THE ZETA-DETERMINANTS OF DIRAC LAPLACIANS WITH BOUNDARY CONDITIONS ON THE SMOOTH, SELF-ADJOINT GRASSMANNIAN

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Abstract. In this paper we describe the difference of log of two zeta-determinants of Dirac Laplacians subject to the Dirichlet boundary condition and a boundary condition on the smooth, self-adjoint Grassmannian $Gr^{*}_{\infty}(D)$ on a compact manifold with boundary. Using this result we extend the result of Scott and Wojciechowski ([SW], [S2]) about the quotient of two zeta-determinants of Dirac Laplacians with boundary conditions on $Gr^{*}_{\infty}(D)$. We apply these results to the BFK-gluing formula to obtain the gluing formula for the zeta-determinants of Dirac Laplacians with respect to boundary conditions on $Gr^{*}_{\infty}(D)$. We next discuss the zeta-determinants of Dirac Laplacians subject to the Dirichlet or APS boundary condition on a finite cylinder and finally discuss the relative zeta-determinant on a manifold with cylindrical end when the APS boundary condition is imposed.

§1. Introduction and results

The zeta-determinants of Laplacians subject to the Dirichlet boundary condition have been studied by many authors in different contexts. For instance, Burghelea, Friedlander and Kappeler ([BFK]) proved the gluing formula for the zeta-determinants of Laplacians on a closed manifold with respect to the Dirichlet boundary condition. The relative zeta-determinant of Laplacians on a manifold with cylindrical end was described by P. Loya, J. Park ([LP1]) and J. Müller, W. Müller ([MM]) independently when the Dirichlet boundary condition is imposed on the cylinder part. One way of extending these results to the cases of other boundary conditions is to compare the zeta-determinants of Laplacians subject to the Dirichlet boundary condition with the ones subject to given boundary conditions.

In this paper we first describe the difference of log of two zeta-determinants of Dirac Laplacians subject to the Dirichlet boundary condition and a boundary condition on the smooth, self-adjoint Grassmannian $Gr^{*}_{\infty}(D)$ on a compact manifold with boundary. Using this result we extend the result of S. Scott and K. Wojciechowski ([SW], [S2]) about the quotient of two zeta-determinants of Dirac Laplacians subject to boundary conditions P_1, P_2 on $Gr^{*}_{\infty}(D)$. We next apply these results to the BFK-gluing formula to obtain the gluing formula for the zeta-determinants of Dirac Laplacians with respect to boundary conditions on $Gr^{*}_{\infty}(D)$. In fact, P. Loya and J. Park ([LP3], [LP4]) have already obtained the same result but their method is different from the one that we

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present here. Moreover, it is an advantage of this approach to be able to see the relation between the result of this paper and the BFK-gluing formula. Obviously, the Atiyah-Patodi-Singer (APS) boundary condition belongs to this class and we discuss the zeta-determinants of Dirac Laplacians subject to the Dirichlet or APS boundary condition on a finite cylinder and finally discuss the relative zeta-determinant on a manifold with cylindrical end when the APS boundary condition is imposed, which extends the result of [MM] (or [LP1]).

Now we introduce the basic settings. Let (M, g) be a compact oriented m-dimensional Riemannian manifold ($m > 1$) with boundary Y and $E \to M$ be a Clifford module bundle. Choose a collar neighborhood N of Y which is diffeomorphic to $[0,1)\times Y$. We assume that the metric g is a product one on N and the bundle E has the product structure on N, which means that $E|_N = p^*E|_Y$, where $p : [0, 1) \times Y \to Y$ is the canonical projection. Suppose that D_M is a compatible Dirac operator acting on smooth sections of E. We assume that D_M has the following form on N

$$D_M = G(\partial_u + B),$$

where $G : E|_Y \to E|_Y$ is a bundle automorphism, ∂_u is the inward normal derivative to Y on N and B is a Dirac operator on Y. We further assume that G and B are independent of the normal coordinate u and satisfy

$$\begin{aligned} &G^* = -G, \qquad G^2 = -I, \qquad B^* = B, \qquad GB = -BG, \\ &dim\ (ker(G-i) \cap kerB) = dim\ (ker(G+i) \cap kerB)\,. \end{aligned} \tag{1.1}$$

Then we have, on N, the Dirac Laplacian

$$D_M^2 = -\partial_u^2 + B^2.$$

We next introduce the boundary conditions on Y. The Dirichlet boundary condition on Y is defined by the restriction map $\gamma_0 : C^\infty(M) \to C^\infty(Y), \gamma_0(\phi) = \phi|_Y$, and the realization D^2_{M,γ_0} is defined to be the operator D_M^2 with the following domain

$$Dom(D^2_{M,\gamma_0}) = \{\phi \in C^\infty(E) \mid \quad \phi|_Y = 0\}.$$

Then D^2_{M,γ_0} is an invertible operator by the unique continuation property of D_M (*cf.* [B]).

The APS boundary condition $\Pi_>$ (or $\Pi_<$) is defined to be the orthogonal projection to the space spanned by positive (or negative) eigensections of B. If $kerB \neq \{0\}$, we need an extra condition to obtain a self-adjoint operator, say, a unitary involution on $kerB$ anticommuting with G. Suppose that $\sigma : kerB \to kerB$ is a unitary operator satisfying

$$\sigma G = -G\sigma, \qquad\qquad \sigma^2 = Id_{kerB}.$$

We put $\sigma^\pm = \frac{I\pm\sigma}{2}$ and denote by $\Pi_{<,\sigma^-}$, $\Pi_{>,\sigma^+}$

$$\Pi_{<,\sigma^-} = \Pi_< + \frac{1}{2}(I-\sigma)|_{kerB}, \quad \Pi_{>,\sigma^+} = \Pi_> + \frac{1}{2}(I+\sigma)|_{kerB}.$$

Then the realizations $D_{M,\Pi_{<,\sigma^-}}$ and $D^2_{M,\Pi_{<,\sigma^-}}$ are defined by D_M and D_M^2 with the following domains

$$\begin{aligned} Dom\left(D_{M,\Pi_{<,\sigma^-}}\right) &= \{\phi \in C^\infty(E) \mid \Pi_{<,\sigma^-}(\phi|_Y) = 0\}, \\ Dom\left(D^2_{M,\Pi_{<,\sigma^-}}\right) &= \{\phi \in C^\infty(E) \mid \Pi_{<,\sigma^-}(\phi|_Y) = 0,\ \Pi_{>,\sigma^+}((\partial_u + B)\phi)|_Y = 0\}. \end{aligned}$$

$D_{M,\Pi_{>,\sigma+}}$ and $D^2_{M,\Pi_{>,\sigma+}}$ are defined similarly.

As a generalization of the APS boundary condition we introduce the self-adjoint Grassmannian $Gr^*(D)$, which is the set of all orthogonal pseudodifferential projections P such that

$$-GPG = Id - P, \quad P - \Pi_> \text{ is a classical pseudodifferential operator of order } -1.$$

As a dense subset of $Gr^*(D)$, we define $Gr^*_\infty(D)$ by

$$Gr^*_\infty(D) = \{P \in Gr^*(D) \mid P - \Pi_> \text{ is a smoothing operator }\}.$$

Then Wojciechowski ([W]) showed that $\eta_{D_P}(s)$ and $\zeta_{D^2_P}(s)$ for $P \in Gr^*_\infty(D)$ have regular values at $s = 0$. Clearly, $\Pi_{>,\sigma}$ belongs to $Gr^*_\infty(D)$. The Calderón projector $\mathfrak{C}$ is defined to be the orthogonal projection from $L^2(E|_Y)$ onto $\overline{\{\phi|_Y \mid D_M(\phi) = 0\}}$, the Cauchy data space. Then $\mathfrak{C}$ is known to be an element of $Gr^*_\infty(D)$ by S. Scott ([S1]) and G. Grubb ([Gr]). The realization $D^2_{M,P}$ is defined to be the operator D^2_M with the following domain.

$$Dom\left(D^2_{M,P}\right) \;=\; \{\phi \in C^\infty(M) \mid P\gamma_0\phi = 0, \;\; (I - P)\gamma_0(\partial_u + B)\phi = 0\}.$$

The purpose of this paper is to describe the relative zeta-determinant $\log Det D^2_{M,P} - \log Det D^2_{M,\gamma_0}$ and discuss some of its applications including the gluing formula for the zeta-determinants of Dirac Laplacians.

To describe the main result we define $Q : C^\infty(Y) \to C^\infty(Y)$ as follows. For $f \in C^\infty(Y)$ there exists a unique section $\phi \in C^\infty(M)$ satisfying $D^2_M\phi = 0, \; \phi|_Y = f$. Then we define

$$Q(f) = -(\partial_u\phi)|_Y. \tag{1.2}$$

The Green formula shows that $Q-B$ is a non-negative operator and $ker(Q-B) = Im\mathfrak{C}$, the Cauchy data space (Lemma 2.3). We regard $(I - P)\,(Q - B)\,(I - P)$ as an operator on $Im\,(I - P)$, *i.e.*,

$$(I - P)\,(Q - B)\,(I - P) : C^\infty(Y) \cap Im\,(I - P) \to C^\infty(Y) \cap Im\,(I - P).$$

Since $Q - |B|$ ([L3]) and $P - \Pi_>$ are smoothing operators, $(I - P)\,(Q - B)\,(I - P)$ differs from $2\Pi_<|B|$ by a smoothing operator and hence the zeta-determinant of $(I - P)\,(Q - B)\,(I - P)$ is well-defined. It is not difficult to show that $ker\,(I - P)\,(Q - B)\,(I - P) = \{\psi|_Y \mid \psi \in ker D_{M,P}\}$ (Lemma 2.3). Let $\{h_1, h_2, \cdots, h_q\}$ be an orthonormal basis for $ker\,((I - P)\,(Q - B)\,(I - P))$, $q = dimker D_{M,P}$. Then there exist $\psi_1, \psi_2, \cdots, \psi_q$ such that

$$D_{M,P}\psi_i = 0, \quad \psi_i|_Y = h_i.$$

We define a $q \times q$ positive definite Hermitian matrix $V_{M,P}$ by

$$V_{M,P} = (v_{ij}), \quad v_{ij} = \langle\psi_i, \; \psi_j\rangle_M. \tag{1.3}$$

If $\mathfrak{P}$ is an invertible elliptic operator of order > 0 with discrete spectrum $\{\lambda_j \mid j = 1, 2, 3, \cdots\}$, we define the zeta function $\zeta_{\mathfrak{P}}(s) = \sum_{\lambda_j \in Spec(\mathfrak{P})} \lambda_j^{-s}$ and the zeta-determinant $Det\mathfrak{P}$ by $e^{-\zeta'_{\mathfrak{P}}(0)}$. If $\mathfrak{P}$ has a non-trivial kernel, we define the modified zeta-determinant $Det^*\mathfrak{P}$ by

$$Det^*\mathfrak{P} := Det(\mathfrak{P} + pr_{ker\mathfrak{P}}).$$

Similarly, if α is a trace class operator, we define the modified Fredholm determinant by

$$det^*_{Fr}(I + \alpha) = det_{Fr}(I + \alpha + pr_{ker(I+\alpha)}).$$

Equivalently, $Det^*\mathfrak{P}$ and $det^*_{Fr}(I + \alpha)$ are the determinants of $\mathfrak{P}$ and $I + \alpha$ when restricted to the orthogonal complement of $ker\mathfrak{P}$ and $ker(I + \alpha)$, respectively.

Then the following is the main result of this paper.

Theorem 1.1. *Suppose that M is a compact Riemannian manifold with boundary Y having the product structure near the boundary and D_M is a compatible Dirac operator which has the form (1.1) near the boundary. Then for $P \in Gr^*_\infty(D)$ and the Dirichlet boundary condition γ_0 on Y, we have the following equality.*

$$\log Det^* D^2_{M,P} - \log Det D^2_{M,\gamma_0} \;=\; \log det V_{M,P} + \log Det^*\left((I-P)(Q-B)(I-P)\right),$$

where $((I-P)(Q-B)(I-P))$ is considered to be an operator defined on $Im(I-P)$.

Remark : (1) We take the negative real axis as a branch cut for logarithm.
(2) If we parametrize the collar neighborhood N by $(-1,0]\times Y$ with the boundary $\{0\}\times Y$ and write the Dirac operator D_M on N by $D_M = G(\partial_u + B)$ with ∂_u the outward unit normal derivative, $Q(f)$ is defined by

$$Q(f) := (\partial_u \phi)|_Y, \quad \text{where} \quad D^2_M\phi = 0 \quad \text{and} \quad \phi|_Y = f. \tag{1.4}$$

Then $(Q+B)$ is a non-negative operator and in this case Theorem 1.1 can be written as follows.

$$\log Det^* D^2_{M,I-P} - \log Det D^2_{M,\gamma_0} \;=\; \log det V_{M,I-P} + \log Det^*\left(P(Q+B)P\right). \tag{1.5}$$

(3) Even if the boundary of M consists of two components Y and Z, Theorem 1.1 still holds as far as M has the product structures near Y and a boundary condition $\mathfrak{B}$ is imposed on Z so that $D^2_{M,\mathfrak{B},\gamma_0}$ is an invertible operator. For example, if $\mathfrak{B}$ is the Dirichlet boundary condition on Z, both $D^2_{M,\mathfrak{B},P}$ and $D^2_{M,\mathfrak{B},\gamma_0}$ are invertible operators. In this case, Q is defined as follows. For $f \in C^\infty(Y)$, choose $\phi \in C^\infty(M)$ such that $D^2_M\phi = 0$, $\phi|_Z = 0$ and $\phi|_Y = f$. Then $Q(f) := -(\partial_u\phi)|_Y$. Since the term $\log det V_{M,P}$ does not appear in this case, Theorem 1.1 can be written by

$$\log Det D^2_{M,\mathfrak{B},P} - \log Det D^2_{M,\mathfrak{B},\gamma_0} \;=\; \log Det\left((I-P)(Q-B)(I-P)\right). \tag{1.6}$$

Since G is a bundle automorphism with $G^2 = -I$, the restriction $E|_Y$ splits into $\pm i$-eigenspaces $E^\pm_Y$, say, $E|_Y = E^+_Y \oplus E^-_Y$ and the Dirac operator D_M can be written by

$$D_M = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\left(\partial_u + \begin{pmatrix} 0 & B^- \\ B^+ & 0 \end{pmatrix}\right),$$

where $B^\pm : C^\infty(E^\pm_Y) \to C^\infty(E^\mp_Y)$ and $(B^\pm)^* = B^\mp$. Then there exists the unitary operator $K : L^2(Y, E^+_Y) \to L^2(Y, E^-_Y)$ satisfying $Im\mathfrak{C} = graph(K)$. For $P \in Gr^*_\infty(D)$, there exists a unitary operator $T : L^2(Y, E^+_Y) \to L^2(Y, E^-_Y)$ such that

$$ImP = graph(T), \quad T = K \;+\; \textit{a smoothing operator}. \tag{1.7}$$

As the first application of Theorem 1.1 we extend the result of S. Scott and K. Wojciechowski ([SW], [S1]) as follows.

Theorem 1.2. *Suppose that P is a pseudodifferential projection in $Gr^*_\infty(D)$. Then :*

$$\frac{Det^* D^2_{M,P}}{Det D^2_{M,\mathfrak{C}}} = \left(det V_{M,P}\right)^2 \cdot |det^*_{Fr}\left(\frac{1}{2}\left(I + T^{-1}K\right)\right)|^2.$$

We next apply Theorem 1.1 and 1.2 to the BFK-gluing formula for the zeta-determinants of Dirac Laplacians. Let $(\widetilde{M}, \widetilde{g})$ be a closed Riemannian manifold and Y be a hypersurface of $\widetilde{M}$ such that $\widetilde{M} - Y$ has two components. We denote by M_1, M_2 the closure of each component. We choose a collar neighborhood of Y which is diffeomorphic to $(-1, 1) \times Y$ and assume that $\widetilde{g}$ is a product metric on N. Let $\widetilde{E} \to \widetilde{M}$ be a Clifford module bundle having the product structure on N and $\widetilde{D}$ be a compatible Dirac operator acting on smooth sections of $\widetilde{E}$ which has the form, on N, $\widetilde{D} = G(\partial_u + B)$ satisfying (1.1) as before. We denote by D_{M_1}, D_{M_2} the restrictions of $\widetilde{D}$ to M_1, M_2 and by γ_0 the Dirichlet boundary condition on Y. Suppose that $\{h_1, h_2, \cdots, h_q\}$ is an orthonormal basis for $\left(ker\widetilde{D}\right)|_Y := \{\Phi|_Y \mid \widetilde{D}\Phi = 0\}$, where $q = dimker\widetilde{D}$. Then there exist $\Phi_1, \cdots, \Phi_q$ in $ker\widetilde{D}$ with $\Phi_i|_Y = h_i$. We define a positive definite Hermitian matrix A_0 by

$$A_0 = (a_{ij}), \qquad \text{where} \qquad a_{ij} = \langle \Phi_i, \Phi_j \rangle_{\widetilde{M}}. \tag{1.8}$$

The BFK-gluing formula can be stated as follows (*cf.* [BFK], [L3]).

$$\begin{aligned} \log Det^* \widetilde{D}^2 - \log Det D^2_{M_1,\gamma_0} - \log Det D^2_{M_2,\gamma_0} = \\ - \log 2 \cdot (\zeta_{B^2}(0) + l) + \log det A_0 + \log Det^*(Q_1 + Q_2), \end{aligned} \tag{1.9}$$

where $l =$ $dimkerB$ and Q_1 is defined by (1.4), Q_2 by (1.2). Theorem 1.1 and 1.2 together with (1.9) lead to the following result, which is the main mativation for Theorem 1.1.

Theorem 1.3. *Let $\mathfrak{C}_1$, $\mathfrak{C}_2$ be Calderón projectors for D_{M_1}, D_{M_2} and P_1, P_2 be orthogonal pseudodifferential projections belonging to $Gr^*_\infty(D_{M_1})$, $Gr^*_\infty(D_{M_2})$, respectively. Suppose that for $i = 1, 2$, K_i, $T_i : L^2(Y, E^+_Y) \to L^2(Y, E^-_Y)$ are unitary maps such that $graph(K_i) = Im\mathfrak{C}_i$ and $graph(T_i) = ImP_i$. Then the following equalities hold.*

$$\begin{aligned} (1)\ \log Det^* \widetilde{D}^2 - \log Det D^2_{M_1,\mathfrak{C}_1} - \log Det D^2_{M_2,\mathfrak{C}_2} = \\ - \log 2 \cdot (\zeta_{B^2}(0) + l) + 2 \log det A_0 + 2 \log |det^*_{Fr} \left(\frac{1}{2} \left(I - K_1^{-1} K_2 \right) \right) |. \end{aligned}$$

$$\begin{aligned} (2)\ \log Det^* \widetilde{D}^2 - \log Det^* D^2_{M_1,P_1} - \log Det^* D^2_{M_2,P_2} = - \log 2 \cdot (\zeta_{B^2}(0) + l) + 2 \log det A_0 \\ - 2 \sum_{i=1}^{2} \log det V_{M_i,P_i} + 2 \log |det^*_{Fr} \left(\frac{1}{2} \left(I - K_1^{-1} K_2 \right) \right) | - 2 \sum_{i=1}^{2} \log |det^*_{Fr} \left(\frac{1}{2} \left(I + T_i^{-1} K_i \right) \right) |. \end{aligned}$$

Remark : The result of Theorem 1.3 was obtained earlier by P. Loya and J. Park in [LP3] (or [LP4]) in a different way.

Note that there exists a unitary map $T_{\sigma^+} : C^\infty(E^+_Y) \to C^\infty(E^-_Y)$ satisfying (1.7) so that $Im\left(\Pi_{>,\sigma^+}\right) = graph\left(T_{\sigma^+}\right)$ and $Im\left(\Pi_{<,\sigma^-}\right) = graph\left(-T_{\sigma^+}\right)$. In this case Theorem 1.1 and 1.3 can be written as follows.

Corollary 1.4. *Under the same notations as in Theorem 1.3 the following equalities hold.*

(1) $\log Det^* D^2_{M_1,\Pi_{<,\sigma^-}} - \log Det D^2_{M_1,\gamma_0} = \log det V_{M_1,\Pi_{<,\sigma^-}} + \log Det^*(\Pi_{>,\sigma^+}(Q_1 + |B|)\Pi_{>,\sigma^+})$.

(2) $\log Det^* D^2_{M_2,\Pi_{>,\sigma^+}} - \log Det D^2_{M_2,\gamma_0} = \log det V_{M_2,\Pi_{>,\sigma^+}} + \log Det^*(\Pi_{<,\sigma^-}(Q_2 + |B|)\Pi_{<,\sigma^-})$.

(3) $\log Det^* \widetilde{D}^2 - \log Det^* D^2_{M_1,\Pi_{<,\sigma^-}} - \log Det^* D^2_{M_2,\Pi_{>,\sigma^+}} = - \log 2 \cdot (\zeta_{B^2}(0) + l)$

$$+ 2\log det A_0 - 2\left(\log det V_{M_1,\Pi_{<,\sigma^-}} + \log det V_{M_2,\Pi_{>,\sigma^+}}\right) + 2\log |det^*_{Fr}\left(\frac{1}{2}\left(I - K_1^{-1}K_2\right)\right)|$$

$$- 2\log |det^*_{Fr}\left(\frac{1}{2}\left(I - T_{\sigma^+}^{-1}K_1\right)\right)| - 2\log |det^*_{Fr}\left(\frac{1}{2}\left(I + T_{\sigma^+}^{-1}K_2\right)\right)|.$$

In general, the operator Q_1 or Q_2 (with the same notation as in Theorem 1.3) is not easy to describe except in a cylinder case. If M is a cylinder, Corollary 1.4 can be reduced to a much simpler form. We denote $N_{0,r} := [0, r] \times Y$ and impose the Dirichlet boundary condition γ_0 on $Y_0 := \{0\} \times Y$. We denote by $(-\partial_u^2 + B^2)_{N_{0,r},\gamma_0,\Pi_{<,\sigma^-}}$ $((-\partial_u^2 + B^2)_{N_{0,r},\gamma_0,\gamma_r})$ the Dirac Laplacian subject to the Dirichlet bundary condition on Y_0 and $\Pi_{<,\sigma^-}$ on $Y_r := \{r\} \times Y$ (the Dirichlet boundary condition γ_0, γ_r on Y_0, Y_r). One can check easily that $(-\partial_u^2 + B^2)_{N_{0,r},\gamma_0,\gamma_r}$ and $(-\partial_u^2 + B^2)_{N_{0,r},\gamma_0,\Pi_{<,\sigma^-}}$ are invertible operators and hence the kernels are trivial. Moreover, one can show by direct computation (*cf.* [L4]) that Q_1 can be expressed by

$$Q_1 = \frac{1}{r} pr_{kerB} + |B| + \frac{2|B|e^{-r|B|}}{e^{r|B|} - e^{-r|B|}} pr_{(kerB)^\perp}. \tag{1.10}$$

Then the first and second assertations of Corollary 1.4 can be stated in this case as follows, which was obtained in [L4] and [L5].

Corollary 1.5. *Suppose that* $l := dimkerB$ *and* $N_{0,r} := [0, r] \times Y$. *Then :*

$$\begin{aligned} &\log Det(-\partial_u^2 + B^2)_{N_{0,r},\gamma_0,\Pi_{<,\sigma^-}} - \log Det(-\partial_u^2 + B^2)_{N_{0,r},\gamma_0,\gamma_r} \\ = \; &\log Det(-\partial_u^2 + B^2)_{N_{0,r},\Pi_{>,\sigma^+},\gamma_r} - \log Det(-\partial_u^2 + B^2)_{N_{0,r},\gamma_0,\gamma_r} \\ = \; &-\frac{l}{2} \cdot \log r + \frac{1}{2}\log 2 \cdot \zeta_{B^2}(0) + \frac{1}{4}\log Det^* B^2 + \frac{1}{2}\log det_{Fr}\left(I + \frac{e^{-r|B|}}{e^{r|B|} - e^{-r|B|}} pr_{(kerB)^\perp}\right). \end{aligned}$$

We next consider the Dirac Laplacian $(-\partial_u^2 + B^2)_{N_{0,r},\Pi_{>,\tau^+},\Pi_{<,\sigma^-}}$ on $N_{0,r}$ with the boundary conditions $\Pi_{>,\tau^+}$ on Y_0 and $\Pi_{<,\sigma^-}$ on Y_r, where σ and τ are unitary involutions on $kerB$ anticommuting with G. Then it is not difficult to see that

$$ker(-\partial_u^2 + B^2)_{N_{0,r},\Pi_{>,\tau^+},\Pi_{<,\sigma^-}} = \{f \in C^\infty(Y) \mid f \in (Im\, \tau^- \cap Im\, \sigma^+)\}. \tag{1.11}$$

We also introduce the boundary condition $(\partial_u + |B|)$ on Y_r and denote by $(-\partial_u^2 + B^2)_{N_{0,r},\gamma_0,(\partial_u+|B|)}$ the Dirac Laplacian subject to the Dirichlet condition γ_0 on Y_0 and $(\partial_u + |B|)$ on Y_r, *i.e.*

$$Dom\left((-\partial_u^2 + B^2)_{N_{0,r},\gamma_0,(\partial_u+|B|)}\right) = \{\phi \in C^\infty(N_{0,r}) \mid \phi|_{Y_0} = 0,\ ((\partial_u + |B|)\phi)\,|_{Y_r} = 0\}.$$

Then $(-\partial_u^2 + B^2)_{N_{0,r},\gamma_0,(\partial_u+|B|)}$ is an invertible operator. To describe the next result we introduce a constant α_1 as follows. We first consider the asymptotic expansion of the heat kernel of B^2. As $t \to 0^+$,

$$Tr e^{-tB^2} \sim \sum_{j=0}^{\infty} b_j t^{-\frac{m-1}{2}+j}.$$

This series shows that $\zeta_{B^2}(s)$ is analytic at $s=-\frac{1}{2}$ if $dimY = m-1$ is even. However, if $dimY$ is odd, $\zeta_{B^2}(s)$ has a simple pole at $s=-\frac{1}{2}$. We define α_1 by

$$\alpha_1 = \begin{cases} \zeta_{B^2}(-\frac{1}{2}), & \text{if } dimY \text{ is even} \\ \frac{d}{ds}\left(s\cdot\zeta_{B^2}(s-\frac{1}{2})\right)|_{s=0} + \frac{1}{\sqrt{\pi}}(\log 2 - 1)\cdot b_{\frac{m}{2}}, & \text{if } dimY \text{ is odd.} \end{cases} \tag{1.12}$$

Then we have the following result.

Theorem 1.6. *Suppose that $l = dimkerB$ and $k_+ = dim\ (Im\ \sigma^+ \cap Im\ \tau^-)$. Then :*

(1) $\log Det^*(-\partial_u^2+B^2)_{N_{0,r},\Pi_{>,\tau^+},\Pi_{<,\sigma^-}} = \alpha_1\cdot r + 2k_+\log r + \log 2\cdot(\zeta_{B^2}(0)+l) + \log|det^*\left(\frac{\sigma+\tau}{2}\right)|,$

where $det^*\left(\frac{\sigma+\tau}{2}\right) = det\left(\frac{\sigma+\tau}{2} + pr_{ker(\sigma+\tau)}\right)$.

(2) $\log Det(-\partial_u^2 + B^2)_{N_{0,r},\gamma_0,(\partial_u+|B|)} = \alpha_1\cdot r + \log 2\cdot(\zeta_{B^2}(0)+l)$.

Remark : The first equality in Theorem 1.6 was proved first by P. Loya and J. Park in [LP2].

Finally, we are going to apply Corollary 1.4 to the relative zeta-determinant on a manifold with cylindrical end studied by J. Müller, W. Müller in [MM] and P. Loya, J. Park in [LP1]. Let $M_{1,\infty} = M_1 \cup_Y [0,\infty)\times Y$ and $N_{0,\infty} = [0,\infty)\times Y$. We denote by $D_{M_{1,\infty}}$ the extension of D_{M_1} to $M_{1,\infty}$ and by $(-\partial_u^2+B^2)_{N_{0,\infty},\gamma_0}$ the Dirac Laplacian on $N_{0,\infty}$ subject to the Dirichlet boundary condition on $\{0\}\times Y$. Let μ_1 be the smallest positive eigenvalue of B. Then the scattering theory for a Dirac operator on a manifold with cylindrical end ([Gu], [M1]) shows that $D_{M_{1,\infty}}$ determines a regular one-parameter family of unitary operators $C(\lambda)$, called on-shell scattering operators, with $\lambda\in\mathbb{R}$, $|\lambda| < \mu_1$, which act on $kerB$ and satisfy

$$C(\lambda)C(-\lambda) = I, \quad C(\lambda)G = GC(\lambda).$$

They showed independently in [MM] and [LP1] that for $l = dimkerB$,

$$\log Det\left(D^2_{M_{1,\infty}}, (-\partial_u^2+B^2)_{N_{0,\infty},\gamma_0}\right) - \log Det(D^2_{M_1,\gamma_0}) = \\ -\log 2\cdot(\zeta_{B^2}(0)+l) + \log Det^*(Q_1+|B|) - \log detA_1, \tag{1.13}$$

where A_1 is a positive definite Hermitian matrix defined as follows. Let $\{\psi_1,\cdots,\psi_{q'}\}$ be an orthonormal basis of the space of L^2-solutions of $D_{M_{1,\infty}}$ on $M_{1,\infty}$ and $\{f_1,\cdots,f_{\frac{l}{2}}\}$ be an orthonormal basis of $ImC(0)^+$, the space of the limiting values of the extended L^2-solutions of $D_{M_{1,\infty}}$. We put $\psi_{q'+j} = \frac{1}{2}E(f_j,0)$ for $1\le j\le\frac{l}{2}$, where $\frac{1}{2}E(f_j,0)$ is the extended L^2-solution of $D_{M_{1,\infty}}$ on $M_{1,\infty}$ whose limiting value is f_j (see [M1] or [MM] for notations and definitions). Then we define

$$A_1 = (a_{ij}), \quad \text{where } a_{ij} = \langle\psi_i|_Y, \psi_j|_Y\rangle_Y, \ \ 1\le i,j\le q'+\frac{l}{2}. \tag{1.14}$$

Setting $q'+\frac{l}{2} = q$, we define another $q\times q$ positive definite Hermitian matrix $\widetilde{V}$ as follows. We denote by $\psi_{i,0}$ the limiting value of ψ and $\psi_{i,0}=0$ if ψ_i is an L^2-solution. We also define ψ_{i,L^2} by

$$\psi_{i,L^2} = \begin{cases} \psi_i & \text{on } M_1 \\ \psi_i - \psi_{i,0} & \text{on } N_{0,\infty}. \end{cases} \tag{1.15}$$

Then we define

$$\widetilde{V} = (\widetilde{v}_{ij})_{1\le i,j\le q}, \quad \text{where} \quad \widetilde{v}_{ij} = \langle\psi_{i,0},\psi_{j,0}\rangle_Y + \langle\psi_{i,L^2},\psi_{j,L^2}\rangle_{M_{1,\infty}}.$$

Applying Corollary 1.4 to (1.13), we have the following result.

Theorem 1.7.

$$
\begin{aligned}
& \log Det\left(D^2_{M_{1,\infty}}, (-\partial_u^2 + B^2)_{N_{0,\infty},\Pi_{>,\tau^+}}\right) - \log Det(D^2_{M_1,\Pi_{<,\sigma^-}}) \\
= \; & -\log 2\cdot(\zeta_{B^2}(0)+l) - 2\log det A_1 + \log det\widetilde{V} - \log det\left(I + \frac{i}{2}\frac{I-C(0)}{2}C'(0)\frac{I-C(0)}{2}\right) \\
& - 2\log det V_{M_1,\Pi_{<,\sigma^-}} + 2\log|det^*_{Fr}\left(\frac{1}{2}\left(I - K_1^{-1}T_0\right)\right)| - 2\log|det^*_{Fr}\left(\frac{1}{2}\left(I - K_1^{-1}T_{\sigma^+}\right)\right)|,
\end{aligned}
$$

where $graph(T_0) = Im\Pi_{>,C(0)^+}$, $graph(T_{\sigma^+}) = Im\Pi_{>,\sigma^+}$ *and* $C'(0) = \frac{d}{d\lambda}C(\lambda)|_{\lambda=0}$.

Remark : Lemma 5.1 in §5 shows that the left hand side of the above equality does not depend on the choice of a unitary involution τ anticommuting with G.

§2. **The Proof of Theorem 1.1**

In this section we are going to prove Theorem 1.1 by using the method used in [BFK], [C] and [F]. Let P be an orthogonal pseudodifferential projection in $Gr^*_\infty(D)$ and ν be a positive integer $> \frac{m-1}{2}$ with $m = dim M$. Then for $t > 0$ both $(D^2_{M,P}+t)^{-\nu}$ and $(D^2_{M,\gamma_0}+t)^{-\nu}$ are trace class operators. Taking the derivative ν times with respect to t,

$$
\begin{aligned}
& \frac{d^\nu}{dt^\nu}\left\{\log Det\left(D^2_{M,P}+t\right) - \log Det\left(D^2_{M,\gamma_0}+t\right)\right\} \\
= \; & Tr\left\{\frac{d^{\nu-1}}{dt^{\nu-1}}\left(\left(D^2_{M,P}+t\right)^{-1} - \left(D^2_{M,\gamma_0}+t\right)^{-1}\right)\right\}. \qquad (2.1)
\end{aligned}
$$

We introduce the Poisson operator for the Dirichlet condition $P_{\gamma_0}(t) : C^\infty(Y) \to C^\infty(M)$, which is characterized as follows. For any $f \in C^\infty(Y)$,

$$
\left(D^2_M + t\right)P_{\gamma_0}(t)f = 0, \qquad \gamma_0 P_{\gamma_0}(t)f = f. \qquad (2.2)
$$

Then we have

$$
\left(D^2_{M,P}+t\right)^{-1} - \left(D^2_{M,\gamma_0}+t\right)^{-1} = P_{\gamma_0}(t)\gamma_0\left(D^2_{M,P}+t\right)^{-1}. \qquad (2.3)
$$

Combining (2.1) with (2.3) leads to

$$
\begin{aligned}
& \frac{d^\nu}{dt^\nu}\left\{\log Det\left(D^2_{M,P}+t\right) - \log Det\left(D^2_{M,\gamma_0}+t\right)\right\} \\
= \; & Tr\left\{\frac{d^{\nu-1}}{dt^{\nu-1}}\left(P_{\gamma_0}(t)\gamma_0\left(D^2_{M,P}+t\right)^{-1}\right)\right\} \\
= \; & Tr\left\{\frac{d^{\nu-1}}{dt^{\nu-1}}\left(\gamma_0\left(D^2_{M,P}+t\right)^{-1}P_{\gamma_0}(t)\right)\right\} \\
= \; & Tr\left\{\frac{d^{\nu-1}}{dt^{\nu-1}}\left((I-P)\gamma_0\left(D^2_{M,P}+t\right)^{-1}P_{\gamma_0}(t)(I-P)\right)\right\}. \qquad (2.4)
\end{aligned}
$$

According to the method suggested in [F], we define $Q(t) : C^\infty(Y) \to C^\infty(Y)$ by

$$
Q(t) = -\gamma_0\partial_u P_{\gamma_0}(t),
$$

and define $R_P(t) : C^\infty(Y) \to C^\infty(Y)$ by

$$R_P(t) = (I-P)(Q(t)-B)(I-P) + P|B|P + pr_{(kerB \cap ImP)}.$$

Then $R_P(t)$ is a positive definite, essentially self-adjoint elliptic operator. Taking the derivative of $R_P(t)$ with respect to t, we have

$$\frac{d}{dt} R_P(t) \ = \ -(I-P)\gamma_0 \partial_u \left(\frac{d}{dt} P_{\gamma_0}(t) \right) (I-P). \tag{2.5}$$

Lemma 2.1.

$$\frac{d}{dt} P_{\gamma_0}(t) = -\left(D^2_{M,\gamma_0} + t \right)^{-1} P_{\gamma_0}(t).$$

Proof : Taking the derivative in (2.2) with respect to t, we have

$$\left(D^2_M + t \right) \frac{d}{dt} P_{\gamma_0}(t) = -P_{\gamma_0}(t), \qquad \gamma_0 \frac{d}{dt} P_{\gamma_0}(t) = 0,$$

which implies the result. $\square$

Since $(I-P)\gamma_0(\partial_u + B)(D^2_P + t)^{-1} = 0$, the equation (2.5) and Lemma 2.1 lead to

$$\begin{aligned}
\frac{d}{dt} R_P(t) &= (I-P)\gamma_0 \partial_u \left(D^2_{M,\gamma_0} + t \right)^{-1} P_{\gamma_0}(t)(I-P) \\
&= -(I-P)\gamma_0 \left(\partial_u + B \right) \left(\left(D^2_{M,P} + t \right)^{-1} - \left(D^2_{M,\gamma_0} + t \right)^{-1} \right) P_{\gamma_0}(t)(I-P) \\
&= -(I-P)\gamma_0 \left(\partial_u + B \right) P_{\gamma_0}(t) \gamma_0 \left(D^2_{M,P} + t \right)^{-1} P_{\gamma_0}(t)(I-P) \\
&= -(I-P) \left(\gamma_0 \cdot \partial_u \cdot P_{\gamma_0}(t) + B \right) (I-P)\gamma_0 \left(D^2_{M,P} + t \right)^{-1} P_{\gamma_0}(t)(I-P) \\
&= -(I-P) \left(-Q(t) + B \right) (I-P)\gamma_0 \left(D^2_{M,P} + t \right)^{-1} P_{\gamma_0}(t)(I-P) \\
&= R_P(t)(I-P)\gamma_0 \left(D^2_{M,P} + t \right)^{-1} P_{\gamma_0}(t)(I-P),
\end{aligned} \tag{2.6}$$

which shows that

$$R_P(t)^{-1} \frac{d}{dt} R_P(t) = (I-P)\gamma_0 \left(D^2_{M,P} + t \right)^{-1} P_{\gamma_0}(t)(I-P). \tag{2.7}$$

Combining (2.4) with (2.7), we have

$$\begin{aligned}
\frac{d^\nu}{dt^\nu} \left\{ \log Det \left(D^2_{M,P} + t \right) - \log Det \left(D^2_{M,\gamma_0} + t \right) \right\} &= Tr \left\{ \frac{d^{\nu-1}}{dt^{\nu-1}} \left(R_P(t)^{-1} R_P(t) \right) \right\} \\
&= \frac{d^\nu}{dt^\nu} \log Det R_P(t).
\end{aligned} \tag{2.8}$$

Since $P - \Pi_>$ and $Q(t) - \sqrt{B^2 + t}$ are smoothing operators ([L3]), the zeta-determinant of $(P|B|P)$ and $((I-P)(Q(t)-B)(I-P))$ are well-defined and hence

$$\log Det R_P(t) = \log Det \left((I-P)\left(Q(t) - B \right)(I-P) \right) + \log Det^* \left(P|B|P \right).$$

This observation and (2.8) lead to the following result.

Theorem 2.2. *For some real numbers $a_0, a_1, \cdots, a_{\nu-1}$, the following equality holds.*

$$\log Det\left(D^2_{M,P}+t\right) - \log Det\left(D^2_{M,\gamma_0}+t\right) = \sum_{j=0}^{\nu-1} a_j \; + \; \log Det\left((I-P)\left(Q(t)-B\right)(I-P)\right) + \log Det^*\left(P|B|P\right),$$

where $((I-P)(Q(t)-B)(I-P))$ and $P|B|P$ are considered as operators defined on $Im(I-P)$ and ImP, respectively.

We next discuss the constant a_0 appearing in Theorem 2.2. It was shown in the Appendix of [BFK] that $\log Det\left(D^2_{M,P}+t\right)$, $\log Det\left(D^2_{M,\gamma_0}+t\right)$ and $\log Det\left((I-P)(Q(t)-B)(I-P)\right)$ have asymptotic expansions as $t \to \infty$ and coefficients are determined by the symbols of operators. Moreover, it is a well-known fact that the zero coefficients of the asymptotic expansions of $\log Det\left(D^2_{M,P}+t\right)$, $\log Det\left(D^2_{M,\gamma_0}+t\right)$ are zeros (*cf.* [V] or Proposition 2.7 in [L2]). Hence, $-a_0$ is, in fact, the sum of $\log Det^*\left(P|B|P\right)$ and the zero coefficient of the asymptotic expansion of $\log Det\left((I-P)(Q(t)-B)(I-P)\right)$. Since $P - \Pi_>$ and $Q(t) - \sqrt{B^2+t}$ are smoothing operators ([L3]), $\log Det\left((I-P)(Q(t)-B)(I-P)\right)$ has the same asymptotic expansion as $\log Det\left(\Pi_<\left(\sqrt{B^2+t}-B\right)\Pi_<\right)$. It was shown in Section 3 of [L4] that the zero coefficient of the asymptotic expansion, as $t \to \infty$, of $\log Det\left(\sqrt{B^2+t}+|B|\right)$ is zero and hence we conclude that

$$a_0 + \log Det^*\left(P|B|P\right) = 0.$$

Therefore, Theorem 2.2 together with this observation lead to the following equality.

$$\log Det\left(D^2_{M,P}+t\right) - \log Det\left(D^2_{M,\gamma_0}+t\right) = \sum_{j=1}^{\nu-1} a_j t^j + \log Det\left((I-P)\left(Q(t)-B\right)(I-P)\right). \tag{2.9}$$

Finally, we are going to discuss the behavior of (2.9) as $t \to 0$. We denote by $q = dimker D^2_{M,P}$. Since D^2_{M,γ_0} is an invertible operator, we have

$$\begin{aligned} \log Det\left(D^2_{M,\gamma_0}+t\right) &= \log Det D^2_{M,\gamma_0} + o(t), \\ \log Det\left(D^2_{M,P}+t\right) &= q \cdot \log t + \log Det^* D^2_{M,P} + o(t). \end{aligned} \tag{2.10}$$

The following lemma shows the relation between $ker D^2_{M,P}$ and $ker\left((I-P)(Q-B)(I-P)\right)$.

Lemma 2.3. *(1) $ker\,(Q-B) = \{\phi|_Y \mid D_M\phi = 0\} = Im\mathfrak{C}$, and hence $(Q-B)$ maps $Im\,(I-\mathfrak{C})$ onto $Im\,(I-\mathfrak{C})$.*
(2) $ker\left((I-P)(Q-B)(I-P)\right) = ker\,(Q-B) \cap Im(I-P) = \{\phi|_Y \mid \phi \in ker D_{M,P}\}$, and $dimker\left((I-P)(Q-B)(I-P)\right) = dimker D_{M,P}$.

Proof: The second assertion follows from the first assertion and the unique continuation property of D_M. If $\phi \in C^\infty(M)$ satisfies $D_M\phi = 0$, $Q(\phi|_Y) = -\left(\partial_u\phi\right)|_Y = B\left(\phi|_Y\right)$, and hence $\phi|_Y \in ker\,(Q-B)$. Conversely, suppose that $f \in ker\,(Q-B)$. We choose the unique section $\phi \in C^\infty(M)$ so that

$$D^2_M\phi = 0, \quad \phi|_Y = f.$$

By the Green Theorem (*cf.* Lemma 3.1 in [CLM])),

$$\begin{aligned} 0 = \langle D^2_M\phi, \phi\rangle_M &= \langle D_M\phi, D_M\phi\rangle_M + \langle (D_M\phi)|_Y, G\phi|_Y\rangle_Y \\ &= \langle D_M\phi, D_M\phi\rangle_M + \langle -Q(f) + Bf, f\rangle_Y, \end{aligned}$$

which implies that $D_M\phi = 0$ and hence $f \in Im\,\mathfrak{C}$. Since $(Q-B)$ is self-adjoint, it maps $Im\,(I-\mathfrak{C})$ onto itself. $\square$

Now let us denote the eigenvalues of $(I-P)\,(Q(t)-B)\,(I-P)$ on $Im(I-P)$ by

$$0 \;<\; \kappa_1(t) \;\le\; \cdots \;\le\; \kappa_q(t) \;<\; \kappa_{q+1}(t) \;\le\; \cdots$$

and the corresponding orthonormal eigensections by

$$h_1(t), \;\; \cdots, \;\; h_q(t), \;\; h_{q+1}(t), \;\; \cdots .$$

Then for $1 \le j \le q$,

$$\lim_{t\to 0} \kappa_j(t) = 0, \qquad \lim_{t\to 0} h_j(t) = h_j,$$

where $\{h_1, h_2, \cdots, h_q\}$ is an orthonormal basis of $ker\,((I-P)\,(Q-B)\,(I-P))$. This leads to

$$\begin{aligned}\log Det\,((I-P)\,(Q(t)-B)\,(I-P)) \;=& \\ \log \kappa_1(t)\cdots\kappa_q(t) \;+\; \log Det^{*}\,((I-P)\,(Q-B)\,(I-P)) \;+\; o(t).& \qquad (2.11)\end{aligned}$$

The second assertion in Lemma 2.3 shows that each h_j can be extended to a global section $\psi_j \in C^{\infty}(M)$ such that

$$D_{M,P}\psi_j = 0, \quad \psi_j|_Y = h_j. \qquad (2.12)$$

The next result shows the behavior of $\kappa_j(t)$ as $t \to 0$ and for its proof we follow the proof of Theorem B in [L1].

Lemma 2.4.

$$\lim_{t\to 0} \frac{\kappa_j(t)}{t} = \langle \psi_j, \psi_j \rangle_M, \qquad \textit{and} \quad \langle \psi_i, \psi_j \rangle_M = 0 \quad \textit{for} \quad i \ne j, \;\; 1 \le i, j \le q,$$

and hence

$$\log \kappa_1(t)\cdots\kappa_q(t) \;=\; q \log t + \log det\,(\langle \psi_i, \psi_j \rangle_M) + o(t).$$

Proof : Since $(I-P)(h_k(t)) = h_k(t)$ and $(I-P)(h_k) = h_k$ for $1 \le k \le q$, we have

$$\begin{aligned}\kappa_j(t)\,\langle h_j(t),\; h_k\rangle_Y \;&=\; \langle ((I-P)\,(Q(t)-B)\,(I-P))\,h_j(t),\; h_k\rangle_Y \\ &=\; \langle ((Q(t)-B))\,h_j(t),\; h_k\rangle_Y. \qquad (2.13)\end{aligned}$$

Let $\psi_j(t)$ be the smooth section on M such that

$$(D_M^2 + t)\psi_j(t) = 0, \quad \psi_j(t)|_Y = h_j(t).$$

Using the Green formula and (2.12), we have

$$\begin{aligned}0 \;&=\; \langle \big(D_M^2 + t\big)\,(\psi_j(t)),\; \psi_k\rangle_M \;=\; t\,\langle \psi_j(t),\; \psi_k\rangle_M \;+\; \langle D_M^2\psi_j(t),\; \psi_k\rangle_M \\ &=\; t\,\langle \psi_j(t),\; \psi_k\rangle_M \;+\; \langle D_M\psi_j(t),\; D_M\psi_k\rangle_M \;+\; \int_Y (D_M\psi_j(t)|_Y,\; G\psi_k|_Y)\, dvol(Y) \\ &=\; t\,\langle \psi_j(t),\; \psi_k\rangle_M \;+\; \langle ((\partial_u + B)\psi_j(t))\,|_Y,\; h_k\rangle_Y\end{aligned}$$

and hence

$$\langle (Q(t)-B)\,h_j(t),\; h_k\rangle_Y \;=\; t\,\langle \psi_j(t),\; \psi_k\rangle_M. \qquad (2.14)$$

The equations (2.13) and (2.14) show that

$$\kappa_j(t)\langle h_j(t),\ h_k\rangle_Y \ = \ t\ \langle \psi_j(t),\ \psi_k\rangle_M. \tag{2.15}$$

Since $\lim_{t\to 0}\psi_j(t)|_Y = \psi_j|_Y$, the unique continuation property of D_M implies $\lim_{t\to 0}\psi_j(t) = \psi_j$. Since $\langle h_j,\ h_k\rangle_Y = \delta_{jk}$, the result follows. $\square$

Lemma 2.4 with (2.9), (2.10) and (2.11) imply Theorem 1.1.

§3. **The Proof of Theorem 1.2 and 1.3**

In this section we are going to prove Theorem 1.2 and 1.3. Note that $Im\mathfrak{C} = graph(K)$ and $Im\,(I-\mathfrak{C}) = graph(-K)$. Since $(I-K)$ is a map from $C^\infty(Y, E_Y^+)$ onto $Im\,(I-\mathfrak{C})$, Lemma 2.3 shows that $(I-\mathfrak{C})\,(Q-B)\,(I-\mathfrak{C})$ has the same spectrum as $(I-K)^{-1}\,(Q-B)\,(I-K)$ and hence

$$\log Det\,((I-\mathfrak{C})\,(Q-B)\,(I-\mathfrak{C})) = \log Det\left((I-K)^{-1}\,(Q-B)\,(I-K)\right). \tag{3.1}$$

We note again $Im(I-P) = graph(-T)$ and define U, L by

$$\begin{aligned} U &= Im\,(I-P)\cap Im\mathfrak{C} = ker(I-P)(Q-B)(I-P) = \{\phi|_Y \mid D_{M,P}\phi = 0\}, \\ L &= (I-T)^{-1}\,(U) = (I+K)^{-1}\,(U) = \{x\in L^2(E_Y^+) \mid Tx = -Kx\}. \end{aligned} \tag{3.2}$$

We also denote by $Im\,(I-P)^*$ and $L^2(E_Y^+)^*$ the orthogonal complements of U, L so that

$$Im\,(I-P) = Im\,(I-P)^*\oplus U, \quad L^2(E_Y^+) = L^2(E_Y^+)^*\oplus L.$$

Then it is not difficult to see that

$$ker(I+K^{-1}T) = L, \quad \text{and} \quad (I+K^{-1}T)|_{L^2(E_Y^+)^*} : L^2(E_Y^+)^* \to L^2(E_Y^+)^*$$

is invertible. For simplicity, we write $\left((I+K^{-1}T)|_{L^2(E_Y^+)^*}\right)^{-1}$ by $\left(I+K^{-1}T\right)^{-1}$ and define

$$\begin{aligned} &\Psi : L^2(E_Y^+)^*\oplus L \to Im(I-P)^*\oplus U \quad \text{by} \\ &\Psi \ = \ 2(I-T)(I+K^{-1}T)^{-1}pr_{L^2(E_Y^+)^*} + (I-T)pr_L \\ &\quad = \ \left((I-K)-(I+K)(I+T^{-1}K)^{-1}(I-T^{-1}K)\right)pr_{L^2(E_Y^+)^*} + (I-T)pr_L. \end{aligned}$$

Then we have the following commutative diagram.

$$\begin{array}{ccc} Im\,(I-P)^*\oplus U & \xrightarrow{(I-P)(Q-B)(I-P)} & Im\,(I-P)^*\oplus U \\ \Psi\Big\uparrow & & \Big\downarrow\Psi^{-1} \\ L^2(E_Y^+)^*\oplus L & \xrightarrow{\Psi^{-1}(I-P)(Q-B)(I-P)\Psi} & L^2(E_Y^+)^*\oplus L \end{array} \tag{3.3}$$

Using the first assertion of Lemma 2.3 and the following identity

$$(I-K) \ = \ \frac{1}{2}(I+T)(I-T^{-1}K) \ + \ \frac{1}{2}(I-T)(I+T^{-1}K),$$

we have

$$\Psi^{-1}\left((I-P)(Q-B)(I-P)+pr_U\right)\Psi \;=\; \frac{1}{4}(I+K^{-1}T)(I+T^{-1}K)(I-K)^{-1}(Q-B)(I-K)pr_{L^2(E_Y^+)^*}+pr_L. \quad (3.4)$$

Hence, (3.2) and (3.4) lead to

$$\begin{aligned}
&\log Det^*\left((I-P)(Q-B)(I-P)\right)=\log Det\left((I-P)(Q-B)(I-P)+pr_U\right)\\
&=\log Det\left(\frac{1}{4}(I+K^{-1}T)(I+T^{-1}K)(I-K)^{-1}(Q-B)(I-K)\,pr_{L^2(E_Y^+)^*}+pr_L\right)\\
&=\log Det\left(\frac{1}{4}(I+K^{-1}T)(I+T^{-1}K)(I-K)^{-1}(Q-B)(I-K)+pr_L\right)\\
&=\log Det\left(\frac{1}{4}(I+K^{-1}T)(I+T^{-1}K)+pr_L(I-K)^{-1}(Q-B)^{-1}(I-K)\right)\left((I-K)^{-1}(Q-B)(I-K)\right)\\
&=\log det_{Fr}\left(\frac{1}{4}(I+K^{-1}T)(I+T^{-1}K)+pr_L(I-K)^{-1}(Q-B)^{-1}(I-K)pr_L\right)\\
&\qquad+\log Det\left((I-K)^{-1}(Q-B)(I-K)\right)\\
&=\log|det^*_{Fr}\frac{1}{2}(I+T^{-1}K)|^2+\log det\left(pr_L(I-K)^{-1}(Q-B)^{-1}(I-K)pr_L\right)\\
&\qquad+\log Det\left((I-\mathfrak{C})(Q-B)(I-\mathfrak{C})\right). \qquad (3.5)
\end{aligned}$$

Lemma 3.1.

$$det\left(pr_L(I-K)^{-1}(Q-B)^{-1}(I-K)pr_L\right) \;=\; detV_{M,P},$$

where $V_{M,P}$ is a $q\times q$ matrix defined in (1.3).

Proof: Since $(I-K):L\to GU=Im(I-\mathfrak{C})\cap ImP$ is an isomorphism (*cf.* (3.2)), we have

$$det\left(pr_L(I-K)^{-1}(Q-B)^{-1}(I-K)pr_L\right) \;=\; det\left(pr_{GU}(Q-B)^{-1}pr_{GU}\right).$$

Let $\{h_1,\cdots,h_q\}$ be an orthonormal basis for U. Then $\{Gh_1,\cdots,Gh_q\}$ is an orthonormal basis for GU. Suppose that $(Q-B)^{-1}Gh_i=f_i$ and choose ϕ_i such that $D_M^2\phi_i=0$ and $\phi_i|_Y=f_i$. Using the Green formula, we have

$$\begin{aligned}
0 &= \langle D_M^2\phi_i,\ \phi_j\rangle_M=\langle D_M\phi_i,\ D_M\phi_j\rangle_M+\langle D_M\phi_i|_Y,\ G\phi_j|_Y\rangle_Y\\
&=\langle D_M\phi_i,\ D_M\phi_j\rangle_M+\langle(\partial_u+B)\phi_i|_Y,\ f_j\rangle_Y=\ \langle D_M\phi_i,\ D_M\phi_j\rangle_M+\langle(-Q+B)f_i,\ f_j\rangle_Y,
\end{aligned}$$

which shows that

$$\langle(Q-B)^{-1}Gh_i,\ Gh_j\rangle_Y=\langle f_i,\ (Q-B)f_j\rangle_Y=\langle(Q-B)f_i,\ f_j\rangle_Y=\langle D_M\phi_i,\ D_M\phi_j\rangle_M.$$

We note that

$$D_M\left(D_M\phi_i\right)=0,\quad \left(D_M\phi_i\right)|_Y=G(\partial_u+B)\phi_i|_Y=G(-Q+B)f_i=-GGh_i=h_i,$$

which completes the proof of the lemma. $\square$

Theorem 1.2 follows from Theorem 1.1, (3.5) and Lemma 3.1.

Next, we are going to prove Theorem 1.3 by using the similar method. Theorem 1.1 and (1.9) lead to the following equality.

$$\begin{aligned}
&\log Det^*\widetilde{D}^2-\log DetD^2_{M_1,\mathfrak{C}_1}-\log DetD^2_{M_2,\mathfrak{C}_2} \;=\; -\log 2\cdot(\zeta_{B^2}(0)+l)+\log detA_0+\\
&\log Det^*(Q_1+Q_2)-\log Det\left((I-\mathfrak{C}_1)(Q_1+B)(I-\mathfrak{C}_1)\right)-\log Det\left((I-\mathfrak{C}_2)(Q_2-B)(I-\mathfrak{C}_2)\right). \qquad (3.6)
\end{aligned}$$

The following lemma can be checked by the same way as Lemma 2.3.

Lemma 3.2.

$$ker\,(Q_1+B)=\{\phi|_Y \mid D_{M_1}\phi=0\}=Im\mathfrak{C}_1, \quad ker\,(Q_2-B)=\{\psi|_Y \mid D_{M_2}\psi=0\}=Im\mathfrak{C}_2,$$
$$ker(Q_1+Q_2)=Im\mathfrak{C}_1\cap Im\mathfrak{C}_2=\{\tilde{\phi}|_Y \mid \widetilde{D}\tilde{\phi}=0\}.$$

Lemma 3.2 implies that

$$C^\infty(E|_Y)=ker(Q_1+Q_2)\oplus\left(Im\,(I-\mathfrak{C}_1)+Im\,(I-\mathfrak{C}_2)\right),$$

where $dim\ (Im\,(I-\mathfrak{C}_1)\cap Im\,(I-\mathfrak{C}_2)) = dim\ ker\,(Q_1+Q_2) = dim\ ker\widetilde{D} = q$. The following lemma is straightforward.

Lemma 3.3.

$$I-K_1=(I+K_2)\,\frac{I-K_2^{-1}K_1}{2}+(I-K_2)\,\frac{I+K_2^{-1}K_1}{2}.$$
$$I-K_2=(I+K_1)\,\frac{I-K_1^{-1}K_2}{2}+(I-K_1)\,\frac{I+K_1^{-1}K_2}{2}.$$

Using Lemma 3.2 and Lemma 3.3, for $x\in C^\infty(Y,E_Y^+)$ we have

$$\begin{aligned}&(Q_1+Q_2)\,(I-K_1)\,x\\ =&\,(Q_1+B)\,(I-K_1)\,x+(Q_2-B)\,(I-K_1)\,x\\ =&\,(I-\mathfrak{C}_1)\,(Q_1+B)\,(I-\mathfrak{C}_1)\,(I-K_1)x+(I-\mathfrak{C}_2)\,(Q_2-B)\,(I-\mathfrak{C}_2)\,(I-K_2)\,\frac{I+K_2^{-1}K_1}{2}x.\end{aligned}$$

Similarly,

$$\begin{aligned}&(Q_1+Q_2)\,(I-K_2)\,y\\ =&\,(I-\mathfrak{C}_1)\,(Q_1+B)\,(I-\mathfrak{C}_1)\,(I-K_1)\,\frac{I+K_1^{-1}K_2}{2}y+(I-\mathfrak{C}_2)\,(Q_2-B)\,(I-\mathfrak{C}_2)\,(I-K_2)y.\end{aligned}$$

Recall that $ker\,(Q_1+Q_2)=\{(I+K_1)x \mid K_1x=K_2x\}$ and denote it by H. We now define subspace $\widetilde{H}_\pm$ of $Im\,(I-\mathfrak{C}_1)\oplus Im\,(I-\mathfrak{C}_2)$ by

$$\widetilde{H}_+=\{(I-K_1)x,\ (I-K_2)x \mid K_1x=K_2x\}, \quad \widetilde{H}_-=\{(I-K_1)x,\ -(I-K_2)x \mid K_1x=K_2x\},$$

and consider the following diagram.

$$\begin{array}{ccc} Im\,(I-\mathfrak{C}_1)\oplus Im\,(I-\mathfrak{C}_2) & \xrightarrow{\widetilde{R}} & Im\,(I-\mathfrak{C}_1)\oplus Im\,(I-\mathfrak{C}_2) \\ \Phi\big\downarrow & & \big\downarrow\Phi \\ (Im\,(I-\mathfrak{C}_1)+Im\,(I-\mathfrak{C}_2))\oplus\widetilde{H}_- & \xrightarrow{\widetilde{Q}} & (Im\,(I-\mathfrak{C}_1)+Im\,(I-\mathfrak{C}_2))\oplus\widetilde{H}_-, \end{array} \tag{3.7}$$

where Φ, $\widetilde{Q}$ and $\widetilde{R}$ are defined as follows.

$$\Phi\left((I-K_1)x,\ (I-K_2)y\right)=\Big((I-K_1)x+(I-K_2)y,\ pr_{\widetilde{H}_-}\left((I-K_1)x,\ (I-K_2)y\right)\Big),$$

$$\widetilde{Q}\,(a,b)=\Big(\left(Q_1+Q_2\right)(a)\,,\ pr_{\widetilde{H}_-}\ \widetilde{R}\,\Phi^{-1}(a,b)\Big),$$

$$\widetilde{R}=\begin{pmatrix}\mathfrak{S}_1 & \mathfrak{S}_1(I-K_1)\frac{I+K_1^{-1}K_2}{2}(I-K_2)^{-1}\\ \mathfrak{S}_2(I-K_2)\frac{I+K_2^{-1}K_1}{2}(I-K_1)^{-1} & \mathfrak{S}_2\end{pmatrix}pr_{(\widetilde{H}_-)^{\perp}}+pr_{\widetilde{H}_-}$$
$$=\begin{pmatrix}\mathfrak{S}_1 & 0\\ 0 & \mathfrak{S}_2\end{pmatrix}\begin{pmatrix}I & (I-K_1)\frac{I+K_1^{-1}K_2}{2}(I-K_2)^{-1}\\ (I-K_2)\frac{I+K_2^{-1}K_1}{2}(I-K_1)^{-1} & I\end{pmatrix}+pr_{\widetilde{H}_-},$$

where $\mathfrak{S}_1=(I-\mathfrak{C}_1)\left(Q_1+B\right)(I-\mathfrak{C}_1)$ and $\mathfrak{S}_2=(I-\mathfrak{C}_2)\left(Q_2-B\right)(I-\mathfrak{C}_2)$. Then all maps are invertible and the diagram (3.7) commutes. Hence,

$$\log Det\widetilde{Q}\ =\ \log Det^{*}\left(Q_1+Q_2\right)\ =\ \log Det\left(I-\mathfrak{C}_1\right)\left(Q_1+B\right)\left(I-\mathfrak{C}_1\right)$$
$$+\ \log Det\left(I-\mathfrak{C}_2\right)\left(Q_2-B\right)\left(I-\mathfrak{C}_2\right)+\log det_{Fr}\left(\alpha+\beta\right),\quad (3.8)$$

where

$$\alpha=\begin{pmatrix}I & (I-K_1)\frac{I+K_1^{-1}K_2}{2}(I-K_2)^{-1}\\ (I-K_2)\frac{I+K_2^{-1}K_1}{2}(I-K_1)^{-1} & I\end{pmatrix},\quad \beta=\begin{pmatrix}{\mathfrak{S}_1}^{-1} & 0\\ 0 & {\mathfrak{S}_2}^{-1}\end{pmatrix}pr_{\widetilde{H}_-}.$$

We note that $H=Im\mathfrak{C}_1\cap Im\mathfrak{C}_2$ implies $GH=Im\left(I-\mathfrak{C}_1\right)\cap Im\left(I-\mathfrak{C}_2\right)$ and hence

$$Im\left(I-\mathfrak{C}_1\right)\oplus Im\left(I-\mathfrak{C}_2\right)=\left(\left(Im\left(I-\mathfrak{C}_1\right)\ominus GH\right)\oplus\left(Im\left(I-\mathfrak{C}_2\right)\ominus GH\right)\right)\oplus\widetilde{H}_+\oplus\widetilde{H}_-.$$

Since α maps $(\widetilde{H}_-)^{\perp}$ onto $(\widetilde{H}_-)^{\perp}$ and $\alpha|_{\widetilde{H}_+}=2Id|_{\widetilde{H}_+}$,

$$\log det_{Fr}\left(\alpha+\beta\right)=q\log 2+\log det_{Fr}\left(\alpha|_{\oplus_{i=1}^{2}\left(Im(I-\mathfrak{C}_i)\ominus GH\right)}\right)+\log det\left(pr_{\widetilde{H}_-}\beta\right)$$
$$=q\log 2+\log|det^{*}_{Fr}\left(\frac{1}{2}\left(I-K_1^{-1}K_2\right)\right)|^2+\log det\left(pr_{\widetilde{H}_-}\begin{pmatrix}\mathfrak{S}_1^{-1} & 0\\ 0 & \mathfrak{S}_2^{-1}\end{pmatrix}pr_{\widetilde{H}_-}\right),\quad (3.9)$$

where $q=dim\widetilde{H}_+=dimker\widetilde{D}$. Let $\{h_1,\cdots,h_q\}$ be an orthonormal basis of $Im\mathfrak{C}_1\cap Im\mathfrak{C}_2$. Then $\{Gh_1,\cdots,Gh_q\}$ is an orthonormal basis for $Im\ (I-\mathfrak{C}_1)\cap Im\ (I-\mathfrak{C}_2)$ and this gives an orthonormal basis $\left\{\frac{1}{\sqrt{2}}(Gh_1,-Gh_1),\cdots,\frac{1}{\sqrt{2}}(Gh_q,-Gh_q)\right\}$ for $\widetilde{H}_-$. We note that

$$\left\langle pr_{\widetilde{H}_-}\begin{pmatrix}\mathfrak{S}_1^{-1} & 0\\ 0 & \mathfrak{S}_2^{-1}\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}Gh_i\\ -Gh_i\end{pmatrix},\ \frac{1}{\sqrt{2}}\begin{pmatrix}Gh_j\\ -Gh_j\end{pmatrix}\right\rangle\ =\ \frac{1}{2}\left(\langle\mathfrak{S}_1^{-1}Gh_i,Gh_j\rangle+\langle\mathfrak{S}_2^{-1}Gh_i,Gh_j\rangle\right),$$

which shows that

$$\log det\left(pr_{\widetilde{H}_-}\begin{pmatrix}\mathfrak{S}_1^{-1} & 0\\ 0 & \mathfrak{S}_2^{-1}\end{pmatrix}pr_{\widetilde{H}_-}\right)=-q\log 2+\log det\left(pr_{GH}\left(\mathfrak{S}_1^{-1}+\mathfrak{S}_2^{-1}\right)pr_{GH}\right).\quad (3.10)$$

Lemma 3.4.
$$det\left(pr_{GH}\left((Q_1+B)^{-1}+(Q_2-B)^{-1}\right)pr_{GH}\right) \;=\; detA_0,$$
where A_0 is a $q\times q$ matrix defined in (1.8).

Proof : Suppose that $\{Gh_1,\cdots,Gh_q\}$ is an orthonormal basis for $Im\ (I-\mathfrak{C}_1)\cap Im\ (I-\mathfrak{C}_2)$ and denote $(Q_1+B)^{-1}Gh_i = f_i$, $(Q_2-B)^{-1}Gh_j = g_j$. We choose $\phi_1,\cdots,\phi_q\in C^\infty(M_1)$, $\psi_1,\cdots,\psi_q\in C^\infty(M_2)$ such that
$$D^2_{M_1}\phi_i=0,\quad D^2_{M_2}\psi_i=0,\quad \phi_i|_Y=f_i,\quad \psi_j|_Y=g_j.$$
Using the Green formula,
$$\begin{aligned}0=\langle D^2_{M_1}\phi_i,\phi_j\rangle_{M_1} &= \langle D_{M_1}\phi_i, D_{M_1}\phi_j\rangle_{M_1}-\langle (D_{M_1}\phi_i)|_Y,(G\phi_j)|_Y\rangle_Y\\ &= \langle D_{M_1}\phi_i, D_{M_1}\phi_j\rangle_{M_1}-\langle ((\partial_u+B)\phi_i)|_Y, f_j\rangle_Y\\ &= \langle D_{M_1}\phi_i, D_{M_1}\phi_j\rangle_{M_1}-\langle (Q_1+B)f_i, f_j\rangle_Y.\end{aligned}$$
As the same way as in Lemma 3.1,
$$\begin{aligned}\langle (Q_1+B)^{-1}Gh_i, Gh_j\rangle_Y \;&=\; \langle f_i,(Q_1+B)f_j\rangle_Y \;=\; \langle (Q_1+B)f_i, f_j\rangle_Y\\ &=\; \langle D_{M_1}\phi_i, D_{M_1}\phi_j\rangle_{M_1} \;=\; \langle -D_{M_1}\phi_i, -D_{M_1}\phi_j\rangle_{M_1}.\end{aligned}\tag{3.11}$$

Note that
$$D_{M_1}(-D_{M_1}\phi_i)=0,\quad (-D_{M_1}\phi_i)|_Y=-G(\partial_u+B)\phi_i|_Y=-G(Q_1+B)f_i=-GGh_i=h_i.\tag{3.12}$$
As the same way,
$$\langle (Q_2-B)^{-1}Gh_i, Gh_j\rangle_Y=\langle D_{M_2}\psi_i, D_{M_2}\psi_j\rangle_{M_2},\tag{3.13}$$
where
$$D_{M_2}(D_{M_2}\psi_i)=0,\quad (D_{M_2}\psi_i)|_Y=G(\partial_u+B)\psi_i|_Y=G(-Q_2+B)g_i=-GGh_i=h_i.\tag{3.14}$$
Setting
$$\Phi_i=(-D_{M_1}\phi_i)\cup_Y(D_{M_2}\psi_i),$$
Lemma 3.2 with (3.12) and (3.14) shows that Φ_i is a smooth section and belongs to $ker\widetilde{D}$. Hence, (3.11) and (3.13) show that
$$\langle (Q_1+B)^{-1}Gh_i, Gh_j\rangle_Y+\langle (Q_2-B)^{-1}Gh_i, Gh_j\rangle_Y=\langle \Phi_i,\Phi_j\rangle_{\widetilde{M}},$$
which completes the proof of the lemma. □
Theorem 1.3 is obtained by the above lemma with (3.6), (3.8), (3.9) and (3.10).

§4. **The Proof of Theorem 1.6**

In this section we are going to prove Theorem 1.6. To prove the first assertion of Theorem 1.6 we begin with the following fact
$$ker(-\partial_u^2+B^2)_{N_{0,r},\Pi_{>,\tau^+},\Pi_{<,\sigma^-}} \;=\; \{f\in C^\infty(Y)\mid f\in Im\ \tau^-\cap Im\ \sigma^+\}.\tag{4.1}$$

By Corollary 1.5 we have

$$\begin{aligned}
&\log Det^*(-\partial_u^2+B^2)_{N_{0,r},\Pi_{>,\tau^+},\Pi_{<,\sigma^-}} - \log Det(-\partial_u^2+B^2)_{N_{0,r},\gamma_0,\gamma_r}\\
=&\log Det^*(-\partial_u^2+B^2)_{N_{0,r},\Pi_{>,\tau^+},\Pi_{<,\sigma^-}} - \log Det(-\partial_u^2+B^2)_{N_{0,r},\Pi_{>,\tau^+},\gamma_r}\\
&+\log Det(-\partial_u^2+B^2)_{N_{0,r},\Pi_{>,\tau^+},\gamma_r} - \log Det(-\partial_u^2+B^2)_{N_{0,r},\gamma_0,\gamma_r}\\
=&\log Det^*(-\partial_u^2+B^2)_{N_{0,r},\Pi_{>,\tau^+},\Pi_{<,\sigma^-}} - \log Det(-\partial_u^2+B^2)_{N_{0,r},\Pi_{>,\tau^+},\gamma_r}\\
&-\frac{l}{2}\cdot\log r+\frac{1}{2}\log 2\cdot\zeta_{B^2}(0) + \frac{1}{4}\log Det^* B^2 + \frac{1}{2}\log det_{Fr}\left(I+\frac{e^{-r|B|}}{e^{r|B|}-e^{-r|B|}}pr_{(ker B)^\perp}\right).
\end{aligned} \tag{4.2}$$

To eatablish the analogous formula as Corollary 1.5 for $\log Det^*(-\partial_u^2+B^2)_{N_{0,r},\Pi_{>,\tau^+},\Pi_{<,\sigma^-}} - \log Det(-\partial_u^2+B^2)_{N_{0,r},\Pi_{>,\tau^+},\gamma_r}$, we consider, as in the proof of Theorem 1.1,

$$\log Det\left(-\partial_u^2+B^2+t\right)_{N_{0,r},\Pi_{>,\tau^+},\Pi_{<,\sigma^-}} - \log Det\left(-\partial_u^2+B^2+t\right)_{N_{0,r},\Pi_{>,\tau^+},\gamma_r}.$$

To define the operator $R_{cyl}(t) : C^\infty(Y_r)\to C^\infty(Y_r)$ corresponding to $R_P(t)$ in Theorem 1.1, we introduce the Poisson operator $P_{cyl}(t) : C^\infty(Y_r)\to C^\infty(N_{0,r})$ associated with the boundary condition $\Pi_{>,\tau^+}$ on Y_0, which is characterized as follows.

$$\begin{aligned}
(-\partial_u^2+B^2+t)P_{cyl}(t)=0, &\qquad \gamma_r P_{cyl}(t)=Id_{Y_r},\\
\Pi_{>,\tau^+}\gamma_0 P_{cyl}(t)=0, &\qquad \Pi_{<,\tau^-}\gamma_0(\partial_u+B)P_{cyl}(t)=0.
\end{aligned}$$

We define the operator $Q_{cyl}(t): C^\infty(Y_r)\to C^\infty(Y_r)$ by $Q_{cyl}(t)=\gamma_r\partial_u P_{cyl}(t)$ and finally define

$$\begin{aligned}
R_{cyl}(t) &= \Pi_{>,\sigma^+}Q_{cyl}(t)\Pi_{>,\sigma^+}+|B|+\sigma^-\\
&= \Pi_{>,\sigma^+}(Q_{cyl}(t)+|B|)\Pi_{>,\sigma^+}+|B|\Pi_<+\sigma^-.
\end{aligned}$$

Then $\left(\Pi_{>,\sigma^+}\left(Q_{cyl}(t)+|B|\right)\Pi_{>,\sigma^+}\right)$ is described as follows, which can be checked by direct computation.

Lemma 4.1. *Suppose that $Bf=\lambda f$ and $\widetilde{Q}_{cyl}(t)=\left(\Pi_{>,\sigma^+}\left(Q_{cyl}(t)+|B|\right)\Pi_{>,\sigma^+}\right)$.*

(1) If $\lambda>0$,

$$\widetilde{Q}_{cyl}(t)f=\left(\sqrt{\lambda^2+t}+\lambda+\frac{2\sqrt{\lambda^2+t}\;e^{-r\sqrt{\lambda^2+t}}}{e^{r\sqrt{\lambda^2+t}}-e^{-r\sqrt{\lambda^2+t}}}\right)f.$$

(2) If $\lambda=0$ and $f\in Im\,\sigma^+\cap Im\,\tau^-$,

$$\widetilde{Q}_{cyl}(t)f=\left(\frac{\sqrt{t}\,\left(e^{r\sqrt{t}}-e^{-r\sqrt{t}}\right)}{e^{r\sqrt{t}}+e^{-r\sqrt{t}}}\right)f.$$

(3) If $\lambda=0$ and $f\in Im\,\sigma^+\cap(Im\,\sigma^+\cap Im\,\tau^-)^\perp$,

$$\widetilde{Q}_{cyl}(t)f=\left(\frac{\sqrt{t}\,\left(e^{r\sqrt{t}}-e^{-r\sqrt{t}}\right)}{e^{r\sqrt{t}}+e^{-r\sqrt{t}}}\frac{I+\sigma}{2}+\frac{4\sqrt{t}}{e^{2r\sqrt{t}}-e^{-2r\sqrt{t}}}\frac{I+\sigma}{2}\frac{I-\tau}{2}\frac{I+\sigma}{2}\right)f,$$

Proof : (1) is straightforward. If $Bf = 0$ and $f \in Im\ \sigma^+$, $P_{cyl}(t)(f)$ is given by

$$P_{cyl}(t)(f)(u,y) = \frac{e^{\sqrt{t}u} + e^{-\sqrt{t}u}}{e^{r\sqrt{t}} + e^{-r\sqrt{t}}} \frac{I+\sigma}{2} f(y) + \frac{2\left(e^{\sqrt{t}(u-r)} - e^{-\sqrt{t}(u-r)}\right)}{e^{2r\sqrt{t}} - e^{-2r\sqrt{t}}} \frac{I+\tau}{2} \frac{I+\sigma}{2} f(y).$$

Taking the derivative of $P_{cyl}(t)(f)(u,y)$ with respect to u at $u = r$ gives (2) and (3). $\square$

Corollary 4.2.

$$\Pi_{>,\sigma^+} \left(Q_{cyl}(t) + |B|\right) \Pi_{>,\sigma^+} \ = \ \Pi_{>,\sigma^+} \left(\sqrt{B^2+t} + |B|\right) \Pi_{>,\sigma^+} + \textit{a smoothing operator.}$$

Proceeding as in the proof of Theorem 1.1, we obtain the following result.

Lemma 4.3.

$$\begin{aligned}\log Det \left(-\partial_u^2 + B^2 + t\right)_{N_{0,r},\Pi_{>,\tau^+},\Pi_{<,\sigma^-}} &- \log Det \left(-\partial_u^2 + B^2 + t\right)_{N_{0,r},\Pi_{>,\tau^+},\gamma_r} \\ &= \sum_{j=0}^{\nu-1} a_j t^j + \log Det \left(\Pi_{>,\sigma^+}(Q_{cyl}(t) + |B|)\Pi_{>,\sigma^+}\right) + \log Det^*(|B|\Pi_<).\end{aligned}$$

Using the same argument as in the proof of Theorem 1.1, it is not difficult to see that the zero coefficients of the asymptotic expansions, as $t \to \infty$, of $\log Det \left(-\partial_u^2 + B^2 + t\right)_{N_{0,r},\Pi_{>,\tau^+},\Pi_{<,\sigma^-}}$, $\log Det \left(-\partial_u^2 + B^2 + t\right)_{N_{0,r},\Pi_{>,\tau^+},\gamma_r}$ and $\log Det \left(\Pi_{>,\sigma^+}(Q_{cyl}(t) + |B|)\Pi_{>,\sigma^+}\right)$ are zero, which implies that $a_0 + \log Det^*(|B|\Pi_<) = 0$. We next discuss the behavior of each term in Lemma 4.3 as $t \to 0$. We denote $\mathfrak{M} = Im\ \sigma^+ \cap (Im\ \sigma^+ \cap Im\ \tau^-)^{\perp}$,

$$k_+ = dim\ (\ Im\ \sigma^+ \cap Im\ \tau^-) \quad \text{ and } \quad \frac{l}{2} - k_+ = dim\ \mathfrak{M}.$$

The equality (4.1) and the invertibility of $(-\partial_u^2 + B^2)_{N_{0,r},\Pi_{>,\tau^+},\gamma_r}$ imply that

$$\begin{aligned}\log Det \left(-\partial_u^2 + B^2 + t\right)_{\Pi_{>,\tau^+},\Pi_{<,\sigma^-}} &= k_+ \log t + \log Det^* \left(-\partial_u^2 + B^2\right)_{\Pi_{>,\tau^+},\Pi_{<,\sigma^-}} + o(t), \\ \log Det \left(-\partial_u^2 + B^2 + t\right)_{\Pi_{>,\tau^+},\gamma_r} &= \log Det \left(-\partial_u^2 + B^2\right)_{\Pi_{>,\tau^+},\gamma_r} + o(t).\end{aligned} \tag{4.3}$$

Simple computation shows that, as $t \to 0$,

$$\log \left(\frac{\sqrt{t}\ \left(e^{r\sqrt{t}} - e^{-r\sqrt{t}}\right)}{e^{r\sqrt{t}} + e^{-r\sqrt{t}}} \right) = \log r + \log t + o(t) \quad \text{ and } \quad \log \left(\frac{4\sqrt{t}}{e^{2r\sqrt{t}} - e^{-2r\sqrt{t}}} \right) = -\log r + o(t)\ ,$$

which lead to

$$\begin{aligned}\log Det \left(\Pi_{>,\sigma^+}(Q_{cyl}(t) + |B|)\Pi_{>,\sigma^+}\right) &= \log Det^* \left(\left(2|B| + \frac{2|B|e^{-r|B|}}{e^{r|B|} - e^{-r|B|}} \right) \Pi_> \right) \\ &\quad + k_+(\log r + \log t) + \log det \left(\frac{1}{r} \frac{I+\sigma}{2} \frac{I+\tau}{2} \frac{I+\sigma}{2} \right) |_{\mathfrak{M}} + o(t)\end{aligned}$$

$$= \frac{1}{2}\log Det^*(2|B|) + \frac{1}{2}\log det_{Fr}\left(I + \frac{e^{-r|B|}}{e^{r|B|}-e^{-r|B|}} pr_{(kerB)^\perp}\right) + (2k_+ - \frac{l}{2})\log r + k_+ \log t$$
$$+ \log det\left(\frac{I+\sigma}{2}\frac{I+\tau}{2}\frac{I+\sigma}{2}\right)|_{\mathfrak{M}} + o(t). \quad (4.4)$$

Letting $t \to 0$, Lemma 4.3 together with (4.3) and (4.4) imply that

$$\log Det^*\left(-\partial_u^2 + B^2\right)_{N_{0,r},\Pi_{>,\tau^+},\Pi_{<,\sigma^-}} - \log Det\left(-\partial_u^2 + B^2\right)_{N_{0,r},\Pi_{>,\tau^+},\gamma_r} = \frac{1}{2}\log Det^*(2|B|)$$
$$+ (2k_+ - \frac{l}{2})\log r + \frac{1}{2}\log det_{Fr}\left(I + \frac{e^{-r|B|}}{e^{r|B|}-e^{-r|B|}} pr_{(kerB)^\perp}\right) + \log det\left(\frac{I+\sigma}{2}\frac{I+\tau}{2}\frac{I+\sigma}{2}\right)|_{\mathfrak{M}}. \quad (4.5)$$

Lemma 4.4.
$$det\left(\frac{I+\sigma}{2}\frac{I+\tau}{2}\frac{I+\sigma}{2}\right)|_{\mathfrak{M}} \;=\; |det^*\left(\frac{\sigma+\tau}{2}\right)|,$$

where $det^*\left(\frac{\sigma+\tau}{2}\right) = det\left(\frac{\sigma+\tau}{2} + pr_{ker\frac{\sigma+\tau}{2}}\right)$.

Proof : If we denote $\Sigma^{\pm} = (\,Im\ \sigma^{\pm} \cap Im\ \tau^{\mp})$, we have

$$det\left(\frac{I+\sigma}{2}\frac{I+\tau}{2}\frac{I+\sigma}{2}\right)|_{\mathfrak{M}} \;= det\left(\frac{I-\sigma}{2} + pr_{\Sigma^+} + \frac{I+\sigma}{2}\frac{I+\tau}{2}\frac{I+\sigma}{2}\right).$$

Since $detG = 1$ and $G \circ\ pr_{\Sigma^+} = pr_{\Sigma^-}\ \circ G$, we have

$$det\left(\frac{I-\sigma}{2} + pr_{\Sigma^+} + \frac{I+\sigma}{2}\frac{I+\tau}{2}\frac{I+\sigma}{2}\right) = det\left(G\left(\frac{I-\sigma}{2} + pr_{\Sigma^+} + \frac{I+\sigma}{2}\frac{I+\tau}{2}\frac{I+\sigma}{2}\right)\right)$$
$$= det\left(\left(\frac{I+\sigma}{2} + pr_{\Sigma^-} + \frac{I-\sigma}{2}\frac{I-\tau}{2}\frac{I-\sigma}{2}\right)G\right) = det\left(\frac{I+\sigma}{2} + pr_{\Sigma^-} + \frac{I-\sigma}{2}\frac{I-\tau}{2}\frac{I-\sigma}{2}\right).$$

Therefore, we have

$$\left(det\left(\frac{I-\sigma}{2} + pr_{\Sigma^+} + \frac{I+\sigma}{2}\frac{I+\tau}{2}\frac{I+\sigma}{2}\right)\right)^2$$
$$= det\left(\frac{I-\sigma}{2} + pr_{\Sigma^+} + \frac{I+\sigma}{2}\frac{I+\tau}{2}\frac{I+\sigma}{2}\right)\left(\frac{I+\sigma}{2} + pr_{\Sigma^-} + \frac{I-\sigma}{2}\frac{I-\tau}{2}\frac{I-\sigma}{2}\right)$$
$$= det\left(pr_{\Sigma^+} + pr_{\Sigma^-} + \left(\frac{\sigma+\tau}{2}\right)^2\right)$$
$$= det\left(pr_{ker(\sigma+\tau)} + \left(\frac{\sigma+\tau}{2}\right)^2\right) = \left(det^*\left(\frac{\sigma+\tau}{2}\right)\right)^2.$$

Since the determinant of an operator that we want to compute is positive, the result follows. $\square$

Since $\log Det(2|B|) = \log 2 \cdot \zeta_{B^2}(0) + \frac{1}{2}\log Det^* B^2$, (4.5) leads to the following result.

Theorem 4.5.

$$\log Det^* \left(-\partial_u^2 + B^2\right)_{N_{0,r},\Pi_{>,\tau^+},\Pi_{<,\sigma^-}} - \log Det \left(-\partial_u^2 + B^2\right)_{N_{0,r},\Pi_{>,\tau^+},\gamma_r} = (2k_+ - \frac{l}{2}) \log r +$$
$$\frac{1}{2}\log 2 \cdot \zeta_{B^2}(0) + \frac{1}{4}\log Det^* B^2 + \log |det^* \left(\frac{\sigma+\tau}{2}\right)| + \frac{1}{2}\log det_{Fr}\left(I + \frac{e^{-r|B|}}{e^{r|B|} - e^{-r|B|}} pr_{(kerB)^\perp}\right).$$

Corollary 4.6.

$$\log Det^* \left(-\partial_u^2 + B^2\right)_{N_{0,r},\Pi_{>,\tau^+},\Pi_{<,\sigma^-}} - \log Det \left(-\partial_u^2 + B^2\right)_{N_{0,r},\gamma_0,\gamma_r} = (2k_+ - l) \log r +$$
$$\log 2 \cdot \zeta_{B^2}(0) + \frac{1}{2}\log Det^* B^2 + \log |det^* \left(\frac{\sigma+\tau}{2}\right)| + \log det_{Fr}\left(I + \frac{e^{-r|B|}}{e^{r|B|} - e^{-r|B|}} pr_{(kerB)^\perp}\right).$$

It is a well-known fact (*cf.* [LP1] or [MM]) that

$$\log Det(-\partial_u^2 + B^2)_{N_{0,r},\gamma_0,\gamma_r} = l \cdot \log 2 + l \cdot \log r + \alpha_1 \cdot r - \frac{1}{2}\log Det^* B^2 + \log det_{Fr}(I - e^{-2r|B|} pr_{(kerB)^\perp}), \quad (4.6)$$

where α_1 is the constant defined in (1.12). For any positive real number μ, we note that

$$\left(1 - e^{-2r\mu}\right)\left(1 + \frac{e^{-r\mu}}{e^{r\mu} - e^{-r\mu}}\right) = 1.$$

Corollary 4.6 and (4.6) with this observation lead to

$$\log Det^* \left(-\partial_u^2 + B^2\right)_{N_{0,r},\Pi_{>,\tau^+},\Pi_{<,\sigma^-}} = \alpha_1 \cdot r + 2k_+ \log r + \log 2 \cdot (\zeta_{B^2}(0) + l) + \log |det^* \left(\frac{\sigma+\tau}{2}\right)|,$$

which completes the proof of the first equality in Theorem 1.6.

To prove the second equality, we play the same game with $(-\partial_u^2 + B^2 + t)_{N_{0,r},\gamma_0,(\partial_u+|B|)}$ and $(-\partial_u^2 + B^2 + t)_{N_{0,r},\gamma_0,\gamma_r}$. We define $R_{(\partial_u+|B|)}(t) : C^\infty(Y_r) \to C^\infty(Y_r)$ corresponding to $R_P(t)$ in Theorem 1.1 as follows.

$$R_{(\partial_u+|B|)}(t) = \gamma_r(\partial_u + |B|)P_{\gamma_r}(t) = Q_1(t) + |B|,$$

where $P_{\gamma_r}(t)$ is the Poisson operator defined on Y_r characterized as follows.

$$(-\partial_u^2 + B^2 + t)P_{\gamma_r}(t) = 0, \quad \gamma_0 P_{\gamma_r}(t) = 0, \quad \gamma_r P_{\gamma_r}(t) = Id.$$

Then proceeding as in the proof of Theorem 1.1, we have the following equality.

$$\log Det(-\partial_u^2 + B^2 + t)_{N_{0,r},\gamma_0,(\partial_u+|B|)} - \log Det(-\partial_u^2 + B^2 + t)_{N_{0,r},\gamma_0,\gamma_r} = \sum_{j=0}^{\nu-1} a_j t^j + \log Det(Q_1(t) + |B|).$$

In the above equality, $a_0 = 0$ because the zero coefficients of the asymptotic expansions of $\log Det(-\partial_u^2 + B^2 + t)_{N_{0,r},\gamma_0,(\partial_u+|B|)}$, $\log Det(-\partial_u^2 + B^2 + t)_{N_{0,r},\gamma_0,\gamma_r}$ and $\log Det(Q_1(t) + |B|)$, as $t \to \infty$, are zero. Moreover, $(-\partial_u^2 + B^2)_{N_{0,r},\gamma_0,(\partial_u+|B|)}$, $(-\partial_u^2 + B^2)_{N_{0,r},\gamma_0,\gamma_r}$ and $(Q_1 + |B|)$ are invertible operators, which yields the following equality.

$$\log Det(-\partial_u^2 + B^2)_{N_{0,r},\gamma_0,(\partial_u+|B|)} - \log Det(-\partial_u^2 + B^2)_{N_{0,r},\gamma_0,\gamma_r} = \log Det(Q_1 + |B|).$$

Since $Q_1 = \frac{1}{r} pr_{kerB} + |B| + \frac{2|B|e^{-r|B|}}{e^{r|B|} - e^{-r|B|}} pr_{(kerB)^\perp}$ (*cf.* (1.10)), we have the following result, from which the second equality of Theorem 1.6 follows by (4.6).

Theorem 4.7.

$$\log Det(-\partial_u^2 + B^2)_{N_{0,r},\gamma_0,(\partial_u+|B|)} - \log Det(-\partial_u^2 + B^2)_{N_{0,r},\gamma_0,\gamma_r} =$$
$$- l \cdot \log r + \log 2 \cdot \zeta_{B^2}(0) + \frac{1}{2} \log Det^* B^2 + \log det_{Fr} \left(I + \frac{e^{-r|B|}}{e^{r|B|} - e^{-r|B|}} pr_{(ker B)^{\perp}} \right).$$

§5. **The Proof of Theorem 1.7**

In this section, we are going to prove Theorem 1.7. For simplicity, we denote $(-\partial_u^2+B^2)_{N_{0,\infty},\gamma_0}$, $(-\partial_u^2+B^2)_{N_{0,\infty},\Pi_{>,\tau^+}}$ by Δ_{∞,γ_0}, $\Delta_{\infty,\Pi_{>,\tau^+}}$, respectively. Then the equation (1.13) imply that

$$\log Det\left(D^2_{M_1,\infty}, \Delta_{\infty,\Pi_{>,\tau^+}}\right) - \log Det^*\left(D^2_{M_1,\Pi_{<,\sigma^-}}\right) = -\log 2 \cdot (\zeta_{B^2}(0) + l) - \log det A_1 +$$
$$+ \log Det\left(\Delta_{\infty,\gamma_0}, \Delta_{\infty,\Pi_{>,\tau^+}}\right) + \log Det^*(Q_1 + |B|) + \log Det D^2_{M,\gamma_0} - \log Det^*\left(D^2_{M_1,\Pi_{<,\sigma^-}}\right). \tag{5.1}$$

We now compute the relative zeta-determinant $\log Det\left(\Delta_{\infty,\gamma_0}, \Delta_{\infty,\Pi_{>,\tau^+}}\right)$. The relative zeta-function $\zeta(s, \Delta_{\infty,\gamma_0}, \Delta_{\infty,\Pi_{>,\tau^+}})$ is defined by

$$\zeta(s, \Delta_{\infty,\gamma_0}, \Delta_{\infty,\Pi_{>,\tau^+}}) =$$
$$\frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \int_{N_{0,\infty}} \left(e^{-t\Delta_{\infty,\gamma_0}}(t,(u,y),(u,y)) - e^{-t\Delta_{\infty,\Pi_{>,\tau^+}}}(t,(u,y),(u,y))\right) dvol(y) du dt$$

and $\log Det\left(\Delta_{\infty,\gamma_0}, \Delta_{\infty,\Pi_{>,\tau^+}}\right) = -\frac{d}{ds}|_{s=0}\zeta(s, \Delta_{\infty,\gamma_0}, \Delta_{\infty,\Pi_{>,\tau^+}})$. It is a well-known fact (*cf.* [APS] or [BW]) that the heat kernel $e^{-t\Delta_{\infty,\gamma_0}}(t,(u,y),(v,z))$ and $e^{-t\Delta_{\infty,\Pi_{>,\tau^+}}}(t,(u,y),(v,z))$ are given as follows.

$$e^{-t\Delta_{\infty,\gamma_0}}(t,(u,y),(v,z)) = \sum_{\mu_j \in Spec(B)} \frac{e^{-\mu_j^2 t}}{\sqrt{4\pi t}} \left\{ e^{-\frac{(u-v)^2}{4t}} - e^{-\frac{(u+v)^2}{4t}} \right\} \varphi_j(y) \otimes \varphi_j(z).$$

$$e^{-t\Delta_{\infty,\Pi_{>,\tau^+}}}(t,(u,y),(v,z)) = \sum_{0<\mu_j \in Spec(B)} \frac{e^{-\mu_j^2 t}}{\sqrt{4\pi t}} \left\{ e^{-\frac{(u-v)^2}{4t}} - e^{-\frac{(u+v)^2}{4t}} \right\} \varphi_j(y) \otimes \varphi_j(z)$$

$$+ \sum_{\phi_j \in \tau^-} \frac{1}{\sqrt{4\pi t}} \left\{ e^{-\frac{(u-v)^2}{4t}} - e^{-\frac{(u+v)^2}{4t}} \right\} \phi_j(y) \otimes \phi_j(z) + \sum_{\psi_j \in \tau^+} \frac{1}{\sqrt{4\pi t}} \left\{ e^{-\frac{(u-v)^2}{4t}} + e^{-\frac{(u+v)^2}{4t}} \right\} \psi_j(y) \otimes \psi_j(z)$$

$$+ \sum_{0<\mu_j \in Spec(B)} \left\{ \frac{e^{-\mu_j^2 t}}{\sqrt{4\pi t}} \left\{ e^{-\frac{(u-v)^2}{4t}} + e^{-\frac{(u+v)^2}{4t}} \right\} - \mu_j e^{\mu_j(u+v)} erfc(\frac{u+v}{2\sqrt{t}} + \mu_j \sqrt{t}) \right\} G\varphi_j(y) \otimes G\varphi_j(z),$$

where $B\varphi_j = \mu_j \varphi_j$ and $erfc(x)$ is the error function defined by $erfc(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt$. Then direct computation shows that

$$Tr\left(e^{-t\Delta_{\infty,\gamma_0}} - e^{-t\Delta_{\infty,\Pi_{>,\tau^+}}}\right) = -\frac{l}{4} - \frac{1}{2} \sum_{\mu_j > 0} erfc(\mu_j \sqrt{t}).$$

According to [M2] we split $\zeta(s,\Delta_{\infty,\gamma_0},\Delta_{\infty,\Pi_{>,\tau^+}})$ into two parts.

$$\zeta(s,\Delta_{\infty,\gamma_0},\Delta_{\infty,\Pi_{>,\tau^+}})=\zeta_1(s,\Delta_{\infty,\gamma_0},\Delta_{\infty,\Pi_{>,\tau^+}})+\zeta_2(s,\Delta_{\infty,\gamma_0},\Delta_{\infty,\Pi_{>,\tau^+}}),$$

where

$$\zeta_1(s,\Delta_{\infty,\gamma_0},\Delta_{\infty,\Pi_{>,\tau^+}})=\frac{1}{\Gamma(s)}\int_0^1 t^{s-1}Tr\left(e^{-t\Delta_{\infty,\gamma_0}}-e^{-t\Delta_{\infty,\Pi_{>,\tau^+}}}\right)dt,$$
$$\zeta_2(s,\Delta_{\infty,\gamma_0},\Delta_{\infty,\Pi_{>,\tau^+}})=\frac{1}{\Gamma(s)}\int_1^\infty t^{s-1}Tr\left(e^{-t\Delta_{\infty,\gamma_0}}-e^{-t\Delta_{\infty,\Pi_{>,\tau^+}}}\right)dt.$$

For $Res>0$,

$$\begin{aligned}\zeta_1(s,\Delta_{\infty,\gamma_0},\Delta_{\infty,\Pi_{>,\tau^+}}) &= -\frac{l}{4}\frac{1}{\Gamma(s+1)}-\frac{1}{2}\frac{1}{\Gamma(s+1)}\sum_{\mu_j>0}erfc(\mu_j)-\frac{1}{4\sqrt{\pi}}\,\frac{\Gamma(s+\frac{1}{2})}{\Gamma(s+1)}\,\zeta_{B^2}(s)\\ &\quad+\frac{1}{\sqrt{4\pi}}\frac{1}{\Gamma(s+1)}\sum_{\mu_j>0}\int_1^\infty t^{s-\frac{1}{2}}\mu_j e^{-t\mu_j^2}dt. \qquad (5.2)\end{aligned}$$

For $Res<0$,

$$\begin{aligned}\zeta_2(s,\Delta_{\infty,\gamma_0},\Delta_{\infty,\Pi_{>,\tau^+}}) &= \frac{l}{4}\frac{1}{\Gamma(s+1)}+\frac{1}{2}\frac{1}{\Gamma(s+1)}\sum_{\mu_j>0}erfc(\mu_j)\\ &\quad-\frac{1}{\sqrt{4\pi}}\frac{1}{\Gamma(s+1)}\sum_{\mu_j>0}\int_1^\infty t^{s-\frac{1}{2}}\mu_j e^{-t\mu_j^2}dt. \qquad (5.3)\end{aligned}$$

Since the last terms in (5.2) and (5.3) are entire functions, they give the meromorphic continuations of $\zeta_1(s,\Delta_{\infty,\gamma_0},\Delta_{\infty,\Pi_{>,\tau^+}})$ and $\zeta_2(s,\Delta_{\infty,\gamma_0},\Delta_{\infty,\Pi_{>,\tau^+}})$ to the whole complex plane, having regular values at $s=0$. Therefore, we have

$$\zeta(s,\Delta_{\infty,\gamma_0},\Delta_{\infty,\Pi_{>,\tau^+}})=-\frac{1}{4\sqrt{\pi}}\,\frac{\Gamma(s+\frac{1}{2})}{\Gamma(s+1)}\,\zeta_{B^2}(s).$$

Since $\Gamma'(\frac{1}{2})=-\sqrt{\pi}(\gamma+2\log 2)$ (*cf.* p. 15 in [MOS]) for $\gamma=-\Gamma'(1)$ the Euler constant, we have the following result.

Lemma 5.1.

$$\log Det\left(\Delta_{\infty,\gamma_0},\Delta_{\infty,\Pi_{>,\tau^+}}\right)=-\frac{1}{2}\log 2\cdot\zeta_{B^2}(0)-\frac{1}{4}\log Det^*B^2.$$

The above lemma together with (5.1) leads to the following equality.

$$\begin{aligned}\log Det\left(D^2_{M_{1,\infty}},(-\partial_u^2+B^2)_{N_{0,\infty},\Pi_{>,\tau^+}}\right)-\log Det(D^2_{M_1,\Pi_{<,\sigma^-}}) &= -\log detA_1-\frac{1}{4}\log Det^*B^2\\ -\log 2\cdot(\frac{3}{2}\zeta_{B^2}(0)+l)+\log Det^*(Q_1+|B|)+\log DetD^2_{M_1,\gamma_0}&-\log Det^*\left(D^2_{M_1,\Pi_{<,\sigma^-}}\right). \qquad (5.4)\end{aligned}$$

Next, we are going to analyze the term $\log Det^*(Q_1+|B|)$. Let $L_{2,M_{1,\infty}}$, $L^{ext}_{2,M_{1,\infty}}$ be the spaces of all L^2- and extended L^2-solutions of $D_{M_{1,\infty}}$ on $M_{1,\infty}$. Then it is not difficult to show (*cf.* [L3] or [L4]) that

$$ker\,(Q_1+|B|)=\left\{\phi|_Y\mid\phi\in\left(L_{2,M_{1,\infty}}+L^{ext}_{2,M_{1,\infty}}\right)\right\}=Im\mathfrak{C}_1\cap Im\Pi_{>,C(0)^+}. \qquad (5.5)$$

Using (5.5) we decompose $L^2(Y, E|_Y)$ by

$$L^2(Y, E|_Y) = ker\,(Q_1 + |B|) \oplus \left(Im\,(I - \mathfrak{C}_1) + Im\left(I - \Pi_{>, C(0)^+}\right)\right), \tag{5.6}$$

where $dim\ \left(Im\,(I - \mathfrak{C}_1) \cap Im\left(I - \Pi_{>, C(0)^+}\right)\right) = dim\ ker\,(Q_1 + |B|)$. Let $K_1,\ T_0 : L^2(E_Y^+) \to L^2(E_Y^-)$ be unitary maps whose graphs are $Im\ \mathfrak{C}_1,\ Im\ \Pi_{>, C(0)^+}$, respectively. We now consider $\left(Q_1 + |B| + pr_{ImC(0)^-}\right)$ rather than $(Q_1 + |B|)$. Using Lemma 3.3,

$$\begin{aligned}\left(Q_1 + |B| + pr_{ImC(0)^-}\right)(I - K_1)\,x &= (I - \mathfrak{C}_1)\,(Q_1 + B)\,(I - \mathfrak{C}_1)\,(I - K_1)x \\ &+\ \left(I - \Pi_{>, C(0)^+}\right)\left(|B| - B + pr_{ImC(0)^-}\right)\left(I - \Pi_{>, C(0)^+}\right)(I - T_0)\,\frac{I + T_0^{-1}K_1}{2}x.\end{aligned}$$

$$\begin{aligned}\left(Q_1 + |B| + pr_{ImC(0)^-}\right)(I - T_0)\,y &= (I - \mathfrak{C}_1)\,(Q_1 + B)\,(I - \mathfrak{C}_1)\,(I - K_1)\,\frac{I + K_1^{-1}T_0}{2}y \\ &+\ \left(I - \Pi_{>, C(0)^+}\right)\left(|B| - B + pr_{ImC(0)^-}\right)\left(I - \Pi_{>, C(0)^+}\right)(I - T_0)y.\end{aligned}$$

Recall that $ker\,(Q_1 + |B|) = ker\left(Q_1 + |B| + pr_{ImC(0)^-}\right) = \{(I + K_1)x \mid K_1x = T_0x\}$ and we denote it by H. We now define a subspace $\widetilde{H}_-$ of $Im\,(I - \mathfrak{C}_1) \oplus Im\left(I - \Pi_{>, C(0)^+}\right)$ by

$$\widetilde{H}_- = \{(I - K_1)x, -(I - K_2)x \mid K_1x = K_2x\},$$

and consider the following diagram.

$$\begin{array}{ccc} Im\,(I - \mathfrak{C}_1) \oplus Im\left(I - \Pi_{>, C(0)^+}\right) & \xrightarrow{\widetilde{R}} & Im\,(I - \mathfrak{C}_1) \oplus Im\left(I - \Pi_{>, C(0)^+}\right) \\ \Phi\big\downarrow & & \big\downarrow\Phi \\ \left(Im\,(I - \mathfrak{C}_1) + Im\left(I - \Pi_{>, C(0)^+}\right)\right) \oplus \widetilde{H}_- & \xrightarrow{\widetilde{Q}} & \left(Im\,(I - \mathfrak{C}_1) + Im\left(I - \Pi_{>, C(0)^+}\right)\right) \oplus \widetilde{H}_-, \end{array} \tag{5.7}$$

where Φ, $\widetilde{Q}$ and $\widetilde{R}$ are defined as follows.

$$\begin{aligned}\Phi\left((I - K_1)x,\ (I - T_0)y\right) &= \Big((I - K_1)x + (I - T_0)y,\ pr_{\widetilde{H}_-}\left((I - K_1)x,\ (I - T_0)y\right)\Big), \\ \widetilde{Q}\,(a, b) &= \Big(\left(Q_1 + |B| + pr_{ImC(0)^-}\right)(a),\ pr_{\widetilde{H}_-}\ \widetilde{R}\ \Phi^{-1}(a, b)\Big), \\ \widetilde{R} &= \begin{pmatrix} \mathfrak{S}_1 & \mathfrak{S}_1(I - K_1)\frac{I + K_1^{-1}T_0}{2}(I - T_0)^{-1} \\ \mathfrak{S}_2(I - T_0)\frac{I + T_0^{-1}K_1}{2}(I - K_1)^{-1} & \mathfrak{S}_2 \end{pmatrix} pr_{(\widetilde{H}_-)^{\perp}} + pr_{\widetilde{H}_-} \\ &= \begin{pmatrix} \mathfrak{S}_1 & 0 \\ 0 & \mathfrak{S}_2 \end{pmatrix}\begin{pmatrix} I & (I - K_1)\frac{I + K_1^{-1}T_0}{2}(I - T_0)^{-1} \\ (I - T_0)\frac{I + T_0^{-1}K_1}{2}(I - K_1)^{-1} & I \end{pmatrix} + pr_{\widetilde{H}_-},\end{aligned}$$

where $\mathfrak{S}_1 = (I - \mathfrak{C}_1)\,(Q_1 + B)\,(I - \mathfrak{C}_1)$, $\mathfrak{S}_2 = \Pi_{<, C(0)^-}\left(|B| - B + pr_{ImC(0)^-}\right)\Pi_{<, C(0)^-}$. Then all maps are invertible and the diagram (5.7) commutes. As the same way as in Section 2, we have

$$\begin{aligned}&\log Det\widetilde{Q} = \log Det^*\left(Q_1 + |B| + pr_{ImC(0)^-}\right) \\ &= \log Det\left((I - \mathfrak{C}_1)\,(Q_1 + B)\,(I - \mathfrak{C}_1)\right) + \log Det^*\left(2|B|\Pi_<\right) + q\log 2 + \log|det^*\left(\frac{I - K_1^{-1}T_0}{2}\right)|^2 \\ &\quad - q\log 2 + \log det\left(pr_{GH}\left((Q_1 + B)^{-1} + (|B| - B + pr_{ImC(0)^-})^{-1}\right)pr_{GH}\right) \\ &= \log DetD^2_{M_1, \mathfrak{C}_1} - \log DetD^2_{M_1, \gamma_0} + \frac{1}{2}\log 2 \cdot \zeta_{B^2}(0) + \frac{1}{4}\log Det^*B^2 + \log|det^*\left(\frac{I - K_1^{-1}T_0}{2}\right)|^2 \\ &\quad + \log det\left(pr_{GH}\left((Q_1 + B)^{-1} + (|B| - B + pr_{ImC(0)^-})^{-1}\right)pr_{GH}\right),\end{aligned} \tag{5.8}$$

where $q = dimker\,(Q_1 + |B|)$.

We next discuss the relation between $\log Det^*\left(Q_1 + |B| + pr_{ImC(0)^-}\right)$ and $\log Det^*\,(Q_1 + |B|)$. Since $ker(Q_1 + |B|) = ker(Q_1 + |B| + pr_{ImC(0)^-})$, we have

$$\begin{aligned}
&\log Det^*\left(Q_1 + |B| + pr_{ImC(0)^-}\right) = \log Det\left(Q_1 + |B| + pr_{ImC(0)^-} + pr_{ker(Q_1+|B|)}\right) \\
&= \log Det\left(Q_1 + |B| + pr_{ker(Q_1+|B|)}\right) + \log det_{Fr}\left(I + \left(Q_1 + |B| + pr_{ker(Q_1+|B|)}\right)^{-1} pr_{ImC(0)^-}\right) \\
&= \log Det\left(Q_1 + |B| + pr_{ker(Q_1+|B|)}\right) \\
&\qquad + \log det_{Fr}\left(I + pr_{ImC(0)^-}\left(Q_1 + |B| + pr_{ker(Q_1+|B|)}\right)^{-1} pr_{ImC(0)^-}\right).
\end{aligned}$$

It was shown in Lemma 5.2 of [L6] that

$$pr_{kerB}\left(Q_1 + |B| + pr_L + pr_{ImC(0)^+}\right)^{-1} \frac{I - C(0)}{2} = \frac{i}{2} C'(0) \frac{I - C(0)}{2},$$

where L is the space of restriction of L^2-solutions of $D_{M_{1,\infty}}$ to Y, *i.e.* $L = \left(L_{2,M_{1,\infty}}\right)|_Y$ and $C'(0) = \frac{d}{d\lambda} C(\lambda)|_{\lambda=0}$. Using this result, it is not difficult to show that

$$\frac{I - C(0)}{2}\left(Q_1 + |B| + pr_{ker(Q_1+|B|)}\right)^{-1} \frac{I - C(0)}{2} \;=\; \frac{i}{2}\frac{I - C(0)}{2} C'(0) \frac{I - C(0)}{2},$$

which leads to the following lemma.

Lemma 5.2.

$$\log Det^*\left(Q_1 + |B| + pr_{ImC(0)^-}\right) = \log Det^*\,(Q_1 + |B|) + \log det\left(I + \frac{i}{2}\frac{I - C(0)}{2} C'(0) \frac{I - C(0)}{2}\right).$$

Finally, we are going to analyze the last term in the last equality of (5.8). Let $\{h_1, \cdots, h_q\}$ be an orthonormal basis for $Im\,\mathfrak{C}_1 \cap Im\,\Pi_{>,C(0)^+}$. Then $\{Gh_1, \cdots, Gh_q\}$ is an orthonormal basis for $Im\,(I - \mathfrak{C}_1) \cap Im\,\Pi_{<,C(0)^-}$. Let $\varphi_1, \cdots, \varphi_q$ be elements in $\left(L_{2,M_{1,\infty}} + L^{ext}_{2,M_{1,\infty}}\right)$ such that $\varphi_i|_Y = h_i$. Then as the same way as in Lemma 3.1 and 3.4, we can show that

$$\langle (Q_1 + B)^{-1} Gh_i,\; Gh_j \rangle_Y = \langle \varphi_i|_{M_1},\; \varphi_j|_{M_1} \rangle_{M_1}. \tag{5.9}$$

We denote by $\varphi_{i,0}$ the limiting value of φ_i and $\varphi_{i,0} = 0$ if φ_i is an L^2-solution. We define φ_{i,L^2} by (1.15). Then using the same argument as in Lemma 3.1, we can show that

$$\langle (|B| - B + pr_{ImC(0)^-})^{-1} Gh_i,\; Gh_j \rangle_Y = \langle \varphi_{i,0},\; \varphi_{j,0} \rangle_Y + \langle \varphi_{i,L^2}|_{N_{0,\infty}},\; \varphi_{j,L^2}|_{N_{0,\infty}} \rangle_{N_{0,\infty}}, \tag{5.10}$$

where $N_{0,\infty} := [0, \infty) \times Y$. Hence, (5.9) and (5.10) imply that

$$\begin{aligned}
\langle \left((Q_1 + B)^{-1} + (|B| - B + pr_{ImC(0)^-})^{-1}\right) Gh_i, Gh_j \rangle_Y &= \langle \varphi_{i,0}, \varphi_{j,0} \rangle_Y + \langle \varphi_{i,L^2}, \varphi_{j,L^2} \rangle_{M_{1,\infty}} \\
&=: \mathfrak{w}_{ij},
\end{aligned} \tag{5.11}$$

where we denote $\mathfrak{W} = (\mathfrak{w}_{ij})$. On the other hand, let $\{\psi_1, \cdots, \psi_{q'}\}$ be an orthonormal basis for L^2-solutions of $D_{M_{1,\infty}}$ and $\{f_1, \cdots, f_{\frac{l}{2}}\}$ be an orthonormal basis for $Im\,C(0)^+$, where $q' + \frac{l}{2} = q$. We put $\psi_{q'+i} = \frac{1}{2}E(f_i, 0)$ $(1 \le i \le \frac{l}{2})$, where $\frac{1}{2}E(f_i, 0)$ is the extended L^2-solution of $D_{M_{1,\infty}}$ whose limiting value is f_i. Then $\psi_i = \sum_{j=1}^{q} c_{ij}\varphi_j$ for some $c_{ij} \in \mathbb{C}$ and we define a matric $C = (c_{ij})$. Note that

$$\psi_i|_Y = \sum_{j=1}^{q} c_{ij}\varphi_j|_Y = \sum_{j=1}^{q} c_{ij}h_j.$$

Setting $A_1 = \left(\langle \psi_i|_Y, \psi_j|_Y\rangle_Y\right)_{1\le i,j\le q}$, we have

$$A_1 = CC^*.$$

We denote by $\psi_{i,0}$ the limiting value of ψ_i and define ψ_{i,L^2} by the same way as (1.15). Then we have

$$\widetilde{V} := \left(\langle \psi_{i,0}, \psi_{j,0}\rangle_Y + \langle \psi_{i,L^2}, \psi_{j,L^2}\rangle_{M_{1,\infty}}\right)_{1\le i,j\le q} = C\mathfrak{W}C^*,$$

which shows that

$$\log det\left(pr_{GH}\left((Q_1+B)^{-1} + (|B| - B + pr_{ImC(0)^-})^{-1}\right)pr_{GH}\right) \;=\; -\log det A_1 + \log det\widetilde{V}. \quad (5.12)$$

Theorem 1.7 follows from (5.4), (5.8), (5.12), Lemma 5.2 and Theorem 1.2.

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