Algebraic cycles and algebraic models of smooth manifolds

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1 Introduction and main results

Throughout this paper the term *real algebraic variety* designates a locally ringed space isomorphic to an algebraic subset of $\mathbb{R}^n$, for some n, endowed with the Zariski topology and the sheaf of $\mathbb{R}$-valued regular functions. Every real algebraic variety carries also the *Euclidean topology*, which is determined by the usual metric topology on $\mathbb{R}$. Unless explicitly stated otherwise, all topological notions related to real algebraic varieties will refer to the Euclidean topology.

Given a real algebraic variety X, we denote by $H_p^{\mathrm{alg}}(X, \mathbb{Z}/2)$ the subgroup of the homology group $H_p(X, \mathbb{Z}/2)$ generated by the homology classes of compact p-dimensional Zariski closed subvarieties of X [3, 7, 8, 14, 15]. Assuming that X is compact and nonsingular (as in [3, 8], nonsingular means that the irreducible components of X are pairwise disjoint, nonsingular, and of the same dimension), we let $H^q_{\mathrm{alg}}(X, \mathbb{Z}/2)$ denote the inverse image of $H_p^{\mathrm{alg}}(X, \mathbb{Z}/2)$ under the Poincaré duality isomorphism $H^q(X, \mathbb{Z}/2) \to H_p(X, \mathbb{Z}/2)$, where $p + q = \dim X$. The groups $H_p^{\mathrm{alg}}(X, \mathbb{Z}/2)$ and $H^q_{\mathrm{alg}}(X, \mathbb{Z}/2)$ have the expected functorial properties and are important invariants of compact nonsingular real algebraic varieties, cf. [14] for a short survey of their properties and applications. An interesting question arises at the borderline between real algebraic geometry and differential topology: to what extent are the groups $H_p^{\mathrm{alg}}(X, \mathbb{Z}/2)$ and $H^q_{\mathrm{alg}}(X, \mathbb{Z}/2)$ determined by the underlying topological or differentiable

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structures of X? Of course, it is equivalent to work with either homology or cohomology groups. We choose the latter because this will slightly simplify exposition.

Notation. Henceforth let M be a smooth (of class $\mathcal{C}^\infty$) compact m-dimensional manifold without boundary and let G be a subgroup of $H^k(M, \mathbb{Z}/2)$, where $1 \leq k \leq m-1$.

By Tognoli's theorem [8, 31], M has an *algebraic model*, that is, there exists a nonsingular real algebraic variety X diffeomorphic to M. In fact, one can find an uncountable family of pairwise nonisomorphic algebraic models of M [12]. If there is a smooth diffeomorphism $\varphi : M \to X$ such that $G = \varphi^*(H^k_{\text{alg}}(X, \mathbb{Z}/2))$, where $\varphi^* : H^k(X, \mathbb{Z}/2) \to H^k(M, \mathbb{Z}/2)$ is the isomorphism induced by φ, then X is called an *algebraic model of M of type G*. Thus we have a convenient way for expressing how the groups $H^k_{\text{alg}}(X, \mathbb{Z}/2)$ vary as X runs through the class of algebraic models of M. Furthermore, we have the following.

Proposition 1.1 *If there exist a nonsingular real algebraic variety Y and a homotopy equivalence $\psi : M \to Y$ such that $G = \psi^*(H^k_{\text{alg}}(Y, \mathbb{Z}/2))$, then M has an algebraic model of type G.*

In view of this result the topological and differentiable aspects of the problem raised in the question above are equivalent and reduced to the problem of existence of algebraic models of type G.

Recall that if X is a compact nonsingular real algebraic variety, then every cup product $w_{i_1}(X) \cup \ldots \cup w_{i_r}(X)$ of the Stiefel-Whitney classes of X is in $H^k_{\text{alg}}(X, \mathbb{Z}/2)$, where $i_1 + \cdots + i_r = k$ [7, 14, 15]. Hence a necessary condition in order for M to have an algebraic model of type G is that every cup product $w_{i_1}(M) \cup \ldots \cup w_{i_r}(M)$, where $i_1, \ldots, i_r$ are nonnegative integers satisfying $i_1 + \cdots + i_r = k$, be in G. This condition is not sufficient in general, even if $k = 2$ and $G = H^2(M, \mathbb{Z}/2)$ [4, 13, 29]. For $k = 1$ or 2, and under some mild assumptions on M, necessary and sufficient conditions are given in [11, 13]. These results are special cases of the present paper, which deals also with $k \geq 3$. Previously for $k \geq 3$ it has only been known how to construct, in some exceptional situations, algebraic models X of M with $H^k_{\text{alg}}(X, \mathbb{Z}/2) \neq H^k(X, \mathbb{Z}/2)$ [1, 3, 7, 8, 14, 22].

Before stating our results we need some preparation.

Definition 1.2. We say that M has property (G) if there exist a compact connected nonsingular real algebraic variety A and a smooth map $a : M \to A$ such that $a^*(H^\ell(A, \mathbb{Z}/2)) = 0$ for $1 \le \ell \le k-1$, $a^*(H^k(A, \mathbb{Z}/2)) = G$, and $H^q_{\mathrm{alg}}(A, \mathbb{Z}/2) = H^q(A, \mathbb{Z}/2)$ for $q \ge 0$.

This technical and somewhat mysterious notion will play a crucial role in our main theorems. Therefore we shall first make a list of cases in which M has property (G). It is known that some manifolds M do not have property (G) with $G = H^2(M, \mathbb{Z}/2)$, cf. Remark 4.9.

A cohomology class v in $H^\ell(M, \mathbb{Z}/2), \ell \ge 1$, is said to be *spherical* if $v = f^*(c)$, where $f : M \to S^\ell$ is a continuous (or equivalently smooth) map into the unit ℓ-sphere S^ℓ and c is the unique generator of the group $H^\ell(S^\ell, \mathbb{Z}/2) \simeq \mathbb{Z}/2$. Recall that v is spherical if and only if the homology class in $H_{m-\ell}(M, \mathbb{Z}/2)$ Poincaré dual to v can be represented by a closed smooth submanifold of M with trivial normal vector bundle [30, Théorème II.2] (M orientable and cohomology with coefficients in $\mathbb{Z}$ are considered in [30], but the statement here is a straightforward modification). This easily yields many examples of manifolds M with $H^\ell(M, \mathbb{Z}/2)$ generated by spherical cohomology classes. It is well-known that the set

$$S^\ell(M) = \{v \in H^\ell(M, \mathbb{Z}/2) \mid v \text{ is spherical}\}$$

need not be a subgroup of $H^\ell(M, \mathbb{Z}/2)$.

Let us define

$$W^\ell(M) = \{v \in H^\ell(M, \mathbb{Z}/2) \mid v = w_\ell(\xi) \text{ for some real vector}$$
$$\text{bundle } \xi \text{ on } M \text{ with } w_i(\xi) = 0 \text{ for } 1 \le i \le \ell - 1\},$$

where $w_j(\xi)$ is the jth Stiefel-Whitney class of ξ. Clearly, $W^\ell(M)$ is a subgroup of $H^\ell(M, \mathbb{Z}/2)$. A classical result asserts that $W^1(M) = H^1(M, \mathbb{Z}/2)$ [21, p. 236,Theorem 3.4]. In general, $W^2(M) \neq H^2(M, \mathbb{Z}/2)$ [4, 13, 29]. Furthermore, $W^\ell(M) = 0$ if ℓ is not a power of 2 [24, p. 94, Problem 8-B].

In order to have a convenient reference, we record the following fact.

Proposition 1.3 (I) *If $2\ell - 1 \geq m$ or if $\ell = 1, 3$ or 7, then $S^\ell(M)$ is a subgroup of $H^\ell(M, \mathbb{Z}/2)$.*

(II) *If $\ell = 1, 2, 4$ or 8, then $S^\ell(M) \subseteq W^\ell(M)$.*

The next statement provides a partial clarification of Definition 1.2.

Proposition 1.4 *The manifold M has property (G) in each of the following cases:*

(i) *G is generated by some elements of $S^k(M) \cup W^k(M)$,*

(ii) *$G = 0$,*

(iii) *$k = 1$,*

(iv) *$H^k(M, \mathbb{Z}/2)$ is generated by spherical cohomology classes and either $2k - 1 \geq m$ or $k = 2, 3, 4, 7$ or 8.*

Our first main result is as follows.

Theorem 1.5 *Assume that M has property (G), M is connected, and $w_1(M) = 0, \dots, w_{k-1}(M) = 0$. If $2k + 2 \leq m$ or if $2k + 1 \leq m$ with either k odd or $k = 2$, then the following conditions are equivalent:*

(a) *M has an algebraic model of type G,*

(b) *$w_k(M)$ is in G.*

We now head toward our second main result. Let $[M]$ denote the fundamental class of M in $H_m(M, \mathbb{Z}/2)$. As usual, $< -, - >$ will stand for the scalar (Kronecker) product of cohomology and homology classes.

Theorem 1.6 *Assume that M has property (G) and the subgroup*

$$\Gamma = \{v \in H^{m-k}(M, \mathbb{Z}/2) \mid < u \cup v, [M] >= 0 \; for \; all \; u \; in \; G\}$$

of $H^{m-k}(M,\mathbb{Z}/2)$ is generated by spherical cohomology classes. Furthermore, assume that $w_{j_1}(M)\cup\ldots\cup w_{j_s}(M)=0$ for all nonnegative integers $j_1,\ldots,j_s$ satisfying $j_1+\cdots+j_s=m-\ell(m-k)$ for some integer ℓ, $1<\ell<m/(m-k)$. If m is not divisible by $m-k$ or if

$$\dim_{\mathbb{Z}/2} G \geq \dim_{\mathbb{Z}/2} H^k(M,\mathbb{Z}/2)+1-m/(m-k),$$

then the following conditions are equivalent:

(a) *M has an algebraic model of type G,*

(b) *$w_{i_1}(M)\cup\ldots\cup w_{i_r}(M)$ is in G for all nonnegative integers $i_1,\ldots,i_r$ satisfying $i_1+\cdots+i_r=k$.*

The hypothesis of Theorem 1.6 look complicated, however several of them are automatically satisfied if $2k+1\leq m$. In particular, we have the following.

Corollary 1.7 *Assume that M has property (G) and the group $H^{m-k}(M,\mathbb{Z}/2)$ is generated by spherical cohomology classes. If $2k+1\leq m$, then the following conditions are equivalent:*

(a) *M has an algebraic model of type G,*

(b) *$w_{i_1}(M)\cup\ldots\cup w_{i_r}(M)$ is in G for all nonnegative integers $i_1,\ldots,i_r$ satisfying $i_1+\cdots+i_r=k$.*

Proof. Since $2k+1\leq m$, Proposition 1.3 (I) implies that every cohomology class in $H^{m-k}(M,\mathbb{Z}/2)$ is spherical. Hence the conclusion follows from Theorem 1.6. □

Proposition 1.4 demonstrates that Theorem 1.5 and Corollary 1.7 are applicable in numerous cases. Recall that if M is *stably parallelizable* (that is, the direct sum of the tangent bundle to M and some trivial vector bundle on M is a trivial vector bundle on M), then $w_i(M)=0$ for $i\geq 1$, and hence the hypothesis in Theorems 1.5, 1.6 and Corollary 1.7 concerning the Stiefel-Whitney classes of M are all satisfied. This allows for a great simplification of the statement of Theorem 1.6.

Corollary 1.8 *Assume that M has property (G), the subgroup*

$$\Gamma = \{v \in H^{m-k}(M, \mathbb{Z}/2) \mid < u \cup v, [M] >= 0 \, for \, all \, u \, in \, G\}$$

of $H^{m-k}(M, \mathbb{Z}/2)$ is generated by spherical cohomology classes, and M is stably parallelizable. If m is not divisible by $m-k$ or if

$$\dim_{\mathbb{Z}/2} G \geq \dim_{\mathbb{Z}/2} H^k(M, \mathbb{Z}/2) + 1 - m/(m-k),$$

then M has an algebraic model of type G.

Proof. It is a special case of Theorem 1.6 since $w_i(M) = 0$ for $i \geq 1$. □

Propositions 1.3 (I) and 1.4 make Corollary 1.8 easily applicable also in cases where $2k + 1 \geq m$ with, for instance, $G = 0$ or $m - k = 1, 3$ or 7. More about this point can be found in Section 2.

The author conjectures that if M has property (G), then conditions (a) and (b) in Theorem 1.6 are always equivalent, without any extra assumptions.

The paper is organized as follows. In Section 2, assuming that the results stated above are true, we derive several corollaries. First we concentrate on cases $k = 1$ or 2 to show that Theorem 1.5 contains most of previously known results on algebraic models of type G (Corollaries 2.1 and 2.2, cf. [11, 13, 14]). A new result for $k = 2$ is also included (Corollary 2.3, Example 2.5). The remaining part of Section 2 contains some consequences of Theorems 1.5 and 1.6 for $k \geq 3$. Propositions 1.1, 1.3, and 1.4 are proved in Section 3. As might have been expected, Proposition 1.1 is completely independent of the rest of the paper. Although Proposition 1.3 is certainly well-known, we offer a short proof for sake of completeness. Section 4 contains proofs of Theorems 1.5 and 1.6. Let us mention that besides algebraic models of type G there is also algebraic approximation of type G [13]. It is explained in Remark 4.8 how these two notions are connected.

Early papers [2, 4, 5, 6, 7, 25, 26] have been stimulating in leading to the problem investigated here. The results devoted to the topic under consideration as well as related contributions are surveyed in article [14], which also contains a more complete list of references. Recent publications include [1, 23].

Convention. All smooth manifolds that appear in this paper are paracompact. They are without boundary unless explicitly stated to the contrary.

2 Special cases of the main results

The first three corollaries below will demonstrate that Theorems 1.5 and 1.6 take a particularly attractive form for $k = 1$ or 2. As a consequence of Theorem 1.5 we immediately obtain the following result of [11].

Corollary 2.1 *If M is connected, $k = 1$, and $m \geq 3$, then the following conditions are equivalent:*

(a) *M has an algebraic model of type G,*

(b) *$w_1(M)$ is in G.*

Proof. By Proposition 1.4 (iii), M has property (G). Hence Theorem 1.5 implies the conclusion. $\square$

It follows from Grothendieck's formula [18, Example 15.3.6] that $H^2_{\mathrm{alg}}(X, \mathbb{Z}/2) \subseteq W^2(X)$ for every compact nonsingular real algebraic variety X, cf. also [10, Theorem 1.1]. Thus if $k = 2$ and M has an algebraic model of type G, then $G \subseteq W^2(M)$. We can now easily derive the main result of [13].

Corollary 2.2 *If M is connected and orientable, $k = 2$, and $m \geq 5$, then the following conditions are equivalent:*

(a) *M has an algebraic model of type G,*

(b) *$w_2(M)$ is in G and $G \subseteq W^2(M)$.*

Proof. We already know that (a) implies (b). Since M is orientable, we have $w_1(M) = 0$ [21, p. 246, Theorem 12.1]. If $G \subseteq W^2(M)$, then M has property (G) in view of Proposition 1.4 (i), and hence applying Theorem 1.5 we see that (b) implies (a). $\square$

A result complementing Corollary 2.2 can be obtained by specializing to $k = 2$ in Corollary 1.7.

Corollary 2.3 *If $k = 2$, $m \geq 5$, and the group $H^{m-2}(M, \mathbb{Z}/2)$ is generated by spherical cohomology classes, then the following conditions are equivalent:*

(a) *M has an algebraic model of type G,*

(b) *$w_1(M) \cup w_1(M)$ and $w_2(M)$ are in G, and $G \subseteq W^2(M)$.*

Proof. We already know that (a) implies (b). If $G \subseteq W^2(M)$, then M has property (G) by virtue of Proposition 1.4 (i). Hence (b) implies (a) in view of Corollary 1.7. $\square$

Remark 2.4. Condition $G \subseteq W^2(M)$ in Corollaries 2.2 and 2.3 is essential. Indeed, given an integer $m \geq 6$, there exists a compact connected orientable smooth manifold M of dimension m such that for every algebraic model X of M one has $H^2_{\mathrm{alg}}(X, \mathbb{Z}/2) \neq H^2(X, \mathbb{Z}/2)$; in particular, $W^2(M) \neq H^2(M, \mathbb{Z}/2)$ [29]. Such phenomenon cannot occur if $m \leq 5$ [14, Theorem 3.6].

We shall illustrate Corollary 2.3 with an example.

Example 2.5. Assume that $M = N \times S^{\ell_1} \times \cdots \times S^{\ell_p}$, where N is a compact connected smooth surface, $\ell_i \geq 2$ for $1 \leq i \leq p$, and $\ell_1 + \cdots + \ell_p \geq 3$. One easily sees that the group $H^{m-2}(M, \mathbb{Z}/2)$ is generated by spherical cohomology classes. In particular, Corollary 2.3 is applicable. Let us also note that $W^2(M) = H^2(M, \mathbb{Z}/2)$ since $H^2(M, \mathbb{Z}/2)$ is generated by spherical cohomology classes, $W^2(M)$ is a subgroup of $H^2(M, \mathbb{Z}/2)$, and $S^2(M) \subseteq W^2(M)$ in view of Proposition 1.3 (II). Furthermore, $w_j(M) = 0$ for $j \geq 1$, provided that N is orientable. If N is nonorientable of even genus, then $w_1(M) \neq 0$ and $w_1(M) \cup w_1(M) = w_2(M) = 0$.

Assuming that N is nonorientable of odd genus we have $w_1(M) \neq 0$, $w_1(M) \cup w_1(M) = w_2(M) \neq 0$. Clearly, Corollary 2.2 is not applicable if N is nonorientable.

We shall now consider algebraic models of real projective m-space $\mathbb{P}^m(\mathbb{R})$. Recall that $\mathbb{P}^m(\mathbb{R})$ is a real algebraic variety in the sense of this paper [8, Theorem 3.4.4] (in other words, using terminology of [8], $\mathbb{P}^m(\mathbb{R})$ is an *affine* real algebraic variety). Obviously, $H^\ell_{\mathrm{alg}}(\mathbb{P}^m(\mathbb{R}), \mathbb{Z}/2) = H^\ell(\mathbb{P}^m(\mathbb{R}), \mathbb{Z}/2) \backsimeq \mathbb{Z}/2$ for $0 \leq \ell \leq m$. Furthermore, if m is even and X is any algebraic model of $\mathbb{P}^m(\mathbb{R})$, then $H^\ell_{\mathrm{alg}}(X, \mathbb{Z}/2) = H^\ell(X, \mathbb{Z}/2)$ for $0 \leq \ell \leq m$ (this is obvious for $\ell = 0$, while for $\ell \geq 1$ it follows from the fact that the ℓ-fold cup product $w_1(X) \cup \ldots \cup w_1(X)$ generates $H^\ell(X, \mathbb{Z}/2)$). The situation is nontrivial for m odd.

Corollary 2.6 *Assume that the binomial coefficient $\binom{m+1}{i}$ is even for $1 \leq i \leq k$. If $2k+2 \leq m$ or if $2k+1 \leq m$ with either k odd or $k = 2$, then there exists an algebraic model X of $\mathbb{P}^m(\mathbb{R})$ with $H^k_{\mathrm{alg}}(X, \mathbb{Z}/2) = 0$.*

Proof. The assumption implies $w_i(\mathbb{P}^m(\mathbb{R})) = 0$ for $1 \leq i \leq k$, cf. [24, p. 45,Theorem 4.5], and hence the conclusion follows from Theorem 1.5. □

Our next result has a similar flavor.

Corollary 2.7 *If M is stably parallelizable, $H^{m-k}(M, \mathbb{Z}/2)$ is generated by spherical cohomology classes, and m is not divisible by $m-k$, then there exists an algebraic model X of M with $H^k_{\mathrm{alg}}(X, \mathbb{Z}/2) = 0$.*

Proof. It suffices to apply Corollary 1.8 with $G = 0$. □

We shall conclude this section by giving two additional consequences of Theorem 1.6.

Corollary 2.8 *Assume that $M = N_1 \times \cdots \times N_p$, where each N_j is either a compact connected orientable smooth surface or a sphere of positive dimension for $1 \leq j \leq p$. Then M has an algebraic model of type G in each of the following cases:*

(i) $2k+1 \leq m$ *and* $k = 1, 2, 3, 4, 7$ *or* 8,

(ii) *m is not divisible by $m-k$, $2k-1 \geq m$ and $m-k=3$ or 7.*

Proof. The group $H^i(M, \mathbb{Z}/2)$ is generated by spherical cohomology classes for all $i \geq 1$. Suppose now that either (i) or (ii) holds. By Proposition 1.4 (iii), (iv), M has property (G) and by Proposition 1.3 (I), every cohomology class in $H^{m-k}(M, \mathbb{Z}/2)$ is spherical. Since M is stably parallelizable, the conclusion is a consequence of Corollary 1.8. □

Corollary 2.9 *Assume that $M = S^\ell \times \cdots \times S^\ell$ is the p-fold product and $k = p\ell - \ell$. If G is a nonzero subgroup of $H^k(M, \mathbb{Z}/2)$ and $\ell = 1, 3$ or 7, then M has an algebraic model of type G.*

Proof. As in Corollary 2.8, M has property (G), every cohomology class in $H^{m-k}(M, \mathbb{Z}/2)$ is spherical, and M is stably parallelizable. Furthermore, $\dim_{\mathbb{Z}/2} G \geq 1$ and $\dim_{\mathbb{Z}/2} H^k(M, \mathbb{Z}/2) = p = m/(m-k)$. It suffices to apply Corollary 1.8. □

3 Proofs of the propositions

The groups $H_p^{\mathrm{alg}}(-, \mathbb{Z}/2)$ and $H^p_{\mathrm{alg}}(-, \mathbb{Z}/2)$ behave in a functorial fashion. If $f : X \to Y$ is a regular map (that is, a morphism) between compact nonsingular real algebraic varieties, then the induced homomorphisms

$$f_* : H_*(X, \mathbb{Z}/2) \to H_*(Y, \mathbb{Z}/2),\ f^* : H^*(Y, \mathbb{Z}/2) \to H^*(X, \mathbb{Z}/2)$$

satisfy

$$f_*(H_p^{\mathrm{alg}}(X, \mathbb{Z}/2)) \subseteq H_p^{\mathrm{alg}}(Y, \mathbb{Z}/2),\ f^*(H^p_{\mathrm{alg}}(Y, \mathbb{Z}/2)) \subseteq H^p_{\mathrm{alg}}(X, \mathbb{Z}/2)$$

for all $p \geq 0$ [15, Section 5] or [7].

Lemma 3.1 *With the notation as above, if*

$$f_* : H_*(X, \mathbb{Z}/2) \to H_*(Y, \mathbb{Z}/2)$$

is an isomorphism, then $f^*(H^p_{\mathrm{alg}}(Y, \mathbb{Z}/2)) = H^p_{\mathrm{alg}}(X, \mathbb{Z}/2)$ for all $p \geq 0$.

Proof. Let $D_X : H^*(X, \mathbb{Z}/2) \to H_*(X, \mathbb{Z}/2)$ and $D_Y : H^*(Y, \mathbb{Z}/2) \to H_*(Y, \mathbb{Z}/2)$ be the Poincaré duality isomorphisms. We assert that given u in $H^p(X, \mathbb{Z}/2)$, we get

$$f^*(v) = u \text{ for } v = (D_Y^{-1} \circ f_* \circ D_X)(u).$$

First note that $\dim X = \dim Y$ and $f_*([X]) = [Y]$ since f_* is an isomorphism. We now compute using standard properties of the cap product $\cap$, cf. [17]:

$$\begin{aligned} f_*(D_X(f^*(v))) &= f_*(f^*(v) \cap [X]) \\ &= v \cap f_*([X]) \\ &= v \cap [Y] = D_Y(v) \\ &\qquad\qquad = D_Y((D_Y^{-1} \circ f_* \circ D_X)(u)) \\ &\qquad\qquad = f_*(D_X(u)). \end{aligned}$$

Since f_* and D_X are isomorphisms, the assertion follows. We know that $f_*(H_q^{\text{alg}}(X, \mathbb{Z}/2)) \subseteq H_q^{\text{alg}}(Y, \mathbb{Z}/2)$ for $q \geq 0$, and therefore the definition of v implies that v is in $H^p_{\text{alg}}(Y, \mathbb{Z}/2)$, provided that u is in $H^p_{\text{alg}}(X, \mathbb{Z}/2)$. By applying the assertion, we obtain the inclusion $H^p_{\text{alg}}(X, \mathbb{Z}/2) \subseteq f^*(H^p_{\text{alg}}(Y, \mathbb{Z}/2))$. The reciprocal inclusion also holds, f being a regular map. Hence the proof is complete. $\square$

Remark 3.2. Under the same hypothesis as in Lemma 3.1 one also gets $f_*(H_p^{\text{alg}}(X, \mathbb{Z}/2)) = H_p^{\text{alg}}(Y, \mathbb{Z}/2)$ for all $p \geq 0$, but we will not need it.

Proof of Proposition 1.1. Let $\psi : M \to Y$ be as in the hypothesis. Without loss of generality we may assume that ψ is a smooth map. Note that (M, ψ) and (Y, id_Y), where id_Y is the identity map of Y, are bordant in the unoriented sense [16, p. 20] (this is so since the Stiefel-Whitney classes of compact smooth manifolds are invariant under homotopy equivalence [24, p. 132, Theorem 11.14] and therefore the Stiefel-Whitney numbers of (M, ψ) and (Y, id_Y) are identical, implying thereby the existence of a required bordism, cf. [16, (17.3)], which is recalled here as Theorem 4.1 in Section 4). By [6, Proposition 4.1] (the reader may just as well see Theorem 4.2 in Section 4 for a sharper version), there exist a nonsingular real algebraic variety X, a regular map $g : X \to Y$, and a smooth diffeomorphism $\varphi : M \to X$

such that $g \circ \varphi$ is homotopic to ψ. In particular, g is a homotopy equivalence. Furthermore,

$$\varphi^* \circ g^* = (g \circ \varphi)^* = \psi^* : H^*(Y, \mathbb{Z}/2) \to H^*(M, \mathbb{Z}/2),$$

and hence $\varphi^*(g^*(H^k_{\mathrm{alg}}(Y, \mathbb{Z}/2))) = G$ in view of $\psi^*(H^k_{\mathrm{alg}}(Y, \mathbb{Z}/2)) = G$. In order to prove that X is an algebraic model of M of type G it suffices to show $g^*(H^k_{\mathrm{alg}}(Y, \mathbb{Z}/2)) = H^k_{\mathrm{alg}}(X, \mathbb{Z}/2)$. The latter equality follows from Lemma 3.1 since the induced homomorphism $g_* : H_*(X, \mathbb{Z}/2) \to H_*(Y, \mathbb{Z}/2)$ is an isomorphism, g being a homotopy equivalence. □

Our next task is a simple and for most readers familiar exercise.

Proof of Proposition 1.3. Let us consider (I). We shall use the characterization, recalled in Section 1, of spherical cohomology classes in terms of the Poincaré dual homology classes. Thus for $2\ell - 1 \geq m$ the assertion in (I) follows by a trivial application of transversality (alternatively, one can use [20, p. 223, Proposition 11.1]). If $\ell = 1, 3$ or 7, then S^ℓ is parallelizable (the tangent bundle to S^ℓ is trivial) and hence the normal bundle of the diagonal in $S^\ell \times S^\ell$ is trivial, which readily yields $S^\ell(S^\ell \times S^\ell) = H^\ell(S^\ell \times S^\ell, \mathbb{Z}/2)$, while this in turn implies that the subset $S^\ell(M)$ is closed under addition in $H^\ell(M, \mathbb{Z}/2)$ and therefore a subgroup.

In each case considered in (II) we have $W^\ell(S^\ell) = H^\ell(S^\ell, \mathbb{Z}/2)$, which immediately implies the inclusion $S^\ell(M) \subseteq W^\ell(M)$. □

Given two real vector bundles ξ_1 and ξ_2 on a topological space S, we will write $\xi_1 \simeq \xi_2$ to indicate that they are isomorphic. If $f : S \to T$ is a continuous map from S into a topological space T and η is a real vector bundle on T, then $f^*\eta$ will denote the pullback vector bundle on S. We let $\gamma_{p,q}$ denote the universal vector bundle on the Grassmannian $\mathbb{G}_{p,q}$ of q-dimensional vector subspaces of $\mathbb{R}^p$. Recall that $\mathbb{G}_{p,q}$ is endowed with a canonical structure sheaf which makes it into a real algebraic variety in the sense of this paper [8, Theorem 3.4.4] ($\mathbb{G}_{p,q}$ is an affine real algebraic variety according to terminology used in [8]). Furthermore, $\mathbb{G}_{p,q}$ is nonsingular and $H_i^{\mathrm{alg}}(\mathbb{G}_{p,q}, \mathbb{Z}/2) = H_i(\mathbb{G}_{p,q}, \mathbb{Z}/2)$ for all $i \geq 0$ [8, Propositions 3.4.3, 11.3.3]. It follows that $W = S^{k_1} \times \cdots \times S^{k_r} \times \mathbb{G}_{p_1,q_1} \times \cdots \times \mathbb{G}_{p_s,q_s}$ is a compact connected nonsingular real algebraic variety with $H_i^{\mathrm{alg}}(W, \mathbb{Z}/2) = H_i(W, \mathbb{Z}/2)$ for all $i \geq 0$.

Proof of Proposition 1.4. Assume that (i) holds and G is generated by elements $v_1, \ldots, v_r$ of $S^k(M)$ and elements $w_1, \ldots, w_s$ of $W^k(M)$. Let $f_i : M \to S^k$ be a smooth map satisfying $v_i = f_i^*(c)$, where c is the unique generator of $H^k(S^k, \mathbb{Z}/2) \simeq \mathbb{Z}/2$, and let ξ_j be a real vector bundle on M such that $w_j = w_k(\xi_j)$ and $w_\ell(\xi_j) = 0$ for $1 \leq \ell \leq k-1$. Let $g_j : M \to \mathbb{G}_{p_j,q_j}$ be a smooth classifying map for ξ_j, that is, $\xi_j \backsimeq g_j^* \gamma_{p_j,q_j}$. Since the cohomology ring $H^*(\mathbb{G}_{p_j,q_j}, \mathbb{Z}/2)$ is generated (as a ring) by the Stiefel-Whitney classes of γ_{p_j,q_j} [24] and since we have $w_d(\xi_j) = w_d(g_j^* \gamma_{p_j,q_j}) = g_j^*(w_d(\gamma_{p_j,q_j}))$ for all integers $d \geq 0$, it follows that $g_j^*(H^\ell(\mathbb{G}_{p_j,q_j}, \mathbb{Z}/2)) = 0$ for $1 \leq \ell \leq k-1$ and $g_j^*(H^k(\mathbb{G}_{p_j,q_j}, \mathbb{Z}/2))$ is the subgroup of $H^k(M, \mathbb{Z}/2)$ generated by $w_k(\xi_j) = w_j$. Let us consider the smooth map

$$a = (f_1, \ldots, f_r,\ g_1, \ldots, g_s) : M \to S^k \times \cdots \times S^k \times \mathbb{G}_{p_1,q_1} \times \cdots \times \mathbb{G}_{p_s,q_s} = A.$$

By construction and Künneth's theorem for cohomology [27], $a^*(H^\ell(A, \mathbb{Z}/2)) = 0$ for $1 \leq \ell \leq k-1$ and $a^*(H^k(A, \mathbb{Z}/2)) = G$. Thus M has property (G) in view of the remark immediately preceding the proof.

Clearly, (ii) is a special case of (i), but also follows directly from Definition 1.2 with A consisting of one point. Similarly, (iii) is a special case of (i) since $W^1(M) = H^1(M, \mathbb{Z}/2)$.

Suppose that (iv) holds. If either $2k-1 \geq m$ or $k = 3$ or 7, then $S^k(M) = H^k(M, \mathbb{Z}/2)$ in view of Proposition 1.3 (I), whereas for $k = 2, 4$ or 8 we have $W^k(M) = H^k(M, \mathbb{Z}/2)$ by virtue of Proposition 1.3 (II) and the fact that $W^k(M)$ is a subgroup of $H^k(M, \mathbb{Z}/2)$. Now (i) implies that M has property (G). $\square$

4 Proofs of the theorems

We begin with some preliminaries, collected here for the convenience of the reader. Recall our convention from Section 1 that all smooth manifolds are without boundary unless explicitly stated otherwise. Smooth manifolds with boundary will play however some role behind the scenes. Given a smooth manifold P, let $\mathcal{N}_*(P)$ denote the *unoriented bordism group* of P [16]. The following fundamental result will be frequently used.

Theorem 4.1 *Let $f : N \to P$ be a smooth map from a compact smooth manifold N into a smooth manifold P. Then (N, f) represents 0 in $\mathcal{N}_*(P)$ if and only if for every nonnegative integer ℓ and every cohomology class v in $H^\ell(P, \mathbb{Z}/2)$ one has*

$$< f^*(v) \cup w_{i_1}(N) \cup \ldots \cup w_{i_r}(N),\, [N] >= 0$$

for all nonnegative integers $i_1, \ldots, i_r$ satisfying $i_1 + \cdots + i_r = \dim N - \ell$.

Reference for the proof: [16, (17.3)]. □

Observe that if P consists of one point, then the map $f : N \to P$ is irrelevant and Theorem 4.1 gives necessary and sufficient conditions for N to be the boundary of a compact smooth manifold with boundary.

Our basic tools also include an algebraic approximation theorem for smooth manifolds and smooth maps, whose formulation requires some preparation.

If W is a nonsingular real algebraic variety, then a bordism class in $\mathcal{N}_*(W)$ is said to be *algebraic* if it can be represented by a pair (Y, h), where Y is a compact nonsingular real algebraic variety and $h : Y \to W$ is a regular map [3, 6].

A topological real vector bundle on a real algebraic variety Y is said to admit an *algebraic structure* if it is topologically isomorphic to an algebraic subbundle of the trivial vector bundle with total space $Y \times \mathbb{R}^p$ for some p, cf. [9] for various characterizations of such vector bundles and for their basic properties.

Given smooth manifolds N and P, we endow the set $\mathcal{C}^\infty(N, P)$ of all smooth maps from N into P with the $\mathcal{C}^\infty$ topology [19] (in our applications N is always compact so it does not matter whether we take the weak $\mathcal{C}^\infty$ topology or the strong one). The tangent vector bundle to N will be denoted by $\tau(N)$. We wish to draw the reader's attention to the following, henceforth frequently used, notational convention.

Notation. For any smooth embedding $e : N \to P$ we will write $\bar{e}$ to denote the smooth diffeomorphism from N onto $e(N)$ defined by $\bar{e}(x) = e(x)$ for all x in N.

Theorem 4.2 *Let N be a compact smooth submanifold of $\mathbb{R}^n$ and let W be a nonsingular real algebraic variety. Let $f : N \to W$ be a smooth map such that the bordism class of (N, f) in $\mathcal{N}_*(W)$ is algebraic. Suppose that N contains a (possibly empty) Zariski closed nonsingular subvariety L of $\mathbb{R}^n$, the restriction $f \mid L : L \to W$ is a regular map, and the restriction $\tau(N) \mid L$ of the tangent bundle $\tau(N)$ admits an algebraic structure. If $2 \dim N + 1 \leq n$, then there exist a smooth embedding $e : N \to \mathbb{R}^n$, a Zariski closed nonsingular subvariety X of $\mathbb{R}^n$, and a regular map $g : X \to W$ such that $L \subseteq X = e(N)$, $e \mid L : L \to \mathbb{R}^n$ is the inclusion map, $g \mid L = f \mid L$, and $g \circ \bar{e}$ is homotopic to f. Furthermore, given a neighborhood $\mathcal{U}$ in $\mathcal{C}^\infty(N, \mathbb{R}^n)$ of the inclusion map $N \hookrightarrow \mathbb{R}^n$ and a neighborhood $\mathcal{V}$ of f in $\mathcal{C}^\infty(N, W)$, the objects e, X, and g can be chosen in such a way that e is in $\mathcal{U}$ and $g \circ \bar{e}$ is in $\mathcal{V}$.*

Reference for the proof: Almost identical, but slightly different, versions of this theorem are contained in [2, 3, 5, 6]. For sake of clarity we offer the following comments. If ν (resp. μ) is the normal vector bundle of L in N (resp. $\mathbb{R}^n$), then $\nu \oplus \tau(L) \simeq \tau(N) \mid L$ and hence $\nu \oplus \tau(L) \oplus \mu \simeq (\tau(N) \mid L) \oplus \mu$. The vector bundle $\tau(L) \oplus \mu$ is trivial, whereas $(\tau(N) \mid L) \oplus \mu$ admits an algebraic structure (each direct summand has such property). In particular, ν is stably equivalent to a vector bundle admitting an algebraic structure and therefore ν also admits an algebraic structure [8, Proposition 12.3.5] (note that L is compact). In view of this conclusion and [6, Theorem 3.2], we may assume that N is a Zariski closed nonsingular subvariety of $\mathbb{R}^n$. We can now apply [3, p. 65, Theorem 2.8.4]. Thus there exist a nonnegative integer ℓ, a smooth embedding $\epsilon : N \to \mathbb{R}^n \times \mathbb{R}^\ell$, a Zariski closed nonsingular subvariety Y of $\mathbb{R}^n \times \mathbb{R}^\ell$, and a regular map $h : Y \to W$ such that $L \times \{0\} \subseteq Y = \epsilon(N)$, $\epsilon(y) = (y, 0)$ for all y in L, and $h \circ \bar{\epsilon}$ is in a small neighborhood, that can be specified in advance, of f in $\mathcal{C}^\infty(N, W)$. Furthermore, we may assume that ϵ belongs to a specified in advance neighborhood in $\mathcal{C}^\infty(N, \mathbb{R}^n \times \mathbb{R}^\ell)$ of the map $N \to \mathbb{R}^n \times \mathbb{R}^\ell$, $x \to (x, 0)$ (this is not explicitly stated in [3, p. 65, Theorem 2.8.4], but is actually contained in the proof; cf. also [2, Proposition 2.8]). Since $2 \dim Y + 1 \leq n$, a standard result on general projections allows one to find a linear map $\pi : \mathbb{R}^n \times \mathbb{R}^\ell \to \mathbb{R}^n$ such that π is arbitrarily close to the canonical projection $\mathbb{R}^n \times \mathbb{R}^\ell \to \mathbb{R}^n$, $\pi(x, 0) = x$ for all x in $\mathbb{R}^n$, the image $X = \pi(Y)$ is a Zariski closed nonsingular subvariety of $\mathbb{R}^n$, and the restriction $\rho : Y \to X$ of π is a biregular isomorphism. Define $e : N \to \mathbb{R}^n$ and $g : X \to W$ by $e(x) = \rho(\bar{\epsilon}(x))$ for x in N and $g = h \circ \rho^{-1}$. It follows from the construction that e, X, and g satisfy all the requirements. Note that $g \circ \bar{e}$ is always homotopic to f, provided that $g \circ \bar{e}$ is sufficiently close to f. □

Henceforth we shall abbreviate the last statement in Theorem 4.2 by saying that e is close to the inclusion map and $g \circ \bar{e}$ is close to f.

For applications of Theorem 4.2 the following fact will be important.

Proposition 4.3 *Given a nonsingular real algebraic variety W, one has $H_i^{\mathrm{alg}}(W, \mathbb{Z}/2) = H_i(W, \mathbb{Z}/2)$ for all $i \geq 0$ if and only if every homogeneous element in the graded group $\mathcal{N}_*(W)$ is an algebraic bordism class.*

Reference for the proof: [3, Lemma 2.7.1] □

Let us note that the equivalent conditions of Proposition 4.3 are satisfied for $W = S^{k_1} \times \cdots \times S^{k_r} \times \mathbb{G}_{p_1,q_1} \times \cdots \times \mathbb{G}_{p_s,q_s}$, cf. Section 3.

We shall also need a result of a different nature. Let X be a compact nonsingular real algebraic variety. Define $\mathrm{Alg}^p(X)$ to be the subset of $H^p(X, \mathbb{Z}/2)$ that consists of all elements u for which there exist a compact irreducible nonsingular real algebraic variety T (depending on u), two points t_0 and t_1 in T, and a cohomology class z in $H^p_{\mathrm{alg}}(X \times T, \mathbb{Z}/2)$ such that

$$u = i_{t_1}^*(z) - i_{t_0}^*(z),$$

where given t in T, we let $i_t : X \to X \times T$ denote the map defined by $i_t(x) = (x, t)$ for all x in X (this definition does not force $u = 0$, the parameter space T being possibly disconnected). Since i_t is a regular map, it follows that $\mathrm{Alg}^p(X)$ is contained in $H^p_{\mathrm{alg}}(X, \mathbb{Z}/2)$. In fact, $\mathrm{Alg}^p(X)$ is a subgroup of $H^p_{\mathrm{alg}}(X, \mathbb{Z}/2)$. If u is in $\mathrm{Alg}^p(X)$, v is in $H^q_{\mathrm{alg}}(X, \mathbb{Z}/2)$ and w is in $H^q_{\mathrm{alg}}(Y, \mathbb{Z}/2)$, where Y is a compact nonsingular real algebraic variety, then the cup product $u \cup v$ belongs to $\mathrm{Alg}^{p+q}(X)$ and the cross product $u \times w$ belongs to $\mathrm{Alg}^{p+q}(X \times Y)$. Furthermore, for every regular map $f : X \to Y$, one has $f^*(\mathrm{Alg}^p(Y)) \subseteq \mathrm{Alg}^p(X)$. All these assertions can be either easily directly verified or deduced from the alternative description of $\mathrm{Alg}^p(-)$ given below.

Another way of looking at things is as follows. Let $\mathcal{X}$ be a reduced quasiprojective scheme over $\mathbb{R}$. The set $\mathcal{X}(\mathbb{R})$ of $\mathbb{R}$-rational points of $\mathcal{X}$ is contained in an affine open subset of $\mathcal{X}$. Thus if $\mathcal{X}(\mathbb{R})$ is dense in $\mathcal{X}$, we can and will regard $\mathcal{X}(\mathbb{R})$ as a real algebraic variety whose structure sheaf is the restriction of the structure sheaf of $\mathcal{X}$; up to isomorphism, each real algebraic variety is of this form.

If the irreducible components of $\mathcal{X}$ are of the same dimension, $\mathcal{X}$ is nonsingular, and $\mathcal{X}(\mathbb{R})$ is dense in $\mathcal{X}$, then $\mathcal{X}(\mathbb{R})$ is nonsingular in the sense used in this paper. Assuming in addition that $\mathcal{X}(\mathbb{R})$ is compact in the Euclidean topology, we have the cycle homomorphism

$$\mathrm{cl}_{\mathbb{R}} : Z^p(\mathcal{X}) \to H^p(\mathcal{X}(\mathbb{R}), \mathbb{Z}/2)$$

defined on the group $Z^p(\mathcal{X})$ of algebraic cycles on $\mathcal{X}$ of codimension p : for any integral subscheme $\mathcal{V}$ of $\mathcal{X}$ of codimension p the cohomology class $\mathrm{cl}_{\mathbb{R}}(\mathcal{V})$ is Poincaré dual to the homology class represented by the subvariety $\mathcal{V}(\mathbb{R})$ of $\mathcal{X}(\mathbb{R})$ if $\mathcal{V}(\mathbb{R})$ has codimension p in $\mathcal{X}(\mathbb{R})$ and otherwise $\mathrm{cl}_{\mathbb{R}}(\mathcal{V}) = 0$. By definitions, $\mathrm{cl}_{\mathbb{R}}(Z^p(\mathcal{X})) = H^p_{\mathrm{alg}}(\mathcal{X}(\mathbb{R}), \mathbb{Z}/2)$, whereas $\mathrm{Alg}^p(\mathcal{X}(\mathbb{R}))$ is the image under $\mathrm{cl}_{\mathbb{R}}$ of the subgroup of $Z^p(\mathcal{X})$ that consists of the cycles *algebraically equivalent* to 0. The assertions concerning $\mathrm{Alg}^p(-)$ made above follow from the standard properties of cycles algebraically equivalent to 0, which can be found in [18, Chapter 10].

The former description of $\mathrm{Alg}^p(-)$ is more natural in the context of this paper, whereas the latter one has been used in paper [22] to which we refer below.

Theorem 4.4 *Let X be a compact nonsingular real algebraic variety of dimension m. If u is in $\mathrm{Alg}^p(X)$ and v is in $H^{m-p}_{\mathrm{alg}}(X, \mathbb{Z}/2)$, then $< u \cup v,\ [X] >= 0$.*

Reference for the proof: [22, Theorem 2.1] $\square$

The main point is that if $\mathrm{Alg}^p(X) \neq 0$, then certain specific cohomology classes in $H^{m-p}(X, \mathbb{Z}/2)$ cannot be in $H^{m-p}_{\mathrm{alg}}(X, \mathbb{Z}/2)$.

If X is as in Theorem 4.4 and, in addition, irreducible, then $\mathrm{Alg}^m(X)$ contains every cohomology class Poincaré dual to the homology class in $H_0(X, \mathbb{Z}/2)$ represented by two distinct points (referring to the first definition of $\mathrm{Alg}^p(-)$, take $T = X$ and consider the cohomology class in $H^m_{\mathrm{alg}}(X \times X, \mathbb{Z}/2)$ corresponding to the diagonal in $X \times X$). This simple fact will be at the base of several forthcoming constructions. More specifically, the following example will be needed.

Example 4.5. Given a positive integer ℓ let B^ℓ be an irreducible nonsingular real algebraic variety with precisely two connected components B^ℓ_0 and B^ℓ_1, each diffeomorphic to S^ℓ (for example $B^\ell = \{(x_0, \dots, x_\ell) \in \mathbb{R}^{\ell+1} \mid x_0^4 - 4x_0^2 + 1 + x_1^2 + \cdots + x_\ell^2 = 0\}$). Let p_j be a point in B^ℓ_j for $j = 0, 1$. Let κ_ℓ be the cohomology class in $H^\ell(B^\ell, \mathbb{Z}/2)$ Poincaré dual to the homology class in $H_0(B^\ell, \mathbb{Z}/2)$ represented by $\{p_0, p_1\}$. We already know that κ_l belongs to $\mathrm{Alg}^\ell(B^\ell)$.

Let now $B = B^\ell \times \cdots \times B^\ell$ and $B_0 = B^\ell_0 \times \cdots \times B^\ell_0$ be the d-fold products for some positive integer d, and let $\delta : B_0 \hookrightarrow B$ be the inclusion map. We assert that

$$H^q(B_0, \mathbb{Z}/2) = \delta^*(\mathrm{Alg}^q(B)) \text{ for all } q \geq 0.$$

If q is not a multiple of ℓ, then $H^q(B_0, \mathbb{Z}/2) = 0$ and the assertion is obvious. It is also trivial for $q = 0$.

Suppose then that $q = r\ell$ for some positive integer r. Since κ_ℓ is in $\mathrm{Alg}^\ell(B^\ell)$, the cohomology classes

$$u_1 = \kappa_\ell \times 1 \times 1 \times \cdots \times 1, u_2 = 1 \times \kappa_\ell \times 1 \times \cdots \times 1, \dots, u_d = 1 \times 1 \times \cdots \times 1 \times \kappa_\ell$$

of $H^\ell(B, \mathbb{Z}/2)$ are all in $\mathrm{Alg}^\ell(B)$ (here 1 stands for the multiplicative identity element in the cohomology ring $H^*(B^\ell, \mathbb{Z}/2)$). By construction, $\delta^*(u_1), \dots, \delta^*(u_d)$ generate $H^\ell(B_0, \mathbb{Z}/2)$ and hence the elements $\delta^*(u_{i_1} \cup \ldots \cup u_{i_r})$, where $i_1, \dots, i_r$ are in the set $\{1, \dots, d\}$, generate $H^q(B_0, \mathbb{Z}/2)$. The assertion holds since the $u_{i_1} \cup \ldots \cup u_{i_r}$ are in $\mathrm{Alg}^q(B)$.

The following technical result will be crucial in the proof of Theorem 1.5.

Lemma 4.6 *Let K be a compact smooth submanifold of $\mathbb{R}^n$ and let V be a connected nonsingular real algebraic variety. Let $f : K \to V$ and $h : K \to S^1$ be smooth maps such* *that $f^*(H^p(V,\mathbb{Z}/2)) = 0$ for $p \geq 1$ and (K,h) represents 0 in the unoriented bordism group $\mathcal{N}_*(S^1)$. If $2\dim K + 1 \leq n$, then there exist a smooth embedding $\epsilon : K \to \mathbb{R}^n$, a Zariski closed nonsingular subvariety Y of $\mathbb{R}^n$, and a regular map $g : Y \to V$ such that $\epsilon(K) = Y$, ϵ is close to the inclusion map $K \hookrightarrow \mathbb{R}^n$, $g \circ \bar{\epsilon}$ is close to f, and $\sigma^*(h^*(c))$ is in $\mathrm{Alg}^1(Y)$, where c is the unique generator of $H^1(S^1,\mathbb{Z}/2) \simeq \mathbb{Z}/2$ and $\sigma = \bar{\epsilon}^{-1}$.*

Proof. We assert that (K,F) represents 0 in $\mathcal{N}_*(V \times S^1)$, where $F = (f,h) : K \to V \times S^1$. By Theorem 4.1 and Künneth's theorem for cohomology, it suffices to show that given v in $H^p(V,\mathbb{Z}/2)$ and w in $H^q(S^1,\mathbb{Z}/2)$, we have

$$(*) \qquad < F^*(v \times w) \cup w_{i_1}(K) \cup \ldots \cup w_{i_r}(K),\ [K] >= 0$$

for all nonnegative integers $i_1, \ldots, i_r$ satisfying $i_1 + \cdots + i_r = \dim K - (p+q)$. Note that $F^*(v \times w) = f^*(v) \cup h^*(w)$. Since $f^*(v) = 0$ for $p \geq 1$ and $w = 0$ for $q \geq 2$, we have to consider only two cases: $(p,q) = (0,0)$ and $(p,q) = (0,1)$.

Suppose that $(p,q) = (0,0)$. Since $V \times S^1$ is connected, we have $F^*(v \times w) = 0$ or $F^*(v \times w) = 1$. If $F^*(v \times w) = 0$, then $(*)$ is obvious. If $F^*(v \times w) = 1$, then $(*)$ holds in view of Theorem 4.1 and the fact that K is the boundary of a compact smooth manifold with boundary (recall that (K,h) represents 0 in $\mathcal{N}_*(S^1)$).

Suppose that $(p,q) = (0,1)$. Then $f^*(v) = 0$ or $f^*(v) = 1$. In the former case $(*)$ is obvious, while in the latter case $(*)$ follows from Theorem 4.1 since (K,h) represents 0 in $\mathcal{N}_*(S^1)$.

Hence the assertion is proved and we can continue. Let B^1 and κ_1 be as in Example 4.5 (with $\ell = 1$). Let $\tau : S^1 \to B^1$ be a smooth map which transforms diffeomorphically S^1 onto B^1_0. The assertion implies that (K,Φ) represents 0 in $\mathcal{N}_*(V \times B^1)$, where $\Phi = (f, \tau \circ h) : K \to V \times B^1$. In particular, the hypothesis of Theorem 4.2 are satisfied for $\Phi : K \to V \times B^1$ with $L = \emptyset$. Hence there exist a smooth embedding $\epsilon : K \to \mathbb{R}^n$, a Zariski closed nonsingular subvariety Y of $\mathbb{R}^n$, and a regular map $\Psi = (g,b) : Y \to V \times B^1$ such that $\epsilon(K) = Y$, ϵ is close to the inclusion map $K \hookrightarrow \mathbb{R}^n$, $\Psi \circ \bar{\epsilon} = (g \circ \bar{\epsilon},\ b \circ \bar{\epsilon})$ is close to $\Phi = (f, \tau \circ h)$, and $\Psi = (g,b)$ is homotopic to $\Phi \circ \sigma = (f \circ \sigma,\ \tau \circ h \circ \sigma)$, where $\sigma = \bar{\epsilon}^{-1}$. We shall now show that

$\sigma^*(h^*(c))$ is in $\mathrm{Alg}^1(Y)$, which will complete the proof.

Since $c = \tau^*(\kappa_1)$ and b is homotopic to $\tau \circ h \circ \sigma$, we have

$$\sigma^*(h^*(c)) = \sigma^*(h^*(\tau^*(\kappa_1))) = (\tau \circ h \circ \sigma)^*(\kappa_1) = b^*(\kappa_1).$$

On the other hand, $b : Y \to B^1$ is a regular map and κ_1 is in $\mathrm{Alg}^1(B^1)$, which implies that $b^*(\kappa_1)$ is in $\mathrm{Alg}^1(Y)$. Hence $\sigma^*(h^*(c))$ is in $\mathrm{Alg}^1(Y)$ as required. □

We need one more auxiliary fact.

Lemma 4.7 *Let N be a compact connected smooth manifold and let k be a positive integer satisfying $2k \leq \dim N$. Then each homology class in $H_k(N, \mathbb{Z}/2)$ can be represented by a compact connected smooth submanifold of N that is also the boundary of some compact smooth manifold with boundary.*

Proof. It is well-known that given a homology class in $H_k(N, \mathbb{Z}/2)$, one can find a compact smooth submanifold Q of N representing it [30, Théorème II.26]. Since N is connected, we may assume that Q is also connected. Let U be an open subset of $N \backslash Q$ diffeomorphic to $\mathbb{R}^n$, where $n = \dim N$. Whitney's embedding theorem implies that there exists a compact smooth submanifold Q' of N which is contained in U and diffeomorphic to Q. Let P be a compact smooth submanifold of N obtained by joining Q and Q' with a k-dimensional tube. Thus P is connected and represents the same homology class in $H_k(N, \mathbb{Z}/2)$ as Q does. Furthermore, P is the boundary of a compact smooth manifold with boundary since the union $Q \cup Q'$ has this property. □

We now have all machinery necessary for the proofs of Theorems 1.5 and 1.6 ready at hand. The proofs will be preceded by a general remark, which has been already alluded to in Section 1.

Remark 4.8. Suppose that M is a smooth submanifold of $\mathbb{R}^n$. Then M is said to admit an *algebraic approximation of type G* if for any neighborhood $\mathcal{U}$ in $\mathcal{C}^\infty(M, \mathbb{R}^n)$ of the inclusion map $M \hookrightarrow \mathbb{R}^n$, there exist a smooth embedding $e : M \to \mathbb{R}^n$ and a Zariski closed nonsingular subvariety X of $\mathbb{R}^n$ such that e is in $\mathcal{U}$, $e(M) = X$, and $G = \bar{e}^*(H^k_{\mathrm{alg}}(X, \mathbb{Z}/2))$. Clearly, M has an algebraic model of type G, provided that it admits an algebraic approximation of type G. The converse is true if $2m + 1 \leq n$ [13, Proposition 1.2] and remains undecided for $2m + 1 > n$.

In the proofs of Theorems 1.5 and 1.6 given below we assume, which can be done without any loss of generality, that M is a smooth submanifold of $\mathbb{R}^n$, where $2m + 1 \leq n$, and show that condition (b) implies that M admits an algebraic approximation of type G. As noted, this is precisely equivalent to showing that (b) implies (a), however, the approximation comes out naturally with no extra effort and makes the structure of the proofs more transparent.

Proof of Theorem 1.5. We already know that (a) implies (b). Suppose then that (b) holds. Assuming that M is a smooth submanifold of $\mathbb{R}^n$, where $2m + 1 \leq n$, we prove below that M admits an algebraic approximation of type G. Given a subset S of M, we let ϵ_S denote the inclusion map $S \hookrightarrow M$.

Let us set

$$\Gamma = \{v \in H^{m-k}(M, \mathbb{Z}/2) \mid \; u \cup v = 0 \text{ for all } u \text{ in } G\}, \Gamma = \{v_1, \dots, v_s\}.$$

Let P_i be a compact smooth submanifold of M, whose homology class in $H_k(M, \mathbb{Z}/2)$ is Poincaré dual to v_i. In view of Lemma 4.7, we may assume that P_i is connected and is the boundary of a compact smooth manifold with boundary. Furthermore, $P_1, \dots, P_s$ may be assumed to be pairwise disjoint since $2k + 1 \leq m$. Note that

$$G = \{u \in H^k(M, \mathbb{Z}/2) \mid \; \epsilon^*_{P_i}(u) = 0 \text{ for } i = 1, \dots, s\}. \tag{1}$$

Indeed, since the bilinear map

$$H^k(M, \mathbb{Z}/2) \times H^{m-k}(M, \mathbb{Z}/2) \to H^m(M, \mathbb{Z}/2) \simeq \mathbb{Z}/2, \; (u, v) \to u \cup v$$

is a dual pairing [17, p. 300, Proposition 8.13], we get

$$G = \{u \in H^k(M, \mathbb{Z}/2) \mid \; u \cup v_i = 0 \text{ for } i = 1, \dots, s\}.$$

Standard properties of $\cup$ and $\cap$ products, and the definition of the Poincaré duality yield

$$\begin{aligned}\epsilon_{P_i*}(\epsilon_{P_i}^*(u)\cap[P_i]) &= u\cap\epsilon_{P_i*}([P_i])\\ &= u\cap(v_i\cap[M])\\ &= (u\cup v_i)\cap[M]\end{aligned}$$

for any u in $H^k(M,\mathbb{Z}/2)$. The homomorphism $\epsilon_{P_i*}: H_0(P_i,\mathbb{Z}/2)\to H_0(M,\mathbb{Z}/2)$ is injective, P_i being connected, and therefore $\epsilon_{P_i}^*(u)=0$ is equivalent to $u\cup v_i=0$. Thus (1) holds.

Let T_i be an open tubular neighborhood of P_i in M for $i=1,\dots,s$. These tubular neighborhoods can be chosen small enough so that they are pairwise disjoint. Since the inclusion map $P_i\hookrightarrow T_i$ is a homotopy equivalence and $\dim P_i=k$, we have

$$\epsilon_{T_i}^*(H^p(M,\mathbb{Z}/2))=0 \text{ for } p\geq k+1 \text{ and } i=1,\dots,s. \tag{2}$$

Furthermore, applying (1), we obtain

$$G=\{u\in H^k(M,\mathbb{Z}/2)\mid\ \epsilon_{T_i}^*(u)=0 \text{ for } i=1,\dots,s\}. \tag{3}$$

For the subsequent constructions we first have to investigate the normal vector bundle ν_i of P_i in M. We assert that

$$\nu_i \text{ splits off a trivial vector bundle of rank } 2 \text{ for } i=1,\dots,s. \tag{4}$$

Since $\dim P_i=k$, $\operatorname{rank}\nu_i=m-k$, and in all cases $2k+1\leq m$, it follows that ν_i can be represented as the direct sum of a vector bundle μ_i of rank k and a trivial vector bundle of rank $m-2k$. Thus (4) holds if $2k+2\leq m$. If merely $2k+1\leq m$ an additional argument, involving the Stiefel-Whitney class $w_k(\nu_i)$, is required.

We claim that $w_k(\nu_i)=0$. Indeed, for every $j\geq 1$ we have

$$w_j(\tau(M)\mid P_i)=\epsilon_{P_i}^*(w_j(\tau(M)))=\epsilon_{P_i}^*(w_j(M))=0,$$

where the last equality follows since $w_\ell(M)=0$ for $1\leq\ell\leq k-1$, $w_k(M)$ belongs to G and (1) holds, and $\dim P_i=k$. Observe that $\tau(P_i)\oplus\nu_i\simeq\tau(M)\mid P_i$, and hence $w_j(\tau(P_i)\oplus\nu_i)=0$ for all $j\geq 1$. This, together with Whitney's product formula, implies that $w_k(\nu_i)$ is a linear combination of the cohomology classes $w_{i_1}(\tau(P_i)) \cup \ldots \cup w_{i_r}(\tau(P_i)) = w_{i_1}(P_i) \cup \ldots \cup w_{i_r}(P_i)$, where $i_1, \ldots, i_r$ are nonnegative integers satisfying $i_1 + \cdots + i_r = k$. Since P_i is connected and is the boundary of a compact smooth manifold with boundary, by virtue of Theorem 4.1, we have $w_{i_1}(P_i) \cup \ldots \cup w_{i_r}(P_i) = 0$. Hence $w_k(\nu_i) = 0$ as claimed. Note that the claim implies $w_k(\mu_i) = 0$.

If k is odd, then interpreting $w_k(\mu_i)$ as an obstruction, we conclude that μ_i splits off a trivial line bundle [24, p. 140, Theorem 12.1], [28, Theorem 38.11], which implies (4).

Suppose finally that $k = 2$. Let ξ_i be a real line bundle on P_i with $w_1(\xi_i) = w_1(\nu_i)$. Since $w_2(\nu_i) = 0$, it follows that ν_i and ξ_i are stably equivalent, cf. for example [10, Remark 1.4]. This implies that ν_i is isomorphic to the direct sum of ξ_i and a trivial vector bundle of rank $m - 3$ [21, p. 100, Theorem 1.5] (if $\dim P_i = k = 2$, we have $\operatorname{rank} \nu_i = m - 2 \geq 3$). Hence (4) holds also in this last case.

Let us indentify T_i with the total space of the bundle ν_i. It follows from (4) that there exists a smooth embedding of $P_i \times \mathbb{R}^2$ into T_i which maps $(x, 0)$ to x for all x in P_i. Thus choosing a point t_0 in S^1 one can find a compact smooth submanifold K_i of M and a smooth diffeomorphism $\tau_i : P_i \times S^1 \to K_i$ such that $P_i \subseteq K_i \subseteq T_i$ and $\tau_i(x, t_0) = x$ for all x in P_i. By applying (2), we get

$$\epsilon_{K_i}^*(H^p(M, \mathbb{Z}/2)) = 0 \text{ for } p \geq k+1 \text{ and } i = 1, \ldots, s, \tag{5}$$

while (1) and (3) imply

$$G = \{u \in H^k(M, \mathbb{Z}/2) \mid \epsilon_{K_i}^*(u) = 0 \text{ for } i = 1, \ldots, s\}. \tag{6}$$

Let $\pi_i : P_i \times S^1 \to S^1$ be the canonical projection. We assert that if $\epsilon_{P_i}^*(w) \neq 0$ for some w in $H^k(M, \mathbb{Z}/2)$ and some i, then

$$\tau_i^*(\epsilon_{K_i}^*(w)) \cup \pi_i^*(c) \neq 0, \tag{7}$$

where c is the unique generator of $H^1(S^1, \mathbb{Z}/2) \simeq \mathbb{Z}/2$. First note that

$$\pi_i^*(c) = 1 \times c.$$

It follows from Künneth's theorem for cohomology that

$$\tau_i^*(\epsilon_{K_i}^*(w)) = w' \times 1 + w'' \times c$$

for some w' in $H^k(P_i, \mathbb{Z}/2)$ and w'' in $H^{k-1}(P_i, \mathbb{Z}/2)$. We now have

$$\begin{aligned}\tau_i^*(\epsilon_{K_i}^*(w)) \cup \pi_i^*(c) &= (w' \times 1 + w'' \times c) \cup (1 \times c)\\ &= w' \times (1 \cup c) + w'' \times (c \cup c)\\ &= w' \times c\end{aligned}$$

Thus in order to justify (7) we have to show $w' \neq 0$. We proceed as follows. Define $\delta_i : P_i \to P_i \times S^1$ by $\delta_i(x) = (x, t_0)$ for x in P_i. Then $\tau_i \circ \delta_i : P_i \to K_i$ is the inclusion map, $\epsilon_{K_i} \circ \tau_i \circ \delta_i = \epsilon_{P_i}$, and therefore

$$\delta_i^*(\tau_i^*(\epsilon_{K_i}^*(w))) = (\epsilon_{K_i} \circ \tau_i \circ \delta_i)^*(w) = \epsilon_{P_i}^*(w).$$

On the other hand,

$$\delta_i^*(\tau_i^*(\epsilon_{K_i}^*(w))) = \delta_i^*(w' \times 1 + w'' \times c) = w' \cup 1 + w'' \cup 0 = w'.$$

Since $\epsilon_{P_i}^*(w) \neq 0$, we get $w \neq 0$, which proves (7).

Define $h_i : K_i \to S^1$ by $h_i = \pi_i \circ \tau_i^{-1}$. If $\epsilon_{P_i}^*(w) \neq 0$ for some w in $H^k(M, \mathbb{Z}/2)$ and some i, then (7) implies $\epsilon_{K_i}^*(w) \cup h_i^*(c) \neq 0$ and, using connectedness of K_i, we obtain

$$< \epsilon_{K_i}^*(w) \cup h_i^*(c),\ [K_i] >\neq 0. \tag{8}$$

Directly from the definition of h_i and the fact that P_i is the boundary of a compact smooth manifold with boundary we deduce

$$(K_i, h_i) \text{ represents } 0 \text{ in } \mathcal{N}_*(S^1) \text{ for } i = 1, \dots, s. \tag{9}$$

Since M has property (G), there exist a compact connected (connectedness will play no role in this proof) nonsingular real algebraic variety A and a smooth map $a : M \to A$ such that $a^*(H^\ell(A, \mathbb{Z}/2)) = 0$ for $1 \leq \ell \leq k-1$, $a^*(H^k(A, \mathbb{Z}/2)) = G$, and $H^q_{\mathrm{alg}}(A, \mathbb{Z}/2) = H^q(A, \mathbb{Z}/2)$ for $q \geq 0$. Clearly, $a \mid K_i = a \circ \epsilon_{K_i}$, and hence (5) and (6) imply

$$(a \mid K_i)^*(H^p(A, \mathbb{Z}/2)) = 0 \text{ for } p \geq 1 \text{ and } i = 1, \dots, s. \tag{10}$$

Let $b : M \to \mathbb{G}_{d,m}$ be a smooth classifying map for the tangent vector bundle $\tau(M)$ to M, that is, $\tau(M) \simeq b^*\gamma_{d,m}$. Then $b^*(H^*(\mathbb{G}_{d,m}, \mathbb{Z}/2))$ is generated as a ring by the Stiefel-Whitney classes $w_j(M)$ of M (keep in mind $w_j(M) = b^*(w_j(\gamma_{d,m}))$ and that $H^*(\mathbb{G}_{d,m}, \mathbb{Z}/2)$ is generated as a ring by the Stiefel-Whitney classes $w_j(\gamma_{d,m})$ of $\gamma_{d,m}$ [24]). Since $b \mid K_i = b \circ \epsilon_{K_i}$, $w_j(M) = 0$ for $1 \leq j \leq k-1$, and $w_k(M)$ is in G, by applying (5) and (6), we get

$$(b \mid K_i)^*(H^p(\mathbb{G}_{d,m}, \mathbb{Z}/2)) = 0 \text{ for } p \geq 1 \text{ and } i = 1, \ldots, s. \tag{11}$$

Setting $V = A \times \mathbb{G}_{d,m}$ and $f = (a, b) : M \to V$, we deduce from (10), (11), and Künneth's theorem for cohomology that the restrictions $f \mid K_i : K_i \to V$ satisfy

$$(f \mid K_i)^*(H^p(V, \mathbb{Z}/2)) = 0 \text{ for } p \geq 1 \text{ and } i = 1, \ldots, s. \tag{12}$$

In view of (9) and (12), Lemma 4.6 can be applied to $f \mid K_i : K_i \to V$ and $h_i : K_i \to S^1$. Thus there exist a smooth embedding $\epsilon_i : K_i \to \mathbb{R}^n$, a Zariski closed nonsingular subvariety Y_i of $\mathbb{R}^n$, and a regular map $g_i : Y_i \to V$ such that $Y_i = \epsilon_i(K_i)$, ϵ_i is close to the inclusion map $K_i \hookrightarrow \mathbb{R}^n$, $g_i \circ \bar{\epsilon}_i$ is close to $f \mid K_i$, and $\sigma_i^*(h_i^*(c))$ belongs to $\mathrm{Alg}^1(Y_i)$, where $\sigma_i = \bar{\epsilon}_i^{-1}$. It follows that one can find a smooth embedding $\epsilon : M \to \mathbb{R}^n$ and a smooth map $H : M \to V$ such that ϵ is close to the inclusion map $M \hookrightarrow \mathbb{R}^n$, H is close to f, and $\epsilon \mid K_i = \epsilon_i, H \mid K_i = g_i \circ \bar{\epsilon}_i$ for $i = 1, \ldots, s$. Set $N = \bar{\epsilon}(M)$ and $\sigma = \bar{\epsilon}^{-1}$. Define $F = (\alpha, \beta) : N \to V$ to be the composite map $F = H \circ \sigma$. By construction, $Y_i \subseteq N$ and $g_i = F \mid Y_i$ for $i = 1, \ldots, s$. Furthermore, we may assume that the maps $F = (\alpha, \beta)$ and $f \circ \sigma = (a \circ \sigma, b \circ \sigma)$ are homotopic (this is always satisfied if H is sufficiently close to f). Thus α is homotopic to $a \circ \sigma$, and β is homotopic to $b \circ \sigma$.

The last fact established above and $b^*\gamma_{d,m} \simeq \tau(M)$ imply

$$\beta^*\gamma_{d,m} \simeq (b \circ \sigma)^*\gamma_{d,m} \simeq \sigma^*(b^*\gamma_{d,m}) \simeq \sigma^*\tau(M) \simeq \tau(N).$$

This yields $(\beta \mid Y_i)^*\gamma_{d,m} \simeq \tau(N) \mid Y_i$, which implies that the vector bundle $\tau(N) \mid Y_i$ on Y_i admits an algebraic structure (the restriction $\beta \mid Y_i : Y_i \to \mathbb{G}_{d,m}$ is a regular map since $g_i = F \mid Y_i = (\alpha \mid Y_i, \beta \mid Y_i)$ is a regular map).

The union $Y = Y_1 \cup \ldots \cup Y_s$ is a Zariski closed nonsingular subvariety of $\mathbb{R}^n$, the sets $Y_1, \ldots, Y_s$ being pairwise disjoint. Furthermore, $Y \subseteq N$, the restriction $\tau(N) \mid Y$ admits an algebraic structure, and $\alpha : N \to A$ is a smooth map whose restriction $\alpha \mid Y : Y \to A$ is a regular map. By Proposition 4.3, every homogeneous element in the bordism group $\mathcal{N}_*(A)$ is algebraic; in particular, the bordism class of (N, α) in $\mathcal{N}_*(A)$ is algebraic. Hence we can apply Theorem 4.2 to $\alpha : N \to A$ and $Y \subseteq N$. It follows that there exist a smooth embedding $\delta : N \to \mathbb{R}^n$, a Zariski closed nonsingular subvariety X of $\mathbb{R}^n$, and a regular map $\lambda : X \to A$ such that $Y \subseteq X = \delta(N)$, $\delta \mid Y : Y \to \mathbb{R}^n$ is the inclusion map, δ is close to the inclusion map $N \hookrightarrow \mathbb{R}^n$, and $\lambda \circ \bar{\delta}$ is homotopic to α (according to Theorem 4.2, $\lambda \circ \bar{\delta}$ is close to α and $\lambda \mid Y = \alpha \mid Y$, but these properties will not be used).

By construction, the map $e = \delta \circ \bar{\epsilon} : M \to \mathbb{R}^n$ is a smooth embedding close to the inclusion map $M \hookrightarrow \mathbb{R}^n$ and $X = e(M)$. We shall now show the final property: $G = \bar{e}^*(H^k_{\text{alg}}(X, \mathbb{Z}/2))$, proving thereby that M admits an algebraic approximation of type G. It will be more convenient to set $\psi = \bar{e}^{-1}$ and prove the equivalent condition: $\psi^*(G) = H^k_{\text{alg}}(X, \mathbb{Z}/2)$.

Since $\lambda \circ \bar{\delta}$ is homotopic to α, and α is homotopic to $a \circ \sigma$ (the latter fact has been established earlier in the proof), it follows that λ is homotopic to $a \circ \sigma \circ \bar{\delta}^{-1} = a \circ \psi$. Hence the induced homomorphisms satisfy

$$\psi^* \circ a^* = (a \circ \psi)^* = \lambda^* : H^k(A, \mathbb{Z}/2) \to H^k(X, \mathbb{Z}/2),$$

which in view of $a^*(H^k(A, \mathbb{Z}/2)) = G$ implies

$$\psi^*(G) = \lambda^*(H^k(A, \mathbb{Z}/2)).$$

The inclusion $\lambda^*(H^k_{\text{alg}}(A, \mathbb{Z}/2)) \subseteq H^k_{\text{alg}}(X, \mathbb{Z}/2)$ holds since λ is a regular map. Thus recalling $H^k_{\text{alg}}(A, \mathbb{Z}/2) = H^k(A, \mathbb{Z}/2)$, we obtain

$$\lambda^*(H^k(A, \mathbb{Z}/2)) \subseteq H^k_{\text{alg}}(X, \mathbb{Z}/2).$$

In conclusion,

$$\psi^*(G) \subseteq H^k_{\text{alg}}(X, \mathbb{Z}/2). \tag{13}$$

We shall now complete the proof by analyzing the homomorphisms in cohomology induced by the inclusion maps $e_i : Y_i \hookrightarrow X$ for $i = 1, \dots, s$.

Let w be an element of $H^k(M, \mathbb{Z}/2) \backslash G$. By (1), $\epsilon^*_{P_i}(w) \neq 0$ for some i satisfying $1 \leq i \leq s$. Since $\sigma_i : Y_i \to K_i$ is a diffeomorphism, we have $\sigma_{i*}([Y_i]) = [K_i]$ and hence (8) yields

$$< \sigma_i^*(\epsilon^*_{K_i}(w)) \cup \sigma_i^*(h_i^*(c)),\ [Y_i] > \neq 0.$$

By applying the equality $\psi \circ e_i = \epsilon_{K_i} \circ \sigma_i$, we get

$$< e_i^*(\psi^*(w)) \cup \sigma_i^*(h_i^*(c)),\ [Y_i] > \neq 0.$$

As has been noted earlier in the proof, $\sigma_i^*(h_i^*(c))$ belongs to $\mathrm{Alg}^1(Y_i)$, which in view of Theorem 4.4 implies that $e_i^*(\psi^*(w))$ is not in $H^k_{\mathrm{alg}}(Y_i, \mathbb{Z}/2)$. Since $e_i : Y_i \hookrightarrow X$ is a regular map, we have $e_i^*(H^k_{\mathrm{alg}}(X, \mathbb{Z}/2)) \subseteq H^k_{\mathrm{alg}}(Y_i, \mathbb{Z}/2)$. In summary, $\psi^*(w)$ is not in $H^k_{\mathrm{alg}}(X, \mathbb{Z}/2)$.

We just proved

$$\psi^*(H^k(M, \mathbb{Z}/2) \backslash G) \subseteq H^k(X, \mathbb{Z}/2) \backslash H^k_{\mathrm{alg}}(X, \mathbb{Z}/2).$$

By comparing this with (13), we get $\psi^*(G) = H^k_{\mathrm{alg}}(X, \mathbb{Z}/2)$. Thus M admits an algebraic approximation of type G. In particular, X is an algebraic model of M of type G and therefore (a) holds. $\square$

Remark 4.9. Suppose that M has property (G). A short argument, like the one in the proof of formula (13) above, shows that there exist an algebraic model X of M and a smooth diffeomorphism $\psi : X \to M$ satisfying $\psi^*(G) \subseteq H^k_{\mathrm{alg}}(X, \mathbb{Z}/2)$. It follows that the manifold M in Remark 2.4 cannot have property (G) with $G = H^2(M, \mathbb{Z}/2)$.

Proof of Theorem 1.6. We already know that (a) implies (b). Suppose that (b) holds. Furthermore, assume that M is a smooth submanifold of $\mathbb{R}^n$, where $2m + 1 \leq n$. We show below that M admits an algebraic approximation of type G.

Since M has property (G) there exist a compact connected nonsingular real algebraic variety A and a smooth map $a : M \to A$ such that $a^*(H^\ell(A, \mathbb{Z}/2)) = 0$ for $1 \leq \ell \leq k - 1$, $a^*(H^k(A, \mathbb{Z}/2)) = G$, and $H^q_{\mathrm{alg}}(A, \mathbb{Z}/2) = H^q(A, \mathbb{Z}/2)$ for $q \geq 0$. By Proposition 4.3, every homogeneous element in the bordism group $\mathcal{N}_*(A)$ is algebraic; in particular, the bordism class of (M, a) in $\mathcal{N}_*(A)$ is algebraic. In view of Theorem 4.2 (applied to $a : M \to A$ and $L = \emptyset$), one can find a nonsingular real algebraic variety Y, a regular map $\alpha : Y \to A$, and a smooth diffeomorphism $\sigma : M \to Y$ such that $\alpha \circ \sigma$ is homotopic to a. In particular, $\alpha^*(H^\ell(A, \mathbb{Z}/2)) = 0$ for $1 \le \ell \le k - 1$ and

$$(M, a) \text{ and } (Y, \alpha) \text{ represent the same bordism class in } \mathcal{N}_*(A). \tag{1}$$

Let B^{m-k} be as in Example 4.5 (with $\ell = m - k$). Let $B = B^{m-k} \times \cdots \times B^{m-k}$ and $B_0 = B_0^{m-k} \times \cdots \times B_0^{m-k}$ be the d-fold products. Choosing d large enough and using the fact that Γ is generated by spherical cohomology classes we can find a smooth map $b : M \to B$ satisfying

$$b(M) \subseteq B_0 \text{ and } b^*(H^{m-k}(B, \mathbb{Z}/2)) = \Gamma. \tag{2}$$

Define $\beta : Y \to B$ to be a constant map sending Y to a point in B_0.

We shall compare the smooth map $(a, b) : M \to A \times B$ and the regular map $(\alpha, \beta) : Y \to A \times B$. More precisely, we assert that

$$(M, (a, b)) \text{ and } (Y, (\alpha, \beta)) \text{ represent the same bordism class in } \mathcal{N}_*(A \times B). \tag{3}$$

By Theorem 4.1 and Künneth's theorem in cohomology, in order to prove (3) it suffices to show that given cohomology classes ξ in $H^p(A, \mathbb{Z}/2)$ and η in $H^q(B, \mathbb{Z}/2)$, we have

$$\begin{aligned} &< (a, b)^*(\xi \times \eta) \cup w_{i_1}(M) \cup \ldots \cup w_{i_r}(M), [M] > = \\ &\qquad\qquad < (\alpha, \beta)^*(\xi \times \eta) \cup w_{i_1}(Y) \cup \ldots \cup w_{i_r}(Y), [Y] > \end{aligned}$$

for all nonnegative integers $i_1, \ldots, i_r$ satisfying $i_1 + \cdots + i_r = m - (p + q)$. Since $(a, b)^*(\xi \times \eta) = a^*(\xi) \cup b^*(\eta)$ and $(\alpha, \beta)^*(\xi \times \eta) = \alpha^*(\xi) \cup \beta^*(\eta)$, the last displayed equality is equivalent to

$$\begin{aligned} &< a^*(\xi) \cup b^*(\eta) \cup w_{i_1}(M) \cup \ldots \cup w_{i_r}(M), [M] > = \\ &\qquad\qquad < \alpha^*(\xi) \cup \beta^*(\eta) \cup w_{i_1}(Y) \cup \ldots \cup w_{i_r}(Y), [Y] > . \end{aligned} \tag{4}$$

We shall prove (4) in seven steps.

Step 1. Suppose that either $1 \leq p \leq k-1$ or q is not a multiple of $m-k$. In the former case $a^*(\xi) = 0$ and $\alpha^*(\xi) = 0$, while in the latter case $\eta = 0$. Thus (4) holds.

Step 2. Suppose $q = 0$. By construction, $b(M) \subseteq B_0$ and $\beta(M) \subseteq B_0$. Hence either $b^*(\eta) = \beta^*(\eta) = 0$ or $b^*(\eta) = \beta^*(\eta) = 1$. In the former case (4) is obvious, whereas in the latter case (4) holds in view of (1) and Theorem 4.1.

Step 3. Suppose $p \geq k+1$ and $q = \ell(m-k)$, $\ell \geq 1$. Then $p+q \geq m+1$ and hence (4) is void.

Step 4. Suppose $p = k$ and $q = \ell(m-k)$, $\ell \geq 1$. If $\ell \geq 2$, then $p+q \geq m+1$ and (4) is void. Suppose then $\ell = 1$, that is, $q = m-k$. Since $a^*(\xi)$ is in G and, by (2), $b^*(\eta)$ is in Γ, we have $< a^*(\xi) \cup b^*(\eta), [M] >= 0$. On the other hand, β is a constant map and $m-k > 0$, which implies $\beta^*(\eta) = 0$. Thus (4) holds also for $\ell = 1$.

Step 5. Suppose $p = 0$ and $q = m-k$. Since β is a constant map and $m-k > 0$, we have $\beta^*(\eta) = 0$, and hence the right-hand-side of (4) is equal to 0. Since A is connected, we have either $a^*(\xi) = 0$ or $a^*(\xi) = 1$. In the former case, the left-hand-side of (4) is equal to 0, and hence (4) is satisfied. In the latter case, $a^*(\xi) \cup b^*(\eta) = b^*(\eta)$ is in Γ (cf. (2)), which implies that the left-hand-side of (4) is equal to 0 since $w_{i_1}(M) \cup \ldots \cup w_{i_r}(M)$ belongs to G for all nonnegative integers $i_1, \ldots, i_r$ with $i_1 + \cdots + i_r = k$. Thus (4) holds in all the cases considered in this step.

Step 6. Suppose $p = 0$ and $q = \ell(m-k)$, $\ell \geq 2$, $\ell \neq m/(m-k)$. If $\ell < m/(m-k)$, then $w_{i_1}(M) \cup \ldots \cup w_{i_r}(M) = 0$ (cf. the hypothesis of Theorem 1.6) and since Y is diffeomorphic to M, we also have $w_{i_1}(Y) \cup \ldots \cup w_{i_r}(Y) = 0$ for $i_1 + \cdots + i_r = m-q$. Thus in this case (4) holds. If $\ell > m/(m-k)$, then $q \geq m+1$ and (4) is void.

Step 7. Suppose $p = 0$, m is divisible by $m-k$, and $q = m = h(m-k)$. Since A is connected we have either $\xi = 0$ or $\xi = 1$. In the former case (4) trivially holds. Suppose then that $\xi = 1$, that is, $a^*(\xi) = 1$ and $\alpha^*(\xi) = 1$. Since β is a constant map and $m-k > 0$, we have $\beta^*(\eta) = 0$, and therefore in the case under consideration, (4) is equivalent to

$$< b^*(\eta), [M] >= 0. \tag{5}$$

Note that each connected component of B is diffeomorphic to the d-fold product $S^{m-k} \times \dots \times S^{m-k}$. It follows that $H^{h(m-k)}(B, \mathbb{Z}/2)$ is generated by cohomology classes of the form $\eta_1 \cup \ldots \cup \eta_h$, where $\eta_1, \ldots, \eta_h$ are in $H^{m-k}(B, \mathbb{Z}/2)$. Thus it suffices to prove (5) for $\eta = \eta_1 \cup \ldots \cup \eta_h$. We have $b^*(\eta) = b^*(\eta_1) \cup \ldots \cup b^*(\eta_h)$, where $b^*(\eta_1), \ldots, b^*(\eta_h)$ are in Γ (cf. (2)). Thus the proof of (5) will be complete if we show that given $v_1, \ldots, v_h$ in Γ, we get $v_1 \cup \ldots \cup v_h = 0$. This certainly holds if $\dim_{\mathbb{Z}/2} \Gamma \leq h - 1$. We shall now prove the last inequality. Since

$$H^k(M, \mathbb{Z}/2) \times H^{m-k}(M, \mathbb{Z}/2) \to \mathbb{Z}/2, \ (u, v) \to < u \cup v, [M] >$$

is a dual pairing [17, p. 300, Proposition 8.13] and Γ is dual to G with respect to this pairing, we obtain

$$\dim_{\mathbb{Z}/2} \Gamma = \dim_{\mathbb{Z}/2} H^k(M, \mathbb{Z}/2) - \dim_{\mathbb{Z}/2} G.$$

By assumption,

$$\dim_{\mathbb{Z}/2} G \geq \dim_{\mathbb{Z}/2} H^k(M, \mathbb{Z}/2) - 1 + h.$$

Hence $\dim_{\mathbb{Z}/2} \Gamma \leq h - 1$ as required and the proof of Step 7 is complete.

We have considered all cases and therefore (4) is proved. In other words, (3) holds and the bordism class of $(M, (a, b))$ in $\mathcal{N}_*(A \times B)$ is algebraic. By Theorem 4.2, there exist a smooth embedding $e : M \to \mathbb{R}^n$, a Zariski closed subvariety X of $\mathbb{R}^n$, and a regular map $(f, g) : X \to A \times B$ such that $e(M) = X$, e is close to the inclusion map $M \hookrightarrow \mathbb{R}^n$, and $(f, g) \circ \bar{e} = (f \circ \bar{e}, \, g \circ \bar{e})$ is homotopic to (a, b) (Theorem 4.2 asserts also that $(f, g) \circ \bar{e}$ is close to (a, b), but this will not be used). In particular, the induced homomorphisms satisfy

$$a^* = (f \circ \bar{e})^* = \bar{e}^* \circ f^* : H^k(A, \mathbb{Z}/2) \to H^k(M), \mathbb{Z}/2),$$

$$b^* = (g \circ \bar{e})^* = \bar{e}^* \circ g^* : H^{m-k}(B, \mathbb{Z}/2) \to H^{m-k}(M, \mathbb{Z}/2).$$

Keeping in mind $a^*(H^k(A, \mathbb{Z}/2)) = G$, we obtain $G = \bar{e}^*(f^*(H^k(A, \mathbb{Z}/2)))$. Since f is a regular map, we have $f^*(H^k_{\mathrm{alg}}(A, \mathbb{Z}/2)) \subseteq H^k_{\mathrm{alg}}(X, \mathbb{Z}/2)$. Hence using $H^k_{\mathrm{alg}}(A, \mathbb{Z}/2) = H^k(A, \mathbb{Z}/2)$, we get

$$G \subseteq \bar{e}^*(H^k_{\mathrm{alg}}(X, \mathbb{Z}/2)). \tag{6}$$

In view of (2),

$$\Gamma = b^*(H^{m-k}(B, \mathbb{Z}/2)) = \bar{e}^*(g^*(H^{m-k}(B, \mathbb{Z}/2))).$$

Furthermore, the first part of (2) implies $g(X) \subseteq B_0$, and hence Example 4.5 yields

$$g^*(H^{m-k}(B, \mathbb{Z}/2)) = g^*(\mathrm{Alg}^{m-k}(B)).$$

Since g is a regular map, we have $g^*(\mathrm{Alg}^{m-k}(B)) \subseteq \mathrm{Alg}^{m-k}(X)$. It follows that

$$\Gamma \subseteq \bar{e}^*(\mathrm{Alg}^{m-k}(X)). \tag{7}$$

We can now easily complete the proof. Since $\bar{e} : M \to X$ is a diffeomorphism, Theorem 4.4 implies

$$< \bar{e}^*(w) \cup \bar{e}^*(z), [M] > = 0$$

for all w in $H^k_{\mathrm{alg}}(X, \mathbb{Z}/2)$ and all z in $\mathrm{Alg}^{m-k}(X)$. Hence (6), (7), and the fact that Γ is dual to G yield $G = \bar{e}^*(H^k_{\mathrm{alg}}(X, \mathbb{Z}/2))$ (and, incidentally, also $\Gamma = \bar{e}^*(\mathrm{Alg}^{m-k}(X))$). This means that M admits an algebraic approximation of type G; in particular, X is an algebraic model of M of type G, and hence (a) holds. □

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