On pairs of locally flat 2-spheres in simply connected 4-manifolds

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ABSTRACT. We prove a necessary and sufficient condition for the representation of a pair of mutually orthogonal primitive classes by disjoint locally flat spheres with nice complement. We show the necessity of the condition following V. Rochlin ([R]) and the sufficiency following the basic outline and extending ideas of Hambleton-Kreck and Lee-Wilczyński ([H-K], [L-W1], [L-W2]. The condition is an inequality which depends on homological data.

1. INTRODUCTION

We study the problem of representing a pair $\alpha_1, \alpha_2 \in H_2(X^4)$ of 2-homology classes in a simply connected 4-manifold X^4 by disjoint topological locally flat spheres. When $\alpha_i \;\; i = 1, 2$ are disjointly representable by (smooth or locally flat) spheres we draw immediately two conclusions: $\alpha_1 \pm \alpha_2$ are representable by (smooth or locally flat) spheres and $\alpha_1 \cdot \alpha_2 = 0$. So we define:

Definition 1.1. *If $\alpha_i \;\; i = 1, 2$ are such that: $\alpha_1, \;\; \alpha_2, \; \alpha_1 \pm \alpha_2$ are (smoothly or topologically) representable by spheres and $\alpha_1 \cdot \alpha_2 = 0$, then we say that they satisfy the* ***obvious (smooth or topological) conditions****.*

Assume that $\alpha_1, \alpha_2 \in H_2(X^4)$ are linearly independent and that they cannot be completed into an integral basis because in this case our problem collapses to a triviality. Associated to such a pair there is an integer $d \geq 2$ which can be interpreted as the order of the torsion of $H_2(X^4)/ \prec \alpha_1, \alpha_2 \succ$, and a pair $u = (u_1, u_2)$ of units $\pmod d$ such that $u_1 u_2 \equiv 1 \bmod d$ which are defined as follows. Since each α_i is primitive there exist duals $\alpha_i^{'}$ (i.e. $\alpha_i \cdot \alpha_i^{'} = 1$). Then $u_1 \equiv \alpha_1^{'} \cdot \alpha_2 \bmod d$ and similarly for u_2. Our main theorem then is:

Theorem 1.2. *Suppose $\pi_1(X^4) = 1$. Let $\alpha_i \in H_2(X^4)$ i=1,2 be two primitive classes which satisfy the obvious topological conditions. Then the following is a* ***necessary and sufficient*** *condition for disjointly representing them by a simply embedded pair of locally flat spheres:*

$$b_2(X^4) \geq \max_{1 \leq l \leq d-1} |\sigma(X^4) - \frac{2}{d^2}(l(d-l)\alpha_1^2 + \bar{l}(d-\bar{l}))\alpha_2^2)| + 2 \tag{1}$$

where $\bar{l} = (\alpha_1 \cdot \alpha_2^{'})l \;\; mod \;\; d$ and $1 \leq \bar{l} \leq d-1$.

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The organization of this paper is then as follows. In Section 1.2 we prove that the obvious conditions suffice stably to produce a pair of disjoint spheres which is simple (ie the fundamental group of the complement is abelian). In Section 1.3 we find a further obstruction. This obstruction assumes the form of the inequality in theorem 1.2 above. Its proof involves looking at a certain branched cover[1] and computing various homological data as in [R]. In Section 1.4 we formalize the relationship between simple embeddings of spheres and cyclic group actions. In Section 1.5 we study the ZG-module structure of the second homology group of a manifold which supports a cyclic (G) group action with fixed point set two disjoint topological spheres. In Section 1.6 we discuss the topological realization of such modules. In Section 1.7 we show how to split hyperbolic summands from such modules and in Section 1.8 we put together the proof of the main theorem. I would like to thank my thesis advisor Allan Edmonds for his support, Darek Wilczyński for a usefull exchange of email messages regarding splittings of modules and Ian Hambleton for his generous help in understanding part of his work which relates to this paper and in resolving various issues that arose in relation to it.

2. Stable Embeddings

In this section we prove that the (smooth or topological) obvious conditions suffice stably to produce (smooth or topological) embeddings of spheres. Furthermore the resulting embeddings can be made simple. Before we state the stable theorem we list below a number of facts needed in its proof. This will allow for a more efficient right-up of the proof.

FACT 1 Norman's Trick: Let A be an embedded sphere in some 4-manifold, B any surface and S a sphere embedded with trivial normal bundle such that S∩B is a singleton and S∩A=∅. Then by the Norman trick A#S can be taken in such a way that (A#S)∩B has one less element than A∩B. Using parallel copies of S (as many as the number n of points in A∩B) and iterating the process we can take A$\#_n$S disjoint from B. If A∩B=$\{x_i^+ : i=1,...,k\}\cup\{x_i^- : i=1,...,l\}$ and B∩S=$\{x^-\}$ then A$\#_{k+l}$S can be taken disjoint from B and on homology we have[A$\#_{k+l=n}$S]=[A]+(k-l)[S]. So, if A·B=0 then k=l and hence [A$\#_n$S]=[A].

FACT 2 : ([W2]) Let (X,λ) be an integral lattice with a unimodular quadratic form λ. Then given that the signature $\sigma(\lambda)$ and the rank r(X) satisfy $|\sigma(\lambda)| \leq r(X) - 4$ the orthogonal group of λ operates transitively on primitive elements of a given square.

FACT 3: ([W3]) If M^4 is indefinite then every automorphism of the quadratic form of $M^4\#S^2 \times S^2$ is induced by an autodiffeomorphism of $M^4\#S^2 \times S^2$.

FACT 4: Up to stabilization the hypothesis of both facts 2 and 3 can be satisfied.

FACT 5: Every primitive ordinary element of $H_2(M^4\#S^2 \times S^2)$ (as in fact 3) above is smoothly S^2-representable.

FACT 6: The effect of surgering a 4-manifold along a nullhomotopic circle is easily seen to be that of taking connected sum with a copy of $S^2 \times S^2$ if the framing

[1] P. Gilmer in his thesis [G] proves, in the case d is a prime power, a similar inequality, as a necessary condition, using different methods. The inequalities are equivalent where they overlap but his is more general in that it makes no simple embedding assumption. It is more restrictive in that it needs d to be a prime power.

is chosen properly. We have:

$$H_2(X^4 \# S^2 \times S^2) = H_2(X^4) \oplus H_2(S^2 \times S^2)$$

We say that a homology class $\alpha_1 \in H_2(X^4)$ is stably S^2- representable if there is an embedding $f : S^2 \to X^4 \#_s S^2 \times S^2$ for some s such that : $H_2(f)(S^2) = \alpha_1 \oplus 0 \in H_2(X^4) \oplus_s H_2(S^2 \times S^2)$. Now here is a brief description of how surgery on a circle has the effect mentioned above. Let T be a tubular neighborhood of a circle embedded in X^4. We delete it and we see that: $\partial(X^4 - T) = S^1 \times S^2$. We then glue back in $D^2 \times S^2$. Pick a $D^2 \times q \subset D^2 \times S^2$ cap it off with a disk in the interior of $X^4 - T$ and then look at the wedge of two spheres: The latter one (call the homology class it represents ζ_1) and some $p \times S^2 \subset D^2 \times S^2$(call its homology class ζ_2). Take an open regular neighborhood of the wedge. Its closure has boundary a three sphere and the tubular neighborhood is actually a punctured $S^2 \times S^2$. Now let's assume that we have an embedded surface F which represents a given homology class, say $\gamma \in H_2(X^4)$. When doing surgery on a circle disjoint from F the latter represents in general some homology class $\gamma \oplus \delta \in H_2(X^4 \# S^2 \times S^2) = H_2(X^4) \oplus H_2(S^2 \times S^2)$. We would like to see what the extra summand is. The surface F might only possibly have a non-zero algebraic intersection number with the sphere representing ζ_1. Then suppose that $[F] = \gamma \oplus (k\zeta_1 + l\zeta_2)$. We have: $[F] \cdot \zeta_1 = \gamma \cdot \zeta_1 + (k\zeta_1 + l\zeta_2) \cdot \zeta_1 = l$ and $0 = [F] \cdot \zeta_2 = \gamma \cdot \zeta_2 + (k\zeta_1 + l\zeta_2) \cdot \zeta_2 = k$. So we have $[F] = \gamma \oplus ([F] \cdot \zeta_1)\zeta_2$. Now the possible intersection $([F] \cdot \zeta_1)$ depends on the choice of the capping-off disk in the interior of $X^4 - T$ mentioned above. If it is chosen so as to have zero algebraic intersection number with F then F represents $\gamma \oplus 0 \in H_2(X^4 \# S^2 \times S^2) = H_2(X^4) \oplus H_2(S^2 \times S^2)$. Notice that the latter can always be achieved by spinning ([F-K]) if the surgery circle is a push-off of a circle on F. Also if the surgery circle is a Whitney circle consisting of two arcs on two surfaces then both surfaces still represent the same homology class in the above sense.

FACT 7: Let $A = \cup_i A_i \hookrightarrow X_4$ be disjoint embeddings of surfaces. The commutator subgroup of $\pi_1(X^4 - A)$ can be stably surgered in such a way that the embedded surfaces still represent the same homology classes in the sense of Fact 6 above. We need only show that given any nullhomologous circle $S \hookrightarrow X^4 - A$ one can find a disk $D \hookrightarrow X^4$ with $\partial D = S$ such that $[D] \cdot [A_i] = 0$ for all i. The latter will ensure that the spheres still represent the "same" homology classes. Here is how we find D:
Let $D \hookrightarrow X^4$ be some disk with $\partial D = S$. Since S is nullhomologous in $X^4 - A$ the algebraic intersection number of D with each of A_i is well defined because S also bounds a surface embedded in $X^4 - A$ hence the union of this surface and the disk produce an immersed closed surface in X^4. Let α be the homology classes this surface represents. Let $-A$ be an immersed sphere representing $-\alpha$. Trade D for $D' = D\#(-A)$. Observe now that $[D'] \cdot [A_i] = 0$. Modify the immersed D' into an embedded D'' which maintains the property above using finger moves. This shows as in Fact 6 that after doing surgery on S none of the $[A_i]$'s pick up any extra summands.

Theorem 2.1. *Let X^4 be a 1-connected, closed, compact, (smooth or topological) 4-manifold. Suppose α_1, $\alpha_2 \in H_2(X^4)$ are primitive and they satisfy: α_1,α_2,$\alpha_1 \pm \alpha_2$ are smoothly (topologically) stably S^2-representable and $\alpha_1 \cdot \alpha_2 = 0$. Then α_1, α_2*

are stably (smoothly or topologically) disjointly S^2-representable. Furthermore the fundamental group of the complement of the embeddings can be made abelian.

CASE I: $\alpha_1 + \alpha_2$ is characteristic.
Let $A_1, A_2 \to X^4$ be smoothly embedded 2-spheres representing α_1,α_2 respectively. Since $\alpha_1 \cdot \alpha_2 = 0$ we have: $A_1 \cap A_2 = \{x_i^\pm : i = 1, ..., k\}$ where $x_i^\pm$ is $\pm$ - signed. Then we can find pairwise disjoint arcs b_i on A_2 from x_i^- to x_i^+. We then replace two small disks on A_1 around x_i^- *and* x_i^+ for all i by the linking annuli of the arcs b_i, thus obtaining a surface A_1' of genus k which is disjoint from A_2 and such that $[A_1'] = [A_1] = \alpha_1$. Then $[A_1' \# A_2] = \alpha_1 + \alpha_2$ is characteristic and S^2-representable and so by theorem 1 of [Ke-M]: $\alpha^2 \equiv \sigma(X^4) + 8KS(X^4) \pmod 2$ and therefore by theorem 2 of [F-K] we can stably surger $A' \# A_2$ to a sphere. We can obviously choose the surgery curves on $(A_1' \# A_2) - pt$. So we have A_1' surgered to A_1'' a 2-sphere such that $[A_1''] = \alpha_1$ and $A_1'' \cap A_2 = \emptyset$. We ensure $[A_1''] = \alpha_1$ by choosing the cup-off disk (cf. Fact 6 above) appropriately. What makes such a choice possible is that α_2 is primitive.

CASE II: $\alpha_1 + \alpha_2$ is ordinary.

subcase II(a): X^4 is even. (i.e. for all x $\in H_2(X^4)$ x·x $\in$ 2Z).

Let $\alpha_1 + \alpha_2 = q\gamma$, where q$\in Z$ and $\gamma \in H_2(X^4)$ is primitive. Since $\alpha_1 + \alpha_2$ is ordinary and X^4 is even it follows that q is odd and γ is ordinary. In some X_k^4 by FACTS 1 and 2 we can think of γ as pu+v for some pair u,v $\in H_2(M_k^4)$ of hyperbolic elements. (i.e. $u^2 = v^2 = 0$ and u·$v = 1$). Since u is primitive ($u \cdot v = 1$) and ordinary ($u{\cdot}v = 1 \neq v^2 = 0$ mod 2) by FACT 3 it is S^2-representable in X_k^4. Let U$\to X_k^4$ be a an embedded sphere representing it. Let $A_1, A_2 \to X_k^4$ be embedded 2-spheres representing α_1 and α_2 respectively. We have $u \cdot (\alpha_1 + \alpha_2) = q$ *odd and* $u^2 = 0$. Since $\alpha_1 \cdot \alpha_2$=0 just as in CASE I, $A_1 \cap A_2 = \{x_i^\pm : i = 1, ..., l\}$. Let a_i (resp.b_i) be pairwise disjoint smooth curves on A_1 (resp. A_2) with end-points $x_i^\pm$. Let $c_i = a_i \cap b_i$ denote the loops thus formed (Whitney circles). Doing framed surgery along curves nearby every c_i has the effect of taking connected sums with l-many copies of $S^2 \times S^2$. (since $\pi_1(X^4) = \{1\}$, and also provides disks D_i such that $D_i \cap (A_1 \cup A_2) = c_i$ (Whitney disks). For each of these l-many pairs $x_i^\pm$ of intersections there are two obstructions to applying the Whitney trick in order to eliminate them. One is the obstruction e_i to extending the obvious normal vectorfield on c_i to D_i and the other is the algebraic intersection number of D_i with $A_1 \cup A_2$. If e_i is even we can pass to: $(X_k^4, D_i)\#(S^2 \times S^2, S)$ where S$\to S^2 \times S^2$ is a sphere: [S]=$(-e_i/2, 1) \in H_2(S^2 \times S^2)$ and get $D_i' = D_i \# S$ with $e_i' = 0$ and $D_i' \cap (A_1 \cup A_2) = \emptyset$. Now we can do the Whitney trick. In case some e_i is odd we can change the original disk so that the framing becomes even and there are no intersections of the new disk with $A_1 \cup A_2$ as follows: Let $D_i' = D_i \# U$. Then $e_i' = e_i$ (because $U \cdot U = 0$) and $d_i' = d_i + U \cdot (\alpha_1 + \alpha_2) = d$. Obviously $d_i' = d'^{A_1}_i + d'^{A_2}_i$. Since d_i' is odd we can assume w.l.o.g. that $d'^{A_1}_i$ is even and $d''^{A_2}_i$ is odd. Spin D_i' around c_i, $|d'^{A_1}_i|$-many times on a_i and $|d'^{A_2}_i|$-many times on b_i so that we get: D_i'' with $e_i'' = e_i' \pm d'^{A_1}_i \pm d'^{A_2}_i$ even and $d''^{A_1}_i = 0$ and $d''^{A_2}_i = 0$. We have now achieved that e_i'' is even but we did that at the expense of introducing intersections of D_i'' with possibly both A_1 and A_2. Now stabilize once more into $(X_{k+1}^4, D_i'')\#(S^2 \times S^2, S^2 \times *)$ to change D_i'' to $D_i''' = D_i'' \# S^2 \times *$ with $e_i''' = e_i''$

and $d_i^{'''} = d_i^{''}$. Then $[* \times S^2]^2 = 0$ and $* \times S^2 \cap D_i^{'''} = \{pt\}$. So using the Norman trick as in observation 1 we eliminate $\mathrm{int}(D_i^{'''}) \cap (A_1 \cap A_2)$, by replacing A_1, A_2 by $A_1 \# \pm (* \times S^2)$ and $A_2 \# \pm (* \times S^2)$, respectively where obviously $[A_1] = [A_1']$ and $[A_2] = [A_2']$. This completes subcase II(a).

subcase II(b): X^4 is odd. (i.e. $\exists x \in H_2(X^4) : x^2 \notin 2Z$)
Since $\partial X^4 = \emptyset$, its intersection form is unimodular. Since X^4 is odd, by stabilizing, it also becomes odd and indefinite. Hence it decomposes as a direct sum:$\lambda = \oplus_p(1) \oplus_q (-1)$. (See [Ki] pg. 25 Theorem 3.2 for a proof). So there exists a basis γ_1,...,γ_n of $H_2(X^4)$ such that:$\gamma_i \cdot \gamma_j = \pm\delta_{ij}$. Let $\alpha_1 + \alpha_2 = \sum_{i=1}^{n} m_i\gamma_i$. Since $\alpha_1 + \alpha_2$ is ordinary andX^4 is odd m_i is even for some i. Now γ_i is primitive ($\gamma^2 = 1$) and ordinary $\gamma_i \cdot \gamma_j$=0$\neq \gamma_i^2 = \pm 1 \mathrm{mod}\ 2$ $j \neq i$). So γ_i is S^2-representable by Fact 3. Let $C_i \to X_k^4$ be a smoothly embedded sphere representing γ_i. Let A_1, A_2, a_i,b_i,c_i, D_i,e_i,$d_i = d_i^{A_1} + d_i^{A_2}$ be as in subcase II(a). When e_i is even we proceed as before. Suppose for D_1,...,D_m , e_1,...,e_m are all odd. Let C_i^1,...,C_i^m be parallel copies of C_i any two of which intersect at a point which is common only to those two. Replace D_j by $D_j \# C_i^j$ j=1,...,m; where $e'_j = e_j \pm 1$ is now even and $d'_j = d_j + m_i = m_i$ is even. Spin appropriately as before to get D''_j with e''_j even and $d''^{A_1}_j = 0 = d''^{A_2}_j$ j=1,...,m. Now the disks D''_j intersect pairwise at a point. Use finger moves to eliminate these points. (Each such point gives a pair of intersection points of some of the disks with A_1 or A_2-we can choose-of opposite sign. But$d''^{A_1}_j$ and $d''^{A_2}_j$ do not change. Now use the Norman trick as before. This completes the proof of subcase II(b). That the spheres we get still represent the same homology classes is easy to see and we need only appeal to FACT 6. □

3. An Obstruction

Let X^4 be a 1-connected, compact, orientable 4-manifold without boundary. Let $\alpha_i \in H_2(X^4)$ i=1,2 be two linearly independent homology classes which are primitive. Suppose there exist topological embeddings $A_i \hookrightarrow X^4$ i=1,2 of disjoint spheres such that $[A_i] = \alpha_i$. Let $\nu_i \hookrightarrow X^4$ be their tubular neighborhoods and $\nu = \cup_i\nu_i$. Let e_i i=1,...,n be a basis of $H_2(X^4)$. Also let $\alpha_{ij}^{\perp}$ j=1,...,n- 1 be a basis of $\{\alpha_i\}^{\perp} \subset H_2(X^4)$ i=1,2. Then we have:

Proposition 3.1. *The group $H_1(X^4 - \nu)$ is cyclic of order d given in the following three alternative ways:*

3.2.

$$d = \gcd_{j\in\{1,\dots,n-1\}} (\alpha_{1j}^{\perp} \cdot \alpha_2) = \gcd_{j\in\{1,\dots,n-1\}} (\alpha_{2j}^{\perp} \cdot \alpha_1) =$$

$$\gcd_{1\leq k\neq l\leq n} \begin{vmatrix} \alpha_1 \cdot e_k & \alpha_1 \cdot e_l \\ \alpha_2 \cdot e_k & \alpha_2 \cdot e_l \end{vmatrix} = |Tor(H_2(X^4)/ \prec \alpha_1, \alpha_2 \succ |$$

Furthermore the integers $\alpha_1' \cdot \alpha_2$ and $\alpha_1 \cdot \alpha_2'$ are well defined modulo d and ,when $d \geq 2$, $(\alpha_1' \cdot \alpha_2)(\alpha_1 \cdot \alpha_2') \equiv 1 \bmod d$.

Proof: The proof easily follows by looking at the exact sequences below. We only should point out that the generators of $H_1(X^4-\nu)=Z_d$ are the fiber circles of the tubular neighborhoods of the spheres.

3.3. $H_2(X^4-\nu)\rightarrowtail H_2(X^4-\nu_1) \longrightarrow H_2(X^4-\nu_1, X^4-\nu) \twoheadrightarrow H_1(X^4-\nu)$

3.4. $H_2(X^4-\nu)\rightarrowtail H_2(X^4-\nu_2) \longrightarrow H_2(X^4-\nu_2, X^4-\nu) \twoheadrightarrow H_1(X^4-\nu)$

3.5. $H_2(X^4-\nu)\rightarrowtail H_2(X^4) \longrightarrow H_2(X^4, X^4-\nu) \twoheadrightarrow H_1(X^4-\nu)$

3.6. $H_2(\nu)\rightarrowtail H_2(X^4) \twoheadrightarrow H_2(X^4,\nu)$

Where one can easily see by excision, homotopy, Alexander duality and dual universal coefficient theorem that $H_2(X^4-\nu_1, X^4-\nu) = Hom_Z(H_2(A_2)\to Z) = Z$ and $H_2(X^4-\nu_2, X^4-\nu) = Hom_Z(H_2(A_1)\to Z) = Z$ and $H_2(X^4, X^4-\nu) = Hom_Z(H_2(A_1)\oplus H_2(A_2)\to Z) = Z\oplus Z$. □

Now assume that $\pi_1(X^4-\nu)$ is abelian. Corresponding to the isomorphism $\pi_1(X^4-\nu)\to Z_d$ that sends the generator given by the fiber circle of ν_1 to $1\in Z_d$, (hence the one given by the fiber circle of ν_2 to $\alpha_2'\cdot\alpha_1\in Z_d$), there is a d-fold cover of $X^4-\nu$ which can then be extended in the usual linear manner to a d-fold branched cover (M^4,π) of X^4 branched along ν such that $\pi_1(M_0 = M^4 - \pi^{-1}(\nu)) = 0$ (M_0 is the universal cover of $X^4-\nu$) and hence by Van-Kampen $\pi_1(M^4)=0$. Since both fiber circles are generators implies that the boundary Lens spaces are covered by two Lens spaces. We can easily compute $H_2(X^4-\nu) = Z^{b_2-2}$, $H_3(X^4-\nu) = H^1(X^4-\nu,\partial\nu) = Z$, $H_3(M_0)=Z$. Using the obvious equation of Euler characteristics: $\chi(M_0) = d\chi(X^4-\nu) = d(b_2-2)$ we can then easily compute $H_2(M_0) = Z^{d(b_2-2)}$, so that $H_2(M) = Z^{d(b_2-2)+2}$. Letting $\omega = e^{\frac{2\pi i}{d}}$ one then looks at E_r the ω^r-eigenspace of g_* as it acts on $H_2(M^4)\otimes\mathbb{C}$ $0\le r\le d-1$ where g is the generator of the Z_d-action on M^4. One easily checks then that the splitting $H_2(M^4)\otimes\mathbb{C} = E_0\oplus E_1\oplus\ldots\oplus E_{d-1}$ is orthogonal. Hence one then gets:

$$\sigma(M^4,g^s) = \sum_{r=0}^{d-1}\omega^{rs}\sigma(E_r),\quad \sigma(E_0)=\sigma(X^4)$$

Then one solves:

$$\sigma(E_r) = \sigma(X^4) + \frac{1}{d}\sum_{s=1}^{d-1}(\omega^{-rs}-1)\sigma(M^4,g^s)$$

By the G-signature theorem

$$\sigma(M^4,g^s) = \frac{\alpha_1^2}{d}\csc^2(\frac{\pi s}{d}) + \frac{\alpha_2^2}{d}\csc^2(\frac{u_2\pi s}{d})$$

where $u_2\equiv\alpha_2'\cdot\alpha_1 \bmod d$ is a unit and $\alpha_2'\cdot\alpha_2 = 1$. It then follows that (see p. 332 of [K]):

$$\sigma(E_r) = |\sigma(X^4) - \frac{2}{d^2}(j(d-j)\alpha_1^2 + \bar{j}(d-\bar{j}))\alpha_2^2)|$$

where $\bar{j} = (\alpha_1\cdot\alpha_2')j \bmod d$ and $1\le\bar{j}\le d-1$. One also computes that $rank_Z E_0 = b_2$ and $rank_Z E_i = b_2-2$ for $i=1\ldots d-1$.

Proposition 3.7. *The following is a necessary condition for the representation of two linearly independent, primitive homology classes by a simple pair of topological spheres:*

3.8. $$b_2(X^4) \geq \max_{1\leq j\leq d-1} |\sigma(X^4) - \frac{2}{d^2}(j(d-j)\alpha_1^2 + \bar{j}(d-\bar{j}))\alpha_2^2)| + 2$$

where $\bar{j} \equiv (\alpha_1 \cdot \alpha_2^{'})j \bmod d$, *and* $1 \leq \bar{j} \leq d-1$

Proof: Condition 3.8 simply states that $|\sigma(E_r)| \leq rank_Z(E_r)$ □

The proposition below is an easy corollary of Freedman's work and it can be regarded as the base case (d=1) to our main theorem. This is the easy case where necessary and sufficient conditions can be found. The rest of the paper should be thought of as a way to make Freedman's work apply in the more general setting with arbitrary d.

Proposition 3.9. *Let* $\alpha_i \in H_2(X^4)$ $i = 1, 2$ *satisfy the obvious topological conditions. Then a necessary and sufficient condition for their representation by a pair of disjoint topological spheres with simply connected complement is the existence of* $\alpha_i^{'} \in H_2(X^4)$ $i = 1, 2$ *such that:* $\alpha_i^{'} \cdot \alpha_j = \delta_{ij}$

Proof: The obvious conditions suffice to solve the problem stably. The existence of $\alpha_i^{'} \in H_2(X^4)$ $i = 1, 2$ such that $\alpha_i^{'} \cdot \alpha_j = \delta_{ij}$ implies d=1. Now apply Freedman's disk theorem (see for instance [F-Q] p. 85) with $\pi_1 = 0$. □

4. Group Actions

In the previous section we showed that given a pair of simply embedded spheres which represent primitive, linearly independent elements on homology one can construct a cover branched along the two spheres which then supports a cyclic group action whose fixed point set is the two spheres. We wish here to formalize this relationship. We show that our embedding problem can be translated in terms of group actions. We need some terminology before we state the proposition.

Definition 4.1. *A dyad* $D = (X^4, \{A_1, A_2\})$ *consists of a 1-connected, closed, compact 4- manifold* X^4 *and disjointly embedded locally flat spheres* $A_i \hookrightarrow X^4$ *i=1,2 representing primitive homology classes* α_i *with* $\pi_1(X^4 - A_1 \cup A_2)$ *abelian.*

Recall that associated to such a dyad there is an integer d given by 3.2 and a pair $u = (u_1, u_2)$, where $u_1 \equiv \alpha_1^{'} \cdot \alpha_2$, $u_2 \equiv \alpha_1 \cdot \alpha_2^{'}$ are multiplicative inverses mod d. For this reason we will decorate a dyad by writing $D_{d,u}$. On the other hand for a semifree, locally linear, cyclic group action (Z_d, M^4) with $M^G = S_1 \cup S_2$ two disjoint spheres representing non trivial primitive homology classes, $(G, \nu(S_1))$ and $(G, \nu(S_2))$ are related by a pair $u = (u_1, u_2)$ of multiplicative inverses mod d. (i.e. the choice of generator for which $(G, \nu(S_1))$ is rotation by $\frac{2\pi}{d}$ turns $(G, \nu(S_2))$ into rotation by $\frac{2\pi u_2}{d}$ etc). We call this *the pair of units associated to the action.*

We are now ready to state:

Proposition 4.2. *There is a one to one correspondence between isomorphism classes of dyads $D_{d,u}$ and isomorphism classes of semifree, locally linear, cyclic group actions (Z_d, M^4) with $\pi_1(M^4) = 0$, M^G a pair of spheres representing non-trivial, primitive homology classes with $\pi_1(M^4 - M^G) = 0$ and u as its associated pair of units.*

Proof: Starting with a dyad $D_{d,u}$ we already saw in the previous section how to get the desired action. Going the other direction if (Z_d, M^4) is a cyclic group action as in the statement of the theorem pass to the quotient to get the dyad. Notice that the fact that both classes in $H_2(X^4)$ are primitive makes both fiber circles of $\nu(A_1)$ and $\nu(A_2)$ generators and that induces two G-fixed spheres in M^4. □

5. The Z[G]-module structure of $H_2(M^4)$

Let $G = Z_d = \pi_1(X^4 - \nu)$, notation as in Section 3. We now wish to study the ZG-module structures of $H_2(M_0)$ and $H_2(M)$, where M^4 is ramified cover of X^4 branched along the core spheres of $\nu = \nu_1 \cup \nu_2$. (We assume from now on that $b_2(X) \geq 3$ because the representation problem is trivial for lower ranks) G acts on M_0 via deck transformations. We will show that stably $H_2(M_0) \cong \prec m, g-1 \succ$, where g is a generator of G and m is some non zero integer (stably isomorphic here simply means that for some integers k, l $H_2(M_0) \oplus ZG^k \cong \prec m, g-1 \succ \oplus ZG^l$). We will also show that $H_2(M^4) = Z \oplus Z \oplus F$ where F is a free ZG- module.

Let $\mathbf{C}_* = \mathbf{C}_*(M_0)$ be a G-cellular chain complex. As in [Wil] the ZG-modules of 2 and 3-cycles fit into the exact sequences:

5.1. $$\mathbf{Z}_2 \rightarrowtail \mathbf{C}_2 \longrightarrow \mathbf{C}_1 \longrightarrow \mathbf{C}_0 \twoheadrightarrow Z$$

5.2. $$\mathbf{Z}_3 \rightarrowtail \mathbf{C}_3 \xrightarrow{\partial_3} \mathbf{C}_2 \twoheadrightarrow coker(\partial_3)$$

It is also easy to check that the following sequences are exact:

5.3. $$\mathbf{Z}_2 \rightarrowtail H_2(M_0) \oplus \mathbf{C}_2 \twoheadrightarrow coker(\partial_3)$$

5.4. $$\mathbf{C}_4 \rightarrowtail \mathbf{Z}_3 \twoheadrightarrow H_3(M_0)$$

The sequence (5.4) splits. (This is due to the fact that $\mathbf{C}_4$ is a free ZG-module and projective modules over ZG are weakly injective (cf. [C-R] p. 778, 791). As in [Wil] we use the "loop-suspension" notation to encode in a brief shorthand notation the information contained in (5.1) and (5.2). For a detailed explanation as well as its origins see [W1]. From (5.1) we have $\mathbf{Z}_2 = \Omega^3 Z$ and from (5.2) we have $coker(\partial_3) = S^2 Z_3$. So (5.3) can now be written in the form:

5.5. $$\Omega^3 Z \rightarrowtail H_2(M_0) \oplus C_2 \twoheadrightarrow S^2 H_3(M_0)$$

But by virtue of the usual standard resolutions, $\Omega^3(Z)$ is represented by I the augmentation ideal of ZG and $S^2(H_3(M_0))$ is represented by Z. So (5.5) now limits the possibilities for the ZG-module $H_2(M_0)$. We compute[2]:

$$Ext^1_{\mathbf{ZG}}(S^2 Z, \Omega^3 \mathbf{Z}) = Ext^1_{\mathbf{Z}[G]}(Z, I) = H^1(G, I) = Z_d$$

[2]References for all the standard material on homological algebra, cohomology of groups and the like are [Rot], [B], [Ev], [H-S].

Hence, by virtue of 5.5, there are at most d-many possibilities for $H_2(M_0) \oplus C_2$. By the computation above we easily see that we only need to find representatives of $Ext^1_{ZG}(Z, I)$. The generator here is given by the standard resolution: $I \rightarrowtail ZG \twoheadrightarrow Z$. More precisely all other resolutions are obtained by picking homomorphisms $\phi : I \to I$ and then completing the diagram below:

$$\begin{tikzcd} I \arrow[r, tail] \arrow[d, "\phi"'] & ZG \arrow[r, two heads] & Z \arrow[d, "1_Z"] \\ I & & Z \end{tikzcd}$$

Furthermore the modules that fit in the middle of such resolutions are obtained as push-outs of:

$$\begin{tikzcd} I \arrow[r, tail] \arrow[d, "\phi"'] & ZG \\ I & \end{tikzcd}$$

for some homomorphism $\phi : I \to I$. Hence they look like: $I \oplus ZG$ modulo $(\phi(x), x)$ $x \in I$. In a conversation with R. Swan he pointed out to me that a complete set of these resolutions is given by the split one and the ones obtained by setting ϕ to be multiplication by $m \in Z - \{0\}$. Let A_m denote these modules. We will be simply writing A instead when there is no chance of confusion regarding m. The modules can be written as the ideals of ZG generated by m and g-1. These are easily seen to depend only on m mod d. All of these assertions are easy to check and we leave the proofs as an exercise for the reader. We therefore have:

Proposition 5.6. *The module $H_2(M_0)$ is stably isomorphic to one of $\prec m, g-1 \succ$ $m \neq 0$.*

We now begin dealing with the ZG-module structure of $H_2(M^4)$. We first prove:

Lemma 5.7. *$H_2(M^4) \cong Z \oplus Z \oplus P$, for some ZG-module P.*

Proof Let $\pi : M^4 \longrightarrow X^4$ be the branch map. Let π_* be the map it induces on second homology. Notice that there is a transfer map tr on homology going the opposite direction such that the composition $tr \circ \pi_*$ is multiplication by the norm element $N \in ZG$. There is a choice of dual $\alpha_1{}'$, (i.e. $\alpha_1' \cdot \alpha_1 = 1$), such that $\beta_1 = \alpha_1 - \alpha_1{}^2 \cdot \alpha_1{}' \in \prec \alpha_1 \succ^{\perp}$ is primitive. Let $a_i \in H_2(M^4)$ be such that $\pi_*(a_i) = \alpha_i$. Recall that for $a, b \in H_2(M^4)$, $a \cdot (Nb) = (Na) \cdot b = \pi_*(a) \cdot \pi_*(b)$. More generally $tr(\alpha) \cdot tr(\beta) = d\alpha \cdot \beta$. Then $a_1{}' = tr(\alpha_1{}')$ is a G-fixed dual to a_1 (in fact it can be represented by a G-invariant surface). Similarly if we let $\beta_1' \in H_2(X^4)$ be a dual to β_1 then $b_1' = tr(\beta_1')$ is a G-fixed dual to $b_1 = a_1 - a_1^2 a_1'$. There is a Z-splitting $H_2(M^4) = \prec a_1' \succ \oplus \prec a_1 \succ^{\perp}$. Notice that there is a Z-subspace B of $\prec a_1 \succ^{\perp}$ such that $\prec a_1 \succ^{\perp} = \prec b_1 \succ \oplus B$. Let b_i, $i = 2, .., s$ be a basis for B. Substituting, for each $i \geq 2$, every b_i by $c_i = b_i - (b_i \cdot b_1')b_1$ and letting C be the Z-span of the c_i's we get a new Z-splitting $H_2(M^4) = \prec a_1' \succ \oplus \prec b_1 \succ \oplus C$ such that $C \perp b_1', a_1$. Now the ZG-trivial summands are generated by a_1' and b_1 whereas $P = C$. This is seen as follows. Notice that a_1', b_1 are both G-fixed. Also $\prec a_1 \succ^{\perp}$ is G-invariant hence a_1' splits. Furthermore the existence of the G-fixed dual b_1' easily implies that C is invariant as well. $\square$

We now prepare the ground for showing that the summand P in the lemma above is projective. For that we only need to show that it is cohomologically trivial (see for example [B], Ch. VI, S. 8).

It is easy to verify that $H^i(Z_d, A_m)$ is a cyclic group of order $\gcd(m,d)$ for all i. To see this one first proves $H^i(Z_d, A) = H^{i+1}(Z_d, A)$ for all i, by looking at the long exact cohomology sequence obtained by hitting the defining resolution of A_m with the functor $H^*(Z_d, -)$. Then one simply computes odd or even cohomology.

To compute the even cohomology of A_m just observe $N \prec m, g-1 \succ = mN$ and $\prec m, g-1 \succ^G = (m/gcd(m,d)) \prec N \succ$.

Let p be a divisor of d and h a generator for Z_p. Then any ZG-module L can be thought of as a ZZ_p-module. This reduction module is usually denoted by L_{Z_p}. We simply write L_p. It is easy to see that:

$$\prec m, g-1 \succ_p = \prec m, h-1 \succ \oplus_{(d-p)/p} Z[Z_p]$$

In our particular case (ZG with G abelian) we have

$$H^i(Z_d, L) = \oplus_{p|d} H^i(Z_{p^n}, L)$$

the summation taken over all primes $p|d$ with p^n the maximum power of p dividing d. So now let p be any such prime.

Hitting $I \rightarrowtail A = \prec m, g-1 \succ \twoheadrightarrow Z$ with $H^*(Z_{p^n}, -)$, $H^*(Z_p, -)$ results in:

$$\begin{array}{ccccccc}
Z_{\gcd(m,p^n)} = H^2(Z_{p^n}, A_{p^n}) & \rightarrowtail & Z_{p^n} & \xrightarrow{p^{\gcd(m,p^n)}} & Z_{p^n} & \twoheadrightarrow & H^1(Z_{p^n}, A_{p^n}) = Z_{\gcd(m,p^n)} \\
res_1 \downarrow & & \downarrow res_2 & & \downarrow res_3 & & \downarrow res_4 \\
Z_{\gcd(m,p)} = H^2(Z_p, A_p) & \rightarrowtail & Z_p & \xrightarrow[a]{} & Z_p & \twoheadrightarrow & H^1(Z_p, A_p) = Z_{\gcd(m,p)}
\end{array}$$

Where res_2, res_3 are projections and the lower middle horizontal map a is zero or an isomorphism depending on whether p divides m or not. Furthermore if $p^n|m$ then res_1, res_4 are projections whereas if $p|m$ but $\gcd(m,d) < p^n$ then $res_1 = 0$ and res_4 is a projection. We gather these observations into the following:

Lemma 5.8. *The cohomology group $H^i(Z_d, A)$ is cyclic of order $\gcd(m,d)$, for $i \geq 1$. Moreover, for any prime p with n its maximum power dividing d, if $p|m$ but $\gcd(m,p^n) < p^n$, then the map $H^i(Z_{p^n}, A_{p^n}) \to H^i(Z_p, A_p)$ given by restriction of coefficients, is zero or surjective according as i is odd or even.*

We now set out to compute $H^i(Z_d, H_2(M^4)$. Since

$$H^i(Z_d, H_2(M^4) = \oplus_{p|d} H^i(Z_{p^n}, H_2(M^4)$$

where the direct sum is taken over all primes p such that p^n is the maximum power dividing d, it suffices to compute all such $H^i(Z_{p^n}, H_2(M^4)$. We hit the short exact sequence

$$H_2(M^4 - \nu) \rightarrowtail H_2(M^4) \twoheadrightarrow H_2(M^4, M^4 - \nu)$$

with $H^*(Z_{p^n}, -)$, $H^*(Z_p, -)$. This, taking into account that Edmonds' results in [Ed], imply $H^i(Z_p, H_2(M^4)) = H^i(Z_p, Z \oplus Z)$, results in the following commutative diagram with exact rows and vertical maps given by restriction:

$$\begin{tikzcd}
H^2(Z_{p^n},A) \arrow[r, tail] \arrow[d] & H^2(Z_{p^n},H_2(M)) \arrow[r] \arrow[d] & Z_{p^n} \oplus Z_{p^n} \arrow[r] \arrow[d] & H^3(Z_{p^n},A) \arrow[r, two heads] \arrow[d] & H^3(Z_{p^n},H_2(M)) \\
H^2(Z_p,A) \arrow[r, tail] & Z_p \oplus Z_p \arrow[r] & Z_p \oplus Z_p \arrow[r, two heads] & H^3(Z_p,A) &
\end{tikzcd}$$

A close inspection of this diagram easily yields:

Proposition 5.9. $H^i(Z_d, H_2(M^4)) = H^i(Z_d, Z \oplus Z)$

Proof: It suffices to show $H^i(Z_{p^n}, H_2(M^4)) = H^i(Z_{p^n}, Z \oplus Z)$ for any prime $p|d$. If p and m are relatively prime this is trivial since $H^i(Z_{p^n}, A) = H^i(Z_p, A) = 0$. If p^n divides m then $H^{odd}(Z_{p^n}, H_2(M)) = 0$. This easily follows by examining the right hand side of the diagram. Then by examining the top row alone and recalling that $kH^*(Z_k, *) = 0$ we see that $H^{even}(Z_{p^n}, H_2(M)) = Z_{p^n} \oplus Z_{p^n}$. Suppose now that $\gcd(m, p^n) = p^k$ with $1 \leq k \leq n-1$. Then in the diagram above the far left-hand side vertical map is zero whereas the far right hand side one is the projection. Examining the right-hand side of the diagram again we get $H^{odd}(Z_{p^n}, H_2(M)) = 0$. Now finish as before. □

Notice that now the cohomological triviality of P is evident and hence we have:

Proposition 5.10. *$H_2(M^4)$ is isomorphic to $Z \oplus Z \oplus P$ with P projective.*

Actually P is stably free as one can easily see and hence free as is always the case for modules over ZG.

Theorem 5.11. *$H_2(M^4)$ is of the form $Z \oplus Z \oplus Free$.*

Proof: We show that P is stably free. This is equivalent to showing that $P \otimes ZG/N$ is a stably free ZG/N-module (cf. [L-W1]). Tensoring:

$$H_2(\nu) \rightarrowtail H_2(M^4) \twoheadrightarrow H_2(M^4, \nu)$$

with ZG/N results in:

$$Z_d{}^2 \to Z_d{}^2 \oplus (P \otimes ZG/N) \to Z_{gcd(m,d)} \oplus ZG/N^{d(b_2-2)} \to 0$$

Since $P \otimes ZG/N$ is Z-torsion free the latter clearly implies that: $P \otimes ZG/N \equiv (ZG/N)^{d(b_2-2)}$ □

6. Realizing 2-pointed Hermitian Forms

In Section 1.4 we gave a translation of our embedding problem in terms of group actions with a certain fixed-point set. In Section 1.5 we identified the ZG-module structure of the second homology of such actions. We wish here to examine closely the relationship between the algebra arising in the two incarnations of the problem. We recall some terminology. Let $D_{d,u} = (X^4, \{A_1, A_2\})$ be a dyad as in Section 1.4 and let (M^4, Z_d) be its corresponding group action. Let $\lambda_M : H_2(M^4) \times H_2(M^4) \to Z$ be its intersection pairing. Define:

6.1. $$h(x, y) = \sum_{g \in G} \lambda_M(x, gy)g \in ZG$$

for all $x, y \in H_2(M^4)$.

Then $(H_2(M^4), h)$ is a hermitian ZG-module with respect to the obvious involution of ZG. Since λ_M is unimodular, h is easily seen to be non-singular (i.e. the adjoint h^* is bijective). We have seen in the previous section that for a suitable choice of basis $H_2(M^4) \cong Z \oplus Z \oplus P$ with P ZG-free. For a suitable choice of basis for $H_2(X^4) \cong Z \oplus Z \oplus P_0$ we have:

6.2. $$\pi_* = 1_Z \oplus d \oplus ((-) \otimes_{ZG} Z)$$

where $\pi_* : H_2(M^4) \to H_2(X^4)$ is induced by the projection map. By means of geometric considerations we can easily see that:

6.3. $$\lambda_M(x, Ny) = \lambda(\pi_*(x), \pi_*(y)), \quad h(x, y) \otimes_{ZG} Z = \lambda(\pi_*(x), \pi_*(y))$$

Remark 6.4. *The discussion above can be carried through for the 2-pointed hermitian form $(H_2(M^4), h, [M^G])$. Notice that the embedding of $\pi_*(H_2(M^4), h, [M^G])$ in $(H_2(X^4), \lambda, \{\alpha_1, \alpha_2\})$ completely determines (by 6.2) the latter up to isomorphism. Also notice that $[M^G] = \{a_1, a_2\}$ and $a_1 = (1, 0; 0)$, $a_2 = (u_1, \frac{-u_1\alpha_1^2}{d}; N) \in H_2(M^4) = Z \oplus Z \oplus P$.*

One can check that after suitable choices of basis the commutative diagram of ZG-modules below

$$\begin{tikzcd}
H_2(\nu(M^G)) \arrow[r, tail] \arrow[d] & H_2(M^4) \arrow[r, two heads] \arrow[d] & H_2(M^4, \nu(M^G)) \arrow[d] \\
H_2(\nu) \arrow[r, tail] & H_2(X^4) \arrow[r, two heads] & H_2(X^4, \nu)
\end{tikzcd}$$

is generated by:

$$\begin{tikzcd}[column sep=huge, row sep=huge]
Z \arrow[r, tail, "\begin{pmatrix} -u_1\alpha_1 2/d \\ N \end{pmatrix}"] \arrow[d, "1_Z"'] & Z \oplus ZG \arrow[r, two heads] \arrow[d, "d\oplus(-\otimes_{ZG} Z)"] & (Z \oplus ZG)/e \arrow[d, "\begin{pmatrix} d & u_1\alpha_1^2/d\epsilon(\ ) \\ 0 & \epsilon(\ ) \end{pmatrix}"] \\
Z \arrow[r, tail, "\begin{pmatrix} -u_1\alpha_1 2/d \\ d \end{pmatrix}"'] & Z \oplus Z \arrow[r, two heads, "\begin{pmatrix} 1 & u_1\alpha_1^2/d \\ 0 & 1 \end{pmatrix}"'] & Z \oplus Z_d
\end{tikzcd}$$

where $e = (-u_1\alpha_1^2/d, N)$, by adding

$$\begin{tikzcd}
(ZG)^{(b_2(X^4)-3)} \arrow[r, "1"] \arrow[d, "\epsilon"'] & (ZG)^{(b_2(X^4)-3)} \arrow[d, "\epsilon"] \\
Z^{(b_2(X^4)-3)} \arrow[r, "1"'] & Z^{(b_2(X^4)-3)}
\end{tikzcd}$$

to the right-hand side square and,

$$\begin{array}{ccc} Z & \xrightarrow{1} & Z \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle 1} \\ Z & \xrightarrow[1]{} & Z \end{array}$$

to the left-hand side one.

We now turn to one of the central ingredient of the proof of our main theorem. We need some definitions first. Let $A = (Z \oplus Z \oplus P, h, \{x,y\})$ be a 2-pointed hermitian module. Let $H(ZG^s) = (ZG^s, \oplus_s \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \{0,0\})$ For two such modules A_i we say that they are *stably equivalent* if for some s,r integers we have: $A_1 \oplus H(ZG^s) \cong A_2 \oplus H(ZG^s)$. A module A as above is said to be *realizable* (resp. *stably realizable*) if there exists (M^4, Z_d) such that $A = (H_2(M^4), h, [M^G])$ (resp. $A \oplus H(ZG^s) = (H_2(M^4), h, [M^G])$). We are now, in analogy to [L-W1], ready to state:

Theorem 6.5. *A 2-pointed module $(Z \oplus Z \oplus P, h, \{x,y\})$ is realizable iff it is stably realizable.*

The proof is easy, (one utilizes Freedman's Disk theorem with $\pi_1 = Z_d$) and it is the same as in [L-W1]. One only needs to show that stably realizable implies realizable. Suppose $A = (Z \oplus Z \oplus P, h, \{x,y\})$ is stably realizable. This means that for some positive integer s there exists a cyclic group action (M^4, Z_d) such that: $A \oplus H(ZG^s) \cong (H_2(M^4), h, [M^G])$. The s orthogonal ZG-hyperbolic summands form a subspace of $H_2(M - M^G)$. Now apply ([F-Q], p. 85, Theorem 5.1A) to $M/G - M^G/G$ with $\pi_1(M/G - M^G/G) = Z_d$. So one gets that M/G is homeomorphic to $Y^4 \#_s S^2 \times S^2$. Then (M^4, G) is easily seen to be homeomorphic to $(\tilde{Y}^4, G) \#_s (G \times S^2 \times S^2, G)$ where $\tilde{Y}^4$ is the d-fold branched cover of Y^4. The latter homeomorphism then induces a diagonal ZG-isomorphism on second homology.

7. Splitting 2-pointed Hermitian Forms

We present a splitting theorem about hermitian forms over ZG whose underlying module is of the form $Z^n \oplus P$ with P a ZG- projective module. The proof relies on [L-W2] as well as on [H-K] where a splitting theorem is proven when the underlying module is projective. The authors in [H-K] (Remark 3.9) remark that their proof can be adapted to work for modules of the kind we consider here. We focus and follow [L-W2] arranging things as follows. A careful reading of the proof of theorem 6.1 of [L-W2] reveals a proof of 7.9 below. (This statement over ZG/N can be used to reorganize the proof of their theorem). It then allows us to state theorem 7.21 (which includes their 6.1) under looser, in principle, conditions (all of them present in their proof however). Although it is possible to state our theorem in exact analogy to theirs, something which tailors it better for the topological application, we prefer to restate things taking advantage of 7.9 thus making the algebra, at least conceptually more approachable as well as achieving what seems to be a more general purely algebraic statement. We do everything below without mention of the two points since by Remark 6.4 the application of the results to the 2-pointed case is entirely obvious.

We now present a few facts that are either contained in the literature or are easy to see. Let M be a ZG-module. Define $M^G = \{x \in M : (g-1)x = 0\}$ and $M^N = \{x \in M : Nx = 0\}$ Notice the inclusions $IM \subseteq M^N$ and $NM \subseteq M^G$ with equality holding iff M is projective. Obviously $A = M/M^N$ is a module over $\Lambda_0 = ZG/I = Z$ and $B = M/M^G$ is a module over $\Lambda_1 = ZG/N$. Both A and B are easily seen to be Z-torsion free. Define $C = M/(M^G + M^N)$; it is easily seen to be a module over $\Lambda_d = ZG/ \prec N, g-1 \succ$. Obviously $\Lambda_d = Z_d$. For a projective module P notice that $P/P^N = P/IP \cong P \otimes_{ZG} Z$ and $P/P^G = P/NP \cong P \otimes_{ZG} (ZG/N)$. If $M = Z^n \oplus P$ then: $A = Z^n \oplus P/P^N \cong Z^n \oplus (P \otimes_{ZG} Z)$ $B = P/P^G \cong P \otimes_{ZG} (ZG/N)$, $C = P/(P^G + P^N) \cong P \otimes_{(ZG)} Z_d$. As in [Sw]:

$$
\begin{array}{ccc}
Z^n \oplus P & \longrightarrow & B \\
\downarrow & & \downarrow \\
A & \longrightarrow & C,
\end{array}
$$

is both a pull-back as well as a push-out. As in [H-R] $(Z^n \oplus P, h)$ can be obtained as a pull-back in this manner:

$$
\begin{array}{ccc}
(M = Z^n \oplus P, h) & \longrightarrow & (P_1, h_1) \\
\downarrow & & \downarrow \\
(Z^n \oplus P_0, h_0) & \longrightarrow & (P_d, h_d),
\end{array}
$$

where (P_1, h_1) is a non-singular Λ_1-module, $(Z^n \oplus P_0, h_0)$ is a non-degenerate Λ_0-module and (P_d, h_d) is a non- singular Λ_d-module. The following is then clear categorically:

Lemma 7.1. *Let P be a ZG-projective module and $(Z^n \oplus P, h)$ be a non- singular hermitian ZG-module. Suppose there exist splittings ϕ_0 over Z, ϕ_1 over Λ_1 and an isometry ψ_d satisfying* 7.5 *below:*

7.2. $$\phi_0 : (Z^n \oplus P_0, h_0) \cong (Z^n \oplus P_0', h_0') \oplus H(Z^s)$$

7.3. $$\phi_1 : (P_1, h_1) \cong (P_1', h_1') \oplus H(\Lambda_1^s),$$

7.4. $$\psi_d : (P_1', h_1') \otimes_{\Lambda_1} Z_d \cong (P_0', h_0'/P_0') \otimes_Z Z_d,$$

7.5. $$(\phi_0/P_0) \otimes_Z Z_d = (\psi_d \oplus 1_{H(Z_d^s)}) \circ (\phi_1 \otimes_{\Lambda_1} Z_d)$$

Then there exist isometries τ and τ' over Z satisfying 7.8 *below:*

7.6. $$\tau : (Z^n \oplus P, h) \cong (Z^n \oplus P', h') \oplus H(Z^s)$$

7.7. $$\tau' : (Z^n \oplus P', h') \otimes_{ZG} Z \cong (Z^n \oplus P_0', h_0'),$$

7.8. $$\phi_0 = (\tau' \oplus 1_{H(Z^s)}) \circ (\tau \otimes_{ZG} Z)$$

Let $\Gamma = \prod_{1<n|d} Z[\zeta_n]$, $\zeta_n = e^{\frac{2\pi i}{n}}$, be the usual maximal order in $\Lambda_1 \otimes_Z Q$ containing Λ_1. The involution on Λ_1 extends to the obvious involution on Γ. The following is a pullback diagram of rings with involutions

$$\begin{tikzcd} \Lambda_1 \arrow[r] \arrow[d] & \Gamma \arrow[d] \\ \widehat{\Lambda_1} \arrow[r] & \widehat{\Gamma} \end{tikzcd}$$

where $\widehat{\Lambda_1} = \Lambda_1 \otimes_Z \widehat{Z}$, $\widehat{\Gamma} = \Gamma \otimes_Z \widehat{Z}$, and $\widehat{Z} = \prod_{q|d} \widehat{Z}_q$ is the product of the q-adic integers. Thus by letting $(P_{1\Gamma}, h_{1\Gamma}) = (P_1, h_1) \otimes_{\Lambda_1} \Gamma$, $(\widehat{P}_1, \widehat{h}_1) = (P_1, h_1) \otimes_Z \widehat{Z}$ and $(\widehat{P}_{1\Gamma}, \widehat{h}_{1\Gamma}) = (P_{1\Gamma}, h_{1\Gamma}) \otimes_Z \widehat{Z}$,

$$\begin{tikzcd} (P_1, h_1) \arrow[r] \arrow[d] & (P_{1\Gamma}, h_{1\Gamma}) \arrow[d] \\ (\widehat{P}_1, \widehat{h}_1) \arrow[r] & (\widehat{P}_{1\Gamma}, \widehat{h}_{1\Gamma}) \end{tikzcd}$$

is a pullback diagram of hermitian modules. Hence to split Λ_1- hyperbolics off (P_1, h_1) one needs to do so over Γ and $\widehat{\Lambda_1}$ in a manner that makes their induced splittings over $\widehat{\Gamma}$ compatible. From the general ring-theoretic discussion in [L-W2] we only quote what is minimally necessary for what follows to make sense. Since $\widehat{\Lambda_1}$ and Z_d are semilocal rings, the natural projection induced by the augmentation map:

$$\widehat{\Lambda_1}/rad\widehat{\Lambda_1} \to Z_d/radZ_d$$

is a split surjection. That is there is an isomorphism of rings with involutions:

$$\widehat{\Lambda_1}/rad\widehat{\Lambda_1} \cong Z_d/radZ_d \times \prod_j F_j$$

with each F_j being either a finite field with involution or a product of two fields interchanged by the involution and $F_j \neq F_2, F_2 \times F_2$. Also $Z_d/radZ_d = \prod_{q|d} F_q$ is a product of prime fields with the trivial involution. Let $A = Z_d \times \prod_j F_j$. As explained in [L-W2] there exists a map of $\widehat{\Lambda_1}$-modules

$$\mu = \Theta_{\widehat{h}_{\Lambda_1}} : \widehat{P}_1 \to \widehat{\Lambda_1}^+/\widehat{\Lambda_1}_+$$

given by $x \to \widehat{h}_1(x, x)$ for all $x \in \widehat{P}_1$. $\widehat{\Lambda_1}^+/\widehat{\Lambda_1}_+$ is 0 or Z_2 according as d is odd or even. Set $K = ker\mu$ and let $j : K \to \widehat{P}_1$ be the inclusion and h_K the restriction of $\widehat{h}_{\Lambda_1}$ to K. K is then just the maximal submodule of $\widehat{P}_1$ on which $\widehat{h}_1$ is even hermitian.

The following lemma can be extracted from the proof of theorem 6.1 of [L-W2]

Lemma 7.9. *Let (P_1, h_1) be a non-singular hermitian module over $\Lambda_1 = Z[G]/N$. Assume P_1 is a* ***projective*** *Λ_1- module. Suppose that there exist isometries ϕ_d, ϕ_Γ satisfying* 7.12 *below, where $p : P_1 \to P_1 \otimes_{\Lambda_1} Z_d$:*

7.10. $$\phi_d : (P_1, h_1) \otimes_{\Lambda_1} Z_d \cong (P_d, h_d) \oplus H(Z_d^k)$$

7.11. $$\phi_\Gamma : (P_1, h_1) \otimes_{\Lambda_1} \Gamma \cong (P_{1\Gamma}, h_1\Gamma) \oplus H(\Gamma^k)$$

7.12. $$p^{-1}(\phi_d^{-1}(H(Z_d^k))) \otimes_Z \widehat{Z} \subseteq K$$

Then there is a splitting of (K, h_K) over A:

7.13. $$\phi_A : (K, h_K) \otimes_{\widehat{\Lambda_1}} A \cong (K_A, h_A) \oplus BH(A^k)$$

Furthermore if W is the preimage of $\phi_A^{-1}(BH(A^k))$ via $K \to K \otimes_{\widehat{\Lambda_1}} A$ and $h_W = h_K/W$ then (W, h_W) has a unique quadratic refinement $(W, [f])$. In the case d is even assume further that $(W, [f]) \otimes_{\widehat{\Lambda_1}} F_2$, (which is non-singular) has zero Arf invariant. Then there exist ψ, θ_d satisfying 7.16 *below:*

7.14. $$\psi : (P_1, h_1) \cong (P_1', h_1') \oplus H(\Lambda_1^k)$$

7.15. $$\theta_d : (P_1', h_1') \otimes_{\Lambda_1} Z_d \cong (P_d, h_d)$$

7.16. $$\phi_d = (\theta_d \oplus 1_{H(Z_d^k)}) \circ (\psi \otimes_{\Lambda_1} Z_d)$$

Proof: We sketch a proof as essentially contained in [L-W2] referring the reader there for the details. We do nothing but adapt their proof suitably for our statement. We quote from their proof. Assumption 7.10 induces a splitting:

$$\phi_d : (\widehat{P}_1, \widehat{h}_1) \otimes_{\widehat{\Lambda_1}} Z_d \cong (P_d, h_d) \oplus H(Z_d^k)$$

Assumption 7.11 provides a splitting over Γ. Tensoring the latter over Z with $\widehat{Z}$ one gets a splitting over $\widehat{\Gamma}$:

7.17.
$$\widehat{\phi}_\Gamma : (\widehat{P}_{1\Gamma}, \widehat{h}_{1\Gamma}) \cong (\widehat{P}_{1\Gamma}', \widehat{h}_{1\Gamma}') \oplus H(\widehat{\Gamma}^k)$$

Next one considers $(\widehat{P}_1, \widehat{h}_1)$ As shown in [L-W2] it suffices to show that there exists a hermitian $\widehat{\Lambda_1}$-module $(\widehat{P}_1', \widehat{h}_1')$ and a splitting

7.18. $$\widehat{\alpha} : (\widehat{P}_1, \widehat{h}_1) \cong (\widehat{P}_1', \widehat{h}_1') \oplus BH(\widehat{\Lambda_1}^k)$$

such that:

7.19. $$\widehat{\alpha}^{-1} BH(\widehat{\Lambda_1}^k) \otimes_{\widehat{\Lambda_1}} Z_d = \phi_d^{-1} BH(Z_d^k)$$

Assumption 7.12 implies that there exists an isometry over Z_d:

7.20. $$\phi_K : (K, h_K) \otimes_{\widehat{\Lambda_1}} Z_d \cong (P_d', h_d') \oplus BH(Z_d^k)$$

such that $(j \otimes Z_d)\phi_K^{-1}BH(Z_d^k) = \phi_d^{-1}BH(Z_d^k)$. K can be seen to be a $\widehat{\Lambda_1}$-free module abstractly isomorphic to $\widehat{P}_{\Lambda_1}$.

Next one shows that there exists a splitting for $(K, h_K) \otimes_{\widehat{\Lambda_1}} A$.

$$\phi_A : (K, h_K) \otimes_{\widehat{\Lambda_1}} A \cong (P_A, h_A) \oplus BH(A^k)$$

This is done using 7.20 and the obvious isometry 7.17 over $\widehat{\Gamma}$. Clearly $(W, h_W) \otimes_{\widehat{\Lambda_1}} A = \phi_A H(A^k)$ in free and non-singular over A and hence so is (W, h_W) over $\widehat{\Lambda_1}$. In fact $(W, h_W) \cong H(\widehat{\Lambda_1}^k)$. To see this observe that since $W \subseteq K$ the form h_W is even hermitian so, by Corollary 5.2 of [L-W2], (W, h_W) has a $\widehat{\Lambda_1}_-$ - quadratic refinement $(W, [f])$ which is unique because $\widehat{\Lambda_1}_- = \widehat{\Lambda_1}^-$. It suffices to show $(W, [f]) \cong QH(\widehat{\Lambda_1}^k)$. By Wall's lifting lemma [W4], it suffices to show such an isomorphism exists over $\widehat{\Lambda_1}/rad\widehat{\Lambda_1}$. If F denotes one of F_j or F_q with $q \geq 3$ then $F_- = F^-$ and so $(W, h_W) \otimes_{\widehat{\Lambda_1}} F \cong H(F^k)$ implies $(W, [f]) \otimes_{\widehat{\Lambda_1}} F \cong QH(F^k)$. In the case $F = F_2$ assumption 7.13 does the trick. Therefore $(W, [f]) \cong QH(\widehat{\Lambda_1}^k)$ as desired.
Since the map $U(W, [f]) \to U((W, [f]) \otimes_{\widehat{\Lambda_1}} A)$ by [W4] Thm. 2, there exists a splitting

$$\widehat{\alpha}_K : (K, h_K) \cong (K', h_{K'}) \oplus BH(\widehat{\Lambda_1}^k)$$

such that:

$$\widehat{\alpha}_K^{-1} BH(\widehat{\Lambda_1}^k) \otimes_{\widehat{\Lambda_1}} A = \phi_A^{-1} BH(A^k)$$

Therefore $(\widehat{P}_{\Lambda_1}, \widehat{h}_{\Lambda_1})$ contains the based hyperbolic submodule $j\widehat{\alpha}_K^{-1}BH(\widehat{\Lambda_1}^k)$. Hence there is a splitting:

$$\widehat{\alpha} : (\widehat{P}_1, \widehat{h}_1) \cong (\widehat{P}_1', \widehat{h}_1') \oplus BH(\widehat{\Lambda_1}^k)$$

such that $\widehat{\alpha}^{-1}BH(\widehat{\Lambda_1}^k) = j\widehat{\alpha}_K^{-1}BH(\widehat{\Lambda_1}^k)$ □

The following easily then follows from lemmas 7.1, 7.9 above:

Theorem 7.21. *Let $Z^n \oplus P$ be a hermitian ZG-module where P is ZG-projective. Suppose over Z and Γ there are splittings ϕ_0 and ϕ_Γ satisfying 7.24 below:*

7.22. $$\phi_0 : (Z^n \oplus P_0, h_0) \cong (Z^n \oplus P_0', h_0') \oplus H(Z^s),$$

7.23. $$\phi_\Gamma : (P_1, h_1) \otimes_{\Lambda_1} \Gamma \cong (P_\Gamma, h_\Gamma) \oplus H(\Gamma^s),$$

7.24. $$p^{-1}(J) \otimes_Z \widehat{Z} \subseteq K$$

where $J \subset (P_1, h_1) \otimes_{\Lambda_1} Z_d$ is the inverse image of $\phi_d^{-1}(H(Z_d^s)) \subseteq (P_0, h_0) \otimes_Z Z_d$ via the canonical isometry $(P_1, h_1) \otimes_{\Lambda_1} Z_d \cong (P_0, h_0) \otimes_Z Z_d$ and ϕ_d is the induced from 7.22 isometry $(\phi_0/P_0) \otimes_Z Z_d : (P_0, h_0) \otimes_Z Z_d \cong (P_0', h_0') \otimes_Z Z_d \oplus H(Z_d^s)$ Then there is a splitting of (K, h_K) over A:

7.25. $$\phi_A : (K, h_K) \otimes_{\widehat{\Lambda_1}} A \cong (K_A, h_A) \oplus BH(A^k)$$

Furthermore if W is the preimage of $\phi_A^{-1}(BH(A^k))$ via $K \to K \otimes_{\widehat{\Lambda_1}} A$ and $h_W = h_K/W$ then (W, h_W) has a unique quadratic refinement $(W, [f])$. In the case d is even assume further that $(W, [f]) \otimes_{\widehat{\Lambda_1}} F_2$, (which is non-singular) has zero Arf invariant.

Then there exist isometries τ, τ' satisfying 7.28 below:

7.26. $$\tau : (Z^n \oplus P, h) \cong (Z^n \oplus P', h') \oplus H(ZG^s)$$

7.27. $$\tau' : (Z^n \oplus P', h') \otimes_{ZG} Z \cong (Z^n \oplus P_0', h_0'),$$

7.28. $$\phi_0 = (\tau' \oplus 1_{H(Z^s)}) \circ (\tau \otimes Z)$$

Proof: The induced isometry ϕ_d together with assumptions 7.23 and 7.24 imply 7.25. This is easily seen using lemma 7.9 above. The assumption over F_2 implies 7.14, 7.15, 7.16. Assumptions 7.22 and 7.14 provide splittings over Z and Λ_1 which via 7.16 are compatible over Z_d. Now apply 7.1. QED □

8. Proof of the Main Theorem

We are now ready to prove the main theorem. We will need the following lemma:

Lemma 8.1. *Let $x \in H_2(M^4)$. Then $h(x,x) \in ZG_+$ iff $\pi_*(x) \cdot \pi_*(x) \equiv 0 \mod 2$ and, when d is even, $\frac{\pi_* x \cdot \alpha_1}{d} \equiv \frac{\pi_* x \cdot \alpha_2}{d} \mod 2$.*

Proof: Recall that $h(x,y) = \sum_{g \in G} \lambda_M(x, gy)g \in ZG$ for all $x, y \in H_2(M^4)$ and that G acts on $H_2(M^4)$ as an isometry with respect to λ_M. Hence $\lambda_M(x, gx) = \lambda_M(x, g^{-1}x)$. Therefore $h(x,x) \in ZG_+$ iff $\lambda_M(x, gx) \equiv 0 \pmod 2$ for all $g \in G$ such that $g^2 = 1$. On the other hand: $\lambda(\pi_* x, \pi_* x) = \lambda_M(x, Nx) \equiv \sum_{g^2=1} \lambda_M(x, gx) \pmod 2$. Hence $\lambda(\pi_* x, \pi_* x) \equiv \lambda_M(x,x) \pmod 2$ if d is odd, and $\lambda(\pi_* x, \pi_* x) \equiv \lambda_M(x,x) + \lambda_M(x, g^{d/2}x) \pmod 2$ if d is even, where $g \in G$ is a generator. This easily finishes the proof if d is odd. For the case d is even notice that since the action of G on M^4 is semifree

$$\lambda_M(x, [M^G]) \equiv \lambda_M(x, g^{d/2}x) \pmod 2$$

Hence $\lambda(\pi_* x, \pi_* x) \equiv \lambda_M(x, g^{d/2}x) \pmod 2$. Observing that $d\lambda_M(x, [M^G]) = \lambda_M(x, N[M^G]) = \lambda(\pi_* x, \alpha_1 + \alpha_2)$ finishes the lemma. □

For the convenience of the reader we restate the main theorem.

Theorem 8.2. *Let $\alpha_i \in H_2(X^4)$ i=1,2 be two primitive classes which satisfy all the obvious conditions. Let d be as in 3.2. Then the following is a* ***necessary and sufficient*** *condition for disjointly representing them by simple locally flat embeddings of spheres:*

8.3. $$b_2(X^4) \geq \max_{1 \leq l \leq d-1} |\sigma(X^4) - \frac{2}{d^2}(l(d-l)\alpha_1^2 + \bar{l}(d-\bar{l}))\alpha_2^2)| + 2$$

where $\bar{l} = (\alpha_1 \cdot \alpha_2')l \mod d$ and $1 \leq \bar{l} \leq d-1$.

Proof: The necessity of the obvious conditions of course is trivial. The necessity of 8.3 is proven in theorem 3.7. We now turn to the sufficiency of the conditions. Suppose that $\alpha_i \in H_2(X^4)$ i=1,2 are two primitive, linearly independent classes which satisfy the obvious topological conditions as well as 8.3. (Recall that the topological obvious conditions state that $\alpha_1, \alpha_2, \alpha_1 \pm \alpha_2$ are all individually topologically S^2-representable and that $\alpha_1 \cdot \alpha_2 = 0$). As in section 1.2 use the obvious conditions to solve the problem stably i.e. for some positive integer s there exists a simply embedded pair of topological locally flat spheres $A_i \hookrightarrow X_s = X\#_s S^2 \times S^2$ such that:

$$[A_i] = \alpha_i \oplus 0 \in H_2(X_s) = H_2(X) \oplus_s H_2(S^2 \times S^2)$$

with $\pi_1(X_s - \nu(A_1 \cup A_2)) = Z_d$, d as calculated in 3.2. What follows is a process of distabilizing so to speak. Let M_s be the corresponding d-fold remified cover over X_s branched along the pair of embedded spheres. We use the notation of section 1.7. Let $(Z^2 \oplus P, h, \{a_1, a_2\}) = (H_2(M_s), h_s, [M_s^G])$ where P is ZG- free (cf. section 1.5) Then $(Z^2 \oplus P_0, h_0, \{a_1 \otimes Z, a_2 \otimes Z\})$ embeds in $(H_2(X_s), \lambda, \{\alpha_1, \alpha_2\})$ in a fashion described in section 1.6 and remark 6.4. Hence there exists a splitting over the integers:

$$\text{8.4.} \qquad \phi_0 : (Z^2 \oplus P_0, h_0, \{a_1 \otimes Z, a_2 \otimes Z\}) \cong (Z^2 \oplus P_0^{'}, h_0^{'}, \{\alpha_1, \alpha_2\}) \oplus H(Z^s),$$

where $Z^2 \oplus P_0^{'}$ embeds in $H_2(X)$ in the same fashion as $Z^2 \oplus P_0$ in $H_2(X_s)$ and $h_0^{'}$ is the restriction of λ on the image of that embedding. A splitting of $(P_1, h_1, 0) \otimes_{\Lambda_1} \Gamma$,

$$\text{8.5.} \qquad \phi_\Gamma : (P_1, h_1) \otimes_{\Lambda_1} \Gamma \cong (P_\Gamma, h_\Gamma) \oplus H(\Gamma^s),$$

is obtained by means of 8.3. To see this observe that $(E = Z^2 \oplus P, h) \otimes_Z C$ decomposes over CG into the orthogonal direct sum of (E_l, h_l) (cf. section 1.3) where E_l is the subspace of $(Z^2 \oplus P) \otimes_Z C$ on which the generator of $G = Z_d$ acts as multiplication by $e^{\frac{2\pi i l}{d}}$. (In section 1.3 E_l were defined to be the eigenspace of the l^{th} power of the generator of G). Let T be the kernel of:

$$H_2(M_s) = Z^2 \oplus P \xrightarrow{\pi_*} Z^2 \oplus P_0 \subset H_2(X_s) = H_2(X) \oplus Z^{2s} \rightarrow H_2(X)$$

Lemma 8.1 above implies trivially that the restriction of h_s on T is even hermitian. One can check, using the latter, that (see 7.21):

$$p^{-1}(J) \otimes_Z \widehat{Z} \subseteq K$$

as well as that the condition of theorem 7.21 over F_2 when d is even is satisfied. Hence by 7.21 there exist isometries:

$$\text{8.6.} \qquad \tau : (Z^n \oplus P, h) \cong (Z^n \oplus P^{'}, h^{'}) \oplus H(ZG^s)$$

$$\text{8.7.} \qquad \tau^{'} : (Z^n \oplus P^{'}, h^{'}) \otimes_{ZG} Z \cong (Z^n \oplus P_0^{'}, h_0^{'}),$$

such that:

$$\text{8.8.} \qquad \phi_0 = (\tau^{'} \oplus 1_{H(Z^s)}) \circ (\tau \otimes Z)$$

As explained in section 1.6 this enables us to surger the ZG hyperbolic summands of $H_2(M_s)$ produced by 8.6 equivariantly thus producing $(H_2(M), Z_d)$ with the corresponding 2-pointed hermitian form given by $(Z^2 \oplus P', h', \{b_1, b_2\})$. Passing to $(M/G, M^G/G)$ we get a manifold with a pair of simply embedded spheres. By 8.7 above and 6.4 the 2-pointed forms of (X^4, A_1, A_2) and $(M/G, M^G/G)$ are isomorphic. Since M/G and X obviously have the same Kirby- Siebenmann invariant this isomorphism of 2-pointed forms can be realized by a homeomorphism. This follows by Freedman and a correction in [C-H] (Every isometry of the intersection form of a 4-manifold is realizable by a homeomorphism.) QED □

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