METRICAL SHELLINGS OF SIMPLICIAL COMPLEXES

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1. Introduction

A theorem of L. Solomon and J. Tits [5] states that a Tits building is homologically a wedge of spheres, that is, it has non-trivial homology only in dimension zero and the highest dimension. This result enabled Solomon to recognize the Steinberg character of a finite group with a BN-pair as the character of a homology representation. On the other hand, there exists a quite elaborate theory of so-called Cohen Macauly complexes. A simplicial complex is called Cohen Macauly if the star (or link) of every simplex, including the whole complex, is homologically a wedge of spheres. A basic theorem of G. Reisner relates this property to the Cohen Macauly property of a certain ring associated with the complex. An important combinatorial tool in this theory is the notion of shellability. This is a property of complexes which easily implies the Cohen Macauly property, on the homological as well as on the ring theoretic side. The reader may consult [1] and [2] for further information.

In the paper [1], A. Björner introduces the notion of shellability into the theory of buildings. He shows that a building is shellable and thus reproves the Solomon-Tits theorem.

The main purpose of this note is to point out that the proof only depends on a metrical property of the complex, namely the existence of certain "projection maps" from the set of all chambers (maximal simplices) onto the set of chambers containing a fixed simplex. For buildings, the projections can be found in [6], 3.19.

After I had completed this note, A. Björner has kindly informed me of a new version of his paper [1]. Following a suggestion of J. Tits he gives a proof for the shellability of buildings which is very similar to the proof given below. Both proofs only use the projection maps and nothing more about buildings. It should be pointed out that also the Solomon-Tits result only relies on the projection maps.

A. Dress has given a new approach to the projection maps [4] which was inspired by a more recent paper [7] of Tits. The spirit of this work of Tits is to reformulate and generalize a large part of the "abstract theory" of buildings without using the notion of an apartement. As an intermediate step in his proof, Dress uses a certain "exchange condition" for numbered complexes (in fact, for "chamber systems") which generalizes the well known [3] exchange condition for Coxeter groups. This condition is easily shown to be true for buildings (using the criterion of [7], not using apartements). In a second step it is shown that the exchange property implies the existence of the projection maps.

Comparing Björner's statement to the more general statements in sections 2 and 3 below, the following question naturally arises: Do there exist many complexes with the exchange property or with projection maps which are not buildings? There are two answers.

The answer is no for complexes which are "homogeneous" in the sense that the diameter of the star of a simplex of codimension 2 only depends on the "type" (see below) of that simplex. This homogeneity holds for buildings, more generally for complexes of type M for some Coxeter matrix M (cf. [7]), and for complexes admitting a chamber transitive automorphism group. We shall show in a subsequent paper that a homogenous complex with projection maps actually is a building.

In general, the answer to the above question is yes. Firstly, there exist non homogeneous complexes satisfying the exchange property. Secondly, in the general case the existence of projections does not imply the exchange property.

It is known, for example from the theory of tilings or tesselations, that also complexes with a lower degree of homogeneity than buildings deserve attention from a geometrical point of view. They can even be classified in certain cases. Because of the importance of the exchange condition and the projection maps in the theory of buildings, it seems reasonable to formulate the existence of projection maps as a geometric axiom of its own.

Apart from presenting still another proof of the Tits-Solomon theorem, this note is intended to be a contribution towards characterizing the class of complexes with projection maps also in the non homogeneous case.

2. The result

We recall some standard definitions (see e. g. [3], Chap. IV, Exercises No. 15, 20). A numbered complex is a simplicial complex Δ together with a function "type" from the vertex set of Δ to a set I such that the restriction of type to every chamber (maximal simplex) is bijective. (The elements of I should be viewed as names or colors of the vertices.) The map type induces a morphism of complexes, also called type, from Δ to the power set $P(I)$ of I. We set

$$\text{cotype } A = I \setminus \text{type } A \quad , A \in \Delta .$$

For $i \in I$, two chambers C, C' are i-adjacent if $\text{cotype}(C \cap C') = i$. We write $C \overset{i}{\sim} C'$ if $C = C'$ or C, C' are i-adjacent. A gallery of length m is a sequence $(C_o, \ldots, C_m, i_1, \ldots, i_m)$ such that $C_{\nu-1} \overset{i_\nu}{\sim} C$ for $\nu = 1, \ldots, m$. (Usually, a gallery is defined just as a sequence $(C_o, \ldots, C_m)$ such that $C_{\nu-1} \overset{i}{\sim} C_\nu$ for all ν and some $i = i_\nu$. However, we do not assume that $C \overset{i}{\sim} C'$ holds for at most one i, and if $C \overset{i}{\sim} C' \overset{j}{\sim} C$, $i \neq j$ we have to regard $(C, C'; i)$ as different from $(C, C'; j)$.) A set of chambers is called connected if any two of its elements can be joined by a gallery inside that set. We assume Δ to be strongly connected, i. e., for every simplex $A \in \Delta$ (including $A = \emptyset$), the set $C(A)$ of chambers containing A,

$$C(A) = \{C \mid C \text{ a chamber}, C \supseteq A\}$$

is connected. A gallery $(C, \ldots, D; \ldots)$ is called a geodesic if it has smallest possible length among all galleries from C to D. This length $d(C, D)$ is called the distance between C and D.

The crucial condition for our purposes is the following.

Exchange condition. Let $(C_o, \ldots, C_m; i_1, \ldots, i_m)$ be a geodesic and D a chamber, $i \in I$ such that $C_m \overset{i}{\sim} D$, $C_m \neq D$. If $(C_o, \ldots, C_m, D; i_1, \ldots, i_m, i)$ is not a geodesic, then there exists a gallery of the form $(C'_o, \ldots, C'_{m-1}, i_1, \ldots, \hat{i}_\nu, \ldots, i_m)$ (i_ν omitted) such that $C'_o = C_o$, $C'_{m-1} \overset{i}{\sim} C_m$.

If Δ is the Coxeter complex of a Coxeter group W, the set of chambers is identified with W, we can assume $C_o = 1$, furthermore we necessarily have $C'_{m-1} = D$, because D is the only chamber i-adjacent to C_m. These facts imply that for Δ a Coxeter complex, the above condition is equivalent to the usual exchange condition for the corresponding Coxeter group (cf. [3], Chap. IV, Exer. No. 16 a)-d)).

The exchange property has important implications about the sets $C(A)$ of chambers containing a fixed simplex A.

Firstly, any $C(A)$ is convex, i. e. $C_\nu \in C(A)$ for any two elements $C,D \in C(A)$ and any geodesic $(C_o=C,C_1,\dots,C_m=D;i_1,\dots,i_m)$, and all ν. For buildings, this is [6], 3.14.

Secondly, the sets $C(A)$ have the following property.

Gate property. Given a chamber E and a simplex A, let C be a chamber such that

$$C \in C(A)\ ,\ d(E,C) \leq d(E,D) \quad \text{for all} \quad D \in C(A)\ .$$

Then $d(E,D) = d(E,C) + d(C,D)$ for all $D \in C(A)$. (In particular, C is unique. It is called by Tits the projection of E onto A.)

For buildings, this is [6], 3.19.6, in the general case, it is [4], § 5, Satz 8.

Finally we recall the notion of shellability (cf. [1], 1.1 and 4.14). A shelling of Δ is a well ordering $\leq$ of its set of chambers such that the complex

$$P(C) \cap \bigcup_{\substack{C' \leq C \\ C' \neq C}} P(C')$$

is pure of codimension 1 (in $P(C)$) for all chambers C. Here $P(C)$ denotes the power set of C, i. e. the complex consisting of all simplices contained in C. A subcomplex $\Gamma \subseteq P(C)$ is pure of codimension 1 if it is the union of $P(C \setminus \{x\})$, where x ranges over certain vertices of C. If R denotes the simplex consisting of these vertices then Γ consists exactly of those simplices contained in C that do not contain R. This remark leads to the following criterion (see [1], Proposition 1.2 and Remark 4.14).

Lemma. A well ordering $\leq$ of the set $\mathcal{C}$ of chambers of Δ is a shelling if and only if there exists a map

$$R : \mathcal{C} \to \Delta$$

such that

$$\text{(o)} \quad R(C) \subseteq C \quad \text{for all} \quad C \in \mathcal{C}$$

$$\text{(i)} \quad \Delta = \bigcup_{C \in \mathcal{C}} \{A \in \Delta \mid R(C) \subseteq A \subseteq C\}$$

$$\text{(ii)} \quad R(C) \subseteq D \Rightarrow C \leq D \ .$$

For the proof of the "if"-part, one shows that (o),(i),(ii) implies

$$\bigcup_{C' \leq C} P(C') = \bigcup_{C' \leq C} \{A \in \Delta \mid R(C') \subseteq A \subseteq C'\}$$

for all chambers C. Notice that the union in (i) is necessarily disjoint by (ii). The map R is uniquely determined by $\leq$.

Now we are ready to state and easily prove our result.

Proposition. A strongly connected numbered complex which satisfies the exchange condition is shellable.

Proof: Fix a chamber E and consider the following relation on the set of chambers $\mathcal{C}$:

$$d(E,D) = d(E,C) + d(C,D) \quad , \ C,D \in \mathcal{C} \ .$$

Is is readily checked that this is a partial ordering, i. e. transitive and antisymmetric. In fact, this holds for any metric space $(\mathcal{C},d)$ and $E \in \mathcal{C}$.

Choose a well ordering $\underset{m}{\leq}$ on $\{C \in \mathcal{C} \mid d(E,C) = m\}$, for all natural numbers m. Define a relation $\leq$ on all of $\mathcal{C}$ by

$$\begin{aligned} C \leq D \ :\Leftrightarrow \ & d(E,C) < d(E,D) \\ & \text{or } d(E,C) = d(E,D) =: m \\ & \quad \text{and } C \underset{m}{\leq} D \ . \end{aligned}$$

Then $\leq$ is a well ordering and extends the above partial ordering.

<u>Claim</u>: Any well ordering $\leq$ of $\mathcal{C}$ such that $d(E,D) = d(E,C) + d(C,D)$ implies $C \leq D$ is a shelling of Δ. The corresponding map $R : \mathcal{C} \to \Delta$ (see the lemma) is given by

$$R(C) = \text{face of } C \text{ of cotype } J(C) ,$$

where $J(C) := \{i \in I \mid d(E,C) \leq d(E,D) \text{ for all } D \in \mathcal{C} \text{ such that } C \overset{i}{\sim} D\}$.

We first show that for fixed C we have

$$d(E,D) = d(E,C) + d(C,D) \quad \text{for all} \quad D \in \mathcal{C}(R(C)) .$$

By the gate property of $\mathcal{C}(R(C))$, there exists $C' \in \mathcal{C}(R(C))$ having this property. We have to show $C' = C$. Suppose $C' \neq C$ and choose a geodesic $(C' = D_o,\ldots,D_{m-1},D_m = C ; i_1,\ldots,i_m)$. The convexity of $\mathcal{C}(R(C))$ implies that $D_\nu \supseteq R(C)$ for all ν , i. e. $i_\nu \in J(C)$ for all ν . In particular, for $D = D_{m-1}$ we have

$$\begin{aligned} d(E,D) &= d(E,C') + d(C',D) \\ &= d(E,C') + d(C',C) - 1 \\ &= d(E,C) - 1 , \end{aligned}$$

and $C \overset{i}{\sim} D$, $i \in J(C)$. This contradicts the definition of $J(C)$.

We have just shown that property (ii) of the above lemma holds.

For the proof of (i), let $A \in \Delta$ be given. Choose $C \in \mathcal{C}(A)$ such that $d(E,C) \leq d(E,D)$ for all chambers $D \in \mathcal{C}(A)$. In particular, $d(E,C) \leq d(E,D)$ for all D such that $C \overset{i}{\sim} D$, $i \in$ cotype A. This means cotype $A \subseteq J(C)$, i. e., $R(C) \subseteq A$.

3. A remark about the notion of shellability

In the preceeding proof, the particular choice of the well ordering of $\mathcal{C}$ was of no importance. Only the map R and a certain partial ordering of $\mathcal{C}$ had to be constructed explicitly. Inspection of Björner's lemma above shows that this is a general fact about shellings. We shall now explicate this remark.

In condition (o) and (i), only the map R occurs and not the ordering of $\mathcal{C}$. If in addition to such an R a partial ordering $\leq$ (or any relation on $\mathcal{C}$) satisfying (ii) is given, then any extension of $\leq$ to a well ordering also satisfies (ii) and therefore is a shelling. Thus we are led to the following alternative definition.

<u>**Definition.**</u> A shelling of Δ is a map $R : \mathcal{C} \to \Delta$ together with a partial ordering $\leq$ of $\mathcal{C}$ such that (o), (i) and (ii) above hold and such that $\leq$ admits an extension to a well ordering. A map $R : \mathcal{C} \to \Delta$ resp. a partial ordering of $\mathcal{C}$ occuring in a shelling is called a <u>shelling operator</u> resp. a <u>shelling order</u>.

Note that R is uniquely determined by $\leq$. This holds in the well ordered case, as was remarked above, and the general case immediately follows.

Conversely, given a shelling operator R, there exists a smallest partial ordering (obviously unique) $\underset{R}{\leq}$ such that $(R,\underset{R}{\leq})$ is a shelling. Of course, $\underset{R}{\leq}$ is the transitive relation generated by the relation "$C \subseteq R(D), C,D \in \mathcal{C}$", i. e., $C \underset{R}{\leq} D$ if and only if there exist $C_o = C, C_1, \ldots, C_m = D$ in $\mathcal{C}$ such that $R(C_{\nu-1}) \subseteq C_\nu$ for all $\nu = 1,\ldots,m$.

A map $R : \mathcal{C} \to \Delta$ is a shelling operator if and only if it satisfies (o), (i) and

(ii') The relation "$C \subseteq R(D)$" on $\mathcal{C}$ can be extended to a well ordering.

In particular, the transitive relation $\underset{R}{\leq}$ generated by "$C \subseteq R(D)$" must be antisymmetric, i. e. a partial ordering. Of course, not every partial ordering can be extended to a well ordering. The following criterion was pointed out to the author by A. Dress.

A partial ordering on a set X can be extended to a well ordering if and only if every nonempty subset Y of X contains at least one minimal element y, that is $z \in Y$, $z \leq y$ implies $z = y$.

We propose the name <u>partial well ordering</u> for such a partial ordering. Note that the "only if"-part is trivial.

The criterion is fulfilled in particular if there exists a map $e : X \to \mathbb{N}$ (or any well ordered set instead of $\mathbb{N}$) such that $x \leq y$, $x \neq y$ implies $e(x) < e(y)$. This was the case in the above proposition, with $e(C) = d(E,C)$.

We now can formulate a proposition which includes the above proposition and also a converse statement. We call Δ metrically shellable if for any chamber E the partial ordering given by $d(E,D) = d(E,C) + d(C,D)$, $C,D \in \mathcal{C}$, is a shelling order.

Proposition'. A strongly connected numbered complex Δ is metrically shellable if and only if all $\mathcal{C}(A)$, $A \in \Delta$ have the gate property.

Proof: The proof of the above proposition only used the gate property and the convexity of the sets $\mathcal{C}(A)$. Now it is immediately checked that the former implies the latter. Thus, the "if"-part is already proved. For the converse, let $A \in \Delta$ and $E \in \mathcal{C}$ be given. Consider the shelling operator $R = R_E$ of the shelling order "$d(E,D) = d(E,C) + d(C,D)$". By property (i), there exists $C \in \mathcal{C}$ such that $R(C) \subseteq A \subseteq C$. Property (ii) says that $d(E,D) = d(E,C) + d(C,D)$ holds for all $D \in \mathcal{C}(R(C))$, in particular, for all $D \in \mathcal{C}(A)$. Thus C is the projection of E onto A whose existence was to be shown.

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