic growth (near the boundary component belonging to B) is formulated as a growth condition with respect to certain coordinates which describe a neighborhood of the cusp. This condition can be translated into a growth condition for ϕ when we view it as an element in

$$\mathrm{Hom}_{K_\infty^B}(\Lambda^\bullet(\mathfrak{b}_\infty/\mathfrak{k}_\infty^K \oplus \check{\mu}_\infty), \mathcal{C}_\infty(\Gamma_B \backslash B_\infty)) \quad .$$

We shall give the result of this translation. We have a map

$$B_\infty \xrightarrow{\pi} T_\infty/K_\infty^T \cdot Z = T_\infty^{(1)} \times A$$

where $A = \mathbb{R}^*_{>0}$ and the decomposition is obtained from the decomposition of $T = B/U$ into its split and its anisotropic part. We write

$$\pi(b_\infty) = (\pi_1(b_\infty), |\alpha|_\infty(b_\infty)) \quad .$$

For a relatively compact set $\Omega \subset T_\infty^{(1)}$ we write

$$X(\Omega,c) = \{b_\infty \in B_\infty \mid \pi_1(b_\infty) \in \Omega, |\alpha|_\infty(b_\infty) \geq c\} \quad .$$

Then the condition for ϕ to be of logarithmic growth is equivalent to

For any Ω and c large enough we have a constant C_Ω and an $N \in \mathbb{N}$ such that for $X_{i_1},\dots,X_{i_e}$ from a fixed basis in $\mathfrak{b}_\infty/\mathfrak{b}_\infty^K$ we have

$$|\phi(\mathrm{ad}(b_\infty^{-1})X_{i_1},\dots,\mathrm{ad}(b_\infty^{-1})X_{i_e})(b_\infty)| \le$$
$$C_\Omega(\ \log|\alpha|_\infty(b_\infty))^N \quad \text{for all} \quad b_\infty \in X(\Omega,c) \quad .$$

It is rather straightforward to check that this definition is equivalent to the definition of Borel ([Bo1], loc. cit.). Now we choose a basis for $\mathfrak{g}_\infty = \bigoplus_{v\in S_\infty} \mathfrak{g}_v$ where the basis is made out of basises of the $\check{\mathfrak{u}}_v$ and a basis of $\mathfrak{z}_\infty/\mathfrak{z}_\infty^K$. Then we may write

$$b = (\dots b_v \dots)$$

and if we modify by an element in the centre then

$$b_v = \begin{pmatrix} t_v & * \\ 0 & 1 \end{pmatrix}$$

(up to conjugation). Then for $X_i \in \mathfrak{z}_\infty/\mathfrak{z}_\infty^K$ we have $\mathrm{ad}(b_v)^{-1}X_i = X_i$ and for $X_i \in \check{\mathfrak{u}}_v$ we have

$$\mathrm{ad}(b_v)^{-1}X_i = t_v^{-1}X_i \quad .$$

Hence we get

$$\phi(\mathrm{ad}(b_\infty)^{-1}X_{i_1},\dots,\mathrm{ad}(b_\infty^{-1})X_{i_e})(b_\infty) =$$
$$\prod t_v^{m_v} \cdot \phi(X_{i_1},\dots,X_{i_e})(b_\infty)$$

where m_v is the number of basis elements $X_{i_v} \in \check{\mathfrak{u}}_v$. Hence the growth condition can also be written as

$$|\phi(X_{i_1},\dots,X_{i_e})(b_\infty)| \le C_\Omega(\log|\alpha|_\infty(b_\infty))^N \cdot \prod_{v\in S_\infty} |t_v|^{m_v}$$

where $|t_v| = |t_v|_v$ for v real and $|t_v| = |t_v|_v^{1/2}$ for v complex. Since we required that $\pi_1(b_\infty) \in \Omega$ we find easily that

$$\Pi \ |t_v|^{m_v} \leq c_2(|\alpha|_\infty(b_\infty))^{(\Sigma m_v)/n}$$

where $n = |F:\mathbb{Q}|$, hence the growth depends on the "number of tangent vectors in direction of U_∞" in $X_{i_1},\ldots,X_{i_e}$.

Now it is easy to prove (*). We use the same argument as in the end of the proof of (ii). Since $I' \neq \emptyset$, the form $\omega_{I'}$ "uses up some tangent vectors in direction U_∞" and therefore

$$|(\phi \wedge \omega_{I'})(X_1,\ldots,X_n)(b_\infty)| \leq C_\Omega \cdot (\log|\alpha|_\infty(b_\infty))^N \cdot |\alpha|_\infty(b_\infty)$$

where $0 \leq \beta < 1$, and the (*) follows since the volume of $\partial(\Gamma\backslash X)(c)$ shrinks faster than $\phi \wedge \omega_{I'}$ grows.

<u>3.2.5</u> If $\dim M_{\mathbb{C}} = 1$ and if ω is its central character then we put

$$H^{\bullet}_{res}(\tilde{S},\tilde{M}_{\mathbb{C}}) = J_d\Big(\bigoplus_{\substack{\tilde{\phi} \\ \mathrm{type}(\tilde{\phi})|_Z = \omega^{-1}}} \mathbb{C}\tilde{\phi} \otimes \big(\bigoplus_{\nu=0}^{m-1} \Lambda^{\nu}(\mathfrak{g}_\infty/k_\infty)\big)$$

and we put $\tilde{H}^{\bullet}_{res}(\tilde{S},\tilde{M}_{\mathbb{C}}) = \tilde{H}^{\bullet}(\tilde{S},\tilde{M}_{\mathbb{C}}) \cap H^{\bullet}_{res}(\tilde{S},\tilde{M}_{\mathbb{C}})$, our proposition 3.2.4 tells us that for this module we have to sum from $\nu = 1$ to $m - 1$. Now our previous discussion yields

If $\dim M_{\mathbb{C}} > 1$ then

$$H^{\bullet}_{cusp}(\tilde{S},\tilde{M}_{\mathbb{C}}) = \tilde{H}^{\bullet}(\tilde{S},\tilde{M}_{\mathbb{C}}) \quad .$$

If $\dim M_{\mathbb{C}} = 1$ then

$$\tilde{H}^{\bullet}(\tilde{S},\tilde{M}_{\mathbb{C}}) = H^{\bullet}_{cusp}(\tilde{S},\tilde{M}_{\mathbb{C}}) + \tilde{H}^{\bullet}_{res}(\tilde{S},\tilde{M}_{\mathbb{C}}) \quad .$$

One important consequence of this result is that the cuspidal cohomology which is actually defined by transcendental means can be defined internally in the cohomology theory of sheaves. To see this we pass to a finite level and discuss the decomposition

$$\tilde{H}^{\bullet}(\tilde{S}_K,\tilde{M}_{\mathbb{C}}) = H^{\bullet}_{cusp}(S_K,\tilde{M}_{\mathbb{C}}) + \tilde{H}^{\bullet}_{res}(S_K,\tilde{M}_{\mathbb{C}}) \quad .$$

We introduce the Hecke algebra

$$\mathcal{H}_{K_f} = \mathcal{H}_c(K_f \backslash G(\mathbb{A}_f)/K_f)$$

of compactly supported $\bar{\mathbb{Q}}$-valued functions which are biinvariant under K_f . This algebra acts by convolution on the cohomology

$$H^{\bullet}(S_K,\tilde{M}) = H^{\bullet}(\tilde{S},\tilde{M})^{K_f}$$

(see 1.2.1 , [J-L], § 9). It follows easily from well known theorems in the theory of automorphic forms (for instance multiplicity one, |J-L|, prop. 11.1.1) that we can find an operator $P_c \in \mathcal{H}_{K_f}$ which is a projector to the subspace $H^{\bullet}_{cusp}(\tilde{S}_K,\tilde{M}_{\mathbb{C}})$ in the above decomposition of $\tilde{H}^{\bullet}(\tilde{S}_K,\tilde{M}_{\mathbb{C}})$. Since $\tilde{H}^{\bullet}(\tilde{S}_K,\tilde{M}_{\mathbb{C}})$ is defined as the kernel of r we get that $H^{\bullet}_{cusp}(S_K,\tilde{M}_{\mathbb{C}})$ is internally defined.

This implies that we have a decomposition of $\bar{\mathbb{Q}}$-vector spaces

$$\tilde{H}^{\bullet}(\tilde{S},\tilde{M}) = H^{\bullet}_{cusp}(\tilde{S},\tilde{M}) \oplus \tilde{H}^{\bullet}_{res}(\tilde{S},\tilde{M}) \quad .$$

Moreover if we put Λ to be the set of weights parametrizing our representations M , so $M = M(\lambda)$ for some λ then the systems

$$\{\tilde{H}^{\bullet}(\tilde{S},M(\lambda))\}_{\lambda\in\Lambda}, \{\tilde{H}^{\bullet}_{cusp}(\tilde{S},M(\lambda))\}_{\lambda\in\Lambda}, \{H^{\bullet}_{res}(\tilde{S},M(\lambda))\}_{\lambda\in\Lambda}, \{\tilde{H}^{\bullet}_{res}(\tilde{S},M(\lambda))\}_{\lambda\in\Lambda}$$

inherit the $\mathbb{Q}$-structure of the system

$$\{H^{\bullet}(\tilde{S},M(\lambda))\}_{\lambda \in \Lambda} \quad .$$

From the transcendental description of $H^{\bullet}_{cusp}(\tilde{S},\tilde{M}_{\mathbb{C}})$ we get that this cohomology is a semisimple module under the action of the group $\pi_o(G_\infty) \times G(A_f)$, we get

$$H^{\bullet}_{cusp}(\tilde{S},M(\lambda)_{\mathbb{C}}) = \bigoplus_{\pi \in \underline{Coh}_o(M(\lambda))} H^{\bullet}_{cusp}(\tilde{S},M(\lambda)_{\mathbb{C}})(\pi_f) \quad .$$

If we go to a finite level then we get a decomposition of $\mathcal{H}_{K_f}$-modules

$$\tilde{H}^{\bullet}_{cusp}(S_K,M(\lambda)_{\mathbb{C}}) = \bigoplus_{\pi \in \underline{Coh}_o(M(\lambda))} H^{\bullet}_{cusp}(S_K,M(\lambda)_{\mathbb{C}})(\pi_f^{K_f}) \quad .$$

There is only a finite number of π for which

$$H^{\bullet}_{cusp}(S_K,M(\lambda)_{\mathbb{C}})(\pi_f^{K}) \neq 0$$

and one knows that π_f is equivalent to π_f' if and only if $\pi_f^{K_f} \sim (\pi_f')^{K_f}$ provided $\pi_f^{K_f}$ is not trivial. Hence by the same argument as above we get a decomposition defined over $\bar{\mathbb{Q}}$

$$H^{\bullet}_{cusp}(\tilde{S},M(\lambda)) = \bigoplus_{\pi \in \underline{Coh}_o(M)} H^{\bullet}(\tilde{S},M(\lambda))(\pi_f)$$

and if we introduce the set

$$\underline{M} = \{(\lambda,\pi_f) \mid \lambda \in \Lambda, \pi_f \text{ finite part of a cuspidal automorphic form } \pi \in \underline{Coh}_o(M(\lambda))$$

then the system

$$\{H^{\bullet}(\tilde{S},M(\lambda))(\pi_f)\}_{(\lambda,\pi_f)}$$

gets a $\mathbb{Q}$-structure s.t. the system of projections

$$H^{\bullet}(\tilde{S},M(\lambda)) \rightarrow H^{\bullet}(\tilde{S},M(\lambda))(\pi_f)$$

is defined over $\mathbb{Q}$.

This fact has been used in |Ha5| to prove some rationaltiy results for special values of L-functions. It is clear that one does not need the full strength of proposition 3.2.4.

3.3. We have to recall some facts concerning the cohomology of Harish-Chandra modules for the two groups $GL_2(\mathbb{R})$ and $GL_2(\mathbb{C})$. We shall also need some results on the cohomology of non-unitary modules.

If we study the $(\mathfrak{g}_\infty,K)$-cohomology of a Harish-Chandra module H_{π_∞} then we will always know that

$$H_{\pi_\infty} = \bigotimes_{v\in S_\infty} H_{\pi_v} \quad .$$

(If π_∞ is the infinite component of an automorphic form this is in |Fl|, |Go|, § 3 .) Then we have a Künneth-formula

$$H^{\bullet}(\mathfrak{g}_\infty,K,H_{\pi_\infty}^{(K_\infty)} \otimes M_{\mathbb{C}}) = \bigotimes_{v\in S_\infty} H^{\bullet}(\mathfrak{g}_v,K_v,H_{\pi_v}^{(K_v)} \otimes M_v) \quad .$$

Here $\mathfrak{g}_v$ is the Lie-algebra of the real group $R_{F_v/\mathbb{R}}(GL_2/F_v) = G_v/\mathbb{R}$, we have

$$G \times_{\mathbb{Q}} \mathbb{R} = \prod_{v\in S_\infty} G_v \quad .$$

If v is a real place then it corresponds to an embedding $\tau : F \rightarrow \mathbb{R}$ and if v is complex then it corresponds to a pair of conjugate embeddings

$$\tau, \bar{\tau} \ : \ F \to \mathbb{C}$$

which can also be viewed on the two continuous isomorphisms $\tau, \bar{\tau} : F_v \tilde{\to} \mathbb{C}$.

In the first case we have $M_v = M_\tau \otimes_{\bar{\mathbb{Q}}} \mathbb{C}$ and in the second case

$$M_v \simeq M_{\tau,\mathbb{C}} \otimes M_{\bar{\tau},\mathbb{C}}$$

as a module for $G_v \times_{\mathbb{R}} \mathbb{C} = \prod_{\substack{\tau : F_v \to \mathbb{C} \\ \tau \text{ continuous}}} GL_2$.

We consider Harish-Chandra modules for $(\mathfrak{g}_v, K_v)$ and recall the properties of those which have cohomology with coefficients in M_v .

We give $T_v/\mathbb{R}$, $B_v/\mathbb{R}$ etc. the obvious meaning (1.0.2). We start from a character

$$\lambda \ : \ T_v(\mathbb{R}) \ \to \ \mathbb{C}^*$$

which is also considered to be a character on $B_v(\mathbb{R})$; then $\mathbb{C} \cdot \lambda$ will be the one dimensional $T_v(\mathbb{R})$-module on which $T_v(\mathbb{R})$ acts by λ . If $\mathfrak{t}_v = \mathrm{Lie}(T_v/\mathbb{R})$ and $K_v^T = T_v(\mathbb{R}) \cap K_v = B_v(\mathbb{R}) \cap K_v$, then we get by differentiation a $(\mathfrak{t}_v, K_v^T)$-module also denoted by $\mathbb{C}\lambda$. We consider the induced Harish-Chandra module

$$V_\lambda = \left\{ f : G_v(\mathbb{R}) \to \mathbb{C} \;\middle|\; \begin{array}{l} f(bg) = \lambda(b) f(g) \quad \text{for} \quad b \in B_v(\mathbb{R}) \\ g \in G_v(\mathbb{R}) \quad \text{and} \quad f \ \text{is} \ K_v\text{-finite} \end{array} \right\}$$

We are interested in the cohomology

$$H^\bullet(\mathfrak{g}_v, K_v, V_\lambda \otimes M_v)$$

and recall the procedure developed by A. Borel and P. Delorme which gives us a formula for it, since we are interested in some informations falling out in course of the proof we reproduce it in some detail (|B-W|, III, 3.3, |Del|,).

We decompose the Lie-algebra

$$\mathfrak{g}_v = \mathrm{Lie}(K_v) + \mathrm{Lie}(B_v(\mathbb{R})) = (\mathfrak{k}_v + \mathfrak{b}_v) \oplus \check{\mathfrak{u}}_v \quad .$$

We arranged our data in such a way that we get an identification

$$\mathfrak{g}_v/\mathfrak{k}_v \simeq \mathfrak{b}_v/\mathfrak{b}_v^K \oplus \check{\mathfrak{u}}_v \tag{3.3.1}$$

which is compatible with the action of K_v^T on both sides. This gives us an identification

$$\mathrm{Hom}_{K_v^T}(\Lambda^\bullet(\mathfrak{b}_v/\mathfrak{b}_v^K \oplus \check{\mathfrak{u}}_v), \mathbb{C}\lambda \otimes M_v) \simeq \mathrm{Hom}_{K_v}(\Lambda^\bullet(\mathfrak{g}_v/\mathfrak{k}_v), V_\lambda \otimes M_v) \tag{3.3.2}$$

We want to describe this identification explicitely. Let $M_v^\vee = \mathrm{Hom}(M_v,\mathbb{C})$ be the contragredient module. The element ω in the space on the left can be evaluated on

$$D \otimes \Phi \in \Lambda^\bullet(\mathfrak{b}_v/\mathfrak{b}_v^K \oplus \check{\mathfrak{u}}_v) \otimes M_v^\vee$$

and the result is a number $\omega(D \otimes \Phi) \in \mathbb{C} = \mathbb{C}\lambda$. The corresponding element on the right is given by the rule

$$\tilde{\omega}(D \otimes \Phi)(g) = \tilde{\omega}(D \otimes \Phi)(b_\infty k) = \lambda(b_\infty)\omega(\mathrm{ad}(k^{-1})D \otimes \rho_v^\vee(k^{-1})\Phi) \tag{3.3.3}$$

and it is clear that $\tilde{\omega}(D \otimes \Phi)(g) \in V_\lambda$. Here of course $g = bk$ with $b \in B_v(\mathbb{R})$, $k_v \in K_v$.

The Lie-algebra $\check{\mathfrak{u}}_v$ acts trivially on $\mathbb{C}\lambda$ and exploiting the arguments of Kostant ([Ko], Thm. 5.14) Borel and Delorme show that we have a degenerating spectral sequence

$$H^\bullet(\mathfrak{a}_v, K_v^T, H^\bullet(\check{\mathfrak{u}}_v, M_v) \otimes \mathbb{C}\lambda) \Rightarrow H^\bullet(\mathfrak{g}_v, K_v, V_\lambda \otimes M_v) \quad .$$

The Lie-algebra $\mathfrak{a}_v$ acts semi-simple on $H^\bullet(\check{\mathfrak{u}}_v, M_v)$ and K_v^T acts trivially on $\mathfrak{a}_v$, hence we get

$$\mathrm{Hom}(\Lambda^\bullet(\mathfrak{a}_v/\mathfrak{a}_v^K), (H^\bullet(\check{\mathfrak{u}}_v, M_v) \otimes \mathbb{C}\lambda)(0)) = H^\bullet(\mathfrak{g}_v, K_v, V_\lambda \otimes M_v)$$

where (0) indicates that we take the weight zero space. This is the Borel-Delorme formula.

We now separate the two cases v real and v complex

<u>3.4. $F_v = \mathbb{R}$:</u> We have $M_v = M(d,\nu)$ and then we know

$$H^0(\check{\mathfrak{u}}_v, M_v) = \mathbb{C}e_d \quad , \quad H^1(\check{\mathfrak{u}}_v, M_v) = \mathbb{C}(e_{-d} \otimes u_\alpha^v)$$

(see 2.3.). We choose the two characters

$$\lambda_0 \; : \; \begin{pmatrix} t_1 & * \\ 0 & t_2 \end{pmatrix} \to t_1^{-d} \cdot (t_1 t_2)^{-\nu}$$

$$\lambda_1 \; : \; \begin{pmatrix} t_1 & * \\ 0 & t_2 \end{pmatrix} \to t_2^{-d-1} t_1 (t_1 t_2)^{-\nu}$$

(see 2.8.2). Then the above formula of Borel and Delorme tells us

$$H^q(\mathfrak{g}_v, K_v, V_{\lambda_i} \otimes M_v) = \begin{cases} \mathbb{C} & q = i, i+1 \\ 0 & \text{otherwise} \end{cases} \quad .$$

But actually we know a little bit more, we represent the classes generating the cohomology $H(\check{\mathfrak{u}}_v, M_v)$ by explicit classes

$$e_d \in \mathrm{Hom}(\Lambda^0 \check{\mathfrak{u}}_v, M_v) \quad , \quad e_{-d} \otimes u_\alpha^v \in \mathrm{Hom}(\Lambda^1 \check{\mathfrak{u}}_v, M_v)$$

and hence we constructed explicit generators

$$e^{(i)} \in \mathrm{Hom}_{K_v^T}(\Lambda^o(\mathcal{A}_v/\mathcal{A}_v^K) \otimes \Lambda^i \check{\mu}_v, \mathbb{C}\lambda_i \otimes M_v) \quad \mathrm{Hom}(\Lambda^i(\mathfrak{g}_v, k_v), V_{\lambda_i} \otimes M_v)$$

representing the generators of $H^i(\mathfrak{g}_v, K_v, V_{\lambda_i} \otimes M_v)$. We define the character

$$\alpha_v^s \ : \ \begin{pmatrix} t_1 & * \\ o & t_2 \end{pmatrix} \to |\alpha_v(b)|^s = |t_1/t_2|_v^s$$

and it is standard that we have an intertwining operator

$$T_s \ : \ V_{\lambda_1\alpha_v^s} \to V_{\lambda_o\alpha_v^s}$$

which for $Re(s) > -1/2$ is given by

$$T_s \ (g) = \int_{\mathbb{R}} f((\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix})(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix})g)du \quad .$$

The following facts concerning these intertwining operators are well known and important for us and the general cohomology theory of $(\mathfrak{g}_v, K_v)$-modules.

(i) For $s = o$ the operator

$$T_o \ : \ V_{\lambda_1} \to V_{\lambda_o}$$

has an infinite dimensional kernel. This kernel is the direct sum of two irreducible $(\mathfrak{g}_v, K_v)$-modules

$$\ker(T_o) = D_{\lambda_1}^+ \oplus D_{\lambda_1}^- \quad .$$

(ii) The differential form

$$e^{(i)} \in \mathrm{Hom}_{K_v}(\Lambda^1(\mathfrak{g}_v/k_v), V_{\lambda_1} \otimes M_v)$$

constructed above maps to zero under T_o .

(iii) The two modules $D^{\pm}_{\lambda_1}$ are the only irreducible infinite dimensional unitary $(\mathfrak{g}_v, K_v)$-modules which have cohomology with coefficients in M_v.

All these assertions can be derived from |Go|, § 2, Thm. 2 or |B-W|, VI, § 4. The assertion (iii) can be found in |Vo| Chap. 5.

3.5. $F_v \simeq \mathbb{C}$. In this case we have

$$M_v = M(d(\tau),\nu(\tau)) \otimes M(d(\bar{\tau}),\nu(\bar{\tau})) = M(d_1,\nu_1) \otimes M(d_2,\nu_2) \quad .$$

Let us assume that $d_1 \geq d_2$ and let us identify $F_v \Longrightarrow \mathbb{C}$ by means of the map τ. Again we understand the cohomology

$$H^\bullet(\check{\mu}_v, M_v) = H^\bullet(\check{\mu}_o, M(d_1,v_1)) \otimes H^\bullet(\check{\mu}_o, M(d_2,v_2))$$

and the generating cohomology classes are given by explicit representatives

$$e_{d_1} \otimes e_{d_2} \quad , \quad u^v_{\alpha,\tau} \otimes e_{-d_1} \otimes e_{d_2} \quad , \quad e_{d_1} \otimes (u^v_{\alpha,\bar{\tau}} \otimes e_{-d_2}) \quad ,$$
$$u^v_{\alpha,\tau} \otimes e_{-d_1} \otimes u^v_{\alpha,\bar{\tau}} \otimes e_{-d_2}$$

let us call these classes

$$[e^{(i,j)}] \in H^i(\check{\mu}_o, M(d_1,v_1)) \otimes H^j(\check{\mu}_o, M(d_2,v_2)) \quad .$$

We know that these classes are eigenclasses for the action of $T_v(\mathbb{R})$ and the action of

$$T_v(\mathbb{R}) = \left\{ \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \middle| \ t_1,t_2 \in \mathbb{C}^* = F_v^* \right\}$$

on these classes is given by the characters (listed in the same order)

$$\{t_1^{d_1}(t_1t_2)^{\nu_1}\bar{t}_1^{d_2}\overline{(t_1t_2)}^{\nu_2},\, t_2^{d_1+1}t_1^{-1}(t_1t_2)^{\nu_1}\bar{t}_1^{d_2}\overline{(t_1t_2)}^{\nu_2},\, t_1^{d_1}(t_1t_2)^{\nu_1}\bar{t}_2^{d_2+1}\bar{t}_1^{-1}\overline{(t_1t_2)}^{\nu_2},$$

$$t_2^{d_1+1}t_1^{-1}(t_1t_2)^{\nu_1}\bar{t}_2^{d_2+1}\bar{t}_1^{-1}\overline{(t_1t_2)}^{\nu_2}\} \; .$$

Let $\{\lambda_{o,o},\lambda_{1,o},\lambda_{o,1},\lambda_{1,1}\} = \Lambda$ be the list of inverses of the above characters listed again in the same order. Then the Borel-Delorme formula tells us

$$H^q(\mathfrak{g}_v,K_v,V_{\lambda_{i,j}} \otimes M_v) = \mathbb{C}[e^{(i,j)}] \otimes \mathrm{Hom}(\Lambda^\bullet(\mathfrak{a}_v/\mathfrak{a}_v^K),\mathbb{C})$$

where $[e^{(i,j)}]$ has degree $i+j$ and is represented by the above form

$$e^{(i,j)} \in \mathrm{Hom}_{K_v^T}(\Lambda^o(\mathfrak{a}_v/\mathfrak{a}_v^K) \otimes \Lambda^{i+j}\check{\mu}_v,\mathbb{C}\lambda_{i,j} \otimes M_v)$$

$$\mathrm{Hom}_{K_v}(\Lambda^{i+j}(\mathfrak{g}_v/k_v),V_{\lambda_{i,j}} \otimes M_v) \quad .$$

Again we have intertwining operators

$$T_s : V_{\lambda_{1,1}\cdot\alpha_v^s} \to V_{\lambda_{o,o}\cdot\alpha_v^{-s}}$$

$$T_s : V_{\lambda_{1,o}\cdot\alpha_v^s} \to V_{\lambda_{o,1}\cdot\alpha_v^{-s}}$$

which again given by the integral

$$T_s(f)(g) = \int_{\mathbb{C}} f(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}g)\,du$$

which certainly converges for $\mathrm{Re}(s)$ large enough and which has meromorphic continuation into the entire complex plane. For the measure du we choose the Tamagawa measure. Again we collect some facts:

(i) The operator

$$T \; : \; V_{\lambda_{1,1}} \to V_{\lambda_{o,o}}$$

has an infinite dimensional kernel which is isomorphic to $V_{\lambda_{1,o}} \simeq V_{\lambda_{o,1}}$. The image of T is $M_v^V = \mathrm{Hom}(M_v, \mathbb{C})$. The operator induces the zero map on the cohomology

$$H^{\bullet}(\mathfrak{g}_v, K_v, V_{\lambda_{1,1}} \otimes M_v) \to H^{\bullet}(\mathfrak{g}_v, K_v, V_{\lambda_{o,o}} \otimes M_v) \quad .$$

(ii) The operator

$$T \; : \; V_{\lambda_{1,o}} \to V_{\lambda_{o,1}}$$

is an isomorphism. It yields

$$T(e^{(1,o)}) \; = \; - \frac{2\pi}{d_1 - d_2 + 1} \cdot e^{(o,1)}$$

on the level of differential forms.

(iii) The modules $V_{\lambda_{1,o}}$, $V_{\lambda_{o,1}}$ are the only irreducible infinite dimensional $(\mathfrak{g}_v, K_v)$-modules which have cohomology with coefficients in M_v . If $_{v}^{(1)}$ is the derived Lie-algebra then the restriction of $V_{\lambda_{1,o}}$ to $(\mathfrak{g}_v^{(1)}, K_v^{(1)})$ is unitary if and only if $d_1 = d_2$.

The first assertion (i) is stated in [Go], § 2, 4 , the fact that the operator is trivial on the cohomology follows from the Borel-Delorme formula, the left hand side has cohomology in degree 2 and 3 , the right hand side has cohomology in degree 0 and 1 .

The first assertion in (ii) is also in [Go] loc. cit.. We shall prove the formula stated there after the following lines.

For the assertion (iii) we refer to [En], [Du] and [Vo]. This has been pointed out to me by G. Zuckerman.

We have to prove the formula stated in (ii). We have to analyse the K_v-types occuring in $V_{\lambda_{1,o}}$ and $V_{\lambda_{o,1}}$ first. To do this we consider $K_v^{(1)} = K_v \cap SL_2(\mathbb{C})$. If we restrict $\lambda_{1,o}$ to the compact torus

$$T_c^{(1)} = \left\{ \begin{pmatrix} e^{i\theta} & o \\ o & e^{-i\theta} \end{pmatrix} \right\} = T_v(\mathbb{R}) \cap K_v^{(1)}$$

we get the character

$$\begin{pmatrix} e^{i\theta} & o \\ o & e^{-i\theta} \end{pmatrix} \to e^{i\theta(-d_1-d_2-2)} .$$

This implies that

$$V_{\lambda_{1,o}|K_v^{(1)}} = \bigoplus_{\substack{\nu = d_1+d_2+2 \\ \nu \bmod 2}} \vartheta(\nu)$$

where $\vartheta(\nu)$ is the irreducible representation of $K_v^{(1)}$ which has highest weight ν. If we restrict M_v to $K_v^{(1)}$ we get a Clebsch-Gordon decomposition

$$M_{v|K_v^{(1)}} = \bigoplus_{\substack{\nu = d_1+d_2 \\ \nu \bmod 2}}^{d_1-d_2} \vartheta(\nu)$$

(remember that we assumed $d_1 \geq d_2$) and hence we find

$$\dim \mathrm{Hom}_{K_v}(\Lambda^1(\mathfrak{g}_v/k_v), V_{\lambda_{1,o}} \otimes M_v) = 1 .$$

Then it becomes clear that T maps $e^{(1,o)}$ to a multiple of $e^{(o,1)}$. We have to compute the proportionality factor.

This means that we have to compute the integral

$$\int_{\mathbb{C}} e^{(1,o)}(X_{\overline{\alpha}})\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\right) du \quad .$$

Here we recall $\breve{\mu}_v \otimes \mathbb{C} = \bigoplus_{\tau : F_v \tilde{\to} \mathbb{C}} \breve{\mu}_o = \mathbb{C}X_\alpha \oplus \mathbb{C}X_{\overline{\alpha}}$ where $X_{\overline{\alpha}} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \breve{\mu}_o$, and the result of this integration is a vector in M_v which is proportional to $e_{d_1} \otimes e_{-d_2}$. Hence we can say

$$\int_{\mathbb{C}} e^{(1,o)}(X_{\overline{\alpha}})\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\right)du = c \cdot e_{d_1} \otimes e_{-d_2}$$

and we have to compute the proportionality factor c . Now we follow very closely the computation in |Ha3| p. 72 . We introduce polar coordinates and write $u = r \cdot e^{i\theta}$ and get

$$2 \int_0^\infty \int_0^{2\pi} e^{(1,O)}(X_{\overline{\alpha}})\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} e^{i\theta} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}\begin{pmatrix} e^{-i\theta} & 0 \\ 0 & 1 \end{pmatrix}\right)d\theta \; rdr \quad .$$

If we carry out the integration against θ we get zero for all components $e_a \otimes e_b$ except for the component $e_{d_1} \otimes e_{-d_2}$ where the integrand does not depend on θ . Hence we have to compute the factor c in

$$4\pi \int_0^\infty e^{(1,O)}(X_{\overline{\alpha}})\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}\right)rdr = c \cdot e_{d_1} \otimes e_{-d_2} \quad .$$

We decompose

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} = b(r) \cdot k(r) = \begin{pmatrix} (1+r^2)^{-1/2} & r \\ 0 & (1+r^2)^{1/2} \end{pmatrix} \cdot \begin{pmatrix} -r(1+r^2)^{-1/2} & -(1+r^2)^{-1/2} \\ (1+r^2)^{-1/2} & -r(1+r^2)^{-1/2} \end{pmatrix}$$

and we consider the matrix coefficients

$$\mathrm{ad}\, k(r) \cdot X_\alpha = \ldots + C(r) \cdot X_{\overline{\alpha}}$$

$$\rho(k(r))e_{-d_1} \otimes e_{+d_2} = \ldots D(r) \cdot e_{d_1} \otimes e_{-d_2}$$

(comp. |Ha3|, p. 72) where we take the standard basis on $M_v = M(d_1,\nu_1) \oplus M(d_2,\nu_2)$ built by the weight vectors with respect to $T_v/\mathbb{R}$ and the basis X_α, $X_{\bar\alpha}$, H in $\mathfrak{g}_v/k_v$ (comp. |Ha3|, p. 66). We have to compute

$$4\pi \int_0^\infty \lambda_{1,0}(b(r)) \cdot C(r)\, D(r)\, r dr \quad .$$

We have

$$\lambda_{1,0}(b(r)) = (1+r^2)^{-1/2(d_1+2-d_2)}$$

$$C(r) = (1+r^2)^{-1}$$

To compute $D(r)$ we recall that

$$k(r) = b(r)^{-1} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}$$

and we find easily that

$$D(r) = (1+r^2)^{-1/2(d_1-d_2)} \quad .$$

Hence we have to evaluate

$$- 4\pi \int_0^\infty (1+r^2)^{-d_1+d_2-1}\, r dr = - \frac{2\pi}{d_1-d_2+1}$$

and this gives the desired formula.

<u>Remark</u>: a) This formula differs by a factor 2 from the result in |Ha3|, p. 72 , if we take $d_1 = d_2 = 0$. This is due to the different choice of the measures on $\mathbb{C}$ (|Ha3|, p. 70, line 6 bottom).

b) We are here in the domain of convergence for the intertwining operator, we could actually also compute

$$T_s \;:\; \mathrm{Hom}_{K_v}(\Lambda^1(\mathfrak{g}_v/k_v), V_{\lambda_{1,0}\cdot\alpha_v^s}) \to \mathrm{Hom}_{K_v}(\Lambda^1(\mathfrak{g}_v/k_v), V_{\lambda_{0,1}\cdot\alpha_v^{-s}})$$

and define $e_s^{(1,0)}$, $e_{-s}^{(0,1)}$ and get

$$T_s(e_s^{(1,0)}) = -\frac{2\pi}{s+d_1-d_2+1}\, e_{-s}^{(0,1)}$$

(comp. |Ha3|,§2, where $e_s^{(1,0)} = \omega_s$).

3.6. We are now able to say something much more precise about the decomposition

$$H^{\bullet}_{cusp}(\tilde{S},\widetilde{M}(\lambda)) = \bigoplus_{\pi \in \underline{Coh_o}(M(g))} H^{\bullet}(\tilde{S},\widetilde{M}(\lambda))(\pi_f)$$

which we discussed in 3.2.5 . We have $\pi = \pi_\infty \times \pi_f$ and $\pi_\infty = \bigotimes_{v \in S_\infty} \pi_v$. The condition $\pi \in \underline{Coh_o}(M(\lambda))$ is equivalent to

$$H^{\bullet}(\mathfrak{g}_v, K_v, H_{\pi_v}^{(K_v)} \otimes M_v) \neq 0$$

where $H_{\pi_v}^{(K_v)}$ is the Harish-Chandra module attached to π_v . We also know that π_v restricted to the derived group $G_v^{(1)}(\mathbb{R}) = SL_2(F_v)$ has to be unitary. Then we get

3.6.1 If there is a complex place $v = \{\tau,\bar{\tau}\}$ for which $M_v = M(d(\tau),\nu(\tau)) \otimes M(d(\bar{\tau}),\nu(\bar{\tau}))$ and for which $d(\tau) \neq d(\bar{\tau})$ then we get $Coh_o(M(\lambda)) = \emptyset$. Let us call this case the <u>non-selfconjugate case</u>. Since in this case we also cannot have contributions from the one dimensional case we get

$$\tilde{H}^{\bullet}(\tilde{S},\widetilde{M}(\lambda)) = 0$$

in the non-selfconjugate case.

The selfconjugate case is the case where $d(\tau) = d(\bar{\tau})$ for all $v = \{\tau,\bar{\tau}\}$. In this case we get

<u>3.6.2</u> If $M(\lambda)$ is selfconjugate representation then we have for each v exactly one irreducible representation of $GL_2(F_v)$ which is unitary on $SL_2(F_v)$ and has non-trivial $(\mathfrak{g}_v,K_v)$-cohomology with coefficients in $M_v = M_v(\lambda_v)$. This representation is given by the following construction.

<u>3.6.2.1</u> If v is real then we start from a character

$$\lambda_1 \cdot \varepsilon \quad : \quad B(\mathbb{R}) \to \mathbb{C}^*$$

described in 3.4. but eventually twisted by a sign character ε which factors over the determinant. Then we know that the kernel

$$D^+_{\lambda_1\varepsilon} \oplus D^-_{\lambda_1\varepsilon} = \ker(T : V_{\lambda_1\varepsilon} \to V_{\lambda_0\varepsilon})$$

has a unique positive definite invariant scalar product, let $c(\lambda_v) = \pi_v$ the completion of this space with respect to this scalar product. This becomes a representation of $G_v(\mathbb{R}) = GL_2(\mathbb{R})$ which does not depend on the choice of ε . If $H_{\pi_v} = H_{c(\lambda_v)}$ is the space on which $G_v(\mathbb{R})$ acts then

$$H_{c(\lambda_v)}^{(K_v)} = D^+_{\lambda_1\varepsilon} \oplus D^-_{\lambda_1\varepsilon}$$

and

$$H(\mathfrak{g}_v,K_v,H_{c(\lambda_v)}^{(K_v)} \otimes M_v) = \begin{cases} \mathbb{C} + \mathbb{C} & \text{for} \quad q = 1 \\ 0 & \text{otherwise} \end{cases} .$$

<u>3.6.2.2</u> If v is complex, then we choose $c(\lambda_v)$ to be the unitary completion of our representation $V_{\lambda_{1,0}}$ constructed in 3.5. Then

$$H^q(\mathfrak{g}_v, K_v, H_{c(\lambda_v)}^{(K_v)} \otimes M_v) = \begin{cases} \mathbb{C} & q = 1,2 \\ 0 & \text{otherwise} \end{cases} .$$

Therefore we may summarize the Eichler-Shimura isomorphism to

$$H^{\bullet}_{cusp}(\tilde{S}, M(\lambda)_{\mathbb{C}}) = \bigoplus_{\substack{\pi \text{ cuspidal autom. form, } \pi_\infty = \bigotimes_{v \in S_\infty} \pi_v \\ \text{and } \pi_v \sim c(\lambda_v)}} \bigotimes_{v \in S} H^{\bullet}(\mathfrak{g}_v, K_v, H_{\pi_v}^{(K_v)} \otimes M_v) \otimes H_{\pi_f}$$

where H_π is the subspace of $L^2_0(G(\mathbb{Q})\backslash G(\mathbb{A}))(\omega^{-1})$ of type π (see introduction to this section). This holds of course only in the selfconjugate case.

IV The Eisenstein cohomology

We now focus our attention onto the map

$$r : H^{\bullet}(\tilde{S}, \tilde{M}) \to H^{\bullet}(\partial\tilde{S}, \tilde{M}) \quad ,$$

we want to describe the image in terms of the description of the right hand side given in II, Theorem 1, and we want to give a section from the image back by means of Eisenstein series; this section will be defined over $\mathbb{Q}$ in the sense explained before. We put

$$V_\phi = \mathrm{Ind}_{\pi_0(B_\infty) \times B(\mathbb{A}_f)}^{\pi_0(G_\infty) \times G(\mathbb{A}_f)} \bar{\mathbb{Q}} e_\phi$$

(see 2.7.), we have (Thm. 1)

$$H^{\bullet}(\partial\tilde{S}, \tilde{M}) = \bigoplus_{\gamma \in \mathrm{Coh}(M)} \bigoplus_{\phi : \phi \text{ of type } \gamma} V_\phi \otimes \Lambda^{\bullet}(\mathfrak{X}(T/Z))$$

where we have of course to take into account that $V_\phi \subset H^{d(\phi)}(\partial\tilde{S}, \tilde{M})$ where

$d(\phi)$ can be computed in terms of the type of ϕ (see 2.8. and 2.9.). We also recall that for ϕ with type $(\phi) = \gamma \in \mathrm{Coh}(M)$ we have that $s_o \cdot \phi = |\alpha| \cdot \phi^{s_o}$ has type $(s_o\phi) = \alpha + \gamma^{s_o} \in \mathrm{Coh}(M)$ (see 2.9.).

In the above decomposition we group the pairs corresponding to ϕ and $s_o \cdot \phi$ and get

$$H^{\bullet}(\partial\tilde{S},\tilde{M}) = \bigoplus_{\{\phi,s_o\cdot\phi\}} (V_\phi \oplus V_{s_o\cdot\phi}) \otimes \Lambda^{\bullet}(\mathcal{H}(T/Z)) \quad .$$

We have always $\phi \neq s_o \cdot \phi$ (This holds on the level of types.). We define for each $[\phi] = \{\phi, s_o\cdot\phi\}$

$$\mathrm{Im}\, r_{[\phi]} = \mathrm{Im}(r) \cap (V_\phi \oplus V_{s_o\cdot\phi}) \otimes \Lambda^{\bullet}(\mathcal{H}(T/Z)) \quad .$$

Our second main result will give us an exact description of $\mathrm{Im}\, r_{[\phi]}$ in terms of the given datum ϕ and data derived from it and it asserts that

$$\mathrm{Im}(r) = \bigoplus_{[\phi]} \mathrm{Im}\, r_{[\phi]} \quad .$$

<u>4.1.</u> If we restrict the character ϕ to the torus $T^{(1)}$ then we have seen in 2.9. that

$$w(\phi^{(1)}) + w(s_o\cdot\phi^{(1)}) = -4 \quad .$$

If $w(\phi^{(1)}) \neq -2$ then we label the pair $\{\phi, s_o\cdot\phi\}$ in such a way that $w(\phi^{(1)}) < -2$, if $w(\phi^{(1)}) = -2$ then we don't have a specific labeling so we could interchange ϕ and $s_o\cdot\phi$. If the degree of the cohomology corresponding to ϕ is higher than the middle dimension, then we have automatically $w(\phi^{(1)}) < -2$ (see 2.9.2). If we have labeled our pair $\{\phi, s_o\cdot\phi\}$ in such a way that $w(\phi^{(1)}) \leq -2$ then we say that ϕ is in the fundamental chamber.

4.2. Now we assume that ϕ is in the fundamental chamber, we want to construct a system of intertwining operators

$$T_\phi^{loc} : V_\phi \to V_{s_o\cdot\phi} .$$

If we say system we mean of course that we want to have a system of operators defined over $\mathbb{Q}$. In 2.7.1 we constructed the $\mathbb{Q}$-structure on the system $\{V_\phi\}_{\mu_1}$ where $\mu_1 = (\lambda,\gamma,\phi) \in \underline{M}_1$. We want T_ϕ^{loc} to be defined for all ϕ and we want it to define an isomorphism over $\mathbb{Q}$, we observe that s_o defines an automorphism of $\underline{M}_1$.

This construction has been discussed in |Ha4|, 4.7., 4.8. and we can use |Ha4| loc. cit. with only one modification. We put

$$T_\phi^{loc} = \bigotimes_{\mathfrak{y}} T_{\phi_{\mathfrak{y}}}$$

where T_{ϕ} maps the spherical function

$$\psi_{o,\mathfrak{y}}(g_{\mathfrak{y}}) = \psi_{o,\mathfrak{y}}(b_{\mathfrak{y}}k_{\mathfrak{y}}) = \phi_{\mathfrak{y}}(b_{\mathfrak{y}})\phi_{1,\mathfrak{y}}(\det(k_{\mathfrak{y}}))$$

(if $\phi\begin{pmatrix}\underline{t}_1 & o\\ o & \underline{t}_2\end{pmatrix} = \phi_1(\underline{t}_1)\cdot\phi_2(\underline{t}_2)$) in case that the character $\phi^{(1)}$ is unramified. If $\phi^{(1)}$ is ramified we put

$$T_{\phi_{\mathfrak{y}}}(\psi_{\mathfrak{y}})(g_{\mathfrak{y}}) = \int_{U_o(F_{\mathfrak{y}})} \psi_{\mathfrak{y}}(\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}1 & u_{\mathfrak{y}}\\ 0 & 1\end{pmatrix}g_{\mathfrak{y}})du_{\mathfrak{y}}$$

where $du_{\mathfrak{y}}$ is the local component of the Tamagawa measure on U_o/F defined by the differential form dx . The integral is actually a finite sum. It is clear that the system of operators T_ϕ^{loc} has the required properties. The only difference to the case treated in [Ha4], 4.7. is that we may encounter situations where $T_{\phi_{\mathfrak{y}}}$ is not an isomorphism. This can only happen for one special character, namely

$$\phi^{(1)}_{\gamma} = |\alpha|^2_{\gamma}$$

(we assumed that ϕ is in the fundamental chamber). In this case $\phi^{(1)}_{\gamma}$ is of course unramified and we get a diagram

$$\begin{tikzcd} T_{\phi_\gamma} : V_{\phi_\gamma} \arrow[r] \arrow[dr] & \bar{\mathbb{Q}}\,\psi_{o,\gamma} \arrow[d, hook] \\ & V_{s_o \cdot \phi_\gamma} \end{tikzcd}$$

where $\psi_{o,\gamma}$ is the spherical function in $V_{s_o\cdot\phi_\gamma}$. The kernel of T_{ϕ_γ} in this case is denoted by V'_{ϕ_γ}.

We are now able to state our second main theorem

<u>Theorem 2</u>. <u>Let</u> $M = M(\lambda)$ <u>and</u> $\phi \in \mathrm{Coh}(M)$, $[\phi] = \{\phi, s_o\cdot\phi\}$, <u>we assume that</u> ϕ <u>is in the fundamental chamber</u>. <u>Let</u> $t = \dim \mathcal{H}(T/Z)$ = rank <u>of the group of units of the field</u> F, <u>let</u> $n = [F:\mathbb{Q}]$.

1) <u>If</u> ϕ <u>is not balanced and therefore</u> $\deg(\phi) > \deg(s_o\cdot\phi)$ <u>then we have</u>

$$\mathrm{Im}\, r_{[\phi]} = V_\phi \otimes \Lambda^{\bullet}\mathcal{H}(T/Z)$$

<u>unless we have</u> $\phi^{(1)} = |\alpha|^2$.

2) <u>If</u> $\phi^{(1)} = |\alpha|^2$ <u>then</u> $\dim M(\lambda) = 1$ <u>and</u>

$$\mathrm{Im}\, r_{[\phi]} = \bigoplus_{m<t} V_\phi \otimes \Lambda^m\mathcal{H}(T/Z) \oplus \tilde{V}_\phi \otimes \Lambda^t\mathcal{H}(T/Z) \oplus \bar{\mathbb{Q}} e_{s_o\cdot\phi} \otimes \Lambda^t\mathcal{H}(T/Z) .$$

<u>Here</u> $\tilde{V}_\phi \subset V_\phi$ <u>is the subspace generated by those tensors for which at least</u>

one finite component is in V'_ϕ and $e_{s_o \cdot \phi}$ is the spherical function in $V_{s_o \cdot \phi}$.

3) In the balanced case we get

$$\mathrm{Im}\, r_{[\phi]} = X_\phi \otimes \Lambda^\bullet(\mathcal{H}(T/Z))$$

where

$$X_\phi = \left\{(v, |d_F|^{-1/2}(\tfrac{-2\pi}{d+1})^{n/2} \cdot \frac{L(\phi^{(1)},-1)}{L(\phi^{(1)},0)} \cdot T_\phi^{loc}(v)) \mid v \in V_\phi\right\}$$

and $d = |d(\tau) - d(\bar{\tau})|$ for any pair of complex embeddings of F .

Moreover we have

$$\mathrm{Im}(r) = \bigoplus_{[\phi]} \mathrm{Im}\, r_{[\phi]}$$

and we have a decomposition

$$H^\bullet(\tilde{S},\tilde{M}) = \tilde{H}^\bullet(\tilde{S},\tilde{M}) \oplus H^\bullet_{Eis}(\tilde{S},\tilde{M})$$

into $\pi_o(G_\infty) \times G(A_f)$-modules and no irreducible submodule of $\tilde{H}^\bullet(\tilde{S},\tilde{M})$ intertwines with any subquotient of $H^\bullet_{Eis}(\tilde{S},\tilde{M})$.

We have a rational structure on the system $\{V_\phi\}_{\phi \in M_1}$ (see 2.7.1) hence we get a rational structure on the system of the $\mathrm{Im}\, r_{[\phi]}$ where $[\phi] \in M_1/\{s_o\}$. The above theorem gives us

Corollary 4.2.1: The splitting in Theorem 1 gives us a section

$$\mathrm{Eis} : \bigoplus_{\substack{[\phi] \\ \phi \in \mathrm{Coh}(M)}} \mathrm{Im}\, r_{[\phi]} \to H^\bullet(\tilde{S},\tilde{M}) = H^\bullet(\tilde{S},M(\lambda)) \quad .$$

The system of these sections is defined over $\mathbb{Q}$.

This corollary is obvious from Theorem 2. The theorem 2 itself is a generalization of the corresponding theorem in |Ha4| and in a certain sense it is also a special case of the results in my earlier paper |Ha1|. The point is that here the information is much more precise and specific (see |Ha2|, concluding remarks).

We shall now give the proof of theorem 2, and we shall of course pretty much follow the arguments in our earlier papers. We start from an algebraic Hecke character ϕ, let γ be the type of ϕ. We assume that ϕ is in the fundmanetal chamber and $\gamma \in \mathrm{Coh}(M(\lambda))$. We identified

$$V_\phi \otimes \Lambda^\bullet \mathcal{H}(T/Z) \subset H^\bullet(\partial\tilde{S},\tilde{M})$$

where V_ϕ gets mapped into the cohomology of degree $d(\phi)$ (Theorem 1). We recall briefly that this identification was depending on a choice of a generator (see 2.4.1)

$$e(\lambda,\gamma,<) \in H^0(\check{\mu},M(\lambda))(\gamma)$$

which depends on an ordering $<$ of the set $\{\tau \mid \tau : F \to \bar{\mathbb{Q}}\}$. Given this generator and given ϕ we selected an element

$$e_\phi \in H^0(\tilde{S}^T,H^0(\check{\mu},M)(\gamma))$$

(see 2.6.) and this choice provided the above embedding. We can look at this embedding in a different way. We consider the Harish-Chandra module

$$V_{\phi_\infty} = \mathrm{Ind}_{B_\infty}^{G_\infty} \phi_\infty \quad .$$

This Harish-Chandra module is the tensor product of the modules V_{λ_v} considered in 3.3. Then

$$V_{\phi_\infty} \otimes V_{\phi,\mathbb{C}} = \mathrm{Ind}_{B(A)}^{G(A)} \phi = V_\phi^*$$

where we now consider ϕ as a character on $B(A)$ and where we take K_∞-finite functions at the infinite components. Since ϕ is trivial on $B(\mathbb{Q})$ we get an embedding

$$I_\phi : V_\phi^* \to \mathcal{C}_\infty(B(\mathbb{Q})\backslash G(A))$$

and hence we get a map

$$H^\bullet(\mathfrak{g}_\infty, K_\infty, V_{\phi_\infty} \otimes M(\lambda)_{\mathbb{C}}) \otimes V_{\phi,\mathbb{C}} \to H^\bullet(\mathfrak{g}_\infty, K_\infty, \mathcal{C}_\infty(B(\mathbb{Q})\backslash G(A)) \otimes M(\lambda)_{\mathbb{C}}) =$$

$$= H^\bullet(\partial\tilde{S}, M(\lambda)_{\mathbb{C}}) \quad .$$

We keep our ordering $<$ and we assume that pairs of conjugates are consecuting each other.

Then

$$H^\bullet(\mathfrak{g}_\infty, K_\infty, V_{\phi_\infty} \otimes M(\lambda)_{\mathbb{C}}) = \bigotimes_{v \in S_\infty} H^\bullet(\mathfrak{g}_v, K_v, V_{\phi_v} \otimes M(\lambda_v))$$

by the Künneth formula. Using the Borel-Delorme formula we constructed

$$\tilde{e}(\phi_v) \in \mathrm{Hom}_{K_v}(\Lambda^{d(\phi_v)}(\mathfrak{g}_v/k_v), V_{\phi_v} \otimes M(\lambda_v))$$

in 3.4. and 3.5. (The ϕ_v correspond to the λ_i or $\lambda_{i,j}$ and the $\tilde{e}(\phi_v)$ to the $e^{(i)}$ or $e^{(i,j)}$). We then get an identification

$$H^\bullet(\mathfrak{g}_v, K_v, V_{\phi_v} \otimes M(\lambda_v)) = \mathbb{C} \cdot [e(\phi_v)] \otimes \mathrm{Hom}(\Lambda^\bullet(\mathfrak{a}_v/\mathfrak{a}_v^K), \mathbb{C}) \quad .$$

Multiplying the generators together we get

$$H^{\bullet}(\mathfrak{g}_{\infty},K_{\infty},V_{\phi_{\infty}} \otimes M(\lambda)_{\mathbb{C}}) = \mathbb{C}[\tilde{e}(\phi)] \otimes \bigotimes_{v\in S_{\infty}} \mathrm{Hom}(\Lambda^{\bullet}(\mathfrak{t}_v/\mathfrak{t}_v^{K}),\mathbb{C})$$

$$\mathbb{C}[\tilde{e}(\phi)] \otimes \Lambda^{\bullet}\partial\mathcal{L}(T/Z) \quad .$$

Hence the choice of the generator $[e(\phi)]$ provides an embedding

$$I_{\phi}^{\bullet} \;:\; C[\tilde{e}(\phi)] \otimes \Lambda^{\bullet}\partial\mathcal{L}(T/Z) \otimes V_{\phi,\mathbb{C}} \to H^{\bullet}(\partial\tilde{S},M(\lambda)_{\mathbb{C}})$$

and it is simply a question of keeping track of the different identifications to convince oneself that the embedding I_d is simply the complexification of the embedding of Theorem 1, the generator $\tilde{e}(\phi)$ is simply the product of the local generators selected in 3.4. and 3.5..

The identification (3.3.1) gives us an identification

$$\mathfrak{g}_{\infty}/k_{\infty} = \mathfrak{t}_{\infty}/\mathfrak{t}_{\infty}^{K_{\infty}} \otimes \check{\mu}_{\infty}$$

which is compatible with the action of $K_{\infty}^{T} = T_{\infty} \cap K_{\infty} = B_{\infty} \cap K_{\infty}$. We twist the character by a complex power of the Tate character $|\alpha|$ and then we get an identification of type (3.3.2) namely

$$\mathrm{Hom}_{K_{\infty}^{T}}(\Lambda^{\bullet}(\mathfrak{t}_{\infty}/\mathfrak{t}_{\infty}^{K_{\infty}} \otimes \check{\mu}_{\infty}),\mathbb{C}\phi\cdot|\alpha|^{s} \otimes M(\lambda)_{\mathbb{C}}) \otimes V_{\phi,\mathbb{C}} \tilde{\to}$$

$$\tilde{\to} \mathrm{Hom}_{K_{\infty}}(\Lambda^{\bullet}(\mathfrak{g}_{\infty}/k_{\infty}),V^{*}_{\phi\cdot|\alpha|^{s}} \otimes M(\lambda)_{\mathbb{C}}) \quad .$$

Here we notice that the left hand side does not depend on $s \in \mathbb{C}$, since the K_{∞}^{T}-module $\mathbb{C}\phi\cdot|\alpha|^{s}$ does only depend on the restriction of $\phi\cdot|\alpha|^{s}$ to K_{∞}^{T} .
Hence we get for any

$$\xi \in \mathrm{Hom}(\Lambda^{\bullet}(\mathfrak{t}_{\infty}/\mathfrak{t}_{\infty}^{K_{\infty}}),\mathbb{C}) \otimes \mathbb{C}\cdot\tilde{e}(\lambda,\gamma,\omega) \in \mathrm{Hom}_{K_{\infty}}(\Lambda^{\bullet}(\mathfrak{g}_{\infty}/k_{\infty}),V_{\phi_{\infty}\cdot|\alpha|_{\infty}^{s}} \otimes M(\lambda)_{\mathbb{C}})$$

(see 3.3.) and any $\psi \in V_{\phi.\mathbb{C}}$ a differential form

$$\omega(\underline{g},\phi\cdot|\alpha|^s,\xi,\psi) \in \mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}_\infty/k_\infty),V^*_{\phi|\alpha|^s} \otimes M(\lambda))$$

defined by (see 3.3.3)

$$\omega(\underline{g},\phi\cdot|\alpha|^s,\xi,\psi)(D\otimes\Phi) = \omega(b_\infty k_\infty\cdot g_f,\phi|\alpha|^s,\xi,\psi)(D\otimes\Phi) = \tag{4.2.2}$$

$$\phi_\infty\alpha_\infty^s(b_\infty)\cdot\psi(g_f)\xi(\mathrm{ad}(k_\infty^{-1})D\otimes\rho^v(k_\infty^{-1})\Phi)$$

(comp. |Ha4|, p. 123). Here we have to take $D \in \Lambda^\bullet(\mathfrak{g}_\infty/k_\infty)$ and $\Phi \in M(\lambda)^v_{\mathbb{C}} = M(\lambda^v)_{\mathbb{C}}$. Now we know that for $\mathrm{Re}(s) >> 0$ we have an intertwining operator

$$\mathrm{Eis}^* : V^*_{\phi\cdot|\alpha|^s} \to \mathcal{A}(G(\mathbb{Q})\backslash G(A))$$

given by the formula

$$\mathrm{Eis}^*(\psi^*)(\underline{g}) = \sum_{a\in B(\mathbb{Q})\backslash G(\mathbb{Q})} \psi^*(a\underline{g})$$

where the right hand side is the space of automorphic forms on $G(A)$ (comp. [HC] , [La]). If we apply the Eisenstein-intertwining operator to our differential form $\omega(\underline{g},\phi|\alpha|^s,\xi,\psi)$ we get a differential form

$$\mathrm{Eis}(\underline{g},\phi\cdot|\alpha|^s,\xi,\psi) \in \mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}_\infty/k_\infty),\mathcal{A}(G(\mathbb{Q})\backslash G(A))\otimes M(\lambda)_{\mathbb{C}}) \quad .$$

As in [Ha1] and [Ha2] we have to study the behaviour of this differential form at $s = 0$, we have to find out under what conditions it is holomorphic at $s = 0$ and yields a closed form. To do this we have to compute the constant term of the Eisenstein series first, the constant term is given by the integral

$$\int_{U(\mathbb{Q})\backslash U(A)} E(\underline{ug},\phi|\alpha|^s,\psi^*)\, d\underline{u}$$

where $d\underline{u}$ is the Tamagawa measure on $U(A) = U_o(A_F)$. We recall that this means that

$$d\underline{u} = |d_F|^{-1/2} \cdot \Pi\, du_v$$

where $\mathrm{vol}_{du_{\mathfrak{y}}}(\mathcal{O}_{\mathfrak{y}}) = 1$ for all finite places of F and

du_v = Lebesque-measure for v real

$du_v = i dz \wedge d\bar{z} = 2 \times$ Lebesque-measure if $F_v \simeq \mathbb{C}$.

We have $\mathrm{vol}_{d\underline{u}}(U(\mathbb{Q})\backslash U(A)) = \mathrm{vol}_{d\underline{u}}(U(F)\backslash U_o(A_F)) = 1$ and the standard computation of the constant term yields

$$\int_{U(\mathbb{Q})\backslash U(A)} E(\underline{ug}, \phi\cdot|\alpha|^s, \psi^*)\, d\underline{u} = \psi^*(\underline{g}) + \int_{U(A)} \psi^*(s_o \cdot \underline{ug})\, d\underline{u} = \psi^*(\underline{g}) + T^*_{\phi\cdot|\alpha|^s}$$

Here we have $\psi^*(\underline{g}) \in V^*_{\phi\cdot|\alpha|^s} \subset \mathcal{C}_\infty(U(A)T(\mathbb{Q})\backslash G(A))$ and the second term is contained in

$$V_{s_o\cdot\phi\cdot|\alpha|^{-s}} \subset \mathcal{C}_\infty(U(A)T(\mathbb{Q})\backslash G(A)) \quad .$$

The second term is a product of local integrals and hence the operator $T^*_{\phi|\alpha|^s}$ is a product of local operators for $\mathrm{Re}(s)$ large enough. A standard computation yields that

$$T_{\phi\cdot|\alpha|^s} = T_{\phi_\infty\cdot\alpha_\infty^s} \otimes T_{\phi_f\alpha_f^s}$$

and

$$T_{\phi_f\alpha_f^s} = \frac{L(\phi^{(1)}, s-1)}{L(\phi^{(1)}, s)} \cdot T^{loc}_{\phi_f\alpha_f^s}$$

where the operator $T^{loc}_{\phi_f\alpha_f^s}$ is the above local intertwining operator (|Ha4|, III).

The operator

$$T_{\phi_\infty \alpha_\infty^s} = \bigotimes_{v \in S_\infty} T_{\phi_v \alpha_v^s}$$

and the local operators at the infinite places are given by integrals, which are convergent in the neighborhood of $s = 0$.

Now we have to distinguish two cases. The first case is that

$$\frac{L(\phi^{(1)}, s-1)}{L(\phi^{(1)}, s)}$$

is holomorphic at $s = 0$. In this case the Eisenstein operator

$$\mathrm{Eis} \; : \; V^*_{\phi \cdot |\alpha|^s} \to \mathcal{A}(G(\mathbb{Q}) \backslash G(A))$$

is defined at $s = 0$ and the formula for the constant term tells us that it is an injection. If we apply this to our differential form we get

$$\int E(\underline{ug}, \phi|\alpha|^s, \xi, \psi) d\underline{u} =$$

$$= \omega(\underline{g}, \phi|\alpha|^s, \xi, \psi) + |d_F|^{-1/2} \frac{L(\phi^{(1)}, s-1)}{L(\phi^{(1)}, s)} \cdot \omega(\underline{g}, s_0 \cdot \phi|\alpha|^{-s}, T_{\infty,s}(\xi), T^{\mathrm{loc}}_{\phi|\alpha|^s}(\psi))$$

where $T_{\infty,s}(\xi)$ is the effect of $T_{\phi_\infty \alpha_\infty^s}$ on our differential form $\xi \in \mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}_\infty/k_\infty), V_{\phi_\infty \alpha_\infty^s} \otimes M(\lambda)_{\mathbb{C}})$. We may choose $\xi = \bigotimes_{v \in S_\infty} \xi_v$ and we know that (see 3.4., 3.5.) T_{ϕ_v} maps ξ_v to a zero form or at least trivial cohomology class unless we are in the case that v is a complex place and ϕ_v is one of the characters λ_{01} or λ_{10} in 3.5., in which case we have to compute

$$T_{\phi_v \alpha_v^s} \; : \; \mathrm{Hom}_{K_v^T}(\Lambda^\bullet \mathfrak{g}_v/\mathfrak{k}_v^K, \mathbb{C}e_s^{(1,0)}) \to \mathrm{Hom}_{K_v^T}(\Lambda^\bullet \mathfrak{g}_v/\mathfrak{k}_v^K, \mathbb{C}e_{-s}^{(0,1)}) .$$

We claim that this operator is given by multiplication by

$$\frac{-2\pi}{d(\tau)-d(\bar{\tau})+1}$$

where we assume that $d(\tau) \geq d(\bar{\tau})$ since ϕ is in the fundamental chamber. This is the content of 3.5. (ii) on the subspace $\mathrm{Hom}_{K_v^T}(\Lambda^0(\mathfrak{g}_v/\mathfrak{g}_v^K), \mathbb{C}e_s^{(1,0)})$ by the result is the same on the complement

$$\mathrm{Hom}_{K_v^T}(\Lambda^1(\mathfrak{g}_v/\mathfrak{g}_v^K), \mathbb{C}e_s^{(1,0)}) = \mathrm{Hom}_{K_v^T}(\Lambda^2(\mathfrak{g}_v/k_v), V_{\phi_v\alpha_v^s} \otimes M(\lambda)_{\mathbb{C}})$$

since we are dealing with the same K_v-type in $V_{\phi_v\alpha_v^s}$. We get that the contribution of the second term in the formula for the constant term is always zero, unless all $v \in S_\infty$ are imaginary and the local components of ϕ at these places are of the above type, i. e. we are in the balanced case. Then our computations in 3.5. (i) yield the formula in Theorem 2, 3).

In the second case we assume that the ratio

$$\frac{L(\phi^{(1)},s-1)}{L(\phi^{(1)},s)}$$

has a pole at $s = 0$. We analyse the consequence of this assumption for the type $\gamma^{(1)}$ of $\phi^{(1)}$. We have seen, that our assumption, that ϕ is in the fundamental chamber is equivalent to the assumption that $w(\gamma^{(1)}) \leq -2$ (see 4.1). This implies that the Euler product for the L-function $L(\phi^{(1)},s)$ is konvergent in the half-plane $\mathrm{Re}(s) > w(\gamma^{(1)})/2+1$, this is so because $\phi^{(1)}(\pi_{\mathfrak{p}}) = N\mathfrak{p}^{w(\gamma^{(1)})/2}$ for a uniformizing $\pi_{\mathfrak{p}}$ at a place $\mathfrak{p}$ where $\phi^{(1)}$ does not ramify. But this implies that $L(\phi^{(1)},s)$ is certainly holomorphic at $s = 0$ and $L(\phi^{(1)},0) \neq 0$ (|Lg |, XV, § 4), or we have $\phi^{(1)} = |\alpha|$. One checks easily with the results in 2.8. and 2.9. that this last case does not occur. This implies that the numerator in

$$\frac{L(\phi^{(1)},s-1)}{L(\phi^{(1)},s)}$$

has to be responsible for the pole and this means that we must have $\phi^{(1)} = |\alpha|^2$, and in this case we definitely get a pole. Looking into our lists in 2.8. and 2.9. again, we find that $\phi \in \mathrm{Coh}(M)$ implies $d(\tau) = 1$ for all τ and it also implies that $\deg(\phi) = n$ (So we are certainly not in the balanced case and our Theorem is proved in that case.).

Now we have to discuss the effect of the pole to our previous argument. The Eisenstein operator

$$\mathrm{Eis}^{*} : V^{*}_{\phi\alpha^s} \to \mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A}))$$

does not extend to an embedding of V^{*}_{ϕ} at $s = 0$. But the point is that the Eisenstein differential form $E(\underline{g},\phi|\alpha|^s,\xi,\psi)$ may be holomorphic at $s = 0$ and may be even a closed form. This is due to the fact that the local intertwining operators

$$T_{\phi_v\alpha_v^s} : V_{\phi_v\alpha_v^s} \to V_{s_o\cdot\phi_v\cdot\alpha_v^{-s}}$$

have codimension 1 kernels at $s = 0$ under these special circumstances and therefore the local intertwining operators help to cancel the pole of the ζ-function.

The Eisenstein differential form is holomorphic at $s = 0$ if and only if the constant term

$$\omega(\underline{g},\phi|\alpha|^s,\xi,\psi) + |d_F|^{-1/2} \frac{L(\phi^{(1)},s-1)}{L(\phi^{(1)},s)} \cdot \omega(\underline{g},s_o\cdot\phi|\alpha|^{-s},T_{\infty,s}(\xi),T^{\mathrm{loc}}_{\phi|\alpha|^s}(\psi))$$

is holomorphic at $s = 0$. We want to find out when this is the case. We may choose

$$\xi = \bigotimes_{v\in S_\infty} \xi_v$$

where

$$\xi_v \in \operatorname{Hom}_{K_v^T}(\Lambda^{\nu(v)} \mathfrak{p}_v/\mathfrak{p}_v^K, \mathbb{C}e_v)$$

and where e_v is one of the canonical representatives in $\operatorname{Hom}(\Lambda^{\bullet} \check{\mu}_v, \mathbb{C}\phi_v \otimes M(\lambda_v)_{\mathbb{C}})$ which we selected in 3.4. and 3.5.. Unter our assumption we have

$$\deg(e_v) = \begin{cases} 1 & v \text{ is real} \\ 2 & v \text{ is complex} \end{cases}$$

we have seen already that the degree of ϕ must be n. The number $\nu(v)$ may be 0 or 1 and we saw in 3.4. and 3.5. that the intertwining operator $T_{\phi_v \alpha_v^s}$ evaluated at $s = 0$ maps ξ_v to zero if $\nu(v) = 0$ and maps ξ_v not to zero if $\nu(v) = 1$. This means that

$$\bigotimes_{v \in S_\infty} T_{\phi_v \alpha_v^s} \left(\bigotimes_{v \in S_\infty} \xi_v \right)$$

has a zero of order $\#\{v \in S_\infty \mid \nu(v) = 0\}$.

The degree of ξ is $n + \sum_{v \in S_\infty} \nu(v)$. Since we have to map

$$\bigotimes_{v \in S_\infty} (\Lambda^{\bullet}(\mathfrak{p}_v/\mathfrak{p}_v^K) \to \Lambda^{\bullet}\mathcal{H}(T/Z)$$

we see that we always have at least one $\nu(v) = 0$ and therefore we have always a cancellation of the pole, i.e.

$$\frac{L(\phi^{(1)}, s-1)}{L(\phi^{(1)}, s)} \omega(\underline{g}, s_o \cdot \phi|\alpha|^s, T_{\infty,s}(\xi), T^{loc}_{\phi\alpha^s}(\psi))$$

is always holomorphic at $s = 0$, and it is even zero if $\deg(\xi) < n+r_1+r_2-1 = t+n$, i.e. it is not in the top degree. But then we can conclude that the constant term at $s = 0$, which consists only of the first term

$$\omega(\underline{g},\phi,\xi,\psi) \in \mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}_\infty/k_\infty),V_\phi \otimes M(\lambda)_{\mathbb{C}})$$

is closed, because it was constructed so. This proves that

$$\bigoplus_{m<t} V_\phi \otimes \Lambda^m \mathcal{H}(T/Z)$$

is contained in $\mathrm{Im}\, r_{[\phi]}$ because

$$E(\underline{g},\phi,\xi,\psi)$$

is closed and restricts to the given class. But if $\deg(\xi) = n+t$ our above considerations produce only a first order zero of $T_{\infty,s}(\xi)$ at $s = 0$. Then the second term in the formula for the constant term is not necessarily zero and then we cannot conclude that the Eisenstein form is closed at $s = 0$. But we may take a

$$\psi \in V_\phi \qquad\qquad \psi = \otimes_{\mathfrak{y}} \psi_{\mathfrak{y}}$$

where we have $\psi_{\mathfrak{y}} \in V'_{\phi_{\mathfrak{y}}}$ for at least one $\mathfrak{y}$. (We can identify the vector spaces $V_{\phi_{\mathfrak{y}}\cdot\alpha^s_{\mathfrak{y}}}$ (see |G-J|, § 3) since the functions in these spaces are determined by their restrictions to the maximal open compact $GL_2(\mathfrak{o}_{\mathfrak{y}})$ and $\alpha_{\mathfrak{y}}$ is unramified.) Then the operator

$$T_{\phi_{\mathfrak{y}}\cdot\alpha^s_{\mathfrak{y}}}(\psi_{\mathfrak{y}})$$

will produce another zero at $s = 0$ and our previous argument applies again. This proves that the second summand in Theorem 2, 2) is contained in $\mathrm{Im}\, r_{[\phi]}$. But the last summand

$$\overline{\mathbb{Q}}\, e_{s_o\cdot\phi} \in H^0(\partial\tilde{S},\widetilde{M}(\lambda)_{\mathbb{C}})$$

is obviously contained in $\mathrm{Im}\, r_{[\phi]}$ because we simply have to take the right combination of constant functions on the components. Hence we proved

$$\mathrm{Im}\, r \supset \bigoplus_{[\phi]} \mathrm{Im}\, r_{[\phi]}$$

and that $\mathrm{Im}\, r_{[\phi]}$ always contains at least that subspace of $V_\phi \otimes \Lambda^\bullet \mathfrak{H}(T/Z) \oplus V_{s_0 \cdot \phi} \otimes \Lambda^\bullet \mathfrak{H}(T/Z)$ which it is supposed to be equal to. But now we use the standard Poincaré-duality argument (|Se2|, Lemma 11 , |Ha2|, 4.6.). We have the pairing

$$H^\bullet(\partial\tilde{S},\widetilde{M}(\lambda)_{\mathbb{C}}) \times H^\bullet(\partial\tilde{S},\widetilde{M}(\lambda^{\vee})_{\mathbb{C}}) \to H^{n+t}(\partial\tilde{S},\mathbb{C})$$

where $M(\lambda^{\vee})$ is the contragredient representation. It is well known that $\mathrm{Im}(r)$ and $\mathrm{Im}(r^{\vee})$ have to be orthogonal, then it is clear that $\mathrm{Im}(r)$ and $\mathrm{Im}(r^{\vee})$ cannot be larger than the spaces described in Theorem 2. Hence the theorem is proved.

<u>Corollary 4.3.1</u>: If $\phi \in \mathrm{Coh}(M(\lambda))$ is a balanced character then the number

$$C(\phi) = |d_F|^{-1/2} \left(\frac{-2\pi}{d+1}\right)^{n/2} \cdot \frac{L(\phi^{(1)},-1)}{L(\phi^{(1)},0)} \in \bar{\mathbb{Q}}$$

and for any $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ we have that $\phi^{\sigma} \in \mathrm{Coh}(M(\lambda^{\sigma}))$ is also balanced and

$$C(\phi)^{\sigma} = C(\phi^{\sigma}) \quad .$$

<u>Proof</u>: We have identified the sum

$$V_\phi \otimes \Lambda^\bullet \mathfrak{H}(T/Z) \oplus V_{s_0 \cdot \phi} \otimes \Lambda^\bullet \mathfrak{H}(T/Z) \subset H^\bullet(\partial\tilde{S},M(\lambda))$$

in theorem 1. To get this identification of the left hand side as a subspace we had to select generators

$$\delta(\mu) \cdot e(\lambda,\gamma,\omega) \quad , \quad \delta(\mu') \cdot e(\lambda,s_o\cdot\gamma,\omega)$$

in $H^{d(\phi)}(\check{\mu},M(\lambda))(\gamma)$ and $H^{d(s_o\cdot\phi)}(\check{\mu},M(\lambda))(s_o\cdot\gamma)$ where $\gamma = \mathrm{type}\ (\phi)$ (see 2.4.1). Here $\mu = (\lambda,\gamma)$ $\mu' = (\gamma,s_o\cdot\gamma)$ and ω is a total ordering of the set $\{\tau | \tau : F \to \overline{\mathbb{Q}}\}$. With this choice of generators the isomorphism in theorem 1 becomes an isomorphism of $\mathbb{Q}$-structures. We have to recall the definition of $\delta(\mu)$ and $\delta(\mu')$. We have

$$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\mu)) = \mathrm{Stab}(\mu) = \mathrm{Stab}(\mu') = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\mu'))$$

hence $\mathbb{Q}(\mu) = \mathbb{Q}(\mu')$. But now it is clear that the set

$$I(\gamma) = \{\tau \mid \tau : F \to \overline{\mathbb{Q}},\ \deg(\gamma_\tau) = 1\}$$

is a CM-type, i.e. contains exactly one out of any pair of conjugate elements τ. Then $I(s_o\cdot\gamma)$ is the complementary CM-type and $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\mu))$ induces permutations

$$p(\sigma,\gamma) \quad , \quad p(\sigma,s_o\cdot\gamma)$$

on $I(\gamma)$ and $I(s_o\cdot\gamma)$. If $\theta \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is the complex conjugation induced by the embedding $\overline{\mathbb{Q}} \subset \mathbb{C}$ the multiplication by θ induces a bijection

$$\Theta : I(\gamma) \to I(s_o\cdot\gamma) \quad .$$

This implies that $p(\sigma,\gamma)$ and $p(\sigma,s_o\cdot\gamma)$ have the same signature and this implies that we may choose $\delta(\mu) = \delta(\mu')$. The image $\mathrm{Im}\ r_{[\phi]}$ in theorem 2 has been computed with respect to the generators $e(\lambda,\gamma)$, $e(\lambda,s_o\cdot\gamma)$ without the correcting factors $\delta(\mu)$. But this means that for the embedding

$$V_\phi \otimes \Lambda^\bullet \mathcal{H}(T/Z) \oplus V_{s_o\cdot\phi} \otimes \Lambda^\bullet \mathcal{H}(T/Z) \subset H^\bullet(\partial\tilde{S},\tilde{M}(\lambda))$$

which is defined by means of the modified generators and gives an embedding over $\mathbb{Q}$ the image $\operatorname{Im} r_{[\phi]}$ is given by

$$\operatorname{Im} r_{[\phi]} = \{ v, C(\phi)\cdot T^{loc}(v) \mid v \in V_\phi \otimes \mathbb{C}\}$$

and then the corollary is a fact of the $\mathbb{Q}$-rationality in theorem 1.

4.4. In this section we discuss the compatibility of the above corollary with Deligne's conjecture (comp. [D]) on special values of algebraic Hecke characters.

We consider our imaginary quadratic extension $F/\mathbb{Q}$, let

$$\psi : I_F/F^* \to \mathbb{C}^*$$

be an algebraic Hecke character. Here we should view ψ as an algebraic Hecke character on the torus $T^{(1)} = R_{F/\mathbb{Q}}(G_m)$ in the sense of 2.5.2 . Then we have $X(T^{(1)}) = \bigoplus_{\tau : F \to \bar{\mathbb{Q}}} \mathbb{Z}$ and the type of ψ is simply a collection of integers

$$\text{type}(\psi) = (\ldots n_\tau, n_{\bar\tau} \ldots)_\tau .$$

We have two numbers, the weight and the width

$$w = w(\psi) = n_\tau + n_{\bar\tau} = n_\sigma + n_{\bar\sigma} \qquad \delta = \min_\tau |n_\tau - n_{\bar\tau}| .$$

For the Hecke L-function $L(\psi,s)$ we have absolute convergence in the half-space $\operatorname{Re}(s) > \frac{w}{2} + 1$, the centre of the critical strip and the centre for the functional equation is $\frac{w}{2} + \frac{1}{2} = \frac{w+1}{2}$. Deligne looks at special values of this L-function at the critical points. These are the points

$$\frac{w+\delta}{2} - \nu \qquad \nu = 0,\ldots,\delta-1$$

in this special case. He attaches a non-zero complex number $\Omega(\psi)$ to the character ψ, the so called transcendental period and he predicts that

$$\frac{\pi^{\nu\frac{n}{2}}|d_F|^{-\nu/2}}{\Omega(\psi)} \cdot L(\psi, \frac{w+\delta}{2} - \nu) \in \bar{\mathbb{Q}} \qquad \text{for} \qquad \nu = 0,\ldots,\delta-1$$

and that for $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$

$$\left(\frac{\pi^{\nu\frac{n}{2}}|d_F|^{-\nu/2}}{\Omega(\psi)} L(\psi, \frac{w+\delta}{2} - \nu) \right)^{\sigma} = \frac{\pi^{\nu\frac{n}{2}}|d_F|^{-\nu/2}}{\Omega(\psi^{\sigma})} L(\psi^{\sigma}, \frac{w+\delta}{2} - \nu) .$$

The transcendental period is very hard to understand. The factor $\pi^{\nu\frac{n}{2}}$ can be interpreted as the period of the Tate-motiv $\mathbb{Z}(-1)$ over F and passing from ν to $\nu+1$ means twisting by the Tate-motiv.

Deligne's conjecture implies that the ratio of two consecutives of these numbers satisfy

$$\pi^{\frac{n}{2}}|d_F|^{-1/2} \qquad \frac{L(\psi, \frac{w+\delta}{2} - \nu-1)}{L(\psi, \frac{w+\delta}{2} - \nu)} \in \bar{\mathbb{Q}}$$

provided the denominator does not vanish and it also implies that this ratio behaves the right way under the action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. We claim that this consequence of Deligne's conjecture follows from our corollary 4.3.1.

First of all we observe that it suffices to look at the values $\nu = 0,\ldots,\nu_0$ where

$$\nu_0 = \begin{cases} \frac{\delta-1}{2} - 1 & \text{if } \delta \text{ is odd} \\ \frac{\delta}{2} - 1 & \text{if } \delta \text{ is even} \end{cases} .$$

The value ν_0 is the value for which $\frac{w+\delta}{2} - \nu_0 - 1$ is either the centre of the critical strip or the first critical value left of the centre of the critical strip. The rest of the cases can be taken care of by the functional equation. If $|\alpha|$ is the Tate character on I_F then

$$L(\psi, \frac{w+\delta}{2} - \nu) = L(\psi \cdot |\alpha|^{\frac{\delta+w}{2} - \nu}, 0) \quad .$$

The only thing we have to do is to construct a representation $M(\lambda)$ of $G \times_{\mathbb{Q}} \overline{\mathbb{Q}}$ such that we have a balanced algebraic Hecke character $\phi \in \mathrm{Coh}(M(\lambda))$ such that

$$\phi^{(1)} = \psi \cdot |\alpha|^{\frac{\delta+w}{2} - \nu} \quad .$$

This is simply a question of types we must have

$$\mathrm{type}(\psi \cdot |\alpha|^{\frac{\delta+w}{2} - \nu}) = \gamma^{(1)}$$

where $\gamma \in \mathrm{Coh}(M(\lambda))$. We have

$$\mathrm{type}(\psi \cdot |\alpha|^{\frac{\delta+w}{2} - \nu}) = (\ldots n_\tau - \frac{\delta+w}{2} + \nu, n_{\bar\tau} - \frac{\delta+w}{2} + \nu, \ldots) =$$

$$= (\ldots, \frac{n_\tau - n_{\bar\tau}}{2} - \frac{\delta}{2} + \nu, \frac{n_{\bar\tau} - n_\tau}{2} - \frac{\delta}{2} + \nu, \ldots) \quad .$$

For each pair of conjugate embeddings τ , $\bar\tau$ the sum of the two components in the type of $\psi \cdot |\alpha|^{(w+\delta)/2-\nu}$ is ≤ -2 because of our restriction on ν and one of the components is ≥ 0 . This defines a complementary pair of CM-types, one of them is

$$\mathcal{T}_+(\psi) = \{\tau \mid \frac{n_\tau - n_{\bar\tau}}{2} - \frac{\delta}{2} + \nu \geq 0\} \quad .$$

We choose

$$d(\tau) = \frac{n_\tau - n_{\bar\tau}}{2} - \frac{\delta}{2} + \nu$$

for $\tau \in \mathcal{T}_+(\psi)$, then $\bar\tau \in \mathcal{T}_-(\psi)$ and we put

$$-d(\bar{\tau}) - 2 = \frac{n_{\bar{\tau}} - n_{\tau}}{2} - \frac{\delta}{2} + \nu \quad .$$

With this choice of the $d(\tau)$ we put

$$M(\lambda) = \bigotimes_{\tau : F \to \bar{\mathbb{Q}}} M(d(\tau), \nu(\tau))$$

where the $\nu(\tau)$ are still at our disposal. We have to choose them in such a way that the restriction of $\rho = \rho(\lambda)$ to the centre $Z/\mathbb{Q}$ becomes the type of an algebraic Hecke character. The type of this central character is

$$(\ldots, d(\tau) + 2\nu(\tau), \ldots) \quad .$$

One way of choosing the $\nu(\tau)$ is given by

$$\nu(\tau) = \begin{cases} 0 & \text{if} \quad \tau \in \mathcal{T}_+(\psi) \\ -d(\tau)-1 & \text{if} \quad \tau \in \mathcal{T}_-(\psi) \end{cases} \quad .$$

Since with this choice the central character on $Z/\mathbb{Q} = R_{F/\mathbb{Q}}(G_m)$ corresponds to the type of

$$\psi \cdot |\alpha|^{\frac{w+\delta}{2} - \nu}$$

on $T^{(1)}/\mathbb{Q} = R_{F/\mathbb{Q}}(G_m)$, and hence it is the type of an algebraic Hecke character.

V. The period integrals

In this section we want to generalize the results of [Ha4], III to the situation here. This means that we shall evaluate the Eisenstein classes on certain "cycles with coefficients" which are constructed by means of the tori in $G/\mathbb{Q}$ which correspond to quadratic extensions of our field F .

5.1. We select a quadratic extension E/F and an embedding $E^* \to GL_2(F)$. The choice of such an embedding is nothing else than a choice of an embedding

$$i_H \ : \ H/\mathbb{Q} \to G/\mathbb{Q}$$

where $H/\mathbb{Q} = R_{E/\mathbb{Q}}(G_m) = R_{F/\mathbb{Q}}(R_{E/F}(G_m))$. The group $H(\mathbb{R}) = H_\infty$ contains a unique maximal connected compact subgroup $K_\infty'^H$ and we put

$$K_\infty^H = Z_\infty^o \cdot K_\infty'^H \quad .$$

We call a point $\underline{g} \in G(A)$ adapted to the embedding i_H if we have for its infinite component

$$g_\infty^{-1} K_\infty^H g_\infty \subset K_\infty$$

(see |Ha4|, 3.1). If we choose such an adapted point $\underline{g}$, we can construct an embedding

$$J(\underline{g}) \ : \ H(\mathbb{Q})\backslash H(A)/K_\infty^H \to G(\mathbb{Q})\backslash G(A)/K_\infty$$

given by $\underline{h} \to \underline{hg}$. If we select a level subgroup $K_f \subset G(A_f)$ and if we put

$$K_f^H(\underline{g}_f) = H(A_f) \cap \underline{g}_f K_f \underline{g}_f^{-1}$$

then $K_f^H(\underline{g}_f)$ is an open compact subgroup in $H(A_f)$ and we put

$$S_K^H = H(\mathbb{Q})\backslash H(A)/K_\infty^H K_f^H(\underline{g}_f)$$

The $J(\underline{g})$ induces a map (denoted by the same letters)

$$J(\underline{g}) \ : \ S_K^H \to S_K \quad .$$

If we have a sheaf $\tilde{M} = \tilde{M}(\lambda)$ on S_K which is obtained by a representation

ρ of $G \times_{\mathbb{Q}} \overline{\mathbb{Q}}$ then we may restrict this sheaf to S_K^H and this restriction is of course the sheaf on S_K^H which we can construct from the representation ρ restricted to $H \times_{\mathbb{Q}} \overline{\mathbb{Q}}$, let us call this restriction $\tilde{M}$ again. We get a map

$$J^\bullet(\underline{g}) \quad : \quad H^\bullet(S_K,\tilde{M}) \quad \to \quad H^\bullet(S_K^H,\tilde{M})$$

which obviously extends to a map between the limits

$$J^\bullet(\underline{g}) \quad : \quad H^\bullet(\tilde{S},\tilde{M}) \quad \to \quad H^\bullet(\tilde{S}^H,\tilde{M}) \qquad .$$

5.1.1 One checks easily that this map depends only on the connected component of G_∞ in which we select our adapted point. This is clear because the adapted points within a connected component form a contractible set if we project to the symmetric space.

5.2. We apply proposition 2.6.1 to our torus $H/\mathbb{Q}$ and the local system $\tilde{M}$. We recall that we wrote $M = M(\lambda)$ (see 1.4.) and then we have a decomposition of M into one dimensional weight spaces with respect to $H \times_{\mathbb{Q}} \overline{\mathbb{Q}}$

$$M(\lambda) \quad = \quad \bigoplus_{\mu} \; M(\lambda,\mu)$$

and this provides a decomposition

$$H^\bullet(\tilde{S}^H,\tilde{M}(\lambda)) \quad = \quad \bigoplus_{\mu} \; H^\bullet(\tilde{S}^H,\tilde{M}(\lambda,\mu)) \tag{5.2.0}$$

and the above mentioned proposition yields

$$H^\bullet(\tilde{S}^H,\tilde{M}(\lambda,\mu)) \quad = \quad \bigoplus_{\eta\,:\,\mathrm{type}(\eta)\,=\,\mu} \; \overline{\mathbb{Q}}_\eta \otimes \Lambda^\bullet(\mathfrak{h}(H/Z)) \qquad . \tag{5.2.1}$$

Here $\bar{\mathbb{Q}}_\eta$ is a one dimensional $\bar{\mathbb{Q}}$-vector space on which $\pi_o(H_\infty) \times H(A_f)$ acts by the algebraic Hecke character η.

5.2.2 We observe that the systems

$$\{H^\bullet(\tilde{S}^H,\tilde{M}(\lambda))\}_\lambda \;,\; \{H^\bullet(\tilde{S}^H,\tilde{M}(\lambda,\mu))\}_{(\lambda,\mu)} \quad \text{and} \quad \{\vec{\mathbb{Q}}_\eta \otimes \Lambda^\bullet \mathcal{H}(H/Z)\}_{(\lambda,\mu,\eta)}$$

have obviously $\mathbb{Q}$-structures and that the systems of maps $J^\bullet(\underline{g})$ and (5.2.0), (5.2.1) are defined over $\mathbb{Q}$.

5.3. We constructed the system of sections

$$\mathrm{Eis} \;:\; \mathrm{Im}\, r_{[\phi]} \to H^\bullet(\tilde{S},\tilde{M})$$

(Corollary 4.2.1), let us put $\mathrm{Eis}(\phi,\psi) = \mathrm{Eis}(\psi)$ for $\psi \in \mathrm{Im}\, r_{[\phi]}$. Our goal is to restrict these classes to $\tilde{S}^H$, i.e. we want to compute the classes

$$J^\bullet(\underline{g})(\mathrm{Eis}(\phi,\psi))$$

in terms of the description of the cohomology $H^\bullet(\tilde{S}^H,\tilde{M})$ given in 5.2.1.

5.3.1 First of all we analyse which are the cases of interest for us, we look for the degrees d where the cohomology $H^d(\tilde{S}^H,\tilde{M})$ is possibly non-zero and where we have also Eisenstein classes. If $n = [F:\mathbb{Q}] = r_1+2r_2$ in the usual notation then the lowest degree (except 0) where we can have Eisenstein cohomology is n if $r_1 \geq 1$ and r_2 if $r_1 = 0$. On the other hand we have

$$\mathrm{rank}(\mathcal{H}(H/Z)) = r_1 + r_2 - \delta$$

where δ is the number of real places of F which become complex in E. This means that we have only two cases where the degrees match, namely

A) Both fields F and E are totally real,

B) Both fields E and F are totally imaginary,

and the degree we are looking for is the rank of $\mathcal{H}(H/Z)$ which is of course equal to the dimension of $\tilde{S}^H$.

<u>5.3.2</u> If P_η is the projection operator to the η-component in (5.2.1) then we are interested in the evaluation

$$P_\eta(J^\bullet(\underline{g})(\mathrm{Eis}(\phi,\psi)) \;=\; J^\bullet(\underline{g})(\mathrm{Eis}(\phi,\psi))_\eta$$

where

$$J^\bullet(\underline{g})(\mathrm{Eis}(\phi,\psi))_\eta \;\in\; \bar{\Phi}_\eta \otimes \Lambda^d\, \mathcal{H}(H/Z)$$

and $d = \mathrm{rank}(\mathcal{H}(H/Z))$. We may look at this from a different point of view. We have the map

$$\psi \;\to\; \{\underline{g} \to J^\bullet(\underline{g})(\mathrm{Eis}(\phi,\psi)_\eta\} \tag{5.3.3}$$

which maps $\mathrm{Im}\ r_{[\phi]}$ to a space of functions on

$$\pi_o(G_\infty) \times G(A_f)$$

with values in $\bar{\Phi}_\eta \otimes \Lambda^d(\mathcal{H}(H/Z))$. Here we have to remember that g_∞ has to be adapted on $J^\bullet(\underline{g})$ depends only on the image of g_∞ in $\pi_o(G_\infty)$ (see 5.1.1). But it follows from the definitions that

$$J^\bullet(\underline{hg})(\mathrm{Eis}(\phi,\psi))_\eta \;=\; \eta(\underline{h}) \cdot J^\bullet(\underline{g})(\mathrm{Eis}(\phi,\psi))_\eta$$

for $\underline{h} \in \pi_o(H_\infty) \times H(A_f)$. Hence we see that the above map (5.3.3) is an intertwining operator

$$I(\phi,\eta,i_H) \quad : \quad \mathrm{Im}\, r_{[\phi]} \;\to\; \mathrm{Ind}_{\pi_0(H_\infty)\times H(A_f)}^{\pi_0(G_\infty)\times G(A_f)} \bar{\mathfrak{Q}}_\eta \otimes \Lambda^d(\mathcal{H}(H/Z)) \quad ,$$

we want to abbreviate

$$\mathrm{Ind}_{\pi_0(H_\infty)\times H(A_f)}^{\pi_0(G_\infty)\times G(A_f)} \bar{\mathfrak{Q}}_\eta \otimes \Lambda^d(\mathcal{H}(H/Z)) = \tilde{W}_\eta \quad .$$

5.3.4 At this point I want to recall that we have permanently to deal with a problem of keeping track of various identifications. This problem is related to the question: What does it mean to compute the intertwining operator

$$\tilde{I}(\phi,\eta,i_H) \quad : \quad \mathrm{Im}\, r_{[\phi]} \;\to\; \tilde{W}_\eta \quad ?$$

To give a meaning to this we have to relate both spaces to certain reference spaces. We know already (theorem 2) that $\mathrm{Im}\, r_{[\phi]}$ is related to the V_ϕ which is a concrete space of functions on $G(A_f)$ but this relation depends on certain choices of generators (see 4.3.). If we want to relate $\tilde{W}$ to a reference space consisting of functions on $G(A_f)$ we may use Poincaré duality. This will be explained in the next sections.

5.3.5 If we want to specify the class $J^\bullet(\underline{g})(\mathrm{Eis}(\phi,\psi))_\eta$ we may look at the Poincaré-duality pairing

$$H^0(S_K^H,\tilde{M}^\vee) \times H^d(S_K^H,\tilde{M}) \;\to\; H^d(S_K^H,\bar{\mathfrak{Q}})$$

where $M^\vee$ is the dual representation of M and where $K = K_\infty K_f$ is a suitable level. The tangent space of S_K^H at the point

$$y_0 = \underline{e}_H \cdot K_\infty^H K_f^H \bmod K_\infty^H K_f^H$$

($\underline{e}_H$ = identity in $H(A)$) is identified to

$$\mathrm{Lie}(H_\infty)/\mathrm{Lie}(K_\infty^H) = \mathfrak{h}_\infty/\mathfrak{k}_\infty^H .$$

If we choose an orientation Ω_o in $\mathfrak{h}_\infty/\mathfrak{k}_\infty^H$ we may transport it to any other point by translations, hence it defines an orientation on S_K^H. Using this orientation we have the canonical homomorphism

$$\mathrm{tr}_K : H^d(S_K^H, \bar{\mathbb{Q}}) \to \bar{\mathbb{Q}}$$

which maps the fundamental class on each connected component of S_K^H to $1 \in \bar{\mathbb{Q}}$. If we extend the coefficient system from $\bar{\mathbb{Q}}$ to the complex numbers then each class can be represented by a differential form of degree d and

$$\mathrm{tr}_K([\omega]) = \int_{S_K^H} \omega .$$

This homomorphism is not compatible with the change of levels hence we normalize it

$$\mathrm{tr} = \frac{1}{[K_{f,o}^H : K_f^H]} \mathrm{tr}_K$$

where $K_{f,o}^H \subset H(A_f)$ is the maximal compact subgroup of units. With this normalization we get a commutative diagram

$$\begin{array}{ccc} H^d(S_K^H, \bar{\mathbb{Q}}) & \longrightarrow & \bar{\mathbb{Q}} \\ \downarrow & & \| \\ H^d(S_{K'}^H, \bar{\mathbb{Q}}) & \longrightarrow & \bar{\mathbb{Q}} \end{array}$$

if $K = K_\infty^H K_f^H \supset K' = K_\infty^H (K_f^H)'$. Hence we get a pairing which does not depend on the level

$$\begin{tikzcd} \langle\ ,\ \rangle\ :\ H^o(S^H_K,\tilde{M}^\vee)\times H^d(S^H_K,\tilde{M}) \arrow[rr] \arrow[dr] & & \bar{\mathbb{Q}} \\ & H^d(S^H_K,\bar{\mathbb{Q}}) \arrow[ur, "tr"'] & \end{tikzcd} \qquad (5.3.5.1)$$

<u>5.3.5.2</u> We want to give a more explicit formula for the homomorphism tr which will be useful for computations lateron. The problem we have to deal with is a pedantic consideration concerning the normalization of measures.

Let us put $\bar{H} = (H/Z)/\mathbb{Q}$. If $K^{\bar{H}}_\infty$ is the maximal connected compact subgroup in $\bar{H}_\infty$ then we have a map

$$p\ :\ H_\infty/K^H_\infty \ \to\ \bar{H}_\infty/K^{\bar{H}}_\infty \quad .$$

Taking the definition of K^H_∞ (see 5.1) into account we see that p is covering of degree 2^n in case A and an isomorphism in case B . In any case we have an isomorphism of Lie-algebras

$$\mathrm{Lie}(H_\infty)/\mathrm{Lie}(K^H_\infty) = \mathfrak{h}_\infty/k^H_\infty \xrightarrow{\sim} \bar{\mathfrak{h}}_\infty/k^{\bar{H}}_\infty \quad .$$

We choose an ordered basis $Y_1,\ldots,Y_d$ of $\mathfrak{h}_\infty/k^H_\infty$ this can be extended to a d-frame on S^H by translations and it defines an orientation on S^H_K . This basis provides by the exponential map local isomorphisms

$$\begin{tikzcd} & H_\infty/K^H_\infty \arrow[d, "p"] \\ \exp\ :\ \mathbb{R}^d \arrow[ur] \arrow[r] & \bar{H}_\infty/K^{\bar{H}}_\infty \end{tikzcd}$$

where $\exp(t_1,\ldots,t_d) = \exp(\sum\limits_{v=1}^{d} t_v Y_v)$. If we transport the Lebesque measure form $\mathbb{R}^d$ by the exponential map to the two groups we get two invariant measures

$$dh'_\infty \quad \text{on} \quad H_\infty/K_\infty^K \quad \text{and} \quad d\bar{h}'_\infty \quad \text{on} \quad \bar{H}_\infty/K_\infty^{\bar{H}} \quad .$$

These measures are related by

$$dh'_\infty = d\zeta \cdot d\bar{h}'_\infty \qquad (5.3.5.2.1)$$

where $d\zeta$ is the measure on $\ker(p)$ that gives every element the volume 1.

We also may fix a measure $d\bar{h}_\infty$ on $\bar{H}_\infty$ by the requirement

$$d\bar{h}_\infty = d\bar{k}_\infty \cdot d\bar{h}'_\infty \qquad (5.3.5.2.2)$$

where $d\bar{k}_\infty$ gives volume one to $K_\infty^{\bar{H}}$.

If we have a d-form ω on S_K^H which represents the class $[\omega]$ then

$$\omega(Y_1,\ldots,Y_d) = \omega(Y_1 \ \ldots \ Y_d) = \omega(Y_H)$$

becomes a function on S_K^H and hence it is a function on $H(\mathbb{Q})\backslash H(A)/K_\infty^H$. We define a measure

$$d\underline{h}' = dh'_\infty \times d\underline{h}_f$$

on $H(A)/K_\infty^H$ where $d\underline{h}_f$ gives volume one to the open maximal compact subgroup $K_{f,o}^H$ of units in $H(A_f)$. Then it is quite clear that

$$tr([\omega]) = \int_{H(\mathbb{Q})\backslash H(A)/K_\infty^H} \omega(Y_H)(\underline{h}')d\underline{h}' \qquad (5.3.5.3)$$

If $\omega(Y_H)(\underline{h}')$ turns out to be invariant under the action of $Z(A)$. In that case we get

$$tr([\omega]) = vol_{d\underline{z}}(Z(\mathbb{Q})\backslash Z(A)/Z_\infty^o) \cdot \int_{\bar{H}(\mathbb{Q})\backslash \bar{H}(A)/K_\infty^{\bar{H}}} \omega(Y_H)(\underline{\bar{h}}')d\underline{\bar{h}}'$$

where $\underline{dz} = d\zeta \times d\underline{z}_f$ and $\mathrm{vol}_{dz_f}(K^Z_{f,o}) = 1$ (recall that $Z_\infty/Z^o_\infty = \ker(p)$) and

$$d\underline{\bar{h}}' = d\bar{h}'_\infty \times d\underline{\bar{h}}_f$$

where $d\underline{\bar{h}}_f$ gives volume one to the image of $K^H_{f,o}$ in $\bar{H}(A_f)$ (This is not the maximal compact subgroup in general!). This follows from (5.3.5.2.1). Now we have

$$\mathrm{vol}_{d\underline{z}}(Z(\mathbb{Q})\backslash Z(A)/Z^o_\infty) = h_F$$

where h_F is the class number of F (in the narrow sense) and using (5.3.5.2.2) we get the final formula

$$\mathrm{tr}([\omega]) = h_F \cdot \int_{\bar{H}(\mathbb{Q})\backslash \bar{H}(A)} \omega(Y_H)(\underline{\bar{h}})\,d\underline{\bar{h}} \qquad (5.3.5.4)$$

where $d\underline{\bar{h}} = d\bar{h}_\infty \times d\underline{\bar{h}}_f$.

<u>5.3.6</u> To specify the class $J^\bullet(\underline{g})(\mathrm{Eis}(\phi,\psi))$ we may look at $J^\bullet(\underline{g})(\mathrm{Eis}(\phi,\psi))$ as a linear form on $H^o(S^H_K,\tilde{M}^v)$ and compute its restriction to the η^{-1} - component of this cohomology group. To do this we select a generator

$$m^v(\mu^{-1}) \in M^v(\mu^{-1}) = M(\lambda^v,\mu^{-1})$$

and we define an element $m^v(\eta^{-1}) \in H^o(S^H_K,M^v(\mu^{-1}))$ by

$$m^v(\eta^{-1})(\underline{h}) = \eta^{-1}(\underline{h})\rho^v(h_\infty)m^v(\mu^{-1}) = \eta^{-1}(\underline{h})\cdot\mu^{-1}(\underline{h}_\infty)m^v(\mu^{-1}) \qquad (5.3.6.1)$$

(see 2.6 and 2.5.4.2). Using the pairing (5.3.4.1) we may look at

$$\langle J^\bullet(\underline{g})(\mathrm{Eis}(\phi,\psi)),m^v(\eta^{-1})\rangle$$

the result is a number in $\bar{\mathbb{Q}}$. This defines a slightly modified intertwining operator

$$I(\phi,\eta,m^{v}(\mu^{-1}),i_H) \quad : \quad \operatorname{Im} r_{[\phi]} \rightarrow W_\eta$$

where now

$$W_\eta = \operatorname{Ind}_{\pi_o(H_\infty) \times H(A_f)}^{\pi_o(G_\infty) \times G(A_f)} \eta$$

this a space of functions on $\pi_o(G_\infty) \times G(A_f)$ with values in $\bar{\mathbb{Q}}$ and it is one of the reference spaces which we are looking for.

<u>5.3.6.2</u> The system of induced modules

$$\{W_\eta\}_{(\mu,\eta)}$$

has again an obvious $\mathbb{Q}$-structure. Of course we want that the system of intertwining operators

$$I(\phi,\eta,m^{v}(\mu^{-1}),i_H) \quad : \quad \operatorname{Im} r_{[\phi]} \rightarrow W_\eta$$

is defined over $\mathbb{Q}$. It is obvious that the system $I(\phi,\eta,i_H)$ in (5.3.3) is defined over $\mathbb{Q}$ and hence we can say: The system

$$\{M(\lambda^{v},\mu^{-1})\}_{(\lambda,\mu)}$$

has a $\mathbb{Q}$-structure and hence the system $I(\phi,\eta,m^{v}(\mu^{-1}),i_H)$ is defined over $\mathbb{Q}$ if and only if the system of generators

$$\{m^{v}(\mu^{-1})\}_{(\lambda,\mu)}$$

is defined over $\mathbb{Q}$, i.e. is mapped inter itself by the transition maps (see 2.4.). Such a system exists by Hilbert's theorem 90.

Of course there is a more elegant way of looking at this. We may look at Poincaré duality as giving an identification

$$\tilde{W}_\eta \otimes M(\lambda^\vee,\mu^{-1}) \xrightarrow{\sim} W_\eta \quad .$$

If we tensor $I(\phi,\eta,i_H) : \operatorname{Im} r_{[\phi]} \to \tilde{W}_\eta$ by $M(\lambda^\vee,\mu^{-1})$ we get an operator

$$I(\phi,\eta,i_H) \otimes 1 \quad : \quad \operatorname{Im} r_{[\phi]} \otimes M(\lambda^\vee,\mu^{-1}) \to W_\eta \quad .$$

If we vary the data λ,ϕ and μ,η we get systems of spaces on both sides and both systems have natural $\mathbb{Q}$-structures and the system of intertwining operators becomes $\mathbb{Q}$-rational. We have the formula

$$I(\phi,\eta,i_H)(\psi \otimes m^\vee(\mu^{-1})) = I(\phi,\eta,m^\vee(\mu^{-1}),i_H) \quad .$$

<u>5.3.6.2</u> We want to treat the analogous problem for $\operatorname{Im} r_{[\phi]}$, the answer is in principle given by theorem 1 and 2. We have

$$\begin{array}{c} \operatorname{Im} r_{[\phi]} \subset V_\phi \otimes \Lambda^\bullet(\mathfrak{K}(T/Z)) \oplus V_{s_0\cdot\phi} \otimes \Lambda^\bullet(\mathfrak{K}(T/Z)) \\ \quad\quad\big\downarrow I \\ H^\bullet(\partial\tilde{S},M) \end{array}$$

But if we recall Theorem 1 we see that the inclusion map I depends on the identification

$$\operatorname{Ind}_{\pi_0(B_\infty)\times B(A_f)}^{\pi_0(G_\infty)\times G(A_f)} \bar{\mathbb{Q}}e_\phi \xrightarrow{\sim} V_\phi$$

i.e. it depends on the choice of a generator

$$e(\gamma) = e(\lambda,\gamma) \in H^\bullet(\check{\mu},M)(\gamma) \quad .$$

So the natural diagram to write down is

$$\begin{tikzcd} \mathrm{Im}\, r_{[\phi]} \arrow[r] & V_\phi \otimes H^\bullet(\check{\mu},M)(\gamma) \otimes \Lambda^\bullet(\mathcal{H}(T/Z)) \oplus V_{s_o\cdot\phi} \otimes H^\bullet(\check{\mu},M)(s_o\cdot\gamma) \otimes \Lambda^\bullet(\mathcal{H}(T/Z)) \arrow[d, hook] \\ & H^\bullet(\partial\tilde{S},\tilde{M}) \end{tikzcd}$$

where now the inclusion map is canonical. We remember that we are interested in the case where we are in degree $d = \operatorname{rank} \mathcal{H}(H/Z)$ and we have under these circumstances

$$\begin{tikzcd} \mathrm{Im}\, r^{(d)}_{[\phi]} \arrow[r] & V_\phi \otimes H^d(\check{\mu},M)(\gamma) \oplus V_{s_o\cdot\phi} \otimes H^{d'}(\check{\mu},M)(s_o\cdot\gamma) \arrow[d, hook] \\ & H^\bullet(\partial\tilde{S},\tilde{M}) \end{tikzcd}$$

this means especially that the factor $\Lambda^\bullet(\mathcal{H}(T/Z))$ disappears, we are in degree zero in this variable.

We have to distinguish the two cases again.

<u>Case A</u>: Then $d = [F:\mathbb{Q}]$, of course we assume that ϕ is in the fundamental chamber. Then $d' = 0$ and we have

$$E_\phi \;:\; V_\phi \otimes H^d(\check{\mu},M)(\gamma) \xrightarrow{\sim} \mathrm{Im}\, r^{(d)}_{[\phi]} \qquad \text{if } d > 1 \text{ or } \phi^{(1)} \neq |\alpha|^2$$

$$E_\phi \;:\; V'_\phi \otimes H^1(\check{\mu},M)(\gamma) \xrightarrow{\sim} \mathrm{Im}\, r^{(1)}_{[\phi]} \qquad \text{if } d = 1 \text{ and } \phi^{(1)} = |\alpha|^2$$

(see Theorem 2). We compose the maps

$$\mathrm{Eis}\circ E_\phi \;:\; V_\phi^{?}\otimes H^d(\quad,M)(\gamma) \to H^d(\tilde{S},\tilde{M}) \quad .$$

<u>Case B</u>: In this case we have $d = [F:\mathbb{Q}]/2$ and $d' = d$. Assuming that ϕ is in the fundamental chamber we define E_ϕ by the diagram

$$\begin{tikzcd} E_\phi \;:\; V_\phi\otimes H^d(\check{\mu},M)(\gamma) \arrow[r] \arrow[dr, "\mathrm{Id}"] & \mathrm{Im}\, r^{(d)}_{[\phi]} \arrow[d, "pr_\phi"] \\ & V_\phi\otimes H^d(\check{\mu},M)(\gamma) \end{tikzcd}$$

where pr_ϕ is the projection to the ϕ coordinate.

Again we form the composite map

$$\mathrm{Eis}\circ E_\phi \;:\; V_\phi\otimes H^d(\check{\mu},M)(\gamma) \to H^d(\tilde{S},\tilde{M}) \quad .$$

Selecting a generator in $H^d(\check{\mu},M)(\gamma)$ gives us a map

$$E_\phi(e(\gamma)) \;:\; V_\phi \to \mathrm{Im}\, r_{[\phi]}$$

which then be composed with Eis. But on the other hand the datum $e(\gamma)$ is nothing else than an element

$$\xi(e(\gamma)) = \xi \in \mathrm{Hom}_{K_\infty^T}(\Lambda^0(\mathfrak{g}_\infty/\mathfrak{g}_\infty^K),\mathbb{C}) \otimes \mathrm{Hom}(\Lambda^d(\check{\mu}_\infty, M_{\mathbb{C}})$$
$$= \mathrm{Hom}(\Lambda^d(\mathfrak{g}_\infty/k_\infty), V^*_{\phi_\infty}\otimes M_{\mathbb{C}})$$

(see proof of theorem 2) which yields that the class

$$\mathrm{Eis}\quad E_\phi(e(\gamma))(\psi) = E(\phi,e(\gamma),\psi)$$

over $\mathbb{C}$ is given by the Eisenstein series

$$\mathrm{Eis}(\underline{g},\phi,\xi(e(\gamma)),\psi) \in \mathrm{Hom}_{K_\infty}(\Lambda^d(\mathfrak{g}_\infty/\mathfrak{k}_\infty),\mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A}) \otimes M_{\mathbb{C}}) \quad .$$

5.3.6.3 (In the following we want to ignore the case $d = 1$ and $\phi^{(1)} = |\alpha|^2$ we just keep in mind that in that case we have to restrict our attention to V'_ϕ .)

We found that we have an operator

$$E_\phi \quad : \quad V_\phi \otimes H^d(\check{\mu},M)(\gamma) \quad \tilde{\to} \quad \mathrm{Im}\ r^{(d)}_{[\phi]} \quad .$$

Composing this with (5.3.6.1) we get an operator

$$I(\phi,\eta,i_H) \quad : \quad V_\phi \otimes H^d(\check{\mu},M)(\gamma) \otimes M(\lambda^\vee,\mu^{-1}) \quad \to \quad W_\eta$$

and again this gives us a system of linear maps defined over $\mathbb{Q}$ between the two systems. (Note that still $\mathrm{type}(\phi) = \gamma$ and $\mathrm{type}(\eta) = \mu$.) If we evaluate on generators in the second and third variable we get

$$I(\phi,\eta,i_H)(\psi \otimes e(\gamma) \otimes m^\vee(\mu^{-1})) \quad = \quad I(\phi,\eta,e(\gamma),m^\vee(\mu^{-1}),i_H)(\psi)$$

where

$$I(\phi,\eta,e(\gamma),m^\vee(\mu^{-1}),i_H) \quad : \quad V_\phi \quad \to \quad W_\eta \quad .$$

This last operator is an intertwining operator relating two "concrete" spaces of functions.

5.3.7 We want to give the explicit integral formula for the expression

$$I(\phi,\eta,e(\gamma),m^\vee(\eta^{-1}),i_H)(\psi)(\underline{g}) \quad = \quad \langle J^\bullet(\underline{g})\mathrm{Eis}(\phi,E_\phi(\psi \otimes e(\gamma)),m^\vee(\eta^{-1})\rangle$$

which is obtained from (5.3.5.2). It is of course clear that our intertwining operator is zero unless the following condition is fulfilled

$$\eta|Z(A) = \phi|Z(A) \qquad (5.3.7.1)$$

which we shall assume from now on.

In the above expression we have to interprete the Eisenstein series as a differential d-form on S_K^H with values in $\tilde{M}(\lambda)_{\mathbb{C}}$. Here $\tilde{M}(\lambda)_{\mathbb{C}}$ is the flat vector bundle whose fibers are the stalks of $\tilde{M}(\lambda)$ and where the local sections in the sheaf are constant for the connection. (We assume K_f small enough.) This differential form will be restricted via $J^{\bullet}(\underline{g})$ to S_K^H and will be evaluated on the section $m^{\vee}(\eta^{-1})$ in the dual sheaf. The result of this evaluation is a closed d-form on S_K^H which represents a cohomology class. We apply tr to it and the resulting number is the number we want to compute. To apply tr we use (5.3.5.4) and here is the result:

$$\langle J^{\bullet}(\underline{g})\mathrm{Eis}(\phi,E_{\phi}(\psi \otimes e(\gamma)),m^{\vee}(\eta^{-1}))\rangle =$$

$$h_F \int_{\overline{H}(\mathbb{Q})\backslash\overline{H}(A)} \eta^{-1}(\underline{h})\langle \mathrm{Eis}(\underline{hg},\phi,\xi(e(\gamma)),\psi,\mathrm{ad}(g_{\infty}^{-1})(Y_H) \otimes \rho^{\vee}(g_{\infty}^{-1})m^{\vee}(\mu^{-1})\rangle d\underline{\overline{h}} \qquad (5.3.7.1)$$

where we have to take the following convention into account:

The element $Y_H = Y_1 \wedge \ldots \wedge Y_d$ where $Y_1,\ldots,Y_d$ is a basis of $\mathfrak{f}_{\infty}/k_{\infty}^{\overline{H}}$ and $d\underline{\overline{h}}$ is the measure on $\overline{H}(A)$ derived from this choice according to the rules in 5.3.5. (Note that the choice of this basis cancels out up to the sign, only the orientation given by Y_H counts.)

The proof of the above formula is a little bit painful but easy if one uses the dictionary that translates the $(\mathfrak{g}_{\infty},K_{\infty})$-complex into the de-Rham complex. If

$$\tilde{\omega} \in \mathrm{Hom}_{K_{\infty}}(\Lambda^d(\mathfrak{g}_{\infty}/k_{\infty}),\mathcal{A}(G(\mathbb{Q})\backslash G(A)) \otimes M(\lambda)_{\mathbb{C}})$$

we can define a class

$$\omega \in \Omega^d(G(\mathbb{Q})\backslash G(A)/K_\infty, \tilde{M}(\lambda)_{\mathbb{C}})$$

where $\tilde{\tilde{M}}(\lambda)_{\mathbb{C}}$ is the above flat bundle. To define ω we pick a point $\underline{\bar{x}} \in G(\mathbb{Q})\ G(A)/K_\infty$ and we choose tangent vectors $Y_1,\ldots,Y_d$ at the point $\underline{\bar{x}}$. Then we should have

$$\omega(\underline{\bar{x}})(Y_1,\ldots,Y_d) \in M(\lambda)_{\mathbb{C},\underline{\bar{x}}} \quad .$$

This means that for $\underline{x} \in G(A)/K_\infty$ that projects to $\underline{\bar{x}}$ we must have

$$\omega(\underline{x})(Y_{1,\underline{x}},\ldots,Y_{d,\underline{x}}) \in M(\lambda)_{\mathbb{C}}$$

and here are $Y_{\nu,\underline{x}}$ the tangent vectors lying above the Y_ν . And we must have for $\gamma \in G(\mathbb{Q})$

$$\omega(\gamma\underline{x})(Y_{1,\gamma\underline{x}},\ldots,Y_{d,\gamma\underline{x}}) = \rho(\gamma)\omega(\underline{x})(Y_{1,\underline{x}},\ldots,Y_{d,\underline{x}})$$

(see 1.1.2). Now we may write $\underline{x} = \underline{g}\underline{x}_o$ where

$$\underline{x}_o = 1 \cdot K_\infty \bmod K_\infty \in G(A)/K_\infty$$

the differential of the left translation by $\underline{g}^{-1}$ maps the tangent space at $\underline{x}$ isomorphically to the tangent space at $\underline{x}_o$, but this tangent space is $\mathfrak{g}_\infty/k_\infty$. Therefore we define the differential d-form by

$$\omega(\underline{x})(Y_{1,\underline{x}},\ldots,Y_{d,\underline{x}}) = \rho(g_\infty)\tilde{\omega}(\underline{g})(dL_{\underline{g}^{-1}}(Y_{1,\underline{x}},\ldots,Y_{d,\underline{x}}))$$

it obviously has the required properties. This is the procedure that provides the isomorphism between the $(\mathfrak{g}_\infty,K_\infty)$-complex and the de-Rham complex (see |B-W|, VII, 2.7., |Ha2| § 1).

Let us now assume that $\tilde{\omega}$ is our Eisenstein series and ω the corresponding differential form. Then the differential

$$dJ(\underline{g})(\underline{h})$$

maps the tangent space of S_K^H at $\underline{h}$ to the tangent space at $\underline{hg}$ and (5.3.5) yields that the above expression is

$$\int_{\bar{H}(\mathbb{Q})\backslash\bar{H}(\mathbb{A})} \langle\omega(\underline{hg} \bmod K_\infty), dJ(\underline{g})(\underline{h})(Y_H) \otimes m^v(\eta^{-1})\rangle\, d\underline{\bar{h}}$$

where the Y_H is a basis element in $\Lambda^d(\mathfrak{g}_\infty/k_\infty^{\bar{H}})$ and this element and $d\underline{\bar{h}}$ are linked as in (5.3.5.4). (We assume (5.3.7.1) of course.) Using our formulas above and exploiting the obvious formula

$$dL_{\underline{g}^{-1}\underline{h}^{-1}} \circ dJ(\underline{g})(\underline{h})(Y_H) = ad(g_\infty^{-1})(Y_H)$$

we find for our integral

$$\int_{\bar{H}(\mathbb{Q})\backslash\bar{H}(\mathbb{A})} \langle\rho(h_\infty g_\infty)\mathrm{Eis}(\underline{hg},\phi,\xi,\psi)(ad(g_\infty)^{-1}(Y_H)), m^v(\eta^{-1})(\underline{h})\rangle\, d\underline{\bar{h}} =$$

$$\int_{\bar{H}(\mathbb{Q})\backslash\bar{H}(\mathbb{A})} \eta^{-1}(\underline{h})\ \langle\mathrm{Eis}(\underline{hg},\phi,\xi,\psi)(ad(g_\infty^{-1})(Y_H)), \rho^v(g_\infty)^{-1}m^v(\mu^{-1})\rangle\, d\underline{\bar{h}} =$$

$$\int_{\bar{H}(\mathbb{Q})\backslash\bar{H}(\mathbb{A})} \eta^{-1}(\underline{h})\ \langle\mathrm{Eis}(\underline{hg},\phi,\xi,\psi), ad(g_\infty^{-1})(Y_H) \otimes \rho^v(g_\infty^{-1})m^v(\mu^{-1})\rangle\, d\underline{\bar{h}} \quad .$$

For the last two steps we have to take the definition of the section $m^v(\eta^{-1})$ into account (see 5.3.6) $(\mu^{-1}(h_\infty)m^v(\mu^{-1}) = \rho^v(h_\infty)m^v(\mu^{-1})$!). This is the desired formula.

5.4. Before we proceed we want to look a little bit closer on the possible choices of weights $\mu \in X(H \times \bar{\mathbb{Q}})$ for which we may have a non-trivial

$I(\phi,\eta,i_H)$ with $\mathrm{type}(\eta) = \mu$. Of course we get from (5.3.7.1) that

$$\gamma|Z \times_{\mathbb{Q}} \bar{\mathbb{Q}} = \mu|Z \times_{\mathbb{Q}} \bar{\mathbb{Q}} \qquad (5.4.1)$$

where $\gamma = \mathrm{type}(\phi)$. Moreover we have to require that μ is a weight of $H \times_{\mathbb{Q}} \bar{\mathbb{Q}}$ in $M(\lambda)$ and that μ is the type of an algebraic Hecke character.

We want to make these conditions a little bit more explicit. We have

$$X(H \times \bar{\mathbb{Q}}) = \bigoplus_{\tau : E \to \bar{\mathbb{Q}}} \mathbb{Z}$$

so our element μ is given as

$$\mu = (\ldots m_\tau \ldots)_{\tau : E \to \bar{\mathbb{Q}}} \quad .$$

We have $Z/\mathbb{Q} \to H/\mathbb{Q}$ and we have

$$X(Z \times_{\mathbb{Q}} \bar{\mathbb{Q}}) = \bigoplus_{\tau : F \to \bar{\mathbb{Q}}} \mathbb{Z} \quad .$$

If $M = \bigotimes_{\tau : F \to \mathbb{Q}} M(d(\tau),\nu(\tau))$ then the central character is $\delta_o = (\ldots,d(\tau)+2\nu(\tau),\ldots)_{\tau : F \to \bar{\mathbb{Q}}}$ and 5.4.1 can be expressed

$$m_{\tau'} + m_{\tau''} = d(\tau)+2\nu(\tau) \qquad (5.4.2)$$

where $\tau : F \to \bar{\mathbb{Q}}$ and τ' , τ'' are the two embeddings lying above τ .

Now we look at the constraints on μ which follows from the assumption that μ is the type of an algebraic Hecke character. We have the two cases

<u>Case A</u>: We have $m_{\tau'} = m_{\tau''} = m$ for all τ' , τ'' .

<u>Case B</u>: In this case we have that E is totally imaginary we do not have such a simple necessary and sufficient condition. But we have the constraint that for any pair τ , $\bar{\tau}$ of conjugate embeddings of E into $\bar{\mathbb{Q}}$

we must have

$$m_\tau + m_{\bar\tau} = w(\mu) = \frac{d(\tau)+d(\bar\tau)}{2} + \nu(\tau) + \nu(\bar\tau) \qquad (5.4.3)$$

where $w(\mu)$ is the weight of μ. If E is totally imaginary this is the only constraint.

Now we analyse the weight condition. Since we assume (5.4.1) we may restrict μ to $H^{(1)}/\mathbb{Q}$, call this restriction $\mu^{(1)}$. The torus

$$H^{(1)} \times_{\mathbb{Q}} \bar{\mathbb{Q}} \to G^{(1)} \times_{\mathbb{Q}} \bar{\mathbb{Q}} = \bigotimes_{\tau : F \to \bar{\mathbb{Q}}} SL_2$$

can be conjugated (in several different ways) into the torus $T^{(1)} \times_{\mathbb{Q}} \bar{\mathbb{Q}}$. We may identify

$$X(T^{(1)} \times \bar{\mathbb{Q}}) = \bigoplus_{\tau : F \to \bar{\mathbb{Q}}} \mathbb{Z}$$

by selecting the $\frac{1}{2}$ positive root as a generator. Then we have a map induced by the above conjugation

$$X(H \times_{\mathbb{Q}} \bar{\mathbb{Q}}) \to X(H^{(1)} \times_{\mathbb{Q}} \bar{\mathbb{Q}}) \tilde{\to} X(T^{(1)} \times_{\mathbb{Q}} \bar{\mathbb{Q}})$$

which is given by

$$(\ldots m_{\tau'}, m_{\tau''} \ldots)_{\tau' : E \to \bar{\mathbb{Q}}} \to (\ldots, \pm(m_{\tau'} - m_{\tau''}), \ldots)_{\tau : F \to \bar{\mathbb{Q}}}$$

where again τ', τ'' are lying above τ and the sign depends on the choice of the conjugation in the τ-component. The weights of $T^{(1)} \times_{\mathbb{Q}} \bar{\mathbb{Q}}$ in $M = \bigotimes_{\tau : F \to \bar{\mathbb{Q}}} M(d(\tau), \nu(\tau))$ in the τ-component are $d(\tau)$, $d(\tau)-2, \ldots, -d(\tau)$. Hence for all $\tau : F \to \bar{\mathbb{Q}}$ and the two embeddings $\tau', \tau'' : E \to \bar{\mathbb{Q}}$ lying above τ we must have

$$|m_{\tau'} - m_{\tau''}| \leq d(\tau)$$

and

$$m_{\tau'} - m_{\tau''} \equiv d(\tau) \bmod 2 \quad .$$

Case A: In this case we have $m_{\tau'} = m$ for all τ' , $d(\tau) = d$, $\nu(\tau) = \nu$, we have

$$M = \bigotimes_{\tau : F \to \overline{\mathbb{Q}}} M(d,\nu) \qquad\qquad d \equiv 0 \bmod 2$$

and $\mu^{(1)} = \mu \mid H^{(1)} \times_{\mathbb{Q}} \overline{\mathbb{Q}}$ has to be the zero weight.

Case B: Again we have much more flexibility. Over each pair $\tau, \bar{\tau} : F \to \overline{\mathbb{Q}}$ of conjugate embeddings we have four embeddings of E into $\overline{\mathbb{Q}}$. Let us draw a diagramm

$$\begin{tikzcd}
\left.\begin{matrix}\tau' \\ \tau'' \\ \bar{\tau}' \\ \bar{\tau}''\end{matrix}\right\} : E \arrow[rr, shift left=3] \arrow[rr, shift left=1] \arrow[rr, shift right=1] \arrow[rr, shift right=3] & & \overline{\mathbb{Q}} \\
F \arrow[u] \arrow[urr, shift left, "\tau"] \arrow[urr, shift right, "\bar{\tau}"'] & & \\
\mathbb{Q} \arrow[u] & &
\end{tikzcd}$$

We must have

$$m_{\tau'} + m_{\bar{\tau}'} = m_{\tau''} + m_{\bar{\tau}''} = w(\mu)$$

and

$$|m_{\tau''} - m_{\tau'}| \leq d(\tau)$$

$$|m_{\bar{\tau}''} - m_{\bar{\tau}''}| \leq d(\bar{\tau})$$

$$m_{\tau''} - m_{\tau'} \equiv d(\tau) \bmod 2$$

$$m_{\bar{\tau}''} - m_{\bar{\tau}'} \equiv d(\bar{\tau}) \bmod 2 \quad .$$

This implies $d(\tau) \equiv d(\bar{\tau}) \bmod 2$, since $|m_{\tau''} - m_{\tau'}| = |m_{\bar{\tau}''} - m_{\bar{\tau}'}|$ the inequalities have to be fulfilled for the smaller value out of $\{d(\tau),d(\bar{\tau})\}$.

5.5. Our goal is to compute the operator

$$I(\phi,\eta,e(\gamma),m^{v}(\mu^{-1}),i_H) \ : \ V_\phi \quad \to W_\eta$$

and the answer will be given in theorem 3. This theorem is the generalization of the corresponding theorem in |Ha4|. We will give a formula for this operator by comparing it to a local intertwining operator

$$I^{loc}(\phi,\eta,i_H) = \bigotimes_{\mathfrak{y}} I^{loc}_{\mathfrak{y}}(\phi_{\mathfrak{y}},\eta_{\mathfrak{y}}) \quad .$$

(We have to take into account that $\pi_0(B\ ) = \pi_0(G\ ) = \pi_0(H\ )$ the induction at the infinite places is trivial.)

We recall the construction of the local operators

$$I^{loc}_{\mathfrak{y}}(\phi_{\mathfrak{y}},\eta_{\mathfrak{y}},i_H) \ : \ V_{\phi_{\mathfrak{y}}} \to \mathrm{Ind}_{H_0(F_{\mathfrak{y}})}^{G_0(F_{\mathfrak{y}})} \eta_{\mathfrak{y}}$$

given in |Ha4|, 3.2.4 and explain the necessary modifications here. Our ϕ is the local component of a ϕ whose type is in $\mathrm{Coh}(M)$, it is quite clear that the system

$$\{V_{\phi_{\mathfrak{y}}}\}_{\phi_{\mathfrak{y}}}$$

has an obvious $\mathbb{Q}$-structure which is consistent with the obvious $\mathbb{Q}$-structure on the system

$$\{V_\phi\}_\phi$$

which we introduced in 2.7. The same arguments also yield a $\mathbb{Q}$-structure on the system

$$\{W_{\eta_\mathfrak{Y}}\}_{\eta_\mathfrak{Y}}$$

where $W_{\eta_\mathfrak{Y}} = \mathrm{Ind}_{H_o(F_\mathfrak{Y})}^{G_o(F_\mathfrak{Y})} \eta_\mathfrak{Y}$. We want to construct a system of operators

$$I^{loc}(\phi_\mathfrak{Y}, \eta_\mathfrak{Y}, i_H) \ : \ V_{\phi_\mathfrak{Y}} \to W_{\eta_\mathfrak{Y}}$$

which is defined over $\mathbb{Q}$ in the sense of 1.3. Moreover we require that for almost all $\mathfrak{Y}$ for which $\phi_\mathfrak{Y}$ and $\eta_\mathfrak{Y}$ are not ramified we have

$$I^{loc}(\phi_\mathfrak{Y}, \eta_\mathfrak{Y}, i_H)(\psi_{\mathfrak{Y},o})(1) = 1 \qquad (5.5.1)$$

where $\psi_{\mathfrak{Y},o} \in V_{\phi_\mathfrak{Y}}$ is the standard spherical function (see 4.2.). The condition (5.5.1) guarantees the convergence of the tensorproduct of operators. We shall show the existence of such a system of operators by exhibiting a special choice which will enter in the formulation of our theorem 3.

As in |Ha4|, p. 136 we define a system of local intertwining operators as integrals

$$\tilde{I}_\mathfrak{Y}(\phi_\mathfrak{Y}, \eta_\mathfrak{Y}, i_H) \ : \ \psi_\mathfrak{Y} \to \{g_\mathfrak{Y} \to \int_{\bar{H}_o(F_\mathfrak{Y})} \eta_\mathfrak{Y}^{-1}(t_\mathfrak{Y})\psi_\mathfrak{Y}(t_\mathfrak{Y} g_\mathfrak{Y})d^*t\}$$

where the measure d^*t is the quotient measure of those invariant measures on $H_o(F_\mathfrak{Y}) = E_\mathfrak{Y}^*$ and $Z_o(F_\mathfrak{Y}) = F_\mathfrak{Y}^*$ which are normalized to give volume one to the units. (Note the change of sign if we compare with |Ha4|.)

<u>Lemma 5.5.2</u>: a) <u>The above integrals are convergent if $\phi_{\mathfrak{p}}$ is in the positive chamber and $\eta_{\mathfrak{p}}$ satisfies the constraints in 5.4. They define intertwining operators</u>

$$\tilde{I}_{\mathfrak{p}}(\phi_{\mathfrak{p}},\eta_{\mathfrak{p}},i_H) \;:\; V_{\phi_{\mathfrak{p}}} \otimes \mathbb{C} \rightarrow \mathrm{Ind}_{H_o(F_{\mathfrak{p}})}^{G_o(F_{\mathfrak{p}})} \eta_{\mathfrak{p}} \otimes \mathbb{C} \quad .$$

b) <u>These operators are actually defined over $\overline{\mathbb{Q}}$ and the system of operators is defined over $\mathbb{Q}$ (if we vary $\phi_{\mathfrak{p}}$ and $\eta_{\mathfrak{p}}$) in the sense of 1.0.3</u> .

<u>Proof</u>: We follow the argument given in [Ha4], p. 138 - 140. There is no problem for those primes $\mathfrak{p}$ which do not split in E , in that case $H_o(F_{\mathfrak{p}})$ is compact, the integral is a finite sum. In the case that splits into $\mathfrak{p} = \mathfrak{P}\cdot\mathfrak{P}'$ we decompose the integral into a finite sum and a geometric series, we get an explicit formula for the intertwining operator as in |Ha |, p. 140 bottom. This implies that the system of operators is defined over $\mathbb{Q}$, once we settled the question of convergence (see remark below).

We have $E_{\mathfrak{p}}^* = E_{\mathfrak{P}}^* \times E_{\mathfrak{P}'}^*$ where $E_{\mathfrak{P}} = E_{\mathfrak{P}'} = F_{\mathfrak{p}}$. The $\mathfrak{p}$ component of η can be written as

$$\eta_{\mathfrak{p}}(x_{\mathfrak{p}}) = \eta_{\mathfrak{p}}(x_{\mathfrak{P}},x_{\mathfrak{P}'}) = \eta_{\mathfrak{P}}(x_{\mathfrak{P}})\cdot\eta_{\mathfrak{P}'}(x_{\mathfrak{P}'}) \quad .$$

The character ϕ will be written as

$$\phi \;:\; \begin{pmatrix} \underline{t}_1 & \underline{u} \\ 0 & \underline{t}_2 \end{pmatrix} \rightarrow \phi_1(\underline{t}_1)\cdot\phi_2(\underline{t}_2) \quad .$$

The computations in |Ha4| loc. cit. show that for the summation of the integrals we have to sum up the series

$$\sum_{\nu \geq N} (\eta_{\mathfrak{P}}(\pi_{\mathfrak{P}})^{-1}\phi_1(\pi_{\mathfrak{Y}}))^{\nu} \quad \text{and} \quad \sum_{\nu \geq N} (\eta_{\mathfrak{P}'}(\pi_{\mathfrak{P}})^{-1}\phi_1(\pi_{\mathfrak{Y}}))^{\nu}$$

in the case of non-ramification of the characters. (Otherwise we get finite sums again.) So we have to worry on the absolute values

$$|\eta_{\mathfrak{P}}(\pi_{\mathfrak{P}})^{-1}\phi_1(\pi_{\mathfrak{Y}})| \quad \text{and} \quad |\eta_{\mathfrak{P}'}(\pi_{\mathfrak{P}}^{-1})\phi_1(\pi_{\mathfrak{Y}})|$$

(the values are in $\overline{\mathbb{Q}}^* \subset \mathbb{C}^*$ and we have to take the usual absolute values). This comes down to the consideration of the algebraic Hecke character $\chi = \eta^{-1} \cdot \phi_1 \circ N_{E/F}$ on $H(A) = I_E$. It is a well known formula and obvious from the definition that for all primes $\mathfrak{P}$ of E (comp. [Sel], II, § 3, prop. 2)

$$|(\eta^{-1} \cdot \phi_1 \circ N_{E/F}) \quad (\pi_{\mathfrak{P}})| = N\mathfrak{P}^{w(\chi)/v}$$

where $w(\chi)$ is the weight of χ and $v = 1$ in Case A and $v = 2$ in case B. Using the notations of 5.4. and the condition 5.4.3. it follows easily that

$$w(\eta) = -\frac{d+2}{2} \qquad \text{in case A}$$

$$w(\eta) = \frac{d(\bar{\tau})-d(\tau)-2}{2} \qquad \text{in case B} \quad .$$

The condition that ϕ is in the positive chamber means $d(\bar{\tau}) \leq d(\tau)$, hence we have $w(\chi) < 0$ and this proves the lemma.

<u>5.5.3</u> **Remark:** Actually we do not need the convergence, we could certainly also define the intertwining operator by a formula like the one in |Ha4|, p. 140 bottom and use analytic continuation to prove the fact that it is indeed an intertwining operator. But we have to avoid the "pole of the

geometric" series, this means we have to be sure that $\eta_{\mathfrak{P}}^{-1}(\pi_{\mathfrak{P}})\phi_1(\pi_{\mathfrak{Y}}) \neq 1$. The verification of this assertion requires of course the same argument as above.

Now we put as in [Ha4], p. 136

$$I^{loc}(\phi_{\mathfrak{Y}},\eta_{\mathfrak{Y}},i_H) = \frac{L_F(\phi_{\mathfrak{Y}}^{(1)},0)}{L_E(\eta_{\mathfrak{Y}}^{-1}\cdot\phi_{1\mathfrak{Y}}\circ N_{E/F},0)} \cdot \tilde{I}_{\mathfrak{Y}}(\phi_{\mathfrak{Y}},\eta_{\mathfrak{Y}},i_H) .$$

Then this gives us a system of intertwining operators which satisfies (5.5.1) for almost all $\mathfrak{Y}$.

<u>5.6.</u> The theorem 3 which we want to state will have the form

$$I(\phi,\eta,e(\gamma),m^{\vee}(\mu^{-1}),i_H) = c \cdot I^{loc}(\phi,\eta,i_H) .$$

The factor c will have the form

$$c = c_{global} \cdot c_{\infty}$$

where c_{global} will be the ratio of two values of L-functions and c_{∞} will be the contribution from the infinite places. The factor c_{∞} must of course depend on our data

$$c_{\infty} = c_{\infty}(e(\gamma),m^{\vee}(\mu^{-1}),i_H,\Omega_o) ,$$

it has to be bilinear in the first two variables and Ω_o is the orientation which we selected on $\mathfrak{f}_{\infty}/K_{\infty}^H$. I want to define this number but I have to introduce some additional redundant data to fix it.

First of all we want to specify the field extension E/F and the embedding

$$i_H \; : \; H/\mathbb{Q} \;\to\; G/\mathbb{Q} \quad .$$

We recall that this embedding is nothing else than an embedding

$$E^* \;\to\; GL_2(F) \quad .$$

We may write $E = F(\sqrt{\Delta})$ for $\Delta \in F^*$ and we identify H/F to the torus H_o/F for which

$$H_o(F) \;=\; \left\{ \begin{pmatrix} a & b \\ b\Delta & a \end{pmatrix} \;\middle|\; a^2-b^2\Delta \neq 0 \right\} \quad .$$

The next thing we will do is to choose a total order $<$ on the set

$$\{\tau \mid \tau : F \to \mathbb{C}\} \quad .$$

If we are in the case A then this also defines a total order on the set of places of F and moreover for each $v \in S_\infty$ we have $F_v = \mathbb{R}$.

If we are in the case B then a $v \in S_\infty$ is a pair $\{\tau,\bar{\tau}\}$ of conjugate embeddings. But under our present circumstances the rational character γ fixes a CM-type

$$\mathcal{T}(\gamma) \;=\; \{\tau \mid \deg(\gamma_\tau) = 1\}$$

(see 4.3.). Since we assume that γ is in the fundamental chamber we have always $d(\tau) \geq d(\bar{\tau})$ for $\tau \in \mathcal{T}(\gamma)$. This allows us to define an order on the set of places S_∞ , we identify this set with $\mathcal{T}(\gamma)$ and transport the order. But it also selects an identification

$$i_v \; : \; F_v \;\to\; \mathbb{C}$$

where i_v is the identification induced by $\tau \in \mathcal{T}(\gamma)$ if $\{\tau,\bar{\tau}\} = v$.

We go back to the general case. At each place v we select a square root $\delta_v^2 = \Delta$ with $\delta_v \in F_v$. We put

$$g_\infty = (\ldots\begin{pmatrix}\delta_v^{-1} & 0\\ 0 & 1\end{pmatrix}\ldots)_{v\in S_\infty} .$$

Let $C_0/F \to GL_2/F$ the maximal split torus given by

$$C_0(F) = \left\{\begin{pmatrix}a & b\\ b & a\end{pmatrix} \mid a^2-b^2 \neq 0\right\} .$$

Then we have for any $v \in S_\infty$

$$\begin{pmatrix}\delta_v & 0\\ 0 & 1\end{pmatrix} \cdot H_0(F_v) \cdot \begin{pmatrix}\delta_v^{-1} & 0\\ 0 & 1\end{pmatrix} = C_0(F_v) .$$

This tells us first of all that g_∞ is adapted to $H/\mathbb{Q}$, this is so since the maximal connected subgroup of $C_0(F_v)$ is contained in K_v, we put as usual $K_v^C = C_0(F_v) \cap K_v$.

We write

$$\mathrm{Lie}(C_0(F_v)/K_v^C) = \mathbb{R}\cdot\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \mathbb{R}\cdot Y_v^0 .$$

Then the elements $\mathrm{ad}\begin{pmatrix}\delta_v^{-1} & 0\\ 0 & 1\end{pmatrix}\cdot Y_v^0 = Y_v$ will form an ordered basis of $\mathfrak{g}_\infty/k_\infty^H$, and hence they define an orientation

$$\Omega(\{\delta_v\}_v,<)$$

on S_K^H (see 5.3.5). We compare this to the given orientation and put $\Omega(\{\delta_v\}_v,<)/\Omega_0 = \pm 1$ according to whether they are equal or not.

In 2.4.1 we wrote down explicit representatives for the classes in $H^d(\tilde{\mu},M)(\gamma)$ they were given by

$$\xi(\lambda,\gamma,<)(u_{\alpha,\tau_1},\ldots,u_{\alpha,\tau_d}) = E(\gamma) \qquad (5.6.1)$$

where $\tau_1 < \tau_2 < \ldots < \tau_d$ with respect to the given order, where the τ_i are the elements of $\mathcal{T}(\gamma)$ in case B and where

$$E(\gamma) = \begin{cases} \bigotimes_{\tau : F \to \overline{\mathbb{Q}}} e_{-d} & \text{in case A} \\ \bigotimes_{\tau \in \mathcal{T}(\gamma)} e_{-d(\tau)} \otimes \bigotimes_{\tau \notin \mathcal{T}(\gamma)} e_{d(\tau)} & \text{in case B} \end{cases}$$

We called the class represented by the above element $e(\lambda,\gamma,<)$. (Recall that they do not necessarily form a rational system of generators.)

The last datum entering is $m^v(\mu^{-1})$. We saw in 5.4. that $\mu^{(1)}$ has to be the zero weight in case A and in case B the $\mu^{(1)}$ was given as

$$\mu^{(1)} = (\ldots (m_{\tau'}, -m_{\tau''}), \ldots)_{\tau : F \to \overline{\mathbb{Q}}}$$

where τ', τ'' lie above τ . We put $\mu(\tau) = \pm \frac{m_{\tau'} - m_{\tau''}}{2} + \frac{d(\tau)}{2}$. Now we are ready to define c_∞ .

Case A: We have $M = \bigotimes_{\tau : F \to \overline{\mathbb{Q}}} M(d,\nu))$ where $d \equiv 0 \bmod 2$ (see 2.8.1 and 5.4). We write

$$\phi\begin{pmatrix} \underline{t}_1 & * \\ 0 & \underline{t}_2 \end{pmatrix} = \phi_1(\underline{t}_1) \cdot \phi_2(\underline{t}_2) \quad .$$

It is clear that (see 2.8.2)

$$\mathrm{type}(\phi_1) = (\ldots,\nu-1,\ldots)_{\tau : F \to \overline{\mathbb{Q}}}$$

$$\mathrm{type}(\phi_2) = (\ldots,d+\nu+1,\ldots)_{\tau : F \to \overline{\mathbb{Q}}}$$

and hence we have for any $v \in S_\infty$ that

$$\phi_{1,v}(t_v) = t_1^{-\nu+1} \cdot \sigma_{1,v}(t_1) \quad ; \quad \phi_{2,v}(t_v) = t_2^{-d-\nu-1}\, \sigma_{2,v}(t_2) \quad ,$$

where $\sigma_{1,v}$ and $\sigma_{2,v}$ are sign characters.

We saw (comp. 2.6.2) that we must have $\sigma_{1,v} = \sigma_{2,v} = \sigma_v$ otherwise the character ϕ cannot contribute to the cohomology of the boundary. We also know that

$$\mathrm{type}(\eta) = (\ldots,\frac{d}{2} + \nu,\ldots)_{\tau : E \to \bar{\mathbb{Q}}} .$$

This implies that for $v \in S_\infty$

$$\eta_v : H_o(F_v) = \mathbb{R}^* \times \mathbb{R}^* \to \mathbb{C}^*$$

is given by

$$\eta_v(t_1,t_2) = t_1^{-\frac{d}{2} - \nu} \varepsilon_{v'}(t_1) \cdot t_2^{-\frac{d}{2} - \nu} \cdot \varepsilon_{v''}(t_2) .$$

It is clear that 5.3.7.1 implies $\varepsilon_{v'} = \varepsilon_{v''} = \varepsilon_v$, i.e. η must have the same signature at pairs of places of E lying over one place of F . We put

$$c_\infty = 0$$

if there is a place $v \in S_\infty$ for which the product $\sigma_v \cdot \varepsilon_v \neq 1$.

If for all v the product $\sigma_v \varepsilon_v = 1$ then we put

$$c_\infty(e(\lambda,\gamma,<),m^v(\mu^{-1}),i_H,\Omega_o,\{\delta_v\}) =$$

$$= 2^{n(d+2)} \cdot \langle E(\gamma),m^v(\mu^{-1})\rangle \cdot \frac{(d/2!)^{2n}}{((d+1)!)^n} \cdot \frac{\Omega(\{\delta_v\},<)}{\Omega_o} \cdot \prod_{v \in S_\infty} \delta_v^{-1}\sigma_v(\delta_v) .$$

(Here we consider $\delta_v \in F_v$ as elements of $\mathbb{R}^*$.) We notice that a different choice of the δ_v also changes the connected component of the pinpoint g_∞ and hence changes the intertwining operator.

Case B: In this case we put

$$c_\infty(e(\lambda,\gamma,<),m^v(\mu^{-1}),i_H,\Omega_o) =$$

$$\langle E(\gamma),m^v(\mu^{-1})\rangle \cdot \frac{\Omega(\{\delta_v\},<)}{\Omega_o} \cdot \prod_{\tau\in\Sigma(\gamma)} i_v(\delta_v^{-1})\ 2^{d(\tau)+1} \cdot \frac{(d(\tau)-\mu(\tau))!\mu(\tau)!}{(d(\tau)+1)!}$$

We observe that the result does not depend on the choice of the sign in $\mu(\tau) = \pm \frac{m_{\tau''}-m_{\tau'}}{2} + \frac{d(\tau)}{2}$.

We observe that in case B a different choice of the δ_v does not affect the value of this expression, our pinpoint stays in the same component.

Theorem 3: With the above conventions and under the assumption of 5.3.7.1 we have the following formula for the intertwining operator

$$I(\phi,\eta,e(\lambda,\gamma,<),m^v(\mu^{-1})i_H,\{\delta_v\}) =$$

$$h_F \cdot \frac{L_E(\eta\cdot\phi_1 \cdot N_{E/F},0)}{L_F(\phi^{(1)},0)} \cdot c_\infty(e(\lambda,\gamma,<),m^v(\mu^{-1}),i_H,\Omega_o) \cdot I^{loc}(\phi,\eta,i_H) \quad .$$

Proof: We know already that we have to compute the integral

$$I(\phi,\eta,e(\lambda,\gamma,<),m^v(\mu^{-1}),i_H)(\psi)(g_f) =$$

$$h_F \cdot \int_{\bar H(\mathbb{Q})\backslash \bar H(\mathbb{A})} \eta^{-1}(\underline{h})\langle \mathrm{Eis}(\underline{hg},\phi,e(\lambda,\gamma,<),\psi),\mathrm{ad}(g_\infty^{-1})(Y_H) \otimes \rho^v(g_\infty^{-1})m^v(\mu^{-1})\rangle\, d\underline{\bar h}$$

(see 5.3.7.1). Here g_∞ is the element selected above and Y_H is the wedge of the above Y_v in the order $<$ on the set S_∞ . To compute this integral we go along the same lines as in [Ha4] p. 138 ff. We recall that the Eisenstein differential form is obtained by analytic continuation from an infinite series (see 4.2.). By a standard computation we get that the above integral is equal to

$$\int_{\underline{H}(A)} \eta^{-1}(\underline{h})<\omega(\underline{hg},\phi|\alpha|^{s},e(\lambda,\gamma,<),\psi),\mathrm{ad}(g^{-1})(Y_H) \otimes \rho^{v}(g_\infty^{-1})m^{v}(\mu^{-1})> d\underline{\bar{h}}$$

evaluated at the point $s = 0$ (We drop the factor h_F). This integral is a product of local integrals, the finite places can be handled in the same way as we did this in |Ha4| (see also 5.5.). The infinite places present some problem, we have to take the coefficient system into account. To put it in a slightly different form: The contribution of the finite places is the ratio of the two special values of L-functions. At the infinite places we have to compute c_∞ this is new (to some extend). We have to evaluate the integrals

$$\int_{\underline{H}_o(F_v)} \eta_v^{-1}(h_v)\ \omega(h_v g_v,\phi_v\alpha_v^{s},e(\lambda,\gamma,<)),\mathrm{ad}(g_v^{-1})(Y_v) \otimes \rho^{v}(g_v^{-1})m_v(\mu^{-1}))dh_v \quad .$$

Here we write

$$m^{v}(\mu^{-1}) = \bigotimes_{\tau : F\to\bar{\mathbb{Q}}} m_\tau(\mu^{-1})$$

and $m_v(\mu^{-1}) = m_\tau(\mu^{-1}) \otimes m_{\bar{\tau}}(\mu^{-1})$ if $\{\tau,\bar{\tau}\} = v$. Recall that

$$g_v = \begin{pmatrix} \delta_v^{-1} & 0 \\ 0 & 1 \end{pmatrix}$$

we abbreviate $\rho^{v}(g_v^{-1})m_v(\mu^{-1}) = w_v$. The factor c_∞ is the product of these integrals over all infinite places up to the sign factor

$$\Omega(\{\delta_v\}_v,<)/\Omega_o \quad .$$

We write $h_v g_v = g_v g_v^{-1} h_v g_v = g_v h_v'$ where $h_v' \in C_o(F_v)$. Since $g_v \in B_o(F_v)$ we get for our integral

$$\phi_v\alpha_v^{s}(g_v) \int_{\underline{C}_o(F_v)} \eta_v^{-1}(h_v')<\omega(h_v',\phi_v\alpha_v^{s},e(\lambda,\gamma,<))),Y_o \otimes w_v> d\bar{h}_v, \quad .$$

Here we identify $H_o(F_v)$ and $C_o(F_v)$ by means of the conjugation by g_v and transport the character η_v from $H_o(F_v)$ to $C_o(F_v)$. We write for $t_v \in F_v^*$

$$h(t_v) = \begin{pmatrix} \frac{1+t_v}{2} & \frac{1-t_v}{2} \\ \frac{1-t_v}{2} & \frac{1+t_v}{2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} t_v & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

and this provides an identification of F_v and $\overline{C}_o(F_v)$. Our above integral becomes

$$\phi_v \alpha_v^s(g_v) \int_{F_v^*} \eta_v(h(t_v^{-1})) \cdot \langle \omega(h(t_v), \phi_v \alpha_v^s, e(\lambda,\gamma,<))), Y_o \otimes w_v \rangle \, d^* t_v \, .$$

Here $d^* t_v$ is an invariant measure on F_v^* which has been specified as follows:

<u>Case A</u>: We map $Y_o = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ by the exponential map into $\overline{C}_o(F_v) = \mathbb{R}^*$ this gives us

$$x \to \exp x\, Y_o = \begin{pmatrix} \cosh x & \sinh x \\ \sinh x & \cosh x \end{pmatrix}$$

and if we project down to $\overline{C}_o(F_v)$ we get

$$x \to \begin{pmatrix} \frac{e^{-2x}+1}{2} & \frac{1-e^{-2x}}{2} \\ \frac{1-e^{-2x}}{2} & \frac{1+e^{-2x}}{2} \end{pmatrix}$$

i.e. $t = e^{2x}$ and hence the measure is $dx = \frac{1}{2}\frac{dt}{t} = dt_v^*$.

<u>Case B</u>: In this case we have $\overline{C}_o(F_v) = \mathbb{C}^* = \mathbb{R}_{>0}^* \times S^1$. The same reasoning as above yields the measure

$$\frac{1}{2}\frac{dr}{r} \cdot d\theta$$

where $\int_{S^1} d\theta = 1$, and r is the variable in $\mathbb{R}^*_{>0}$.

Is is easy to check that the integrand does not depend on the circular variable θ in case B and hence our integral becomes

$$\frac{\phi_v \alpha_v^s(g_v)}{2} \int \eta_v(h(t^{-1}))\langle\omega(h(t),\phi_v\alpha_v^s,e(\lambda,\gamma,<)),Y_o \otimes w_v\rangle \frac{dt}{|t|}$$

where we integrate over $\mathbb{R}^*$ in case A and over $\mathbb{R}^*_{>0}$ in case B.

We have to unravel the definition of the definition of $\omega(h(t),\phi_v\alpha_v^s,e(\lambda,\gamma,<))$. Recall that we may write (see 5.6.1)

$$e(\lambda,\gamma,<) = \bigwedge_{i=1}^{d} u^v_{\alpha,\tau_i} \otimes E(\gamma) \quad .$$

The element u^v_{α,τ_i} (see 2.3.1, 3.4 and 3.5) has to be viewed as an element

$$u^v_{\alpha,\tau_i} \in \mathrm{Hom}(\mathfrak{g}_{v_i}/k_{v_i},\mathbb{C}) \quad .$$

Then for any v we write

$$h(t) = h_v(t) = b_v(t) \cdot h_v(t)$$

and

$$\omega(h_v(t),\phi_v\alpha_v^s,e(\lambda,\gamma,<)) = \phi_v\alpha_v^s(b_v(t)) \cdot (\mathrm{ad}(k_v(t)^{-1})u^v_{\alpha,\tau} \otimes \rho(k_v(t))w_v)$$

where $v = \{\tau,\bar{\tau}\}$ and $\tau \in \mathcal{T}(\gamma)$. The second factor lies in

$$\mathrm{Hom}(\mathfrak{g}_v/k_v,\mathbb{C}) \otimes M_v$$

and can be evaluated on $Y_o \otimes w_v$, the result is a number in $\mathbb{C}$. Hence we get for our integral

$$\frac{\phi_v\alpha_v^s(g_v)}{2}\int \eta_v(h(t)^{-1})\phi_v\alpha_v^s(g(t))\langle \mathrm{ad}\ k(t)^{-1}u^v_{\alpha,\tau},Y_o\rangle\cdot\langle\rho(k(t)^{-1})e_{\gamma_v},w_v\rangle\frac{dt}{|t|} =$$

$$\frac{\phi_v\alpha_v^s(g_v)}{2}\int \eta_v(h(t)^{-1})\phi_v\alpha_v^s(b(t))\langle u^v_{\alpha,\tau},\mathrm{ad}\ k(t)Y_o\rangle\cdot\langle e_{\gamma_v},\rho^v(k(t)w_v\rangle\frac{dt}{|t|}$$

with the same domain of integration. The Iwasawa decomposition for $h(t)$ is

$$h(t) = \begin{pmatrix} \frac{1+t}{2} & \frac{1-t}{2} \\ \frac{1-t}{2} & \frac{1+t}{2} \end{pmatrix} = \begin{pmatrix} \frac{t}{u(t)} & * \\ 0 & u(t) \end{pmatrix}\begin{pmatrix} \frac{1+t}{2u(t)} & -\frac{1-t}{2u(t)} \\ \frac{1-t}{2u(t)} & \frac{1+t}{2u(t)} \end{pmatrix}$$

with $u(t) = (\frac{1+t^2}{2})^{1/2}$.

Now it seems to be appropriate to separate the cases.

Case A: In this case $M = \bigotimes_{\tau:F\to\bar{\mathbb{Q}}} M(d,\nu)$ (see 5.4.), d has to be even and the dual module $M^v = \bigotimes_{\tau:F\to\bar{\mathbb{Q}}} M(d,-d-\nu)$. In accordance with our discussion of the definition of c_∞ we find for the 4 factors in our integrand:

The first factor is

$$\eta_v(h(t)^{-1}) = t^{+\frac{d}{2}+\nu}\cdot\varepsilon_v(t) \quad . \tag{Ia}$$

The second factor is

$$\phi_v\alpha_v^s\begin{pmatrix} \frac{t}{u(t)} & * \\ 0 & u(t) \end{pmatrix} = t^{-\nu+1}u(t)^{-d-2}\cdot(\frac{|t|}{u(t)^2})^{s/2}\cdot\sigma_v(t) . \tag{Ib}$$

One verifies directly that

$$\langle u^v_{\alpha,\tau},\mathrm{ad}\ k(t)Y_o\rangle = \frac{t}{u(t)^2} \quad . \tag{Ic}$$

The last factor requires a little bit more work, it is

$$\langle e_{-d}, \rho(k(t)w_v\rangle \quad .$$

The Cartan involution $\theta : g \to {}^t g^{-1}$ acts as identity on $K_v^{(1)}$ and this allows us to rewrite the Iwasawa decomposition

$$k(t) = b(t)^{-1} \cdot h(t) = b(t)^{-\theta} \cdot h(t)^{\theta} = b(t)^{-\theta} h(t^{-1}) \quad .$$

Our last term becomes

$$\langle b(t)^{\theta} e_{-d}, h(t)^{-1} w_v\rangle \quad .$$

Now we take into account that e_{-d} is a highest weight vector for the opposite Borel subgroup and that w_v is a weight vector for $h(t)^{-1}$ then we obtain

$$\langle b(t)^{\theta} e_{-d}, h(t^{-1}) w_v\rangle = u(t)^{-d} t^{d/2} \langle e_{-d}, w_v\rangle \quad . \tag{Id}$$

Multiplying all four terms (Ia-d) together we find the value

$$\frac{\phi_v \alpha_v^s(g_v)}{2} \int_{\mathbb{R}^*} \frac{t^{d+2}}{u(t)^{2d+4+s}} |t|^{s/2} \sigma_v \varepsilon_v(t) \frac{dt}{|t|} =$$

$$\phi_v \alpha_v^s(g_v) \cdot 2^{d+1+s/2} \int_{\mathbb{R}^*} \frac{t^{d+2} \cdot |t|^{s/2}}{(1+t^2)^{d+2+s/2}} \sigma_v \varepsilon_v(t) \frac{dt}{|t|} \quad .$$

Since d is even the integral vanishes if $\sigma_v \varepsilon_v$ is an odd character, hence we assume $\sigma_v \varepsilon_v$ to be even and then our integral becomes

$$\phi_v \alpha_v^s(g_v)\ 2^{d+2+s/2} \int_0^{\infty} \frac{t^{d+1+s/2}}{(1+t^2)^{d+2+s/2}} dt \quad .$$

The substitution $1+t^2 = \frac{1}{w}$ transforms the integral into

$$\int_0^1 w^{d/2+s/4} \cdot (1-w)^{d/2+s/4} dw = \frac{\Gamma(d/2+1+s/4)^2}{\Gamma(d+2+s)} \quad .$$

Now we put $s = 0$ and we find that the contribution from the infinite places is

$$\phi_v(g_v) \ \frac{\Omega(\{\delta_v\},<)}{\Omega_o} \cdot 2^{n(d+2)} \cdot \frac{(d/2!)^{2n}}{((d+1)!)^n} \cdot \prod_{v\in S_\infty} \phi_v\left(\begin{pmatrix} \delta_v^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right) <e_{-d},w_v> \ .$$

But we recall that $m^v(\mu^{-1}) = \otimes\, m_\tau(\mu^{-1})$ and $w_v = \rho^v(g_v^{-1})\, m_v(\mu^{-1})$. If we substitute this into the last bracket then

$$<e_{-d},\rho^v(g_v^{-1})\cdot m_v(\mu^{-1})> \ = \ <\rho(g_v)e_{-d},m_v(\mu^{-1})> \ = \ \delta_v^{-\nu}\cdot <e_{-d},m_v(\mu^{-1})> \ .$$

On the other hand we have seen that

$$\phi_v(g_v) \ = \ \phi_v\left(\begin{pmatrix} \delta_v^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right) = \ \delta_v^{\nu-1}\cdot \sigma_v(\delta_v)$$

and the above expression simplifies to

$$\frac{\Omega(\{\delta_v\},<)}{\Omega_o} \cdot 2^{n(d+2)} \ \frac{(d/2!)^{2n}}{((d+1)!)^n} \cdot \prod_{v\in S_\infty} \delta_v^{-1}\cdot \sigma_v(\delta_v)\cdot <E(\gamma),m^v(\mu^{-1})>$$

because

$$<E(\gamma),m^v(\mu^{-1})> \ = \ \prod_{v\in S_\infty} <e_{-d},m_v(\mu^{-1})> \ .$$

This proves theorem 3 in the case A.

<u>Case B:</u> We proceed essentially in the same way as in case A. We have

$$M \ = \bigotimes_{v\in S_\infty} M_v$$

where

$$M_v \ = \ m(d(\tau),\nu(\tau)) \otimes M(d(\bar\tau),\nu(\bar\tau))$$

where $d(\tau) \geq d(\bar{\tau})$ and we assume $\tau \in \mathcal{T}(\gamma)$. The type of η was denoted by

$$\mathrm{type}(\eta) = \mu = (\ldots m_{\tau'}, m_{\bar{\tau}'}, \ldots)_{\tau' : E \to \bar{\mathbb{Q}}} \quad .$$

I claim that this implies that for $t \in \mathbb{R}^{*}_{>0}$

$$\eta_v(h(t)^{-1}) = t^{\nu(\tau)+\nu(\bar{\tau}) + \frac{d(\tau)+d(\bar{\tau})}{2}} \quad . \tag{Ia}$$

To see this we have to recall that η_v was defined on $C_o(F_v)$ by transport from $H_o(F_v)$ by means of the conjugation by $g_v = \begin{pmatrix} \delta_v^{-1} & 0 \\ 0 & 1 \end{pmatrix}$. We identified

$$\mathbb{C}^* = F_v^* \xrightarrow{h} \bar{C}_o(F_v)$$

where

$$h : t \to \begin{pmatrix} \frac{1+t}{2} & \frac{1-t}{2} \\ \frac{1-t}{2} & \frac{1+t}{2} \end{pmatrix} \in C_o(F_v)$$

and $h(t)$ was taken modulo the centre. This means that if we diagonalize $C_o(F_v)$ then

$$h : F_v^* \to C_o(F_v) \xrightarrow{\mathrm{diag}} \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$$

we will

$$\mathrm{diag} \circ h(t) = \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad \mathrm{diag} \circ h(t) = \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} \quad .$$

Hence we have that for $t \in \mathbb{C}^*$

$$\eta_v(h(t)) = t^{-m_{\tau'}} \cdot \bar{t}^{-m_{\bar{\tau}'}}$$

or

$$\eta_v(h(t)) = t^{-m_{\tau''}} \cdot \bar{t}^{-m_{\bar{\tau}''}} \quad .$$

But 5.4.2 combined with the constraint condition $m_{\tau'} + m_{\bar{\tau}'} = w(\mu)$ implies

$$m_{\tau'} + m_{\bar{\tau}'} = \nu(\tau) + \nu(\bar{\tau}) + \frac{d(\tau)+d(\bar{\tau})}{2}$$

and if we restrict to $t \in \mathbb{R}^*_{>0}$ we get (Ia). The next formula follows easily form 2.3.2

$$\phi_v \alpha_v^s(b(t)) = \phi_v \alpha_v^s \left(\begin{pmatrix} \frac{t}{u(t)} & * \\ & u(t) \end{pmatrix}\right) =$$

$$t^{-\nu(\tau)-\nu(\bar{\tau})-d(\bar{\tau})+1} \cdot u(t)^{d(\bar{\tau})-d(\tau)-2} \left(\frac{t}{u(t)^2}\right)^s \quad . \qquad \text{(Ib)}$$

The third factor is again very easy to compute

$$\langle u^v_{\alpha,\tau}, \mathrm{ad}\ k(t) Y_0 \rangle = \frac{t}{u(t)^2} \qquad \text{(Ic)}$$

(comp. |Ha3|, 1.4.1).

The computation of the last term is a little bit more amusing. We have to write $w_v = w_\tau \otimes w_{\bar{\tau}}$ and we have to look at

$$\langle e_{-d(\tau)} \otimes e_{d(\bar{\tau})}, \rho^v(k(t)) w_v \rangle =$$

$$\langle e_{-d(\tau)}, \rho^v_\tau(k(t)) w_\tau \rangle \cdot \langle e_{d(\bar{\tau})}, \rho^v_{\bar{\tau}}(k(t)) w_{\bar{\tau}} \rangle \quad .$$

We exploit the Cartan involution a second time and write

$$k(t) = b(t)^{-1} h(t) = b(t)^{-\theta} \cdot h(t)^{-1} \quad .$$

We get for our factor

$$<\rho_\tau(b(t)^\theta)e_{-d(\tau)},\rho^\vee_\tau(h(t)^{-1})w_\tau> \cdot <\rho_{\bar\tau}(b(t))e_{d(\bar\tau)},\rho^\vee_{\bar\tau}(h(t))w_{\bar\tau}> \quad .$$

Now all vectors are eigenvectors with respect to the transformation which we apply to them. We have

$$\rho^\vee_\tau(h(t)^{-1})w_\tau = t^{m_{\tau'}}w_\tau$$

where τ' is one of the two places above τ and then

$$\rho^\vee_{\bar\tau}(h(t))w_{\bar\tau} = t^{-m_{\bar\tau'}}\, w_{\bar\tau} \quad .$$

Multiplying all terms together we find for the above product (without the scalar product factors)

$$u(t)^{-d(\tau)}t^{-\nu(\tau)} \cdot t^{m_{\tau'}} \cdot u(t)^{-d(\bar\tau)}t^{d(\bar\tau)+\nu(\bar\tau)} \cdot t^{-m_{\bar\tau'}} =$$

$$u(t)^{-d(\tau)-d(\bar\tau)}\, t^{d(\bar\tau)+\nu(\bar\tau)-\nu(\tau)+m_{\tau'}-m_{\bar\tau'}} \qquad \text{(Id)}$$

If we multiply (Ia-d) together we get

$$t^{\frac{d(\tau)+d(\bar\tau)}{2}+2+\nu(\bar\tau)-\nu(\tau)+m_{\tau'}-m_{\bar\tau'}} \cdot u(t)^{-2d(\tau)-4} \cdot \left(\frac{t}{u(t)^2}\right)^s$$

But we have still the relation 5.4.2 and 5.4.3, namely

$$m_{\tau'} + m_{\tau''} = d(\tau) + 2\nu(\tau)$$

$$m_{\tau'} + m_{\bar\tau'} = \frac{d(\tau)+d(\bar\tau)}{2} + \nu(\tau) + \nu(\bar\tau) \quad .$$

Hence

$$m_{\tau'}-m_{\bar\tau'} = m_{\tau'}-m_{\tau''} + \frac{d(\tau)-d(\bar\tau)}{2} + \nu(\tau)-\nu(\bar\tau)$$

and the exponent of t simplifies to

$$t^{d(\tau)+m_{\tau'}-m_{\tau''}+2} = t^{2\mu(\tau)+2} .$$

Hence the contribution at v is

$$\langle e_{-d(\tau)}, w_\tau\rangle \cdot \langle e_{d(\bar{\tau})}, w_{\bar{\tau}}\rangle \cdot 2^{d(\tau)+1+s} \phi_v \alpha_v^s \left(\begin{pmatrix} \delta_v^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right) \times$$

$$\int_0^\infty (1+t^2)^{-d(\tau)-2-s}\, t^{2+\mu(\tau)+s} \frac{dt}{t} .$$

Substituting again $1+t^2 = 1/w$ we get as our local term

$$\langle e_{-d(\tau)}, w_\tau\rangle \cdot \langle e_{d(\bar{\tau})}, w_{\bar{\tau}}\rangle\; 2^{d(\tau)+1+s}\, \phi_v \alpha_v^s \left(\begin{pmatrix} \delta_v^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right) \times$$

$$\frac{\Gamma(d(\tau)+1-\mu(\tau) + \frac{s}{2}) \cdot \Gamma(1+\mu(\tau) + \frac{s}{2})}{\Gamma(d(\tau)+2+s)} .$$

We have to evaluate at $s = 0$, then

$$\phi_v \left(\begin{pmatrix} \delta_v^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right) = i_v(\delta_v)^{\nu(\tau)-1} \cdot \overline{i_v(\delta_v)}^{\,-d(\bar{\tau})-\nu(\bar{\tau})} .$$

Now we recall that $w_\tau = \rho^v(g_v^{-1}) m_\tau(\mu^{-1})$ and hence

$$\langle e_{-d(\tau)}, \rho_\tau^v(g_v^{-1}) m_\tau(\mu^{-1})\rangle \cdot \langle e_{d(\bar{\tau})}, \rho_{\bar{\tau}}(g_v^{-1}) m_{\bar{\tau}}(\mu^{-1})\rangle =$$

$$i_v(\delta_v)^{-\nu(\tau)} \cdot \overline{i_v(\delta_v)}^{\,d(\bar{\tau})+\nu(\bar{\tau})} .$$

So the local contribution at the place v is eventually

$$<e_{-d(\tau)},m_\tau(\mu^{-1})> \cdot <e_{d(\bar{\tau})},m_{\bar{\tau}}(\mu^{-1})> \cdot 2^{d(\tau)+1} \cdot i_v(\delta_v)^{-1}$$

$$\frac{\Gamma(d(\tau)+1-\mu(\tau)) \cdot \Gamma(1+\mu(\tau))}{\Gamma(d(\tau)+2)}$$

Multiplying all terms together we get the assertion of the theorem 3 in case B.

5.7. Theorem 3 has implications for special values of L-functions which are attached to algebraic Hecke characters.

The systems of one dimensional vector spaces

$$\{H^{\bullet}(\check{\mu},M(\lambda))(\gamma)\}_{(\lambda,\gamma)\in\Lambda_1} \quad , \quad \{M^{v}(\lambda)(\mu^{-1})\}_{(\lambda,\mu)\in\Lambda_2}$$

have a naturally defined $\mathbb{Q}$-structure. Hence they admit $\mathbb{Q}$-rational systems of generators

$$e_1(\lambda,\gamma) \in H^{\bullet}(\check{\mu},M(\lambda))(\gamma) \quad , \quad m^{v}(\mu^{-1}) \in M^{v}(\lambda)(\mu^{-1}) \quad .$$

Using these generators we defined the intertwining operators (see 5.3.6.3)

$$I(\phi,\eta,e_1(\lambda,\gamma),m^{v}(\mu^{-1}),i_H) \;:\; V_\phi \to W_\eta \quad .$$

We introduce the sets

$$\Phi = \{(\phi,\lambda,\gamma) \mid \gamma \in \mathrm{Coh}(\lambda), \mathrm{type}(\phi) = \gamma\}^{\cdot}$$

and

$$\underline{H} = \{(\eta,\lambda,\mu) \mid \mu \text{ weight in } M(\lambda), \mathrm{type}(\eta) = \mu\}^{\cdot} \; ,$$

The Galois group acts upon Φ and $\underline{H}$ and we have obvious $\mathbb{Q}$-structures on

$$\{V_\phi\}_{(\phi,\lambda,\gamma)} \qquad \{W_\eta\}_{(\eta,\lambda,\mu)}$$

and the above intertwining operator is a $\mathbb{Q}$-rational system of maps. Since the system of local operators

$$I^{loc}(\phi,\eta) \;:\; V_\phi \to W_\eta$$

is also defined over $\mathbb{Q}$ we get

<u>Corollary 5.7.1</u>: <u>If</u> $(\phi,\lambda,\gamma) \in \Phi$ <u>and</u> $(\eta,\lambda,\mu) \in \underline{H}$ <u>and if in addition</u>

$$\phi|Z(A) = \eta|Z(A)$$

<u>then</u>

$$L(\phi,\eta) = c_\infty(e_1(\lambda,\gamma), m^\vee(\mu^{-1}) i_H, \Omega_0) \cdot \frac{L_E(\eta \cdot \phi_1 \circ N, 0)}{L_F(\phi^{(1)}, 0)} \in \overline{\mathbb{Q}}$$

<u>and for</u> $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ <u>we have</u>

$$L(\phi^\sigma, \eta^\sigma) = L(\phi,\eta)^\sigma \quad .$$

This is an obvious consequence of theorem 3 and our above considerations. But it is not yet the final result we are aiming at. We want to have a more direct result about the ratio of the L-values and clarify the factor c_∞. Of course if we want a result of the above type for the ratio of the L-functions we may ignore rational factors in c_∞.

We recall the construction of a rational system of generators $e_1(\lambda,\gamma) = e_1(\mu)$ (see 2.4.1). We select an order $<$ on the set $\{\tau | \tau : F \to \overline{\mathbb{Q}}\}$ and for $\mu = (\lambda,\gamma)$ we defined $\mathbb{Q}(\mu)$ such that

$$\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}(\mu)) = \mathrm{Stab}(\mu) \quad .$$

Then we defined a class $d(\mu) \in \mathbb{Q}(\mu)^*/(\mathbb{Q}(\mu)^*)^2$ and we selected a root $\delta(\mu) \in \bar{\mathbb{Q}}^*$, i.e. $\delta(\mu)^2 = d(\mu)$. Then we saw that

$$e_1(\lambda,\gamma) = e(\lambda,\gamma,<) \cdot \delta(\mu)$$

is defined over $\mathbb{Q}(\mu)$ and its transforms under the action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ form a rational system of generators, i.e. we put

$$e_1(\mu^\sigma) = e_1(\lambda^\sigma,\gamma^\sigma) = e_1(\lambda,\gamma)^\sigma \quad .$$

Then we define

$$\delta(\mu^\sigma) = e_1(\mu^\sigma)/e(\lambda^\sigma,\gamma^\sigma,<)$$

this means that we selected for all μ^σ in the orbit a root $\delta(\mu^\sigma)$ of $d(\mu^\sigma) = d(\mu)^\sigma$. This system $e_1(\lambda,\gamma)$ will be used from now on.

The above rule of selecting a square root $\delta(\mu^\sigma)$ out of $d(\mu^\sigma) = d(\mu)^\sigma$ can also be described in a slightly different way. We choose an order $<$ on the set $\{\tau | \tau : F \to \bar{\mathbb{Q}}\}$ and a root

$$\delta(\mu)^2 = d(\mu)$$

for one μ . The order induces an order on the set $I(\gamma)$ introduced in 2.4.1. For $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ we may compare the order $<^\sigma$ and $<$ on $I(\gamma)$ and we put

$$\delta(\mu^\sigma) = \delta(\mu)^\sigma$$

provided these two orders have the same sign. In our case the set $I(\gamma)$

is also the CM-type defined by γ , so we do not need to refer to the cohomological interpretation if we want to define the system $\{\delta(\mu^{\sigma})\}$.

We now separate the cases again

<u>Case A</u>: In this case we have of course that the Galois group acts trivially on the types and hence we have $\mathbb{Q}(\mu) = \mathbb{Q}$ and it is clear that $d(\mu) = d_F$ the discriminant of our field. Hence we do not have any problem of "consistent extraction of square roots" and put

$$\sqrt{d_F} = \delta(\mu) > 0 \quad .$$

The element Δ has to be totally positiv, at each real place v we may choose $\delta_v^2 = \Delta_v$ also with $\delta_v > 0$ then

$$\prod_{v \in S_\infty} \delta_v^{-1} = (N_{F/\mathbb{Q}}(\Delta))^{-1/2} > 0 \quad .$$

Hence we obtain from our corollary

$$\frac{\Omega(\{\delta_v\},<)}{\Omega_o} \cdot \langle E(\gamma), m^v(\mu^{-1})\rangle \cdot \sqrt{d_F} \cdot N_{F/\mathbb{Q}}(\Delta)^{-1/2} \cdot \frac{L(\eta^{-1}\cdot\phi_1 \circ N_{E/F}, 0)}{L(\phi^{(1)}, 0)} \in \bar{\mathbb{Q}}$$

provided the character $\eta \cdot \phi_1 \circ N_{E/F}$ is even and this number behaves in the correct way under the Galois group action. But

$$\frac{\Omega(\{\delta_v\},<)}{\Omega_o} = \pm 1$$

and this number does not depend on η , ϕ , so we may drop it.

It is quite clear that $\langle E(\gamma), m^v(\mu^{-1})\rangle$ behaves the right way under the action of the Galois group and we shall see later that it is $\neq 0$. Hence we may also drop it. Hence we get that under our assumptions on ϕ ,

η , under the assumption that $\eta \cdot \phi_1 \circ N_{E/F}$ is even and $\eta|Z(A) = \phi|Z(A)$:

Cor. 5.7.2. A

$$\tilde{L}(\eta,\phi) = \sqrt{d_F} \cdot N_{F/\mathbb{Q}}(\Delta)^{-1/2} \cdot \frac{L(\eta^{-1} \cdot \phi_1 \circ N_{E/F},0)}{L(\phi^{(1)},0)} \in \overline{\mathbb{Q}}$$

and

$$\tilde{L}(\eta^\sigma,\phi^\sigma) = (\tilde{L}(\phi,\eta))^\sigma .$$

This is of course well known and a special case of much more general results (|Sie|). But nevertheless it may be helpful if we translate it a little bit further into classical language. We have seen in 5.4. that

$$M = \bigotimes_{\tau : F \to \overline{\mathbb{Q}}} M(d,\nu)$$

where $d \equiv 0 \bmod 2$. This allows us to choose $\nu = d/2$ then the central character of the representation becomes trivial and we do not loose anything.

In this case we have

$$\text{type}(\phi_1) = (\ldots,-\frac{d}{2} - 1,\ldots)$$

$$\text{type}(\phi_2) = (\ldots, \frac{d}{2} + 1,\ldots)$$

$$\text{type}(\phi^{(1)}) = (\ldots,-d-2,\ldots)$$

If $|\alpha| : F^*\backslash I_F \to \mathbb{R}^*_{>0}$ is the Tate character then we find

$$\phi_1 = |\alpha|^{\frac{d}{2}+1} \chi_1$$

$$\phi_2 = |\alpha|^{-\frac{d}{2}-1} \chi_2$$

where χ_1 , χ_2 are Dirichlet-characters whose parity at the infinite places of F is $\frac{d}{2} + 1$ + parity of ϕ_1 (= parity of ϕ_2).

The character $\eta : E^*\backslash I_E \to \mathbb{C}^*$ is a Dirichlet character on $E^*\backslash I_E$. Our assumptions can be stated in terms of χ_1 , χ_2 , η and d as follows

(i) $\phi|Z(A) = \eta|Z(A) \iff \chi_1\chi_2 = \eta|Z(A) = \eta|I_F$.

(ii) The parity of $\eta \quad \phi_1 \quad N_{E/F}$ is even $\iff$

$$\text{parity of } \eta \cdot \chi_1 \quad N_{E/F} = \text{parity of } \frac{d}{2} + 1 \quad .$$

If these two conditions for η , χ_1 , χ_2 , d are fulfilled then

$$d_F^{1/2} \cdot N_{F/\mathbb{Q}}(\Delta)^{-1/2} \cdot \frac{L_E(\eta^{-1} \cdot \phi_1 \circ N_{E/F}, \frac{d}{2} + 1)}{L_F(\chi_1/\chi_2, d+2)} \in \overline{\mathbb{Q}}$$

and this number transforms the right way under the action of the Galois group. This is now really a classical result.

<u>Case B</u>: We begin by discussing the expression

$$\frac{\Omega(\{\delta_v\},<)}{\Omega_0} \cdot \prod_{v \in S_\infty} i_v(\delta_v^{-1}) \quad .$$

We may look at it in the following way. The CM-type defined by our character γ defines a "half-norm" of Δ

$$N_{\mathcal{T}}(\Delta) = \prod_{\tau \in \mathcal{T}(\gamma)} \tau(\Delta) \in \mathbb{C}^* \quad .$$

Choosing the δ_v means that we extract a square root out of each $\tau(\Delta)$ hence the product of the δ_v is a square root out of $N_{\mathcal{T}}(\Delta)$. If we fix $<$ and Ω_0 then we may normalize this root in such a way that the sign becomes $+1$, i.e. we define

$$N_{\mathcal{T}(\gamma)}(\Delta)^{-1/2} = \mathrm{Sqrt}(N_{\mathcal{T}(\gamma)}(\Delta)^{-1},<,\Omega_o) = \prod_{v\in S_\infty} i_v(\delta_v^{-1})$$

provided we have chosen the δ_v such that

$$\frac{\Omega(\{\delta\}_v,<)}{\Omega_o} = +1 \quad .$$

We shall show later that $\langle E(\gamma), m^\vee(\mu^{-1})\rangle \neq 0$ and since we know that

$$\langle E(\gamma), m^\vee(\mu^{-1})\rangle^\sigma = \langle E(\gamma^\sigma), m(\mu^{-\sigma})\rangle$$

we can conclude that under our above conventions:

<u>Cor. 5.7.2. B</u>:

$$\tilde{L}(\phi,\eta) = \delta(\mu)\cdot N_{\mathcal{T}(\gamma)}(\Delta)^{-1/2} \cdot \frac{L_E(\eta^{-1}\phi_1 \circ N_{E/F},0)}{L_F(\phi^{(1)},0)} \in \bar{\mathbb{Q}}$$

and for $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ we have

$$\tilde{L}(\phi^\sigma,\eta^\sigma) = (\tilde{L}(\phi,\eta))^\sigma \quad .$$

We want to illustrate the implications of this result for the behavior of special L-values under quadratic extensions of the ground field.

So we start from the general situation:

E/F is a quadratic extension of a totally imaginary field F .

$\psi : E^*\backslash I_E \to \mathbb{C}^*$ is an algebraic Hecke character on E for which 0 is critical .

The assumption that 0 is critical for ψ can be reformulated as follows: If we write

$$\mathrm{type}(\psi) = (\ldots n(\tau')\ldots)_{\tau' : E \to \mathbb{C}}$$

then we can arrange each pair of conjugate embedding $\{\tau', \bar{\tau}'\}$ in such a way that

$$n(\tau') < 0 \leq n(\bar{\tau}') \quad .$$

This defines also a CM-type $\mathcal{T}'(\psi)$ on E namely

$$\mathcal{T}'(\psi) = \{\tau' : E \to \bar{\mathbb{Q}} \mid n(\tau') < 0\} \quad .$$

Let us assume that this CM-type is induced by a CM-type $\mathcal{T}$ on F, by this we mean that

$$\mathcal{T}'(\psi) = \{\tau' : E \to \bar{\mathbb{Q}} \mid \tau' | F \in \mathcal{T}\} \quad .$$

This condition is of course equivalent to the condition that for each pair τ', τ'' of embeddings of E which lie over one embedding τ we have that both $\tau', \tau'' \in \mathcal{T}'(\psi)$ or none of them is in $\mathcal{T}'(\psi)$.

We claim that, under the assumption that $\mathcal{T}'(\psi)$ is induced by a CM-type on F, we may write

$$\psi = \eta^{-1} \cdot \phi_1 \circ N_{E/F}$$

with the conventions of corollary 5.7.2 on ϕ_1 and η .

This is simply a question of types. We have to construct a coefficient system

$$M = \bigotimes_{\tau : F \to \overline{\mathbb{Q}}} M(d(\tau), \nu(\tau))$$

s.t. we can find a $\phi \in \mathrm{Coh}(M)$ and a character $\eta : H(\mathbb{Q})\backslash H(A) \to \mathbb{C}^*$ such that $\psi = \eta^{-1} \cdot \phi_1 \circ N_{E/F}$ and the assumptions of theorem 3 are satisfied. Let us denote the central character of the representation $\rho : G \times_{\mathbb{Q}} \overline{\mathbb{Q}} \to GL(M)$ by ζ, so

$$\mathrm{type}(\zeta) = (\ldots z(\tau) \ldots)_{\tau : F \to \overline{\mathbb{Q}}}$$

and $z(\tau) = d(\tau) + 2\nu(\tau)$. To construct M we can also prescribe $d(\tau)$, $z(\tau)$ if we observe the parity condition $d(\tau) \equiv z(\tau) \bmod 2$.

We concentrate on a pair $\tau, \overline{\tau}$ of conjugate embedding of F into $\overline{\mathbb{Q}}$. We have for $\phi \in \mathrm{Coh}(M)$, $\phi = (\phi_1, \phi_2)$

$$\mathrm{type}(\phi_1)_\tau = \frac{z(\tau) - d(\tau)}{2} - 1$$

$$\mathrm{type}(\phi_1)_{\overline{\tau}} = \frac{z(\overline{\tau}) + d(\overline{\tau})}{2} .$$

(Of course we want that $\mathcal{T} = \mathcal{T}(\mathrm{type}(\phi))$ in the sense of 5.6..)
We have also η to our disposal, let us write

$$\mathrm{type}(\eta) = (\ldots e_{\tau'}, e_{\tau''}, e_{\overline{\tau}'}, e_{\overline{\tau}''}, \ldots)$$

where $\tau'|F = \tau''|F = \tau \in \mathcal{T}$. We have by 5.4.2. that

$$e_{\tau'} + e_{\tau''} = z(\tau)$$

$$e_{\overline{\tau}'} + e_{\overline{\tau}''} = z(\overline{\tau}) .$$

So we may write, if we take 5.4.3 into account

$$e_{\tau'} = \frac{z(\tau)}{2} + \alpha \quad , \quad e_{\tau''} = \frac{z(\tau)}{2} - \alpha$$

$$e_{\bar{\tau}'} = \frac{z(\bar{\tau})}{2} - \alpha \quad , \quad e_{\bar{\tau}''} = \frac{z(\bar{\tau})}{2} + \alpha$$

with $\alpha \in \frac{1}{2}\mathbb{Z}$. We have to satisfy

$$|2\alpha| \leq \min(d(\tau), d(\bar{\tau}))$$

(see 5.4.) but beyond that there is no further restriction. Then for $\psi = \eta^{-1} \cdot \phi_1 \circ N_{E/F}$ we find on the level of types

$$n(\tau') = -\frac{d(\tau)}{2} - \alpha - 1$$

$$n(\tau'') = -\frac{d(\tau)}{2} + \alpha - 1$$

$$n(\bar{\tau}') = \frac{d(\bar{\tau})}{2} + \alpha$$

$$n(\bar{\tau}'') = \frac{d(\bar{\tau})}{2} - \alpha$$

So we put $\alpha = \frac{n(\tau'')-n(\tau')}{2}$, $d(\tau) = -n(\tau')-n(\tau'')$ and $d(\bar{\tau}) = n(\bar{\tau}')+n(\bar{\tau}'')$. Then the necessary parity conditions are satisfied and our assumption $n(\tau'), n(\tau'') < 0$ and $n(\bar{\tau}'), n(\bar{\tau}'') \geq 0$ also imply that

$$|2\alpha| \leq \min(d(\tau), d(\bar{\tau})) \quad .$$

We still have to choose the central character ζ of our coefficient system M (which is a rather irrelevant datum). As in 4.3 we choose

$$z(\tau) = -d(\tau)-2 \quad , \quad z(\bar{\tau}) = d(\bar{\tau}) \quad .$$

Then we have that

$$\zeta = \gamma = (\ldots, z(\tau), \ldots)$$

is the type of an algebraic Hecke character, where now $\gamma \in \mathrm{Coh}(M)$. After making this choice we define the type of η by means of the $e_{\tau'}, e_{\tau''} \dots$, this is the type of an algebraic Hecke character because type(ψ) is the type of an algebraic Hecke character. Now we choose ϕ_1 arbitrary with the above specification of the type, then we define η by

$$\psi = \eta^{-1} \cdot \phi_1 \circ N_{E/F}$$

and we adjust ϕ_2 such that $\phi_1\phi_2|I_F = \eta|I_F$. This proves that for the given ψ we can construct a coefficient system M and find an η s.t. $\tilde{L}_E(\psi,0)$ occurs as the numerator of $\tilde{L}(\eta,\phi)$, hence the special value can be computed in terms of a value of a L-function over F.

We have to show that under our assumptions we have

$$\langle E(\gamma), m^{v}(\mu^{-1})\rangle \neq 0 \quad .$$

To see this we have to show that for each $\tau : F \to \bar{\mathbb{Q}}$ we have

$$\langle E_\tau(\gamma), m_\tau^{v}(\mu^{-1})\rangle \neq 0 \quad .$$

The point is that the two components in this product are weight vectors with respect to the two tori

$$T_o \times_{F,\tau} \bar{\mathbb{Q}} \quad , \quad H_o \times_{F,\tau} \bar{\mathbb{Q}} \quad .$$

The torus $T_o \times_{F,\tau} \bar{\mathbb{Q}}$ sits in two Borel subgroups B_o^+ and B_o^- and since H_o/F comes from a field extension we have

$$H_o \times_{F,\tau} \bar{\mathbb{Q}} \cap B_o^{\pm} = Z_o \times_{F,\tau} \bar{\mathbb{Q}} \quad .$$

On the other hand we know that $E_\tau(\gamma)$ is indeed a weight vector for one of the two groups B_o^+ or B_o^-. But then it becomes clear that

$$\langle E_\tau(\gamma), m_\tau(\mu^{-1})\rangle = 0$$

would imply that

$$\langle bE_\tau(\gamma), h\cdot m_\tau(\mu^{-1})\rangle = \langle E_\tau(\gamma), b^{-1}hm_\tau(\mu^{-1})\rangle = 0$$

where b has to be taken in $B_o^+(\overline{\mathbb{Q}})$ or $B_o^-(\overline{\mathbb{Q}})$ and $h \in H_o(\overline{\mathbb{Q}})$. But under our present assumptions we have that

$$B_o^\pm(\overline{\mathbb{Q}}) \cdot H_o(\overline{\mathbb{Q}}) \subset G_o(\overline{\mathbb{Q}})$$

is Zariski dense hence we get

$$\langle E_\tau(\gamma), gm_\tau(\mu^{-1})\rangle = 0$$

for all $g \in G_o(\overline{\mathbb{Q}})$ which is a contradiction.

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