AN INVERSE FORM OF THE BRUNN-MINKOWSKI INEQUALITY WITH APPLICATIONS TO LOCAL THEORY OF NORMED SPACES

V.D. Milman

Tel Aviv University
Tel Aviv, Israel

Max-Planck-Institut
für Mathematik
Gottfried-Claren-Str. 26
D-5300 Bonn 3

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Abstract

For every convex symmetric compact body K_X in $\mathbb{R}^n$ a linear map u_X, $\det u_X = 1$, is constructed. Then, for every two such bodies K_X and K_Y, an inverse form of the classical Brunn-Minkowski inequality for volumes is true up to a numberical constant C for the bodies $u_X K_X$ and $u_Y K_Y$ and also for their polars. The result is applied to study normed linear spaces.

Let $X = (\mathbb{R}^n, \|\cdot\|)$ be a normed space and $K(\|\cdot\|)$ (also $K(X)$ or just K) be its unit ball. We also equip X with a euclidean norm $(\mathbb{R}^n, |\cdot|)$ and, as consequence, with the inner product (x,y) such that $(x,x) = |x|^2$. Let $D(|\cdot|)$ be an ellipsoid in $\mathbb{R}^n$ which is the unit ball of the norm $|\cdot|$. We denote $d_X = d(X, \ell_2^n)$ the Banach-Mazur distance between X and ℓ_2^n and

$$d(K, D(|\cdot|)) = \inf \{ab: a^{-1}|x| \le \|x\| \le b|x| \quad \text{for} \quad x \in \mathbb{R}^n\}.$$

The dual norm $\|\cdot\|^*$ is naturally defined by $\|x\|^* = \sup\{|(x,y)| : \|y\| \le 1\}$. Then $K(\|\cdot\|^*) = K^\circ$ is the polar body of K with respect to the inner product defined by $|\cdot|$. Throughout the paper, we use the same letters c, C for different numerical constants.

A. The main results of this note are the following ones:

<u>Theorem 1:</u> a) There exists a numerical constant C such that for every convex compact symmetric bodies K_X and K_Y in $\mathbb{R}^n$ there exists a linear map $u: \mathbb{R}^n \to \mathbb{R}^n$, $\det u = 1$, such that for every $\varepsilon > 0$

$$[\mathrm{Vol}\,(uK_X + \varepsilon K_Y)]^{1/n} \le C([\mathrm{Vol}\,K_X]^{1/n} + \varepsilon[\mathrm{Vol}\,K_Y]^{1/n})$$

and the same inequality is also true for the polar bodies $(uK_y)^\circ$, K_X° and K_y° .

b) In the case of $K_y = D$ (the euclidean ball) we have a slight improvement in the above inequality: for some numberical sequence $\alpha_n \to 1 (n \to \infty)$

$$[\mathrm{Vol}\ (uK_X + \varepsilon D)]^{1/n} \leq C[\mathrm{Vol}\ K_X]^{1/n} + \alpha_n \varepsilon [\mathrm{Vol}\ D]^{1/n}$$

and the same inequality is satisfied for the polar body $(uK_X)^\circ$.

<u>Corollary 2.</u> There exists a numberical constant $C > 0$ such that every finite dimensional normed space $X = (\mathbb{R}^n, \|\cdot\|)$ has an euclidean structure $(\mathbb{R}^n, |\cdot|)$ such that

$$\mathrm{Vol\ Conv}\ (K(X) \cup D) \leq C^n\ \mathrm{Vol}\ D$$

and

$$\mathrm{Vol\ Conv}\ (K(X^*) \cup D) \leq C^n\ \mathrm{Vol}\ D,$$

where $D = \{x \in \mathbb{R}^n : |x| \leq 1\}$.

<u>Proof.</u> Using Theorem 1 , we may assume that $[\mathrm{Vol}(K(X) + \varepsilon D)]^{1/n} \leq C([\mathrm{Vol}\ K(X)]^{1/n} + \varepsilon(\mathrm{Vol}\ D)^{1/n})$ and similarly for $K(X)^\circ = K(X^*)$. By a proportional normalization of D we may also assume that $\mathrm{Vol}\ K(X) = \mathrm{Vol}\ D$. Then, by Santalo inequality [S], $\mathrm{Vol}\ K(X^*) \leq \mathrm{Vol}\ D$. Take now $\varepsilon = 1$ and note that $\mathrm{Conv}\ (K(X) \cup D) \subset K(X) + D$.

□

By $[M_1]$, Theorem 4.1., Corollary 2 implies

<u>Theorem 3.</u> Let $X = (\mathbb{R}^n, \|\cdot\|)$ be a normed space with a euclidean structrue from Corollary 2. Then for every $\lambda < 1$ there exists a set A of $[\lambda n]$-dimensional subspaces of $\mathbb{R}^n$ of a normalized Haar measure (say, $\mu(A) \geq 1 - 4^{-n}$) and for every $E \in A$ there exists a subspace $F: E \subset F \subset \mathbb{R}^n$, such that the Banach -Mazur distance d_E of $\ell_2^{[\lambda n]}$ from E, equipped with the quotient norm $(F, \|\cdot\|)/_{E^\perp}$, where $E^\perp = \{x \in F, x \perp E\}$, is at most

$$d_E \leq f(1/(1-\lambda)),$$

where $f(t)$ depends on $t > 0$ only.and not on n.

<u>Remark 1.</u> The above theorem is a probabilistic version of the "quotient of a subspace" theorem proved before (see $[M_2]$, $[M_3]$). However, we would like to emphasize that the construction of the euclidean norm in the Theorem 3 heavily uses the previous version (non-probabilistic)of this theorem.

<u>Remark 2.</u> Using Corollary 2 also all other statements of Theorem 4.1.from $[M_1]$ are now applicable for every finite dimensional normed space.

<u>Remark 3.</u> Direct use of Theorem 4.1 from $[M_1]$ gives for a function $f(t)$ an exponential estimate $f(t) \leq C^t$. Using more delicate tools, it can be shown that f has only

polynomial growth, say $f(t) \leq C\, t^2$.

B. <u>Construction of a special ellipsoid related to a given convex symmetric compact body</u> $K \subset \mathbb{R}^n$. We need additional notations. Let $u: (\mathbb{R}^n, |\cdot|) \to (\mathbb{R}^n, \|\cdot\|)$ be a linear map from a euclidean space. We use the ℓ-norm of u defined by $\ell(u) = \sqrt{n}(\int_S \|ux\|^2 d\mu(x))^{1/2}$ where S is the euclidean unit sphere and μ is the normalized rotation invariant measure on S. If u is invertable then the dual operator norm $\ell^*(u^{-1})$ is defined (we consider, following Pietsch [Pi], the trace duality).

For a given euclidean structure $(\mathbb{R}^n, |\cdot|)$ we introduce a transformation qs of a convex symmetric body $K \subset \mathbb{R}^n$ defined by two subspaces $F \subset E \subset \mathbb{R}^n$. Then

$$qsK = P_F(K \cap E) \subset F$$

where P_F is the orthogonal projection on F. We say that qs operates from $\mathbb{R}^n$ on F and $\dim qs = \dim F$. Similarly if $X = (\mathbb{R}^n, \|\cdot\|, |\cdot|)$ is a normed space with a euclidean structure defined by a norm $|\cdot|$, then qsX is the norm space with the unit ball $qsK(X)$ (i.e. qsX is a quotient of a subspace of X). We use also the dual operation qs which is a restriction on a subspace F of the orthogonal projection on a subspace E, $F \subset E$ (i.e. sqX is a subspace of a quotient space of X).

Construction. We start with ℓ-euclidean structure on $X = (\mathbb{R}^n, \|\cdot\|)$, i.e. with such euclidean norm $|\cdot|$ that the identity map $u: (\mathbb{R}^n, |\cdot|) \to (\mathbb{R}^n, \|\cdot\|)$ satisfies $\ell(u)\ell^*(u^{-1}) = n$. (It was introduce in [F.T.] using [L]). Then $d(K(X), D(|\cdot|)) \leq n$. It is known ([$M_2$] and [$M_3$] ; see a short proof also in [B.M], Lemma 4.6) that there exists qs-operation on a subspace $E_{\tilde{n}_1} \subset \mathbb{R}^n$ $(\dim E_{\tilde{n}_1} = \tilde{n}_1 \geq n[1 - c/(\log d_X)^2])$ such that

$$(1) \qquad d(qsK(X), D(E_{\tilde{n}_1}, |\cdot|)) \leq C(\log(d_X + 1))^5 \leq C(\log(n+1))^5 .$$

Let $qsK(X)$ be the unit ball of the space $X_1 = (E_{\tilde{n}_1}, \|\cdot\|_1)$. Take an ℓ-euclidean structure $|\cdot|$ in X_1, i.e. a map $u_1 (E_{\tilde{n}_1}, |\cdot|_1) \to X_1$ satisfies $\ell(u_1)\ell^*(u_1^{-1}) = \tilde{n}_1$, and restrict it on a subspace $E_{n_1} \subset X_1$, $\dim E_{n_1} = n_1 \geq \lambda n$, where $d_1 = d(K(X_1), D(E_{n_1}, |\cdot|_1)) \leq C(1-\lambda)^{-3/2} d_{X_1} \log d_{X_1}$ (combine Lemma 6.1. from [F.T.] with Proposition 2.5. from [M_3]; the same reasons are used in [M_4], section 4). Choose $(1-\lambda)^{-1} \cong$ $\cong c(\log(d_X + 1))^2$ and use the estimate (1) on d_{X_1}. We obtain $d_1 \leq C(\log(d_X + 1))^9$. Combining this estimate with (1) we see that on the subspace $E_{n_1} \subset \mathbb{R}^n$

$$(2) \qquad d(D(|\cdot|), D(|\cdot|_1)) \leq C(\log(d_X + 1))^{14} .$$

We correct at this point our euclidean norm $|\cdot|$ and substitute it on a euclidean norm $|||\cdot|||_1$ such that $|||\cdot|||_1|_{E_{n_1}^\perp} = |\cdot|\,|_{E_{n_1}^\perp}$ and $|||\cdot|||_1|_{E_{n_1}} = |\cdot|_1|_{E_{n_1}}$.

Clearly it may be done in the way that in $\mathbb{R}^n$ we have

$$d(D(|||\cdot|||_1),D(|\cdot|)) \leq C(\log(d_X + 1))^{14}$$

as in (2). Therefore $d(K(X),D(|||\cdot|||_1)) \leq C\, n(\log(d_X + 1))^{14}$. Note now that $\ell(u)$ and $\ell^*(u^{-1})$ are ideal operator norms and so $\ell(u_1|_{E_{n_1}})\ \ell^*(u_1^{-1}|_{u_1E_{n_1}}) \leq \tilde{n}_1 \leq n$. Therefore, we again may use [M_2] and [M_3] to find a qs-operation on a subspace $E_{\tilde{n}_2} \subset \mathbb{R}^n$, $\dim E_{\tilde{n}_2} = \tilde{n}_2 \geq n_1(1 - C/(\log d_{E_{n_1}})^2) \geq n_1(1 - C/(\log\log(d_X + 1))^2)$ and, as in (1),

$$d(qs\ K(E_{n_1}),\ D(E_{\tilde{n}_2},|||\cdot|||_1)) \leq C(\log d_1)^5 \leq C(\log\log(d_X + 1))^5.$$

We continue such procedure $t = t(n)$ times where t is the smallest number such that t-iterated logarithm $\underbrace{\log\ldots\log}_{t} n = \log^{(t)} n \leq 2$. Let $|||\cdot||| = |||\cdot|||_t$ be an euclidean norm constructed on the last step. Then $d(D(|||\cdot|||),D(|\cdot|)) \leq$ $\leq C(\prod_{i=1}^{t} \log^{(i)} n)^{14} \overset{\text{def}}{=} f(n)$, and $d(K(X),D(|||\cdot|||)) \leq cn\cdot f(n) \leq$ $\leq c\, n^2$.

The main property of the constructed euclidean norm $|||\cdot|||$ in $X = (\mathbb{R}^n, \|\cdot\|)$ is the following one:

in the space $X = (\mathbb{R}^n, \|\cdot\|, |||\cdot|||)$ there exist a partial flag of subspaces $\mathbb{R}^n = E_{n_0} \supset E_{n_1} \supset \ldots \supset E_{n_t}$ and a sequence of of qs-operations φ_i, $i = 1,2,\ldots,t$, from $E_{n_{i-1}}$ on E_{n_i} such that

i) $d(K_i, D(E_{n_i}, |||\cdot|||)) \leq C(\log^{(i)} n)^9$, for $i = 1,2,\ldots,t$, where $K_i = \varphi_i K_{i-1} \subset E_{n_i}$ and $K_0 = K(X)$ is the unit ball of X; c is a numerical constant;

ii) dimensions n_i, $i = 1,2,\ldots,t(n)$, depend on n only and for numerical constants c_1 and c_2

$$n_{i-1}\left[1 - \frac{c_1}{(\log^{(i)} n)^2}\right] \geq n_i \geq n_{i-1}\left[1 - \frac{c_2}{(\log^{(i)} n)^2}\right],$$

$i = 1,2,\ldots,t(n) = t$, $n_0 = n$ and $t(n)$ is the first integer such that the t-iterated logarithm $\log^{(t)} n \leq 2$.

Using technique from [B.M.] (see section 4.6) we may also state:

iii) there exist numerical constants c and C such that

$$c\left(\frac{\text{Vol } K}{\text{Vol } D(|||\cdot|||)}\right)^{1/n} \leq \left(\frac{\text{Vol } K_i}{\text{Vol } D(E_{n_i}, |||\cdot|||)}\right)^{1/n_i} \leq C\left(\frac{\text{Vol } K}{\text{Vol } D(|||\cdot|||)}\right)^{1/n}$$

for all $i = 1,2,\ldots,t$ and the same inequalities are satisfied for the dual bodies K° and K_i° with respect to the euclidean norm $|||\cdot|||$. Moreover, constants c and C can be taken as $1 - \varepsilon(n)$ and $1 + \varepsilon(n)$ where $\varepsilon(n) \to 0$ if $n \to \infty$.

C. Plan of the proof of Theorem 1. To prove Theorem 1 we choose u such that the constructed above ellipsoids for

uK_X and K_y are proportional and their partial flags of subspaces (see the property i)) coincide (use the property ii)). We apply now qs-operations φ_i to pass from the convex bodies $uK_X + \varepsilon K_y$, uK_X and K_y to the bodies $(uK_X)_i + \varepsilon(K_y)_i$, $(uK_X)_i$ and $(K_y)_i$ where we denote $(A)_i = \varphi_i((A)_{i-1})$ as in i). (Note that $(uK_X + \varepsilon K_y)_i$ is different from $(uK_X)_i + \varepsilon(K_y)_i$). The Property i) and a technique from [B.M.] (Section 4b) allow us to show that, on the i-th step, the n_i-th root of the ratio of volumes of these bodies to the volume of the n_i-dimensional unit ball $D(|||\cdot|||)$ will not change much (as in the property iii)). Then, after t steps, we come to the bodies, C-isomorphic to euclidean balls (in $|||\cdot|||$-norm) of some radii. In this case the inequality is trivial.

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