Meromorphic connections on $\mathbb{P}^1$ and the multiplicity of Abelian integrals

HOSSEIN MOVASATI [1]

Abstract

In this paper we introduce the concept of Abelian integrals in differential equations for an arbitrary vector bundle on $\mathbb{P}^1$ with a meromorphic connection. In this general context we give an upper bound for the numbers we are looking for.

Let V be a locally free sheaf (vector bundle) of rank α on $\mathbb{P}^1$ and $D = \sum_{i=1}^r m_i c_i$ be a positive divisor in $\mathbb{P}^1$, i.e. all c_i's are positive. We denote by C the set of c_i's. A meromorphic connection ∇ on V with the pole divisor D is a $\mathbb{C}$-linear homomorphism of sheaves

$$\nabla : V \to \Omega^1_{\mathbb{P}^1}(D) \otimes_{\mathcal{O}_{\mathbb{P}^1}} V$$

satisfying the Leibniz identity

$$\nabla(f\omega) = df \otimes \omega + f\nabla\omega,\ f \in \mathcal{O}_{\mathbb{P}^1}, \omega \in V$$

where $\Omega^1_{\mathbb{P}^1}(D)$ is the sheaf of meromorphic 1-forms in $\mathbb{P}^1$ with poles on D (the pole order of a section of $\Omega^1_{\mathbb{P}^1}(D)$ at c_i is less than m_i). For any two meromorphic connection ∇_1 and ∇_2 with the same pole divisor D, $\nabla_1 - \nabla_2$ is a $\mathcal{O}_{\mathbb{P}^1}$-linear map.

Let t be the affine coordinate of $\mathbb{C} = \mathbb{P}^1 - \{\infty\}$, where ∞ is the point at infinity in $\mathbb{P}^1$. By Leibniz rule and by composing ∇ with the holomorphic vector field $\frac{\partial}{\partial t}$ we can define:

$$\nabla_{\frac{\partial}{\partial t}} : H^0(\mathbb{P}^1, V(*\infty)) \to H^0(\mathbb{P}^1, V(D + *\infty))$$

where $*\infty$ means that the pole order at ∞ is arbitrary. Since $\frac{\partial}{\partial t}$ is a holomorphic vector field in $\mathbb{P}^1$ with a zero of multiplicity two at ∞, if $\omega \in H^0(\mathbb{P}^1, V(*\infty))$ has a pole (resp. zero) of order m at ∞ then $\nabla_{\frac{\partial}{\partial t}}\omega$

[1] Supported by MPIM-Germany

Keywords: Meromorphic connection, Gauss-Manin connection
Math. classification: 14F05, 14F43

has a pole (resp. zero) of order $m-1$ (resp. $max\{2, m+1\}$) at ∞. If there is no confusion we write $\nabla = \nabla_{\frac{\partial}{\partial t}}$.

For any point $b \in \mathbb{P}^1 \backslash C$ we can find a frame $\{e_1, e_2, \dots, e_\alpha\}$ of holomrphic sections of V in a neighborhood of b such that $\nabla e_i = 0 \ \forall i$ and any other solution of $\nabla \omega = 0$ is a linear combination of e_i's. Analytic continuations of this frame in $\mathbb{P}^1 \backslash C$ define the monodromy operator

$$T : \pi_1(\mathbb{P}^1 \backslash C, b) \to GL(V_b)$$

We say that ∇ is irreducible if the action of monodromy on a non-zero element of V_b generates the whole V_b.
Let V^* be the dual vector bundle of V. There is defined a natural dual connection $\nabla^* : V^* \to \Omega^1_{\mathbb{P}^1}(D) \otimes_{\mathcal{O}_{\mathbb{P}^1}} V^*$ on V^* as follows

$$< \nabla^* \delta, \omega >= d < \delta, \omega > - < \delta, \nabla \omega >, \ \ \delta \in V^* \omega \in V$$

If $\{e_1, e_2, \dots, e_\alpha\}$ is a base of flat sections in a neighborhood of b then we can define the dual of it as follows: $< \delta_i, e_j >= 0$ if $i \neq j$ and $= 1$ if $i = j$. We can easily check that δ_i's are flat sections. The associated monodromy for ∇^* with respect to this basis is just T^*, where T^* is the composition of T with the transpose operator. We can also define a natural connection on $\wedge^k V = \{\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_k \mid \omega_i \in V\}$ with the pole divisor D as follows:

$$\nabla(\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_k) = \sum_{i=1}^{k} \omega_1 \wedge \omega_2 \wedge \cdots \widehat{\omega_i, \nabla \omega_i} \cdots \wedge \omega_k$$

where $\widehat{\omega_i, \nabla \omega_i}$ means that we replace ω_i by $\nabla \omega_i$.

Proposition 1. *If the connection ∇ over V is irreducible then for any global meromorphic non zero section of V with poles at $C \cup \{\infty\}$, say ω, we have*

1. *$\{\nabla^i \omega \mid i = 0, 1, 2, \dots\}$ generates each fiber $V_b, b \in \mathbb{P}^1 \backslash C \cup \{\infty\}$;*

2. *$\{\nabla^i \omega \mid i = 0, 1, 2, \dots, \alpha - 1\}$ generates a generic fiber V_b.*

Proof. If there exists a $\omega \in H^0(\mathbb{P}^1, V(*\infty))$ such that $\{\nabla^i \omega \mid i = 0, 1, 2, \dots\}$ does not generate V_b then there is a $\delta_b \in V_b^*$ such that

$$< \delta_b, \nabla^i \omega >= 0, \ \ i = 0, 1, 2, \dots$$

Consider the flat section δ passing through δ_b. Since $\frac{\partial^i <\delta,\omega>}{\partial^i t} |_b = < \delta_b, \nabla^i \omega > = 0$, we conclude that $< \delta, \omega >$ is identically zero. Since ∇ is irreducible, we conclude that ω is the zero section.
Now let us prove the second part. Let k be the smallest number such that for all non-zero $\omega \in H^0(\mathbb{P}^1, V(*\infty))$ $A = \omega \wedge \nabla\omega \wedge \cdots \wedge \nabla^k \omega$ is not identically zero. We want to prove that $k = \alpha - 1$. Fix a non zero ω with the property $A \wedge \nabla^{k+1}\omega = 0$. Let $B = \mathbb{P}^1 - C \cup \{\infty\} \cup zero(A)$ and V' be the vector bundle over B generated by $\omega, \nabla\omega, \cdots, \nabla^k\omega$. Since $A \wedge \nabla^{k+1}\omega = 0$, ∇ induces on V' a well-defined holomorphic connection. But this means that V'_b is invariant under monodromy. ∇ is irreducible and so $k + 1 = dim(V'_b) = \alpha$. □

Every line bundle L in $\mathbb{P}^1$ is of the form $L_{a\infty}$, where a is an integer and $L_{a\infty}$ is the line bundle associated to the divisor $a\infty$. We define $c(L) = a$ (Chern class). According to Grothendieck decomposition theorem, every vector bundle V on $\mathbb{P}^1$ can be written as $V = \oplus_{i=1}^{\alpha} L_i$, where L_i's are line bundles. We define $c(V) = \sum_{i=1}^{\alpha} c(L_i)$. In view of Proposition 1 the following definition is natural.

Definition 1. For any meromorphic global section of V with poles at $C \cup \{\infty\}$ define its degree to be the sum of its pole orders. For any natural number n let $H_\nabla(n)$ be the smallest number such that for all ω of degree n the set $\{\nabla^i\omega \mid i = 0, 1, 2, \ldots, H_\nabla(n) - 1\}$ generates each fiber $V_b, b \in \mathbb{P}^1 \backslash C \cup \{\infty\}$.

Of course we have

$$H_\nabla(n) \geq \alpha$$

Let V be a line bundle. In this case $H_\nabla(n)$ is the maximum multiplicity of a zero of a ω of degree n minus one and so $H_\nabla(n) = n + c(V) + 1$. In general case we can only give an upper bound for $H_\nabla(n)$.

Proposition 2. *Let ∇ be an irreducible connection then*

$$H_\nabla(n) \leq (\alpha - 1)(\sum m_i) + \alpha(n + 1) - \frac{\alpha(\alpha - 1)}{2} + c(V)$$

Proof. For any global meromorphic section ω of V with poles at $C \cup \{\infty\}$ we define $A = \omega \wedge \nabla\omega \wedge \cdots \wedge \nabla^{\alpha-1}\omega$. In Proposition 1 we proved that A is a nonzero global meromorphic section of $\wedge^\alpha V$. Let m be the order of the pole of ω at ∞. The sum of pole orders of A at C is at most $(\alpha-1)(\sum m_i) + \alpha(n-m)$.

Each $\nabla^i\alpha$ has a pole (resp. zero if $m-i$ is positive) of order $m-i$ at ∞ and so A has order $m+m-1+m-2+\cdots+m-(\alpha-1)=m\alpha-\alpha(\alpha-1)/2$ at infinity. We conclude that the multiplicity of a zero of A in $\mathbb{P}^1\backslash C\cup\{\infty\}$ is less than

$$(\alpha-1)(\sum m_i)+\alpha(n-m)+m\alpha-\alpha(\alpha-1)/2+c(V)$$

If $b\in\mathbb{P}^1\backslash C\cup\{\infty\}$ be a point such that $\omega,\nabla\omega\cdots,\nabla^i\omega, i\geq\alpha-1$ do not generate V_b then A has a zero of multiplicity $i-(\alpha-2)$. The proposition is proved. □

It does not seem to the author this upper bound to be the best one. More precisely for any vector bundle V and divisor D on $\mathbb{P}^1$ can we find a meromorphic connection ∇ on V with pole divisor D such that $H_\nabla(n)$ is the above number?

Let δ be a flat section of V^* in a small open set U around b and ω be a global meromorphic section of V with poles at $C\cup\{\infty\}$. From now on we use the notation $\int_\delta\omega$ instead of $<\delta,\omega>$. Let us fix the number n and suppose that the degree of ω is less than n. What is the maximum multiplicity of $\int_\delta\omega$ at $t\in U$, say $H_\nabla(n,\delta)$? Let $S(n)$ be the vector space of meromorphic sections of V with poles at $C\cup\{\infty\}$ and degree less than n. Since $S(n)$ is a finite dimensional vector space, $H_\nabla(n,\delta_t)$ is a finite number.

Proposition 3. *If ∇ is irreducible then for all $t\in U$*

$$H_\nabla(n,\delta)\geq dim_{\mathbb{C}}S(n)-1$$

The equality happens except for a finite number of points in U.

Proof. Let $\omega_1,\omega_2,\dots,\omega_b$ be a basis for the vector space $S(n)$. Consider the determinant

$$W_b(t)=det[\frac{\partial^i\int_\delta\omega_j}{\partial t^i}]_{b\times b}$$

It is enough to prove that $W_b(t)$ is not identically zero. Let $a\leq b$ be the smallest number such that $W_a(t)$ is identically zero. There exist holomorphic functions $p_i, i=1,2,\dots,a$ in U such that

$$A_a=\sum_{i=1}^{a-1}A_ip_i/p_a=0$$

where A_i is the i-th column of $[\frac{\partial^i \int_\delta \omega_j}{\partial t^i}]_{a\times a}$. This is a equalities. If we act $\frac{\partial}{\partial t}$ to the i-th equation and subtract the $(i+1)$-th equation we conclude that $[\frac{\partial^i \int_\delta \omega_j}{\partial t^i}]_{b-1\times b-1}.[\frac{\partial (p_i/p_b)}{\partial t}]_{b-1\times 1} = 0$. By hypothesis, this implies that $[\frac{\partial (p_i/p_b)}{\partial t}]_{b_1\times 1} \equiv 0$ or equivalently $\sum_{i=1}^a c_i \int_\delta \omega_i = 0$, where c_i's are constant. Since ∇^* is irreducible, we have $\sum_{i=1}^a c_i\omega_i = 0$ which is a contradiction. □

Proposition 4. *If ∇ is irreducible then*

$$H_\nabla(n) = supp\{H_\nabla(n,\delta)\}$$

where δ runs through all flat sections of V^* in $\mathbb{P}^1\backslash C \cup \{\infty\}$.

Proof. The proof is essentially stated in Proposition 1. If there exists a degree n section ω such that $\{\nabla^i\omega \mid i = 0, 1, 2, \ldots, p-1\}$ does not generate V_b then there is a $\delta_b \in V_b^*$ such that

$$\int_{\delta_b} \nabla^i\omega = 0,\ i = 0, 1, 2, \ldots, p-1$$

Consider the flat section δ passing through δ_b. We conclude that $\int_\delta \omega$ has multiplicity p at b and hence $H_\nabla(n,\delta_b) \le H_\nabla(n)$. The proof of the other part is similar. □

Regular Connections and Linear Equations: Consider the connection ∇^* on V^* as before and fix a trivialization map for V^* around a singular point c_i. ∇^* is called regular at c_i if each flat section of V^* in a sector with the vertex c_i has at most a polynomial growth near c_i (see [**Ku**] p. 36 or [**AB**] p. 8). ∇ is called regular if it is regular in all c_i's.

Let $\omega_1, \omega_2, \ldots, \omega_\alpha$ be global meromorphic sections of V with poles at $C\cup\{\infty\}$. Let also $\delta_1, \ldots, \delta_\alpha$ be a base of flat sections of V^* in a neighborhood of b. The Wornskian function is defined as follows

$$W(t) = W(\omega_1, \cdots, \omega_\alpha)(t) = det[\int_{\delta_j} \omega_i]_{\alpha\times\alpha}$$

The division of two such functions is a one valued meromorphic function in $\mathbb{P}^1\backslash C$ and by regularity of ∇^* we conclude that it extends meromorphically to the whole $\mathbb{P}^1$. Fix an ω. By a similar argument as stated in Proposition 3 and by irreducibility of ∇ we know that $W(\omega, \nabla\omega, \ldots, \nabla^{\alpha-1}\omega)$ is not identically zero. The set $\{\int_\delta \omega \mid \delta \text{ is a flat section of } V^*\}$ is a base for the space of solutions of the following linear equation

$$(1) \qquad \psi : \begin{vmatrix} Y & \int_\delta \omega \\ Y' & \int_\delta \nabla\omega \\ \vdots & \vdots \\ Y^{(\alpha)} & \int_\delta \nabla^\alpha \omega \end{vmatrix}$$

writing in other form

$$(2) \qquad \psi : Y^{(\alpha)} + \sum_{i=1}^{\alpha} (-1)^i P_i Y^{(i)} = 0$$

where

$$P_i = \frac{W(\omega, \nabla\omega, \ldots, \widehat{\nabla^{\alpha-i}\omega}, \ldots, \nabla^\alpha\omega)}{W}, \; W = W(\omega, \nabla\omega, \ldots, \nabla^{\alpha-1}\omega)$$

Since $\int_{\delta_i} \tilde{\omega}$ has polynomial growth at the points of C, ψ is regular therefore it must be Fuchsian i.e. P_i has poles of order at most i(see [**AB**]). The union of poles of P_i's is the singular set of the Picard-Fuchs equation ψ. It has three type of singularities:

1. C; in a $c_i \in C$ the solutions of (2) branch.
2. Z the zeros of W; In these singularities like regular points we have a space of solutions of dimension α. Note that $P_1 = \frac{\frac{\partial W}{\partial t}}{W}$ and so neither of these points is regular. For this reason in [**AB**] these are called apparent singularities. For a zero b of W we can find a flat section δ of V^* such that $\int_\delta \omega$ has multiplicity greater than α at b.
3. ∞; Let m be the order of the pole of ω at ∞. The solutions of (2) in a neighborhood of ∞ are meromorphic functions with poles of order at most m at ∞.

Since $P_1 = \frac{\frac{\partial W}{\partial t}}{W}$ we have

$$Res(P_1 dt, t = c) = mul(W, t = c), \; c \in C \cup Z$$

Now Consider a regular linear equation ψ with singularities at $C \cup Z \cup \{\infty\}$ and suppose that it has apparent singularities in Z and a singularity of type 3 in ∞. Furthermore assume that ψ has the same monodromy representation like as of ∇^*.

Proposition 5. *ψ is obtained by a meromorphic global section of V with poles at $C \cup \{\infty\}$.*

Proof. Consider in a neighborhood of b a base of flat sections δ_i of V^* and a base $e_i, i = 1, 2, \ldots, \alpha$ for solutions of ψ such that the monodromy representation of the both ∇^* and ψ with respect to these bases is the same Define a section of $V = (V^*)^*$ as follows

$$\omega(\delta_t) = e_i(t)$$

this is a one valued holomorphic section of V in $\mathbb{P}^1 \backslash C \cup \{\infty\}$. Since ψ and ∇^* are regular, ω extends meromorphically to C. $\square$

Let ϕ_b be the maximum multiplicity of solutions of ψ at b. If b is a regular point of ψ then $\phi_b = \alpha - 1$ and if it is an apparent singularity of ψ then $\phi_b \geq \alpha$. In the last case by definition of W we can see that W has a zero of order at least $\phi_b - (\alpha - 1)$ at b and by $P_1 = \frac{\frac{\partial W}{\partial t}}{W}$ we have

$$\phi_b \leq Res(P_1 dt, t = b) + (\alpha - 1)$$

Remark: Let us choose a trivialization of V in a small disk D around a singular point c_i of the connection ∇, $V \mid_D \cong D \times \mathbb{C}^\alpha$, and a coordinate z in D. In this coordinate we can write $\nabla v = \frac{\partial v}{\partial z} + \sum_{j=1}^{m_i} \frac{C_j}{z^j} v + A(z)v$, where v is a holomorphic vector in D, $C_j, 1 \leq j \leq m_i$ (resp. $A(z)$) is a constant (resp. holomorphic in z) matrix. C_1 is called the residue of the connection at c_i. Now we can apply the Levelt's theory (see [**AB**] Section 1, 2.2) to understand the local theory of this connection.

Lefschetz Pencil: Let M be a projective compact complex manifold of dimension two, $\{M_t\}_{t \in \mathbb{P}^1}$ a pencil of hyperplane sections of M and f the meromorphic function on M whose level sets are M_t's (see [**La**]). We set $\mathcal{R}$ the indeterminacy points of f, $L_t = M_t - \mathcal{R}$, $C = \{c_1, c_2, c_3, \ldots, c_r\}$ the set of critical values of f, $\beta = dim(H^1(L_t, \mathbb{C}))$ for a $t \in \mathbb{P}^1 - C$ and $\mathbb{C}[t]$ the ring of polynomials in t. Since $f \mid_{M - \mathcal{R}}$ is a C^∞ fibration over $\mathbb{P}^1 - C$ (see [**La**]), β is independent of t. We assume that

1. The axis of the pencil intersects M transversally. This is equivalent to this fact that in a coordinate system (x, y) around each indeterminacy point of f we can write $f = \frac{x}{y}$;

2. The critical points of f are isolated;

3. The pole divisor $D=M_\infty$ of f is a regular fiber, i.e. $\infty \notin C$.

We define $\Omega^i(*D)$ to be the set of meromorphic i-forms in M with poles of arbitrary order along D. The set $\tilde{H}=\cup_{t\in B} H^1(L_t,\mathbb{C})$, where $B=\mathbb{P}^1-C$, has a natural structure of a complex manifold and the natural projection $\tilde{H}^1\to B$ is a holomorphic vector bundle which is called the cohomology vector bundle. The sheaf of holomorphic sections of $\tilde{H}$ is also denoted by $\tilde{H}$. In what follows when we consider f as a holomorphic function we mean its restriction to $M-\mathcal{R}$. Let $\mathbb{C}_{M-\mathcal{R}}$ be the sheaf of constant functions in $M-\mathcal{R}$ and $R^1f_*\mathbb{C}_{M-\mathcal{R}}$ be the 1-th direct image of the sheaf $\mathbb{C}_{M-\mathcal{R}}$ (see [**GrRe**]). Any element of $R^1f_*\mathbb{C}_{M-\mathcal{R}}(U)$, U being an open set in B, is a holomorphic section of the cohomology fiber bundle map. It is easy to verify that

$$\tilde{H}\cong R^1f_*\mathbb{C}_{M-\mathcal{R}}\otimes_{\mathbb{C}}\mathcal{O}_{\mathbb{P}^1}, \text{ in } B$$

Now let us introduce the Gauss-Manin Connection on $\tilde{H}$. Consider a holomorphic coordinate $(t,0)$ in U, a small open disk in $\mathbb{P}^1$. The Gauss-Manin connection is defined as follows:

$$\nabla:\tilde{H}(U)\to\Omega^1_{\mathbb{P}^1}\otimes_{\mathcal{O}_U}\tilde{H}(U)$$

$$\nabla(g\otimes c)=dg\otimes c,\ c\in R^1f_*\mathbb{C}_{M-\mathcal{R}}(U),\ g\in\mathcal{O}_{\mathbb{P}^1}(U)$$

The sheaf of flat sections of ∇ is $R^1f_*\mathbb{C}_{M-\mathcal{R}}$. Let $\frac{\partial}{\partial t}$ be a vector field in U. We write $\nabla_{\frac{\partial}{\partial t}}=\frac{\partial}{\partial t}\circ\nabla$. $(\nabla_{\frac{\partial}{\partial t}}(g\otimes c)=\frac{\partial g}{\partial t}\otimes c)$.

In the same way we can define the cohomology fiber bundle $\tilde{H}_c$ of compact fibers M_t. Since $\tilde{H}_c$ is a ∇-invariant sub-vector bundle of $\tilde{H}$, we have the restriction of ∇ to $\tilde{H}_c$ which we denote it again by ∇.

Let ω be a meromorphic 1-form in M with poles along some fibers of f. Let also $\{\delta_t\}_{t\in\mathbb{P}^1-C},\delta_t\subset L_t$ be a a continuous family of cycles. The Abelian integral $\int_{\delta_t}\omega$ appears in the deformation $df+\epsilon\omega$ of df inside holomorphic foliations (differential equations) and it is related to the number of limit cycles which born from the cycles δ_t (see [**Ho**]). The pair $(\tilde{H},\nabla)$ is defined in $\mathbb{P}^1-C$ and in order to be in the context of this paper we may be interested to prove:

Proposition 6. *Under the assumptions 1,2,3, there is a vector bundle V, a sub-vector bundle $\bar{V}\subset V$ and a meromorphic connection on V with poles in C*

$$\nabla:V\to\Omega^1_{\mathbb{P}^1}(D)\otimes_{\mathcal{O}_{\mathbb{P}^1}}V,\ D=\sum m_ic_i$$

such that:

1. *$\bar{V}$ is ∇-invariant.*

2. *(V, ∇) (resp. $(\bar{V}, \nabla)$) coincides with $(\tilde{H}, \nabla)$(resp. $(\tilde{H}_c, \nabla)$) in $\mathbb{P}^1 - C$;*

3. *The Brieskorn lattices (Petrov module in the context of differential equations) of f $H' = \frac{\Omega^1(*D)}{df \wedge \Omega^0(*D) + d\Omega^0(*D)}$ is $\mathbb{C}[t]$-isomorphism to the module of global sections of V with poles of arbitrary order at ∞.*

This is a task which is done in details in [**Ho1**]. If the singularities of f are non-degenerated, i.e. in a holomorphic coordinate (x, y) around a singularity p_i we can write $f = f(p_i) + x^2 + y^2$, then all the m_i's are equal to one. In other words ∇ is logarithmic.

The pair (V, ∇) is not irreducible but if $H^1(M, \mathbb{C}) = 0$ and f satisfies 1, 2, 3 and has non-degenerated singularities with distinct images then $(\bar{V}, \nabla)$ is irreducible (see [**La**] 7.3). The following proposition justifies the use of $(\bar{V}, \nabla)$ instead of (V, ∇).

Proposition 7. *For an ω a meromorphic 1-form in M with poles of order at most n along D, the integral $\int_{\delta_t} \omega$ is a polynomial of degree n. $\nabla^i_{\frac{\partial}{\partial t}} \omega, i > n$ restricted to each fiber has not residues in $\mathcal{R}$ and hence is a meromorphic section of $\bar{V}$.*

Proof. We have $p(t) := \int_{\delta_t} \omega = t^n \int_{\delta_t} \frac{\omega}{f^n}$. Since the 1-form $\frac{\omega}{f^n}$ has not pole along D, $\frac{p(t)}{t^n}$ has finite growth at $t = \infty$. Since $p(t)$ is holomorphic in $\mathbb{C}$ (even in the points of C), we conclude that $p(t)$ is a polynomial of degree at most n. The second part is a direct consequence of the first one and the fact that $\frac{\partial}{\partial t} \int_{\delta_t} \omega = \int_{\delta_t} \nabla_{\frac{\partial}{\partial t}} \omega$. □

I tried to study the maximum multiplicity of Abelian integrals in the context of meromorphic connections. My motives were the paper [**Ma**] and also a paper of mine, where the extension of cohomology vector bundles and their connections to the critical values of a meromorphic function is discussed. The upper bound obtained in Proposition 2 seems to be far from the best one (at least for Guass-Manin connections). Some works in Differential equations (see [**HoIl**]) suggest that the number $H_\nabla(n)$ must be very sensitive with respect to ∇.

Here I would like to express my thanks to Max-Planck institute for hospitality. Thanks go also to C. Hertling, Y. Holla and S. Archava for many useful conversations.

References

[**AB**] Anosov, D. V.; Bolibruch, A. A. The Riemann-Hilbert problem. Aspects of Mathematics, E22. Friedr. Vieweg & Sohn, Braunschweig, 1994.

[**AGV**] Arnold, V. I.; Gusein-Zade, S. M.; Varchenko, A. N. Singularities of differentiable maps. Vol. II. Monodromy and asymptotics of integrals. Monographs in Mathematics, 83. Birkhäuser Boston, Inc., Boston, MA, 1988.

[**EV**] Esnault, H.; Viehweg, E. Semistable bundles on curves and irreducible representations of the fundamental group. Algebraic geometry: Hirzebruch 70 (Warsaw, 1998), 129–138, Contemp. Math., 241, Amer. Math. Soc., Providence, RI, 1999

[**GrRe**] Grauert, H.; Remmert, R. Coherent analytic sheaves. Grundlehren der Mathematischen Wissenschaften, 265. Springer-Verlag, Berlin-New York, 1984.

[**He**] Hertling, C. Frobenius manifolds and moduli space for hypersurface singularities, Priprint.

[**HoIl**] Horozov E.; Iliev I. Linear estimate for the number of zeros of Abelian integrals with cubic Hamiltonians. Nonlinearity 11 (1998), no. 6, 1521–1537.

[**Ho**] Movasati, H. Abelian integrals in holomorphic foliations, Priprint math.AG/0203062.

[**Ho1**] Movasati H., Relative cohomology with respect to a Lefschetz Pencil, priprint math.AG/0112204.

[**Ku**] Kulikov, V. S. Mixed Hodge structures and singularities. Cambridge Tracts in Mathematics, 132. Cambridge University Press, Cambridge, 1998.

[**La**] Lamotke, K. The topology of complex projective varieties after S. Lefschetz. Topology 20 (1981), no. 1, 15–51.

[**Ma**] Mardešić, P. An explicit bound for the multiplicity of zeros of generic Abelian integrals. Nonlinearity 4 (1991), no. 3, 845–852.