MICROLOCALITY OF THE CAUCHY PROBLEM IN INHOMOGENEOUS GEVREY CLASSES

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§1. INTRODUCTION

1. In this paper we consider the (one-sided) Cauchy problem

$$p(x,t,D_x,D_t)\,u = 0\ , \qquad t > 0\ , \tag{1}$$

$$D_t^j u\,|_{t=0} = f_j\ , \qquad j = 0,\dots,m-1\ , \tag{2}$$

for al class of linear partial differential operators with analytic coefficients.Our main result is that the obstructions to solve this problem are essentially of microlocal nature (cf. theorem 1.5 and the related result on microregularity from theorem 1.18).When microlocalisation is here with respect to the analytic wave front set,this result is essentially known.For constant coefficients it is in fact proved in Liess [3] and for analytic coefficients it can be proved starting from a result of Schapira [1] (also cf. Sjöstrand [1] and Hörmander [7]). We shall therefore add an additional assumption on the operator under consideration and gain additional precision in the results, in that microlocality and microregularity will be obtained with respect to some wave front set which is better adapted to the problem under study. (In particular we should note that this wave front set localizes on sets which are, in general, considerably smaller than those which appear in the analytic wave front set.) Technically our assumptions depend on some weight function on R^n and for a particular choice of this function we recover standard analytic microlocalisation. (Proofs could be simplified significantly in that case.) For all other choices of the weight function our additional assumption implies that the principal part of the operator degenerates on a nontrivial set and that the lower order terms satisfy a condition of Levi type. Typical examples of equations which one then obtains are quasi-elliptic equations, the Schrödinger equation, products of such equations perturbed by low order terms, and the image of such equations under linear changes of coordinates.

To be more precise, we shall assume that $p(x,t,D_x,D_t)$ has the form

$$p(x,t,D_x,D_t) = D_t^j + \sum_{\substack{j < m \\ |\alpha| + j \le m}} q_{j\alpha}(x,t)\, D_x^\alpha D_t^j \quad , \qquad (3)$$

for some real-analytic coefficients $q_{j\alpha}(x,t)$ defined for (x,t) near $0 \in R_x^n \times R_t$. (Later on we will write z for (x,t), $q_{j\alpha}(z)$ for $q_{j\alpha}(x,t)$ and $p(z,D)$ for $p(x,t,D_x,D_t)$. As usual, $D_x^\alpha = (-i)^{|\alpha|}(\partial/\partial x)^\alpha$, $D_t^j = (-i)^j(\partial/\partial t)^j$.) The main assumption is now that we are given some (globally) Lipschitz-continuous function $\phi : R^n \to R$ (by this we mean, that

$| \phi(\xi) - \phi(\eta)| \le C|\xi - \eta|, \forall \xi, \forall \eta \in R^n$, for some C), such that

$$\left| D^{\gamma}_{x,t} \; D^{\beta}_{\xi} \; \sum_{\alpha} q_{j\alpha}(x,t) \; \xi^{\alpha} \right| \; \le \; c^{|\gamma|+1} \; \gamma! \phi(\xi)^{m-j-|\beta|} \;,$$

$$\forall j \,,\; \forall \gamma \,,\; \forall \beta \,,\forall \xi \in R^{n} \,,\text{if } |(x,t)| \le \varepsilon \,, \quad (4)$$

for some $c > 0$ and $\varepsilon > 0$.

For technical reasons, we shall always assume that $\phi(\xi) \ge |\xi|^{\delta}$, for some positive δ , at infinity, and $\phi(\xi) > 0$, $\forall \xi \in R^{n}$.

To give an example, assume that we are given some rational numbers $M_i \ge 1$, $i = 1,\dots,n$, and suppose

$$p(x,t,D_x,D_t) = D^{m}_{t} + \sum_{\substack{j < m \\ \alpha_1 M_1 + \cdots + \alpha_n M_n + j \le m}} q_{j\alpha}(x,t) D^{\alpha}_{x} \, D^{j}_{t} \qquad (5)$$

for some real-analytic functions $q_{j\alpha}$. (4) is then valid with $\phi = \sum_j (1 + |\xi_j|)^{1/M_j}$. Also note that (4) is always valid with $\phi = 1 + |\xi|$.

2. We now return to (1) and (2) . In order to give a meaning to (2), we must at least assume that u is a germ of an extendable distribution defined for $t > 0$ in a neighborhood of $0 \in R^{n+1}$. We may , and shall , then as well assume that u is a germ of a distribution defined in a full neighborhood of $0 \in R^{n+1}$. Likewise , the f_j will be germs of distributions defined near $0 \in R^{n}$. We shall henceforth denote the space of germs of distributions in n or $n+1$ variables, defined near $0 \in R^{n}$ or $0 \in R^{n+1}$ by $\mathcal{D}'$. (The precise meaning of $\mathcal{D}'$ must be clear from the context.)

Our first concern is then to see how much regularity on the f_j is required, if we want to find a solution for the problem (1) and (2).

In the case of operators of form (5) , the answer can be formulated in terms of anisotropic Gevrey classes . Let us in fact denote by G^M the set of germs f of C^∞ functions, defined near $0 \in R^n$ such that for some $\varepsilon > 0$ and $c > 0$ (which may depend on f)

$$|D_x^\alpha f(x)| \le c^{|\alpha|+1} (\alpha_1!)^{M_1} \cdots (\alpha_n!)^{M_n} , \forall\alpha , \forall x , |x| \le \varepsilon .$$

It is then a result (e.g.) of Persson [1] that (1) and (2) have a solution if $f_j \in G^M$, $j = 0 ,\ldots, m-1$. Moreover, this solution solves in fact the two-sided Cauchy problem

$$p(x,t,D_x,D_t)\, u = 0 , \qquad \text{near } 0 \in R^{n+1} , \tag{6}$$

$$D_t^j u \,|_{t=0} = f_j , \qquad j = 0,\ldots, m-1 , \tag{7}$$

and it is well-known (in view of regularity results for quasi-elliptic equations due to Cavalucci [1]) that this result cannot be improved, if no assumption on the type of the operator is made.

To state a similar result for the case of a general ϕ , we introduce :

<u>DEFINITION 1.1</u> (Liess-Rodino [1]) : Consider $x^0 \in R^n$ and let f be a germ of a distribution defined near x^0 . We say that f is of class G_ϕ near x^0 , if there is a neighborhood X of x^0 , $c > 0$, and a bounded sequence of distributions $f_j \in \mathcal{E}'(R^n)$ such that

a) $f = f_1$, in the sense of germs ,

b) $f_j = f_k$ on X , $\forall j$, $\forall k$,

c) $|\hat f_j(\xi)| \le c(cj/\phi(\xi))^j$, $j = 1,2,\ldots$, $\forall\, \xi \in R^n$.

$E'(R^n)$ is here the space of distributions with compact support defined on R^n. (More generally, $E'(U) = \{u \in E'(R^n)$; $\text{supp}\, u \subset U\}$.) When $v \in E'(R^n)$ we denote by $\hat{v}$ the Fourier-Borel transform $\hat{v}(\zeta) = v(\exp(-i\langle x,\zeta\rangle))$, $\zeta \in C^n$, of v . (Of course this definition is modelled on Hörmander's definition of the analytic wave front set. Cf. Hörmander [5]).

We denote by G_ϕ the set of germs of distributions defined near $x^o = 0$ which are of class G_ϕ there. Of course, when $\phi = \sum(1+|\xi_j|)^{1/M_j}$ we just have $G_\phi = G^M$.

Our first result in this paper is now:

THEOREM 1.2. *Assume that* $p(x,t,D_x,D_t)$ *satisfies* (4) *and let* $f_j \in G_\phi$, $j = 0,\dots,m-1$, *be given. Then we can find a germ of a* C^∞ *function, defined near* $0 \in R^{n+1}$, *for which* (6) *and* (7) *are valid.*

Note that this is just the natural formulation of the Cauchy-Kowalewska theorem in G_ϕ classes. A proof of theorem 1.2, in which we use the latter theorem (to which it actually reduces when $\phi \sim |\xi|$) will be given, after some preparations, in §7 below.

3. To state our next result, we shall, once more, place ourselves at first in the quasihomogeneous case from (5). Thus assume that $M_i \geq 1$, $i = 1,\dots,n$, are given and choose $x^o \in R^n$, $\xi^o \in \dot{R}^n$ ($\dot{R}^n = R^n\backslash\{0\}$) .

DEFINITION 1.3. Let f be a germ of a distribution defined near x^o . We shall say that $(x^o,\xi^o) \notin WF^M f$ if we can find an open M-quasicone $\Gamma \subset \dot{R}^n$ which contains ξ^o (a set $A \subset \dot{R}^n$ is called an M-quasicone if $a = (a_1,\dots,a_n) \in A$ implies $(t^{M_1}a_1,\dots,t^{M_n}a_n) \in A$ for all $t > 0$) , $c > 0$, and a bounded sequence $f_j \in E'(R^n)$ such that a) and b) from definition 1.1 are valid and such that

$$|\hat{f}_j(\xi)| \leq c(cj/\phi(\xi))^j \ , \ j = 1,2,\ldots, \ \xi \in \Gamma \ .$$

Here $\phi = \sum(1+|\xi_j|)^{1/M_j}$.

(For similar definitions cf. Lascar [1], Liess-Rodino [1], Rodino [1], Zanghirati [1].)

4. We also need

DEFINITION 1.4. Let $p(x,t,D_x,D_t)$ be as in (5) and consider some germs of distributions $f_o,\ldots,f_{m-1}$, defined near $0 \in R^n$. We say that the one-sided Cauchy problem is solvable in microgerms at $(0,\xi^o)$ if there is a germ of a distribution u_{ξ^o} defined near $0 \in R^{n+1}$ such that

$$p(x,t,D_x,D_t)u_{\xi_o} = 0 \ , \qquad \text{for} \quad t > 0 \ , \tag{8}$$

$$(0,\xi^o) \notin WF^M(D_t^j u_{\xi^o}|_{t=0} - f_j) \ , \quad j = 0,\ldots,m-1 \ . \tag{9}$$

Our main result in this paper is now the following theorem, and its variant in G_ϕ classes from the theorems 1.9 and 1.12 below:

THEOREM 1.5. *Suppose that there are given* $f_o,\ldots,f_{m-1}$ *in* $\mathcal{D}'$ *and assume that the Cauchy problem is solvable in microgerms at* $(0,\xi^o)$ *for any* $\xi^o \in R^n$. *Then we can find a solution* $u \in \mathcal{D}'$ *for* (1) *and* (2) .

5. In the case of a general ϕ it seems difficult to associate some wave front set directly with vectors $\xi^o \in \dot{R}^n$ in a natural way. In fact, as has been first observed in Hörmander [6], wave front sets are associated rather with the points at infinity of a suitable compactification of R^n related to ϕ , than with the points from $\dot{R}^n$. We avoid this difficulty altogether by introducing

DEFINITION 1.6. (Liess-Rodino [1]). Let f be a germ of a distribution defined near $x^o \in R^n$ and consider $\Gamma \subset R^n$. We shall write that $(0,\Gamma) \cap WF_\phi f = \emptyset$, if there is a neighborhood X of x^o, $c_1 > 0$, $c_2 > 0$, and a bounded sequence $f_j \in E'(R^n)$ such that a) and b) from definition 1.1 are valid and such that

$$|\hat{f}_j(\xi)| \le c_1(c_1 j/\phi(\xi))^j, \ \forall j \ , \text{ if } \operatorname{dist}(\xi,\Gamma) \le c_2\phi(\xi) \ .$$

To simplify the notations, we shall henceforth denote $\{\xi \in R^n, \operatorname{dist}(\xi,\Gamma) \le c\phi(\xi)\}$ by $\Gamma_{c\phi}$. If Γ' contains some set of form $\Gamma_{c\phi}$ then we say that Γ' is a ϕ-neigborhood of Γ and write $\Gamma \subset_\phi \Gamma'$.

DEFINITION 1.7. Let $f_o,\ldots,f_{m-1}$ be given germs of distributions defined near $0 \in R^n$ and consider $\Gamma \subset R^n$. We say that the one-sided Cauchy problem is solvable in microgerms at $(0,\Gamma)$, if there is a germ of a distribution u_Γ defined near $0 \in R^{n+1}$ for which

$$p(x,t,D_x,D_t)u_\Gamma = 0 \ , \qquad t > 0 \tag{10}$$

$$(0,\Gamma) \cap WF_\phi(f_j - D_t^j u_\Gamma|_{t=0}) = \emptyset \ , \quad j = 0,\ldots,m-1 \ . \tag{11}$$

REMARK 1.8. If $p(x,t,D_x,D_t)$ has form (5) and if the Cauchy problem is solvable in microgerms at $(0,\xi^o)$ (in G^M), then there is an open M-quasicone Γ, $\xi^o \in \Gamma$, such that the Cauchy problem is solvable in microgerms at $(0,\Gamma)$.

We now have

THEOREM 1.9. *Consider* $\Gamma^1,\ldots,\Gamma^s$, *some sets in* $\dot{R}^n$ *such that* $\cup\Gamma^k = \dot{R}^n$, *and let* $f_o,\ldots,f_{m-1}$ *be given. Assume that the Cauchy problem is solvable in microgerms at* $(0,\Gamma^k)$ *for any* k. *Then we can find a solution* u *for the problem* (1), (2).

Theorem 1.9 will be proved in §10 below.

REMARK 1.10. In view of remark 1.8 it is clear that theorem 1.5 is a consequence of theorem 1.9 .

REMARK 1.11. The definitions 1.4 and 1.7 both refer to the one-sided Cauchy problem. We obtain related definitions for the two-sided Cauchy problem (6), (7), if we just drop the condition "$t > 0$" in (8), respectively (10). In this way we arrive at natural variants of the theorems 1.5 and 1.9, which are also true (and in fact easier to prove. One can also obtain them by using the uniqueness of the solutions.).

6. In the theorems 1.5 and 1.9 we have studied the solvability of (1), (2), in distributions. One may ask if u is a C^∞ funcion, if the solutions for the corresponding microlocal problems (8), (9), respetively (10), (11), are C^∞ functions. This is indeed the case:

THEOREM 1.12. *Consider* $\Gamma^1,\dots,\Gamma^s \subset R^n$ *with* $\cup\Gamma^k = R^n$ *and let* $f_o,\dots,f_{m-1}$ *be given. Assume that for every* k, $1 \le k \le s$, *there is a* C^∞ *function* u_Γ *which satisfies* (10) *and* (11). *Then we can find a germ of a* C^∞ *function, defined near* $0 \in R^{n+1}$, *for which* (1) *and* (2) *are valid.*

Theorem 1.12 will be proved in §8 below.

7. So far we have only analyzed solvability questions for (1) and (2). These questions are naturally related to questions of microlocal uniqueness (or regularity) for the corresponding solutions. To state the relevant results, we must at first introduce a natural notion of boundary wave front set. As is customary we shall define such boundary wave front sets only for a subclass of distributions. Here we shall assume that u is C^∞ in the t-variable. We shall in fact denote by F the space of germs of distributions u defined near $0 \in R^{n+1}$ with the following property: $\exists\varepsilon > 0$, $\forall b \in R$, $\exists b' \in R$, $\exists c > 0$ such that

$|u(v)| \le c$ if $v \in C_o^\infty(R^{n+1})$ satisfies

$$|\hat{v}(\lambda)| \le \exp(\varepsilon|\mathrm{Im}\,\zeta| + \varepsilon\,\mathrm{Im}\,\tau^+ + b\ln(1+|\lambda|) + b'\ln(1+|\zeta|)) .$$

Here $\lambda = (\zeta,\tau)$, $\zeta \in C^n$, $\tau \in C$, are the Fourier-dual variables to $z = (x,t)$. Furthermore, when $a \in R$, then we denote by a^+ its positive part. Finally, we should mention perhaps that in the above we have identified (as we shall also do later on) u with some suitable distribution defining it.

If $u \in \mathcal{D}'$ satisfies $p(x,t,D_x,D_t)\,u = 0$ for $t > 0$, then it follows from theorem 4.3.1 in Hörmander [1], that we can find $u' \in \mathcal{F}$ such that $u = u'$ for $t > 0$.

To justify the notion of a boundary wave front set which we introduce later on, we recall:

PROPOSITION 1.13. (cf. Liess-Rodino [1]). Consider $f \in \mathcal{D}'(R^n)$, $\Gamma \subset R^n$, $x^o \in R^n$. Then there are equivalent:

(i) $(x^o,\Gamma) \cap WF_\phi f = \emptyset$.

(ii) There are $d > 0$, $\varepsilon > 0$, $c > 0$, $c' > 0$ and $b \in R$ such that $|v(f)| \le c$ whenever $v \in C_o^\infty(R^n)$ satisfies

$$|\hat{v}(\zeta)| \le \exp(d\phi(-\mathrm{Re}\,\zeta) + \langle x^o,\mathrm{Im}\,\zeta\rangle + \varepsilon|\mathrm{Im}\,\zeta| + b\ln(1+|\zeta|)) , \text{ if } \zeta \in C^n , \mathrm{Re}\,\zeta \in -\Gamma_{c'\phi} , \quad (12)$$

$$|\hat{v}(\zeta)| \le \exp(\langle x^o,\mathrm{Im}\,\zeta\rangle + \varepsilon|\mathrm{Im}\,\zeta| + b\ln(1+|\zeta|)) , \text{ if } \zeta \in C^n , \mathrm{Re}\,\zeta \notin -\Gamma_{c'\phi} . \quad (13)$$

Moreover, when $f \in C^\infty(R^n)$, then (i) and (ii) are also equivalent to (ii)': There are $d > 0$, $\varepsilon > 0$, $c' > 0$, and for every b some $c > 0$ such that $|v(f)| \le c$, whenever $v \in \mathcal{E}'(R^n)$ satisfies (12) and (13).

In particular, $f \in C^\infty(R^n)$ defines an element in G_ϕ, precisely if we can find $d > 0$, $\varepsilon > 0$, and for every b some $c > 0$ such that $|v(f)| \le c$ whenever $v \in \mathcal{E}'(R^n)$ satisfies

$$|\hat{v}(\zeta)| \leq \exp(d\phi(-\mathrm{Re}\,\zeta) + \varepsilon|\mathrm{Im}\,\zeta| + b\ln(1+|\zeta|)), \ \forall\, \zeta \in C^n .$$

DEFINITION 1.14. Consider $u \in F'$ and $\Gamma \subset R^n$. We say that $(0,\Gamma)$ is not in the boundary wave front set WF^b_ϕ of u ("0" is here the one from R^{n+1}), and write $(0,\Gamma) \cap WF^b_\phi u = \emptyset$, if :

$$\exists d > 0,\ \exists \varepsilon > 0,\ \exists c',\ \forall b \in R,\ \exists b' \in R,\ \exists c,$$

such that $|u(v)| \leq c$ for any $v \in C_0^\infty(R^{n+1})$ which satisfies

$$|\hat{v}(\lambda)| \leq \exp(d\phi(-\mathrm{Re}\,\zeta) + \varepsilon|\mathrm{Im}\,\zeta| + \varepsilon\,\mathrm{Im}\,\tau^+ + b\ln(1+|\lambda|) + b'\ln(1+|\zeta|)), \text{ if } \mathrm{Re}\,\zeta \in -\Gamma_{c'\phi},$$

$$|\hat{v}(\lambda)| \leq \exp(\varepsilon|\mathrm{Im}\,\zeta| + \varepsilon\,\mathrm{Im}\tau^+ + b\ln(1+|\lambda|) + b'\ln(1+|\zeta|)), \text{ if } \mathrm{Re}\,\zeta \notin -\Gamma_{c'\phi}.$$

REMARK 1.15. Various notions of boundary wave front sets have been introduced in the literature, using a variety of definitions and serving different purposes (cf. e.g. Chazarain [1], Melrose-Sjöstrand [1], Sjöstrand [1]). Any notion of boundary regularity should be such that boundary regularity for u implies at least regularity of the traces $D_t^j u_{|t=0}$. In the present situation, this comes to :

" when $(0,\Gamma) \cap WF^b_\phi u = \emptyset$, then $(0,\Gamma) \cap WF_\phi D_t^j u_{|t=0} = \emptyset$, $\forall j$ ".

This is in fact an immediate consequence of proposition 1.13. In particular, $(0,\Gamma) \cap WF^b_\phi u = \emptyset$ implies that $(0,\Gamma) \cap WF^b_a u = \emptyset$ in the sense from Sjöstrand [1], if $\phi = 1 + |\xi|$ and $p(x,t,D_x,D_t)u = 0$. (the latter condition is necessary since WF^b_a is only defined for solutions of equations of type $p(x,t,D_x,D_t)u = 0$.) The converse is also true in view of theorem 1.18 below.

REMARK 1.16. It is possible to give a definition of WF^b_ϕ which is much closer to L.Hörmander's definition of the analytic wave front

set. In fact, if $\chi_j \in C_o^\infty(R^n)$ is a sequence of functions such that $|D_x^{\alpha+\beta} \chi_j(x)| \le c_\alpha c^{|\beta|+1} j^{|\beta|}$, for $|\beta| \le j$,
and if $(0,\Gamma) \cap WF_\phi^b u = \emptyset$, then it follows from proposition 1.13 (and its proof), that we have

$$|g_j(\xi,t)| \le c''(c''j/\phi(\xi))^j, \quad \text{if } \xi \in \Gamma_{c'\phi}, \qquad (14),$$

for $g_j(\xi,t) = \int \chi_j(x)\, u(x,t) \exp(-i \langle x, \xi \rangle)\, dx$, if $0 \le t \le \varepsilon$ for small ε and if the supports of the χ_j are all small. Definition 1.14 is nothing but a quantitatively more precise version of (14), in which also t-derivatives of u are considered :

<u>PROPOSITION 1.17</u>. Consider $u \in F$. Then there are equivalent:

a) $(0,\Gamma) \cap WF_\phi^b u = \emptyset$.

b) Let $\chi_j \in C_o^\infty(R^n)$ be a sequence of functions such that $\chi_j(x)=0$ for $|x| \ge C_1$ and such that $\chi_j(x) = 1$ in some fixed neighborhood of the origin. Moreover assume that

$$|D_x^{\alpha+\beta} \chi_j(x)| \le c_\alpha c^{|\beta|+1} j^{|\beta|}, \qquad \text{if } |\beta| \le j, \text{ and that}$$

if C_1 is small enough, we can find C_2 and for every k some c_k such that

$$\left|\int \chi_j(x)\, D_t^k u(x,t) \exp(-i\langle x, \xi\rangle)\, dx\right| \le c_k (c_k j/\phi(\xi))^j,$$

$$\text{if } \xi \in \Gamma_{C_2\phi}.$$

The proof of proposition 1.17 can be performed with the arguments used in the proof of proposition 1.13. The reason why we prefer here definition 1.14 over the property from part b) in proposition 1.17 is of course that it is the former which we shall directly use in the sequel. The proof of proposition 1.17 will not be given in this paper.

8. We can now state

THEOREM 1.18. *Assume that* $u \in F$ *is a solution of* (1) *and denote by* $f_j = D_t^j u_{|t=0}$, $j = 0,\dots,m-1$. *Let also* $\Gamma \subset R^n$ *be given and assume that* $(0,\Gamma) \cap WF_\phi f_j = \emptyset$, $\forall j$. *Then* $(0,\Gamma) \cap WF_\phi^b u = \emptyset$.

Theorem 1.18 will be proved in §9 below.

REMARK 1.19. In the analytic category this theorem gives a result of Schapira [1]. (Cf. also Sjöstrand [1] and theorem 9.6.9 in Hörmander [7]. For constant coefficients it is also a consequence of the arguments form Liess [3]). In the C^∞ category a similar result appears in de Gosson [1].

9. We mention finally the following result, which may serve as a justification for our notion of boundary wave front set.

PROPOSITION 1.20. Assume that $f \in F$ is such that $(0,R^n) \cap WF_\phi^b f = \emptyset$. Then there is $u \in \mathcal{D}'$ such that $p(x,t,D_x,D_t)u = f$ and such that $D_t^j u_{|t=0} = 0$, $j = 0,\dots,m-1$.

Proposition 1.20 is proved in §7 .

10. Our main concern in this paper is to prove theorem 1.12. The reason why we prefer to concentrate on this result, rather than on theorem 1.9, is that the situation is notationally simpler for C^∞ solutions.

The central idea in the proof of theorem 1.12 is to exploit the analogy (by duality) between the Cauchy-Kowalewska theorem and the Weierstrass preparation theorem. To use this analogy is in fact common practice in constant coefficients (cf. e.g. Kiselman [2]) and it has also been used by L. Ehrenpreis to reprove Petrowsky's theorem for strictly hyperbolic equations with analytic coefficients. (Cf. Ehrenpreis [2]). The main (perhaps new) ingredient which we use here is a non-commutative version for the contour integration formulas which give (what corresponds to) the quotient term in the Weierstrass preparation theorem.These formulas are derived in §4 and we use them in §5 to estimate the aforementioned quotient term. In §6 we then explain,why one needs these estimates to solve Cauchy problems by duality.The §§ 4 - 6 therefore form the core of the paper .The proofs from these paragraphs are based on a number of technical preparations which are collected in the §§ 2 and 3,and the proofs of the results mentioned in the above are brought to an end in the §§ 7 - 10.

Unfortunately,the duality on which everything is based,does not work initially for all distributions with compact support.We shall therefore use a subclass of distributions,which have first been considered by L.Ehrenpreis,who also proved the remarkable fact that these distributions are dense in the set of all distributions with compact support .(Cf. Ehrenpreis [1] and §11 below.) These distributions are closely related to Holmgren's method of deformation of noncharacteristic hypersurfaces and we study this relation in §11 . The paper is concluded with a section of comments.

§ 2 . PREPARATIONS

1. In this paper we shall repeatedly use analytic functionals $u \in A'(C^{n+1})$ which satisfy an estimate of form

$$|\hat{u}(\lambda)| \le c \exp(c'|\mathrm{Im}\,\zeta| + b \ln(1+|\zeta|)) \text{ ,when } |\tau| \le C(1+|\zeta|)),(1)$$

respectively,an estimate of form

$$|\hat{u}(\lambda)| \le c \exp(c'|\mathrm{Im}\,\lambda| + b \ln(1+|\lambda|)) \text{ ,when } |\tau| \ge C(1+|\zeta|)),(2)$$

or similar estimates.
Here and later on , we denote by $A(U)$ the (topologized) space of analytic functions defined on the open set $U \subset C^j$,by X' strong dual of X ,if X is some given locally convex topological

vector space, and by $\hat{u}$ the Fourier-Borel transform of the analytic functional u.

2. The importance of the analytic functionals satisfying (1) for the study (by duality) of the Cauchy problem has first been observed by Ehrenpreis [1], who proved:

PROPOSITION 2.1. Let $\varepsilon' > 0$ and $C > 0$ be given. Then there is $\varepsilon > 0$ with the following property: for any $u \in \mathcal{E}'(z \in R^{n+1}; |z| < \varepsilon)$ there is $c > 0$ and a sequence $v_j \in \mathcal{E}'(|z| < \varepsilon')$ such that

a) $v_j \to u$ in $\mathcal{E}'(|z| < \varepsilon')$

b) $|\hat{v}_j(\lambda)| \le c \exp(\varepsilon'|\mathrm{Im}\ \zeta|)$ if $|\tau| \le C(1+|\zeta|)$,

c) $|\hat{v}_j(\lambda)| \le c \exp(\varepsilon'|\mathrm{Im}\ \lambda|)$ if $|\tau| \ge C(1+|\zeta|)$.

Moreover, if $\mathrm{supp}\ u \subset \{z \in R^{n+1};\ t \ge 0\}$, then we can choose the v_j to have supports also in $t \ge 0$.

3. Using an idea from the proof of proposition 2.1, we can obtain the following variant of a result from Liess [3]:

LEMMA 2.2. Let $C > 0$ be given. Then we can find $C' > 0$, $d' > 0$, and a plurisubharmonic function $\rho : C^{n+1} \to R$ such that

$$\mathrm{Im}\ \tau^+ - |\zeta| \le \rho(\lambda), \tag{3}$$

$$\rho(\lambda) \le d'(1+|\mathrm{Im}\ \zeta| + \mathrm{Im}\ \tau^+), \quad \forall \lambda \in C^{n+1}, \tag{4}$$

$$\rho(\lambda) \le d'(1+|\mathrm{Im}\ \zeta|) \qquad \text{if } |\tau| \le C(1+|\zeta|). \tag{5}$$

A similar statement is valid if we replace everywhere $\mathrm{Im}\ \tau^+$ by $|\mathrm{Im}\ \tau|$.

Proof of lemma 2.2 (scetch). We may assume (we can temporarily add a supplimentary variable) that n is odd. We denote by

$$B_n = \{f \in A(C^n) \; ; \sup_{\substack{x \in C^n \\ |x| \le 1}} |f(x)| \le 1\}$$

and, for $f \in B_n$, by

$$h_f(\lambda) = \int_{\substack{x \in R^n \\ |x| \le 1}} f(x) \exp(-i<x,\zeta> - i(1 - \sum_{j=1}^{n} x_j^2)\tau)dx \,. \qquad (6)$$

Further consider

$$\chi(\lambda) = \sup_{f \in B_n} \ln|h_f(\lambda)| \,, \qquad (7)$$

which is thus plurisubharmonic.

Note that h_f is just the Fourier-Borel transform of a distribution given by an analytic "density"concentrated on $\{(x,t); |x| \le 1, t = 1-|x|^2\}$ and that precisely such distributions were used by L. Ehrenpreis to prove proposition 2.1. It is proved in lemma 2.3 from Liess [3], (using also lemma 9.22 from Ehrenpreis [1]), that

$$\chi(\lambda) \le C_1(1+|\mathrm{Im}\ \zeta| + \mathrm{Im}\ \tau^+) \,,$$

$$\chi(\lambda) \le C_1(1+|\mathrm{Im}\ \zeta|) \qquad \text{if} \quad |\tau| \le C_2(1+|\zeta|) \,,$$

$$\chi(\lambda) \ge C_3\ \mathrm{Im}\ \tau^+ - |\zeta| - 2\ln|\tau| \,,$$

$$\text{for} \quad |\mathrm{Re}\ \zeta| \ge 2\pi+1 \,, \ \mathrm{Im}\ \tau \ge 0 \,,$$

for some positive constants C_i.

Note that the last property was stated in Liess, loc. cit., only for $\zeta \in R^n$, but the proof carries over without any change for $\zeta \in C^n$.

A function ρ with the properties stated in the conclusions of lemma 2.2 is then

$$\rho(\lambda) = \max [0, C_4(\sup_{\substack{\theta\in R \\ |\theta|\le 2\pi+1}} \chi(C_5\, \zeta, \tau+\theta) + 2 \ln|\tau|)] ,$$

for suitable C_4, C_5 .

4. For later purpose we now mention:

<u>LEMMA 2.3.</u> Consider $\varepsilon' > 0$, $C > 0$, $\Gamma'',\Gamma' \subset R^n$, and assume that $\Gamma''_{c\phi} \subset \Gamma'$ for some $c > 0$. Then there are $\varepsilon > 0$, $c'' > 0$, and a plurisubhamronic function $\rho' : C^{n+1} \to R$ such that

$$\rho'(\lambda) \le \varepsilon'|\mathrm{Im}\ \zeta| + \varepsilon'\mathrm{Im}\ \tau^+ + c'' , \tag{8}$$

$$\rho'(\lambda) \le \varepsilon'|\mathrm{Im}\ \zeta| + c''$$

$$\text{if} \quad |\tau| \le C(\phi(-\mathrm{Re}\ \zeta) + |\mathrm{Im}\ \zeta|) \quad \text{and} \quad -\mathrm{Re}\ \zeta \notin \Gamma' , \tag{9}$$

$$\varepsilon|\mathrm{Im}\ \zeta| + \varepsilon\ \mathrm{Im}\ \tau^+ \le \rho'(\lambda) \quad \text{if} \quad -\mathrm{Re}\ \zeta \in \Gamma'' . \tag{10}$$

Moreover, the constants ε, c'' depend here on ε', C, c, but not directly on Γ'' and Γ' .

<u>Proof of lemma 2.3.</u> The situation is here similar to the one from proposition 2.1 in Liess [3]. Before we start the proof, we note that $\eta \notin \Gamma'$, $\theta \in \Gamma''$, implies $|\eta-\theta| \ge c_1\ \phi(\eta)$ for some $c_1 > 0$. If $C_2 > 0$ is given, we can therefore find $\tilde{C}$ so that $|\tau| \le \tilde{C}(\phi(-\mathrm{Re}\ \zeta) + |\mathrm{Im}\ \zeta|)$ implies $|\tau| \le C_2(1+|\mathrm{Re}\ \zeta+\xi| + |\mathrm{Im}\ \zeta|)$ if $-\mathrm{Re}\ \zeta \notin \Gamma'$ and $\xi \in \Gamma''$ (the condition is of course just that $\tilde{C} \le C_2$ and $\tilde{C} \le c_1\ C_2$) .

Let now χ be the function defined in (7) and define

$$\rho''(\lambda) = \sup_{\xi\in\Gamma''} \chi(\zeta+\xi,\tau) + 2|\mathrm{Im}\ \zeta| + 2 \ln|\tau| .$$

With the notations from nr. 3 we thus have

$$\rho''(\lambda) \le C_1(1 + |\mathrm{Im}\ \zeta| + \mathrm{Im}\ \tau^+) \quad \text{and}$$

$$\rho''(\lambda) \geq C_3 \text{ Im } \tau^+ + |\text{Im } \zeta| \quad \text{if} \quad -\text{Re } \zeta \in \Gamma'' ,$$

$$\text{Im } \tau > 0 \quad \text{and} \quad |\text{Re } \tau| \geq 2\pi+1 .$$

Moreover, in view of the discussion from the above, we have $\rho''(\lambda) \leq C_1(1 + |\text{Im } \zeta|)$ for $|\tau| \leq \tilde{C}(\phi(-\text{Re } \zeta) + |\text{Im } \zeta|)$ provided $\tilde{C}$ is small.

The next thing is to replace $\rho''(\lambda)$ by $\rho_1(\lambda) = \max (\rho''(\lambda), |\text{Im } \zeta|)$, for which $\rho_1 \geq C_3 \text{ Im } \tau^+ + |\text{Im } \zeta|$ if $-\text{Re } \zeta \in \Gamma''$ and $|\text{Re } \tau| \geq 2\pi+1$. Finally we set

$$\rho'(\lambda) = c_2 \sup_{|\theta| \leq 2\pi+1} \rho_1(c_3 \zeta, \tau+\theta) ,$$

for suitable c_2, c_3 .

5. In our next result we give a useful decomposition for distributions of the type which appear in the definition of boundary wave front sets.

<u>PROPOSITION 2.4.</u> Consider $\varepsilon' > 0$, $d' > 0$, $c' > 0$ and $\Gamma \subset R^n$. Then we can find $\varepsilon > 0$, $d > 0$, $b' \geq 0$ and for every $b \geq 0$ some c with the following property:

if $u \in E'(R^{n+1})$ is given such that

$$|\hat{u}(\lambda)| \leq \exp(d\phi(-\text{Re } \zeta) + \varepsilon|\text{Im } \zeta| + \varepsilon \text{ Im } \tau^+ + b \ln(1+|\lambda|)) ,$$
$$\text{for} \quad -\text{Re } \zeta \in \Gamma ,$$

and

$$|\hat{u}(\lambda)| \leq \exp(\varepsilon|\text{Im } \zeta| + \varepsilon \text{ Im } \tau^+ + b \ln(1+|\lambda|)) ,$$
$$\text{for} \quad -\text{Re } \zeta \notin \Gamma ,$$

then we can find a sequence $v_1, v_2, \ldots,$ in $E'(R^{n+1})$ and a sequence $\xi^1, \xi^2, \ldots,$ in Γ such that

a) $u = \sum v_j$,

b) $|\hat{v}_j(\lambda)| \le (c/j^2) \exp(d'\phi(-\mathrm{Re}\,\zeta) + \varepsilon'|\mathrm{Im}\,\zeta| + \varepsilon'\mathrm{Im}\,\tau^+$

$+ (b+b')\ln(1+|\lambda|))$ if $|\mathrm{Re}\,\zeta + \xi^j| \le c'\phi(\xi^j)$,

$|\hat{v}_j(\lambda)| \le (c/j^2) \exp(\varepsilon'|\mathrm{Im}\,\zeta| + \varepsilon'\mathrm{Im}\,\tau^+ + (b+b')\ln(1+|\lambda|))$

if $|\mathrm{Re}\,\zeta + \xi^j| \ge c'\phi(\xi^j)$.

Moreover, when $u \in C_0^\infty(R^{n+1})$, then only finitely many v_j are different from 0 .

REMARK 2.5. Proposition 2.4 is a variant of proposition 1.4.5 from Liess-Rodino [1]. The fact that we can work here with $\mathrm{Im}\,\tau^+$ instead of $|\mathrm{Im}\,\tau|$ is a consequence of the fact that the ("weight") function ϕ does not depend on τ at all.

6. We must now also study how distributions like the v_j from proposition 2.4 behave under multiplication with analytic functions. For later purpose we prove a result which is even a little more complicated.

PROPOSITION 2.6. Consider $c_1 > 0$, C, C', $0 < C' < C$, $\varepsilon' > 0$, $d' > 0$, $b \in R$, $\Gamma \subset \Gamma' \subset R^n$ and assume that $\Gamma_{c'\phi} \subset \Gamma'$ for some $c' > 0$. Then there are c, c_2, ε, d, all positive, such that if $\xi^0 \in \Gamma$, $v \in E'(R^{n+1})$ and $g \in A(z \in C^{n+1}, |z| < \varepsilon')$, satisfy

$|g(z)| \le 1$,

$|\hat{v}(\lambda)| \le \exp(\varepsilon|\mathrm{Im}\,\zeta| + b\ln(1+|\zeta|))$

if $-\mathrm{Re}\,\zeta \notin \Gamma$ and $|\tau| \le C(\phi(-\mathrm{Re}\,\zeta)+|\mathrm{Im}\,\zeta|)$, (11)

$|\hat{v}(\lambda)| \le \exp(\varepsilon|\mathrm{Im}\,\zeta| + \varepsilon\,\mathrm{Im}\,\tau^+ + b\ln(1+|\lambda|))$,

if $|\mathrm{Re}\,\zeta + \xi^0| \ge c_2\phi(\xi^0)$, (12)

$|\hat{v}(\lambda)| \le \exp(d\phi(-\mathrm{Re}\,\zeta) + \varepsilon|\mathrm{Im}\,\zeta| + \varepsilon\,\mathrm{Im}\,\tau^+ + b\ln(1+|\lambda|))$,

if $|\mathrm{Re}\,\zeta + \xi^0| \le c_2\phi(\xi^0)$, (13) then it follows that $w = gv$ satisfies

$$|\hat{w}(\lambda)| \le c \exp(\varepsilon'|\mathrm{Im}\,\zeta| + b\ln(1+|\lambda|))$$
$$\text{if} \quad -\mathrm{Re}\,\zeta \notin \Gamma' \quad \text{and} \quad |\tau| \le C'(\phi(-\mathrm{Re}\,\zeta)+|\mathrm{Im}\,\zeta|) \ , \tag{14}$$

$$|\hat{w}(\lambda)| \le c \exp(\varepsilon'|\mathrm{Im}\,\zeta| + \varepsilon'\mathrm{Im}\,\tau^{+} + b\ln(1+|\lambda|)) \ ,$$
$$\text{if} \quad |\mathrm{Re}\,\zeta + \xi^{o}| \ge c_1\phi(\xi^{o}) \ , \tag{15}$$

$$|\hat{w}(\lambda)| \le c \exp(d'\phi(-\mathrm{Re}\,\zeta) + \varepsilon'|\mathrm{Im}\,\zeta| + \varepsilon'\mathrm{Im}\,\tau^{+} + b\ln(1+|\lambda|)),$$
$$\text{if} \quad |\mathrm{Re}\,\zeta + \xi^{o}| \le c_1\phi(\xi^{o}) \ . \tag{16}$$

Moreover, only c depends here on b and the proposition is also true when $\Gamma = \Gamma' = \emptyset$ or R^n .

7. Proof of proposition 2.6. Preparations.

Our first remark is that

$$\hat{w}(\lambda) = \int_{R^{n+1}} F(hg)(\theta)\hat{v}(\lambda-\theta)d\theta \ , \tag{17}$$

if $h \in C_o^{\infty}(R^{n+1})$ has support in $|z| \le \varepsilon'$ and satisfies $hv = v$.

We now fix λ and ξ^{o} and choose for h some function which satisfies

$$|D_{x,t}^{\alpha+\beta+\gamma} h(x,t)| \le c_3^{|\alpha+\beta+\gamma|+1}(c_4\phi(-\mathrm{Re}\,\zeta))^{|\alpha|}(c_4\phi(\xi^{o}))^{|\beta|}$$
$$\text{if} \quad |\alpha| \le c_4\phi(-\mathrm{Re}\,\zeta), \ |\beta| \le c_4\phi(\xi^{o})$$
$$\text{and} \quad |\gamma| \le |b| + n + 2 \ , \tag{18}$$

for some c_3, c_4 . The constant c_4 shall be chosen later on, but c_3 must not depend on c_4 . It is wellknown that such functions exist, if we assume (as we shall) that $\varepsilon < \varepsilon'$. (Cf. e.g. Hörmander [5]). It follows in particular from (18) that

$$|F(hg)(\theta)| \le c_5(1+|\theta|)^{-|b|-n-2} , \tag{19}$$

$$|F(hg)(\theta)| \le c_5(1+|\theta|)^{-|b|-n-2}(c_4\phi(-\mathrm{Re}\ \zeta)/|\theta|)^{c_4\phi(-\mathrm{Re}\ \zeta)} \tag{20}$$

$$|F(hg)(\theta)| \le c_5(1+|\theta|)^{-|b|-n-2}(c_4\phi(\xi^o)/|\theta|)^{c_4\phi(\xi^o)} , \tag{21}$$

for some c_5 .

Note that, if here $|\theta| \ge c_6\phi(-\mathrm{Re}\ \zeta)$, e.g., then we can conclude from (20) that

$$|F(hg)(\theta)| \le c_5(1+|\theta|)^{-|b|-n-2} \exp(-c_4\phi(-\mathrm{Re}\ \zeta)) , \tag{22}$$

if we shrink c_4 until $c_4/c_6 < 1/e$.

Similarily, we will have

$$|F(hg)(\theta)| \le c_5(1+|\theta|)^{-|b|-n-2} \exp(-c_4\phi(\xi^o)) \quad \text{if } |\theta| \ge c_6\phi(\xi^o) , \tag{23}$$

and

$$|F(hg)(\theta)| \le c_5(1+|\theta|)^{-|b|-n-2} \exp(-c_4(\phi(-\mathrm{Re}\ \zeta) + \phi(\xi^o))/2) \quad \text{if } |\theta| \ge c_6(\phi(\xi^o) + \phi(-\mathrm{Re}\ \zeta)) , \tag{24}$$

for c_4 small enough.

8. The next step in the proof of proposition 2.6 is:

<u>LEMMA 2.7.</u> a). Consider $c_7 > 1$. Then there is c_8 so that $|\xi-\eta| < c_8\phi(\xi)$ implies

$$\phi(\eta) \le c_7\phi(\xi) , \quad \phi(\xi) \le c_7\phi(\eta) .$$

b) There is $c_9 \le c_1$ so that $|\eta-\xi| \ge c_9(\phi(\xi) + \phi(\eta))$ for all $\xi \in \Gamma$ and $\eta \notin \Gamma'$.

c) There is $c_{10} \le c_9$ so that $|\theta| \le c_{10}\phi(-\mathrm{Re}\ \zeta)$ together with $|\tau| \le C'(\phi(-\mathrm{Re}\ \zeta) + |\mathrm{Im}\ \zeta|)$ implies

$$|\theta_{n+1}-\tau| < C(\phi(-\mathrm{Re}\,\zeta+\theta') + |\mathrm{Im}\,\zeta|) \ . \tag{25}$$

Moreover,

$$|\mathrm{Re}\,\zeta - \theta' + \xi^o| \geq c_9\phi(\xi^o) \quad \text{if} \quad -\mathrm{Re}\,\zeta \notin \Gamma' \ , \tag{26}$$

then. Here $\theta \in R^{n+1}$, $\theta = (\theta',\theta_{n+1})$.

Proof of lemma 2.7. a) Assume $|\xi-\eta| < c_8\phi(\xi)$. The first inequality in the conclusion follows for small c_8 from $\phi(\eta) \leq \phi(\xi) + \tilde{c}|\xi-\eta| \leq (1+c_8\tilde{c})\phi(\xi)$ and the second from $\phi(\xi) \leq \phi(\eta) + \tilde{c}\, c_8\phi(\xi)$ (which implies $(1-c_8)\phi(\xi) \leq \phi(\eta)$ if $\tilde{c}\, c_8 \leq 1)$. Here $\tilde{c} > 0$ is such that $|\phi(\xi^1) - \phi(\xi^2)| \leq \tilde{c}|\xi^1-\xi^2|$.

b) From the assumption on Γ,Γ', it follows that we can find $c_{11} > 0$ so that $|\xi - \eta| \geq c_{11}\phi(\eta)$ whenever $\xi \in \Gamma$ and $\eta \notin \Gamma'$. If, on the other hand, we had $|\xi-\eta| \leq c_{12}\phi(\xi)$, then we could conclude from part a) that $\phi(\xi) \leq 2\phi(\eta)$ (for example), if c_{12} had been small enough. Thus $|\xi-\eta| < 2c_{12}\phi(\eta)$, then, which would contradict the choice of c_{11} if $2c_{12} < c_{11}$. We conclude that we must also have $|\xi-\eta| \geq c_{12}\phi(\xi)$ for some small c_{12} .

c) A first condition on c_{10} is $c_{10} < C-C'$. If θ and τ are as in c), we conclude that

$$|\theta_{n+1}-\tau| \leq C'|\mathrm{Im}\,\zeta| + (C' + c_{10})\phi(-\mathrm{Re}\,\zeta) \ .$$

We will then obtain (25), if we shrink c_{10} until $|\theta| \leq c_{10}\phi(-\mathrm{Re}\,\zeta)$ implies $(C'+c_{10})\phi(-\mathrm{Re}\,\zeta) \leq C\phi(\mathrm{Re}\,\zeta-\theta')$ (cf. part a)).

To obtain also (26), we apply part b). We conclude for the $\theta,\zeta,$ under consideration that

$$\begin{aligned}|\mathrm{Re}\,\zeta - \theta' + \xi^o| &\geq c_9\phi(\xi^o) + c_9\phi(-\mathrm{Re}\,\zeta) - c_{10}\phi(-\mathrm{Re}\zeta) \\ &\geq c_9\phi(\xi^o) \ , \qquad \text{if } c_{10} \leq c_9 \ .\end{aligned}$$

Before we return to the proof of proposition 2.6 we mention the following corollary to lemma 2.7:

LEMMA 2.8. a) Let $0 < c < c'$ be given. Then we can find c'' so that $[\Gamma_{c\phi}]_{c''\phi} \subset \Gamma_{c'\phi}$, for any $\Gamma \subset R^n$.

b) Let $c > 0$ and $\Gamma \subset R^n$ be given. If c', c'' are suitably small and if

$$\Gamma' = \{\eta \in R^n;\ \exists \xi \in \Gamma \text{ s.t. } |\xi - \eta| < c'\phi(\xi)\} ,$$

then $\Gamma'_{c''\phi} \subset \Gamma_{c\phi}$.

9. We now turn effectively to the proof of proposition 2.6 and fix $c_2 = c_9/2$. There is no loss in generality if we assume $c_1 = c_9$ (we shrink c_1 if necessary) and we introduce (for this proof only) the notations

$$A = \{\theta \in R^{n+1};\ |\theta| < c_{10}\phi(-\text{Re}\ \zeta)\} ,$$

$$B = \{\theta \in R^{n+1};\ |\text{Re}\ \zeta - \theta' + \xi^0| \geq c_2\phi(\xi^0)\} .$$

We also assume that $|\xi - \eta| < c_2\phi(\xi)$ implies $\phi(\xi) \leq 2\phi(\eta) \leq 4\phi(\xi)$.

10. Proof of (14). The assumption is $-\text{Re}\ \zeta \notin \Gamma'$ and $|\tau| < C'(\phi(-\text{Re}\ \zeta) + |\text{Im}\ \zeta|)$. In view of lemma 2.7 we have $A \subset B$ for such λ. We now distinguish in the integral from (17) three cases:

I: $\theta \in A$, II: $\theta \notin A$, $\theta \in B$, III: $\theta \notin (A \cup B)$.

In the case I, we can estimate $F(hg)$ by (19). To estimate $\hat{v}(\lambda - \theta)$ we use (11), which is applicable here in view of lemma 2.7. It follows (since θ is real) that

$$|\hat{v}(\lambda - \theta)| \leq \exp(\varepsilon|\text{Im}\ \zeta| + b\ \ln(1+|\lambda|) + |b|\ \ln(1+|\theta|)) .$$

Integration over A now leads to an estimate of the type (14).

In case II, we estimate $F(hg)$ by (22) (we must shrink c_4 until this is possible.). For $\hat{v}(\lambda - \theta)$ we use (12) and get

$$|\hat{v}(\lambda-\theta)| \le \exp(\varepsilon|\mathrm{Im}\ \zeta| + \varepsilon C'(\phi(-\mathrm{Re}\ \zeta) + |\mathrm{Im}\ \zeta|) + b\ \ln(1+|\lambda|) + |b|\ \ln(1+|\theta|)) .$$

If ε is small, the factor $\exp(-c_4\phi(-\mathrm{Re}\ \zeta))$ from (22) will compensate for the factor $\exp(\varepsilon C'\phi(-\mathrm{Re}\ \zeta))$ in the estimation of $\hat{v}(\lambda-\theta)$, so we can again integrate and obtain an estimate of type (14) if $(\varepsilon+\varepsilon C') \le \varepsilon'$.

It remains to consider the case III, when

$$|\mathrm{Re}\ \zeta - \theta' + \xi^o| \le c_2\phi(\xi^o) \quad \text{and} \quad |\theta| > c_{10}\phi(-\mathrm{Re}\ \zeta) .$$

Since we still have $-\mathrm{Re}\ \zeta \notin \Gamma'$, we also have $|\mathrm{Re}\ \zeta + \xi^o| \ge c_9\phi(\xi^o)$, so $|\theta'| \ge c_2\phi(\xi^o)$ in view of $c_2 = c_9/2$. It follows that $|\theta| > (\phi(-\mathrm{Re}\ \zeta) + \phi(\xi^o))$ all in all, so we can now estimate $F(hg)$ by (24). Furthermore, we estimate $\hat{v}(\lambda-\theta)$ using (13), and we also have $\phi(-\mathrm{Re}\ \zeta + \theta') \le 2\phi(\xi^o)$ (cf.nr. 9), so together with $|\tau| < C'(\phi(-\mathrm{Re}\ \zeta) + |\mathrm{Im}\ \zeta|)$, we obtain

$$|\hat{v}(\lambda-\theta)| \le \exp(2d\phi(\xi^o) + \varepsilon|\mathrm{Im}\ \zeta| + \varepsilon C'\phi(-\mathrm{Re}\ \zeta) + \varepsilon C'|\mathrm{Im}\ \zeta| + b\ \ln(1+|\lambda|) + |b|\ \ln(1+|\theta|)) .$$

If ε and d are small we obtain an estimate of the desired type by just integrating.

11. <u>Proof of (15).</u> The assumption is here $|\mathrm{Re}\ \zeta + \xi^o| \ge c_1\phi(\xi^o)$. This time we only consider the cases $\theta \in B$, $\theta \notin B$. When $\theta \in B$, we estimate $F(hg)$ by (19) and $\hat{v}(\lambda-\theta)$, starting from (12), by $\exp(\varepsilon|\mathrm{Im}\ \zeta| + \varepsilon\ \mathrm{Im}\ \tau^+ + b\ \ln(1+|\lambda|) + |b|\ \ln(1+|\theta|))$. In case $\theta \notin B$, $|\theta| \ge c_{10}\phi(\xi^o)$, so we can estimate $F(hg)$ by (23) and $\hat{v}(\lambda-\theta)$ by $\exp(\varepsilon|\mathrm{Im}\ \zeta| + \varepsilon\ \mathrm{Im}\ \tau^+ + 2d\phi(\xi^o) + b\ \ln(1+|\lambda|) + |b|\ \ln(1+|\theta|))$, etc.

12. <u>Proof of (16).</u> Here $|\mathrm{Re}\ \zeta + \xi^o| < c_1\phi(\xi^o)$. For $\theta \in B$, we use

$$|\hat{v}(\lambda-\theta)| \le \exp(\varepsilon|\mathrm{Im}\ \zeta| + \varepsilon\ \mathrm{Im}\ \tau^+ + b\ \ln(1+|\lambda|) + |b|\ \ln(1+|\theta|)),$$

which leads to an estimation by $c_{14}\exp(\varepsilon|\text{Im}\,\zeta| + \varepsilon\,\text{Im}\,\tau^{+} + b\ln(1+|\lambda|))$ for the contribution of the integral over B in (17), if we estimate $F(hg)$ by (19). When $\theta \notin B$, we use the same estimate for $F(hg)$ and also the fact that

$$|\hat{v}(\lambda-\theta)| \le \exp(\varepsilon|\text{Im}\,\zeta| + \varepsilon\,\text{Im}\,\tau^{+} + d\phi(-\text{Re}\,\zeta + \theta') + b\ln(1+|\lambda|) + |b|\ln(1+|\theta|)\,.$$

It remains to observe that

$$d\phi(-\text{Re}\,\zeta + \theta') \le 2d\phi(\xi^{o}) \le 4d\phi(-\text{Re}\,\zeta) \le d'\phi(-\text{Re}\,\zeta)$$

if $4d \le d'$, and to integrate.

13. We mention some corollaries of the propositions 2.4, 2.6, and of their proofs.

PROPOSITION 2.9. Consider $\varepsilon' > 0$, $d' > 0$, $b \ge 0$, and $\Gamma, \Gamma' \subset R^{n}$ such that $\Gamma \subset_{\phi} \Gamma'$. Then there are $\varepsilon > 0$, $d > 0$, $c > 0$, b', with the following property: whenever $v \in E'(R^{n+1})$ satisfies

$$|\hat{v}(\lambda)| \le \exp(d\phi(-\text{Re}\,\zeta) + \varepsilon|\text{Im}\,\zeta| + \varepsilon\,\text{Im}\,\tau^{+} + b\ln(1+|\lambda|))\,, \quad -\text{Re}\,\zeta \in \Gamma\,,$$

$$|\hat{v}(\lambda)| \le \exp(\varepsilon|\text{Im}\,\zeta| + \varepsilon\,\text{Im}\,\tau^{+} + b\ln(1+|\lambda|))\,, \quad \text{for} \quad -\text{Re}\,\zeta \notin \Gamma\,,$$

it follows that $w = gv$, $g \in A(z \in C^{n+1}; |z| < \varepsilon')$ satisfies

$$|\hat{w}(\lambda)| \le c\exp(d'\phi(-\text{Re}\,\zeta) + \varepsilon'|\text{Im}\,\zeta| + \varepsilon'\,\text{Im}\,\tau^{+} + (b+b')\ln(1+|\lambda|))\,, \quad -\text{Re}\,\zeta \in \Gamma'\,,$$

$$|\hat{w}(\lambda)| \le c\exp(\varepsilon'|\text{Im}\,\zeta| + \varepsilon'\,\text{Im}\,\tau^{+} + (b+b')\ln(1+|\lambda|)) \quad \text{for} \quad -\text{Re}\,\zeta \notin \Gamma'$$

if $|g(z)| \le 1$. (Only c depends here on b.)

When $v \in C_0^\infty(R^{n+1})$ this a a direct consequence of the propositions 2.4 and 2.6: at first we split v according to proposition 2.4, then we multiply each term in the sum with g and then we add the resulting distributions. Note that the constant c which appears in this way does not depend on the number of terms in the decomposition of v, since $\sum 1/j^2 < \infty$. We can therefore obtain the conclusion in the case $v \in \mathcal{E}'(R^{n+1})$ if we just approximate v by a sequence of $C_0^\infty(R^{n+1})$ functions which satisfy similar inequalities. (Convolution with a sequence of $C_0^\infty(R^{n+1})$ functions with small support in $t \geq 0$ which approximate the δ-Dirac distribution will do.) We omit further details. (Actually "approximation" is not necessary: cf. the proof of proposition 2.11 below.)

<u>COROLLARY 2.10.</u> Let u be a germ of a C^∞ function defined near $0 \in R^{n+1}$ and assume that $(0,\Gamma) \cap WF_\phi^b u = \emptyset$. Then $(0,\Gamma) \cap WF_\phi^b gu = \emptyset$ if g is real-analytic near 0.

14. The next result is just a completion of proposition 2.9, which we shall state separately, in order to make references to these propositions more transparent.

<u>PROPOSITION 2.11.</u> Let ε', d', c', $0 < C' < C$, and Γ, $\Gamma' \subset R^n$ be given with $\Gamma \subset_\phi \Gamma'$. Then we can find $\varepsilon > 0$, $d > 0$ and for every $b \geq 0$ some c such that if we add to the assumptions in proposition 2.9 that

$$|\hat{v}(\lambda)| \leq \exp(\varepsilon|\mathrm{Im}\ \zeta| + b\ \ln(1+|\zeta|)),$$
$$\text{for } -\mathrm{Re}\ \zeta \notin \Gamma \text{ and } |\tau| \leq C(\phi(-\mathrm{Re}\ \zeta) + |\mathrm{Im}\ \zeta|),$$

then we can add to the conclusions that

$$|\hat{w}(\lambda)| \leq c\ \exp(\varepsilon'|\mathrm{Im}\ \zeta| + (b+b')\ \ln(1+|\zeta|))$$
$$\text{for } -\mathrm{Re}\ \zeta \notin \Gamma' \text{ and } |\tau| \leq C'(\phi(-\mathrm{Re}\ \zeta) + |\mathrm{Im}\ \zeta|).$$

The proof of this result is similar to that of proposition

2.9. In fact, the main preliminary result, proposition 2.6, is already in the form in which we need it, so we must only observe that we can improve proposition 2.4 to

<u>PROPOSITION 2.12.</u> Let d', ε', c', C, be given and consider $\Gamma \subset_\phi \Gamma' \subset R^n$. If ε, d, b' are suitable we can then find for every $b \geq 0$ some c such that if we add to the assumptions in proposition 2.4 that

$$|\hat{u}(\lambda)| \leq \exp(\varepsilon|\text{Im } \zeta| + b \ln(1+|\zeta|))$$
$$\text{for } -\text{Re } \zeta \notin \Gamma \text{ and } |\tau| \leq C(\phi(-\text{Re } \zeta) + |\text{Im } \zeta|),$$

then we can add to the conclusions that

$$|\hat{v}_j(\lambda)| \leq (c/j^2) \exp(\varepsilon'|\text{Im } \zeta| + (b+b') \ln(1+|\zeta|)),$$
$$\text{if } -\text{Re } \zeta \notin \Gamma' \text{ and } |\tau| \leq C(\phi(-\text{Re } \zeta) + |\text{Im } \zeta|).$$

15. One can prove proposition 2.12 either directly, or else by reduction to proposition 2.4. In the second case we need:

<u>PROPOSITION 2.13.</u> Let $C > 0$, $\varepsilon' > 0$, $c_1 > 0$ and $\Gamma, \Gamma' \subset R^n$ be given with $\Gamma_{c_1\phi} \subset_\phi \Gamma'$. Also choose c_2 so that $\tilde{\Gamma} = \{\eta;\ \exists \xi \in \Gamma \text{ s.t. } |\xi-\eta| < c_2\phi(\xi)\} \subset \Gamma_{c_1\phi}$. Then we can find $b'' \geq 0$, $\varepsilon > 0$, $d > 0$, and for every $b \geq 0$ some c_3 with the following property:

if $v \in E'(R^{n+1})$ and $\xi^o \in \Gamma$ are given such that

$$|\hat{v}(\lambda)| \leq \exp(d\phi(-\text{Re } \zeta) + \varepsilon|\text{Im } \zeta| + \varepsilon \text{ Im } \tau^+ + b \ln(1+|\lambda|)),$$
$$\text{for } |\text{Re } \zeta + \xi^o| \leq c_2\phi(\xi^o),$$

$$|\hat{v}(\lambda)| \leq \exp(\varepsilon|\text{Im } \zeta| + \varepsilon \text{ Im } \tau^+ + b \ln(1+|\lambda|))$$
$$\text{for } |\text{Re } \zeta + \xi^o| \geq c_2\phi(\xi^o),$$

then there are $v^1, v^2 \in E'(R^{n+1})$ such that $v = v^1 + v^2$ and such that

$$|\hat{v}^1(\lambda)| \le c_3 \exp(\varepsilon'|\operatorname{Im}\zeta| + \varepsilon' \operatorname{Im}\tau^+ + (b+b')\ln(1+|\lambda|)) ,$$
$$\forall \lambda \in C^{n+1} ,$$

$$|\hat{v}^2(\lambda)| \le c_3 \exp(d\phi(-\operatorname{Re}\zeta) + \varepsilon'|\operatorname{Im}\zeta| + \varepsilon' \operatorname{Im}\tau^+$$
$$+ (b+b')\ln(1+|\lambda|)) , \quad \text{if} \quad |\operatorname{Re}\zeta + \xi^o| \le c_2\phi(\xi^o) ,$$

$$|\hat{v}^2(\lambda)| \le c_3 \exp(\varepsilon'|\operatorname{Im}\zeta| + \varepsilon' \operatorname{Im}\tau^+ + (b+b')\ln(1+|\lambda|)) ,$$
$$\text{if} \quad |\operatorname{Re}\zeta + \xi^o| \ge c_2\phi(\xi^o) ,$$

$$|\hat{v}^2(\lambda)| \le c_3 \exp(\varepsilon'|\operatorname{Im}\zeta| + (b+b')\ln(1+|\zeta|)),$$
$$\text{if} \quad -\operatorname{Re}\zeta \notin \Gamma' \quad \text{and} \quad |\tau| \le C(\phi(-\operatorname{Re}\zeta) + |\operatorname{Im}\zeta|) .$$

<u>Proof of proposition 2.13.</u> We choose c_4 so that $\tilde{\Gamma}_{c_4\phi} \subset_\phi \Gamma'$ and consider $e \in C^\infty(R^n)$ for which $e(\xi) = 1$ if $|\xi| \ge 1$, $\xi \in \tilde{\Gamma}$, $e(\xi) = 0$ if $\xi \notin \tilde{\Gamma}_{c_4\phi}$, $0 \le e(\xi) \le 1$, $|\operatorname{grad}_\xi e| \le c_5$.
We then define F_1, F_2, by $F_1(\lambda) = (1 - e(-\operatorname{Re}\zeta))\hat{v}(\lambda)$, $F_2(\lambda) = e(-\operatorname{Re}\zeta)\hat{v}(\lambda)$. Let also $\varepsilon > 0$ and a plurisubharmonic function ρ' be associated with $\Gamma'' = \tilde{\Gamma}_{c_4\phi}$, Γ' and ε' as in lemma 2.3. It follows that $\bar{\partial}F_1$ $(= -\bar{\partial}F_2)$ satisfies

$$|\bar{\partial}F_1(\lambda)| \le c_6 \exp(\rho'(\lambda) + b\ln(1+|\lambda|)) ,$$

so we can conclude from results concerning the $\bar{\partial}$-operator proved in Hörmander [2] that there is $H \in C^\infty(C^{n+1})$ with $\bar{\partial}H = \bar{\partial}F_1$ such that

$$\int |H(\lambda)|^2 \exp(-2\rho'(\lambda) - (b+n+3)\ln(1+|\lambda|^2))\, dx \le c_7 .$$

We thus have an L^2-estimate for H and a sup-norm estimate for $\bar{\partial}H$. It is easy to conclude from this (for a related result cf. e.g. Kiselman [1]) that

$$|H(\lambda)| \le c_8 \exp(\varepsilon'|\mathrm{Im}\ \zeta| + \varepsilon'\ \mathrm{Im}\ \tau^+ + (b+n+3)\ \ln(1+|\lambda|)) ,$$

respectively

$$|H(\lambda)| \le c_8 \exp(\varepsilon'|\mathrm{Im}\ \zeta| + (b+n+3)\ \ln(1+|\lambda|))$$
$$\text{if} \quad -\mathrm{Re}\ \zeta \notin \Gamma' \quad \text{and} \quad |\tau| \le C(\phi(-\mathrm{Re}\ \zeta) + |\mathrm{Im}\ \zeta|) .$$

We can therefore define v^1 by $v^1 = F_1 - H$ and v^2 by $v^2 = F_2 + H$.

16. Proof of proposition 2.12. The proposition follows from proposition 2.4, if we also apply proposition 2.13. To see this, consider d', ε', c' and $\Gamma \subset R^n$. We are allowed to shrink c' if necessary, so we may assume (cf. lemma 2.8) that for some c'', $\Gamma''_{c''\phi} \subset \Gamma'$, where

$$\Gamma'' = \{\eta \in R^n;\ \exists \xi \in \Gamma \quad \text{such that} \quad |\xi - \eta| < c'\phi(\xi)\} .$$

Also choose ε'', d'', to be specified later on and consider u as in proposition 2.4. Application of that result for suitable ε, d, b' shows that we can find $w_j \in E'(R^{n+1})$ such that $u = \sum w_j$, and such that the w_j satisfy the inequalities from b), c) in proposition 2.4, if we replace ε' and d' by ε'' and d'' . We now split each w_j in the form $w_j = w_j^1 + w_j^2$, using proposition 2.13. If ε'' and d'' have been chosen suitably, we may assume here that

$$|\hat{w}_j^1(\lambda)| \le (c_9/j^2) \exp(\varepsilon'|\mathrm{Im}\ \zeta| + \varepsilon'\ \mathrm{Im}\ \tau^+$$
$$+ (b+b'')\ \ln(1+|\lambda|)) ,$$

$$|\hat{w}_j^2(\lambda)| \le (c_9/j^2) \exp(d\phi(-\mathrm{Re}\ \zeta) + \varepsilon'|\mathrm{Im}\ \zeta| + \varepsilon'\ \mathrm{Im}\ \tau^+$$
$$+ (b+b'')\ \ln(1+|\lambda|)) \quad \text{if} \quad |\mathrm{Re}\ \zeta + \xi^j| < c'\phi(\xi^j),$$

$$|\hat{w}_j^2(\lambda)| \le (c_9/j^2) \exp(\varepsilon'|\mathrm{Im}\ \zeta| + \varepsilon'\ \mathrm{Im}\ \tau^+$$
$$+ (b+b'')\ \ln(1+|\lambda|)) \quad \text{if} \quad |\mathrm{Re}\ \zeta + \xi^j| \ge c'\phi(\xi^j),$$

$$|\hat{w}_j^2(\lambda)| \le (c_9/j^2)\, \exp(\varepsilon'|\mathrm{Im}\,\zeta| + (b+b")\, \ln(1+|\zeta|))\ ,$$
$$\text{if}\quad -\mathrm{Re}\,\zeta \notin \Gamma' \quad\text{and}\quad |\tau| \le C(\phi(-\mathrm{Re}\,\zeta) + |\mathrm{Im}\,\zeta|)\ .$$

Thus, in particular, $|\sum_j \hat{w}_j^2(\lambda)| \le c_{10} \exp(\varepsilon'|\mathrm{Im}\,\zeta| + (b+b')\cdot \ln(1+|\zeta|))$, if $-\mathrm{Re}\,\zeta \notin \Gamma'$ and $|\tau| < C(\phi(-\mathrm{Re}\,\zeta) + |\mathrm{Im}\,\zeta|)$, and a similar estimate must then also be valid for $\hat{W} = \sum \hat{w}_j^1$, since $\hat{W}$ is just $\hat{v} - \sum w_j^2$. The desired decomposition is therefore obtained, e.g., with $v_1 = w_1^2 + W$, $v_j = w_j^2$, for $j \ge 2$.

17. Finally we need:

<u>PROPOSITION 2.14.</u> Consider $C > 0$, $\varepsilon' > 0$, $c' > 0$, and $\Gamma^1,\dots,\Gamma^s \subset R^n$ such that $\cup\Gamma^k = R^n$. Then we can find $\varepsilon > 0$, $b' \ge 0$, and for every $b \ge 0$ some c such that any $v \in E'(R^{n+1})$ which satisfies

$$|\hat{v}(\lambda)| \le \exp(\varepsilon|\mathrm{Im}\,\zeta| + \varepsilon\,\mathrm{Im}\,\tau^+ + b\,\ln(1+|\lambda|))$$

can be decomposed into the form $v = v^1 + \dots + v^s$, where $v^k \in E'(R^{n+1})$,

$$|\hat{v}^k(\lambda)| \le c\,\exp(\varepsilon'|\mathrm{Im}\,\zeta| + \varepsilon'\mathrm{Im}\,\tau^+ + (b+b')\,\ln(1+|\lambda|))\ ,$$
$$\forall\lambda \in C^{n+1}\ ,$$

and

$$|\hat{v}^k(\lambda)| \le c\,\exp(\varepsilon'|\mathrm{Im}\,\zeta| + (b+b')\,\ln(1+|\lambda|))$$
$$\text{if}\quad -\mathrm{Re}\,\zeta \notin \Gamma^k_{c'\phi} \quad\text{and}\quad |\tau| \le C(\phi(-\mathrm{Re}\,\zeta) + |\mathrm{Im}\,\zeta|)\ .$$

The proof follows by induction from the following result:

<u>PROPOSITION 2.15.</u> Let $\Gamma_1,\Gamma_2 \subset R^n$ and ε' $C > 0$, c, c', $0 < c' < c$ be given. Then we can find C', ε, b' and for every b some $c"$ with the following property:

any $v \in E'(R^{n+1})$ which satisfies

$$|\hat{v}(\lambda)| \le \exp(\varepsilon|\mathrm{Im}\,\zeta| + \varepsilon\,\mathrm{Im}\,\tau^+ + b\,\ln(1+|\lambda|))\ ,\quad \forall\lambda \in C^{n+1}$$

and

$$|\hat{v}(\lambda)| \le \exp(\varepsilon|\mathrm{Im}\ \zeta| + b\ \ln(1+|\zeta|))$$

for $-\mathrm{Re}\ \zeta \notin (\Gamma_1 \cup \Gamma_2)_{c'\phi}$ and $|\tau| \le C'(\phi(-\mathrm{Re}\ \zeta) + |\mathrm{Im}\ \zeta|)$,

can be decomposed into the form $v = v_1 + v_2$, $v_j \in E'(R^{n+1})$,

$$|\hat{v}_j(\lambda)| \le c'' \exp(\varepsilon'|\mathrm{Im}\ \zeta| + \varepsilon'\mathrm{Im}\ \tau^+ + (b+b')\ \ln(1+|\lambda|)) ,$$
$$\forall \lambda \in C^{n+1} ,$$

$$|\hat{v}_j(\lambda)| \le c'' \exp(\varepsilon'|\mathrm{Im}\ \zeta| + (b+b')\ \ln(1+|\zeta|))$$

if $-\mathrm{Re}\ \zeta \notin (\Gamma_j)_{c\phi}$ and $|\tau| \le C(\phi(-\mathrm{Re}\ \zeta) + |\mathrm{Im}\ \zeta|)$.

Proof of proposition 2.15. The situation is similar to that in proposition 2.13. We choose c_1 with $c' < c_1 < c$. If $c_2 > 0$ is very small, we have (cf. lemma 2.8)

$$[(\Gamma_j)_{c_1\phi}]_{c_2\phi} \subset (\Gamma_j)_{c\phi} .$$

Let further $e \in C^\infty(R^n)$ be some function such that $e(\xi) = 1$ for $-\xi \in (\Gamma_1)_{c'\phi}$ if $|\xi| \ge 1$, $e(\xi) = 0$ for $-\xi \notin (\Gamma_1)_{c_1\phi}$, $0 \le e(\xi) \le 1$, $\forall\xi$, and such that $|\mathrm{grad}_\xi\ e| \le c_3$. Define F by $F(\lambda) = e(\mathrm{Re}\ \zeta)\ \hat{v}(\lambda)$. We thus have

$$|\bar{\partial}F(\lambda)| \le c_3 \exp(\varepsilon|\mathrm{Im}\ \zeta| + \varepsilon\ \mathrm{Im}\ \tau^+ + b\ \ln(1+|\lambda|)) ,$$
$$\forall\lambda \in C^{n+1}$$

and

$$|\bar{\partial}F(\lambda)| \le c_3 \exp(\varepsilon|\mathrm{Im}\ \zeta| + b\ \ln(1+|\zeta|))$$

if $-\mathrm{Re}\ \zeta \notin B$ and $|\tau| \le C'(\phi(-\mathrm{Re}\ \zeta) + |\mathrm{Im}\ \zeta|)$,

where $B = [(\Gamma_1)_{c_1\phi} \setminus (\Gamma_1)_{c'\phi}] \cap (\Gamma_2)_{c'\phi}$.

If ε, C', b' have been suitably, we can now find $H \in C^\infty(C^{n+1})$ such that $\bar{\partial}H = \bar{\partial}F$,

$$|H(\lambda)| \leq c_4 \exp(\varepsilon'|\mathrm{Im}\ \zeta| + \varepsilon'\ \mathrm{Im}\ \tau^+ + (b+b')\ \ln(1+|\lambda|)) ,$$
$$\forall\lambda \in C^{n+1} ,$$

respectively

$$|H(\lambda)| \leq c_4 \exp(\varepsilon'|\mathrm{Im}\ \zeta| + (b+b')\ \ln(1+|\zeta|))$$
$$\text{if } -\mathrm{Re}\ \zeta \notin B_{c_2\phi} , \quad |\tau| \leq C(\phi(-\mathrm{Re}\ \zeta) + |\mathrm{Im}\ \zeta|) .$$

To obtain the existence of such an H, we can argue as in the proof of proposition 2.13 (using once more lemma 2.3.). Now we observe that, by the choice of c_2, $B_{c_2\phi} \subset (\Gamma_j)_{c\phi}$ for $j = 1,2$. The proof is therefore complete if we set $\hat{v}_1 = F - H$, $\hat{v}_2 = H + \hat{v}(1 - e(\mathrm{Re}\ \zeta))$.

§3. PRELIMINARIES CONCERNING PSEUDODIFFERENTIAL OPERATORS.

1. In this paragraph we recall some elementary results concerning pseudodifferential operators related to G_ϕ classes.

Let us then choose p of form (3), §1, and assume that (4), §1, is valid for some Lipschitz-continuous ϕ.

Our first remark is:

<u>LEMMA 3.1.</u> Let $\phi : R^n \to R$ be given and assume that

$$|D^\alpha q(\xi)| \leq \phi(\xi)^{j-|\alpha|} \qquad \text{if } \xi \in R^n , \forall\alpha .$$

Then

$$|q(\zeta)| \leq (\phi(\mathrm{Re}\ \zeta) + |\mathrm{Im}\ \zeta|)^j , \qquad \forall\zeta \in C^n .$$

Indeed, in view of Taylor's formula,

$$q(\zeta) = \sum_\alpha [(\partial/\partial\xi)^\alpha\ q(\mathrm{Re}\ \zeta)](\mathrm{Im}\ \zeta)^\alpha/\alpha! ,$$

so

$$|q(\zeta)| \leq \sum_{|\alpha|\leq j} \phi(\mathrm{Re}\ \zeta)^{j-|\alpha|}|\mathrm{Im}\ \zeta|^\alpha/\alpha! \leq (\phi(\mathrm{Re}\ \zeta) + |\mathrm{Im}\ \zeta|)^j .$$

We conclude from (4), §1, that

$$|D^\gamma_{x,t}\, D^\beta_\zeta \sum_{|\alpha|\le m-j} q_{j\alpha}(x,t)\zeta^\alpha|$$

$$\le c^{|\gamma|+1}\gamma!(\phi(\mathrm{Re}\ \zeta) + |\mathrm{Im}\ \zeta|)^{m-j-|\beta|},$$

$$\forall\gamma,\ \forall\beta,\ \forall\zeta\in C^n, \text{ if } |(x,t)|\le\varepsilon\ . \tag{1}$$

Here we may even assume that (x,t) is complex. Let us also note that (1) implies in particular

$$|\tau| \le c_1(\phi(\mathrm{Re}\ \zeta) + |\mathrm{Im}\ \zeta|)\ , \quad \text{if } p(x,t,\zeta,\tau) = 0\ , \tag{2}$$

for some $c_1 > 0$. (Cf. e.g. Malgrange [1, chapt. IV].)

2. Let c_1 be such that (2) is valid. We denote by

$$G = \{\lambda\in C^{n+1},\ |\tau| \ge (c_1+1)(\phi(\mathrm{Re}\ \zeta) + |\mathrm{Im}\ \zeta|)\ .$$

For $\lambda\in G$ we then have $|p(x,t,\zeta,\tau)| \ge c_2|\tau|^m$ if $c_2 > 0$ is small, $(x,t)\in C^{n+1}$, $|(x,t)| < \varepsilon$. Together with (1) we conclude that

$$|D^\gamma_{x,t}\, D^\beta_\lambda\, p(x,t,\zeta,\tau)| \le c_3^{|\gamma|+1}|\tau|^{-|\beta|}\gamma!\ |p(x,t,\zeta,\tau)|$$

$$\text{if } |(x,t)| < \varepsilon \text{ and } \lambda\in G\ . \tag{3}$$

As is standard, we can deduce from this the existence of an inverse symbol. Let us in fact denote by $SF^\mu(U,G)$ the space of formal sums $\sum q_j$, $q_j\in C^\infty(U,G)$, $j = 0,1,2,\dots,$ where U is some neighborhood of $0\in C^{n+1}$ and where:

a) the q_j are analytic functions on $U\times G$,

b) there is $c > 0$ such that

$$|D^\gamma_z\, D^\beta_\lambda\, q_j(z,\lambda)| \le c^{|\gamma|+|\beta|+1}\gamma!\ \beta!\ |\tau|^{\mu-|\beta|}$$

$$\text{if } z\in U\ ,\ \lambda\in G\ . \tag{4}$$

If $\sum q_j \in SF^{\mu}(U,G)$, then we define an element $p \circ \sum q_j$ in $SF^{\mu+m}(U,G)$ by $p \circ \sum q_j = \sum_k r_k$, where

$$r_k = \sum_{|\alpha|+j=k} (i^{-|\alpha|}/\alpha!)[(\partial/\partial\lambda)^{\alpha} p(z,\lambda)](\partial/\partial z)^{\alpha} q_j(z,\lambda) ,$$

and we shall say that $\sum s_j \sim 1$ in $SF^{o}(U,G)$ if

$$|D_z^{\gamma} D_{\lambda}^{\beta} (1 - \sum_{j \leq \kappa} s_j(z,\lambda))|$$

$$\leq c'^{|\gamma|+|\beta|+\kappa+1}\gamma! \ \beta! \ \kappa! \ |\tau|^{-|\beta|-|\kappa|} , \quad \forall z \in U , \ \forall\lambda \in G . \quad (5)$$

<u>LEMMA 3.2.</u> There is $\sum q_j \in SF^{-m}(U,G)$ such that $p \circ \sum q_j \sim 1$ in $SF^{o}(U,G)$.

Lemma 3.2 is standard. (Cf. Hörmander [3] and, for a result closer to this one, Liess-Rodino [1]) . Let us note in fact that $p \circ \sum q_j$ is precisely the standard rule for the composition of symbols and that one can therefore compute the q_j recurently from $q_o = 1/p$,

$$p \cdot q_j = - \sum_{1 \leq |\alpha| \leq j} (1/\alpha!)(\partial/\partial\lambda)^{\alpha} p(z,\lambda) \cdot D_z^{\alpha} q_{j-|\alpha|}(z,\lambda) . \quad (6)$$

One can then easily also prove (4) and (5), either using an induction (which involves an inequality slightly more complicated than (4) itself. Induction directly on (4) does not work) or else using formal norms as in Boutet de Monvel-Kree [1]. We omit further details.

3. We have recalled the construction of the q_j in some detail, since it is now also clear that we have

LEMMA 3.3. Consider p and $\sum q_j$ as in the above and fix $d > 0$. Then

$$\sum_{|\alpha|\le m} (1/\alpha!)(\partial/\partial\lambda)^\alpha p(z,\lambda)\, D_z^\alpha \sum_{j<d} q_j(z,\lambda) - 1$$

$$= \sum_{\substack{\beta,k \\ |\beta|\le m,\, k<d \\ |\beta|+k\ge d}} (1/\beta!)(\partial/\partial\lambda)^\beta p(z,\lambda)\, D_z^\beta q_k(z,\lambda) . \qquad (7)$$

Proof. By the definition of the q_j we have that

$$\sum_{|\alpha|+j=r} (1/\alpha!)(\partial/\partial\lambda)^\alpha p(z,\lambda)\, D_z^\alpha q_j(z,\lambda)$$

is equal to one or to zero, according to wether $r = 0$ or not. In particular, we have that

$$T = \sum_{r<d} \sum_{|\alpha|+j=r} (1/\alpha!)(\partial/\partial\lambda)^\alpha p(z,\lambda)\, D_z^\alpha q_j(z,\lambda) - 1 = 0 .$$

The right hand side of (7) is then what remains from the left hand side in that inequality, if we remove T .

§4 . THE DIVISION ALGORITHM .

1. The proof of theorem 1.12 is by duality . The dual problem is related to the decomposition of $v \in \mathcal{E}'(R^{n+1})$, supp v "small", in the form

$$v = {}^t p(z,D)\, w + \sum_{j=0}^{m-1} w_j \otimes D_t^j \delta_t ,$$

where ${}^{t}p$ is the (formal) adjoint of p, the w_j are in n variables and δ_t is the Dirac distribution on the t-axis at $t = 0$. Unless p is hyperbolic in some sense, we cannot expect that the w and w_j are distributions with compact support, but it is easy to prove the following result:

PROPOSITION 4.1. If d' is sufficiently small, then there is $d > 0$ such that for every $v \in A'(z \in C^{n+1};\ |z| < d)$ there are uniquely determined $w \in A'(z \in C^{n+1};\ |z| < d')$ and $w_j \in A'(x \in C^n;\ |x| < d')$ with (1). Moreover, there is a constant C such that if $|\hat{v}(\lambda)| \le \exp(d|\lambda|)$, then $|\hat{w}(\lambda)| \le C \exp(d'|\lambda|)$, $|\hat{w}_j(\zeta)| \le C \exp(d'|\zeta|)$.

We should note that decompositions which are (apart from the fact that ${}^{t}p$ is replaced by p) formally close to (1) have also been considered by P.Schapira [1] ,in a related context. The decompositions from Schapira,loc.cit.,refer however to a subclass of hyperfunctions and not to (general) analytic functionals.Moreover,the use which is made of the respective decompositions is completely different here when compared with Schapira's paper.

Proposition 4.1 follows by dualization from the Cauchy-Kowalewska theorem, which states, for suitable small choices d, d', that the map

$$T : \begin{array}{c} A(|z| < d') \\ \times \\ \prod_{j=0}^{m-1} A(|x| < d') \end{array} \longrightarrow A(|z| < d),$$

which associates with $(f, f_0, \ldots, f_{m-1})$ the solution u of the Cauchy problem $p(z,D)u = f$, $(-1)^j D_t^j u|_{t=0} = f_j$, $j = 0, \ldots, m-1$, is everywhere defined and continuous. In fact, the dual map ${}^{t}T$ associates with every $v \in A'(|z| < d)$ some analytic functionals $w \in A'(|z| < d')$, $w_j \in A'(|x| < d')$ such that

$$v(u) = w(f) + \sum w_j(f_j)$$

$$= ({}^t p(z,D)w)(u) + \sum (w_j \otimes D_t^j \delta_t)(u) , \qquad (2)$$

and this gives (1) .

2. Now we return to (1). If p has constant coefficients, then it follows from (1), applying Fourier transformation, that

$$\hat{v}(\lambda) = p(-\lambda)\hat{w}(\lambda) + \sum \tau^j \hat{w}_j(\zeta) , \qquad \forall \lambda \in C^{n+1} .$$

This is a global variant of the Weierstrass preparation theorem,and there are global contour integration formulas which give the quotient term w and the remainder terms w_j . This makes the map $v \to (w,w_o,\ldots,w_{m-1})$ very precise and leads to an efficient study of the Cauchy problem (1), (2), §1.

In the variable-coefficient case, the Fourier-transformed of (1) is not simpler than is (1) itself and it is not possible to find explicit formulas for the map $v \to (\hat{w},\hat{w}_o,\ldots,\hat{w}_{m-1})$. Nevertheless, it is possible to approximate this map well enough for the applications which we have in mind in this paper. In all this paragraph we assume that p satisfies (4), §1, for some fixed ϕ .

3. To obtain these formulas, we consider constants c,c',c'', $c > 1$, such that

$$|p(z,\lambda)| \geq c'|\tau|^m \quad \text{for} \quad z \in C^{n+1} , |z| < c'' \quad \text{and}$$

$$\lambda \in C^{n+1} , |\tau| \geq c(\phi(\mathrm{Re}\ \zeta) + |\mathrm{Im}\ \zeta|) . \qquad (3)$$

We denote by

$$G = \{\lambda \in C^{n+1}; |\tau| > c(\phi(\mathrm{Re}\ \zeta) + |\mathrm{Im}\ \zeta|)\} ,$$

and assume that c is so large that we can find

$\sum q_j \in SF^{-m}(|z| < c", G)$ with $p \circ \sum q_j \sim 1$ (cf. §3). $c, c', c"$, shall be, from now on until the end of the paragraph, the ones introcuced here.

Denote further, for $\lambda \in C^{n+1}$, by

$$\Lambda(\lambda) = \{\sigma \in C;\ |\sigma| = c(|\tau| + \phi(-\mathrm{Re}\,\zeta) + |\mathrm{Im}\,\zeta|)\} .$$

Note that $|\tau+\sigma| \geq c(\phi(-\mathrm{Re}\,\zeta) + |\mathrm{Im}\,\zeta|)$ for $\sigma \in \Lambda(\lambda)$ (it is here that we use $c > 1$), so $(-\zeta, -\tau-\sigma) \in G$. It follows that $q_j(x,t,-\zeta,-\tau-\sigma)$ makes sense for $\sigma \in \Lambda(\lambda)$.

Let us further denote by $d\sigma$ the arc-element on $\Lambda(\lambda)$, assuming that $\Lambda(\lambda)$ has anti-clockwise orientation.

We now define for every $\lambda \in C^{n+1}$ an operator $T_\lambda : A'(|z| < c") \to C$ by

$$T_\lambda v = \frac{1}{2\pi i} v[\int_{\Lambda(\lambda)} e^{-i<z,(\zeta,\tau+\sigma)>} \frac{1}{\sigma} \sum_{j<\chi A} q_j(x,t,-\zeta,-\tau-\sigma)d\sigma] , \tag{4}$$

where we have denoted by $A = A(\lambda)$,

$$A = |\tau| + \phi(-\mathrm{Re}\,\zeta) + |\mathrm{Im}\,\zeta| . \tag{5}$$

χ is here some small positive number to be chosen later on.

To simplify notations, we shall henceforth denote by $N = (0,\dots,0,1)$, so that $(\zeta,\tau+\sigma) = \lambda + \sigma N$.

4. The definition of T_λ depends on the choice of c and χ. Of course the dependence on c is only apparent (if c satisfies the above assumptions) since $\sigma \to q_j(z, -\lambda-\sigma N)$ is analytic. On the other hand, T_λ does not depend on χ too much either, as long as χ remains small:

PROPOSITION 4.2. Assume, as we shall always do from now on, that $0 < \varepsilon < c"$. Let also b be given and choose $\chi > 0$, $\chi' > 0$, $\chi^0 > 0$. If $d_1 > 0$, ε and χ^0 are small enough and if $\chi' \leq \chi \leq \chi^0$, then we can find $\tilde{c}$ and $d > 0$ such that

$$|v[\int_{\Lambda(\lambda)} e^{-i<z,\lambda+\sigma N>}\frac{1}{\sigma}\sum_{\chi'A\le j<\chi A} q_j(z,-\lambda-\sigma N)d\sigma]|$$

$$\le \tilde{c}\exp(-dA), \qquad \forall\, \lambda \in C^{n+1}, \tag{6}$$

for any $v \in E'(R^{n+1})$ which satisfies

$$|\hat{v}(\lambda)| \le \exp(d_1\phi(-\mathrm{Re}\ \zeta) + \varepsilon|\mathrm{Im}\ \zeta| + \varepsilon|\mathrm{Im}\ \tau| + b\ \ln(1+|\lambda|)) .$$

(Recall here that A depends also on λ .)

We prepare the proof of proposition 4.2 with

<u>LEMMA 4.3.</u> If ε and χ^o are small enough, and if we fix $0 < \chi' \le \chi \le \chi^o$, then we can find $c_1 > 0$ and $d'' > 0$ for which

$$|q_j(z,-\lambda-\sigma N)| \le c_1 \exp(-d''A)$$

$$\text{if } z \in C^{n+1},\ |z| < \varepsilon\ ,\ \sigma \in \Lambda(\lambda)\ , \text{ and } \chi'A \le j < \chi A\ .$$

<u>Proof of lemma 4.3.</u> At first we observe that

$$|q_j(z,-\lambda-\sigma N)| \le c_2^{j+1}\, j!\ |\tau+\sigma|^{-j} \le c_3(c_3 j/A)^j \tag{7}$$

if $\sigma \in \Lambda(\lambda)$. If $c_3\chi^o \le 1/e$ we can now estimate the right hand side of (7) by

$$c_3(c_3\chi)^{\chi'A} \le c_3\exp(-\chi'A)$$

for χ, χ' and j as in the statement.

5. <u>Proof of proposition 4.2.</u> Since the length of $\Lambda(\lambda)$ is $2\pi cA$ and since the number of terms in the sum from (6) is smaller than χA , which both can be estimated by $c_4(1+|\lambda|)$, it suffices to show that

$$|v(q_j(z,-\lambda-\sigma N)\exp(-i<z,\lambda+\sigma N>)| \le c_5\exp(-dA) \quad \text{for } \sigma \in \Lambda(\lambda)\ .$$

This follows for suitable χ^o, ε and d_1 from lemma 4.3 and proposition 2.9. In fact, we fix $0 < d'' \le d'/2$ and $\varepsilon' > 0$ and can conclude that

$$|v(q_j(z,-\lambda-\sigma N)\ \exp(-i<z,\lambda+\sigma N>))\ |$$

$$\le\ c_6\ \exp(-d''A)\ \exp(d'\phi(-\mathrm{Re}\ \zeta) + \varepsilon'|\mathrm{Im}(\lambda+\sigma N)|$$

$$+ (b + b')\ \ln(1+|\lambda|))\ ,\qquad \sigma \in \Lambda(\lambda)\ ,$$

if d_1, ε and χ^o are small enough. The proof comes to an end if we observe that we could have chosen ε' so that

$$-d''A + d'\phi(-\mathrm{Re}\ \zeta) + \varepsilon'|\mathrm{Im}(\lambda+\sigma N)| + (b + b')\ \ln(1+|\lambda|))$$

$$\le\ c_7 - (d'/2)A\ ,\qquad \text{etc}\ .$$

6. Our next concern is to study how well one can estimate the decomposition from proposition 4.1 if one uses the map T_λ . As a preparation we prove:

PROPOSITION 4.4. There is χ^o such that for any fixed χ, $\chi \le \chi^o$, we can find $C > 0$, $d > 0$, such that

$$|e^{i<z,\lambda>}\ p(z,D)\ [e^{-i<z,\lambda>} \sum_{j<\chi A} q_j(z,-\lambda)] - 1|$$

$$\le\ C\ \exp(-dA)\ ,\qquad \text{if}\ \ z \in C^{n+1}\ ,\ |z| < \varepsilon,\ -\lambda \in G\ . \tag{8}$$

Proof.

$$\exp(i<z,\lambda>)\ p(z,D)\ [\exp(-i<z,\lambda>) \sum_{j<\chi A} q_j(z,-\lambda)]$$

$$= p(z,-\lambda+D) \sum_{j<\chi A} q_j(z,-\lambda)$$

$$= \sum_{|\alpha|\le m} (1/\alpha!)\ p^{(\alpha)}(z,-\lambda)\ D_z^\alpha \sum_{j<\chi A} q_j(z,-\lambda)$$

$$= S(z,\lambda)\ ,$$

where $p^{(\alpha)}(z,\lambda) = (\partial/\partial\lambda)^{\alpha} p(z,\lambda)$.

In view of lemma 3.3 it follows that

$$S(z,\lambda) - 1 = \sum_{\substack{\beta,k \\ |\beta| \le m,\ k < \chi A \\ |\beta|+k \ge \chi A}} (1/\beta!)\, p^{(\beta)}(z,-\lambda)\, D_z^{\beta} q_k(z,-\lambda) . \tag{9}$$

In particular $k \ge \chi A-m$ for k in the sum from the right hand side of (9). Furthermore, the generic term from that sum can be estimated for $-\lambda \in G$ by

$$c_1^{|\beta|+k+1}\, |\tau|^{m-|\beta|}\, k!\, |\tau|^{-m-k} \le c_2|\tau|^{-|\beta|}(c_2 k/|\tau|)^k$$
$$\le c_2|\tau|^{-|\beta|}(c_3\chi)^k ,$$

since $k < \chi A$ and $|\tau| > c_4 A$. We can now conclude the argument as in the proof of proposition 4.2 .

<u>PROPOSITION 4.5.</u> Let b be given. If $d_1 > 0$, ε and χ are small enough, then we can find c_1, $d > 0$, such that

$$|T_\lambda({}^t p(z,D)w) - \hat{w}(\lambda)| \le c_1\, e^{-dA} , \quad \forall \lambda \in C^{n+1} ,$$

for any $v \in \mathcal{E}'(R^{n+1})$ which satisfies

$$|\hat{v}(\lambda)| \le \exp(d_1\phi(-\text{Re}\ \zeta) + \varepsilon|\text{Im}\ \zeta| + \varepsilon|\text{Im}\ \tau| + b\ \ln(1+|\lambda|)) .$$

<u>Proof.</u> It follows from the definition of T_λ that

$$T\,({}^t p(z,D)w) = (1/2\pi i)\ w\Big(\int_{\Lambda(\lambda)} p(z,D)\,[\exp(-i\langle z,\lambda+\sigma N\rangle)(1/\sigma)\ \cdot$$
$$\cdot \sum_{j < \chi A} q_j(z,-\lambda-\sigma N)]d\sigma\Big) .$$

We now use proposition 4.4 and obtain

$$p(z,D)\ [\exp(-i\langle z,\lambda+\sigma N\rangle) \sum_{j<\chi A} q_j(z,-\lambda-\sigma N)]$$

$$= \exp(-i\langle z,\lambda+\sigma N\rangle)(1 + S(z,\lambda+\sigma N)) ,$$

where

$$|S(z,\lambda+\sigma N)| \leq c_2 \exp(-d(|\tau+\sigma| + |\mathrm{Im}\ \zeta| + \phi(-\mathrm{Re}\ \zeta))$$
$$\text{if } z \in C^{n+1} , |z| < \varepsilon , \sigma \in \Lambda(\lambda) .$$

Since $(1/2\pi i)\ w[\int_{\Lambda(\lambda)} \exp(-i\langle z,\lambda+\sigma N\rangle)(1/\sigma)d\sigma] = \hat{w}(\lambda)$ (in view of Cauchy's formula), we can conclude the argument as in the proof of proposition 4.2 .

7. In the next result, we will denote by

$$\gamma^* : A'(x \in C^n;\ |x| < \varepsilon) \to A'(z \in C^{n+1};\ |z| < \varepsilon)$$

the imbedding given by $\gamma^*(w)(f) = w(f_{|t=0})$. Thus $\gamma^*(w) = w \otimes \delta_t$.

<u>PROPOSITION 4.6.</u> Consider $w \in A'(x \in C^n;\ |x| < \varepsilon)$ and $s \in \{0,1,\ldots,m-1\}$. Then

$$T_\lambda(D_t^s\ \gamma^*(w)) = 0 , \quad \forall\lambda .$$

<u>Proof.</u> In view of the definition of T_λ ,

$$T_\lambda(D_t^s\ \gamma^*(w)) = (1/2\pi i)\ w[\int_{\Lambda(\lambda)} (1/\sigma)(-D_t)^s\ (\exp(-i\langle z,\lambda+\sigma N\rangle) \cdot$$
$$\cdot \sum_{j<\chi A} q_j(z,-\lambda-\sigma N))d\sigma_{|t=0}]$$
$$= (1/2\pi i)\ w[\int_{\Lambda(\lambda)} (1/\sigma) \sum_{r<s} (-1)^r(\tau+\sigma)^r \cdot$$
$$\cdot \exp(-i\langle x,\zeta\rangle)D_t^{s-r}(\sum_{j<\chi A} q_j(x,t,-\lambda-\sigma N))_{|t=0}\ d\sigma].$$

The proposition therefore follows if we can show that

$$\int_{\Lambda(\lambda)} (1/\sigma)(\tau+\sigma)^r\ D_t^{s-r}\ q_j(z,-\lambda-\sigma N)d\sigma = 0 , \quad \forall s<m , \forall r\leq s, \forall j, \forall\lambda . \tag{10}$$

To prove this, we need only observe that $\sigma \to D_t^{s-r} q_j(z,-\lambda-\sigma N)$ is analytic for $|\sigma| \geq cA$ and that $|D_t^{s-r} q_j(z,-\lambda-\sigma N)| \leq c_1 |\tau+\sigma|^{-m-j}$.

In fact it follows from this that

$$|(\tau+\sigma)^r (1/\sigma)\ D_t^{s-r} q_j(z,-\lambda-\sigma N)| \leq c(\lambda)|\sigma|^{r-1-m} ,$$

so (10) follows after a contour deformation $\Lambda(\lambda) \to \infty$ (or, alternatively, if we apply the residuum theorem for $\sigma = \infty$.)

§5. ESTIMATES FOR THE QUOTIENT AND FOR THE REMAINDER TERMS IN THE DIVISION ALGORITHM.

1. We can now return to proposition 4.1 and give useful estimates for w and w_j when

$$v = {}^t p(z,D)w + \sum_{j=0}^{m-1} w_j \otimes D_t^j \delta_t . \tag{1}$$

We shall prove such estimates at first when v satisfies some supplimentary inequalities.

PROPOSITION 5.1. Assume that $\varepsilon' > 0$, b, $C' > 0$ are given. Then we can find $\varepsilon > 0$, b', $C > 0$ and $\tilde{c}$ (with ε, b', and C independent of b), with the following property: consider $v \in \mathcal{E}'(R^{n+1})$ such that

$$|\hat{v}(\lambda)| \leq \exp(\varepsilon|\mathrm{Im}\ \zeta| + b\ \ln(1+|\zeta|))$$
$$\text{for } |\tau| \leq C(\phi(-\mathrm{Re}\ \zeta) + |\mathrm{Im}\ \zeta|) , \tag{2}$$

respectively

$$|\hat{v}(\lambda)| \leq \exp(\varepsilon|\mathrm{Im}\ \zeta| + \varepsilon|\mathrm{Im}\ \tau| + b\ \ln(1+|\lambda|))$$
$$\text{for } |\tau| \geq C(\phi(-\mathrm{Re}\ \zeta) + |\mathrm{Im}\ \zeta|) , \tag{3}$$

and assume that v, w, w_j are related by (1). Then

$$|\hat{w}(\lambda)| \le \tilde{c}\,\exp(\varepsilon'|\text{Im}\ \zeta| + (b+b')\ \ln(1+|\zeta|))$$
$$\text{if}\quad |\tau| \le C'(\phi(-\text{Re}\ \zeta) + |\text{Im}\ \zeta|)\ , \tag{4}$$

$$|\hat{w}(\lambda)| \le \tilde{c}\,\exp(\varepsilon'|\text{Im}\ \zeta| + \varepsilon'|\text{Im}\ \tau| + (b+b')\ \ln(1+|\zeta|)) \tag{5}$$

and

$$|\hat{w}_j(\zeta)| \le \tilde{c}\,\exp(\varepsilon'|\text{Im}\ \zeta| + (b+b')\ \ln(1+|\zeta|))\ . \tag{6}$$

Moreover, the same assertion is valid, if we replace everywhere, in the assumption and in the conclusion, $|\text{Im}\ \tau|$ by $\text{Im}\ \tau^+$.

Remark 5.2. Before we turn to the proof of proposition 5.1 we observe that (6) is a consequence of (4) and (5), if we write $\sum \tau^j\ \hat{w}_j(\zeta) = \hat{v}(\lambda) - F({}^t p(z,D)w)(\lambda)$. (At this moment, it is perhaps interesting to note that (6) is already a consequence of (4), if we use arguments related to those from §2 .)

2. Proof of proposition 5.1. Assume that (1) is valid. We apply T_λ on both sides of (1) and get in view of proposition 4.6 that

$$T_\lambda v = T_\lambda({}^t p(z,D)w)\ .$$

Furthermore,

$$T_\lambda({}^t p(z,D)w) - \hat{w}(\lambda) = O(\exp(-d(\phi(-\text{Re}\ \zeta) + |\text{Im}\ \zeta| + |\tau|))$$

for some $d > 0$ (with "uniformity" in the constants) in view of proposition 4.5. Therefore (4) follows from

PROPOSITION 5.3. If ε and χ from the definition of T_λ (cf. §4) are suitably small and if C is sufficiently large, then

$$|T_\lambda v| \le c_1 \exp(\varepsilon'|\mathrm{Im}\ \zeta| + (b+b')\ \ln(1+|\zeta|))$$
$$\text{for} \quad |\tau| \le C'(\phi(-\mathrm{Re}\ \zeta) + |\mathrm{Im}\ \zeta|)\ , \tag{7}$$

if v satisfies (2) and (3).

<u>Proof of proposition 5.3.</u> At first we recall that

$$T_\lambda v = (1/2\pi i) \int_{\Lambda(\lambda)} v[(1/\sigma)\ e^{-i\langle z,\lambda+\sigma N\rangle} \cdot$$
$$\cdot \sum_{j \le \chi A} q_j(z,-\lambda-\sigma N)]d\sigma\ , \tag{8}$$

where, as in §4, $A = |\tau| + \phi(-\mathrm{Re}\ \zeta) + |\mathrm{Im}\ \zeta|$ and where

$$\Lambda(\sigma) = \{\sigma \in C;\ |\sigma| = c(|\tau| + \phi(-\mathrm{Re}\ \tau) + |\mathrm{Im}\ \zeta|)\}\ .$$

Here c is the one from nr. 3 in §4 .

Since the number of terms in the sum from (8) is bounded by χA , it suffices to show that

$$|v(\exp(-i\langle z,\lambda+\sigma N\rangle)\ q_j(z,-\lambda-\sigma N))| \le c_2 \exp(\varepsilon'|\mathrm{Im}\ \zeta| + (b+b_1) \cdot$$
$$\cdot \ln(1+|\zeta|))\ ,$$

$$\text{if } j<\chi A\ ,\ \sigma \in \Lambda(\lambda) \text{ and } |\tau| < C'(\phi(-\mathrm{Re}\ \zeta) + |\mathrm{Im}\ \zeta|)\ . \tag{9}$$

Note that $|\tau| < C'(\phi(-\mathrm{Re}\ \zeta) + |\mathrm{Im}\ \zeta|)$ implies for $\sigma \in \Lambda(\lambda)$ that $|\tau+\sigma| < (C' + C'c + c)(\phi(-\mathrm{Re}\ \zeta) + |\mathrm{Im}\ \zeta|)$. If we choose $C > (C' + C'c + c)$ we can then apply proposition 2.6 and get (9) if χ and ε are small. (The condition on χ is that $|q_j(z,-\lambda-\sigma N)| \le c_3$ for small z and for the j,λ,σ, under consideration. Such a choice is possible, when ε is small, in view of $|q_j(z,-\lambda-\sigma N)| \le c_4(c_4 j/|\tau+\sigma|)^j \le c_4(c_5\chi)^j$.)

3. We have now proved (4) and it remains to prove (5). We may here assume that $C' > 3c$, so it suffices to prove (5) for $|\tau| \ge 3c(\phi(-\mathrm{Re}\ \zeta) + |\mathrm{Im}\ \zeta|)$. Again it suffices to estimate $T_\lambda v$ instead of $\hat{w}$. To do so, we replace the contour $\Lambda(\lambda)$ by

$\Lambda^1(\lambda) \cup \Lambda^2(\lambda)$, where

$$\Lambda^1(\lambda) = \{\sigma;\ |\sigma| = c(\phi(-\mathrm{Re}\ \zeta) + |\mathrm{Im}\ \zeta|)\} ,$$

$$\Lambda^2(\lambda) = \{\sigma;\ |\sigma+\tau| = c(\phi(-\mathrm{Re}\ \zeta) + |\mathrm{Im}\ \zeta|)\}$$

(anticlockwise orientation).

If follows from the residuum theorem that $T_\lambda v = I_1 + I_2$ where

$$I_k = (1/2\pi i) \int_{\Lambda^k(\lambda)} v[(1/\sigma)\ e^{-i<z,\lambda+\sigma N>} \sum_{j \le \chi A} q_j(z,-\lambda-\sigma N)]\ d\sigma . \tag{10}$$

We compute I_1 with the residuum theorem. The only residuum is for $\sigma = 0$ and $\sigma \to q_j(z,-\lambda-\sigma N)$ has no singularity there in view of $|\tau| > 3c(\phi(-\mathrm{Re}\ \zeta) + |\mathrm{Im}\ \zeta|)$. Therefore $I_1 = v(\exp(-i<z,\lambda>) \sum_{j \le \chi A} q_j(z,-\lambda))$ and we can estimate this by $c_6 \exp(\varepsilon|\mathrm{Im}\ \zeta| + \varepsilon|\mathrm{Im}\ \tau| + (b + b_2) \ln(1+|\lambda|))$ for small χ and ε.

It remains to estimate I_2 . Here we can apply proposition 2.6 as in the proof of proposition 5.3. We can then estimate I_2 by the right hand side of (4). We omit further details.

4. We also need

PROPOSITION 5.4. Let $d' > 0$, $\varepsilon' > 0$, be given. Then we can find $d > 0$, $\varepsilon > 0$, b', $\tilde{c}$, with the following property: if $v \in E'(R^{n+1})$ satisfies

$$|\hat{v}(\lambda)| \le \exp(d\phi(-\mathrm{Re}\ \zeta) + \varepsilon|\mathrm{Im}\ \zeta| + \varepsilon\ \mathrm{Im}\ \tau^+ + b \ln(1+|\lambda|)) ,$$

then it follows that

$$|\hat{w}(\lambda)| \le \tilde{c} \exp(d'\phi(-\mathrm{Re}\ \zeta) + \varepsilon'|\mathrm{Im}\ \zeta| + \varepsilon'\ \mathrm{Im}\ \tau^+ + (b + b') \ln(1+|\lambda|)) , \tag{11}$$

$$|\hat{w}_j(\lambda)| \leq \tilde{c} \exp(d'\phi(-\mathrm{Re}\ \zeta) + \varepsilon'|\mathrm{Im}\ \zeta| + (b + b') \ln(1+|\lambda|)) . \tag{12}$$

Here only $\tilde{c}$ depends on b .

Proof. We only consider the case of (11). Once this is proved, (12) is an immediate consequence (cf. remark 5.2. Here we also use proposition 2.9.). Also in this proof we consider the cases $|\tau| \leq 3c(\phi(-\mathrm{Re}\ \zeta) + |\mathrm{Im}\ \zeta|)$ and $|\tau| \geq 3c(\phi(-\mathrm{Re}\ \zeta) + |\mathrm{Im}\ \zeta|)$. In the first case we can argue as in the proof of proposition 5.3, using proposition 2.9 instead of proposition 2.6. In the case $|\tau| \geq 3c(\phi(-\mathrm{Re}\ \zeta) + |\mathrm{Im}\ \zeta|)$ the argument is as in the proof of (5), i.e., we replace $\Lambda(\lambda)$ by the union of $\Lambda^1(\lambda)$ and $\Lambda^2(\lambda)$ and write $T_\lambda v = I_1 + I_2$ with I_k given by (10). We omit further details.

REMARK 5.5. In general, d' cannot be taken zero, even if $d = 0$.

5. We conclude this paragraph with a microlocalization of proposition 5.1. In doing so we shall assume that v satisfies an inequality of type

$$|\hat{v}(\lambda)| \leq c_1 \exp(\varepsilon|\mathrm{Im}\ \zeta| + b \ln(1+|\zeta|))$$
$$\text{for } |\tau| \leq C(\phi(-\mathrm{Re}\ \zeta) + |\mathrm{Im}\ \zeta|) , \tag{13}$$

where ε and C are the constants from proposition 5.1. The constant c_1 may here depend on v and the main point in the conclusion (cf. proposition 5.6 below) will be that the estimates which we will obtain, do not depend on c_1 . The reason why we need (13) at all is that we want to make sure from the very beginning that the w and w_j associated with v as in proposition 4.1 are distributions.

Let now further $\Gamma' \subset R^n$ be some given set. Our main assumptions on v are that

$$|\hat{v}(\lambda)| \leq \exp(\varepsilon|\mathrm{Im}\ \zeta| + b \ln(1+|\zeta|))$$
$$\text{(only) for } |\tau| \leq C(\phi(-\mathrm{Re}\ \zeta) + |\mathrm{Im}\ \zeta|) \text{ and } -\mathrm{Re}\ \zeta \in \Gamma'_{c_2\phi} , \tag{14}$$

respectively that

$$|\hat{v}(\lambda)| \leq \exp(\varepsilon|\mathrm{Im}\ \zeta| + \varepsilon\ \mathrm{Im}\ \tau^{+} + b\ \ln(1+|\lambda|)), \text{ otherwise.} \quad (15)$$

Here c_2 is some given constant.
For w and w_j we have then at least the estimates given by proposition 5.4. We want to improve them when $-\mathrm{Re}\ \zeta$ is in a set of type $\Gamma'_{c_3\phi}$. More precisely, we need:

PROPOSITION 5.6. Let ε', c_2 (and b) be given. Then we can find $\varepsilon > 0$, b', C, c_3 and c_4 with the following property: assume that v satisfies (14), (15) and (13) for some c_1 and let w, w_j, be associated with v as in (1). Then it follows that

$$|\hat{w}(\lambda)| \leq c_4 \exp(\varepsilon'|\mathrm{Im}\ \zeta| + \varepsilon'\ \mathrm{Im}\ \tau^{+} + (b+b')\ \ln(1+|\lambda|))$$
$$\text{if } -\mathrm{Re}\ \zeta \in \Gamma'_{c_3\phi} \quad (16)$$

and

$$|\hat{w}_j(\zeta)| \leq c_4 \exp(\varepsilon'|\mathrm{Im}\ \zeta| + (b+b')\ \ln(1+|\zeta|))$$
$$\text{if } -\mathrm{Re}\ \zeta \in \Gamma'_{c_3\phi} . \quad (17)$$

Proof of proposition 5.6. Once again only (16) is a problem. In fact, if we can prove (16) then we arrive at suitable estimates for $\mathcal{F}({}^{t}p(z,D)w)$ if we use proposition 2.9 and the fact that for $-\mathrm{Re}\ \zeta \notin \Gamma'_{c_3\phi}$ we already have the estimates given by proposition 5.4. (We may have to shrink c_3 when passing from (16) to (17).)

We are thus reduced to the estimation of $\hat{w}$, or, equivalently, to that of $T_\lambda v$ when $-\mathrm{Re}\ \zeta \in \Gamma'_{c_3}\phi$, for some small c_3. Here we can now argue exactly as in the proof of proposition 5.1 (re_lying on proposition 2.11 this time.) We omit further details.

§6. BACK TO RELATION (2), §4. THE CONSTRUCTION OF SOLUTIONS FOR THE CAUCHY PROBLEM STARTING FROM IT.

1. We want to extend here the validity of (2), §4, to larger classes of solutions of $p(z,D)u = f$. Thus let $p(z,D)$ and ϕ be as in §1 (including condition (4), §1) and assume that u is a germ of a C^∞ function which satisfies

$$p(z,D)u = f \tag{1}$$

$$D_t^j u|_{t=0} = f_j, \qquad j = 0,\dots,m-1 \tag{2}$$

in the sense of germs near 0, for some f, f_j. Choosing suitable C^∞ functions to represent u,f,f_j, we may also assume that (1) and (2) are equalities "in functions" for $|z| < 2\varepsilon'$, respectively $|x| < 2\varepsilon'$ for some $\varepsilon' > 0$.

<u>PROPOSITION 6.1.</u> For every ε' there are $\varepsilon'' > 0$, $C > 0$, with the following property: if $v \in \mathcal{E}'(z \in R^{n+1};\ |z| < \varepsilon'')$ satisfies

$$|\hat{v}(\lambda)| \le \exp(\varepsilon''|\mathrm{Im}\ \zeta| + b\ \ln(1+|\lambda|))$$
$$\text{for } |\tau| < C(\phi(-\mathrm{Re}\ \zeta) + |\mathrm{Im}\ \zeta|)$$

and if $w \in A'(C^{n+1})$, $w_j \in A'(C^n)$, are related to v by (1), §4, then we have in fact that $w \in \mathcal{E}'(z \in R^{n+1};\ |z| < \varepsilon')$, $w_j \in \mathcal{E}'(x \in R^n;\ |x| < \varepsilon')$ and

$$v(u) = w(f) + \sum_{j=0}^{m-1} w_j(-f_j)\ . \tag{3}$$

The first assertion is here a consequence of proposition 5.1 and the second follows from the first.

<u>REMARK 6.2.</u> The same conclusions remain valid if u satisfies (1) only for $t \ge 0$, provided that the support of v is concentrated in $t \ge 0$.

2. One remarkable thing about (3) is, that, if we know that a solution u of the problem (1), (2), exists, then we can evaluate $v(u)$ for the subclass of distributions which appears in proposition 6.1, before we know u. Moreover, we can even use this relation to prove the existence of a solution, if the situation is favorable. Let us in fact denote by

$$N = \{v \in E'(z \in R^{n+1};\ |z| < \varepsilon'');\ z \in \operatorname{supp} v \Rightarrow t \geq 0\ ,$$

$$\text{and } \exists c, \exists b \text{ such that } |\hat{v}(\lambda)| \leq c \exp(\varepsilon''|\operatorname{Im} \zeta| + b \ln(1+|\zeta|))$$

$$\text{if } |\tau| < C(\phi(-\operatorname{Re} \zeta) + |\operatorname{Im} \zeta|)\} \tag{4}$$

We endow N with the topology induced from $E'(z \in R^{n+1};\ |z| < \varepsilon'')$, which is the same than is the topology induced from $E'(z \in R^{n+1},\ |z| < \varepsilon'',\ t \geq 0\}$.

If $\varepsilon'' > 0$ is small enough and C is large enough, we can now, if $f_0, \ldots, f_{m-1}$, f, are given (germs of C^∞ functions), define a linear functional

$$L : N \to \mathbf{C} \text{ by } L(v) = w(f) + \sum_{j=0}^{m-1} w_j(-f_j)\ , \tag{5}$$

where v and w_j, w, are related by (1), §4.
(ε'' should here be small and for f, f_j we must choose suitable representatives).

<u>PROPOSITION 6.3.</u> Let $f_0, \ldots, f_{m-1}$, f be given and assume that the map $L : N \to C$ defined in (5) is continuous for suitable ε'', C and suitable representatives for the f_j, f. Then there is a solution u of (1), (2).

<u>Proof.</u> In view of the Hahn-Banach theorem we can find a C^∞ function u, defined for $|z| < \varepsilon''$ such that $v(u) = L(v)$ if $v \in N$. We must check that (1) and (2) are valid. (2) is trivial, since the distributions $w_j \otimes D_t^j \delta_t$ are in N for any C, so $(w_j \otimes D_t^j \delta_t)(u) = L(w_j \otimes D_t^j \delta_t) = w_j(-f_j)$. To check (1), we first

observe that $({}^t p(z,D)w)(u) = L({}^t p(z,D)w) = w(f)$, at least when ${}^t p(z,D)w \in N$. This gives $w(p(z,D)u - f) = 0$ for all w for which ${}^t p(z,D)w \in N$. In view of propositions 2.1 and 2.6 we can conclude that for some suitably small ε any $\tilde{w} \in E'(z \in R^{n+1}, |z| < \varepsilon, t \geq 0)$ can be approximated with elements w for which ${}^t p(z,D)w \in N$. This gives $p(z,D)u = f$ for $|z| < \varepsilon, t \geq 0$.

§7. PROOF OF THEOREM 1.2 AND OF PROPOSITION 1.20 IN THE CASE OF C^∞ FUNCTIONS.

1. Consider $f_0,\dots,f_{m-1} \in G_\phi$ and let f be a germ of a C^∞ function defined near $0 \in R^{n+1}$ such that $(0,R^n) \cap WF_\phi^b f = \emptyset$. We want to construct a C^∞ solution of the problem

$$p(z,D)u = f, \qquad t \geq 0, \tag{1}$$

$$D_t^j u|_{t=0} = f_j, \qquad j = 0,\dots,m-1. \tag{2}$$

Arguing in exactly the same way, we can also construct a C^∞ solution for the problem

$$p(z,D)u = 0, \qquad t \leq 0$$

$$D_t^j u|_{t=0} = f_j, \qquad j = 0,\dots,m-1,$$

and will have thus proved theorem 1.2 (in which $f = 0$) and proposition 1.20 in the case of C^∞ solutions simultaneously.

2. The existence of a solution u for (1) and (2) is an immediate consequence of proposition 6.3. The only preparation which we still need in the case when $f \not\equiv 0$ is

<u>PROPOSITION 7.1.</u> Consider $\Gamma \subset R^n$ and let f be a germ of a C^∞ function defined near $0 \in R^{n+1}$. Assume that

$(0,\Gamma) \cap WF_\phi^b f = \emptyset$. We can then find $d > 0$, $\varepsilon > 0$, c', and for every b some c such that $|v(f)| \le c$ for any $v \in \mathcal{E}'(R^{n+1})$ which satisfies

$$|\hat{v}(\lambda)| \le \exp(d\phi(-\mathrm{Re}\ \zeta) + \varepsilon|\mathrm{Im}\ \zeta| + \varepsilon\ \mathrm{Im}\ \tau^+ + b\ \ln(1+|\lambda|)) \quad \text{if } \mathrm{Re}\ \zeta \in -\Gamma_{c'\phi} ,$$

$$|\hat{v}(\lambda)| \le \exp(\varepsilon|\mathrm{Im}\ \zeta| + \varepsilon\ \mathrm{Im}\ \tau^+ + b\ \ln(1+|\lambda|)) \quad \text{if } \mathrm{Re}\ \zeta \notin -\Gamma_{c'\phi} .$$

3. We postpone the proof of this result until the end of this paragraph and return to the problem (1), (2). Let us then use again the notation N from (4), §6. If $v \in N$ and if $v = {}^t p(z,D)w + \sum w_j \otimes D_t^j \delta_t$, we can here estimate $\hat{w}$ and $\hat{w}_j$ using the propositions 5.1 and 5.4. If we fix ε', d', then we can choose here ε'' so small that, if C has been large, and if v remains in a bounded set from N , then

$$|\hat{w}_j(\zeta)| \le c\ \exp(d'\phi(-\mathrm{Re}\ \zeta) + \varepsilon'|\mathrm{Im}\ \zeta| + b\ \ln(1+|\zeta|)) \tag{3}$$

and

$$|\hat{w}(\lambda)| \le c\ \exp(d'\phi(-\mathrm{Re}\ \zeta) + \varepsilon'|\mathrm{Im}\ \zeta| + \varepsilon'\ \mathrm{Im}\ \tau^+ + b\ \ln(1+|\lambda|)) \tag{4}$$

for some constants c, b. Note that it is precisely on distributions of the type from (3), (4), that suitable representatives for the f_j, f remain bounded if d' and ε' are small. (Cf. the propositions 1.13 and 7.1.) This shows that (if ε'' is small enough), we can find for any bounded set $M \subset N$ some constant c_M with $|L(v)| \le c_M$ if $v \in M$ (C must be sufficiently large), thus giving the desired continuity.

4. We must still prove proposition 7.1. This is in fact a consequence of the following result:

PROPOSITION 7.2. Assume that $\Gamma \subset_\phi \Gamma' \subset R^n$ and let $\varepsilon, \varepsilon', d, d'$, $0 < \varepsilon < \varepsilon'$, $0 < d < d'$ be given. Then there are b', c', and for every $b_1 \geq 0$, $b_2 \in R$, some b_3 and c such that any $v \in E'(R^{n+1})$ which satisfies

$$|\hat{v}(\lambda)| \leq \exp(d\phi(-\mathrm{Re}\,\zeta) + \varepsilon|\mathrm{Im}\,\zeta| + \varepsilon\,\mathrm{Im}\,\tau^+ + b_1 \ln(1+|\lambda|)) \qquad \text{if } -\mathrm{Re}\,\zeta \in \Gamma_{c'\phi/3}, \tag{5}$$

$$|\hat{v}(\lambda)| \leq \exp(\varepsilon|\mathrm{Im}\,\zeta| + \varepsilon\,\mathrm{Im}\,\tau^+ + b_1 \ln(1+|\lambda|)), \qquad \text{if } -\mathrm{Re}\,\zeta \notin \Gamma_{c'\phi/3}, \tag{6}$$

can be decomposed in the form $v = v^1 + v^2$, with v^j satisfying

$$|\hat{v}^1(\lambda)| \leq c\exp(d'\phi(-\mathrm{Re}\,\zeta) + \varepsilon'|\mathrm{Im}\,\zeta| + \varepsilon'\,\mathrm{Im}\,\tau^+ + (b_1+b')\ln(1+|\lambda|) + b_2\ln(1+|\zeta|)), \quad \text{if } -\mathrm{Re}\,\zeta \in \Gamma',$$

$$|\hat{v}^1(\lambda)| \leq c\exp(\varepsilon'|\mathrm{Im}\,\zeta| + \varepsilon'\,\mathrm{Im}\,\tau^+ + (b_1+b')\ln(1+|\lambda|) + b_2\ln(1+|\zeta|)), \quad \text{if } -\mathrm{Re}\,\zeta \notin \Gamma',$$

$$|\hat{v}^2(\lambda)| \leq c\exp(\varepsilon'|\mathrm{Im}\,\zeta| + \varepsilon'\,\mathrm{Im}\,\tau^+ + b_3\ln(1+|\lambda|)).$$

Proof of proposition 7.2 (scetch). We fix c' with $\Gamma_{c'\phi} \subset_\phi \Gamma'$, and consider $e \in C_0^\infty(R^n)$ such that $e(\xi) = 1$ when $\mathrm{dist}(-\xi,\Gamma) \leq c'\phi(-\xi)/3$, $e(\xi) = 0$ when $\mathrm{dist}(-\xi,\Gamma) \geq 2c'\phi(-\xi)/3$, $0 \leq e(\xi) \leq 1$, $\forall \xi \in R^n$, and $|\mathrm{grad}_\xi\, e| \leq c_1$. If v satisfies (5) and if we choose some b_4 (to be specified later on), then $F_1 = e(\mathrm{Re}\,\zeta)\hat{v}$ will satisfy

$$|F_1(\lambda)| \leq c_2\exp(d'\phi(-\mathrm{Re}\,\zeta) + \varepsilon'|\mathrm{Im}\,\zeta| + \varepsilon'\,\mathrm{Im}\,\tau^+ + b_1(1+|\lambda|) + b_4\ln(1+|\zeta|)) \qquad \text{for } \lambda \in \mathrm{supp}\, F_1.$$

This is in fact a consequence of our assumption "$\phi(\xi) \geq c_3|\xi|^\delta$ for some $\delta > 0$" on ϕ. Furthermore, $\bar{\partial}F_1$ is concentrated on

$\{\lambda;\ c'\phi(-\mathrm{Re}\ \zeta)/3 \le \mathrm{dist}(-\mathrm{Re}\ \zeta,\Gamma) \le 2c'\phi(-\mathrm{Re}\ \zeta)/3\}$, so we have

$$|\bar\partial F_1(\lambda)| \le c_4 \exp(\varepsilon|\mathrm{Im}\ \zeta| + \varepsilon\ \mathrm{Im}\ \tau^+ + b_1 \ln(1+|\lambda|)) .$$

If we could find $H \in C^\infty(C^{n+1})$ such that $\bar\partial H = \bar\partial F_1$ and such that

$$|H(\lambda)| \le c_5 \exp(\varepsilon'|\mathrm{Im}\ \zeta| + \varepsilon'\ \mathrm{Im}\ \tau^+ + (b_1+b') \ln(1+|\lambda|)) , \quad \text{if } -\mathrm{Re}\ \zeta \in \Gamma' ,$$

$$|H(\lambda)| \le c_5 \exp(\varepsilon'|\mathrm{Im}\ \zeta| + \varepsilon'\ \mathrm{Im}\ \tau^+ + (b_1+b') \ln(1+|\lambda|) + b_2 \ln(1+|\zeta|)) , \quad \text{if } -\mathrm{Re}\ \zeta \notin \Gamma' ,$$

then the proof would come to an end by setting, $\hat v_1 = F_1 - H$, $v_2 = H + \hat v(1-e(\mathrm{Re}\zeta))$.The existence of such an H is however a consequence of results of Hörmander[2], if we also use the following lemma and start with b_4 much smaller than b_2 (cf. the proof of proposition 2.13 for a similar situation)

<u>LEMMA 7.3.</u> Let $b_4 \ge 0$, $b_5 \le 0$, $A > 0$, and $\Gamma_1 \subset_\phi \Gamma_2 \subset R^n$ be given. Then we can find, b_6, c_1, and a plurisubharmonic function $\rho : C^n \to R$ such that

$$\rho(\zeta) \le b_4 \ln(1+|\zeta|) + A|\mathrm{Im}\ \zeta| + c_1 , \tag{7}$$

$$b_4 \ln(1+|\mathrm{Re}\ \zeta|) - b_6 \ln(1+|\mathrm{Im}\ \zeta|) \le \rho(\zeta) , \text{ if } -\mathrm{Re}\ \zeta \in \Gamma_1 , \tag{8}$$

$$\rho(\zeta) \le b_5 \ln(1+|\zeta|) + A|\mathrm{Im}\ \zeta| + c_1 , \text{ if } -\mathrm{Re}\ \zeta \notin \Gamma_1 . \tag{9}$$

<u>REMARK 7.4.</u> Applying this with A replaced by $A/2$ and adding $A|\mathrm{Im}\ \zeta|/2$ to the resulting ρ , we can replace (8) by

$$b_4 \ln(1+|\zeta|) \le \rho(\zeta) + c_2 \quad \text{if } -\mathrm{Re}\ \zeta \in \Gamma_1 .$$

<u>Proof of lemma 7.3.</u> Let us note at first that $\theta \in \Gamma_1$, $\eta \notin \Gamma_2$ implies that

$$|\theta-\eta| \geq c_3(\phi(\theta) + \phi(\eta)) \geq c_4(|\theta|^\delta + |\eta|^\delta)$$

for some positive δ , for which we may assume $\delta < 1/2$, and c_4 . This is a consequence of lemma 2.7 and of the assumption $\phi(\xi) \geq c_5|\xi|^\delta$. This shows that

$$\ln(1+|\theta-\eta|) \geq (\delta/2)[\ln(1+|\theta|) + \ln(1+|\eta|)] - c_6$$

for such θ, η .

The next thing is to define $A' > 0$ by

$$A = A'(-b_5 + b_4)(2/\delta)$$

and to choose some plurisubharmonic function $\psi : C^n \to R$ such that

$$\psi(\zeta) \leq c_7 , \quad \text{if } |\zeta| \leq 1 , \tag{10}$$

$$\psi(\zeta) \leq A'|\mathrm{Im}\,\zeta| - \ln(1+|\zeta|) + c_7 , \quad \text{if } |\zeta| \geq 1 , \tag{11}$$

$$\psi(i\xi) \geq c_8 - n\ln(1+|\xi|) , \quad \text{if } \xi \in R^n . \tag{12}$$

(When $n = 1$ we may take for ψ the function $f(\zeta) = \ln|[1 - \exp(-iA'\zeta)] / \zeta|$ and for $n > 1$, $\psi(\zeta) = f(\zeta_1/n) + f(\zeta_2/n) + \dots + f(\zeta_n/n)$ will do.)

It remains to set for some suitable c_9:

$$\rho(\zeta) = \sup_{\theta\in\Gamma_1} [b_4 \ln(1+|\theta|) + (-b_5 + b_4)(2/\delta)\psi(\zeta+\theta)] + c_9 .$$

We omit further details. (Since practically all of this paper deals with C^∞ solutions, we could have taken the conclusion from proposition 7.1 as a definition for $(0,\Gamma) \cap WF_\phi^b f = \emptyset$, when f is C^∞ .)

§8. PROOF OF THEOREM 1.12 .

1. In view of proposition 6.3, theorem 1.12 is a consequence of the following result:

PROPOSITION 8.1. Assume that $f_o, \ldots, f_{m-1}$, are as in theorem 1.12. Then there are $C > 0$, $\varepsilon > 0$, and for every $b \geq 0$ some $c > 0$ with the following property:

if $v, w \in E'(z \in R^{n+1};\ |z| < \varepsilon)$, $w_j \in E'(x \in R^n;\ |x| < \varepsilon)$ are given such that

$$v = {}^t p(z,D)w + \sum_{j=0}^{m-1} w_j \otimes D_t^j \delta_t , \tag{1}$$

$$|\hat{v}(\lambda)| \leq \exp(\varepsilon|\mathrm{Im}\ \zeta| + \varepsilon\ \mathrm{Im}\ \tau^+ + b\ \ln(1+|\lambda|)) , \tag{2}$$

respectively

$$|\hat{v}(\lambda)| \leq c'\ \exp(\varepsilon|\mathrm{Im}\ \zeta| + b\ \ln(1+|\lambda|)) ,$$
$$\text{if } |\tau| \leq C(\phi(-\mathrm{Re}\ \zeta) + |\mathrm{Im}\ \zeta|) , \tag{3}$$

for some constant c' , which may depend on v , then

$$|\sum w_j(f_j)| \leq c . \tag{4}$$

2. We start the proof of proposition 8.1 with a result of a similar type, but which is more elementary:

PROPOSITION 8.2. Let $\Gamma \subset R^n$ be given and consider $f_o, \ldots, f_{m-1} \in C^\infty(|x| < \varepsilon)$ for some $\varepsilon > 0$. Assume that we can find $\tilde{\varepsilon} > 0$ and $U \in C^\infty(|z| < \tilde{\varepsilon})$ such that

$$p(z,D)U = 0 , \qquad \text{for } t \geq 0 , \tag{5}$$

$$(0,\Gamma) \cap WF_\phi(D_t^j U|_{t=0} - f_j) = 0 . \tag{6}$$

Then we can find $c_1 > 0$, $C_1 > 0$, $\varepsilon_1 > 0$, and for every $b \geq 0$

some c with the following property:

if v,w,w_j satisfy (1), (2) and also

$$|\hat{v}(\lambda)| \le \exp(\varepsilon_1|\operatorname{Im}\zeta| + b\ln(1+|\zeta|))$$

$$\text{if } -\operatorname{Re}\zeta \notin \Gamma_{c_1\phi} \text{ and } |\tau| \le C_1(\phi(-\operatorname{Re}\zeta) + |\operatorname{Im}\zeta|)\ , \tag{7}$$

respectively,

$$|\hat{v}(\lambda)| \le C'\exp(\varepsilon_1|\operatorname{Im}\zeta| + b\ln(1+|\zeta|))$$

$$\text{if } |\tau| \le C_1(\phi(-\operatorname{Re}\zeta) + |\operatorname{Im}\zeta|) \tag{8}$$

for some C' which may depend on v, then it follows that $|\sum w_j(f_j)| \le c$.

The important point is here of course that c does not depend on C'.

<u>REMARK 8.3.</u> If the conclusions of proposition 8.2 are valid for some $c_1 > 0$, $C_1 > 0$, $\varepsilon_1 > 0$, then they also remain valid if we shrink c_1, ε_1 and enlarge C_1. If follows that if the assumptions from the proposition are satisfied for a finite collection of cones Γ^k, $k = 1,\dots,s$, then we can find c_1, C_1, ε_1, such that the conclusions are valid if we replace Γ by Γ^k, whatever k is.

3. <u>Proof of proposition 8.2.</u> Denote $f_j' = f_j - D_t^j U|_{t=0}$. We can then find c_2, ε', and for every b some c_3 with $|g(f_j')| \le c_3$, $j = 0,\dots,m-1$, for all $g \in E'(R^n)$ which satisfy

$$|\hat{g}(\zeta)| \le \exp(\phi(-\operatorname{Re}\zeta) + \varepsilon'|\operatorname{Im}\zeta| + b\ln(1+|\zeta|))$$

$$\text{if } -\operatorname{Re}\zeta \in \Gamma_{c_2\phi}$$

respectively

$$|\hat{g}(\zeta)| \le \exp(\varepsilon'|\operatorname{Im}\zeta| + b\ln(1+|\zeta|)) \text{ if } -\operatorname{Re}\zeta \notin \Gamma_{c_2\phi}\ .$$

If now the v, w_j, w, are as in the assumption then we can conclude from the propositions 5.4 and 5.6 that the w_j are of the type of the g from before, provided that c_1 and ε_1 is small and that C_1 is large. On the orther hand, we have $\sum w_j(-f_j) = \sum w_j(-f'_j) - v(U)$, if we also use proposition 6.1. This gives the desired conclusion since v is in a bounded set of distributions which are concentrated in $t \geq 0$ and have small support there.

4. Proof of proposition 8.1. Let $f_o,\ldots,f_{m-1}$, and Γ^k , $k = 1,\ldots,s$, be as in the assumption of theorem 1.12. Let also c_1, C_1, ε_1 , be positive constants such that the conclusions from proposition 8.2 hold for c_1, C_1, ε_1 , if we replace Γ by Γ^k , $k = 1,\ldots,s$ (cf. remark 8.3). Also choose $C_2 \geq C_1$, ε_2, $0 < \varepsilon_2 \leq \varepsilon_1$. C_2 and ε_2 will be fixed during the proof, but they have to satisfy some restrictions which we shall introduce only when we need them effectively.

Let now v be given with (2) and (3). If $\varepsilon > 0$, $C > 0$, b_1 and c_3 are suitable, we can then split v (using proposition 2.14) into the form

$$v = \sum v^k ,$$

$$|\hat{v}^k(\lambda)| \leq c_3 \exp(\varepsilon_2|\operatorname{Im} \zeta| + (b+b_1) \ln(1+|\lambda|))$$

$$\text{if} \quad \operatorname{Re}\zeta \notin - \Gamma_{c_1\phi} , \quad |\tau| < C_2(\phi(-\operatorname{Re} \zeta) + |\operatorname{Im} \zeta|) , \tag{9}$$

respectively,

$$|\hat{v}^k(\lambda)| \leq c_3 \exp(\varepsilon_2|\operatorname{Im} \zeta| + \varepsilon_2 \operatorname{Im} \tau^+ + (b+b_1) \ln(1+|\lambda|)) , \tag{10}$$

otherwise.

Let also w_j^k, w^k be such that $v^k = {}^tp(z,D)w^k + \sum w_j^k \otimes D_t^j \delta_t$. Of course we have $\sum\limits_k w_j^k = w_j$. The problem is now that the v^k do not necessarily satisfy (8). In fact, otherwise we could apply proposition 8.2 to conclude that $|\sum w_j^k (f_j)| \leq c_4$ and were done. The idea to overcome this difficulty is here to approximate the v^k by some sequence of distributions v^{ki} , $i = 1,2,\ldots,$ which also satisfy (8). We pause for a moment in the proof of proposition 8.1 to prove:

4. PROPOSITION 8.4. Let ε_1, C_1 be given. Then we can find $\varepsilon_2, C_2, b_2, c_5$, with the following property: fix k and assume that v^k satisfies (9) and (10). Then we can find a sequence of distributions $\{v^{ki}\}_{i=1}^{\infty}$ such that

$$|\hat{v}^k(\lambda) - \hat{v}^{ki}| \le (c_5/i)\exp(\varepsilon_1|\mathrm{Im}\,\zeta| + \varepsilon_1\,\mathrm{Im}\,\tau^+ + (b+b_2)\ln(1+|\lambda|)), \quad \forall\lambda \in C^{n+1}, \tag{11}$$

$$|\hat{v}^k(\lambda) - \hat{v}^{ki}(\lambda)| \le (c_5/i)\exp(\varepsilon_1|\mathrm{Im}\,\zeta| + (b+b_2)\ln(1+|\zeta|))$$
$$\text{if } |\tau| \le C_1(\phi(-\mathrm{Re}\,\zeta) + |\mathrm{Im}\,\zeta|) \text{ and } -\mathrm{Re}\,\zeta \notin -\Gamma^k_{c_1\phi}, \tag{12}$$

$$|\hat{v}^{ki}(\lambda)| \le C(i)\exp(\varepsilon_1|\mathrm{Im}\,\zeta| + (b+b_2)\ln(1+|\zeta|))$$
$$\text{if } |\tau| \le C_1(\phi(-\mathrm{Re}\,\zeta) + |\mathrm{Im}\,\zeta|), \tag{13}$$

for some constants $C(i)$ (which, of course, will tend to infinity).

Note in particular, that

$$|\hat{v}^{ki}(\lambda)| \le c''\exp(\varepsilon_1|\mathrm{Im}\,\zeta| + \varepsilon_1\,\mathrm{Im}\,\tau^+ + (b+b_2)\ln(1+|\lambda|)), \quad \forall\lambda \in C^n, \tag{14}$$

and

$$|\hat{v}^{ki}(\lambda)| \le c''\exp(\varepsilon_1|\mathrm{Im}\,\zeta| + (b+b_2)\ln(1+|\tau|))$$
$$\text{if } |\tau| \le C_1(\phi(-\mathrm{Re}\,\zeta) + |\mathrm{Im}\,\zeta|) \text{ and } \mathrm{Re}\,\zeta \notin -\Gamma^k_{c_1\phi}, \tag{15}$$

as a consequence of (9), (10), (11) and (12).

5. Proof of proposition 8.4. If $\varepsilon_2 > 0$ is small enough, we can find (cf. lemma 2.2) a plurisubharmonic function $\psi : C^{n+1} \to R$ such that

$$\varepsilon_2\,\mathrm{Im}\,\tau^+ - \varepsilon_2|\zeta| \le \psi(\lambda),$$

$$\psi(\lambda) \le \varepsilon_1(1 + |\mathrm{Im}\ \zeta| + \mathrm{Im}\ \tau^+)/2 ,$$

$$\psi(\lambda) \le \varepsilon_1(1+|\mathrm{Im}\ \zeta|)/2 \quad \text{if} \quad |\tau| \le 2C_1(1+|\zeta|) .$$

(We assume tacitely later on that $|\tau| \le C_1(\phi(-\mathrm{Re}\ \zeta) + |\mathrm{Im}\ \zeta|)$ implies $|\tau| \le 2C_1(1+|\zeta|)$.)

We further shrink ε_2 until $\varepsilon_2 \le \varepsilon_1/8$.

Let now $g_i \in C^\infty(C^n)$, $i = 1,2,\ldots$, be a sequence of functions such that:

$$|g_i(\zeta)| \le C'(i)(1+|\zeta|) \exp(-\varepsilon_2|\zeta| + 3\varepsilon_2|\mathrm{Im}\ \zeta|)$$

for some constants $C'(i)$,

$$|\bar\partial g_i(\zeta)| \le (1/i)(1+|\zeta|) \exp(-\varepsilon_2|\zeta| + 3\varepsilon_2|\mathrm{Im}\ \zeta|) ,$$

$$|1 - g_i(\zeta)| \le (1/i)(1+|\zeta|) .$$

Functions g_i with such properties are implicit in Hörmander [4]. An explicit construction is given in Liess [4].

We now define h^{ki} by $h^{ki}(\lambda) = g_i(\zeta)\hat{v}^k(\lambda)$. The h^{ki} are our first step towards constructing $\hat{v}^{ki}$, but they cannot be of form $\hat{v}^{ki}$ by themselves, since they are not analytic on the whole of C^{n+1} . We shall therefore add suitable "small" corrections to the h^{ki} , to make them entire. The functions which we obtain in this way are then the searched - for $\hat{v}^{ki}$. We start by computing the "defect" from analyticity. In fact, in view of $(\partial/\partial\bar\tau)h^{ki} = 0$ and of $(\partial/\partial\bar\zeta)h^{ki} = \hat{v}^k(\partial/\partial\bar\zeta)g_i$, we have

$$|\bar\partial h^{ki}(\lambda)| \le (c_3/i) \exp(\varepsilon_2|\mathrm{Im}\ \zeta| + \varepsilon_2\ \mathrm{Im}\ \tau^+ - \varepsilon_2|\zeta| + 3\varepsilon_2|\mathrm{Im}\ \zeta| + (b+b_1+1)\ \ln(1+|\lambda|))$$

$$\le (c_3/i) \exp(\psi(\lambda) + (\varepsilon_1/2)|\mathrm{Im}\ \zeta| + (b+b_1+1)\ \ln(1+|\lambda|)) .$$

Arguing as in the proof of proposition 2.13 it follows now from the results in chapter IV from Hörmander [2] that we can find f^{ki} in

$C^\infty(\mathbb{C}^{n+1})$ such that $\bar\partial f^{ki} = \bar\partial h^{ki}$ and such that

$$|f^{ki}(\lambda)| \le (c_6/i) \exp(\varepsilon_1|\mathrm{Im}\ \zeta| + \varepsilon_1 \mathrm{Im}\ \tau^+ + (b+b_2) \ln(1+|\lambda|)) , \quad \forall\lambda \in \mathbb{C}^{n+1} ,$$

$$|f^{ki}(\lambda)| \le (c_6/i) \exp(\varepsilon_1|\mathrm{Im}\ \zeta| + (b+b_2) \ln(1+|\lambda|)) ,$$
$$\text{if } |\tau| \le 2C_1(1+|\zeta|) .$$

It remains now to set $\hat v^{ki} = h^{ki} - f^{ki}$. We omit further details.

<u>REMARK 8.5.</u> It follows from the proof that

$$|\hat v(\lambda) - \sum_k \hat v^{ki}(\lambda)| \le (c_7/i) \exp(\varepsilon_1|\mathrm{Im}\ \zeta| + (b+b_2) \ln(1+|\zeta|))$$
$$\text{if } |\tau| \le C(\phi(-\mathrm{Re}\ \zeta) + |\mathrm{Im}\ \zeta|) . \qquad (16)$$

In fact, $\hat v(\lambda) - \sum_k \hat v^{ki} = (1-g_i(\zeta))\ \hat v(\lambda) + \sum_k f^{ki}$, etc.
(c_7 depends here on v .)

6. We have now proved proposition 8.4 and return to the proof of proposition 8.1 . We decompose v^{ki} for each k and i into the form

$$v^{ki} = {}^t p(z,D)\ w^{ki} + \sum_{j=0}^{m-1} w_j^{ki} \otimes D_t^j\, \delta_t .$$

We obtain from (14), (15) and the choice of ε_1, c_1, C_1 , that $\sum_{k,j} w_j^{ki}(f_j)$ is uniformly bounded in i . Proposition 8.1 will therefore be proved, if we can show that

$$\sum w_j(f_j) = \lim_{i\to\infty} \sum_k (\sum_j w_j^{ki}(f_j)) . \qquad (17)$$

To see this, we note at first that

$$|\hat v(\lambda) - \sum_k \hat v^{ki}(\lambda)| \le (c_8/i) \exp(\varepsilon_1|\mathrm{Im}\ \zeta| + \varepsilon_1 \mathrm{Im}\ \tau^+ + (b+b_2) \ln(1+|\lambda|)) , \quad \forall\lambda \in \mathbb{C}^{n+1} .$$

Combining this with (16), we conclude from proposition 5.1 that $w_j - \sum_k w_j^{ki} \to 0$ in $E'(x \in R^n;\ |x| < \varepsilon')$ if ε_1 was small compared to ε'. If ε' is here small enough, this gives (17).

§9. PROOF OF THEOREM 1.18 IN THE CASE OF C^∞ SOLUTIONS

1. Let u be a C^∞ function defined for $|z| < \varepsilon$ such that $p(z,D)u = 0$ if $|z| < \varepsilon$, $t \geq 0$, and such that $(0,\Gamma) \cap WF_\phi D_t^j u_{|t=0} = \emptyset$ for $j = 0,\dots,m-1$. It follows from the proof of proposition 8.2 that we can find d,ε,C,c, and for every b some c' with the following property:

if $v \in E'(z \in R^{n+1};\ |z| < \varepsilon,\ t \geq 0)$ satisfies

$$|\hat{v}(\lambda)| \leq \exp(d\phi(-\text{Re}\ \zeta) + \varepsilon|\text{Im}\ \zeta| + \varepsilon\ \text{Im}\ \tau^+ + b\ \ln(1+|\lambda|)) \quad \text{for } -\text{Re}\ \zeta \in \Gamma_{c\phi}, \tag{1}$$

$$|\hat{v}(\lambda)| \leq \exp(\varepsilon|\text{Im}\ \zeta| + \varepsilon\ \text{Im}\ \tau^+ + b\ \ln(1+|\lambda|)) \quad \text{for } -\text{Re}\ \zeta \notin \Gamma_{c\phi}, \tag{2}$$

and

$$\sup_{\lambda \in V} |\hat{v}(\lambda)| / \exp(\varepsilon|\text{Im}\ \zeta| + b\ \ln(1+|\zeta|)) < \infty,$$

$$V = \{\lambda \in C^{n+1};\ |\tau| < C(\phi(-\text{Re}\ \zeta) + |\text{Im}\ \zeta|)\}, \tag{3}$$

then $|v(u)| \leq c'$.

If it were not for condition (3), this were precisely what we need for $(0,\Gamma) \cap WF_\phi^b u = \emptyset$ (Cf. proposition 7.1). Theorem 1.18 is then for C^∞ solutions a consequence of the following result (after a renotation):

<u>PROPOSITION 9.1.</u> Let ε' and C be given. Then we can find b', d and ε with the following property: for every b there is c_1 such that whenever $v \in E'(R^{n+1})$ is given with (1), (2), we can also find a sequence $v^i \in E'(z \in R^{n+1};\ |x| < \varepsilon',\ t \geq 0)$ such that

$$|\hat{v}(\lambda) - \hat{v}^i(\lambda)| \le (c_1/i)\exp(d\phi(-\mathrm{Re}\,\zeta) + \varepsilon'|\mathrm{Im}\,\zeta| + \varepsilon'\,\mathrm{Im}\,\tau^+ + (b+b')\ln(1+|\lambda|)), \quad \text{for } -\mathrm{Re}\,\zeta \in \Gamma_{c\phi}, \tag{4}$$

$$|\hat{v}(\lambda) - \hat{v}^i(\lambda)| \le (c_1/i)\exp(\varepsilon'|\mathrm{Im}\,\zeta| + \varepsilon'\,\mathrm{Im}\,\tau^+ + (b+b')\ln(1+|\lambda|)), \quad \text{for } -\mathrm{Re}\,\zeta \notin \Gamma_{c\phi}, \tag{5}$$

$$|\hat{v}^i(\lambda)| \le C(i)\exp(\varepsilon'|\mathrm{Im}\,\zeta| + (b+b')\ln(1+|\zeta|)) \quad \text{if } |\tau| \le C(\phi(-\mathrm{Re}\,\zeta) + |\mathrm{Im}\,\zeta|). \tag{6}$$

$$v^i \to v \quad \text{in} \quad E'(z \in R^{n+1};\ |x| < \varepsilon',\ t \ge 0). \tag{7}$$

Proof of proposition 9.1. The proof is parallel to that of proposition 8.4. For small ε we can find a plurisubharmonic function $\psi : C^{n+1} \to R$ such that

$$\varepsilon\,\mathrm{Im}\,\tau^+ - \varepsilon|\zeta|/2 \le \psi(\lambda),$$

$$\psi(\lambda) \le \varepsilon'(1+|\mathrm{Im}\,\zeta| + \mathrm{Im}\,\tau^+)/2,$$

$$\psi(\lambda) \le \varepsilon'(1+|\mathrm{Im}\,\zeta|)/2 \qquad \text{if } |\tau| \le C(\phi(-\mathrm{Re}\,\zeta) + |\mathrm{Im}\,\zeta|)$$

and shrink ε until $\varepsilon \le \varepsilon'/8$. We may of course also assume that $d\phi(-\mathrm{Re}\,\zeta) \le \varepsilon|\mathrm{Re}\,\zeta|/2+1$. Let further $g_i \in C^\infty(C^n)$, $i = 1,2,\ldots$ be a sequence as in the proof of proposition 8.4, with ε_2 replaced by ε. We first set $h^i(\lambda) = \hat{v}(\lambda)\, g_i(\zeta)$ and conclude that

$$|\bar{\partial} h^i(\lambda)| \le (1/i)\exp(4\varepsilon|\mathrm{Im}\,\zeta| + \psi(\lambda) + (b+1)\ln(1+|\lambda|)).$$

We can therefore find $f^i \in C^\infty(C^{n+1})$ with $\bar{\partial} f^i = \bar{\partial} h^i$ and such that

$$|f^i(\lambda)| \le (c_2/i)\exp(\varepsilon'|\mathrm{Im}\,\zeta| + (b+b')\ln(1+|\zeta|)) \quad \text{if } |\tau| \le C(\phi(-\mathrm{Re}\,\zeta) + |\mathrm{Im}\,\zeta|),$$

respectively

$$|f^i(\lambda)| \le (c_2/i)\exp(\varepsilon'|\mathrm{Im}\,\zeta| + \varepsilon'\,\mathrm{Im}\,\tau^+$$
$$+ (b+b')\ln(1+|\lambda|)) \quad \text{for all other } \lambda \ .$$

The searched-for $\hat{v}^i$'s are then $h^i - f^i$. We omit further details.

§10. THE CASE OF DISTRIBUTION SOLUTIONS.

1. Until now we have mainly studied the case of C^∞ solutions of the equation $p(z,D)u = 0$. The reason why this is easier is that $b\ln(1+|\lambda|^2)$ is plurisubharmonic for positive b, while for negative b it is not. In fact, when arguing with $\bar\partial$-estimates, a term of form $b\ln(1+|\lambda|)$ (in the exponent), will be replaced by a term of form $(b+b')\ln(1+|\lambda|)$ if b is positive and a "loss" of estimates of type $b'\ln(1+|\lambda|)$ will not produce any trouble when u is C^∞. On the other hand, if $u \in \mathcal{D}'$, then $u(v)$ will only make sense for those $v \in \mathcal{E}'(R^{n+1})$ which satisfy an estimate of form

$$|\hat{v}(\lambda)| \le c\exp(\delta|\mathrm{Im}\,\lambda| + b\ln(1+|\lambda|))$$

where $b \le b^0$ for some $b^0 \in R$. b^0 is here related to the order near 0 of the distribution u, and it will, in general, be much smaller than 0.

2. In most of the results from this paper it is not difficult to obtain as much control of the b's as is needed later on. The reason is that, although $-\ln(1+|\lambda|^2)$ is not plurisubharmonic, it is not far from a plurisubharmonic function either. One can in fact prove, e.g., the following result (cf. Liess [5]) :

<u>LEMMA 10.1.</u> Let $A > 0$ be given. Then there is $c > 0$ and a plurisubharmonic function $\psi : C^{n+1} \to R$ such that

$$-2(n+1)\ \ln(1+|\lambda|) \le \psi(\lambda)$$
$$\le -\ln(1+|\lambda|) + A|\text{Im}\ \zeta| + A\ \text{Im}\ \tau^+ + c\ , \qquad \forall\lambda \in C^{n+1}\ .$$

Using this result as a $\bar{\partial}$-cohomological tool, one can extend e.g. proposition 2.4 to the case of negative b in the following fashion:

PROPOSITION 10.2. Consider $\varepsilon' > 0$, $d' > 0$, $c' > 0$, and $\Gamma \subset R^n$. We can then find $\varepsilon > 0$, $d > 0$, and for every $b_1 \in R$ some b_2 and c , with the following property:

if u satisfies the inequalities from the statement of proposition 2.4, with b replaced by b_2 , then we can find a sequence $v_1, v_2, \ldots,$ in $E'(R^{n+1})$ and a sequence $\xi^1, \xi^2, \ldots,$ in Γ, which satisfies a) from the statement of proposition 2.4 and such that the inequalities from b) in that statement are valid if we replace $(b+b')$ by b_1 .

2. On the other hand, it is not immediate that one can extend results like proposition 2.14 in the way described just before to the case of negative b . In fact application of lemma 10.1 would reintroduce a factor $\exp(A\ \text{Im}\ \tau^+)$ in the estimates and would ruin the whole construction. One may here try to refine lemma 10.1, but there is no need for doing so, if we use that we only look for solutions $u \in F$ of $p(z,D)u = 0$. Thus for example, the following result is good enough to replace proposition 2.1 4:

PROPOSITION 10.3. Consider $\varepsilon' > 0$, $C > 0$, $c' > 0$ and let $\Gamma^1, \ldots, \Gamma^s$, in R^n , be such that $\cup\Gamma^k = R^n$. Then: $\exists\varepsilon > 0$, $\exists b' \ge 0$, $\forall b_1 \in R$, $\exists b_2 \in R$, $\forall b_3 \ge 0$, $\exists c > 0$ with the following property: any $v \in E'(R^{n+1})$ which satisfies

$$|\hat{v}(\lambda)| \le \exp(\varepsilon|\text{Im}\ \zeta| + \varepsilon\ \text{Im}\ \tau^+ + b_3\ \ln(1+|\lambda| + b_2\ \ln(1+|\zeta|))$$

can be decomposed into the form $v = v^1 + \ldots + v^s$, where $v^k \in E'(R^{n+1})$ satisfies

$$|v^k(\lambda)| \le c \exp(\varepsilon'|\mathrm{Im}\,\zeta| + \varepsilon'\mathrm{Im}\,\tau^+ + (b_3+b')\ln(1+|\lambda|) + b_1 \ln(1+|\zeta|)) , \qquad \forall\lambda \in C^{n+1} ,$$

$$|\hat{v}^k(\lambda)| \le c \exp(\varepsilon'|\mathrm{Im}\,\zeta| + (b_3+b')\ln(1+|\lambda|)) + b_1 \ln(1+|\zeta|))$$
$$\text{if } -\mathrm{Re}\,\zeta \notin \Gamma^k_{c',\phi} \text{ and } |\tau| \le C(\phi(-\mathrm{Re}\,\zeta) + |\mathrm{Im}\,\zeta|) .$$

This result can be proved with the arguments form the proof of proposition 2.14, if we also use the n-dimensional variant of lemma 10.1 (i.e. in lemma 10.1, ψ is now defined on C^n, etc.) . Using this result, we can for example show that the map L introduced in (5), §6, is well-defined on the set N' of all distributions v in $E'(z \in R^{n+1};\ |z| < \varepsilon,\ t \ge 0)$ which satisfy estimates of the form

$$|\hat{v}(\lambda)| \le c \exp(\varepsilon|\mathrm{Im}\,\zeta| + \varepsilon\,\mathrm{Im}\,\tau^+ + b_1 \ln(1+|\lambda|) + b_2 \ln(1+|\zeta|)) , \tag{1}$$

$$|\hat{v}(\lambda)| \le c \exp(\varepsilon|\mathrm{Im}\,\zeta| + b_1 \ln(1+|\lambda|) + b_2 \ln(1+|\zeta|)) ,$$
$$\text{if } |\tau| < C(\phi(-\mathrm{Re}\,\zeta) + |\mathrm{Im}\,\zeta|) \tag{2}$$

for suitable fixed b_1, b_2, and for some c , which may depend on v .

Of course this is of any value only if there are enough distributions in the set N' in order to approximate. The following result gives all we need:

<u>PROPOSITION 10.3.</u> Let $C > 0$ and ε be given. Then: $\exists b'$, $\exists \varepsilon'' > 0$, $\forall b_2$, $\exists b_3$, $\forall b_1$, $\exists c$ such that if

$$|\hat{v}(\lambda)| \le \exp(\varepsilon''|\mathrm{Im}\,\zeta| + \varepsilon''\,\mathrm{Im}\,\tau^+ + b_1 \ln(1+|\lambda|) + b_3 \ln(1+|\zeta|)) ,$$

then we can find a sequence $v^i \in E'(R^{n+1})$ such that the v^i satisfy (1) and (2) with b_1 replaced by $b_1 + b'$ and such that

$$|(\hat{v} - \hat{v}^i)(\lambda)| \le (c/i)\ \exp(\varepsilon|\mathrm{Im}\,\zeta| + \varepsilon\,\mathrm{Im}\,\tau^+ + (b_1 + b')\ln(1 + |\lambda|) + b_2 \ln(1 + |\zeta|)).$$

Proposition 10.3 is a variant of proposition 2.1. It does not seem however adequate to prove it with the methods from Ehrenpreis [1]. What we can do instead, is to use the arguments from the proof of proposition 8.4 and of lemma 10.1. We omit further details.

§ 11. COMPARISION OF SOME ARGUMENTS FROM THIS PAPER WITH HOLMGREN'S METHOD.

1. In this paragraph we discuss the relation of the methods from this paper to the classical method of deformation of noncharacteristic surfaces used in the proof of Holmgren's uniqueness theorem. ("Holmgren's method"). One hint that there might be such a relation is that theorem 1.18 is in fact a result on microlocal uniqueness in the noncharacteristic Cauchy problem. (Actually, results related to theorem 1.18 are sometimes called "micro-Holmgren" theorems.) Moreover, the distributions

$$g \to \int_{\substack{x \in R^n \\ |x| \le 1}} f(x)\ g(x, 1-x^2)\ dx\ , \qquad g \in C_o^\infty(R^{n+1})\ ,$$

of which (6), §2, is nothing but the Fourier-Borel transform, were introduced by L. Ehrenpreis precisely to reprove Holmgren's uniqueness theorem. In our discussion, we shall then also arrive at a class of relevant examples of distributions which are as in proposition 2.1, thus making proposition 6.3 more explicit. Since no precision would be gained by working with some specific weight function ϕ, we shall always assume in this paragraph that $\phi = 1 + |\xi|$. In particular, the present discussion refers to any noncharacteristic Cauchy problem, provided the operator under consideration has analytic coefficients in some complex neighborhood Z of $0 \in C^{n+1}$. All notations (i.e., $p(z,D)$, C, ε'', N, etc.) are here used as in § 6.

2. In this part of the paragraph the arguments will be on a heuristic level. It is in fact not difficult to make them precise, but our main purpose is here to justify the construction from nr.3 below. Let us then consider some real-analytic, real-valued, function h defined in some neighborhood X of $0 \in R^n$, such that $h(0) > 0$. Assume moreover (although less is necessary) that $h(x) < 0$ outside some small neighborhood of the origin, and denote by S, respectively S^+, $S = \{(x,t);\ x \in X,\ t = h(x)\}$, $S^+ = \{(x,t) \in S\ ;\ t \geq 0\}$. We will also assume that $S \subset Z$. If now S is sufficiently flat, it will be noncharacteristic for p. Therefore, if ρ is real-analytic in a neighborhood of S^+, we can solve the Cauchy problem

$$ {}^t p(z,D)\, g = 0, \tag{1} $$

$$ (\partial/\partial\vec{n})^j\, g_{|S^+} = 0\,, \qquad \text{if } 0 \leq j < m-1, \tag{2} $$

$$ (\partial/\partial\vec{n})^{m-1} g_{|S^+} = \rho_{|S^+}\,, \tag{3} $$

in some neighborhood of S^+. Here $\vec{n}$ is the normal to S. Moreover, if all the data of our problem (i.e., ρ, h and the coefficients of p) are analytic on some sufficiently large sets, and if the analytic extension of S on such a set is sufficiently flat, then we may assume that g is a solution of (1) in a neighborhood of

$$ K = \{(x,t) \in R^{n+1}\ ;\ 0 \leq t \leq h(x)\}\ . $$

In fact, a solution of (1), (2), (3) is given by the Cauchy-Kowalewska theorem, so we can for example apply a result of Bony- Schapira [1], concerning the domain of the existence of solutions in that theorem (cf. also Hörmander [7]), but we do not make this more precise here.

Let us now denote by χ_K the characteristic function of K and by $w = g\,\chi_K$. It is then elementary that

$$ {}^{t}p(z,D)\, w = \rho' + \sum_{j=0}^{m-1} w_j \otimes D_t^j\, \delta_t \quad , \tag{4} $$

for some distribution ρ' supported by S^+ and for some w_j in $\mathcal{E}'(R^n)$. We have in fact $\rho' = \rho \cdot p_m(z, \vec{n}(z))\, dS^+$,where p_m is the principal symbol of p and dS^+ is the surface element on S^+ .We can now also rewrite (4) in the form

$$ \rho' = {}^{t}p(z,D)\, w - \sum_{j=0}^{m-1} w_j \otimes D_t^j\, \delta_t \ , \tag{5} $$

which means that we have found a decomposition of type (1),§4, for ρ' .Note that (5) is just

$$ \int_{S^+} u(x,t)\ \rho(x,t) p_m(x,t,\vec{n}(x,t))\, dS^+ = $$
$$ \int_K w(x,t)\ f(x,t)\ dx\, dt - \sum_{j=0}^{m-1} \int w_j(x)\ f_j(x)\ dx \ , $$

if u is a solution of (1),(2),§6, in a neighborhood of K . This is precisely the duality which is used in Holmgren's method.

3.All this has been rather vague,but we can extract from it a method to obtain examples of distributions in N . In fact,what is interesting in (5) ,is that it gives a large class of distributions for which we know immediately that the decomposition from proposition 4.1 is in distributions and not merely in analytic functionals.Moreover,as is clear from the above,the same conclusion will hold for ρ' if we replace p by some other operator which does not differ from p too much.This suggests that the distributions ρ' all lie in N . We shall essentially prove that this is true (cf. remark 11.2) ,but it is more convenient to pass at first to a slightly different class of distributions (in which in particular,no reference to p_m is made) .

Let us in fact assume that W is some given bounded open convex set in R^n and that h is some real-valued,real-analytic, function on W . A number of assumptions on h will be made,which we now introduce .

a) There is some open, convex, bounded set U such that $h > 0$ on U and such that $\{ x \in W ; h(x) = 0 \}$ is the boundary of U (cf. here remark 11.3).

b) h extends analytically onto a complex neighborhood of the set $V \subset C^n$ which is defined by the following two conditions for $x \in V$:

b_1) Re x lies in U.

b_2) For given $x^o \in R^n$, $|x^o| = 1$, denote by $a(x^o,x), b(x^o,x)$, the points in which the line $\mu \to \mathrm{Re}\, x + \mu x^o$, $\mu \in R$, intersects the boundary of U. Then the condition on $\mathrm{Im}\, x$ is that

$$|\langle x^o, \mathrm{Im}\, x\rangle|^2 + |\mathrm{Re}\, x - a(x^o,x)/2 - b(x^o,x)/2|^2 \le |a(x^o,x) - b(x^o,x)|^2/4 .$$

c) The third assumption will be that $\mathrm{grad}_x h$ must be small on V.

The main result from this paragraph is now:

PROPOSITION 11.1. Assume that h satisfies a) and b) and let ρ be an analytic function defined in a neighborhood of $\{(x,h(x)); x \in V\}$. Also fix $C > 0$. Then there are constants c_1, c_2, c_3 such that the distribution $v \in \mathcal{E}'(R^{n+1})$ defined by

$$g \to v(g) = \int_U g(x,h(x))\, \rho(x,h(x))\, dx , \qquad (6)$$

satisfies

$$|\hat{v}(\lambda)| \le c_1 \exp(c_2 |\mathrm{Im}\zeta|) , \qquad \text{for } |\tau| \le C(1 + |\zeta|) , \qquad (7)$$

provided that

$$|\rho(x,h(x))| + |\mathrm{grad}\, h(x)|/c_3 \le 1 , \quad \text{for } x \in V .$$

Moreover, c_2 can here be chosen as small as desired, provided that U lies in a sufficiently small neighborhood of the origin. Finally, v lies in N if $c_2 \le \varepsilon''$ and if the support of v lies in an ε''-neighborhood of the origin.

REMARK 11.2. Thus the distributions defined in proposition 11.1 differ from the ones considered in nr.2 only in the way in which a density is associated with ρ on S^+. If h, ρ and the coefficients of p are defined in a complex neighborhood of S^+, then we can pass from one class to the other by a renotation for ρ, provided grad h is sufficiently small.

REMARK 11.3. As suggested by the argument in nr.2, it is important in our construction that grad h is small (this corresponds geometrically to the fact that the surface $\{(x,t)\ ; x \in V, t=f(x)\}$ is sufficiently flat) and that $t = 0$ on the boundary (in S) of that portion of S on which we effectively integrate. On the other hand, the convexity assumption from a) is only used here to arrive at a simple expression for V.

REMARK 11.4. To give an example, assume that H is an entire function on C^n, which is real valued for real arguments and for which $\{ x\ ;\ H(x) = 0 \}$ is the boundary of some bounded open convex set U. Moreover, assume $H(x) > 0$ in U. All assumptions from proposition 11.1 are then valid for $h = cH$, if only c is sufficiently small.

REMARK 11.5. Let us note here that the proof of proposition 8.4 gives implicitly another tool to construct distributions in N. Although this tool is very flexible from a theoretic point of view, it is not explicit enough for our present discussion.

4. Proof of proposition 11.1. (The ideea of the proof goes back to Ehrenpreis [1]). We note at first that it suffices to prove (7) under the additional assumption that $|\mathrm{Im}\,\zeta| \le |\mathrm{Re}\,\zeta|$. In fact, otherwise, the condition in (7) implies that $|\mathrm{Im}\lambda| \le 2C(1 + |\mathrm{Im}\zeta|)$, so (7) is a consequence of the Paley-Wiener estimates (if c_2 is large enough).
Our next simplification is that, for fixed λ, we may assume, after a rotation, that $\mathrm{Re}\ \zeta_j = 0$, for $j > 1$ and that $\mathrm{Re}\ \zeta_1 > 0$.

We now project along the x_1-axis, and can therefore write that $U = \{ x ; x' = (x_2,\dots,x_n) \in A , \varphi_1(x') < x_1 < \varphi_2(x') \}$ for some $A \subset R^{n-1}$ and some functions $\varphi_1, \varphi_2 : A \to R$. This shows that

$$\hat{v}(\lambda) = \int_A dx' \, [\int_{\varphi_1(x')}^{\varphi_2(x')} \rho(x,h(x)) \, e^{-ih(x)\tau - i<x,\zeta>} dx_1] \; .$$

Here it is convenient to shift the integral in x_1 to an integral on the complex contour

$$\Lambda_{x'} = \{ x_1 \in C ; \; \text{Im } x_1 < 0 , \; | x_1 - \varphi_1(x')/2 - \varphi_2(x')/2 | = |\varphi_1(x') - \varphi_2(x')| /2 \; , \text{anticlockwise orientation} \} .$$

The conclusion of the proposition will follow, if we can show that $\text{Re} (- i h(x)\tau - i<x , \zeta>) \le c_2 |\text{Im}\zeta| + c_4$ if $x' \in A$, $x_1 \in \Lambda_{x'}$ and $|\tau| \le 2C(1 + \text{Re } \zeta_1)$.

The part $- \sum_{j \ge 2} i x_j \zeta_j - i x_1 \text{Im } \zeta_1$ does not produce any problem here, since $\text{Re } \zeta_j = 0$ when $j \ge 2$ and x remains bounded. We now parametrize $\Lambda_{x'}$ by $\theta \to \varphi_1(x')/2 + \varphi_2(x')/2 + | \varphi_1(x')/2 - \varphi_2(x')/2 | \exp -(i\theta)$, $\theta \in (0, \pi)$ (θ runs from π to 0, in order to preserve the orientation) and denote the function $\theta \to h(\varphi_1(x')/2 + \varphi_2(x')/2 + |\varphi_1(x')/2 - \varphi_2(x')/2| \exp-(i\theta) , x')$ by $H(\theta)$. We can then estimate $\text{Re} (-ih(x)\tau - ix_1 \text{Re } \zeta_1)$ by $|H(\theta)\cdot\tau| - \sin\theta \, \text{Re } \zeta_1 \, | \varphi_1(x')/2 - \varphi_2(x')/2|$. All will be proved therefore, if we can show that for c_3 suitably small,

$$|H(\theta)| \le (|\varphi_1(x')/2 - \varphi_2(x')/2| \sin\theta)/(2C) \; . \qquad (8)$$

To analyze this, we observe here at first that $H(\theta)$ vanishes for both $\theta = 0$ and $\theta = \pi$, so (8) is true at the endpoints. Moreover, if we fix some $\delta > 0$, then (8) will be valid on the interval $(\delta, \pi - \delta)$, provided c_3 is small enough, for $|H(\theta)|$ can be estimated in any case by $\pi c_3 |\varphi_1(x') - \varphi_2(x')| /4$ and $\sin\theta$ is strictly positive on $(\delta, \pi - \delta)$. It remains to observe that on the intervals $(0, \delta)$, $(\pi - \delta, \pi)$, the derivative of H can be estimated by the derivative of $|\varphi_1(x')/2 - \varphi_2(x')/2| \cdot \sin\theta/(2C)$ if c_3 is small. This concludes the proof of proposition 11.1.

5. Starting from proposition 11.1,one can construct new distributions in $\mathcal{N}$ by superposition.Assume,e.g.,that h satisfies the assumptions from proposition 11.1.So does then also θh for $0 \le \theta \le 1$ and therefore the distribution

$$w(g) = \int_{\substack{x \in U \\ 0 \le t \le h(x)}} g(x,t)\, f(x,t)\, dx\, dt , \tag{9}$$

is in $\mathcal{N}$,if f is an entire analytic function and if the support of w is in an ε''-neighborhood of the origin.We can in fact introduce coordinates (y,θ) for $K = \{(x,t) ; x \in U, 0 \le t \le h(x)\}$ by $x = y$, $t = \theta h(y)$, so (9) reduces,after a renotation, to

$$w(g) = \int_0^1 \int_U g(x, \theta h(x))\, f(x, \theta h(x))\, h(x)\, dx\, d\theta .$$

For small U and h we see therefore that the characteristic function χ_K of K multiplied with an entire density is in $\mathcal{N}$. With this last remark,we can now make proposition 6.3 somewhat more explicit.Assume in fact that $0 \in R^n$ is in the interior of U and let $\mathcal{M}$ be some bounded set of distributions in $\mathcal{E}'(R^{n+1}; t \ge 0)$ which are all concentrated in a small neighborhood X of 0 .If X is small enough,one can then find a set of entire functions $\mathcal{M}'$ such that the distributions $\mathcal{M}'' = \{ f\chi_K , f \in \mathcal{M}'\}$, form a bounded set in $\mathcal{E}'(K)$ and such that the closure of $\mathcal{M}''$ contains $\mathcal{M}$ .A solution u of (1),(2),§6,will therefore exist if we can find h such that for any set of entire functions $\mathcal{M}'$ for which the set $\mathcal{M}''$ is bounded in $\mathcal{E}'(R^{n+1})$ one can find c for which $|L(w)| \le c$ if $w \in \mathcal{M}''$.

§ 12. COMMENTS AND REMARKS

1. Most of the results from this paper have been implicit in Liess [3] in the case of constant coefficients, when $\phi = 1+|\xi|$. In fact, many proofs from this paper have been based on an analogy to that paper, although the situation is here more difficult.

2. There is an interesting interpretation of the fact that one can define the functional $L(v)$ from §6 for $v \in N$ (all notations are here as in §6) before one has found a solution of the Cauchy problem (1),(2),§6. In fact, for given $f_o,\dots,f_{m-1},f$, a solution of this problem might ultimately fail to exist, but $L(v)$ is defined even in this case. In particular, it always makes sense to speak about the "integral of u" over sufficiently flat, small, analytic hypersurfaces, regardless if (the solution) u exists or not. This is of course also seen easily directly from the duality in Holmgren's method (in fact, this author was told by F.John that he was aware of this fact many years ago.) Moreover, if the solution u actually exists, then one can recover it from suitably chosen classes of such integrals in view of results related to the Radon transform. (After a suitable coordinate transformation one can e.g. apply results from V.G.Romanov [1] .).

3. We have already mentioned that the existence results from §1 have two-sided analogues. The same is valid also for the regularity results. Let us mention explicitly the following corollary of the two sided variant of theorem 1.18 :

<u>PROPOSITION 12.1.</u> Let u be a solution of the two-sided Cauchy problem (6),(7), §1, and assume that $f_j \in G_\phi$. Then we can find $d > 0$, $\varepsilon > 0$, b, c, such that $|u(v)| \le c$ for any $v \in C_o^\infty(R^{n+1})$ which satisfies :

$$|\hat{v}(\lambda)| \le \exp(d\phi(-\mathrm{Re}\,\zeta) + \varepsilon|\mathrm{Im}\lambda| + b\ln(1+|\zeta|)) .$$

4. To give an interpretation of proposition 12.1 let us introduce the Lipschitzian function $\psi : R^{n+1} \to R$ by $\psi(\xi,\tau) = \phi(\xi) + |\tau|$. We can then introduce a regularity class G_ψ and a notion WF_ψ in analogy with the definitions from §1 , for germs of distributions in $n+1$ variables, defined near $0 \in R^{n+1}$.

Let also c be fixed and denote by $\Lambda = \{\lambda \in R^{n+1};\ |\tau| < c\phi(\xi)\}$. On Λ we have that $\phi \sim \psi$ and we get from proposition 12.1:

COROLLARY 12.2. Whatever c is, we have

$$(0,\Lambda) \cap WF_\psi u = \emptyset .$$

5. One can improve the conclusion from this corollary to " $u \in G_\psi$ " (still in the conditions from the beginning of the paragraph). In fact, if c is suitable, then we have that

$$(0,\complement\Lambda) \cap WF_\psi u = \emptyset$$

as a consequence of the Sato-Hörmander regularity theorem (cf. theorem 4.3.1 from Liess-Rodino [1]) in G_ψ classes. This shows that $(0,R^{n+1}) \cap WF_\psi u = \emptyset$ all in all (cf. Liess-Rodino [1]), which means precisely that $u \in G_\psi$.

6. We conclude the paper with a brief discussion of some results for constant coefficient operators which are related to this paper. In fact, in that case one can go much further in the study of the solvability of the Cauchy problem and one also arrives in a natural way at definitions of the type considered in this paper, so this discussion should also serve as a justification for the present approach.

For simplicity, we shall look at the two-sided Cauchy problem, although most of the results have analogues for the one-sided problem. Let us in fact consider some constant coefficient linear partial differential operator $p(D)$, which satisfies the assumptions from §1 for some Lipschitz-continuous function $\phi : R^n \to R$.

(Recall that any $p(x,t,D_x,D_t)$ of form (3), §1, satisfies these assumptions for $\phi = 1 + |\xi|$.) In particular we assume that the operator is of order m . For fixed $\zeta \in C^n$ we denote by $\tau_1(\zeta),\ldots,\tau_m(\zeta)$ the roots of $\tau \to p(\zeta,\tau) = 0$, labelled such that $|\text{Im } \tau_1(\zeta)| \leq \ldots \leq |\text{Im } \tau_m(\zeta)|$. (A k- tuple root is written k times.) The functions $|\text{Im } \tau_i(\zeta)|$ are uniquely defined by this prescription, although the functions τ_i are not.

<u>DEFINITION 12.3.</u> We denote by F_j the set of germs of C^∞ functions defined in neighborhood of $0 \in R^n$ for which $\exists d > 0$, $\exists \varepsilon > 0$, $\forall b \geq 0$, $\exists c > 0$ such that $|v(f)| \leq c$ for any $v \in C_0^\infty(R^n)$ such that

$$|\hat{v}(\zeta)| \leq \exp(d|\text{Im } \tau_{j+1}(-\zeta)| + \varepsilon|\text{Im } \zeta| + b \ln(1+|\zeta|)) . \qquad (1)$$

Thus what was a property in proposition 1.13 is now a definition. One cannot reduce however the study of spaces of type F_j to that of G_ϕ classes, since the functions $|\text{Im } \tau_j(\zeta)|$ are not, in general, Lipschitz-continuous.

The following proposition should now serve as a justification for the introduction of the functions τ_i and the classes F_j .

<u>PROPOSITION 12.4.</u> Let $f \in C^\infty$ be given. Then $f \in F_0$ if and only if we can find a C^∞ solution u of $p(D)u = 0$ such that $u|_{t=0} = f$. Moreover, the condition $f_j \in F_0$, $j = 0,\ldots,m-1$, is enough for the solvability of the Cauchy problem

$$p(D)u = 0 \qquad (2)$$

$$D_t^j u|_{t=0} = f_j , \quad j = 0,\ldots,m-1 \qquad (3)$$

if and only if there is some constant c such that

$$|\text{Im } \tau_m(\zeta)| \leq c(1 + |\text{Im } \zeta| + |\text{Im } \tau_1(\zeta)|) .$$

The first part of this proposition is due to Ehrenpreis [1].

The second is a consequence of the first, if we also use proposition 1 from Liess [1] .

7. The study of the solvability conditions for (2) , (3) can be very often reduced completely to the study of the spaces F_j. (This question has been analyzed in detail in Liess [2] .) Theorems like theorem 1.12 have then their counterparts in theorems concerning microlocal characterizations of the relation " $f \in F_j$ " (the latter are much easier to prove .)

REFERENCES

L. Boutet de Monvel - P.Krée [1] : Pseudo differential operators and Gevrey classes. Ann.Inst.Fourier,17,(1967), 295-323.

A. Cavalucci [1] : Sulle proprietà differenziali delle soluzioni delle equazioni quasi-ellittice.Annali di Matematica pura ed appl. (IV),vol.LXVII,(1965),143-168.

J. Chazarain [1] : Reflection of singularities for a class of operators with multiple characteristics. Publ.R.I.M.S.,Kyoto Univ.,12 Suppl.,(1977),39-53.

L. Ehrenpreis [1] : Fourier analysis in several complex variables. Interscience Publ.,New York,1970.
[2]: Complex Fourier transform technique in variable coefficient partial differential operators.Journ. d'Analyse,XIX, (1967),75-95.

M. de Gosson [1]: Microlocal regularity at the boundary for pseudo-differential operators with the transmission property.Ann.Inst. Fourier, 32:3,(1982),183-213.

L. Hörmander [1]: Linear partial differential operators.Springer Verlag,Berlin Heidelberg New York,Grundlehren der math.Wissenschaften,Bd.116,1963.
[2]: An introduction to complex analysis in several complex variables. D.Van Nostrand Comp.,New York,1966.
[3]: Pseudodifferential operators and hypoelliptic equations, Proc.Symp.Pure Math.,(A.M.S.,Providence,R.I.) ,10,(1967), 138-183.
[4]: Convolution equations in convex domains. Inv.Math.,4:5, (1968),306-318.
[5]: Uniqueness theorems and wave front sets for solutions of linear differential equations with analytic coefficients.Comm. Pure Appl.Math.,24,(1971),671-704.
[6]: On the existence and regularity of solutions of linear pseudodifferential operators.L'enseignement math.,17:2,(1971), 99-163.

[7]: The analysis of linear partial differential operators I. Springer Verlag,Berlin Heidelberg New York Tokyo,Grundlehren der math.Wissenschaften ,256,1983.

C.O.Kiselman [1]: On unique supports for analytic functionals.Ark. Mat., 6 ,(1965),3o7-318.
2 : Existence and approximation theorems for solutions of complex analogues of boundary problems. Ark.Mat.,6:3,(1966), 193-207.

R. Lascar [1]: Propagation des singularités des solutions d'equations pseudo-differentiels quasi homogenes. Ann.Inst.Fourier, XXVII:2,(1977),79-125.

O. Liess [1] :Operators with maximal Cauchy problem. Rev.Roum.Math. Pures et Appl.,17,(1973),359-368.
[2]: Solvability conditions for the Cauchy problem for operators with constant coefficients (in Romanian).Bucharest,1975, (thesis).Unpublished.(Another proof for the results referred to in this paper is scetched in the article :Projections and liftings of exact holomorphic forms and applications to the Cauchy problem.Proc.Roum.Finnish Seminar in complex function theory.Springer Verlag,Lecture Notes in Math. nr.743, 1979 .
[3]: The microlocal Cauchy problem for constant coefficient linear partial differential operators. Rev.Roum.Math.Pures et Appl.,21,(1976),1221-1239.
[4]: Intersection properties for weak analytically uniform classes of functions. Ark.Mat.,14:1,(1976),93-111.
[5]: Boundary behaviour of analytic functions,analytic wave front sets and the intersection problem of Ehrenpreis.Preprint nr.809 at the dept.of math. of the TH Darmstadt,1984.

O. Liess - L.Rodino [1]: Inhomogeneous Grvrey classes and related pseudodifferential operators.Boll.U.M.I.,Serie VI,III-C,(1984), 233-323.

B. Malgrange [1]: Ideals of differentiable functions.Oxford Univ. Press,1966 .

R.B.Melrose - J.Sjöstrand [1]: Singularities of boundary value problems I.Comm.Pure Appl.Math.,31,(1978),593-617.

J. Persson [1]: New proofs and generalizations of two theorems by Lednev for Goursat problems. Math.Ann.,178,(1968),184-208.

L. Rodino [1]: On the Gevrey wave front set of solutions of quasi-elliptic degenerate equations. Rend.Sem.Mat.Univ.Polit. Torino, 40,(1982),221-235.

P.Schapira [1]: Propagation at the boundary and reflection of analytic singularities of solutions of linear partial differential equations. Publ.R.I.M.S.,Kyoto Univ.,12 Suppl.,(1977), 441-453.

J. Sjöstrand [1]: Analytic singularities of solutions of boundary value problems. Nato Adv.Study Inst. on Sing.in Bound.Value Problems,Reidel Publ. Co.,Doordrecht,1981,235-269.

L. Zanghirati [1]: Iterati di operatori e regolarità Gevrey microlocale anisotropa.Rend.Sem.Mat.Padova,67,(1982),85-104.

V.G.Romanov [1] : Integral geometry and inverse problems for hyperbolic equations.Springer Verlag Berlin Heidelberg New York, Springer Tracts in Natural Philosophy,26,1974.