ε-FACTORS OF DISCRETE SERIES REPRESENTATIONS OF CENTRAL SIMPLE ALGEBRAS

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ABSTRACT. We assume a certain construction procedure $t \mapsto \Pi_t^A$ of discrete series representations of A^* where $A \mid F$ is a central simple algebra and we compute the ε-factor $\varepsilon(s, \Pi_t^A, \psi)$ in terms of the parameter t. It turns out that some information depending on A other than the reduced degree is necessary to determine ε such that the explicite constructions do not fit with the Abstract Matching Theorem. The deviations are encapsulated in the Weil representations to be used when constructing Π_t^A which come up for different algebras A in a different way. Using the Weil representation requires to assume F p-adic for $p \neq 2$.

1. DEFINITION OF THE ε-FACTOR

Let F be a p-adic field and $A = M_m(D_d)$ a central simple algebra over F of reduced degree $N = md$ and let $p \neq 2$ (we need this assumtion because we want to use the Weil representation, see sections 8 and 9). Let $\psi : F^+ \to \mathbb{C}^*$ be an additive character.

Then according to Godement, Jacquet [GJ] an ε-factor $\varepsilon(s, \Pi, \psi)$ is assigned to any irreducible admissible representation Π of A^*. It depends on a complex parameter s and appears in a functional equation relating Laurent polynomials i. e. functions of s which are elements of $\mathbb{C}[p^s, p^{-s}]$. Namely let
$\phi \in \mathcal{S}(A)$ be a locally constant compactly supported complex valued function on A,
$f \in \mathfrak{M}(\Pi)$ be a function on A^* which is in the span of matrix coefficients of Π.

There is a well defined Laurent polynomial $\Xi(\phi, f, s) \in \mathbb{C}[p^s, p^{-s}]$ associated to ϕ, f (depending on Π) such that

$$\Xi(\hat{\phi}, \check{f}, 1-s) = (-1)^{N-m} \cdot \varepsilon(s, \Pi, \psi)\Xi(\phi, f, s)$$

where $\hat{\phi} \in \mathcal{S}(A)$ is the Fourier transform of ϕ with respect to a *ψ-selfdual* Haar measure on A, and where $\check{f} \in \mathfrak{M}(\check{\Pi})$ is given as $\check{f}(g) = f(g^{-1})$ for $g \in A^*$. The functional equation implies

$$\varepsilon(s, \Pi, \psi) \cdot \varepsilon(1-s, \check{\Pi}, \psi) = \omega_\Pi(-1)$$

where $\omega_\Pi : F^* \to \mathbb{C}^*$ is the central character of Π, and the root number is defined as

$$W(\Pi, \psi) := \varepsilon(\frac{1}{2}, \Pi, \psi) \in \mathbb{C}$$

which implies $|W(\Pi, \psi)|_{\mathbb{C}} = 1$ for Π unitary. Then we have

$$\varepsilon(s, \Pi, \psi) = W(\Pi, \psi) \cdot q^{(\frac{1}{2}-s)\cdot a(\Pi, \psi)} \tag{1}$$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX where $q = |k_F|$ is the order of the residue field of F and $a(\Pi, \psi)$ is an integer. If we switch to another additive character ψ_b, $b \in F$, given as $\psi_b(x) = \psi(bx)$ for all $x \in F$, then we have the transformation rules:

$$\begin{aligned} W(\Pi, \psi_b) &= \omega_\Pi(b) \cdot W(\Pi, \psi) \\ a(\Pi, \psi_b) &= a(\Pi, \psi) + N \cdot \nu_F(b) \end{aligned} \tag{2}$$

(where N was the reduced degree of our algebra A). Moreover if $\chi : F^* \to \mathbf{C}^*$ is an unramified character and $\tilde{\chi} = \chi \circ \mathrm{Nrd}$ is the corresponding character of A^*, then:

$$W(\tilde{\chi} \otimes \Pi, \psi) = W(\Pi, \psi) \cdot \chi(\pi_F)^{a(\Pi,\psi)} \qquad a(\tilde{\chi} \otimes \Pi, \psi) = a(\Pi, \psi) \tag{3}$$

independent of the choice of π_F because χ is unramified.

Once and for all we normalize the additive character ψ in such a way that it has conducter $\mathfrak{p}_F$, i. e. $\psi(\mathfrak{p}_F) \equiv 1$ and $\psi(\mathfrak{o}_F) \neq 1$. Then the restriction of ψ onto $\mathfrak{o}_F$ is an additive character of the residue field k_F which we denote $\bar{\psi}$.

2. The formula of Bushnell and Fröhlich

We assume now that the irreducible admissible representation Π of A^* is a *discrete series representation.* Then according to Bushnell and Fröhlich [BF85, 3.3.8] the root number $W(\Pi, \psi)$ can be expressed as a Gauss sum. Namely consider a maximal compact modulo center subgroup $\mathfrak{K}$ of A^* and a "nondegenerate" irreducible represention ϱ of $\mathfrak{K}$ which is contained in Π. $\mathfrak{K} = N_{A^*}(\mathfrak{A})$ is the normalizer of a uniquely determined principal order $\mathfrak{A}$ in A. Let $\mathfrak{P}$ be the Jacobson radical of $\mathfrak{A}$ and let $f(\varrho) = \mathfrak{P}^{j+1}$ be the conductor of ϱ, i. e. $1 + f(\varrho) \subseteq \mathrm{Ker}\,\varrho$. "Nondegenerate" especially means that $\mathfrak{A}^* \not\subseteq \mathrm{Ker}\,\varrho$ hence $j \geq 0$. Write $Nf(\varrho) = (\mathfrak{A} : f(\varrho))$ for the absolute norm. Then:

$$W(\Pi, \psi) := (-1)^{N-m} \cdot Nf(\varrho)^{-1/2} \cdot \tau(\check{\varrho}, \psi) \tag{1}$$

where $\tau(\check{\varrho}, \psi)$ is a Gauss sum assigned to the contragredient $\check{\varrho}$ of ϱ. Namely $\tau(\check{\varrho}, \psi)$ is the value of the following scalar operator:

$$T(\check{\varrho}, \psi) = \sum_{u \in \mathfrak{A}^*/1+f(\varrho)} \check{\varrho}(c^{-1}u) \cdot \psi_A(c^{-1}u) \tag{2}$$

where $\psi_A = \psi \circ \mathrm{Trd}_{A|F}$ and c is a generator of the fractional ideal $f(\psi_A)^{-1} \cdot f(\varrho)$ of $\mathfrak{A}$ which is a power of $\mathfrak{P}$. As mentioned before we assume $f(\psi) = \mathfrak{p}_F$ hence $f(\psi_A) = \mathfrak{P}$ and $\nu_\mathfrak{P}(c) = j$.

Remark. Bushnell and Fröhlich (3.3.8) do not make use of the assumption "Π supercuspidal" but of the weaker assumption $L(\Pi) = L(\check{\Pi}) = 1$ which also includes discrete series representations. (See their remark 3.4.(c).) More precisely if Π is essentially discrete series, then $L(\Pi) = L(\check{\Pi}) = 1$ unless Π is an unramified twist of the Steinberg representation. Therefore our methods do not apply to Steinberg representations. Instead one has to use the behaviour of γ- and ε-factors under parabolic induction. For the sake of completeness we quote the result which for $A = M_N(F)$ follows from [GJ, p.97] :

2.1 Proposition. *Let $A\,|\,F$ be a central simple algebra of reduced degree $N = md$, let St^A be the Steinberg representation of A^* (which is the trivial representation if $A = D_N$ is a division algebra) and let χ be an unramified character of F^*. Then:*

$$\varepsilon(s, \tilde{\chi} \otimes St^A, \psi_F) = (-1)^{N-1} \cdot \chi(\pi_F)^{-1} \cdot q^{(\frac{1}{2}-s)(-1)}$$

hence $a(\tilde{\chi} \otimes St^A, \psi_F) = -1$. For $\chi = 1$ we especially see that $\varepsilon(s, St^A, \psi_F) = (-1)^{N-1} \cdot q^{(\frac{1}{2}-s)(-1)}$. And if ψ_F has the conductor $\mathfrak{o}_F$, then from 1.(2)$_2$ we conclude $a(\tilde{\chi} \otimes St^A, \psi_F) = N - 1$.

3. Parameters for discrete series representations

Our aim is to express $\varepsilon(s, \Pi, \psi)$ of a discrete series representation Π of A^* in terms of a certain set of parameters for those representations. We briefly recall what the parameters look like:

Consider $F[T]_{\mathrm{irr}}$ the set of irreducible polynomials of degree ≥ 1 where the highest coefficient is 1, and let $F \hookrightarrow F[T]_{\mathrm{irr}}$, $a \mapsto T - a$ be the natural embedding. Then the exponential distance $\nu_F(a - b) \in \mathbb{Z}$ on F has a well defined extension to an exponential distance $w_F\big(f(T), g(T)\big) \in \mathbb{Q}$ on$F[T]_{\mathrm{irr}}$, i. e.

$$\begin{aligned} w_F\big(f(T), g(T)\big) &\geq \min\{w_F\big(f(T), h(T)\big), w_F\big(h(T), g(T)\big)\} \\ w_F(T - a, T - b) &= \nu_F(a - b) \qquad \text{for } a, b \in F\,. \end{aligned}$$

(see [Zi92]). Moreover there exist approximation procedures on $F[T]_{\mathrm{irr}}$ with respect to the exponential distance w_F.

3.1. *An approximation procedure is a map*

$$F[T]_{\mathrm{irr}} \times \mathbb{Q} \to F[T]_{\mathrm{irr}}\,, \quad \big(f(T), j\big) \mapsto f^j(T) \tag{1}$$

such that:

- (i) *$f^j(T) = T$ for all j if $f(T) = T$*
- (ii) *$w_F(f, f^j) \geq j$ and $f^j(T) = f^{j+\varepsilon}(T)$ if $w_F(f, f^j) \geq j + \varepsilon$ for some $\varepsilon > 0$*
- (iii) *$\deg f^j(T) | \deg f(T)$ and the same divisibility holds for the ramification exponent and inertial degree of the polynomials.*
- (iv) *$w_F(f, g) \geq j$ implies $f^j(T) = g^j(T)$.*

The existence of approximation procedures was proved by H. Koch [Ko81].

3.2. *Note that for $f(T) \in F[T]_{\mathrm{irr}}$, $\nu_F(\alpha) \in \mathbb{Q}$ is the same for all roots α of $f(T)$ in a fixed algebraic closure $\bar{F}|F$, and $f^j(T) = T$ for $j \leq \nu_F(\alpha)$ i. e. the approximation of $f(T)$ starts from the polynomial $T \in F[T]_{\mathrm{irr}}$ (which is the "zero element") and it ends up with $f^\infty(T) = f(T)$.*

There is no p-adic expansion of irreducible polynomials but it is suggestive to think of $f^j(T)$ as of the partial sum of a p-adic expansion. Just as for p-adic numbers there are many approximation procedures and we have to fix one of them. *For later use we describe how to fix the first nontrivial approximation of $f(T)$.* Namely we will fix a complementary group C_F in F^*, i. e. $F^* = C_F \times (1 + \mathfrak{p}_F)$. C_F is generated by a fixed prime element π_F and by the roots of unity of order prime to p in F. Now let $\bar{C} \supseteq C_F$ be a fixed complementary group of $\bar{F}|F$, $\bar{F}^* = \bar{C} \times (1+\mathfrak{p}_{\bar{F}})$ which containes C_F. $\bar{C}$ contains all roots of unity of order prime to p and is fixed by choosing a "string" of roots of π_F.

The approximation procedure on $F[T]_{\mathrm{irr}}$ can be fixed in such a way that the first nontrivial approximation of $f(T)$ is given as follows:

Take a root $\alpha \in \bar{F}$ of $f(T)$ and consider the uniquely determined "symbol" $\mathrm{symb}(\alpha) \in \bar{C}$ such that

(2) $$\alpha \equiv \mathrm{symb}(\alpha) \qquad \mathrm{mod}\ 1 + \mathfrak{p}_{\bar{F}}\,.$$

Then $f^{\nu}(T) = T$ and

(3) $$f^{\nu+\varepsilon}(T) := \text{minimal polynomial of } \mathrm{symb}(\alpha) \text{ over } F\,, \quad \text{if } \nu = \nu_F(\alpha) \text{ and } \varepsilon > 0 \text{ small.}$$

It is possible to see that $\mathrm{symb}(\alpha)$ is conjugated to $\mathrm{symb}(\beta)$ if α is conjugated to β such that $f^{\nu+\varepsilon}(T)$ is well defined. But we note that $\{\,\mathrm{symb}(\beta)\,;\,\beta \text{ conjugated to } \alpha\,\}$ $\subseteq$ {conjugacy class of $\mathrm{symb}(\alpha)$} can be a proper subset.

Now we define

3.3 A polynomial $f(T) \in F[T]_{\mathrm{irr}}$ is called a *minus polynomial* with respect to the fixed approximation procedure if already $f^0(T) = f(T)$. The set of minus polynomials is denoted $F[T]^-_{\mathrm{irr}}$.

3.4 Consider pairs (ϕ, β) where $\beta \in \bar{F}$ is the root of a minus polynomial and $\phi : K^*/1 + \mathfrak{p}_K \to \mathbb{C}^*$ is a tame character of a field K such that:

(i) $K|F(\beta)$ is an unramified extension of fields,

(ii) ϕ is regular over $F(\beta)$, i. e. all conjugate characters are different.

The Galois group $\mathfrak{G}_F = \mathrm{Gal}(\bar{F}|F)$ acts as follows:

$\sigma \circ (\phi, \beta) := \big(\phi \circ \sigma^{-1}, \sigma(\beta)\big)$ for $\sigma \in \mathfrak{G}_F$, and by $t = [\phi, \beta]$ the Galois orbit of the pair (ϕ, β) is denoted. The degree of such a parameter is defined as $\deg t = [K : F]$, and a twist with tamely ramified characters $\chi : F^*/1 + \mathfrak{p}_F \to \mathbb{C}^*$ is given as: $\chi \otimes t := \big[(\chi \circ N_{K|F})\phi, \beta\big]$.

3.5 *If $A|F$ is a central simple algebra of reduced degree N then $\mathcal{T}_N^- = \{t = [\phi, \beta]\,;\, \deg t|N\}$ may serve as a system of parameters for the irreducible discrete series representation of A^*.*

(The minus sign in $\mathcal{T}_N^-$ reminds to the fact that the numbers β are roots of minus polynomials over F).

The parameter set $\mathcal{T}_N^-$ is not canonical because it is necessary to make choices when constructing a discrete series representation Π_t out of a parameter t. In order to obtain a well defined Π_t one has to fix a character $\lambda_\beta : F(\beta)^* \to \mathbb{C}^*$ for all β such that the following compatibility relations are fulfilled.

3.6. (i) $\lambda_\beta \circ \sigma^{-1} = \lambda_{\sigma(\beta)}$ *for all* $\sigma \in \mathfrak{G}_F$.

(ii) $\lambda_\beta \equiv 1$ *the unit character of* F^* *if* $\beta = 0$.

(iii) $(\lambda_\beta[\lambda_b \circ N_{F(\beta)|F}]^{-1})(1+x) = \psi \circ Tr_{F(\beta)|F}((\beta - b)x)$ *for* $x \in \mathfrak{p}_{F(\beta)}^{[j/2]+1}$ *and* $j = -\nu_{F(\beta)}(\beta - b)$ *if* $b \in F$.

(Note that $\nu_{F(\beta)}(\beta - b) = e_{F(\beta)|F} \cdot w_F(f_\beta(T), T - b)$ *is a negative integer because* β*,* b *are roots of minus polynomials and* $b \in F$*).*

(iv) $\lambda_\beta(\beta) = 1$.

Conditions (iii), (iv) are compatible because $\nu_{F(\beta)}(\beta) < 0$ *implies that the cyclic group* $\langle\beta\rangle$ *and the principal units of* $F(\beta)^*$ *have trivial intersection.*

Unfortunately the compatibility relations of 3.6 are not complete because what we need in (iii) is compatibility between λ_β and λ_γ for arbitrary γ whereas we have assumed $\gamma = b \in F$. So far the general compatibility between λ_β and λ_γ can be expressed only in terms of the algebra A at hand such that fixing a compatible system of characters $\{\lambda_\beta\}_\beta$ might depend on A.

3.7 If we have fixed an approximation procedure on $F[T]_{\text{irr}}$ (see 3.1) we say that j is a jump of $f(T)$ if $f^j(T) \neq f^{j+\varepsilon}(T)$ for all $\varepsilon > 0$. When approximating $f(T)$ in general the number of jumps can be infinite but for a minus polynomial it is certainly finite because $f^0(T) = f(T)$, i. e. all jumps are negative. Now the construction of a compatible system of characters $\{\lambda_\beta\}_\beta$ proceeds by induction on the number of jumps of β (that is to say the number of jumps occuring if we approximate the minimal polynomial of β over F). It starts from $\lambda_\beta \equiv 1$ for $\beta = 0$, the only number which has no jumps. According to (2), (3) a number β gives rise to precisely one jump iff $\beta = \text{symb}(\beta) \in \bar{C}$. In this case β is the root of a minus polynomial iff $\nu_F(\beta) < 0$ and to fix λ_β the conditions 3.6 (iii) with $b = 0$ i. e. $\lambda_b \equiv 1$ and 3.6 (iv) will do. Hence *the characters* λ_β *can be chosen independently from* A *if* β *has not more than one jump.*

3.8 Let $A|F$ be central simple of reduced degree N and let $A^{*\wedge}_{\text{discrete}}$ be the set of equivalence classes of irreducible discrete series representations of A^*. Fixing a map $\mathcal{T}_N^- \to A^{*\wedge}_{\text{discrete}}$, $t = [\phi, \beta] \mapsto \Pi_t^A$ means to fix a compatible system $\{\lambda_\beta^A\}_\beta$ of characters $\lambda_\beta^A : F(\beta)^* \to \mathbb{C}^*$, which gives a well defined map

$$t = [\phi, \beta] \mapsto [\phi, \beta, \lambda_\beta^A] \mapsto \Pi_t^A \,. \tag{4}$$

We remark that the construction of Π_t uses all characters λ_γ^A where the minimal polynomial of γ is an approximation polynomial of the minimal polynomial of β.

In the tame case $p \nmid N$ it is known that the system $\{\lambda_\beta^A\}_\beta$ can be chosen independently from A and that the "approximation characters" $\lambda_\gamma$of λ_β are not really necessary to construct Π_t. In this case it happens that Π_t is determined by $\lambda_t = \phi \cdot (\lambda_\beta \circ N_{K|F(\beta)})$ which is a character of K^*. In the general case the situation is less satisfactory but we are able to take some advantage of the remarks in 3.7.

4. The results

We are going to state the results. According to 3.(4) our aim is to describe the ε-factor $\varepsilon(s, \Pi_t^A, \psi)$ in terms of $[\phi, \beta, \lambda_\beta^A]$ in order to see to what extend it depends on A. In a sense this is a test if the explicite constructions $t \mapsto \Pi_t^A$ for the different central simple algebras $A \mid F$ of a fixed reduced degree N reflect the local matching theorem of [BDKV]. Then the ε-factors should only depend on N but not on A. But in fact we will see that $\varepsilon(s, \Pi_t^A, \psi)$ depends on A such that the explicit constructions have to be modified in order to match. We have to exclude Steinberg type representations which require special treatment (see Proposition 2.1. above). The corresponding parameters are $t = [\phi, 0]$ where ϕ is an unramified character of F^* (i. e. ϕ is trivial on the units of F). No unramified extension $K|F$ can occur because unramified characters cannot be regular.

From now we assume $t = [\phi, \beta]$ where $\beta \neq 0$ or the tame character ϕ is ramified.

Case 1: Assume $t = [\phi, 0]$ and consider the finite fields $k_N \supset k_K \supset k_F$, where k_F is the residue field of F and $k_N|k_F$ is the extension of degree N. The tame character ϕ of K^* restricted to the units gives a character $\bar{\phi}$ of k_K^*, and the additive character ψ of F restricted to the integers gives $\bar{\psi}$ of k_F^+. (see 1.).

Theorem 4.1. *For $t = [\phi, 0]$ we obtain:*

(i) $a(\Pi_t^A, \psi) = 0, \quad \varepsilon(s, \Pi_t^A, \psi) = W(\Pi_t^A, \psi)$

(ii) $W(\Pi_t^A, \psi) = (-1)^{N-1} \cdot q^{-N/2} \cdot \tau(\bar{\phi}^{-1} \circ N_{k_N|k_K}, \bar{\psi} \circ Tr_{k_N|k_F})$

where $\tau(\chi, \psi) = \sum_{x \in k^*} \chi(x)\psi(x)$ if χ, ψ are a multiplicative and an additive character of a finite field k.

Especially we see that the ε-factor of Π_t^A does not depend on the algebra A but only on the reduced degree N of $A|F$.

Case 2: Assume $t = [\phi, \beta]$ where $\beta \neq 0$ and the symbol $b = \mathrm{symb}(\beta) \in \bar{C}$ generates a tame extension $F(b)|F$. Then

Theorem 4.2. (i) $a(\Pi_t^A, \psi) = -N \cdot \nu_F(\beta) \quad$ (*see* 1.(1))

(ii) $W(\Pi_t^A, \psi) = (-1)^{N-m+N_b-(m,N_b)} \cdot \chi_{W_b}(b)^{-1} \cdot \left[\frac{\psi_K(\beta)}{\phi(\beta)}\right]^{N_t} \cdot \delta(\lambda_b)^{N_b}$

where $N_b = N/(F(b) : F)$, $N_t = N/degt$, $\psi_K(\beta) = \psi \circ Tr_{K|F}(\beta)$ and the character $\lambda_b = \lambda_b^A$ does not depend on A (see 3.7).

$\delta(\lambda_b)$ is given completely in terms of the field $F(b)$, namely let $\mathfrak{p}$ be the prime ideal of $F(b)$ and ν the $\mathfrak{p}$-exponent. Then:

$$\delta(\lambda_b) = \begin{cases} 1 & \text{if } \nu(b) \text{ is odd} \\ N\mathfrak{p}^{-1/2} \cdot \sum\limits_{x \in \mathfrak{p}^{j/2}/\mathfrak{p}^{j/2+1}} \bar{\lambda}_b(1+x) \cdot \psi_{F(b)}(bx) & \text{if } j = -\nu(b) \text{ is even.} \end{cases}$$

χ_{W_b} is the character of a Weil representation which is given in terms of A, and $\chi_{W_b}(b) \in \{\pm 1\}$ is a sign. This is due to the fact that the ramification exponent of $F(b)|F$ is the denominator of $\nu_F(b) \in \mathbb{Q}$, hence $F(b)|F$ tamely ramified means that the denominator of $\nu_F(b) = \nu_F(\beta)$ is prime to p, hence $\bar{b} \in \bar{C}/C_F$ is of order prime to p, and $\chi_{W_b}(b)$ only depends on $\bar{b}$. According to Howe [Ho], Proposition 2.(iv), $\chi_{W_b}(b)$ is an integer, and it has to be of absolute value 1 because $|W(\Pi_t^A, \psi)|_{\mathbb{C}} = 1$ if the character ϕ is unitary (see section 11 below).

Remarks. 1. From 4.2(ii) we see that in case 2 the root number $W(\Pi_t^A, \psi)$ is the product of a sign namely $(-1)^{N-m+N_b-(m,N_b)}\chi_{W_b}(b)^{-1}$ which depends on A and of the factor $[\psi_K(\beta)/\phi(\beta)]^{N_t} \cdot \delta(\lambda_b)^{N_b}$ which is independent from A.

2. The representation Π_t^A is not in general position if the conductor $a(\Pi_t^A, \psi) = -N \cdot \nu_F(\beta)$ can be reduced by twisting Π_t^A with a character $\chi \circ \mathrm{Nrd}_{A|F}$ where χ is a character of F^*. This is equivalent to the fact that $b = \mathrm{symb}(\beta) \in F$ and implies $(-1)^{N-m+N_b-(m,N_b)}\chi_{W_b}(b)^{-1} = 1$, hence $W(\Pi_t^A, \psi)$ does not depend on A if Π_t^A is a representation which is not in general position.

Case 3: Finally we consider $t = [\phi, \beta]$ where $\beta \neq 0$ and the symbol $b = \mathrm{symb}(\beta) \in \bar{C}$ generates an extension $F(b)|F$ which is not tamely ramified. Then:

Theorem 4.3. (i) $a(\Pi_t^A, \psi) = -N \cdot \nu_F(\beta)$ (*see* 1.(1))
(ii) $W(\Pi_t^A, \psi) = (-1)^{N-m} \cdot \frac{|\chi_{W_b}(b)|}{\chi_{W_b}(b)} \cdot \left[\frac{\psi_K(\beta)}{\phi(\beta)}\right]^{N_t}$

Again χ_{W_b} is the character of a Weil representation which is given in terms of A but now the character value $\chi_{W_b}(b)$ need not to be on the unit circle. Therefore $W(\Pi_t^A, \psi)$ has the factor $(-1)^{N-m} \cdot \frac{|\chi_{W_b}(b)|}{\chi_{W_b}(b)}$ which depends on A and the factor $[\psi_K(\beta)/\phi(\beta)]^{N_t}$ which only depends on $t = [\phi, \beta]$ but not on A. Moreover we note that in case 3 the character λ_β or "approximation characters" of λ_β play no role at all. Therefore in the wild case 3 in a sense the root number formula is simpler than in the tame case 2.

Corollary 4.4. *Assume $t = [\phi, \beta]$ where $\beta \neq 0$, and for a tame character χ of $F(\beta)^*$ let $t_\chi = [\phi \cdot (\chi \circ N_{K|F(\beta)}), \beta]$. Then we get:*

(i) $\varepsilon(s, \Pi_t^A, \psi) = |\beta|_F^{(\frac{1}{2}-s)N} \cdot W(\Pi_t^A, \psi)$
(ii) $W(\Pi_{t_\chi}^A, \psi) = \chi(\beta)^{-N_\beta} \cdot W(\Pi_t^A, \psi)$

where $|\beta|_F = q^{-\nu_F(\beta)}$ is the extension of the normalized absolute value from F to $\bar{F}$ and where $N_\beta = N/(F(\beta) : F)$.

Remarks. 3. The reason for the change from (ϕ, β) to $(\phi \cdot (\chi \circ N_{K|F(\beta)}), \beta)$ is that $\phi \cdot (\chi \circ N_{K|F(\beta)})$ is again regular over $F(\beta)$ because ϕ is regular and $\chi \circ N_{K|F(\beta)}$ is invariant under automorphisms of $K|F(\beta)$.
4. We expect that the maps $t \to \Pi_t^A$ have corrections $t \to \Pi_{t_{\chi(t,A)}}^A$ where $\chi(t, A)$ is an appropriate tame character of $F(\beta)^*$ such that the root number $W(\Pi_{t_{\chi(t,A)}}^A, \psi)$ does not depend on A. In the tame case $p \nmid N$ such corrections were established by C. Bushnell, A. Fröhlich [BF83], A. Moy [M] and H. Reimann [Rei] in the case where A is a division and split algebra respectively.
5. We also note that the formulas in 4.1–4.3 do not change if we replace (ϕ, β) by $\sigma \circ (\phi, \beta) = (\phi\sigma^{-1}, \sigma(\beta))$, hence the formulas only depend on the parameter $t = [\phi, \beta]$.
6. G. Henniart [He] has computed $\varepsilon(s, \Pi^A, \psi) \bmod \mu$ for certain supercuspidal representations of A^*, where μ denotes the complex roots of unity of order a p-power. Henniart's assumptions in terms of our parameters are:
a) $\mathrm{lcm}(\deg t, d) = N$, which means that Π_t^A is supercuspidal,
b) $t = [\phi, \beta]$ such that $b = \mathrm{symb}(\beta) \in \bar{C}$ generates a fully ramified extension of p-power degree.
Therefore either Π_t^A is not in general position (see remark 2) or we are in the situation of 4.3. $b = \mathrm{symb}(\beta)$ plays the role of Henniart's "g_σ" and $\phi(\beta)^{-N_t}$

plays the role of Henniart's $\omega_\Pi^{-1}(g_\sigma)$, because $\Pi = \Pi_t^A$ has the central character $\omega_\Pi = (\phi \cdot \lambda_\beta^A \circ N_{K|F(\beta)})^{N_t}|_{F^*}$. $\psi_K(\beta)$ does not appear in Henniart's formula, be cause it is in μ.

5. Computation of the conductor $a(\Pi_t^A, \psi)$

Because Π_t^A is a discrete series representation of A^* we can use the following formula of Bushnell, Fröhlich [BF85]. Assume that the discrete series representation Π of A^* contains a "nondegenerate" irreducible representation ϱ of $\mathfrak{K} = N_{A^*}(\mathfrak{A})$ (see 2.). Then:

$$\varepsilon(s, \Pi, \psi) = N\big(f(\varrho) \cdot f(\psi_A)^{-1}\big)^{(\frac{1}{2}-s)/N} \cdot W(\Pi, \psi) \tag{1}$$

where the conductor f is a certain power of $\mathfrak{P} = \mathrm{Jac}(\mathfrak{A})$ and $Nf = (\mathfrak{A} : f)$.

Now from the construction of $\varrho \subset \Pi_t^A$ out of $t = [\phi, \beta]$ it will be clear (see below) that

$$f(\varrho) = \mathfrak{P}^{-\nu_{\mathfrak{P}}(\beta)+1} \qquad \text{if } \beta \neq 0 \tag{2}$$

hence $f(\varrho) \cdot f(\psi_A)^{-1} = \mathfrak{P}^{-\nu_{\mathfrak{P}}(\beta)}$ because ψ is of conductor $\mathfrak{p}_F$. Moreover $A = M_m(D_d)$ implies

$$\mathfrak{A}/\mathfrak{P} \cong \big[M_s(k_D)\big]^r \tag{3}$$

where k_D is the residue field of the division algebra and $rs = m$, $\nu_{\mathfrak{P}}(\pi_F) = rd$. Therefore

$$\begin{aligned} (\mathfrak{A} : \mathfrak{P}) &= q^{drs^2} \\ (\mathfrak{A} : \mathfrak{P}^{-\nu_{\mathfrak{P}}(\beta)}) &= q^{-drs^2 \cdot \nu_{\mathfrak{P}}(\beta)} = q^{-N^2 \nu_F(\beta)} \end{aligned} \tag{4}$$

because $N^2 \nu_F(\beta) = (drs)^2 \nu_F(\beta) = dr \cdot s^2 \cdot \nu_{\mathfrak{P}}(\beta)$. Putting (4) into (1) we conclude

$$\varepsilon(s, \Pi_t^A, \psi) = q^{(\frac{1}{2}-s)\cdot(-N\nu_F(\beta))} \cdot W(\Pi_t, \psi)\,, \quad a(\Pi_t, \psi) = -N\nu_F(\beta)$$

In the case $\beta = 0$ the conductor of $\varrho \subset \Pi_t^A$ is $f(\varrho) = \mathfrak{P}$ because we have excluded $t = [\phi, 0]$ where ϕ is an unramified character. For $\varrho \subset \Pi_t^A$ this means that $\mathfrak{A}^* \not\subseteq \mathrm{Ker}\varrho$. Now $f(\varrho) = \mathfrak{P}$ implies $f(\varrho) \cdot f(\psi_A)^{-1} = \mathfrak{A}$, $N(f(\varrho) \cdot f(\psi_A)^{-1}) = 1$ and (1) turns into

$$\varepsilon(s, \Pi_t^A, \psi) = W(\Pi_t^A, \psi)\,, \quad a(\Pi_t^A, \psi) = 0$$

if $t = [\phi, 0]$. Thus we have seen part (i) of Theorems 4.1, 4.2, 4.3.

6. The proof of Theorem 4.1

Assume $t = [\phi, 0]$ and $\phi : K^* \to \mathbb{C}^*$ is a ramified tame character which is regular with respect to the unramified extension $K|F$. $f = [K : F]$ is a divisor of N if

$t \in \mathcal{T}_N^-$. If $A = M_m(D_d)$ we consider $F_d|F$ the unramified extension of degree d in $\bar{F}$ and we let

$$f' := f/(f,d) = [K : K \cap F_d]. \tag{1}$$

$f' = \mathrm{lcm}(f,d)/d$ divides $N/d = m$ and we introduce the numbers $e := N/f$, $e' := m/f'$.

Now let $\mathfrak{A}_{e'}$ be the standard principal order in A which is determined by the divisor e' of m. If $\mathfrak{P}$ is the Jacobson radical of $\mathfrak{A}_{e'}$ then:

$$(\mathfrak{A}_{e'}/\mathfrak{P})^* = \left[\mathrm{GL}_{f'}(k_D)\right]^{e'}, \tag{2}$$

and we use ϕ to distinguish a cuspidal representation of that group. Namely because of $f'd = \mathrm{lcm}(f,d)$ we have $K \subseteq F_{f'd}$ and we let $\chi := \phi \circ N_{F_{f'd}|K}$. This is a tame character of $F^*_{f'd}$ which is regular over F_d hence the "reduction" $\bar{\chi}$ is a character of $(k_D k_K)^*$ which is regular over k_D^*. Thus $\bar{\chi}$ determines (up to equivalence) a cuspidal representation $\sigma = \sigma_{\bar{\chi}}$ of $\mathrm{GL}_{f'}(k_D)$. The tensor power $\sigma^{\otimes e'}$ is a cuspidal representation of $[\mathrm{GL}_{f'}(k_D)]^{e'}$ with central character $\bar{\chi}^{e'}$ on k_D^*. Because $e' = N/f'd = [F_N : F_{f'd}]$ we see that $\bar{\chi}^{e'}$ and $\bar{\chi} \circ N_{F_N|F_{f'd}} = \bar{\phi} \circ N_{F_N|K}$ give rise to the same character of k_D^*. Hence using (2) we can inflate $\sigma^{\otimes e'}$ to a representation of $\mathfrak{A}^*_{e'}$ and then we can extend it to $F^* \cdot \mathfrak{A}^*_{e'}$ in such a way that on F^* the central character is $\phi \circ N_{F_N|K} = \phi^e|_{F^*}$. The resulting representation of $F^* \cdot \mathfrak{A}^*_{e'}$ we denote τ_ϕ.

6.1 Proposition. *For all discrete series representations* Π *of* A^* *which contain* $\tau_\phi|_{\mathfrak{A}^*_{e'}}$ *we get the* ε*-factor as described in* 4.1.

Proof. We are left to compute the root number

$$W(\Pi,\psi) := (-1)^{N-m} \cdot Nf(\varrho)^{-1/2} \cdot \tau(\check{\varrho},\psi) \tag{3}$$

where $\check{\varrho}$ denotes the contragredient of an irreducible representation ϱ of $\mathfrak{K}_{e'}$ such that $\Pi \supset \varrho \supset \tau_\phi|_{\mathfrak{A}^*_{e'}}$.

Because of $f(\varrho) = \mathfrak{P}$ from 2.(2) we conclude that $\tau(\varrho,\psi)$ is the value of the scalar operator $\sum \varrho(u)\psi_A(u)$ where $\psi_A = \psi \circ \mathrm{Trd}_{A|F}$ and where the sum is over all $u \in (\mathfrak{A}_{e'}/\mathfrak{P})^*$. Because of $\varrho \supset \tau_\phi|_{\mathfrak{A}^*_{e'}}$ it is obvious that $\sum \tau_\phi(u)\psi_A(u)$ is a scalar operator of the same value. Now we consider

$$[\mathrm{GL}_{f'}(\mathfrak{O}_D)]^{e'} \twoheadrightarrow (\mathfrak{A}_{e'}/\mathfrak{P})^* \cong [\mathrm{GL}_{f'}(k_D)]^{e'}$$

thinking of $[\mathrm{GL}_{f'}(\mathfrak{O}_D)]^{e'}$ to be diagonally embedded into $\mathfrak{A}^*_{e'}$. Then we can represent $u \in (\mathfrak{A}_{e'}/\mathfrak{P})^*$ by $(x_1,\ldots,x_{e'}) \in [\mathrm{GL}_{f'}(\mathfrak{O}_D)]^{e'}$. Now we use:

$$\begin{aligned}
\psi_A(u) &= \bar{\psi} \circ \mathrm{Tr}_{k_D|k_F}(\mathrm{Tr}\bar{x}_1 + \cdots + \mathrm{Tr}\bar{x}_{e'}) \\
\tau_\phi(u) &= \sigma^{\otimes e'}(u) = \sigma(\bar{x}_1) \otimes \cdots \otimes \sigma(\bar{x}_{e'}) \\
\tau(\varrho,\psi) \cdot I &= \sum_u \tau_\phi(u)\psi_A(u).
\end{aligned}$$

Taking the trace of the last operator equation we find:

$$(\dim\sigma)^{e'} \cdot \tau(\varrho,\psi) = \sum_{(\bar{x}_1,\dots\bar{x}_{e'})} \chi_\sigma(\bar{x}_1)\cdots\chi_\sigma(\bar{x}_{e'}) \cdot \bar{\psi} \circ \mathrm{Tr}_{k_D|k_F}(\mathrm{Tr}\bar{x}_1 + \cdots + \mathrm{Tr}\bar{x}_{e'})$$

$$= \sum_{(\bar{x}_1,\dots\bar{x}_{e'})} \prod_{i=1}^{e'} [\chi_\sigma(\bar{x}_i)\bar{\psi} \circ \mathrm{Tr}_{k_D|k_F}(\bar{x}_i)] = \left[\sum_{\bar{x}} \chi_\sigma(\bar{x})\bar{\psi}_{k_D} \circ \mathrm{Tr}(\bar{x})\right]^{e'}$$

where the sum is over all $(\bar{x}_1,\dots\bar{x}_{e'}) \in [\mathrm{GL}_{f'}(k_D)]^{e'}$ and $\bar{x} \in [\mathrm{GL}_{f'}(k_D)]$ respectively and where we have used the notation $\bar{\psi}_{k_D} = \bar{\psi} \circ \mathrm{Tr}_{k_D|k_F}$. $\sum_{\bar{x}} \sigma(\bar{x})\bar{\psi}_{k_D} \circ \mathrm{Tr}(\bar{x})$ is a scalar operator of value say $\tau(\sigma,\bar{\psi}_{k_D})$. Hence we conclude

$$(\dim\sigma)^{e'} \cdot \tau(\varrho,\psi) = \left[\dim\sigma \cdot \tau(\sigma,\bar{\psi}_{k_D})\right]^{e'}$$
$$\tau(\varrho,\psi) = \tau(\sigma,\bar{\psi}_{k_D})^{e'} .$$

Now we apply Kondo's formula (see Macdonald [Mac]) to the cuspidal representation σ of $\mathrm{GL}_{f'}(k_D)$. It says that $\tau(\sigma,\bar{\psi}_{k_D}) = (-1)^{f'} \cdot q^{d(f'^2-f')/2} \cdot [-\tau(\bar{\chi},\bar{\psi}_{k_D k_K})]$ where $\tau(\bar{\chi},\bar{\psi}_{k_D k_K}) = \sum_{x\in(k_D k_K)^*} \bar{\chi}(x)\bar{\psi}_{k_D k_K}(x)$. Because of $e'f'd = md = N$, the Hasse-Davenport formula yields:

$$\left[-\tau(\bar{\chi},\bar{\psi}_{k_D k_K})\right]^{e'} = -\tau(\bar{\phi} \circ N_{k_N|k_K}, \bar{\psi}_{k_N})$$

hence

$$\tau(\varrho,\psi) = \tau(\sigma,\bar{\psi}_{k_D})^{e'} = (-1)^m q^{N(f'-1)/2}(-1)\tau(\bar{\phi} \circ N_{k_N|k_K}, \bar{\psi}_{k_N})$$

Now we insert this formula into (3) using $f(\varrho) = \mathfrak{P}$ hence $Nf(\varrho) = (\mathfrak{A} : \mathfrak{P}) = q^{e'df'^2} = q^{Nf'}$. Then we obtain the result. $\square$

7. The root number in case $\beta \neq 0$ – a first reduction

Now we consider $t = [\phi,\beta,\lambda_\beta^A] \to \Pi_t^A \in (A^*)^\wedge$ in the case $\beta \neq 0$, and we want to compute the root number $W(\Pi_t^A,\psi)$. We recall some basic facts concerning the construction of Π_t^A. The parameter t contains the fields $K \supset E = F(\beta)$ (up to F-isomorphism), and we fix an embedding of these fields into A. For the centralizer of E we obtain:

> $A_E = M_{m_0}(D_{d_0})$ where $D_{d_0}|E$ is a central division algebra of index $d_0 = d/(d,[E:F])$, and $m_0 = (m, N/[E:F])$ because $m_0 d_0 = N/[E:F]$, $md = N$.

Take a maximal extension $L'|E$ in A_E such that $f_{L'|E} = [K:E]/([K:E],d_0)$. Let $\mathfrak{A}_{L'|F}$ be the principal order in A which is normalized by L'^*. $t = [\phi,\beta,\lambda_\beta^A]$ determines an irreducible representation $\pi_t^\#$ of the group $E^* \cdot \mathfrak{A}_{L'|F}^*$ and there are precisely $N_t = N/[K:F]$ discrete series representations of A^* which contain $\pi_t^\#$. They only differ by unramified character twists, and Π_t^A is one of these. The root number $W(\Pi_t^A, \psi)$ will be independent of that choice. To compute it, we start from the formulas 2.(1), (2) where we choose an irreducible representation ϱ of $\mathfrak{K}_{L'|F}$ such that $\Pi_t^A \supset \varrho \supset \pi_t^{\#}$. In 2.(2) the sum is over $\mathfrak{A}^* = \mathfrak{A}^*_{L'|F}$ which is contained in $E^*\mathfrak{A}^*$. Because $T(\check{\varrho}, \psi)$ is a scalar operator we will get the same value if we replace ϱ by an irreducible component of the restriction $\varrho_{\mathfrak{A}^*}$. Hence $\tau(\check{\varrho}, \psi)$ is also the value of the scalar operator

$$T(\check{\pi}_t^{\#}, \psi) := \sum_{u \in (\mathfrak{A}/f)^*} \check{\pi}_t^{\#}(c^{-1}u)\psi_A(c^{-1}u)\,.$$

The conductor of ϱ and $\pi_t^{\#}$ is the same power of $\mathfrak{P} = \mathrm{Jac}(\mathfrak{A})$ namely

$$(1) \qquad f(\pi_t^{\#}) = \mathfrak{P}^{j+1} \qquad \text{where } j = -\nu_{\mathfrak{P}}(\beta).$$

We have $\beta \in E^* \subseteq \mathfrak{K} = \mathfrak{K}_{L'|F}$ and more precisely β in A is uniquely determined as an element of the fundamental domain $\Delta^-_{L'|F}$ for $\mathrm{Ad}\mathfrak{K}\backslash A(e', f')/\mathfrak{A}_{L'|F}$, where $e' = e_{L'|F}$, $f' = f_{L'|F}$. Because of (1) we may choose $c^{-1} = \beta$ such that:

$$(2) \qquad \tau(\check{\varrho}, \psi) = \text{value of the scalar operator } \sum_{u \in (\mathfrak{A}/f)^*} \check{\pi}_t^{\#}(\beta u)\psi_A(\beta u).$$

Now we use $\pi_t^{\#} = \mathrm{Ind}(\tau_\phi \otimes \tilde{\pi}_\beta)$, where the induction is from $E^*\mathfrak{A}^*_{L'|E}J^1_{\beta,L'|F}$ onto $E^*\mathfrak{A}^*_{L'|F}$. We abbreviate $J^0 = \mathfrak{A}^*_{L'|E}J^1_{\beta,L'|F}$, $\Lambda = \tau_\phi \otimes \tilde{\pi}_\beta$. Note that Λ remains irreducible if it is restricted from E^*J^0 to J^0. Now the value of the scalar operator (2) is preserved under induction (see [Zi93]), hence

$$(3) \qquad \tau(\check{\varrho}, \psi) = \text{value of the scalar operator } \sum_{u \in J^0/1+f(\Lambda)} \check{\Lambda}(\beta u)\psi_A(\beta u).$$

We consider the principal unit subgroups $U^i = 1 + \mathfrak{P}^i$ in $\mathfrak{K} = \mathfrak{K}_{L'|F}$ and the induced subgroups $J^i = U^i \cap J^0$ for $i \geq 1$. The next information we use is that the restrictions Λ_{J^i} are isotypic representations. Moreover for $i = [j/2] + 1$ (see (1)) we have $U^i = J^i$ and

$$\Lambda_{U^i}(1+x) = \psi_A(\beta x) \cdot \mathbf{1}\,.$$

Therefore the same argument as in the proof of [T, §1, Proposition 1] can be applied to obtain

7.1 Proposition.

$$Nf(\Lambda)^{-1/2} \cdot \sum_u \check{\Lambda}(\beta u)\psi_A(\beta u) = \check{\Lambda}(\beta) \cdot \psi_A(\beta) \qquad \textit{if } 2 \nmid j(\Lambda)$$

$$= \check{\Lambda}(\beta) \cdot \psi_A(\beta) \cdot N\mathfrak{P}^{-1/2} \left[\sum_x \check{\Lambda}(1+x)\psi_A(\beta x)\right] \qquad \textit{if } 2 | j(\Lambda)$$

where $j(\Lambda) = j = -\nu_{\mathfrak{P}}(\beta)$ *and where the sum is over* $x \in \mathfrak{P}^{j/2}/\mathfrak{P}^{j/2+1}$.

Note that this formula fits into 2.(1) because of (3) and $f(\varrho) = f(\pi_t^{\#}) = f(\Lambda)$. In the following we shall derive a formula in the more delicate case $2|j$ which will be compatible with $2 \nmid j$ such that the final statement is independent of the parity of j.

8. Further reduction in the case $2|j$

We write the right hand side of 7.1 as the product of two operators, namely:

$$D_1 = \check{\Lambda}(\beta), \quad D_2 = \psi_A(\beta) \cdot N\mathfrak{P}^{-1/2}\left[\sum_x \check{\Lambda}(1+x)\psi_A(\beta x)\right].$$

Because $D_1 D_2 = \mu \cdot \mathbf{1}$ is a scalar operator and because D_1 is invertible, we get

$$\text{(1)} \qquad \operatorname{tr}(D_2) = \mu \cdot \operatorname{tr}(D_1^{-1}).$$

Moreover $\operatorname{tr}(D_1^{-1}) = \chi_\Lambda(\beta)$ is the character value of β with respect to $\Lambda = \tau_\phi \otimes \tilde{\pi}_\beta$. We recall the construction of $\tilde{\pi}_\beta \in (J_{\beta,L'|F})^\wedge$, namely: $\tilde{\pi}_\beta|_{J^1_\beta} = (J^1_\beta, H^1_\beta, \theta_\beta)$ is a Heisenberg representation and $\tilde{\pi}_\beta|_{H_\beta} = \theta_\beta \otimes W_\beta$, where $W_\beta = W(J^1_\beta/H^1_\beta, X_\beta)$ is the Weil representation of $H_\beta/H^1_\beta \xleftarrow{\sim} \mathfrak{K}_{L'|E}/U^1(\mathfrak{K}_{L'|E})$ corresponding to the pair $(J^1_\beta/H^1_\beta, X_\beta)$. Therefore by a result of R. Howe [Ho] we have:

8.1 Proposition. $\tilde{\pi}_\beta \otimes \tilde{\pi}_\beta^\vee = Ind_{H_\beta \uparrow J_\beta}(1)$ $\square$.

For the character χ_β of $\tilde{\pi}_\beta$ this means $\chi_\beta(\beta) \cdot \overline{\chi_\beta(\beta)} \neq 0$ because $\beta \in E^* \subseteq H_\beta$. Now we consider $\Lambda = \tau_\phi \otimes \tilde{\pi}_\beta$ of $E^* J^0$. τ_ϕ is an irreducible representation of the factor $E^* J^0/J^1 \xleftarrow{\sim} E^*\mathfrak{A}^*_{L'|E}/1+\mathfrak{P}_{L'|E}$. Because $\beta \in E^*$ is in the center of that group we see that $\tau_\phi(\beta)$ is a scalar operator. As we know, the central character of τ_ϕ is $\phi^{N_t}|_{E^*}$ where $N_t = N/[K:F] = \frac{N/[E:F]}{[K:E]}$. Further we make use of $\tilde{\pi}_\beta|_{H_\beta} = \theta_\beta \otimes W_\beta$, and $\theta_\beta(\beta) = \lambda_\beta \circ \mathrm{Nrd}_{A_E|E}(\beta) = \lambda_\beta(\beta)^{N/[E:F]} = 1$ because of 3.6(iv), hence $\chi_\beta(\beta) = \chi_{W_\beta}(\beta)$ and $\chi_\Lambda(\beta) = \chi_\beta(\beta) \cdot \chi_{\tau_\phi}(\beta) = \chi_{W_\beta}(\beta) \cdot \phi^{N_t}(\beta) \cdot \dim\tau_\phi \neq 0$. Now the equation (1) yields

$$\text{(2)} \quad \psi_A(\beta) \cdot N\mathfrak{P}^{-1/2}\left[\sum_x \overline{\chi_\Lambda(1+x)}\psi_A(\beta x)\right] = \mu \cdot \chi_\Lambda(\beta) =_{7.1}$$
$$= Nf(\Lambda)^{-1/2} \cdot \tau(\check{\Lambda}, \psi) \cdot \chi_{W_\beta}(\beta) \cdot \phi^{N_t}(\beta) \cdot \dim\tau_\phi$$

Because $\Lambda = \tau_\phi \otimes \tilde{\pi}_\beta$ gives $\Lambda|_{J^1} = \dim\tau_\phi \cdot \tilde{\pi}_\beta|_{J^1}$ we can substitute $\chi_\Lambda(1+x) = \dim\tau_\phi \cdot \chi_\beta(1+x)$, and by a division (2) turns into:

$$\text{(3)} \quad Nf(\Lambda)^{-1/2} \cdot \tau(\check{\Lambda}, \psi) = \frac{\psi_A(\beta)}{\phi^{N_t}(\beta) \cdot \chi_{W_\beta}(\beta)} \cdot N\mathfrak{P}^{-1/2}\left[\sum_x \overline{\chi_\beta(1+x)}\psi_A(\beta x)\right]$$

The sum is over $x \in \mathfrak{P}^{j/2}/\mathfrak{P}^{j/2+1}$. We use that $\tilde{\pi}_\beta|_{J^1} = (J^1, H^1, \theta_\beta)$ is a Heisenberg representation and that $\tilde{\pi}_\beta|_{J^{j/2}}$ is a multiple of the Heisenberg representation $(J^{j/2}, H^{j/2}, \mathrm{res}\theta_\beta)$. From the general expression $H^{[v/2]+1} = 1 + \sum_{i \geq v}(\mathfrak{P}^{[i/2]+1} \cap A_{-i})$, where A_{-i} is the centralizer of the approximation β_{-i} of β, we see that $H^{j/2} = H^{[\frac{j-1}{2}]+1} = 1 + \mathfrak{P}^{j/2} \cap A_b + \mathfrak{P}^{j/2+1}$ where $b = \beta_{-j+1}$ is the first nontrivial approximation of β. Similar: $J^{[\frac{v+1}{2}]} = 1 + \sum_{i \geq v}(\mathfrak{P}^{[\frac{i+1}{2}]} \cap A_{-i})$ i. e. $J^{j/2} = J^{[\frac{j+1}{2}]} =$

$1+\mathfrak{P}^{j/2}$ because $\beta_{-j}=0$, $A_{-j}=A$. Therefore the character of the Heisenberg representation $\eta=(J^{j/2},H^{j/2},\mathrm{res}\theta_\beta)$ is

$$\chi_\eta(1+x)=\begin{cases}0 & \text{if } x\in\mathfrak{P}^{j/2}-A_b'\\ \dim\eta\cdot\theta_\beta(1+x) & \text{if } x\in A_b'.\end{cases}$$

where we have introduced the notation $A_b'=\mathfrak{P}^{j/2}\cap A_b+\mathfrak{P}^{j/2+1}$. Now $\tilde{\pi}_\beta|_{J^{j/2}}$ being a multiple of η, we conclude

$$\chi_\beta(1+x)=\begin{cases}0 & \text{if } x\in\mathfrak{P}^{j/2}-A_b'\\ \dim\tilde{\pi}_\beta\cdot\theta_\beta(1+x) & \text{if } x\in A_b'.\end{cases}$$

We put this into (3) and note that $\dim\tilde{\pi}_\beta=\dim W_\beta$. Then we obtain

(4)

$$Nf(\Lambda)^{-1/2}\cdot\tau(\check{\Lambda},\psi)=\frac{\psi_A(\beta)\cdot\dim W_\beta}{\phi^{N_t}(\beta)\cdot\chi_{W_\beta}(\beta)}\cdot N\mathfrak{P}^{-1/2}\cdot\sum_{x\in A_b'/\mathfrak{P}^{j/2+1}}\overline{\theta_\beta(1+x)}\cdot\psi_A(\beta x)\,.$$

Moreover we know that $\nu_{\mathfrak{P}}(\beta-b)\geq -j+1$ implies $(\theta_\beta\theta_b^{-1})(1+x)=\psi_A\big((\beta-b)x\big)$ for $1+x\in H_\beta^1\cap U^{[\frac{j-1}{2}]+1}$. But the last intersection is nothing else than $1+A_b'$. Therefore in (4) we may replace $\theta_\beta(1+x)=\theta_b(1+x)\cdot\psi_A\big((\beta-b)x\big)$. Now $\theta_b\in X(H_{b,L'|F})$ has been chosen in such a way that $\theta_b=\tilde{\lambda}_b:=\lambda_b\circ\mathrm{Nrd}_{A_b|F(b)}$ on $\mathfrak{K}_{L'|F}\cap A_b\subset H_{b,L'|F}$. Because the sum in (4) is over $A_b'/\mathfrak{P}^{j/2+1}=A_b\cap\mathfrak{P}^{j/2}/A_b\cap\mathfrak{P}^{j/2+1}$ we can replace θ_b by $\tilde{\lambda}_b$ and (4) turns into:

8.2 Proposition. *If $2|j$ then:*

$$Nf(\Lambda)^{-1/2}\cdot\tau(\check{\Lambda},\psi)=\frac{\psi_A(\beta)\cdot dimW_\beta}{\phi^{N_t}(\beta)\cdot\chi_{W_\beta}(\beta)}\cdot N\mathfrak{P}^{-1/2}\cdot\sum\overline{\tilde{\lambda}_b(1+x)}\cdot\psi_A(bx)\,,$$

where the sum is over $x\in A_b\cap\mathfrak{P}^{j/2}/A_b\cap\mathfrak{P}^{j/2+1}$.

9. Handling the Weil representation

The aim of this section is to replace $\dim W_\beta/\chi_{W_\beta}(\beta)$ in 8.2 by $\dim W_b/\chi_{W_b}(b)$ where $b=\beta_{-j+1}$ is the first nontrivial approximation of β. We assume $b\neq\beta$ because otherwise there is nothing to show. $W_\beta=W(J_\beta^1/H_\beta^1,X_\beta)$ is the Weil representation of $\mathrm{Sp}(X_\beta)\leftarrow\mathfrak{K}_\beta/U^1(\mathfrak{K}_\beta)$ which comes from the $\mathbb{F}_p$-vector space J_β^1/H_β^1 provided with the nondegenerate alternating character X_β. Now for W_β we have the basic formula [Zi]:

(1) $$W_\beta(\bar{x})=W_\gamma\big(\iota(\bar{x})\big)\otimes W_{\beta-\gamma}(\bar{x})\,,$$

where ι denotes the following combination of maps:

$$\iota:\mathfrak{K}_\beta/U^1(\mathfrak{K}_\beta)\xrightarrow{\sim}\mathfrak{K}_\beta U^1(\mathfrak{K})/U^1(\mathfrak{K})\hookrightarrow\mathfrak{K}_\gamma U^1(\mathfrak{K})/U^1(\mathfrak{K})\xleftarrow{\sim}\mathfrak{K}_\gamma/U^1(\mathfrak{K}_\gamma)\,,$$

$\gamma=\beta_{-s}$ denoting the last approximation of $\beta\in\Delta_{L'|F}^-\subseteq A(e',f')^\times\subseteq\mathfrak{K}_{L'|F}$. Note the assumption $\gamma\neq 0$, i. e. approximating β we have more than one jump. Hence

$\nu_{\mathfrak{P}}(\beta) = \nu_{\mathfrak{P}}(\gamma) \leq -s-1$, $\gamma-\beta \in \mathfrak{P}^{-s} \subseteq \gamma\mathfrak{P}$ and we see that $\gamma^{-1}\beta \in 1+\mathfrak{P} = U^1(\mathfrak{K})$, which implies $\iota(\bar{\beta}) = \bar{\gamma}$. Therefore from (1) we conclude

$$W_\beta(\beta) = W_\gamma(\gamma) \otimes W_{\beta-\gamma}(\beta)\,. \tag{2}$$

Now we consider the orthogonal with respect to X_β decomposition:

$$J^1_\beta/H^1_\beta \xleftarrow{\sim} J^{[\frac{s+1}{2}]}/H_\beta^{[\frac{s+1}{2}]} = J^{[s/2]+1}\cdot H_\beta^{[\frac{s+1}{2}]}/H_\beta^{[\frac{s+1}{2}]} \oplus H_\gamma^{[\frac{s+1}{2}]}/H_\beta^{[\frac{s+1}{2}]}\,.$$

Note that the second term of the orthogonal decomposition is normalized by $\mathfrak{K}_\beta$ because the first term is normalized and the alternating character X_β is invariant with respect to conjugation by $\mathfrak{K}_\beta$.

9.1 Lemma. *The adjoint action of β on $H_\gamma^{[\frac{s+1}{2}]}/H_\beta^{[\frac{s+1}{2}]}$ is trivial.*

Proof. Consider $H_\gamma^{[\frac{s+1}{2}]} \supseteq H_\beta^{[\frac{s+1}{2}]} \supseteq H^{[s/2]+1}$ where omitting the index means that for γ and β we have the same group. Now $H_\beta = H^{[s/2]+1}\mathfrak{K}_\beta$ and $H_\gamma = H^{[s'/2]+1}\mathfrak{K}_\gamma$ where $-s' < -s$ is the last jump in the approximation of $\gamma = \beta_{-s}$. This implies $H_\gamma = H^{[s/2]+1}\mathfrak{K}_\gamma$ and $H_\gamma^{[\frac{s+1}{2}]} = H^{[s/2]+1}\cdot(\mathfrak{K}_\gamma \cap H_\gamma^{[\frac{s+1}{2}]})$. We conclude that the adjoint action of γ on $H_\gamma^{[\frac{s+1}{2}]}/H^{[s/2]+1} = \mathfrak{K}_\gamma \cap H_\gamma^{[\frac{s+1}{2}]}/\mathfrak{K}_\gamma \cap H_\gamma^{[s/2]+1}$ is trivial because $\mathfrak{K}_\gamma$ is in the centralizer of γ. On the quotient under consideration the groups $\mathfrak{K}_\gamma$ and $\mathfrak{K}_\beta$ both act by conjugation. Especially we consider the action of β. Because γ acts trivially we see:

$$\beta x \beta^{-1} \equiv \beta\gamma^{-1}\cdot x\cdot(\beta\gamma^{-1})^{-1} \qquad \bmod H^{[s/2]+1} \qquad \text{for all } x \in H_\gamma^{[\frac{s+1}{2}]}. \tag{3}$$

But $\gamma \equiv \beta \bmod \mathfrak{P}^{-s}$, $\gamma\beta^{-1} \equiv 1 \bmod \beta^{-1}\mathfrak{P}^{-s} = \mathfrak{P}^{j-s}$. The assumption $\gamma = \beta_{-s} \neq 0$ again implies $j - s \geq 1$, i. e. $\gamma\beta^{-1}$ is a principal unit. Therefore the adjoint action of $\gamma\beta^{-1}$ on $H_\gamma^{[\frac{s+1}{2}]}/H^{[s/2]+1}$ is trivial, and from (3) the asertion follows. $\square$

Now we recall that the Weil representation $W_{\beta-\gamma}$ occuring in (1) was $W_{\beta-\gamma} = W(H_\gamma^{[\frac{s+1}{2}]}/H_\beta^{[\frac{s+1}{2}]}, X_\beta)$. Therefore from (2) and the Lemma we conclude the following relation of character values:

$$\chi_{W_\beta}(\beta) = \chi_{W_\gamma}(\gamma)\cdot \dim W_{\beta-\gamma}\,.$$

Together with $\dim W_\beta = \dim W_\gamma \cdot \dim W_{\beta-\gamma}$ we obtain

$$\frac{\dim W_\beta}{\chi_{W_\beta}(\beta)} = \frac{\dim W_\gamma}{\chi_{W_\gamma}(\gamma)}$$

If the last approximation of γ is nontrivial we can repeat the argument with γ instead of β, and iterating this process we arrive at

$$\frac{\dim W_\beta}{\chi_{W_\beta}(\beta)} = \frac{\dim W_b}{\chi_{W_b}(b)} \tag{4}$$

where $b = \beta_{-j+1}$ is the first nontrivial approximation of $\beta \in \Delta^-_{L'|F}$. We put this into 8.2 and we use two other small remarks, namely:

(5) $$\psi_A(\beta) = \psi \circ \mathrm{Trd}_{A|F}(\beta) = \psi \circ \mathrm{Tr}_{K|F} \circ \mathrm{Trd}_{A_K|K}(\beta) = \psi_K(\beta)^{N_t}$$

because $\beta \in K$ and $A_K|K$ is of reduced degree $N_t = N/[K:F]$,

(6) $$(\dim W_b)^2 \cdot (A_b \cap \mathfrak{P}^{j/2} : A_b \cap \mathfrak{P}^{j/2+1}) = N\mathfrak{P}$$

which is obtained from the formulas following 8.(3) for b instead of β:

$$J_b^1 = 1 + \mathfrak{P} \cap A_b + \mathfrak{P}^{j/2}, \quad H_b^1 = 1 + \mathfrak{P} \cap A_b + \mathfrak{P}^{j/2+1}, \quad \text{hence}$$
$$(\dim W_b)^2 = (J_b^1 : H_b^1) = (\mathfrak{P}^{j/2} : \mathfrak{P}^{j/2} \cap A_b + \mathfrak{P}^{j/2+1}).$$

Then from 2.(1), 7.(3), 8.2 together with (4), (5), (6) we obtain:

9.2 Proposition. *If $t = [\phi, \beta, \lambda_\beta^A]$ and $2|j\ (= \nu_{\mathfrak{P}}(\beta))$ then:*

$$W(\Pi_t^A, \psi) = (-1)^{N-m} \left[\frac{\psi_K(\beta)}{\phi(\beta)}\right]^{N_t} \cdot \chi_{W_b}(b)^{-1} \cdot M^{-1/2} \cdot S$$
$$\textit{where} \qquad M := (A_b \cap \mathfrak{P}^{j/2} : A_b \cap \mathfrak{P}^{j/2+1})$$
$$S := \sum_{x \in A_b \cap \mathfrak{P}^{j/2}/A_b \cap \mathfrak{P}^{j/2+1}} \overline{\tilde{\lambda}_b(1+x)} \cdot \psi_A(bx).$$

Our next aim is to simplify the expression for S which requires to split into the subcases where the extension $F(b)|F$ is tamely ramified or not.

10. The case when the first approximation is tame

$\tilde{\lambda}_b$ is a character of A_b^* and of $\mathfrak{K}_b := \mathfrak{K}_{L'|F} \cap A_b$. Moreover for the centralizer A_b we obtain:

> $A_b \cong M_{m_b}(D_{d_b})$, where $m_b d_b = N_b := N/[F(b):F]$, $m_b = (m, N_b)$ and where $D_{d_b}|F(b)$ is a central division algebra of index $d_b = d/([F(b):F], d)$.

Our main observation is that $\psi_\beta = \psi \circ \mathrm{Tr}_{F(b)|F}$ is an additive character of $F(b)$ of conductor $\mathfrak{p} = \mathfrak{p}_{F(b)}$ because we assume that $F(b)|F$ is tamely ramified. The one dimensional representation $\tilde{\lambda}_b$ of A_b^* is not discrete series, unless A_b is a division algebra. Nevertheless we define $W(\tilde{\lambda}_b, \psi_b)$ as in 2.(1), (2) with respect to the central simple algebra $A_b|F(b)$ and the maximal compact mod center subgroup $\mathfrak{K} = \mathfrak{K}_b$ in A_b:

(1) $$W(\tilde{\lambda}_b, \psi_b) := (-1)^{N_b - m_b} \cdot Nf(\tilde{\lambda}_b)^{-1/2} \cdot \tau(\tilde{\lambda}_b^{-1}, \psi_b).$$

Note that we have $\Pi = \varrho = \tilde{\lambda}_b$ because $\tilde{\lambda}_b$ is of dimension one. We apply Proposition 7.1 to the character $\tilde{\lambda}_b$ of $\mathfrak{K}_b$ (instead of the irreducible representation Λ of $E^* J_\beta^0$) and we use $\tilde{\lambda}_b(b) = \lambda_b(b)^{N_b} = 1$ (see 3.6(iv)). Then we obtain:

(2) $$Nf(\tilde{\lambda}_b)^{-1/2} \cdot \tau(\tilde{\lambda}_b^{-1}, \psi_b) = \psi_A(b) \cdot M^{-1/2} \cdot S$$

where the notations are as in 9.2. Note that $M = S = 1$ if $\mathfrak{P}^{j/2} \cap A_b = \mathfrak{P}^{j/2+1} \cap A_b$. Now we can apply [BF85, 2.8.13 (ii)] for the central simple algebra $A_b|F(b)$, where the group is $G = \mathfrak{K}_b$. This yields

$$W(\tilde{\lambda}_b, \psi_b) = W(\lambda_b, \psi_b)^{N_b}, \tag{3}$$

where $W(\lambda_b, \psi_b)$ is Tate's root number for the character λ_b of $F(b)^*$. We note that $\lambda_b(U^1_{F(b)}) \neq 1$ because $\nu_{\mathfrak{P}}(b) = \nu_{\mathfrak{P}}(\beta) = -j < 0$. From (1), (2), (3) and $\psi_A(b) = \psi_b(b)^{N_b}$ we deduce:

$$\begin{aligned} M^{-1/2}S &= (-1)^{N_b - m_b} W(\tilde{\lambda}_b, \psi_b)\psi_A(-b) = \\ &= (-1)^{N_b - m_b}[W(\lambda_b, \psi_b)\psi_b(-b)]^{N_b}. \end{aligned} \tag{4}$$

On the other hand the root number computation in [T] and the fact $\lambda_b(b) = 1$ imply:

(5)

$$\begin{aligned} W(\lambda_b, \psi_b) \cdot \psi_b(-b) &= 1 \qquad \text{if } j_0 = -\nu(b) \text{ is odd} \\ &= (N\mathfrak{p})^{-1/2} \cdot \sum_{x \in \mathfrak{p}^{j_0/2}/\mathfrak{p}^{j_0/2+1}} \overline{\lambda_b(1+x)}\psi_b(bx) \quad \text{if } j_0 \text{ is even,} \end{aligned}$$

where $\mathfrak{p}$ denotes the prime ideal of $F(b)$ and $\nu = \nu_{\mathfrak{p}}$ is the corresponding exponent. We abbreviate the right hand side of (5) by $\delta(\lambda_b)$ such that from (4), (5) we obtain:

(6) $M^{-1/2}S = (-1)^{N_b - m_b} \cdot \delta(\lambda_b)^{N_b}$, if j is even and $F(b)|F$ is tamely ramified.

Remark. We know $|W(\lambda_b, \psi_b)|_{\mathbb{C}} = 1$ because λ_b is a unitary character. Hence from (5) we see that $\delta(\lambda_b)$ is a complex number of absolute value 1.

11. The case if wild ramification occurs

11.1 Lemma. *If in the extension $F(b)|F$ wild ramification occurs, then*

$$1 + x \mapsto \overline{\tilde{\lambda}_b(1+x)} \cdot \psi_A(bx)$$

is a character on $1 + (A_b \cap \mathfrak{P}^{j/2})/1 + (A_b \cap \mathfrak{P}^{j/2+1})$.

Proof. We use the abbrevation $\omega(1+x) := \overline{\tilde{\lambda}_b(1+x)} \cdot \psi_A(bx)$. Then we find:

$$\frac{\omega(1+x+y+xy)}{\omega(1+x)\cdot\omega(1+y)} = \psi_A(bxy).$$

With the same notation as in 10. we may write $\psi_A(bxy) = \psi_b \circ \mathrm{Trd}_{A_b|F(b)}(bxy) = \psi_b(b \cdot \mathrm{Trd}_{A_b|F(b)}(xy))$ for $x, y \in A_b \cap \mathfrak{P}^{j/2}$. Because $A_b \cap \mathfrak{P}$ is the Jacobson radical of $A_b \cap \mathfrak{A}$ and $A_b \cap \mathfrak{P}^j$ is a certain power of $A_b \cap \mathfrak{P}$, we see that $\mathrm{Trd}_{A_b|F(b)}(A_b \cap \mathfrak{P}^j) =$

$F(b) \cap \mathfrak{P}^j$. But $b \in F(b) \cap \mathfrak{P}^{-j}$ such that $b \cdot \mathrm{Trd}_{A_b|F(b)}(xy) \in \mathfrak{o}_{F(b)}$ is an integer of $F(b)$ if $x, y \in A_b \cap \mathfrak{P}^{j/2}$.

Now we make use of our assumption that in $F(b)|F$ wild ramification occurs. For the differente of $F(b)|F$ this means

$$\mathcal{D}_{\mathfrak{o}_{F(b)}|\mathfrak{o}_F} = \mathfrak{p}^{e_b - 1 + \delta},$$

where $\mathfrak{p}$ is the prime ideal of $F(b)$, $e_b = e_{F(b)|F}$ and $\delta > 0$ is a natural number. As a consequence we obtain

$$\mathrm{Tr}_{F(b)|F}(\mathfrak{p}^{1-\delta}) = \mathfrak{p}_F, \qquad \mathrm{Tr}_{F(b)|F}(\mathfrak{o}_{F(b)}) \subsetneq \mathfrak{p}_F$$

such that $\psi_A(bxy) = \psi \circ \mathrm{Tr}_{F(b)|F}\big(b \cdot \mathrm{Trd}_{A_b|F(b)}(xy)\big) \equiv 1$ for $x, y \in A_b \cap \mathfrak{P}^{j/2}$. $\square$

Because of the Lemma S is a sum over the values of a character on $1 + (A_b \cap \mathfrak{P}^{j/2})/1 + (A_b \cap \mathfrak{P}^{j/2+1})$. Hence either $\omega(1+x) \equiv 1$ for $x \in A_b \cap \mathfrak{P}^{j/2}$ or $S = 0$ which is impossible because $W(\Pi_t^A, \psi) \neq 0$ (see 9.2). Therefore we conclude $S = (A_b \cap \mathfrak{P}^{j/2} : A_b \cap \mathfrak{P}^{j/2+1})$, hence

(1) $\quad M^{-1/2} S = M^{1/2}$ if j is even and if wild ramification occurs in $F(b)|F$.

Further we note that the discrete series representation Π_t^A has the central character

$$\omega(x) = [\phi \cdot \lambda_\beta^A \circ N_{K|F(\beta)}]^{N_t}(x) \qquad \text{for } x \in F^*,$$

[Zi]. Therefore ω is unitary iff ϕ is. On the other hand it is well known that Π_t^A is unitary iff its central character ω is unitary and this implies $|W(\Pi_t^A, \psi)|_{\mathbb{C}} = 1$. Now putting (1) into 9.2 and assuming ϕ unitary we conclude

(2) $\quad M^{-1/2} S = M^{1/2} = |\chi_{W_b}(b)|$ if j is even and if wild ramification occurs in $F(b)|F$.

12. The case where j is odd

Putting 10.(6) and 11.(2) into 9.2 we get 4.2(ii) and 4.3(ii) respectively if $j = -\nu_{\mathfrak{P}}(\beta)$ is even. Finally we want to see how this matches with the case where j is odd. From the first equation in 7.1 we conclude that $\Lambda(\beta)$ is a scalar operator if j is odd, and

$$W(\Pi_t^A, \psi) = (-1)^{N-m}\{\Lambda(\beta)\}^{-1} \cdot \psi_A(\beta) \tag{1}$$

where $\{\Lambda(\beta)\}$ denotes the value of the scalar operataor. We recall $\Lambda = \tau_\phi \otimes \tilde{\pi}_\beta$, $\tau_\phi(\beta)$ is a scalar operator of value $\phi^{N_t}(\beta)$ (see the remarks following 8.1), $\tilde{\pi}_\beta|_{H_\beta} = \theta_\beta \otimes W_\beta$ and $\theta_\beta(\beta) = \tilde{\lambda}_\beta(\beta) = 1$. This implies that $W_\beta(\beta)$ is a scalar operator too, and:

$$\{\Lambda(\beta)\} = \phi^{N_t}(\beta) \cdot \{W_\beta(\beta)\}. \tag{2}$$

Moreover $\{W_\beta(\beta)\} = \frac{\chi_{W_\beta}(\beta)}{\dim W_\beta} = \frac{\chi_{W_b}(b)}{\dim W_b}$, where the second equation follows from section 9.

But now $W_b = W(J_b^1/H_b^1, X_b)$ is the unit representation because $j = -\nu_{\mathfrak{P}}(\beta) = -\nu_{\mathfrak{P}}(b)$ is odd, hence a similar computation as in 9.(6)ff. yields $J_b^1 = H_b^1 = 1 + (\mathfrak{P} \cap A_b) + \mathfrak{P}^{(j+1)/2}$. Therefore from (1), (2) we obtain:

12.1 Proposition. $W(\Pi_t^A,\psi)=(-1)^{N-m}\left[\frac{\psi_K(\beta)}{\phi(\beta)}\right]^{N_t}$ *if* $j=-\nu_{\mathfrak{P}}(\beta)$ *is odd.* □

This Proposition fits into 4.3(ii) which has been proved for j even, because W_b is the unit representation if j is odd. What we are left to show is that it also fits into 4.2(ii). We have defined $\delta(\lambda_b)$ as to be the right hand side of 10.(5). Because $\nu_{\mathfrak{P}}(\beta)=\nu_{\mathfrak{P}}(b)$ odd implies that $\nu(b)$ is odd, we can smoothly extend the definition of $\delta(\lambda_b)$ by $\delta(\lambda_b)=1$ if $\nu_{\mathfrak{P}}(b)$ is odd. Comparing 4.2(ii) and 10.1 it is now enough to verify that

$$(3)\qquad N_b\equiv(m,N_b)\mod 2\qquad \text{if } 2\nmid\nu_{\mathfrak{P}}(b).$$

Let e_b, f_b be the ramification exponent and the inertial degree of $F(b)|F$. We know that:

$$e_b=\text{denominator}\left(\frac{j}{rd}\right)=rd/(j,rd)\,,$$

where $r=r(\mathfrak{A})$ is the period of the order $\mathfrak{A}=\mathfrak{A}_{L'|F}$ which we have fixed in section 7. Namely $\nu_F(b)=\nu_F(\beta)=-j/rd$ is the unique jump of b (see 3.(7)). Because $N=md=srd$, this implies:

$$N_b=\frac{N}{f_b\cdot e_b}=\frac{s\cdot(j,rd)}{f_b}\,.$$

Furthermore $s=(m,f_{L'|F})$ is divisible by (m,f_b). Hence:

$$N_b\cdot f_b/(m,f_b)=\frac{s}{(m,f_b)}\cdot(j,rd)\,.$$

Assume that N_b is even. From the last equation we conclude that s is even because j is odd. Hence $m=rs$ and (m,N_b) are even numbers too. On the other hand (m,N_b) odd if N_b is odd, and we are done.

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