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Isotropy representations, eigenvalues of a Casimir element, and commutative Lie subalgebras

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Introduction

In this paper, we work in the following setting. Let $\mathfrak{g}$ be a semisimple complex Lie algebra and $\mathfrak{h}$ a reductive algebraic subalgebra. The representation of $\mathfrak{h}$ in the complementary $\mathfrak{h}$-stable subspace $\mathfrak{m}$ of $\mathfrak{g}$ is called the isotropy representation of $(\mathfrak{g},\mathfrak{h})$. Let Φ be the Killing form on $\mathfrak{g}$ and $\Phi|_{\mathfrak{h}}$ its restriction to $\mathfrak{h}$. Associated with $\Phi|_{\mathfrak{h}}$, one has the Casimir element $C_{\mathfrak{h}} \in U(\mathfrak{h})$.

Our first aim is to describe eigenvalues of $C_{\mathfrak{h}}$ on $\mathfrak{h}$ and $\mathfrak{m}$. A simple but useful observation is that interrelation between these eigenvalues depends on the Dynkin indices of $\mathfrak{m}$ as $\mathfrak{h}$-module. Recall that to every simple ideal $\mathfrak{h}_i \subset \mathfrak{h}$ one may attach a non-negative rational number $\mathrm{ind}_{\mathfrak{h}_i}\mathfrak{m}$, the (Dynkin) index of the $\mathfrak{h}_i$-module $\mathfrak{m}$. If $\mathrm{ind}_{\mathfrak{h}_i}\mathfrak{m} = \mu_i$, then the eigenvalue of $C_{\mathfrak{h}}$ on $\mathfrak{h}_i$ is equal to $1/(1+\mu_i)$. Because $\mathrm{tr}_{\mathfrak{g}}(C_{\mathfrak{h}}) = \dim\mathfrak{h}$, this is already sufficient for calculating the remaining eigenvalue of $C_{\mathfrak{h}}$ in the isotropy-irreducible case or, more generally, if $C_{\mathfrak{h}}$ acts scalarly on $\mathfrak{m}$. It is also shown that in some cases one can quickly find the eigenvalues making use of the dual Coxeter numbers for $\mathfrak{g}$ and $\mathfrak{h}$.
In general, we have the following estimate. Let $\gamma_{\mathfrak{m}}$ be the maximal eigenvalue of $C_{\mathfrak{h}}$ on $\mathfrak{m}$ and $\mathfrak{m}_{\clubsuit}$ the corresponding eigenspace of $\mathfrak{m}$.

0.1 Theorem. 1. $\gamma_{\mathfrak{m}} \le 1/2$. *Moreover,* $\gamma_{\mathfrak{m}} = 1/2$ *if and only if* $\mathfrak{h}\oplus\mathfrak{m}_{\clubsuit}$ *is a* $\mathbb{Z}_2$*-graded reductive Lie subalgebra of* $\mathfrak{g}$.
2. *If the decomposition* $\mathfrak{g} = \mathfrak{h}\oplus\mathfrak{m}$ *is a* $\mathbb{Z}_2$*-grading, then* $C_{\mathfrak{h}}$ *always acts scalarly on* $\mathfrak{m}$ *(with eigenvalue 1/2).*

With a mild constraint, this can be extended to periodic gradings of $\mathfrak{g}$ of higher order. To avoid technical details, assume that $\mathfrak{g}$ is simple in the next theorem.

0.2 Theorem. *Let* Θ *be an automorphism of* $\mathfrak{g}$ *of order* m *and* $\mathfrak{g} = \oplus_{\mathbb{Z}_m}\mathfrak{g}_i$ *the corresponding periodic grading. Let* $C_0 \in U(\mathfrak{g}_0)$ *be the Casimir element with respect to* $\Phi|_{\mathfrak{g}_0}$*. Suppose* $\mathfrak{g}_0$ *is semisimple, if* $m \ge 3$*. Then the eigenvalue of* C_0 *on* $\mathfrak{g}_i$ *is equal to* $1/m$*, if* $\mathrm{g.c.d.}(i,m)=1$*, and (strictly) less than* d/m*, if* $\mathrm{g.c.d.}(i,m)=d>1$.

The second aim is to study eigenvalues of C_0 on $\wedge\mathfrak{g}_1$. This was inspired by an old result of B. Kostant on eigenvalues of the Casimir element C with respect to Φ on $\wedge\mathfrak{g}$ [Ko65]. The main result of that paper establishes a relation between eigenvalues of C on $\wedge^k\mathfrak{g}$ and commutative Lie subalgebras in $\mathfrak{g}$. In many aspects, the adjoint representation is convenient to regard as an isotropy representation of a symmetric space (= IRSS). This

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idea has led us to an extension of Kostant's results to arbitrary $\mathbb{Z}_2$-gradings and $\mathbb{Z}_m$-gradings ($m \geq 3$) with semisimple $\mathfrak{g}_0$. Adapting his proofs, we tried to argue in an "algebraically closed" fashion whenever possible. Still, at some crucial point we were forced to use (as Kostant did) a compact real form of $\mathfrak{g}$. One of the main results is:

0.3 Theorem. *Let ℓ_k be the maximal eigenvalue of C_0 on $\wedge^k\mathfrak{g}_1$. Then $\ell_k \leq k/m$. Moreover one has $\ell_k = k/m$ if and only if there exists a k-dimensional commutative Lie subalgebra of $\mathfrak{g}_1$. In such a case, the eigenspace of C_0 in $\wedge^k\mathfrak{g}_1$ corresponding to k/m is generated by $\wedge^k\mathfrak{a}$, where $\mathfrak{a}$ runs through all commutative k-dimensional subalgebras of $\mathfrak{g}_1$.*

We also show that the $\mathfrak{g}_0$-submodule of $\wedge\mathfrak{g}_1$ generated by $\wedge^{\dim\mathfrak{a}}\mathfrak{a}$, where $\mathfrak{a}$ ranges over all commutative subalgebras, is multiplicity free.

Our third aim is to determine the maximal eigenvalue ℓ of C_0 on $\wedge\mathfrak{g}_1$, if Θ is involutory, i.e., $m=2$. Here we exploit the fact that $\mathfrak{g}_1$ is an orthogonal $\mathfrak{g}_0$-module and therefore there exists a $\mathfrak{g}_0$-module, denoted by $Spin_0(\mathfrak{g}_1)$, such that $\wedge\mathfrak{g}_1$ and $Spin_0(\mathfrak{g}_1)^{\otimes 2}$ are equal up to a scalar factor. Using our description of $Spin_0(\mathfrak{g}_1)$ for all IRSS (see [Pa00] and Sect. 5), we give a formula for ℓ (see 5.4). Actually, we present a finite set of scalar products parameterized by some elements of W, ℓ being the maximal number in this set. We explicitly compute ℓ for the Hermitian involutions and show that, in this case, ℓ is attained on commutative subalgebras of maximal dimension.

Main Notation. Unless otherwise stated, $\mathfrak{g}$ is a semisimple Lie algebra. Then Φ is the Killing form on $\mathfrak{g}$ and G is the corresponding connected and simply-connected algebraic group. If $\mathfrak{h}$ is a reductive subalgebra of $\mathfrak{g}$, then $\Phi_{\mathfrak{h}}$ is the Killing form on $\mathfrak{h}$ and $\Phi|_{\mathfrak{h}}$ is the restriction of Φ to $\mathfrak{h}$. Note that $\Phi|_{\mathfrak{h}}$ is non-degenerate, whereas $\Phi_{\mathfrak{h}}$ is degenerate if $\mathfrak{h}$ is not semisimple.
We fix a triangular decomposition $\mathfrak{g} = \mathfrak{u}^+ \oplus \mathfrak{t} \oplus \mathfrak{u}^-$. Then Δ (resp. Δ^+) is the set of roots (resp. positive roots) of $(\mathfrak{g},\mathfrak{t})$ and $\Pi \subset \Delta^+$ is the set of simple roots. For any $M \subset \Delta$, let $|M|$ denote $\sum_{\alpha\in M}\alpha$, e.g. $\rho = \frac{1}{2}|\Delta^+|$.

W is the Weyl group of $(\mathfrak{g},\mathfrak{t})$;

$\mathfrak{t}^*_{\mathbb{Q}}$ is $\mathbb{Q}$-vector subspace of $\mathfrak{t}^*$ generated by the lattice of integral weights and $(\ ,\ )_\Phi$ is the W-invariant positive-definite inner product on $\mathfrak{t}^*_{\mathbb{Q}}$ induced by Φ.
The irreducible module with highest weight λ is denoted by V_λ.
For the simple Lie algebras, we follow the numeration of the simple roots from [VO88] and [On95]; then φ_i denotes the fundamental weight corresponding to $\alpha_i \in \Pi$; Λ is the highest root.

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1 Generalities on Casimir elements, the Dynkin index, and the dual Coxeter numbers

Most of the formulas in this section does not depend on the normalization of $\mathfrak{g}$-invariant bilinear form on $\mathfrak{g}$. To stress this fact, we shall also use an arbitrary $\mathfrak{g}$-invariant non-degenerate bilinear form $(\ ,\ )$ on $\mathfrak{g}$ and the induced form on $\mathfrak{t}^*_{\mathbb{Q}}$.

Casimir elements. Let $e_1, \dots, e_n$ be a basis for $\mathfrak{g}$ and $\{e'_i\}_i$ the dual basis relative to $(\ ,\ )$. The Casimir element with respect to $(\ ,\ )$ is $C = \sum_{i=1}^n e_i e'_i \in U(\mathfrak{g})$. It is well-known that C does not depend on the choice of basis and that C is a scalar operator in any irreducible representation. If $\pi_\lambda : \mathfrak{g} \to \mathfrak{gl}(V_\lambda)$ is irreducible, then

(1.1) $$\pi_\lambda(C) = (\lambda, \lambda + 2\rho){\cdot}\mathrm{id}_{V_\lambda}\ .$$

Of course, all this holds for any reductive Lie algebra equipped with a non-degenerate invariant symmetric bilinear form.
It is easily seen that the Casimir element with respect to the Killing form gives rise to the identity operator on $\mathfrak{g}$. Therefore, if $\mathfrak{g}$ is simple, which will be assumed until the end of the section, then

(1.2) $$(\Lambda, \Lambda + 2\rho)_\Phi = 1\ .$$

The dual Coxeter number. Let $\mathcal{O}_{min} \subset \mathfrak{g}$ be the unique adjoint G-orbit of minimal nonzero dimension. As usual, $\Lambda^\vee = 2\Lambda/(\Lambda, \Lambda)$.

1.3 Lemma. *The following numbers are equal*

$$1 + (\rho, \Lambda^\vee), \quad \tfrac{1}{2}\dim \mathcal{O}_{min} + 1, \quad \frac{1}{(\Lambda, \Lambda)_\Phi}.$$

Proof. 1. Noting that the first number does not depend on the choice of an invariant bilinear form, we deduce the equality $1 + (\rho, \Lambda^\vee)_\Phi = \dfrac{1}{(\Lambda, \Lambda)_\Phi}$ from Eq. (1.2)

2. Since $\dim \mathcal{O}_{min} = 1 + \#\{\alpha \in \Delta^+ \mid (\alpha, \Lambda) \neq 0\}$ and $(\alpha, \Lambda^\vee) \in \{0, 1\}$ for $\alpha \in \Delta^+ \setminus \Lambda$, we obtain

$$\dim \mathcal{O}_{min} = 1 + \Big(\sum_{(\alpha, \Lambda^\vee)=1} (\alpha, \Lambda^\vee)\Big) + 1 = \sum_{\alpha \in \Delta^+} (\alpha, \Lambda^\vee) = 2(\rho, \Lambda^\vee). \qquad \square$$

This common number is called the *dual Coxeter number* of $\mathfrak{g}$, denoted by $h^*(\mathfrak{g})$. In the simply-laced case, $h^*(\mathfrak{g}) = h(\mathfrak{g})$—the usual Coxeter number. For the other simple Lie algebras we have $h^*(\mathbf{B}_n) = 2n-1$, $h^*(\mathbf{C}_n) = n+1$, $h^*(\mathbf{F}_4) = 9$, $h^*(\mathbf{G}_2) = 4$.

Some interesting properties of the dual Coxeter numbers are found in [Su98].

The Dynkin index. Let V be a $\mathbb{C}$-vector space. We denote by tr_V the trace function on $\mathrm{End}\, V$. If V is a $\mathfrak{g}$-module, then the function $x \mapsto \mathrm{tr}_V(x^2)$ $(x \in \mathfrak{g})$ is a $\mathfrak{g}$-invariant quadratic form on $\mathfrak{g}$, hence a multiple of the Killing form $x \mapsto \mathrm{tr}_{\mathfrak{g}}(x^2)$. The multiplication factor is called the *(Dynkin) index* of V, denoted by $\mathrm{ind}_{\mathfrak{g}} V$. The index was introduced by E.B. Dynkin in the seminal paper [Dy52]. (He used however another normalization of invariant form on $\mathfrak{g}$. In place of the Killing form, he exploited the form such that $(\alpha, \alpha) = 2$ for a long root α.) Substantial applications of the index to Invariant

Theory were demonstrated in [AVE] and then in the work of A.G. Elashvili on "stabilizers in general position".

Clearly, the index is an additive function of V. It is therefore sufficient to know the indices for the irreducible representations. The following result is essentially contained in [Dy52, Thm. 2.5]:

(1.4)
$$\mathrm{ind}_{\mathfrak g} V_\lambda = \frac{\dim V_\lambda}{\dim \mathfrak g} \cdot \frac{(\lambda, \lambda+2\rho)}{(\Lambda, \Lambda+2\rho)} .$$

Thus, the index of any non-trivial representation is a positive rational number and the adjoint representation has index 1.

Remark. In [On95, §3.10], A.L. Onishchik uses another normalization for the index, which goes back to Dynkin's paper. In that interpretation, the index is always an integer. In view of Lemma 1.3, it is not hard to realize that the transition factor between the two versions is $2h^*(\mathfrak g)$.

2 On a Casimir element and isotropy representations

Let $\mathfrak h$ be a reductive subalgebra of $\mathfrak g$. Then $\Phi|_{\mathfrak h}$ is non-degenerate and $\mathfrak g = \mathfrak h \oplus \mathfrak m$, where $\mathfrak m = \mathfrak h^\perp$. Then $\mathfrak m \simeq \mathfrak g/\mathfrak h$ is an orthogonal $\mathfrak h$-module and the representation $\mathfrak h \to \mathfrak{so}(\mathfrak m)$ is called the *isotropy representation* of the pair $(\mathfrak g, \mathfrak h)$. It is assumed that

($\diamond$) $\mathfrak h$ does not contain simple ideals of $\mathfrak g$.

Choose a basis $(e_i)_{i=1}^n$ for $\mathfrak g$ so that $\langle e_1, \dots, e_s\rangle = \mathfrak h$ and $\langle e_{s+1}, \dots, e_n\rangle = \mathfrak m$. Let $(e_i')_{i=1}^n$ be the dual basis relative to Φ. Consider the following elements in $U(\mathfrak g)$:

$$C_{\mathfrak h} = \sum_{i=1}^s e_i e_i', \quad C_{\mathfrak m} = \sum_{i=s+1}^n e_i e_i' .$$

Then $C_{\mathfrak h} \in U(\mathfrak h)$ is the Casimir element with respect to $\Phi|_{\mathfrak h}$, and $C := C_{\mathfrak h} + C_{\mathfrak m}$ is the Casimir element for $\mathfrak g$. Therefore

$$C(x) = x \text{ for all } x \in \mathfrak g .$$

Our goal is to obtain some relations for the eigenvalues of $C_{\mathfrak h}$. If $x, y \in \mathfrak g$ and $V \subset \mathfrak g$ is an $(\mathrm{ad}\, x{\cdot}\mathrm{ad}\, y)$-stable subspace, then we set $\mathrm{tr}_V(x{\cdot}y) := \mathrm{tr}\,(\mathrm{ad}\, x{\cdot}\mathrm{ad}\, y\,|_V)$.

2.1 Lemma.
1. $\mathrm{tr}_{\mathfrak g}(C_{\mathfrak h}) = \dim \mathfrak h$;
2. *Let* $x, y \in \mathfrak h$. *Then*
   (i) $\Phi(C_{\mathfrak h}(x), y) = \mathrm{tr}_{\mathfrak h}(x{\cdot}y) = \Phi_{\mathfrak h}(x, y)$;
   (ii) $\Phi(C_{\mathfrak m}(x), y) = \mathrm{tr}_{\mathfrak m}(x{\cdot}y)$.

Proof. 1. $\mathrm{tr}_{\mathfrak g}(C_{\mathfrak h}) = \sum_{j=1}^n \Phi(C_{\mathfrak h}(e_j), e_j') = \sum_{i=1}^s \sum_{j=1}^n \Phi([e_i', e_j], [e_j', e_i]) = \sum_{i=1}^s \Phi(e_i', C(e_i)) = \dim \mathfrak h$.

2. (i) $\Phi(C_{\mathfrak h}(x), y) = \sum_{i=1}^s \Phi([e_i, [e_i', x]], y]) = \sum_{i=1}^s \Phi((\mathrm{ad}\, y{\cdot}\mathrm{ad}\, x)e_i', e_i) = \mathrm{tr}_{\mathfrak h}(y{\cdot}x)$.
(ii) Similar. □

Let $\mathfrak{h} = \mathfrak{h}_1 \oplus \ldots \oplus \mathfrak{h}_k \oplus \mathfrak{z}$, where $\mathfrak{z}$ is the centre and the $\mathfrak{h}_i$'s are the simple ideals. Using an appropriate basis for $\mathfrak{h}$, we may further decompose $C_\mathfrak{h}$ as $C_{\mathfrak{h}_1} + \ldots + C_{\mathfrak{h}_k} + C_\mathfrak{z}$. If $x, y \in \mathfrak{h}_i$, then $C_\mathfrak{h}(x) = C_{\mathfrak{h}_i}(x)$ and $\Phi_\mathfrak{h}(x,y) = \Phi_{\mathfrak{h}_i}(x,y)$. It then follows from Lemma 2.1(2) that the ratio $\Phi(C_\mathfrak{m}(x), y)/\Phi(C_\mathfrak{h}(x), y)$ is nothing but the Dynkin index of the $\mathfrak{h}_i$-module $\mathfrak{m}$. Note that the $\mathfrak{h}_i$-modules $\mathfrak{m}$ and $\mathfrak{g}/\mathfrak{h}_i$ have the same non-trivial summands and therefore equal indices. Let γ_i be the eigenvalue of $C_\mathfrak{h}$ on $\mathfrak{h}_i$ $(i = 1, \ldots, k)$. Then $C_\mathfrak{h}(x) = \gamma_i x$, $C_\mathfrak{m}(x) = (1 - \gamma_i)x$ for $x \in \mathfrak{h}_i$. Hence

(2.2)
$$\mathrm{ind}_{\mathfrak{h}_i}(\mathfrak{g}/\mathfrak{h}_i) = \mathrm{ind}_{\mathfrak{h}_i}\mathfrak{m} = \frac{1 - \gamma_i}{\gamma_i} =: \mu_i \ .$$

Hypothesis $(\diamond)$ guarantees us that $\mathfrak{g}/\mathfrak{h}_i$ is a non-trivial $\mathfrak{h}_i$-module. Whence $\mu_i > 0$ and $\gamma_i < 1$. It follows that

(2.3)
$$\frac{\Phi(x,y)}{\Phi_{\mathfrak{h}_i}(x,y)} = \frac{\Phi(C_\mathfrak{h}(x), y) + \Phi(C_\mathfrak{m}(x), y)}{\Phi_\mathfrak{h}(x,y)} = 1 + \mu_i = \frac{1}{\gamma_i} \ .$$

Let us now give practical formulas for computing the index of isotropy representations and related eigenvalues of Casimir elements. Without loss of generality, one may assume that $\mathfrak{t} \cap \mathfrak{h}_i$ is a Cartan subalgebra of $\mathfrak{h}_i$. Then, identifying $\mathfrak{t}$ and $\mathfrak{t}^*$, we may regard the roots for $\mathfrak{h}_i$ as elements of $\mathfrak{t}^*$. Thus, we can compare length of roots for $\mathfrak{g}$ and $\mathfrak{h}_i$.

2.4 Proposition. *Suppose $\mathfrak{g}$ is simple. Let r be the ratio of the square of length of a long root for $\mathfrak{g}$ to that for $\mathfrak{h}_i$. Then the eigenvalue of $C_{\mathfrak{h}_i}$ on $\mathfrak{h}_i$ is equal to $\dfrac{h^*(\mathfrak{h}_i)}{r{\cdot}h^*(\mathfrak{g})}$ and* $\mathrm{ind}_{\mathfrak{h}_i}\mathfrak{m} = r{\cdot}\dfrac{h^*(\mathfrak{g})}{h^*(\mathfrak{h}_i)} - 1$.

Proof. In view of Eq. (2.2), it is enough to deduce one of the two relations.
By Eq. (2.3), $1 + \mathrm{ind}_{\mathfrak{h}_i}\mathfrak{m} = \Phi(x,x)/\Phi_{\mathfrak{h}_i}(x,x)$ for all $x \in \mathfrak{h}_i$, i.e., $(1 + \mathrm{ind}_{\mathfrak{h}_i}\mathfrak{m})\Phi_{\mathfrak{h}_i} = \Phi|_{\mathfrak{h}_i}$. On the other hand, let α be a long root for $\mathfrak{h}_i$. By the definition of r and the third expression for the dual Coxeter number (see 1.3), we have $(\alpha,\alpha)_{\Phi_{\mathfrak{h}_i}} = 1/h^*(\mathfrak{h}_i)$ and $(\alpha,\alpha)_\Phi = 1/r{\cdot}h^*(\mathfrak{g})$. Whence $1 + \mathrm{ind}_{\mathfrak{h}_i}\mathfrak{m} = r{\cdot}h^*(\mathfrak{g})/h^*(\mathfrak{h}_i)$. $\square$

Let $\gamma_\mathfrak{m}$ be the maximal eigenvalue of $C_\mathfrak{h}$ on $\mathfrak{m}$. If $C_\mathfrak{h}$ acts scalarly on $\mathfrak{m}$, then $\gamma_\mathfrak{m}$ can be computed using Lemma 2.1(1): $\dim\mathfrak{h} = \mathrm{tr}\,(C_\mathfrak{h}) = \sum_{i=1}^k \gamma_i \dim\mathfrak{h}_i + \gamma_\mathfrak{m}\dim\mathfrak{m}$. Therefore

(2.5)
$$\gamma_\mathfrak{m}\dim\mathfrak{m} = \sum_{j=1}^k (1 - \gamma_j)\dim\mathfrak{h}_j + \dim\mathfrak{z} = \sum_{j=1}^k \frac{\mu_j}{1 + \mu_j}\dim\mathfrak{h}_j + \dim\mathfrak{z} \ .$$

A sufficient (but not necessary) condition for $C_\mathfrak{h}$ to act scalarly on $\mathfrak{m}$ is that $\mathfrak{m}$ has an irreducible real form. Let $\mathfrak{g}_c$ be a compact form of $\mathfrak{g}$ such that $\mathfrak{g}_c \cap \mathfrak{h} =: \mathfrak{h}_c$ is a compact form of $\mathfrak{h}$. Set $\mathfrak{m}_c = \mathfrak{m} \cap \mathfrak{g}_c$. Then the $\mathfrak{h}_c$-module $\mathfrak{m}_c$ can be irreducible, whereas the $\mathfrak{h}$-module $\mathfrak{m}$ is not.
Let $\mathfrak{m}_\clubsuit$ denote the eigenspace of $C_\mathfrak{h}$ in $\mathfrak{m}$ belonging to $\gamma_\mathfrak{m}$.

2.6 Theorem. 1. $\gamma_{\mathfrak{m}} \leq 1/2$. *Moreover, $\gamma_{\mathfrak{m}} = 1/2$ if and only if $\mathfrak{h} \oplus \mathfrak{m}_{\clubsuit}$ is a $\mathbb{Z}_2$-graded reductive Lie algebra.*
2. *If the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is a $\mathbb{Z}_2$-grading (i.e. $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$), then $C_{\mathfrak{h}}$ always acts scalarly on $\mathfrak{m}$ (with eigenvalue 1/2).*

Proof. 1. Let $u \in \mathfrak{m}$. Using the above basis for $\mathfrak{g}$, we obtain the matrix presentation $\mathrm{ad}\, u = \begin{pmatrix} 0 & M \\ N & X \end{pmatrix}$, where M is an $s \times (n-s)$-matrix, N an $(n-s) \times s$-matrix, and X an $(n-s) \times (n-s)$-matrix. Then

$$(\mathrm{ad}\, u)^2 = \begin{pmatrix} MN & MX \\ XN & NM + X^2 \end{pmatrix} .$$

Since $\mathrm{tr}\,(MN) = \mathrm{tr}\,(NM)$, we conclude that $\Phi(u,u) = \mathrm{tr}\,((\mathrm{ad}\, u)^2) = 2\mathrm{tr}\,(MN) + \mathrm{tr}\,(X^2)$. As in Lemma 2.1, we obtain

$$\Phi(C_{\mathfrak{h}}(u), u) = \sum_{i=1}^{s} \Phi((adu)^2 e_i', e_i) = \mathrm{tr}\,(MN) \ .$$

Now, if $u \in \mathfrak{m}_{\clubsuit}$, then $C_{\mathfrak{h}}(u) = \gamma_{\mathfrak{m}} u$. Therefore $\mathrm{tr}\,(MN) = \gamma_{\mathfrak{m}} \Phi(u,u)$ and

$$(1 - 2\gamma_{\mathfrak{m}})\Phi(u,u) = \mathrm{tr}\,(X^2) \ .$$

Without loss of generality, one may assume that the basis (e_i) is chosen so that $e_1, \dots, e_n \in \mathfrak{g}_c$ and $\Phi(e_i, e_j) = -\delta_{i,j}$. Taking $0 \neq u \in \mathfrak{m}_c \cap \mathfrak{m}_{\clubsuit}$, we see that $\Phi(u,u) < 0$ and X is a *real* skew-symmetric matrix. Hence $\mathrm{tr}\,(X^2) \leq 0$ and $1 - 2\gamma_{\mathfrak{m}} \geq 0$.

If $\gamma_{\mathfrak{m}} = 1/2$, then $\mathrm{tr}\,(X^2) = 0$ for all $u \in \mathfrak{m}_{\clubsuit}$. Whence $X = 0$ for all $u \in \mathfrak{m}_c \cap \mathfrak{m}_{\clubsuit}$. This implies that $X = 0$ for all $u \in \mathfrak{m}_{\clubsuit}$ and hence $[\mathfrak{m}_{\clubsuit}, \mathfrak{m}] \subset \mathfrak{h}$. In particular, $\mathfrak{h} \oplus \mathfrak{m}_{\clubsuit}$ is a $\mathbb{Z}_2$-grading. Since $\Phi|_{\mathfrak{h} \oplus \mathfrak{m}_{\clubsuit}}$ is non-degenerate, this Lie algebra is reductive.

2. If $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is a $\mathbb{Z}_2$-grading, then $X = 0$. Therefore $\Phi(u,u) = 2\mathrm{tr}\,(MN) = 2\Phi(C_{\mathfrak{h}}(u), u)$. Since this holds for all $u \in \mathfrak{m}$, we are done. □

2.7 Corollary. *Let $\mathfrak{h}$ be a maximal reductive subalgebra of $\mathfrak{g}$. Then either $\mathfrak{h}$ is symmetric or $\gamma_{\mathfrak{m}} < 1/2$.*

Under a mild constraint, Theorem 2.6(2) can be generalized to higher periodic gradings. Let Θ be an automorphism of $\mathfrak{g}$ of order m. Letting $\mathfrak{g}_i = \{x \in \mathfrak{g} \mid \Theta(x) = \zeta^i x\}$, where ζ is a fixed primitive m^{th} root of unity, we obtain a periodic grading $\mathfrak{g} = \oplus_{i \in \mathbb{Z}_m} \mathfrak{g}_i$. Here $\mathfrak{g}_0$ is a reductive Lie subalgebra and each $\mathfrak{g}_i$ is a $\mathfrak{g}_0$-module. Consider the Casimir element $C_0 \in U(\mathfrak{g}_0)$, with respect to $\Phi|_{\mathfrak{g}_0}$. For $\mathfrak{h} = \mathfrak{g}_0$, hypothesis $(\diamond)$ means that Θ is non-trivial on each simple ideal of $\mathfrak{g}$.

2.8 Theorem. *Suppose $\mathfrak{g}_0$ is semisimple. Then the eigenvalue of C_0 on $\mathfrak{g}_i$ is equal to $1/m$ whenever i and m are coprime.*

Proof. It clearly suffices to handle the case $i=1$. Take $y \in \mathfrak{g}_1$ and $z \in \mathfrak{g}_{m-1}$. Having chosen a basis for $\mathfrak{g}$ compatible with the decomposition $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_{m-1}$, we obtain the following block presentations:

$$\operatorname{ad} y = \begin{pmatrix} 0 & \ldots & 0 & A_0 \\ A_1 & \ldots & 0 & 0 \\ \ldots & \ddots & \ldots & \ldots \\ 0 & \ldots & A_{m-1} & 0 \end{pmatrix}, \quad \operatorname{ad} z = \begin{pmatrix} 0 & B_1 & \ldots & 0 \\ \ldots & \ldots & \ddots & \ldots \\ 0 & 0 & \ldots & B_{m-1} \\ B_0 & 0 & \ldots & 0 \end{pmatrix} .$$

Then

$$\operatorname{ad} y{\cdot}\operatorname{ad} z = \begin{pmatrix} A_0B_0 & 0 & \cdots & & \\ 0 & A_1B_1 & 0 & & \cdots \\ \cdots & \cdots & \ddots & & \cdots \\ & \cdots & 0 & & A_{m-1}B_{m-1} \end{pmatrix}, \quad \operatorname{ad} z{\cdot}\operatorname{ad} y = \begin{pmatrix} B_1A_1 & 0 & \cdots & \\ 0 & B_2A_2 & 0 & \\ \cdots & \cdots & \ddots & \cdots \\ \cdots & \cdots & 0 & B_0A_0 \end{pmatrix} .$$

Since $[y,z] \in \mathfrak{g}_0$ and $\mathfrak{g}_0$ is semisimple, $\operatorname{tr}_{\mathfrak{g}_i}[y,z] = 0$ for all i. Hence $\operatorname{tr}(A_iB_i) = \operatorname{tr}(B_{i+1}A_{i+1})$ $\forall i \in \mathbb{Z}_m$. On the other hand, $\operatorname{tr}(A_iB_i) = \operatorname{tr}(B_iA_i)$. Thus, $\operatorname{tr}(A_0B_0) = \operatorname{tr}(A_1B_1) = \ldots = \operatorname{tr}(A_{m-1}B_{m-1})$ and $\operatorname{tr}(A_iB_i) = \frac{1}{m}\operatorname{tr}(\operatorname{ad} y{\cdot}\operatorname{ad} z) = \frac{1}{m}\Phi(y,z)$. Finally,

$$\Phi(C_0(y),z) = \operatorname{tr}_{\mathfrak{g}_0}(y{\cdot}z) = \operatorname{tr}(A_0B_0) = \frac{1}{m}\Phi(y,z) .$$

Whence $C_0(y) = \frac{1}{m}y$. □

Remark. For $m=2$, the semisimplicity of $\mathfrak{g}_0$ is not important, see Theorem 2.6. However simple examples show that it cannot be dropped, if $m \geq 3$.

2.9 Corollary. *If m is prime and $\mathfrak{g}_0$ is semisimple, then C_0 acts scalarly on $\mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_{m-1}$, with eigenvalue $1/m$, and*

$$\operatorname{tr}_{\mathfrak{g}_0}(C_0) = \frac{(m+1)\dim \mathfrak{g}_0 - \dim \mathfrak{g}}{m} .$$

In particular, $(m+1)\dim \mathfrak{g}_0 > \dim \mathfrak{g}$.

Proof. To get the equality, use Lemma 2.1(1) with $\mathfrak{h} = \mathfrak{g}_0$. □

Given a finite order automorphism Θ of $\mathfrak{g}$, let us say that a Θ-stable ideal of $\mathfrak{g}$ is Θ-*indecomposable*, if it is not a sum of two proper Θ-stable ideals. It is easily seen that $\mathfrak{g}$ is (uniquely) the sum of Θ-indecomposable ideals, and every such ideal is either simple, or is the sum of isomorphic simple ideals that are cyclically permuted by Θ. Consider the second possibility in more details.

(2.10) Let $\mathfrak{g} = \underbrace{\mathfrak{h} \oplus \ldots \oplus \mathfrak{h}}_{m}$ and $\Theta(x_1,\ldots,x_m) = (x_m,x_1,\ldots,x_{m-1})$. Then $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}_m} \mathfrak{g}_i$, where $\mathfrak{g}_0 \simeq \mathfrak{h}$ is the diagonal in $\mathfrak{g}$ and $\mathfrak{g}_i \simeq \mathfrak{h}$ as $\mathfrak{h}$-module for all i. Here $\Phi|_{\mathfrak{g}_0} = m\Phi_{\mathfrak{h}}$. Therefore C_0 acts scalarly, with eigenvalue $1/m$, on the whole $\mathfrak{g}$. That is, the condition that i, m should be coprime does not arise in this case.

If $\mathfrak{g}$ is not Θ-indecomposable, then it may happen that the restriction of Θ to some Θ-indecomposable ideals has order $< m$. To avoid reservations related to this, it is convenient to give

Definition. $\Theta \in \mathrm{Aut}\,\mathfrak{g}$ is said to be of *pure* order m, if the restriction of Θ to any Θ-indecomposable ideal of $\mathfrak{g}$ has order m.

2.11 Theorem. *Let Θ be an automorphism of $\mathfrak{g}$ of pure order m. Suppose $\mathfrak{g}_0$ is semisimple, if $m \geq 3$. Assume that* g.c.d.$(k, m) =: d > 1$. *Then all the eigenvalues of C_0 on $\mathfrak{g}_k$ are less than d/m.*

Proof. It follows from (2.10) that we may restrict ourselves with the case, where $\mathfrak{g}$ is simple.

(a) $\underline{k = d = m}$. Here $\mathfrak{g}_m = \mathfrak{g}_0$. The eigenvalue of C_0 on the centre of $\mathfrak{g}_0$ (if any) is equal to 0. If $\mathfrak{h}_i$ is a simple ideal of $\mathfrak{g}_0$, then the eigenvalue of C_0 on $\mathfrak{h}_i$ is equal to $1/(1 + \mathrm{ind}_{\mathfrak{h}_i}(\mathfrak{g}/\mathfrak{g}_0))$, see Eq. (2.3). Since $\mathfrak{g}/\mathfrak{g}_0$ is a faithful $\mathfrak{g}_0$-module, $\mathrm{ind}_{\mathfrak{h}_i}(\mathfrak{g}/\mathfrak{g}_0) > 0$.

(b) $\underline{1 < d < m}$. For the reductive "Veronese" subalgebra $\mathfrak{g}(k) = \bigoplus_i \mathfrak{g}_{ki}$, we have $\Theta|_{\mathfrak{g}(k)}$ is an automorphism of order m/d, $\mathfrak{g}(k)_0 = \mathfrak{g}_0$, and $\mathfrak{g}(k)_1 = \mathfrak{g}_k$. Because $\mathfrak{g}(k)$ can be not semisimple, take $\mathfrak{s} = [\mathfrak{g}(k), \mathfrak{g}(k)]$. This step does not change $\mathfrak{g}_0 = \mathfrak{s}_0$ and eventually kills some trivial $\mathfrak{g}_0$-submodules of $\mathfrak{g}_k$. But the nonzero eigenvalues of C_0 on $\mathfrak{s}_1$ and $\mathfrak{g}_k$ remains the same. We are thus reduced to considering a semisimple subalgebra $\mathfrak{s}$ together with $\overline{\Theta} := \Theta|_{\mathfrak{s}}$. Consider $\overline{\Theta}$-indecomposable ideals of $\mathfrak{s}$. Of course, we are only interested in those of them that have nonzero intersection with $\mathfrak{s}_1$.

Let $\mathfrak{s}^{(j)}$ be a $\overline{\Theta}$-indecomposable ideal of $\mathfrak{s}$. The condition $\mathfrak{s}_1^{(j)} \neq 0$ means that $\overline{\Theta}|_{\mathfrak{s}^{(j)}}$ still has the order m/d. Since $\mathfrak{s}_0^{(j)}$ is an ideal of $\mathfrak{s}_0$, it is also semisimple. Clearly, the Casimir element C_0 has the j-component $C_0^{(j)}$, determined by $\mathfrak{s}_0^{(j)}$, which is the only component responsible for the eigenvalue on $\mathfrak{s}_1^{(j)}$. If $C_0^{(j)}$ were defined with respect to the Killing form on $\mathfrak{s}^{(j)}$, then, by Theorem 2.8, the eigenvalue on $\mathfrak{s}_1^{(j)}$ was equal to $1/(m/d) = d/m$. But $C_0^{(j)}$ is defined with respect to the Killing form coming from $\mathfrak{g}$, and the above quantity gains a correcting factor. This factor is just the ratio of the two Killing forms in question and, by Eq. (2.3), it is equal to $1/(1 + \mathrm{ind}_{\mathfrak{s}^{(j)}}(\mathfrak{g}/\mathfrak{s}^{(j)})$. Thus the true eigenvalue is $\dfrac{1}{1 + \mathrm{ind}_{\mathfrak{s}^{(j)}}(\mathfrak{g}/\mathfrak{s}^{(j)})} \cdot \dfrac{d}{m} < \dfrac{d}{m}$. $\square$

This results in the following general statement:

2.12 Corollary. *Let Θ be an automorphism of order m of a semisimple Lie algebra $\mathfrak{g}$ which acts non-trivially on each simple ideal of $\mathfrak{g}$. Suppose $\mathfrak{g}_0$ is semisimple, if $m > 2$. Then the eigenvalues of C_0 on $\mathfrak{g}$ are less than 1.* $\square$

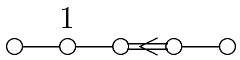
2.13 Examples. Recall that the number $r = r(\mathfrak{h}_i)$ is defined in Prop. 2.4.

1. $\mathfrak{g} = \mathbf{E}_8$, $\mathfrak{h} = \mathfrak{sl}_8$. Here $\mathfrak{h}$ is the fixed-point subalgebra for an automorphism of order 3 and the $\mathfrak{h}$-module $\mathfrak{m}$ is $V_{\varphi_3} \oplus V_{\varphi_3}^*$. Since $\mathrm{rk}\,\mathfrak{g} = \mathrm{rk}\,\mathfrak{h}$ and $\mathfrak{g}$ is simply-laced, $r = 1$.

By Prop. 2.4, $\gamma_{\mathfrak{h}}$, the eigenvalue of $C_{\mathfrak{h}}$ on $\mathfrak{h}$ is $30/9$ and $\mathrm{ind}_{\mathfrak{sl}_8}\mathfrak{m} = 30/9 - 1$. Hence $\mathrm{ind}_{\mathfrak{sl}_8}(V_{\varphi_3}) = 7/6$. Using Eq. (2.5), one computes that $\gamma_{\mathfrak{m}} = 1/3$, which is not surprising in view of Theorem 2.8.

2. $\mathfrak{g} = \mathfrak{sp}_{2n}$, $\mathfrak{h} = \mathfrak{sl}_n$. Here $\mathfrak{m} = V_{2\varphi_1} \oplus V^*_{2\varphi_1} \oplus V_0$ and $r = 2$. Hence
$1 + \mathrm{ind}_{\mathfrak{sl}_n}(\mathfrak{m}) = 2(n+1)/n$ and $\mathrm{ind}_{\mathfrak{sl}_n}(V_{2\varphi_1}) = (n+2)/2n$. Since the eigenvalue of $C_{\mathfrak{h}}$ on $\mathfrak{h}$ is $\dfrac{n}{2(n+1)}$, one computes, using Lemma 2.1(1), that the eigenvalue of $C_{\mathfrak{h}}$ on $V_{2\varphi_1}$ is $\dfrac{1}{2} - \dfrac{1}{n(n+1)}$. It is worth noting that $\tilde{\mathfrak{h}} := \mathfrak{n}_{\mathfrak{g}}(\mathfrak{h}) \simeq \mathfrak{gl}_n$ is a symmetric subalgebra of $\mathfrak{sp}_{2n}$. This small modification results in a modification for the eigenvalues. The eigenvalue of $C_{\tilde{\mathfrak{h}}}$ on $V_{2\varphi_1}$ to be $1/2$.

3. $\mathfrak{g} = \mathfrak{so}_7$, $\mathfrak{h} = \mathbf{G}_2$. Here $\mathfrak{m} = V_{\varphi_1}$. Making use of Dynkin's formula (see (1.4)), we obtain $\mathrm{ind}_{\mathbf{G}_2} V_{\varphi_1} = 1/4$. One then calculates that $\gamma_{\mathfrak{h}} = 4/5$ and $\gamma_{\mathfrak{m}} = 2/5$. Since G/H is not symmetric here, $\gamma_{\mathfrak{m}} < 1/2$. Note that the equality $\dfrac{1}{4} = r \cdot \dfrac{5}{4} - 1$ implies that $r = 1$. However one can begin with proving that $r = 1$ and therefore avoid using Eq. (1.4).

4. The exceptional algebra $\mathfrak{g} = \mathbf{E}_6$ has an outer automorphism of order 4 such that $\mathfrak{g}_0 = \mathfrak{so}_7 \times \mathfrak{sl}_2$. This easily can be detected using Kac's description of finite order automorphisms (see [Ka90, ch. 8] or [Vi76, § 8]). The automorphism is determined by the labelled twisted Dynkin diagram $E_6^{(2)}$: 

.

Here $\mathfrak{g}_1 \simeq \mathfrak{g}_3 = V_{\varphi_3} \otimes V_{\varphi'}$ and $\mathfrak{g}_2 = V_{\varphi_1} \otimes V_{2\varphi'} \oplus V_0$ (φ' refers to the fundamental weight of $\mathfrak{sl}_2$). Having stared at the above diagram, it is not hard to realize that $r_1 := r(\mathfrak{so}_7) = 1$ and $r_2 := r(\mathfrak{sl}_2) = 2$. Therefore the eigenvalue of C_0 on $\mathfrak{so}_7$ is $h^*(\mathbf{B}_3)/h^*(\mathbf{E}_6) = 5/12$ and on $\mathfrak{sl}_2$ is $h^*(\mathbf{A}_1)/2h^*(\mathbf{E}_6) = 1/12$. By Theorem 2.8, the eigenvalues on $\mathfrak{g}_1$ and $\mathfrak{g}_3$ are equal to $1/4$. To compute the eigenvalue of C_0 on the non-trivial representation in $\mathfrak{g}_2$, one may exploit Lemma 2.1(1). We proceed however in a round-about way.
Since $\mathfrak{g}_0 \oplus \mathfrak{g}_2$ is a symmetric subalgebra of $\mathbf{E}_6$, of dimension 46, it is nothing but $\mathfrak{so}_{10} \oplus$ {1-dim torus}. Hence $V_{\varphi_1} \otimes V_{2\varphi'}$ is just the isotropy representation for the symmetric space $SO(10)/SO(7) \times SO(3)$. Therefore, if C_0 were defined w.r.t. the Killing form on $\mathfrak{so}_{10}$, the eigenvalue was equal to $1/2$. But since we are exploiting the restriction of the Killing form of $\mathbf{E}_6$, the correcting factor $h^*(\mathbf{D}_5)/h^*(\mathbf{E}_6) = 8/12$ is needed (we have used here the obvious fact that $r(\mathfrak{so}_{10}) = 1$). Thus, the eigenvalue on $V_{\varphi_1} \otimes V_{2\varphi'}$ is $1/3$.
Now, one can check that the trace of C_0 on $\mathfrak{g}$ is $24 = \dim \mathfrak{g}_0$, as required.

3 Preliminaries on exterior algebras

A nice exposition of the standard facts on $\wedge\mathfrak{g}$ is found Koszul's paper [Kos50]. We identify $\mathfrak{g}$ with $\mathfrak{g}^*$, using Φ, and consider $\wedge\mathfrak{g}$ with the usual differentials: $d : \wedge^l\mathfrak{g} \to \wedge^{l+1}\mathfrak{g}$ and

$\partial : \wedge^l \mathfrak{g} \to \wedge^{l-1}\mathfrak{g}$. Recall that

$$\partial(x_1 \wedge \ldots \wedge x_l) := \sum_{i<j} (-1)^{i+j-1}[x_i, x_j] x_1 \wedge \ldots \hat{x}_i \ldots \hat{x}_j \ldots \wedge x_l$$

for $l \geq 2$ and $\partial(x_1) = 0$. In particular, $\partial(x_1 \wedge x_2) = [x_1, x_2]$.
We regard Φ as having been extended, in the usual way, via determinants, from $\mathfrak{g}$ to $\wedge\mathfrak{g}$. Then $d = -\partial^t$. More precisely, denoting the extension of Φ to $\wedge^l\mathfrak{g}$ by $\Phi^{(l)}$, we have

$$\Phi^{(l)}(x_1 \wedge \ldots \wedge x_l, y_1 \wedge \ldots \wedge y_l) := \det \|\Phi(x_i, y_j)\| \ .$$

For $x \in \mathfrak{g}$, $\varepsilon(x)$ is the exterior product operator and $i(x)$ is the interior product operator in $\wedge\mathfrak{g}$. That is,

$$\varepsilon(x){\cdot}x_1 \wedge \ldots \wedge x_l := x \wedge x_1 \wedge \ldots \wedge x_l,$$

$$i(x){\cdot}x_1 \wedge \ldots \wedge x_l := \sum_{i=1}^{l} (-1)^{i-1}\Phi(x, x_i) x_1 \wedge \ldots \hat{x}_i \ldots \wedge x_l \ .$$

Denote by θ the natural extension of the adjoint representation of $\mathfrak{g}$ to $\wedge\mathfrak{g}$:

$$\theta(x){\cdot}x_1 \wedge \ldots \wedge x_l := \sum_{i=1}^{l} x_1 \wedge \ldots \wedge [x, x_i] \ldots \wedge x_l \ .$$

These operators satisfy the following relations:

(3.1)
$$\left.\begin{array}{r} [d, \theta(x)] = 0, \\ [\partial, \theta(x)] = 0, \\ \varepsilon(x)\partial + \partial\varepsilon(x) = \theta(x) \end{array}\right\} \quad \text{for all } x \in \mathfrak{g}.$$

After Koszul [Kos50, 3.4], it is known that

(3.2)
$$d = \frac{1}{2}\sum_{i=1}^{n} \varepsilon(e_i')\theta(e_i) \ .$$

Combining Eq. (3.1) and (3.2) yields

$$2(d\partial + \partial d) = \sum_{i=1}^{n} \theta(e_i')\theta(e_i) = \theta(C) \ .$$

Below, we write merely C in place of $\theta(C)$. Let m_k be the maximal eigenvalue of C on $\wedge^k\mathfrak{g}$. The following result was obtained by B. Kostant.

3.3 Theorem ([Ko65, § 4], see also [Ko98, 1.1]).
For $k = 0, \ldots, n$, one has $m_k \leq k$. Moreover, $m_k = k$ if and only if there exists a commutative k-dimensional Lie subalgebra of $\mathfrak{g}$. Furthermore, in such a case, the k-eigenspace of C in $\wedge^k\mathfrak{g}$ is spanned by the decomposable k-vectors corresponding to k-dimensional commutative subalgebras.
(m_k here is twice the value assigned to m_k in [Ko65] and the same as in [Ko98].)

The adjoint representation is only a member of the large family of the isotropy representations of symmetric spaces. It is therefore natural to look at the same problem for all symmetric spaces. In the next section, we show that Theorem 3.3 can be generalized to this and even more general setting.

In [Ko65, § 4], Kostant proved an assertion concerning the eigenspace of C in $\wedge^k\mathfrak{g}$ belonging to the maximal eigenvalue. It was an essential step for his proof of Theorem 3.3. Actually, his argument applies to the exterior algebra of any $\mathfrak{g}$-module. Since this more general formulation is needed below, we recall the necessary setup and outline a proof.

In the remainder of the section, $\mathfrak{g}$ is a reductive Lie algebra and C is a Casimir element with respect to a non-degenerate invariant bilinear form $(\ ,\ )$ on $\mathfrak{g}$. Let V be a vector space. An element $u \in \wedge^k V$ is called *decomposable*, if $u = v_1 \wedge \ldots \wedge v_k$ for some $v_i \in V$. If V is a $\mathfrak{g}$-module, then a submodule $W \subset \wedge^k V$ is called *decomposably-generated*, if it is spanned by decomposable elements.

3.4 Theorem (Kostant). 1. *Suppose $W \subset \wedge^k V$ is irreducible; then W is decomposably-generated if and only if a (any) highest weight vector is decomposable.*
2. *Let m_k be the maximal eigenvalue of C on $\wedge^k V$ and M_k the corresponding eigenspace. Then M_k is a sum of irreducible decomposably-generated $\mathfrak{g}$-submodules.*

Proof. 1. Let $u \in W$ be a decomposable element. Consider the line $\langle u\rangle := \mathbb{C}u$ as element of the Grassmannian $\mathrm{Gr}_k(W)$. The closure of the orbit $G\langle u\rangle$ in $\mathrm{Gr}_k(W)$ contains a closed orbit, say $\mathcal{O} = G\langle w\rangle$. Since $\mathcal{O}$ is projective, the stabilizer of $\langle w\rangle$ is a parabolic subgroup of G and hence $w \in W$ is a decomposable highest weight vector.
2. Let P be the set of all weights of $\wedge^k V$. Given $\mu \in P$, let $(\wedge^k V)^\mu$ denote the corresponding weight space. Using Eq (1.1) and the standard properties of weights of representations, one sees that $m_k = \max_{\mu\in P}(\mu, \mu{+}2\rho)$. Set $P^{max} = \{\mu \in P \mid (\mu, \mu{+}2\rho) = m_k\}$ and $(\wedge^k V)^{max} = \oplus_{\mu\in P^{max}}(\wedge^k V)^\mu$. It follows from the construction that each weight vector in $(\wedge^k V)^{max}$ is a highest weight vector and that these are exactly the highest weight vectors for M_k. Since each $(\wedge^k V)^\mu$ has a basis consisting of decomposable elements, we are done. $\square$

It follows that in order to determine the maximal eigenvalue of C on some exterior power of V, it suffices to test C only on the decomposable elements.

4 On eigenvalues of C_0 on $\wedge\mathfrak{g}_1$ and commutative subalgebras

Throughout this section, Θ is an automorphism of $\mathfrak{g}$ of pure order m and $\mathfrak{g} = \mathfrak{g}_0 \oplus \ldots \oplus \mathfrak{g}_{m-1}$ is the corresponding $\mathbb{Z}_m$-grading. The indices are regarded as elements of $\mathbb{Z}_m$, so that $\mathfrak{g}_{-i} = \mathfrak{g}_{m-i}$. Let $\{e_i\}$ be a basis for $\mathfrak{g}$ compatible with $\mathbb{Z}_m$-grading, i.e., consisting of Θ-eigenvectors. Consider the respective decomposition for the coboundary operator:

$d = d_0 + \ldots + d_{m-1}$, where

(4.1)
$$d_i = \frac{1}{2} \sum_{j\,:\,e_j \in \mathfrak{g}_i} \varepsilon(e_j)\theta(e'_j)\ .$$

Notice that here $e'_j \in \mathfrak{g}_{-i}$. Similarly, we have the decomposition $C = C_0 + \ldots + C_{m-1}$, where

(4.2)
$$C_i = \sum_{j\,:\,e_j \in \mathfrak{g}_i} e_j e'_j\ .$$

In particular, C_0 is the Casimir element for $\mathfrak{g}_0$, with respect to $\Phi|_{\mathfrak{g}_0}$. It is easily verified that d_i and C_i do not depend on a choice of basis for $\mathfrak{g}_i$.
Combining Equations (4.1) and (4.2) with (3.1), we obtain

$$\theta(C_i) = 2(d_i\partial + \partial d_i), \quad i = 0, 1, \ldots, m-1.$$

In what follows, we shall write C_i in place of $\theta(C_i)$. Notice that

(4.3)
$$d_i(\mathfrak{g}_j) \subset \left\{ \begin{array}{c} \mathfrak{g}_i \otimes \mathfrak{g}_{j-i}, \text{ if } i \neq j/2 \\ \wedge^2(\mathfrak{g}_i), \quad \text{if } i = j/2 \end{array} \right\} \subset \wedge^2\mathfrak{g}\ .$$

We shall repeatedly use the fact that $\Phi(\mathfrak{g}_i, \mathfrak{g}_j) = 0$ unless $i + j = 0$, and consequences of this relation for higher exterior powers of $\mathfrak{g}$. The following assertion is a refinement of Lemma 2 in [Ko65].

4.4 Proposition. *Let $y, z \in \mathfrak{g}_1$ be arbitrary. Then*
$$\sum_{i\,:\,e_i \in \mathfrak{g}_0} [e_i, y] \wedge [e'_i, z] = -d_1([y, z]).$$

Proof. Let L denote the left hand side. Take arbitrary $u, v \in \mathfrak{g}_{-1}$. Using the extension of Φ to $\wedge\mathfrak{g}$, we obtain

$$\Phi^{(2)}(L, u \wedge v) = \sum_{e_i \in \mathfrak{g}_j} \det \begin{pmatrix} \Phi([e_i, y], u) & \Phi([e'_i, z], u) \\ \Phi([e_i, y], v) & \Phi([e'_i, z], v) \end{pmatrix} =$$
$$= \sum_{e_i \in \mathfrak{g}_j} \left(\Phi(e_i, [y, u])\Phi(e'_i, [z, v]) - \Phi(e_i, [y, v])\Phi(e'_i, [z, u])\right)\ .$$

Since $[y, u], \ldots, [z, v] \in \mathfrak{g}_0$, the last sum is equal to

$$\Phi([y, u], [z, v]) - \Phi([y, v], [z, u]) \overset{Jacobi}{=} \Phi([y, z], [u, v]) = \Phi([y, z], \partial(u \wedge v))\ .$$

Hence $\Phi^{(2)}(L, u \wedge v) = \Phi([y, z], \partial(u \wedge v)) = \Phi^{(2)}(-d([y, z]), u \wedge v)$. Thus, L is equal to the component of $-d([y, z])$ lying in $\wedge^2\mathfrak{g}_1$. Since $[y, z] \in \mathfrak{g}_2$, it follows from Eq. (4.3) with $j = 2$ that the component in question is $-d_1([y, z])$. □

With a little bit more care, one can prove that $\sum_{i\,:\,e_i \in \mathfrak{g}_j} [e_i, y] \wedge [e'_i, z] = -d_{j+1}([y, z])$. But we do not need this equality. For a future use, we record a by-product of the proof. If $x \in \mathfrak{g}_2$ and $u, v \in \mathfrak{g}_{-1}$, then

(4.5)
$$\Phi^{(2)}(d_1(x), u \wedge v) = -\Phi(x, \partial(u \wedge v))\ .$$

4.6 Proposition. *Suppose $\mathfrak{g}_0$ is semisimple whenever $m \geq 3$. Then the eigenvalues of C_0 on $\wedge^2\mathfrak{g}_1$ are less than or equal to $2/m$. The eigenspace belonging to $2/m$ is spanned by the bi-vectors $y \wedge z$ such that $[y,z]=0$.*

Proof. Take any $y \wedge z \in \wedge^2\mathfrak{g}_1$. Recall from Theorems 2.6(2) and 2.8 that $C_0(y) = y/m$ for all $y \in \mathfrak{g}_1$. Using the definition of C_0 and Prop. 4.4, we obtain:

$$C_0(y\wedge z) = C_0(y)\wedge z + y\wedge C_0(z) + \sum_{i=1}^{s}[e_i,y]\wedge[e_i',z] + \sum_{i=1}^{s}[e_i',y]\wedge[e_i,z] = \frac{2}{m}y\wedge z - 2d_1([y,z]) \ .$$

This already implies that if $[y,z]=0$, then $C_0(y\wedge z) = \frac{2}{m}y\wedge z$.
Assume now that $y\wedge z$ is an eigenvector of C_0 and $[y,z]\neq 0$. Since $[y,z] = \partial(y\wedge z)$ and $\partial : \wedge^2\mathfrak{g}_1 \to \mathfrak{g}_2$ is $\mathfrak{g}_0$-equivariant, the C_0-eigenvalues of $y\wedge z$ and $[y,z]$ are equal. Then Theorem 2.11 implies that the eigenvalue of $y \wedge z$ is less than $2/m$.
Thus, we can conclude the proof by invoking Theorem 3.4. □

Before proceeding to the general case, we draw a conclusion (cf. [Ko65, Cor. 5.1]). Let $\mathcal{A}_2 \subset \wedge^2\mathfrak{g}_1$ be the linear span of the decomposable bi-vectors with commuting components.

4.7 Proposition. *Under the assumptions of Prop. 4.6, we have*

1. $\mathcal{A}_2 = \mathrm{Ker}\,(\partial|_{\wedge^2\mathfrak{g}_1})$;
2. $\wedge^2\mathfrak{g}_1 = d_1(\mathfrak{g}_2) \oplus \mathcal{A}_2$.

Proof. 1. Let $u = \sum_i y_i \wedge z_i$. Then, as in the above proof, we obtain

$$C_0(u) = \frac{2}{m}u - 2\sum_i d_1([y_i,z_i]) = \frac{2}{m}u - 2d_1\partial(u) \ .$$

Hence if $\partial(u)=0$, then $u \in \mathcal{A}_2$ by Prop. 4.6. The opposite containment is obvious.
2. Clearly, the map $d_1 : \mathfrak{g}_2 \to \wedge^2\mathfrak{g}_1$ is $\mathfrak{g}_0$-equivariant. Therefore the eigenvalues of C_0 on $d_1(\mathfrak{g}_2)$ are less than $2/m$ and hence the sum of $d_1(\mathfrak{g}_2)$ and $\mathcal{A}_2$ is direct. Assume $\wedge^2\mathfrak{g}_1 = d_1(\mathfrak{g}_2)\oplus\mathcal{A}_2\oplus W$ for some W. Then, by the first part, $\partial(\wedge^2\mathfrak{g}_1) = \partial d_1(\mathfrak{g}_2)\oplus \partial W$ and $\dim \partial d_1(\mathfrak{g}_2) = \dim d_1(\mathfrak{g}_2)$. Using Eq. (4.5), we obtain $\dim d_1(\mathfrak{g}_2) = \dim\partial(\wedge^2\mathfrak{g}_{-1}) = \dim\partial(\wedge^2\mathfrak{g}_1)$. Therefore $\dim\partial W = 0$ and hence $W=0$. □

Our aim is to demonstrate an analogue of Prop. 4.6 for $\wedge^k\mathfrak{g}_1$ with arbitrary k. Unfortunately, I was unable to come up with an "algebraically closed" proof (as for the case $k=2$). Therefore it will merely be shown that Kostant's argument (see [Ko65]) can be adapted to the $\mathbb{Z}_m$-graded setting.

Let $\mathfrak{g}_c$ be a compact form of $\mathfrak{g}$. Define a $\star$-conjugation in $\mathfrak{g}$, relative to $\mathfrak{g}_c$, by the following formula: if $x = u + \sqrt{-1}\,v$ ($u,v \in \mathfrak{g}_c$), then $x^\star = -u + \sqrt{-1}\,v$. Then $\{x,y\} := \Phi(x,y^\star)$ is a positive definite Hermitian inner product on $\mathfrak{g}$. Therefore $x \in \mathfrak{g}_i$ if and only if $x^\star \in \mathfrak{g}_{-i}$.
Note also that

$$[x^\star, y^\star] = -[x,y]^\star \text{ for all } x,y \in \mathfrak{g}.$$

4.8 Theorem. *Suppose $\mathfrak{g}_0$ is semisimple whenever $m \ge 3$. Let ℓ_k be the maximal eigenvalue of C_0 on $\wedge^k\mathfrak{g}_1$. Then $\ell_k \le k/m$. Moreover one has $\ell_k = k/m$ if and only if $\mathfrak{g}_1$ contains a k-dimensional commutative Lie subalgebra. In such a case, the eigenspace of C_0 in $\wedge^k\mathfrak{g}_1$ corresponding to k/m is generated by $\wedge^k\mathfrak{a}$, where $\mathfrak{a}$ runs through all commutative k-dimensional subalgebras of $\mathfrak{g}_1$.*

Proof. Let $u = y_1 \wedge \ldots \wedge y_k$ be an arbitrary decomposable element in $\wedge^k\mathfrak{g}_1$. Set $\mathfrak{a} = \langle y_1, \ldots, y_k\rangle$. We may (and shall) assume that $y_1, \ldots, y_k$ is an orthonormal basis for $\mathfrak{a}$ relative to $\{\ ,\ \}$, i.e., $\Phi(y_i, y_j^\star) = \delta_{ij}$.

A straightforward computation (cf. the proof of Prop. 4.6) shows that

$$C_0(u) = \frac{k}{m}u - 2\sum_{i<j}(-1)^{i+j-1}d_1([y_i, y_j]) \wedge y_1 \ldots \hat{y}_i \ldots \hat{y}_j \ldots \wedge y_k =: \frac{k}{m}u - 2\sum_{i<j} v_{ij}\ .$$

Put $u^\star = y_1^\star \wedge \ldots \wedge y_k^\star$. Then $\Phi^{(k)}(u, u^\star) = 1$ and

$$\Phi^{(k)}(v_{ij}, u^\star) = \Phi^{(2)}(d_1([y_i, y_j]), y_i^\star \wedge y_j^\star) = -\Phi([y_i, y_j], [y_i^\star, y_j^\star]) = \{[y_i, y_j], [y_i, y_j]\} \ge 0\ .$$

Hence

$$\Phi^{(k)}(C_0(u), u^\star) = \frac{k}{m} - 2\sum_{i<j}\{[y_i, y_j], [y_i, y_j]\} \le \frac{k}{m}\ .$$

Thus, if u is an eigenvector of C_0, then the respective eigenvalue $\le k/m$. Moreover, the eigenvalue is equal to k/m if and only if all $[y_i, y_j]$ vanish, i.e., $\mathfrak{a}$ is commutative. By Theorem 3.4, the eigenspace belonging to the maximal eigenvalue is generated by decomposable highest weight polyvectors. This completes the proof. □

Remark. Theorem 3.3 is a formal consequence of the last result. Indeed, take $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{g}$ and $\Theta \in \operatorname{Aut}\tilde{\mathfrak{g}}$ such that $\Theta(x, y) = (y, x)$. Then $\tilde{\mathfrak{g}}_0$ is the diagonal ($\simeq \mathfrak{g}$) and $\tilde{\mathfrak{g}}_1 \simeq \mathfrak{g}$ as $\mathfrak{g}$-module. Let $\tilde{C}$ be the Casimir operator for $\tilde{\mathfrak{g}}$ and $\tilde{C} = \tilde{C}_0 + \tilde{C}_1$ the decomposition corresponding to Θ. By Theorem 4.8, the maximal eigenvalue of $\tilde{C}_0$ on $\wedge^k\tilde{\mathfrak{g}}_1 \simeq \wedge^k\mathfrak{g}$ is at most $k/2$. It remains to observe that $\tilde{C}_0 = \frac{1}{2}C$ and the isomorphism $\tilde{\mathfrak{g}}_1 \simeq \mathfrak{g}$ respects commutative subalgebras.

Let $\mathcal{A}_k$ be the linear span in $\wedge^k\mathfrak{g}_1$ of the lines $\wedge^k\mathfrak{a}$, where $\mathfrak{a}$ ranges over all commutative k-dimensional Lie subalgebras of $\mathfrak{g}_1$. In other words, $\mathcal{A}_k$ is the eigenspace of C_0 in $\wedge^k\mathfrak{g}_1$ belonging to the eigenvalue k/m. Our next goal is to get more information on $\mathcal{A} := \oplus_k\mathcal{A}_k$ as $\mathfrak{g}_0$-module. To this end, fix a Cartan subalgebra $\mathfrak{t}_0 \subset \mathfrak{g}_0$. Recall some properties of $\mathfrak{t}_0$.

4.9 Lemma. 1. *$\mathfrak{z}_{\mathfrak{g}}(\mathfrak{t}_0)$ is a Cartan subalgebra of $\mathfrak{g}$;*
2. *The nonzero weights of $\mathfrak{t}_0$ on each $\mathfrak{g}_i$ are of multiplicity 1.*

Proof. 1. By [Ka90, 8.1(b)], $\mathfrak{t}_0$ contains a regular semisimple element of $\mathfrak{g}$.
2. Let G_0 be the connected subgroup of G corresponding to $\mathfrak{g}_0$. According to [Vi76], the linear group $G_0 \to GL(\mathfrak{g}_i)$ is *visible*, i.e. each fibre of the quotient map $\mathfrak{g}_i \to \mathfrak{g}_i/\!\!/G_0$ contains finitely many G_0-orbits. We then conclude by [Ka80, 3.4]. □

Without loss, one may assume that $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{t}_0)$ is $\mathfrak{t}$, the fixed Cartan subalgebra of $\mathfrak{g}$. Set $\mathfrak{t}_i := \mathfrak{t}\cap\mathfrak{g}_i$. Then $\mathfrak{t} = \mathfrak{t}_0 \oplus \ldots \oplus \mathfrak{t}_{m-1}$ and $\mathfrak{t}_i$ is exactly the zero weight space in $\mathfrak{g}_i$. Let Δ_i be the set of nonzero weights of $\mathfrak{t}_0$ on $\mathfrak{g}_i$. This yields the weight decompositions

$$\mathfrak{g}_i = \mathfrak{t}_i \oplus \Big(\bigoplus_{\alpha\in\Delta_i} \mathfrak{g}_i^\alpha\Big) \ ,$$

where $\dim \mathfrak{g}_i^\alpha = 1$ $(i \in \mathbb{Z}_m)$. The totality of weights $\bigcup_i \Delta_i$ lies in $(\mathfrak{t}_0^*)_{\mathbb{Q}}$, and the last space will be identified with a subspace of $\mathfrak{t}^*_{\mathbb{Q}}$. Note that $\Delta_{-i} = -\Delta_i$. There are two different possibilities for $\{\Delta_i\}$:

- Θ is inner. Then $\mathfrak{t}_0 = \mathfrak{t}$ and $\mathfrak{t}_i = 0$ for $i \ne 0$. Here $\Delta_i \cap \Delta_j = \varnothing$ $(i \ne j)$, and $\bigsqcup_{i=0}^{m-1} \Delta_i = \Delta$ is the whole root system for $\mathfrak{g}$;
- Θ is outer. Then $\mathfrak{t}_0 \ne \mathfrak{t}$ and it may happen that $\Delta_i \cap \Delta_j \ne \varnothing$. Here $\overline{\Delta} := \bigcup_i \Delta_i$ consists of all restricted roots $\bar\gamma$ $(\gamma\in\Delta)$, where $\bar\gamma := \gamma|_{\mathfrak{t}_0}$.

To avoid ambiguities, especially in the second case, we let $v_\alpha^{(i)}$ denote a nonzero weight vector for $\alpha \in \Delta_i$. The following lemma contains two different auxiliary results on periodic gradings.

4.10 Lemma.
1. *Let $\alpha\in\Delta_i$, $\beta\in\Delta_j$ and $(\alpha,\beta)_\Phi < 0$. Then $[v_\alpha^{(i)}, v_\beta^{(j)}] \ne 0$ and $[[v_\alpha^{(i)}, v_\beta^{(j)}], v_{-\beta}^{(-j)}] \ne 0$.*
2. *If $h \in \mathfrak{g}_1$ is semisimple and $[\mathfrak{g}_0, h] = 0$, then $h = 0$.*

Proof. 1. If Θ is inner, this is a standard property of the root decomposition.
A general argument goes as follows. Since $\Phi|_{\mathfrak{g}_j\oplus\mathfrak{g}_{-j}}$ is non-degenerate, there exist $v_{-\beta}^{(-j)}$ such that $\Phi(v_\beta^{(j)}, v_{-\beta}^{(-j)}) = 1$. Then the equality $\Phi(h, [v_\beta^{(j)}, v_{-\beta}^{(-j)}]) = \beta(h)\Phi(v_\beta^{(j)}, v_{-\beta}^{(-j)})$ for $h \in \mathfrak{t}_0$ shows that $h_\beta := [v_\beta, v_{-\beta}]$ is a nonzero element of $\mathfrak{t}_0$. It is easily verified that $\langle v_\beta^{(j)}, h_\beta, v_{-\beta}^{(-j)}\rangle$ is a three-dimensional simple subalgebra of $\mathfrak{g}$. Since $[h_\beta, v_\alpha^{(i)}] = (\alpha,\beta)_\Phi{\cdot}v_\alpha^{(i)}$, we see that $v_\alpha^{(i)}$ cannot be a highest weight vector, i.e., $[v_\alpha^{(i)}, v_\beta^{(j)}] \ne 0$. It also follows from the theory of $\mathfrak{sl}_2$-representations that since $[v_\alpha^{(i)}, v_\beta^{(j)}] \in \mathrm{Im}\,(\mathrm{ad}\, v_\beta^{(j)})$, it cannot belong to $\mathrm{Ker}\,(\mathrm{ad}\, v_{-\beta}^{(-j)})$.

2. It is enough to show that $[\mathfrak{g}_i, h] = 0$ for all i. We argue by induction on i, the case $i = 0$ being given by the hypothesis. Assume that $[\mathfrak{g}_i, h] = 0$. The equality $\Phi([\mathfrak{g}_{-i-1}, h], \mathfrak{g}_i) = \Phi(\mathfrak{g}_{-i-1}, [\mathfrak{g}_i, h]) = 0$ shows that $[\mathfrak{g}_{-i-1}, h] = 0$. Since the centralizer of h is reductive, one also must have $[\mathfrak{g}_{i+1}, h] = 0$, which completes the inductive step. $\square$

Choose a system of positive roots $\Delta_0^+ \subset \Delta_0$, and let $\mathfrak{b}_0$ denote the corresponding Borel subalgebra of $\mathfrak{g}_0$. The following is obvious.

4.11 Lemma. *$y_1\wedge\ldots\wedge y_k \in \wedge^k\mathfrak{g}_1$ is a $\mathfrak{b}_0$-eigenvector (i.e., a highest weight vector) if and only if $\langle y_1,\ldots,y_k\rangle$ is a $\mathfrak{b}_0$-stable subspace of $\mathfrak{g}_1$.* $\square$

Let $\mathfrak{a} \subset \mathfrak{g}_1$ be a $\mathfrak{b}_0$-stable subspace. In particular, $\mathfrak{a}$ is $\mathfrak{t}_0$-stable. Therefore

$$\mathfrak{a} = (\mathfrak{a} \cap \mathfrak{t}_1) \oplus \Big(\bigoplus_{\alpha \in \Delta_\mathfrak{a}} \mathfrak{g}_1^\alpha \Big)$$

for some subset $\Delta_\mathfrak{a} \subset \Delta_1$.

4.12 Proposition. *Suppose $\mathfrak{a}$ is $\mathfrak{b}_0$-stable and $[\mathfrak{a}, \mathfrak{a}] = 0$. Then*

1. *$\mathfrak{a}$ contains no semisimple elements. In particular, $\mathfrak{a} \cap \mathfrak{t}_1 = 0$;*
2. *$\Delta_\mathfrak{a}$ is contained in an open half-space of $(\mathfrak{t}_0^*)_\mathbb{Q}$.*

Proof. 1. Suppose $0 \neq h \in \mathfrak{a}$ is semisimple. The equality $[\mathfrak{b}_0, h] = 0$ cannot hold, since we then would also have $[\mathfrak{g}_0, h] = 0$, which implies that $h = 0$, see Lemma 4.10(2). So, take $e \in \mathfrak{b}_0$ such that $[h, e] \neq 0$. Since $\mathfrak{a}$ is $\mathfrak{b}_0$-stable, $[e, h] \in \mathfrak{a}$. On the other hand, h is semisimple, and $[h, e] \neq 0$ forces that $[h, [h, e]] \neq 0$. Thus, $\mathfrak{a}$ is not commutative.
2. From Lemma 4.10(1) it follows that $(\alpha, \beta)_\Phi \geq 0$ for any pair $\alpha, \beta \in \Delta_\mathfrak{a}$. Letting $|\Delta_\mathfrak{a}| := \sum_{\alpha \in \Delta_\mathfrak{a}} \alpha$, we obtain $(|\Delta_\mathfrak{a}|, \alpha)_\Phi > 0$ for all α in $\Delta_\mathfrak{a}$. Thus, $|\Delta_\mathfrak{a}|$ determines a required open-half space. □

4.13 Corollary. *Let Θ be an involutory automorphism. Suppose $\mathfrak{a}$ is a commutative Lie subalgebra of $\mathfrak{g}_1$. Then*

$$\dim \mathfrak{a} \leq (\dim \mathfrak{g}_1 - \dim \mathfrak{t}_1)/2 .$$

Proof. For any commutative subalgebra $\mathfrak{a}$ of $\mathfrak{g}_1$ there exists a $\mathfrak{b}_0$-stable commutative subalgebra of the same dimension (consider $\wedge^k \mathfrak{a}$ as element in $\mathcal{A}_k$, $k = \dim \mathfrak{a}$, and take the subalgebra corresponding to a highest weight vector). Assume therefore that $\mathfrak{a}$ is $\mathfrak{b}_0$-stable. In case $m = 2$, we have $\Delta_1 = -\Delta_1$. Since $\Delta_\mathfrak{a}$ lies in an open half-space, $\#\Delta_\mathfrak{a} \leq \frac{1}{2} \#\Delta_1$. Because also $\mathfrak{a} \cap \mathfrak{t}_1 = 0$, the assertion follows. □

Remarks. 1. Notice that Lemma 4.10(1) and Prop. 4.12(2) are not really needed for the proof of the Corollary. The only required thing is the relation $\Delta_\mathfrak{a} \cap (-\Delta_\mathfrak{a}) = \varnothing$, which follows from the fact that $[v_\beta^{(1)}, v_{-\beta}^{(1)}] \neq 0$.
2. A weaker inequality $\dim \mathfrak{a} \leq \dim \mathfrak{g}_1/2$ was proved by J.A. Carlson and D. Toledo [CT89, 3.1].

4.14 Proposition. *Let $\mathfrak{a}, \mathfrak{a}'$ be $\mathfrak{b}_0$-stable commutative subalgebras of $\mathfrak{g}_1$. If $|\Delta_\mathfrak{a}| = |\Delta_{\mathfrak{a}'}|$, then $\Delta_\mathfrak{a} = \Delta_{\mathfrak{a}'}$ and $\mathfrak{a} = \mathfrak{a}'$.*

Proof. Obviously, we also have $|\Delta_\mathfrak{a} \setminus \Delta_{\mathfrak{a}'}| = |\Delta_{\mathfrak{a}'} \setminus \Delta_\mathfrak{a}|$. Assume $\Delta_\mathfrak{a} \setminus \Delta_{\mathfrak{a}'} \neq \varnothing$. Then $|\Delta_\mathfrak{a} \setminus \Delta_{\mathfrak{a}'}| \neq 0$, because $\Delta_\mathfrak{a}$ lies in an open half-space. Since $(|\Delta_\mathfrak{a} \setminus \Delta_{\mathfrak{a}'}|, |\Delta_{\mathfrak{a}'} \setminus \Delta_\mathfrak{a}|)_\Phi > 0$, one may conclude that there exist $\alpha \in \Delta_\mathfrak{a} \setminus \Delta_{\mathfrak{a}'}$ and $\beta \in \Delta_{\mathfrak{a}'} \setminus \Delta_\mathfrak{a}$ such that $(\alpha, \beta)_\Phi > 0$. Applying Lemma 4.10 to $(\alpha, -\beta)$ shows that $\gamma := \alpha - \beta \in \Delta_0$ and $0 \neq [e_\gamma^{(0)}, v_\beta^{(1)}] \in \mathfrak{g}_1^\alpha$. Without loss of generality, one may assume that $\gamma \in \Delta_0^+$. (Otherwise consider $(-\alpha, \beta)$.) As $\mathfrak{a}'$ is $\mathfrak{b}_0$-stable, we must then have $\alpha \in \Delta_{\mathfrak{a}'}$. A contradiction! □

Now, we can summarize our considerations in the following

4.15 Theorem. 1. *Let $S = \{\alpha_1, \dots, \alpha_k\}$ be any set of (distinct) weights in Δ_1 and $\mathfrak{a}_S := \langle v^{(1)}_{\alpha_1}, \dots, v^{(1)}_{\alpha_k}\rangle$ the corresponding subspace of $\mathfrak{g}_1$. Set $2\rho_0 := |\Delta_0^+|$. Then one always has the inequality*

$$(\alpha_1 + \dots + \alpha_k,\ \alpha_1 + \dots + \alpha_k + 2\rho_0)_\Phi \le k/m \ , \tag{4.16}$$

and the equality holds if and only if $\mathfrak{a}_S$ is a $\mathfrak{b}_0$-stable commutative Lie subalgebra of $\mathfrak{g}_1$. Moreover, every commutative $\mathfrak{b}_0$-stable subalgebra is (uniquely) of this form;
2. *$\mathcal{A}$ is a multiplicity-free $\mathfrak{g}_0$-module;*
3. *Let $S_1, \dots, S_q$ be all k-element subsets of Δ_1 such that the equality in Eq. (4.16) holds. Then the non-zero elements in $\wedge^k \mathfrak{a}_{S_1}, \dots, \wedge^k \mathfrak{a}_{S_q}$ are the only highest weight vectors in $\mathcal{A}_k$.*

Proof. 1. The sum $\mu := \alpha_1 + \dots + \alpha_k$ is a weight of $\mathfrak{g}_0$-module $\wedge^k \mathfrak{g}_1$. Therefore this part is a consequence of Theorem 4.8 and the following property of weights: If μ occurs in an irreducible $\mathfrak{g}_0$-module V_λ $(\subset \wedge^k \mathfrak{g}_1)$, then $(\mu, \mu + 2\rho_0)_\Phi \le (\lambda, \lambda + 2\rho_0)_\Phi$ and the equality holds if and only if $\lambda = \mu$. Recall that $(\lambda, \lambda + 2\rho_0)_\Phi$ is the eigenvalue of C_0 on V_λ.
2. In view of Theorem 4.8 and Lemma 4.11, the mapping $\mathfrak{a} \mapsto \wedge^{\dim \mathfrak{a}} \mathfrak{a}$ establishes a one-to-one correspondence between the $\mathfrak{b}_0$-stable commutative subalgebras of $\mathfrak{g}_1$ and the $\mathfrak{b}_0$-stable lines in $\mathcal{A}$. Since the corresponding highest weight is $|\Delta_\mathfrak{a}|$, the assertion follows from Prop. 4.14.
3. This follows from parts 1,2 and Lemma 4.11. □

5 $Spin_0(\mathfrak{g}_1)$ and the maximal eigenvalue of C_0

In this section, Θ is an involutory automorphism of $\mathfrak{g}$. In other words, $m = 2$ and $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$. We also assume that $\mathfrak{g}$ is simple. Let $\ell = \max_k \ell_k$ be the maximal eigenvalue of C_0 on $\wedge \mathfrak{g}_1$. We know that $\ell_k \le k/2$, but the maximal ℓ_k may fail to be equal to $k/2$. That is, ℓ is not necessarily attained on "commutative subalgebras". In this section, we give a precise formula for ℓ and show that for the Hermitian involutory automorphisms ℓ does correspond to commutative subalgebras of maximal dimension.

It is a well-known result of Kostant that there exists an isomorphism of $\mathfrak{g}$-modules $\wedge \mathfrak{g} \simeq 2^{\dim \mathfrak{t}}(V_\rho \otimes V_\rho)$. In [Pa00], we obtained a generalization of this to any $\mathbb{Z}_2$-grading of $\mathfrak{g}$. Roughly speaking, the theory of Clifford algebras implies that there exists a $\mathfrak{g}_0$-module, denoted by $Spin_0(\mathfrak{g}_1)$, such that

$$\wedge \mathfrak{g}_1 \simeq 2^{\dim \mathfrak{t}_1} \cdot Spin_0(\mathfrak{g}_1)^{\otimes 2} \ , \tag{5.1}$$

and we gave an explicit general description of $Spin_0(\mathfrak{g}_1)$. We retain the notation of Section 4 concerning the weights of $\mathfrak{t}_0$, adapted to the case $m = 2$. Recall that $\Delta_0 \cup \Delta_1 = \{\bar\gamma \mid \gamma \in \Delta\} =: \overline{\Delta}$. For our present purpose, it is important that Δ_0^+ and Δ^+ to be compatibly chosen. This is being achieved by putting $\Delta_0^+ = \overline{\Delta^+} \cap \Delta_0$.

Let W (resp. W_0) be the Weyl group of $(\mathfrak{g},\mathfrak{t})$ (resp. $(\mathfrak{g}_0,\mathfrak{t}_0)$). Since $\mathfrak{t}_0$ contains regular elements of $\mathfrak{t}$, W_0 can be identified with a subgroup of W. It was shown in [Pa00, Sect. 5 & 6] that there exists an intermediate subgroup $W_0\subset W_{\bar 0}\subset W$, which is responsible for the description of $Spin(\mathfrak{g}_1)$. Namely, $W_{\bar 0}$ is a Weyl group in its own right, acting as reflection group on $\mathfrak{t}_0$; the root system for $W_{\bar 0}$ is $\overline{\Delta}$. Next, each coset in $W_{\bar 0}/W_0$ contains a unique element of minimal length, the length function on $W_{\bar 0}$ being defined by the set of positive roots $\overline{\Delta^+}$. Let $W^0\subset W_{\bar 0}$ be the set of such "minimal" elements. A more geometric description of it is:

(5.2)
$$W^0=\{w\in W_{\bar 0}\mid w(\Delta_0^+)\subset\overline{\Delta^+}\}\ .$$

At the rest of the section, notation V_λ refers to a $\mathfrak{g}_0$-module.

5.3 Theorem [Pa00, 5.2, 6.5] **.**

(i) *For each $w\in W^0$, $\lambda_w:=w^{-1}\rho-\rho_0$ is dominant relative to Δ_0^+;*

(ii) *$Spin_0(\mathfrak{g}_1)=\oplus_{w\in W^0}V_{\lambda_w}$.*

Remark. It is worth mentioning two possibilities. If Θ is inner, then $W_{\bar 0}=W$ and both the construction and the proof become much simpler. At the other extreme, for some outer Θ, we have $W_0=W_{\bar 0}$. Then $W^0=\{id\}$ and $Spin_0(\mathfrak{g}_1)$ appears to be irreducible. This occurs precisely if $\mathfrak{g}_0$ is non-homologous to zero in $\mathfrak{g}$. One obtains here an exact analogue of Kostant's formula for the adjoint representation. (Note that the adjoint representation, as isotropy representation, corresponds to the involution Θ permuting the summands of $\mathfrak{g}\oplus\mathfrak{g}$. Here the diagonal is non-homologous to zero in $\mathfrak{g}\oplus\mathfrak{g}$.)

Although $Spin_0(\mathfrak{g}_1)$ can be highly reducible $\mathfrak{g}_0$-module, the formula for λ_w implies that C_0 acts scalarly on $Spin_0(\mathfrak{g}_1)$, with eigenvalue $(\rho,\rho)_\Phi-(\rho_0,\rho_0)_\Phi$ (see [Pa00, 7.7]).

Combining Theorem 5.3 with Eq. (5.1), we get

5.4 Proposition. *The maximal eigenvalue of C_0 on $\wedge\mathfrak{g}_1$ is equal to*
$$\ell=\max_{w\in W^0}4(\rho,\rho-w^{-1}\rho_0)_\Phi\,.$$

Proof. We use the following easy properties of the function $\lambda\overset{f}{\mapsto}(\lambda,\lambda+2\rho_0)_\Phi=$ {the eigenvalue of C_0 on V_λ}:

- The maximal eigenvalue of C_0 on $V_{\lambda_1}\otimes V_{\lambda_2}$ is being attained on the Cartan (highest) component of this product;
- f is a concave function of λ.

It follows that the maximal eigenvalue of C_0 on $(V_{\lambda_1}\oplus\ldots\oplus V_{\lambda_n})^{\otimes 2}$ is equal to $\max_i f(2\lambda_i)$. Applying this to the collection of highest weights λ_w ($w\in W^0$), one obtains the required formula. (The equality $(w^{-1}\rho,w^{-1}\rho)_\phi=(\rho,\rho)_\Phi$ can be used, since $w\in W_{\bar 0}\subset W$.) $\Box$

Remarks. 1. By Theorem 4.8, dimension of any commutative Lie subalgebra of $\mathfrak{g}_1$ is at most $[2\ell]$. Examples show that this usually gives a better bound than that in Corollary 4.13.

2. It is tempting to suggest that the maximal eigenvalue always corresponds to $w = id$. Indeed, this is true if $W^0 = \{id\}$ or for all Hermitian involutions, see below. But for $\mathfrak{g} = \mathbf{G}_2$ this is not the case. It might be interesting to describe the corresponding w for all involutory automorphisms.

5.5 Example. Let $\mathfrak{g} = \mathbf{E}_6$ and $\mathfrak{g}_0 = \mathbf{F}_4$. In this example, the notation α_i, φ_i refers to simple roots and fundamental weights of $\mathbf{F}_4$. For instance, φ_4 is the highest root and $\rho_0 = \varphi_1 + \ldots + \varphi_4$.
Here $\mathfrak{g}_1 = V_{\varphi_1}$ and $W^0 = \{id\}$. Therefore $\ell = 4(\rho, \rho - \rho_0)_\Phi$. It is convenient to perform all computations in terms of weights and roots of $\mathbf{F}_4$. We have $\rho - \rho_0 = \varphi_1 + \varphi_2 = 5\alpha_1 + 9\alpha_2 + 6\alpha_3 + 3\alpha_4$ and $\rho = 2\varphi_1 + 2\varphi_2 + \varphi_3 + \varphi_4$, whence $4(\rho, \rho - \rho_0)_\Phi = 46(\alpha_4, \alpha_4)_\Phi$. It is easily seen that any long root for $\mathbf{F}_4$ is also a long root for $\mathbf{E}_6$. Therefore $(\alpha_4, \alpha_4)_\Phi = 1/h^*(\mathbf{E}_6) = 1/12$ and $\ell = 23/6$. This implies that dimension of any commutative subalgebra of $\mathfrak{g}_1$ is at most $[23/3] = 7$, whereas Corollary 4.13 provides the bound 12. However, it is not hard to calculate that $\ell_2 = 1, \ell_3 = 3/2$, and $\ell_4 = 23/12 < 2$. Thus, maximal commutative subalgebras are of dimension 3.

In the remainder of the section, we assume that Θ is a *Hermitian involution*, i.e., $\mathfrak{g}_0$ is not semisimple. Then Θ is necessarily inner, the centre of $\mathfrak{g}_0$ is 1-dimensional, and $\mathfrak{g}_1$ is the sum of two irreducible dual $\mathfrak{g}_0$-modules. Since now $\mathfrak{t} = \mathfrak{t}_0$, we have $\Delta = \overline{\Delta}$ and $\Delta = \Delta_0 \sqcup \Delta_1$. Given $\alpha \in \Pi$ and μ in the root lattice, let $\mathrm{ht}_\alpha \mu$ denote the coefficient of α in the expression of μ through the simple roots. There is a one-to-one correspondence between the conjugacy classes of Hermitian involutions and the simple roots α such that $\mathrm{ht}_\alpha \Lambda = 1$. If $\alpha = \alpha_\Theta \in \Pi$ corresponds to a Hermitian Θ, then the sets Δ_0, Δ_1 are the following. We have $\mathrm{ht}_\alpha(\Delta) = \{-1, 0, 1\}$. Put $\Delta(i) = (\mathrm{ht}_\alpha)^{-1}(i)$. Then $\Delta_0 = \Delta(0)$ and $\Delta_1 = \Delta(1) \cup \Delta(-1)$. The compatibility of Δ_0^+ and Δ^+ required for Theorem 5.3 means that $\Delta^+ = \Delta_0^+ \cup \Delta(1)$. Note that $\alpha, \Lambda \in \Delta(1)$. Let φ be the fundamental weight corresponding to α. Set $2\rho_1 = |\Delta(1)|$, so that $\rho = \rho_0 + \rho_1$.

5.6 Lemma. 1. $2\rho_1 = h^*(\mathfrak{g})\varphi$;
2. $\mathrm{ht}_\alpha 2\rho = \#\Delta(1)$.

Proof. 1. This is easily checked case-by-case. However a conceptual proof is available. Since $|\Delta(1)|$ is the sum of all weights of a $\mathfrak{g}_0$-module, it must be orthogonal to all simple roots of $\mathfrak{g}_0$, t.e., to any $\beta \in \Pi \setminus \{\alpha\}$. Hence $2\rho_1$ is proportional to φ, say $2\rho_1 = x\varphi$. Then $x = (2\rho_1, \alpha^\vee)_\Phi$. It is easily seen that the longest element in W_0 takes ρ_1 to itself and α to Λ. Therefore

$$x = (2\rho_1, \Lambda^\vee)_\Phi = \sum_{\gamma \in \Delta(1)} (\gamma, \Lambda^\vee)_\Phi = \#\{\gamma \in \Delta(1) \mid (\gamma, \Lambda) \neq 0,\ \gamma \neq \Lambda\} + 2\ .$$

Obviously, the set R of *all* positive roots γ such that $(\gamma, \Lambda) \neq 0$ and $\gamma \neq \Lambda$ breaks into pairs $\{\gamma_1^{(i)}, \gamma_2^{(i)}\}_{i \in I}$, where $\gamma_1^{(i)} + \gamma_2^{(i)} = \Lambda$. (This is a manifestation of the fact that the nilpotent radical of the centralizer of a highest root vector is a Heisenberg Lie algebra.)

Since $\mathrm{ht}_\alpha(\gamma_1^{(i)} + \gamma_2^{(i)}) = 1$, exactly one element in each pair belongs to $\Delta(1)$. Hence $x = \#I + 2 = (\#R)/2 + 2$. Remembering Lemma 1.3, we see that $x = (\dim \mathcal{O}_{min} - 2)/2 + 2 = h^*(\mathfrak{g})$.

2. This is obvious, since $\mathrm{ht}_\alpha \mu = 1$ if and only if $\mu \in \Delta(1)$. □

5.7 Theorem.

1. *If Θ is Hermitian, then the maximal eigenvalue of C_0 on $\wedge\mathfrak{g}_1$ is attained with $w = id \in W^0$ (in the formula of Prop. 5.4).*

2. *This maximal eigenvalue is equal to* $\dim \mathfrak{g}_1/4$.

Proof. 1. The difference between the value corresponding to $w = id$ and the arbitrary w is equal to $4(w^{-1}\rho - \rho, \rho_0)_\Phi = 2\sum_{\gamma\in\Delta_0^+}(w^{-1}\rho - \rho, \gamma)_\Phi$. Let us show that every summand on the right hand side is non-negative. It is enough to check this for the set Π_0 of all simple roots in Δ_0^+. But the Hermitian case can be characterized by the property that $\Pi_0 \subset \Pi$. Indeed, $\Pi_0 = \Pi \setminus \{\alpha\}$. Thus, if $\gamma \in \Pi_0$ then

$$(w^{-1}\rho - \rho, \gamma^\vee)_\Phi = (\rho, (w\gamma)^\vee)_\Phi - (\rho, \gamma^\vee)_\Phi = (\rho, (w\gamma)^\vee)_\Phi - 1 \ .$$

By Eq. (5.2), we have $w\gamma$ is positive. Whence $(\rho, (w\gamma)^\vee)_\Phi \geq 1$.

2. By part 1, the maximal eigenvalue is equal to $4(\rho, \rho) - 4(\rho, \rho_0) = (2\rho, 2\rho_1)$. By Lemma 5.6, the latter is equal to

$$\#\Delta(1)\cdot h^*(\mathfrak{g})\cdot(\alpha, \varphi)_\Phi = \frac{\dim \mathfrak{g}_1}{2}\cdot h^*(\mathfrak{g})\cdot\frac{(\alpha, \alpha)_\Phi}{2}.$$

Since α is long, we conclude by the definition of the dual Coxeter number. □

Remarks. 1. Since $\mathfrak{g}(1) = \oplus_{\alpha\in\Delta(1)}\mathfrak{g}^\alpha$ is a commutative Lie subalgebra and $\dim \mathfrak{g}(1) = \dim \mathfrak{g}_1/2 =: a$, we have $\mathfrak{g}(1)$ is maximal commutative by Corollary 4.13. This subalgebra generates a submodule of $\mathcal{A}_a \subset \wedge^a \mathfrak{g}_1$, where, by Theorem 4.8, C_0 takes value $\dim \mathfrak{g}_1/4$. So, the maximal eigenvalue of C_0 is attained on commutative subalgebras. Example 5.5 shows that this is not a general phenomenon, see also a discussion for the adjoint representation in [Ko98, 1.1].

2. Because $\wedge^i \mathfrak{g}_1 \simeq \wedge^{2a-i}\mathfrak{g}_1$ and $\mathfrak{g}_1$ contains a-dimensional commutative subalgebras, the second claim of Theorem 5.7 follows from Theorem 4.8 as well.

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